Title
The uniformization of the moduli space of principally polarized abelian 6-folds

Journal
Journal für die reine und angewandte Mathematik (Crelles Journal), 2020(761)

ISSN
0075-4102

Authors
Alexeev, Valery
Donagi, Ron
Farkas, Gavril
et al.

Publication Date
2020-04-01

DOI
10.1515/crelle-2018-0005

Peer reviewed

THE UNIFORMIZATION OF THE MODULI SPACE OF PRINCIPALLY POLARIZED ABELIAN 6-FOLDS

VALERY ALEXEEV, RON DONAGI, GAVRIL FARKAS, ELHAM IZADI, AND ANGELA ORTEGA

Contents

Introduction

It is a classical idea that general principally polarized abelian varieties (ppavs) and their moduli spaces are hard to understand, and that one can use algebraic curves to study some special classes, such as Jacobians and Prym varieties. This works particularly well in small dimension, where in this way one reduces the study of all abelian varieties to the rich and concrete theory of curves. For $g \leq 3$, a general ppav is a Jacobian, and the Torelli map $\mathcal{M}_g \to \mathcal{A}_g$ between the moduli spaces of curves and ppavs respectively, is birational. For $g \leq 5$, a general ppav is a Prym variety by a classical result of Wirtinger [Wir95]. In particular, for $g = 5$, this gives a uniformization of $\mathcal{A}_5$ by curves, as follows. We denote by $\mathcal{R}_g$ the Prym moduli space of pairs $[C, \eta]$ consisting of a smooth curve C of genus g and a non-trivial 2-torsion point $\eta \in \mathrm{Pic}^0(C)$; in particular, the projection $\mathcal{R}_g \to \mathcal{M}_g$ is a finite map of degree $2^{2g} - 1$. Then by Donagi-Smith [DS81], the Prym map $P\colon \mathcal{R}_6 \to \mathcal{A}_5$

is generically of degree 27, with fibers corresponding to a configuration of the 27 lines on a cubic surface.

The uniformization of $\mathcal{A}_g$ for $g \leq 5$ via the Prym map for $P : \mathcal{R}_{g+1} \to \mathcal{A}_g$ has been used, among many other things, to determine the birational type of $\mathcal{A}_g$ in this range. The Prym moduli space $\mathcal{R}_{g+1}$ is known to be unirational for $g \leq 5$, hence it follows that $\mathcal{A}_g$ is unirational for $g \leq 5$ as well. The case $g = 5$ was proved in [Don84]. Other proofs followed in [MM83] and [Ver84]. The first proof of the unirationality of $\mathcal{A}_4$ uses intermediate Jacobians of double solids and appeared in [Cle83]. It was shown in [ILS09b] that this proof actually implies the unirationality of $\mathcal{R}_5$.

The purpose of this paper is to prove a similar uniformization result for the moduli space $\mathcal{A}_6$ of principally polarized abelian varieties of dimension 6. The idea of this construction is due to Kanev [Kan89b] and it uses the rich geometry of the 27 lines on a cubic surface. Suppose $\pi \colon C \to \mathbb{P}^1$ is a cover of degree 27 whose monodromy group equals the Weyl group $W(E_6) \subset S_{27}$ of the E_6 lattice. In particular, each smooth fibre of π can be identified with the set of 27 lines on an abstract cubic surface and, by monodromy, this identification carries over from one fibre to another. Assume furthermore that π is branched over 24 points and that over each of them the local monodromy of π is given by a reflection in $W(E_6)$. A prominent example of such a covering $\pi : C \to \mathbb{P}^1$ is given by the *curve of lines* in the cubic surfaces of a Lefschetz pencil of hyperplane sections of a cubic threefold $X \subset \mathbb{P}^4$, see [Kan89a], as well as Section 1 of this paper. Since $\deg(X^\vee) = 24$, such a pencil contains precisely 24 singular cubic surfaces, each having exactly one node.

By the Hurwitz formula, we find that each such E_6-cover C has genus 46. Furthermore, C is endowed with a symmetric correspondence D of degree 10, compatible with the covering π and defined using the intersection form on a cubic surface. Precisely, a pair $(x, y) \in C \times C$ with $x \neq y$ and $\pi(x) = \pi(y)$ belongs to D if and only if the lines corresponding to the points x and y are incident. The correspondence D is disjoint from the diagonal of $C \times C$. The associated endomorphism $D : JC \to JC$ of the Jacobian of C satisfies the quadratic relation $(D - 1)(D + 5) = 0$. Using this, Kanev [Kan87] showed that the associated *Prym-Tyurin-Kanev variety*

$$PT(C, D) := \operatorname{Im}(D - 1) \subset JC$$

of this pair is a 6-dimensional ppav of exponent 6. Thus, if Θ_C denotes the Riemann theta divisor on JC, then $\Theta_{C|P(C,D)} \equiv 6 \cdot \Xi$, where Ξ is a principal polarization on $P(C, D)$.

Since the map π has 24 branch points corresponding to choosing 24 roots in E_6 specifying the local monodromy at each branch point, the Hurwitz scheme Hur parameterizing E_6-covers $\pi \colon C \to \mathbb{P}^1$ as above is 21-dimensional (and also irreducible, see [Kan06]). The geometric construction described above induces the *Prym-Tyurin-Kanev* map

$$PT\colon \operatorname{Hur} \to \mathcal{A}_6$$

between two moduli spaces of the same dimension. It is tempting to conjecture that this map is generically finite. The following theorem answers a conjecture raised by Kanev in his lectures given in 1987 and thereabouts (see also [LR08, Remark 5.5]):

Theorem 0.1. *The Prym-Tyurin-Kanev map $PT : \operatorname{Hur} \to \mathcal{A}_6$ is generically finite. It follows that the general principally polarized abelian variety of dimension 6 is a Prym-Tyurin variety of exponent 6 corresponding to a E_6-cover $C \to \mathbb{P}^1$.*

This result, which is the main achievement of this paper, gives a structure theorem for general abelian varieties of dimension 6 and offers a uniformization for $\mathcal{A}_6$ by curves with additional discrete data. Just like the classical Prym map $P : \mathcal{R}_6 \to \mathcal{A}_5$, it is expected that the Prym-Tyurin-Kanev map PT will open the way towards a systematic study of the geometry of $\mathcal{A}_6$. Potentially, this theorem also gives a way to determine the birational type of $\mathcal{A}_6$, which is a notorious open problem. Recall that $\mathcal{A}_g$ is a variety of general type for $g \geq 7$, see [Mum83] and [Tai82]; as already indicated, $\mathcal{A}_g$ is unirational for $g \leq 5$. It has been recently established in [FV14] that the boundary divisor $\partial\overline{\mathcal{A}}_6$ of the perfect cone compactification $\overline{\mathcal{A}}_6$ of $\mathcal{A}_6$ is unirational, which induces a lower bound on the slope of the effective cone of $\overline{\mathcal{A}}_6$.

The main idea of the proof of Theorem 0.1 is to study degenerations of Prym-Tyurin-Kanev varieties as the branch locus $(\mathbb{P}^1, p_1 + \cdots + p_{24})$ of the cover $\pi\colon C \to \mathbb{P}^1$ approaches a maximally degenerate point of $\overline{\mathcal{M}}_{0,24}$. The map PT becomes toroidal and its essential properties can be read off a map of fans. Then to show that PT is dominant, it is sufficient to show that the rays in the fan describing the image span a 21-dimensional vector space, i.e. that a certain (21×21)-matrix has full rank. This can be done by an explicit computation, once the general theory is in place. The theory of degenerations of Jacobians is well known, see e.g. [Ale04]. It was extended to Prym varieties by Alexeev-Birkenhake-Hulek in [ABH02]. One of the main goals of the present paper is a further extension of this theory to the case of Prym-Tyurin-Kanev varieties.

In view of the structure Theorem 0.1 it is of compelling interest to understand the birational geometry of the space $\mathcal{H}$ classifying E_6-covers $[\pi : C \to \mathbb{P}^1, p_1, \ldots, p_{24}]$ together with a labeling of the set of their 24 branch points. The space $\mathcal{H}$ admits a compactification $\overline{\mathcal{H}}$ by admissible coverings. Precisely, $\overline{\mathcal{H}}$ is the moduli space of *twisted stable maps* from curves of genus zero into the classifying stack $\mathcal{B}W(E_6)$, that is, the normalization of the stack of admissible covers with monodromy group $W(E_6)$ having as source a nodal curve of genus

46 and as target a stable 24-pointed curve of genus 0 (see Section 5 for details). One has a finite branched morphism

$$\mathfrak{b} : \overline{\mathcal{H}} \to \overline{\mathcal{M}}_{0,24}.$$

In Section 5, we show that $\overline{\mathcal{H}}$ is a variety of general type (Theorem 5.26). From the point of view of $\mathcal{A}_6$, it is more interesting to study the global geometry of the quotient space

$$\overline{\mathrm{Hur}} := \overline{\mathcal{H}}/S_{24},$$

compactifying the Hurwitz space Hur of E_6-covers (without a labeling of the branch points). The Prym-Tyurin-Kanev map PT extends to a regular morphism $PT^{\mathrm{Sat}} : \overline{\mathrm{Hur}} \to \overline{\mathcal{A}}_6^{\mathrm{Sat}}$ to the Satake compactification $\overline{\mathcal{A}}_6^{\mathrm{Sat}}$ of $\mathcal{A}_6$. We establish the following result on the birational geometry of $\overline{\mathrm{Hur}}$, which we regard as a compact master space for ppav of dimension 6:

Theorem 0.2. *The exists a boundary divisor E of $\overline{\mathrm{Hur}}$ that is contracted by the Prym-Tyurin map $PT : \overline{\mathrm{Hur}} \dashrightarrow \overline{\mathcal{A}}_6$, such that $K_{\overline{\mathrm{Hur}}} + E$ is a big divisor class.*

The proof of Theorem 0.2 is completed after numerous preliminaries at the end of Section 8. We stop short of stating that $\overline{\mathrm{Hur}}$ is of general type, for we do not prove that the singularities of $\overline{\mathrm{Hur}}$ do not impose *adjunction conditions*, that is, that pluricanonical forms defined on the smooth part of $\overline{\mathrm{Hur}}$ extend to any resolution of singularities of $\overline{\mathrm{Hur}}$. Such a result has been established for $\overline{\mathcal{M}}_g$ in [HM82] and for $\overline{\mathcal{R}}_g$ in [FL10], but not yet on any Hurwitz space.

In the course of proving Theorem 0.2, we establish numerous facts concerning the geometry of the space $\overline{\mathrm{Hur}}$. One of them is a surprising link between the splitting of the rank 46 Hodge bundle $\mathbb{E}$ on $\overline{\mathrm{Hur}}$ into Hodge eigenbundles and the Brill-Noether theory of E_6-covers, see Theorem 8.3. For a point $[\pi : C \to \mathbb{P}^1] \in \mathrm{Hur}$, we denote by $D : H^0(C, K_C) \to H^0(C, K_C)$ the map induced at the level of cotangent spaces by the Kanev endomorphism of JC and by

$$H^0(C, K_C) = H^0(C, K_C)^{(+1)} \oplus H^0(C, K_C)^{(-5)},$$

the decomposition into the $(+1)$ and the (-5)-eigenspaces of holomorphic differentials respectively. Setting $L := \pi^*(\mathcal{O}_{\mathbb{P}^1}(1)) \in W^1_{27}(C)$, for a general point $[\pi : C \to \mathbb{P}^1] \in \mathrm{Hur}$, we show that the following canonical identifications hold:

$$H^0(C, K_C)^{(+1)} = H^0(C, L) \otimes H^0(C, K_C \otimes L^\vee)$$

and

$$H^0(C, K_C)^{(-5)} = \left(\frac{H^0(C, L^{\otimes 2})}{\mathrm{Sym}^2 H^0(C, L)}\right)^\vee \otimes \bigwedge^2 H^0(C, L).$$

In particular, the $(+1)$-Hodge eigenbundle is fibrewise isomorphic to the image of the Petri map $\mu(L) : H^0(C, L) \otimes H^0(C, K_C \otimes L^\vee) \to H^0(C, K_C)$, whenever the Petri map is injective (which happens generically along Hur, see Theorem 8.2). The identifications above are instrumental in expressing in Section 8 the class of the (-5)-Hodge eigenbundle $\mathbb{E}^{(-5)}$ on a partial compactification $\mathcal{G}_{E_6}$ of Hur in terms of boundary divisors. The moduli space $\mathcal{G}_{E_6}$ differs from $\overline{\mathrm{Hur}}$ only along divisors that get contracted under the Prym-Tyurin-Kanev map. Note that the class $\lambda^{(-5)} = c_1(\mathbb{E}^{(-5)})$ is equal to the pull-back $PT^*(\lambda_1)$ of the Hodge class λ_1 on $\overline{\mathcal{A}}_6$. The explicit realization of the class $\lambda^{(-5)}$ is then used to establish positivity properties of the canonical class $K_{\overline{\mathrm{Hur}}}$.

We are also able to describe the ramification divisor of the Prym-Tyurin-Kanev map in terms of the geometry of the *Abel-Prym-Tyurin curve* $\varphi_{(-5)} = \varphi_{H^0(K_C)^{(-5)}} : C \to \mathbb{P}^5$ given by the linear system of (-5)-invariant holomorphic forms on C.

Theorem 0.3. *An E_6-cover $[\pi : C \to \mathbb{P}^1] \in \mathrm{Hur}$ such that the Petri map $\mu(L)$ is injective lies in the ramification divisor of the map $PT : \mathrm{Hur} \to \mathcal{A}_6$ if an only if the Abel-Prym-Tyurin curve $\varphi_{(-5)}(C) \subset \mathbb{P}^5$ lies on a quadric.*

The conclusion of Theorem 0.3 can be equivalently formulated as saying that the map

$$\mathrm{Sym}^2 H^0(C, K_C)^{(-5)} \longrightarrow H^0(C, K_C^{\otimes 2})$$

given by multiplication of sections is not injective. Note the striking similarity between this description of the ramification divisor of the Prym-Tyurin-Kanev map and that of the classical Prym map $P : \mathcal{R}_{g+1} \to \mathcal{A}_g$, see [Bea77]: A point $[C, \eta] \in \mathcal{R}_{g+1}$ lies in the ramification divisor of P if and only if the multiplication map for the Prym-canonical curve

$$\mathrm{Sym}^2 H^0(C, K_C \otimes \eta) \to H^0(C, K_C^{\otimes 2})$$

is not injective. An important difference must however be noted. While the general Prym-canonical map $\varphi_{K_C \otimes \eta} : C \to \mathbb{P}^{g-2}$ is an embedding when $g \geq 5$, the Abel-Prym-Tyurin curve $\varphi_{(-5)}(C) \subset \mathbb{P}^5$ has 24 points of multiplicity 6, corresponding to the ramification points lying over each branch point of the cover $\pi : C \to \mathbb{P}^1$ (see Section 9 below).

It is natural to ask how the E_6-Prym-Tyurin varieties considered in this paper generalize classical Prym varieties. It is well-known [Don92], [FL10] that the Prym variety of the *Wirtinger cover* of a 1-nodal curve of genus g is the Jacobian of its normalization. Thus, if $\Delta_0'' \subset \overline{\mathcal{R}}_{g+1}$ is the boundary divisor of such covers, then $P(\Delta_0'') = \overline{\mathcal{M}}_g \subset \overline{\mathcal{A}}_g$, thus showing that Jacobians arise as limits of Prym varieties. We generalize this situation and explain how ordinary Prym varieties appear as limits of Prym-Tyurin varieties.

Via the Riemann Existence Theorem, a general E_6-cover $\pi : C \to \mathbb{P}^1$ is determined by a branch divisor $p_1 + \cdots + p_{24} \in \mathrm{Sym}^{24}(\mathbb{P}^1)$ and discrete data involving a collection of roots $r_1, \ldots, r_{24} \in E_6$ which describe the local monodromy of π at the points $p_1, \ldots, p_{24}$. Letting two branch points, say p_{23} and p_{24}, coalesce such that $r_{23} = r_{24}$, whereas the reflections in the remaining roots $r_1, \ldots, r_{22}$ span the Weyl group $W(D_5) \subset W(E_6)$, gives rise to a boundary divisor D_{D_5} of $\overline{\mathrm{Hur}}$. We show in Section 7 that the general point of D_{D_5} corresponds to the following geometric data:

(i) A genus 7 Prym curve $[Y, \eta] \in \mathcal{R}_7$, together with a degree 5 pencil $h : Y \to \mathbb{P}^1$ branched simply along the divisor $p_1 + \cdots + p_{22}$; the unramified double cover $F_1 \to Y$ gives rise to a degree 10 map $\pi_1 : F_1 \to \mathbb{P}^1$ from a curve of genus 13.

(ii) A genus 29 curve $F_2 \subset F_1^{(5)}$, which is pentagonally related to F_1, and is thus completely determined by F_1. Precisely, F_2 is one of the two irreducible components of the locus $\bigl\{x_1 + \cdots + x_5 \in F_1^{(5)} : \pi_1(x_1) = \cdots = \pi_1(x_5)\bigr\}$ inside the fifth symmetric product $F_1^{(5)}$ of F_1. One has a degree 16 cover $\pi_2 : F_2 \to \mathbb{P}^1$ induced by π_1.

(iii) A distinguished point $q_1 + \cdots + q_5 \in F_2$, which determines 5 further pairs of points $\bigl(q_i, q_1 + \cdots + \iota(q_i) + \cdots + q_5\bigr) \in F_1 \times F_2$ for $i = 1, \ldots, 5$, which get identified. To F_2 we attach a rational curve F_0 at the point $q_1 + \cdots + q_5$. The resulting nodal curve $C_1 = F_0 \cup F_1 \cup F_2$ has genus 46 and admits a map $\pi : C_1 \to \mathbb{P}^1$ of degree 27 having $\pi_{|F_i} = \pi_i$ for $i = 0, 1, 2$, where π_0 is an isomorphism. The map π can easily be turned into an E_6-admissible cover having as source a curve stably equivalent to C_1. A general point of the divisor D_{D_5} is realized in this way.

We show in Section 7 that $PT([C_1, \pi]) = P([F_1/Y]) = P([Y, \eta]) \in \mathcal{A}_6$; furthermore, each ordinary Prym variety from $P(\mathcal{R}_7) \subset \mathcal{A}_6$ appears in this way. We summarize the above discussion, showing that the restriction $PT_{D_{D_5}}$ of the Prym-Tyurin map factors via the (generically injective) Prym map $P : \overline{\mathcal{R}}_7 \dashrightarrow \overline{\mathcal{A}}_6$ in the following way.

Theorem 0.4. *If $D_{D_5} \subset \overline{\mathrm{Hur}}$ is the boundary divisor of $W(D_5)$-covers defined above, one has the following commutative diagram:*

(0.1)
$$\begin{tikzcd} D_{D_5} \arrow[r] \arrow[d, "PT_{D_5}"'] & \overline{\mathrm{Hur}} \arrow[d, dashed, "PT"] \\ \overline{\mathcal{R}}_7 \arrow[r, dashed, "P"] & \overline{\mathcal{A}}_6 \end{tikzcd}$$

The fibres of the map $PT_{D_5} : D_{D_5} \to \mathcal{R}_7$ are 2-dimensional and $PT_{D_5}^{-1}\bigl(P[F_1/Y]\bigr)$ is the fibration over the curve $W^1_5(Y)$, having as fibre over a pencil $A \in W^1_5(Y)$ the curve F_2 obtained by applying the 5-gonal construction to A.

We close the introduction by discussing the structure of the paper. In Section 1 we discuss Kanev's construction, whereas in Section 2 we collect basic facts about the E_6 lattice and the group $W(E_6)$ that are used throughout the paper. After recalling the theory of degenerations of Jacobians and ordinary Prym varieties in Section 3, we complete the proof of Theorem 0.1 in Section 4, by describing the Prym-Tyurin map in the neighborhood of maximally degenerate point of the space $\overline{\mathrm{Hur}}$ of E_6-admissible covers. Section 5 is devoted to the birational geometry of this Hurwitz space, whereas in Section 6 we completely describe the Prym-Tyurin map $PT : \overline{\mathrm{Hur}} \dashrightarrow \overline{\mathcal{A}}_6$ at the level of divisors. After proving Theorem 0.4 in Section 7, we complete in Section 8 the proof of Theorem 0.2 after a detailed study of the divisors D_{azy} and D_{syz} of azygetic and syzygetic E_6-covers respectively on a partial compactification $\mathcal{G}_{E_6}$ of Hur. The ramification divisor of the Prym-Tyurin map is described in Section 9. Finally, in Section 10 we construct effective divisors on $\overline{\mathrm{Hur}}$ and link the computation of the Kodaira dimension of $\mathcal{A}_6$ to a version of the Slope Conjecture concerning effective divisors on $\overline{\mathcal{M}}_{46}$.

Acknowledgments: We owe a great debt to the work of Vassil Kanev, who first constructed the Prym-Tyurin map PT and raised the possibility of uniformizing $\mathcal{A}_6$ in this way. The authors acknowledge partial support by the NSF: VA under grant DMS 1200726, RD under grant DMS 1304962, EI under grant DMS-1103938/1430600. The work of GF and AO has been partially supported by the DFG Sonderforschungsbereich 647 "Raum-Zeit-Materie".

1. Kanev's construction and Prym-Tyurin varieties of E_6-type

Consider a cubic threefold $X \subset \mathbb{P}^4$ and a smooth hyperplane section $S \subset X$. The cubic surface S contains a set of 27 lines $\Lambda := \{\ell_s\}_{1\leq s\leq 27}$ forming a famous classical configuration, which we shall review below in Section 2. Consider the lattice $\mathbb{Z}^\Lambda = \mathbb{Z}^{27}$ with the standard basis corresponding to ℓ_s's, and let $\deg\colon \mathbb{Z}^\Lambda \to \mathbb{Z}$ be the degree homomorphism, so that $\deg(\ell_s) = 1$ for all $s = 1, \ldots, 27$.

1.1. By assigning to each line ℓ_s the sum $\sum_{\{s'\colon\, \ell_s\cdot\ell_{s'}=1\}} \ell_{s'}$ of the 10 lines on S intersecting ℓ_s, we define a homomorphism $D'_\Lambda\colon \mathbb{Z}^{27} \to \mathbb{Z}^{27}$ of degree 10. It is easy to check that D'_Λ satisfies the following quadratic equation:

$$(D'_\Lambda + 5)(D'_\Lambda - 1) = 5\left(\sum_{s=1}^{27} \ell_s\right) \cdot \deg$$

In particular, the restriction D_Λ of D'_Λ to the subgroup Ker(deg) satisfies the equation $(D_\Lambda + 5)(D_\Lambda - 1) = 0$.

Consider a generic pencil $\{S_t\}_{t\in\mathbb{P}^1}$ of cubic hyperplane sections of X. This defines:

- a degree 27 smooth curve cover $\pi\colon C \to \mathbb{P}^1$; the points in the fiber $\pi^{-1}(t)$ correspond to the lines lying on S_t;
- a symmetric *incidence* correspondence $\widetilde{D} \subset C \times C$ of degree 10;

- a homomorphism $D' = p_{2*} \circ p_1^*\colon \operatorname{Pic}(C) \to \operatorname{Pic}(C)$ (where $p_i\colon \widetilde{D} \to C$ are the two projections) satisfying the quadratic equation (see also [Kan89a, Proposition 3.1]) $(D'+5)(D'-1) = 5\pi^{-1}(0)\cdot \deg$;
- the restriction D of D' to $JC = \operatorname{Pic}^0(C)$, satisfying $(D+5)(D-1) = 0$.

For a generic such pencil the map $\pi\colon C \to \mathbb{P}^1$ has 24 branch points on $\mathbb{P}^1$, corresponding to singular cubic surfaces in the pencil, each with one node. Over each of the 24 points, the fibre consists of 6 points of multiplicity two and 15 single points. By the Riemann-Hurwitz formula, we compute $g(C) = 46$.

1.2. We refer to [Kan89b, LR08], for the following facts. The cover $\pi\colon C \to \mathbb{P}^1$ is not Galois. The Galois group of its Galois closure is $W(E_6)$, the reflection group of the E_6 lattice. As we shall review in Section 2, the lattice E_6 appears as the lattice $K_S^{\perp} \subset \operatorname{Pic}(S)$. The 27 lines can be identified with the $W(E_6)$-orbit of the fundamental weight ω_6, and one has a natural embedding $W(E_6) \subset S_{27}$. The intermediate non-Galois cover $C \to \mathbb{P}^1$ is associated with the stabilizer subgroup of ω_6 in $W(E_6)$, that is, with the subgroup $W(E_6) \cap S_{26} \cong W(D_5)$.

1.3. By Riemann's Existence Theorem, a 27-sheeted cover $C \to \mathbb{P}^1$ ramified over 24 points is defined by a choice of 24 elements $w_i \in S_{27}$ satisfying $w_1 \cdots w_{24} = 1$. For a cover coming from a pencil of cubic surfaces, each $w_i \in W(E_6)$ is a reflection in a root of the E_6. It is a *double-six*, that is, viewed as an element of S_{27}, it is a product of 6 disjoint transpositions.

Definition 1.4. Let Hur be the Hurwitz space parametrizing irreducible smooth Galois $W(E_6)$-covers $C' \to \mathbb{P}^1$ ramified in 24 points, such that the monodromy over each point is a reflection in a root of the E_6 lattice.

1.5. Note that points in the space Hur correspond to covers where we do not choose a labeling of the branch points. The data for the cover C' consists of the branch divisor $p_1 + \ldots + p_{24}$ on $\mathbb{P}^1$, and, for each of these points, the monodromy $w_i \in W(E_6)$ given by a reflection in a root, once a base point $p_0 \in \mathbb{P}^1$ and a system of arcs γ_i in $\pi_1(\mathbb{P}^1 \setminus \{p_1, \ldots, p_{24}\}, p_0)$ with $\gamma_1 \cdots \gamma_{24} = 1$ has been chosen. The elements $\{w_i\}_{i=1}^{24}$ generate $W(E_6)$ and satisfy the relation $w_1 \cdots w_{24} = 1$. The monodromy data being finite, the space Hur comes with a finite unramified cover

$$\mathfrak{br} : \mathrm{Hur} \to \mathcal{M}_{0,24}/S_{24}$$

to the moduli space of 24 unordered points on $\mathbb{P}^1$. Thus $\dim(\mathrm{Hur}) = 21$. An important fact about this space is the following result of Kanev [Kan06]:

Theorem 1.6. *For any irreducible root system R, the Hurwitz scheme parameterizing Galois $W(R)$-covers such that the monodromy around any branch point is a reflection in $W(R)$, is irreducible.*

1.7. In particular, the space Hur is irreducible. If $[\pi' : C' \to \mathbb{P}^1] \in$ Hur, let $\pi : C \to \mathbb{P}^1$ be an intermediate non-Galois cover of degree 27, that is, the quotient of C' by a subgroup $W(E_6) \cap S_{26} \cong W(D_5)$ in S_{27}. Since $W(E_6)$ acts transitively on the set $\{1, \dots, 27\}$, the 27 subgroups $S_{26} \subset S_{27}$ are conjugate, and the corresponding curves C are isomorphic. Thus, Hur is also a coarse moduli space for degree 27 non-Galois covers $\pi \colon C \to \mathbb{P}^1$, branched over 24 points such that the monodromy at each branch point is a reflection of $W(E_6)$.

Remark 1.8. Up to conjugation, $W(E_6) \cap S_{26}$ is the unique subgroup of index 27 in $W(E_6)$.

1.9. Let $\pi \colon C \to \mathbb{P}^1$ be an E_6-cover as above. Each fiber of π can be identified consistently with the set of 27 lines on a cubic surface. The incidence of lines, in the same way as for the correspondence D_Λ in 1.1, induces a symmetric correspondence $\widetilde{D} \subset C \times C$ of valence 10, which is disjoint from the diagonal $\Delta \subset C \times C$. In turn, $\widetilde{D}$ induces a homomorphism $D' \colon \operatorname{Pic}(C) \to \operatorname{Pic}(C)$, whose restriction $D : JC \to JC$ satisfies the quadratic relation

$$(D - 1)(D + 5) = 0 \in \operatorname{End}(JC). \tag{1.1}$$

Definition 1.10. The *Prym-Tyurin-Kanev variety* $PT(C, D)$ is defined as the connected component of the identity $PT(C, D) := \big(\operatorname{Ker}(D + 5)\big)^0 = \operatorname{Im}(D - 1) \subset JC$.

1.11. Using [Kan87], Equation (1.1) implies that the restriction of the principal polarization Θ_C of JC to $PT(C, D)$ is a multiple of a principal polarization. Precisely, $\Theta_{C|PT(C,D)} = 6 \cdot \Xi$, where $(PT(C, D), \Xi)$ is a ppav. Since

$$0 = \widetilde{D} \cdot \Delta = 2\deg(\widetilde{D}) - 2\operatorname{tr}\big\{D : H^0(C, K_C) \to H^0(C, K_C)\big\},$$

we obtain that

$$\dim PT(C, D) = \frac{1}{6}\big(g(C) - \deg(\widetilde{D})\big) = \frac{1}{6}(46 - 10) = 6,$$

see also [LR08, Proposition 5.3]. We have the morphism of moduli stacks

$$\begin{array}{rccc} PT\colon & \text{Hur} & \longrightarrow & \mathcal{A}_6 \\ & [C, D] & \longmapsto & [PT(C, D), \Xi]. \end{array}$$

Both stacks are irreducible and 21-dimensional. The main result of this paper (Theorem 0.1) is that PT is a dominant, i.e., generically finite, map.

1.12. Our main concrete examples of E_6-covers of $\mathbb{P}^1$ are the *curves of lines* in Lefschetz pencils of cubic surfaces. The subvariety $\mathcal{T} \subset$ Hur corresponding to pencils $\{S_t\}_{t \in \mathbb{P}^1}$ of hyperplane sections of cubic 3-folds $X \subset \mathbb{P}^4$ has expected dimension

$$\binom{7}{3} - 1 + \dim \operatorname{Gr}(2, 5) - \dim \operatorname{PGL}_5 = (35 - 1) + 6 - (25 - 1) = 16.$$

1.13. We now describe the restriction of the map PT to the locus $\mathcal{T} \subset \mathrm{Hur}$ parametrizing such covers. Let V be a 5-dimensional vector space over $\mathbb{C}$ whose projectivization contains X and let $F \in \mathrm{Sym}^3(V^\vee)$ be a defining equation for X. Denote by $\mathcal{F} := \mathcal{F}(X)$ the Fano variety of lines in X. Let $JX := H^{2,1}(X)^\vee / H_3(X, \mathbb{Z})$ be the intermediate Jacobian of X. It is well known [CG72] that the Abel-Jacobi map defines an isomorphism $JX \cong \mathrm{Alb}\,\mathcal{F}$, where $\mathrm{Alb}\,\mathcal{F}$ is the Albanese variety of $\mathcal{F}$. Let Λ be a Lefschetz pencil of hyperplane sections of X and denote by E its base curve. The curve C classifying the lines lying on the surfaces contained in Λ lives naturally in $\mathcal{F}$. The map sending a line to its point of intersection with E induces a degree 6 cover $C \to E$. Furthermore, the choice of a base point of C defines a map $C \to JX$. So we obtain a well-defined induced map $JC \to E \times JX$. The transpose $E \times \mathrm{Pic}^0(\mathcal{F}) = E \times JX \to JC$ of this map is given by pull-back on divisors on each of the factors, using the map $C \to E$ and the embedding $C \hookrightarrow \mathcal{F}$ respectively. On the locus $\mathcal{T}$ we can explicitly determine the Prym-Tyurin variety, see also [Kan89a]:

Lemma 1.14. *The map $JC \to E \times JX$ (or its transpose $E \times JX \to JC$) induces an isomorphism of ppav $PT(C, D) \xrightarrow{\cong} E \times JX$.*

Proof. We first show that the correspondence D restricts to multiplication by (-5) on both factors E and JX. For $\ell \in C$, let $\widetilde{D}(\ell)$ be the sum of the lines incident to ℓ and E inside X. We denote by H_ℓ the hyperplane spanned by E and ℓ and put $S_\ell := H_\ell \cap X$. The lines incident to E and ℓ form 5 pairs $(\ell_1, \ell_1'), \ldots, (\ell_5, \ell_5')$, with $\ell + \ell_i + \ell_i' \in |-K_{S_\ell}|$ for $i = 1, \ldots, 5$.

Consider first the intermediate Jacobian JX. We have

$$\widetilde{D}(\ell) = \sum_{i=1}^{5} (\ell_i + \ell_i') \equiv 5|-K_{S_\ell}| - 5\ell,$$

where $\equiv$ denotes linear equivalence in S_ℓ. Since $|-K_{S_\ell}|$ is constant as ℓ varies, it follows that D restricts to multiplication by (-5) on JX.

Consider the elliptic curve E. Then $\widetilde{D}(\ell)$ in E is the sum of the intersection points of ℓ_i, ℓ_i' with E. Thus $(\ell + \ell_i + \ell_i')|_E$ is also the intersection of the plane $\Pi_i := \langle \ell, \ell_i, \ell_i' \rangle$ with E. Hence $\sum_{i=1}^5 (\ell + \ell_i + \ell_i')|_E$ is the intersection of the 5 planes $\Pi_1, \ldots, \Pi_5$ with E. Projecting from ℓ, we see that the union of these planes is the intersection of H_ℓ with the inverse image Q of the plane quintic in $\mathbb{P}^2 = \mathbb{P}(V/\ell)$ parametrizing singular conics (the discriminant curve for the projection of X from ℓ). Therefore $\sum_{i=1}^5 (\ell + \ell_i + \ell_i')|_E$ is contained in the intersection $Q \cap E$ and since the two divisors have the same degree, we obtain that $\sum_{i=1}^5 (\ell + \ell_i + \ell_i')|_E = Q \cap E$ is constant. This implies that D is multiplication by (-5) on E as well.

So the Prym-Tyurin variety is isogenous to $E \times JX$. To show that they are isomorphic, we show that the pull-back of the polarization of JC to $E \times JX$ is 6 times a principal polarization. This is immediate on the factor E, since the map $C \to E$ has degree 6. To see it on the JX factor as well, we again use the Abel-Jacobi embedding $C \hookrightarrow \mathcal{F} \hookrightarrow JX$ and recall the fact [CG72] that one model of the theta divisor in JX is the image of the degree 6 difference map $\varphi : \mathcal{F} \times \mathcal{F} \to \operatorname{Alb} \mathcal{F} = JX$, defined by $\varphi(\ell, \ell') = \ell - \ell'$. $\square$

We denote by $\mathcal{J}_5 \subset \mathcal{A}_5$ the moduli space of intermediate Jacobians of cubic threefolds.

Lemma 1.15. *The dimension of* $\mathcal{T} \subset \mathrm{Hur}$ *equals its expected dimension, which is* 16.

Proof. First note that $\dim(\mathcal{J}_5) = 10$; the Grassmann bundle over $\mathcal{J}_5$ parametrizing pencils of hyperplane sections has dimension 16, so $\dim \mathcal{T} \leq 16$.

We degenerate the pencil so that the base curve E has 1, 2 or 3 ordinary double points. The number of moduli for $\mathcal{T}$ is 3 more than the number of moduli for the pencils where E has 3 double points, that is, it is a union of three lines.

We count the number of parameters for the curves of lines C corresponding to pencils with $E = \ell_1 \cup \ell_2 \cup \ell_3$ being a union of three lines. The stable model of C breaks into three components $\widetilde{Q}_1, \widetilde{Q}_2$, and $\widetilde{Q}_3$, parametrizing lines incident to ℓ_1, ℓ_2 and ℓ_3 respectively. Each $\widetilde{Q}_i$ is an étale double cover of a plane quintic Q_i which is the discriminant curve for the projection of X from ℓ_i. For a general choice of $X \in |\mathcal{O}_{\mathbb{P}^3}(3)|$ and of a line $\ell_1 \subset X$, the curve Q_1 is a general plane quintic, hence depends on 12 moduli. The choice of lines ℓ_2 and ℓ_3 coplanar with ℓ_1 depends on the choice of a point of Q_1, giving 1 parameter. The moduli of the curves Q_2, Q_3 varies with the choice of the lines ℓ_2, ℓ_3. Therefore, we have a total of 13 moduli for the union $\widetilde{Q}_1 \cup \widetilde{Q}_2 \cup \widetilde{Q}_3$. The number of moduli for $\mathcal{T}$ is thus $3 + 13 = 16$. $\square$

Corollary 1.16. *We have the following equality of* 11*-dimensional irreducible cycles in* $\mathcal{A}_6$*:*

$$\overline{PT(\mathcal{T})} = \mathcal{J}_5 \times \mathcal{A}_1 \subset \mathcal{A}_5 \times \mathcal{A}_1 \subset \mathcal{A}_6.$$

Remark 1.17. One might think that Corollary 1.16 offers potentially a way of proving the dominance of the map PT in the style of [DS81]. It would suffice to show that the extended Prym-Tyurin map on blow-ups $\widetilde{PT} : \mathrm{Bl}_{\mathcal{T}}(\mathrm{Hur}) \to \mathrm{Bl}_{\mathcal{J}_5 \times \mathcal{A}_1}(\mathcal{A}_6)$ maps the exceptional divisor of the source *onto* the exceptional divisor of the image. Surprisingly however it turns out that for a general product $[JX \times E] \in \mathcal{J}_5 \times \mathcal{A}_1$, the union

$$\bigcup_{z \in PT^{-1}[JX \times E]} \mathrm{Im}\Big\{(dPT)_z : \mathbb{P}\big(N_{\mathcal{T}/\mathrm{Hur}}\big)_z \to \mathbb{P}\big(N_{\mathcal{J}_5 \times \mathcal{A}_1/\mathcal{A}_6}\big)_{[JX \times E]}\Big\}$$

is a divisor in the target projective space. In other words, the first order information encoded in this first blow-up does not suffice to prove dominance of PT. So at least one further blowup is needed. The situation seems similar to that for the ordinary Prym map P near an elliptic tail curve, cf. [DS81]: the boundary divisor of elliptic tails is blown down to a codimension 2 locus of cuspidal curves, which is in the closure of another component, the locus of Wirtinger covers. In the present work we therefore follow a different path to establish the dominance of the map PT.

2. The E_6 lattice

In this section we recall basic facts about the E_6 lattice. Our reference for these is [Dol12, Chapters 8,9].

2.1. Let $I^{1,6}$ be the standard Lorenzian lattice with the quadratic form $x_0^2 - \sum_{i=1}^6 x_i^2$. The negative definite E_6 lattice is identified with $k^\perp$, where $k = (-3, 1, \ldots, 1)$. Its dual $E_6^\vee$ is identified with $I^{1,6}/\mathbb{Z}k$. Let us denote the standard basis of $I^{1,6}$ by $f_0, f_1, \ldots, f_6$, to avoid confusion with the edges e_i in a graph.

The roots of E_6 are the vectors with square -2. There are $\binom{6}{2} + \binom{6}{3} + 1 = 36$ pairs of roots corresponding to $\alpha_{ij} = f_i - f_j$, $\alpha_{ijk} = f_0 - f_i - f_j - f_k$ and $\alpha_{\max} = 2f_0 - f_1 - \ldots - f_6$. Obviously, if $r \in E_6$ is a root then $-r$ is a root as well. The simple roots, corresponding to the E_6 Dynkin diagram can be chosen to be $r_1 = \alpha_{123}$, $r_2 = \alpha_{12}$, $r_3 = \alpha_{23}$, $r_4 = \alpha_{34}$, $r_5 = \alpha_{45}$ and $r_6 = \alpha_{56}$.

2.2. The Weyl group $W(E_6)$ is the group generated by the reflections in the roots. It has 51,840 elements. The fundamental weights $\omega_1, \ldots, \omega_6$ are the vectors in $E_6^\vee$ with $(r_i, \omega_j) = \delta_{ij}$.

The exceptional vectors are the vectors in the $W(E_6)$-orbit of ω_6. They can be identified with vectors ℓ in $I^{1,6}$ satisfying $\ell^2 = k\ell = -1$. There are $6 + 6 + 15 = 27$ of them, namely:

$$
\begin{aligned}
&a_i = f_i, \quad \text{for} \quad i = 1, \ldots, 6;\\
&b_i = 2f_0 - f_1 - \cdots - f_6 + f_i, \quad \text{for} \quad i = 1, \ldots, 6;\\
&c_{ij} = f_0 - f_i - f_j, \quad \text{for} \quad 1 \le i < j \le 6.
\end{aligned}
$$

2.3. For each root $r \in E_6$, there are 15 exceptional vectors that are orthogonal to it, 6 exceptional vectors with $r \cdot \ell = 1$ and 6 vectors with $r \cdot \ell = -1$. The collections of the 6 pairs of exceptional vectors non-orthogonal to a root vector are called *double-sixes*. The elements in each pair are exchanged by the reflection $w_r \in W(E_6)$ in the root r.

There are 36 double-sixes, one for each pair $\pm r$ of roots. For example, the double-six for the root $r = \alpha_{\max}$ is $\{a_1, a_2, \ldots, a_6\}$, $\{b_1, b_2, \ldots, b_6\}$. The reflection group acts transitively on the set of the exceptional vectors. This gives rise to an embedding $W(E_6) \subset S_{27}$. Under this embedding, each reflection corresponds to a product of 6 transpositions. For example, the reflection in the root $r = \alpha_{\max}$ is the permutation $(a_1, b_1) \cdots (a_6, b_6) \in S_{27}$.

Note that the choice of a root is equivalent to an ordering of the pair: when we write the same element of $W(E_6)$ as a product $(b_1, a_1) \cdots (b_6, a_6)$, it corresponds to the root $-\alpha_{\max}$.

The $W(E_6)$-action by conjugation is transitive on the set of double sixes, so to study their properties it is usually sufficient to make computations for this special representative.

2.4. For a smooth cubic surface S, the above objects have the following incarnation:

- $I^{1,6} = \mathrm{Pic}(S)$ together with the intersection form,
- $k = K_S$ and $E_6 = K_S^{\perp} \subset \mathrm{Pic}(S)$,
- the exceptional vectors are identified with the lines $\ell_1, \dots, \ell_{27}$ on S,
- a sixer is a set of 6 mutually disjoint lines, a double-six is the set of two sixers corresponding to the opposite roots.

The relationship between the $W(E_6)$-action and the correspondence given by the line incidence is as follows.

Definition 2.5. The correspondence on the set of exceptional vectors is defined by setting $D(\ell) := \sum_{\{\ell' \colon\, \ell' \cdot \ell = 1\}} \ell'$.

Remark 2.6. For further use, we retain the following computation:

$$D(a_1) = b_2 + \cdots + b_6 + c_{12} + \cdots + c_{16}$$

$$D(b_1) = a_2 + \cdots + a_6 + c_{12} + \cdots + c_{16}$$

$$D(a_1 - b_1) = (b_2 - a_2) + \dots (b_6 - a_6).$$

2.7. The group $W(E_6)$ has 25 irreducible representations corresponding to its 25 conjugacy classes, which will appear several times in this paper. For conjugacy classes we use the ATLAS or GAP notation 1a, 2a, 2b, 2c, ..., 12a, (command 'ConjugacyClasses(WE6)'). The number refers to the order of the elements in the conjugacy class. For instance, the reflections in $W(E_6)$ (products of six transpositions) belong to the conjugacy class 2c, the product of two syzygetic reflections belongs to the class 2b, whereas the product of two azygetic reflections belongs to the class 3b (see Section 5 for precise definitions).

3. Degenerations of Jacobians and Prym varieties

3.1. By a theorem of Namikawa and Mumford, the classical Torelli map $\mathcal{M}_g \to \mathcal{A}_g$ sending a smooth curve to its Jacobian extends to a regular morphism $\overline{\mathcal{M}}_g \to \overline{\mathcal{A}}_g^{\mathrm{vor}}$ from the Deligne-Mumford compactification of $\mathcal{M}_g$ to the toroidal compactification of $\mathcal{A}_g$ for the second Voronoi fan. See [AB12] for a transparent modern treatment of this result, and extension results for other toroidal compactifications of $\mathcal{A}_g$. The result applies equally to the stacks and to their coarse moduli spaces. Here, we will work with stacks, so that we have universal families over them.

3.2. At the heart of the result of Namikawa and Mumford is the Picard-Lefschetz formula for the monodromy of Jacobians in a family of curves, see e.g. [Nam73, Proposition 5]. The map of fans for the toroidal morphism $\overline{\mathcal{M}}_g \to \overline{\mathcal{A}}_g^{\mathrm{vor}}$ is described as follows. Fix a stable curve $[C] \in \overline{\mathcal{M}}_g$, and let Γ be its dual graph, with a chosen orientation. Degenerations of Jacobians are described in terms of the groups

$$C_0(\Gamma,\mathbb{Z}) = \bigoplus_{\text{vertices } v} \mathbb{Z}v, \quad C_1(\Gamma,\mathbb{Z}) = \bigoplus_{\text{edges } e} \mathbb{Z}e, \quad H_1(\Gamma,\mathbb{Z}) = \mathrm{Ker}\big\{\partial\colon C_1(\Gamma,\mathbb{Z}) \to C_0(\Gamma,\mathbb{Z})\big\}.$$

The Jacobian $JC = \mathrm{Pic}^0(C)$ is a semiabelian group variety that is an extension

$$1 \to H^1(\Gamma,\mathbb{C}^*) \to \mathrm{Pic}^0(C) \to \mathrm{Pic}^0(\widetilde{C}) \to 0, \tag{3.1}$$

where $\widetilde{C}$ is the normalization of C. In particular, $\mathrm{Pic}^0(C)$ is a multiplicative torus if and only if $\widetilde{C}$ is a union of $\mathbb{P}^1$'s, or equivalently, if $b_1 = h^1(\Gamma) = g$.

The monodromy of a degenerating family of Jacobians is described as follows. Fix a lattice $\Lambda \simeq \mathbb{Z}^g$ and a surjection $\Lambda \twoheadrightarrow H_1(\Gamma,\mathbb{Z})$. The rational polyhedral cone for a neighborhood of $[C] \in \overline{\mathcal{M}}_g$ lives in the space $\Lambda^\vee \otimes \mathbb{R}$ with the lattice $\Lambda^\vee$. It is a simplicial cone of dimension $b_1 = h^1(\Gamma)$ with the rays e_i^* corresponding to the edges of Γ. Here, e_i^* is the linear function on $H_1(\Gamma,\mathbb{Z}) \subset C_1(\Gamma,\mathbb{Z})$ taking the value δ_{ij} on the edge $e_j \in C_1(\Gamma,\mathbb{Z})$.

The rational polyhedral cone corresponding to a neighborhood of $[JC] \in \overline{\mathcal{A}}_g^{\mathrm{vor}}$ lives in the space $\Gamma^2(\Lambda^\vee) \otimes \mathbb{R} = (\mathrm{Sym}^2(\Lambda) \otimes \mathbb{R})^\vee$, where the lattice $\Gamma^2(\Lambda^\vee)$ is the second divided power of $\Lambda^\vee$. It is a simplicial cone with the rays $(e_i^*)^2$ for all $e_i^* \neq 0$, which means that e_i is not a bridge of the graph Γ. We explain what this means in down to earth terms. In an open analytic neighborhood U of $[C]$, one can choose local analytic coordinates $z_1, \dots, z_{3g-3}$ so that the first N coordinates correspond to smoothing the nodes of C, labeled by the edges e_i of the graph Γ. Thus, we have a family of smooth curves over the open subset $V = U - \bigcup_{i=1}^N \{z_i = 0\}$.

Then a complex-analytic map $V \to \mathcal{H}_g$ to the Siegel upper half-plane is given by a formula (see [Nam73, Thm.2] or [Nam76, 18.7])

$$(z_i) \mapsto \sum_{i=1}^N M_i \cdot \frac{1}{2\pi\sqrt{-1}} \log z_i + (\text{a bounded holomorphic function}),$$

where M_i are the $g \times g$ integral matrices corresponding to the quadratic functions $(e_i^*)^2$ on $\Lambda \twoheadrightarrow H_1(\Gamma,\mathbb{Z})$. After applying the exponential map

$$\mathbb{C}^{\frac{g(g+1)}{2}} \to (\mathbb{C}^*)^{\frac{g(g+1)}{2}}, \quad u_{ij} \mapsto \exp(2\pi\sqrt{-1}\, u_{ij}),$$

the matrices $M_i \cdot (\log z_i / 2\pi\sqrt{-1})$ become Laurent monomials in z_i, and one obtains a complex-analytic map from U to an appropriate toroidal neighborhood of $[JC] \in \overline{\mathcal{A}}_g^{\mathrm{vor}}$.

3.3. The following weak form of Torelli's theorem is a sample of our degeneration technique. This is far from being the easiest way to prove the Torelli theorem, but it gives a good illustration of our method which we later apply to Prym-Tyurin-Kanev varieties.

Lemma 3.4. *The image of the Torelli map $\mathcal{M}_g \to \mathcal{A}_g$ has full dimension $3g-3$.*

Proof. For every g, there exists a 3-edge connected trivalent graph Γ of genus g (exercise in graph theory). By Euler's formula, it has $3g-3$ edges. Recall that a connected graph is *2-edge connected* if it has no bridges, i.e. the linear functions e_i^* on $H_1(\Gamma,\mathbb{Z})$ are all nonzero, and it is *3-edge connected* if for $i \neq j$ one has $e_i^* \neq \pm e_j^*$, i.e. $(e_i^*)^2 \neq (e_j^*)^2$.

Let C be a stable curve whose dual graph is Γ and whose normalization is a disjoint union of $\mathbb{P}^1$'s. Then the $3g-3$ matrices M_i in the formula (3.2), i.e. the functions $(e_i^*)^2$, are linearly independent in $\mathrm{Sym}^2(\mathbb{Z}^g)$, cf. [AB12, Remark 3.6]. By looking at the leading terms as $z_i \to 0$, this easily implies that the image has full dimension $3g-3$.

Equivalently, after applying the exponential function, the map becomes

$$(z_1,\dots,z_{3g-3}) \mapsto (\text{monomial map}) \times (\text{invertible function}),$$

Since the monomial part is given by monomials generating an algebra of transcendence degree $3g-3$, the image is full-dimensional. $\square$

Remark 3.5. Note that the regularity of the extended Torelli map $\overline{\mathcal{M}}_g \to \overline{\mathcal{A}}_g^{\mathrm{vor}}$ played no role in the proof of Lemma 3.4. All we need for the conclusion is the fact that the monodromy matrices M_i are linearly independent.

3.6. The theory for Jacobians was extended to the case of Prym varieties in [ABH02]. We briefly recall it. Let $\overline{\mathcal{R}}_g$ be the stack of Prym curves of genus g, classifying admissible pairs $[C,\iota]$ consisting of a stable curve with involution $\iota : C \to C$, so that C/ι is a stable curve of genus g and the map $C \to C/\iota$ is an admissible map of stable curves. We refer to [Bea77] and [FL10] for background on $\overline{\mathcal{R}}_g$. Consider one pair $[C,\iota] \in \overline{\mathcal{R}}_g$ and a small analytic neighborhood U of it. As before, Γ is the dual graph of C.

Then the space $H_1(C,\mathbb{Z})$ of the Jabobian case is replaced by the lattice H_1/H_1^+. Here, H_1^+ and H_1^- are the $(+1)$- and the (-1)-eigenspaces of the involution action ι^* on $H_1(C,\mathbb{Z})$ respectively. Via the natural projection $H_1 \twoheadrightarrow H_1/H_1^+$, we identify H_1^- with a finite index sublattice of H_1/H_1^+.

The degeneration of Prym varieties as groups is

$$P(C,\iota) = \mathrm{Ker}(1+\iota^*)^0 = \mathrm{Im}(1-\iota^*), \qquad \iota^*\colon \mathrm{Pic}^0(C) \to \mathrm{Pic}^0(C).$$

The monodromy of a degenerating family of Prym varieties is obtained by restricting the monodromy map for JC to the (-1)-eigenspace. Combinatorially, it works as follows: For every edge e_i of Γ we have a linear function e_i^* on the group H_1^-, the restriction of the linear function on $H_1(C, \mathbb{Z})$. For the divisor $\{z_i = 0\}$ on U corresponding to smoothing the node P_i of C, the monodromy is given by the quadratic form $(e_i^*)^2$ restricted to $H_1(\Gamma, \mathbb{Z})^-$. Similarly to Lemma 3.4, this can be used to prove various facts about the Prym-Torelli map, but we will not pursue it here.

4. Degenerations of Prym-Tyurin-Kanev varieties

We choose a concrete boundary point in a compactification of the Hurwitz scheme Hur. We start with a single cubic surface S and the set $\{\ell_1, \dots, \ell_{27}\}$ of 27 lines on it. Sometimes we shall use the Schläfli notation $\{a_i, b_i, c_{ij}\}$ for them, as in Section 2. We fix an embedding of $W(E_6)$ into the symmetric group S_{27} permuting the 27 lines on S.

4.1. We choose 12 roots r_i which generate the root system E_6. Let $w_i \in W(E_6)$ be the reflections in r_i; they generate $W(E_6)$. As we saw in Section 2, each w_i is a double-six. Fixing the root r_i gives it an orientation.

4.2. Consider a nodal genus 0 curve E whose normalization is a union of $\mathbb{P}^1$'s and whose dual graph is the tree T shown in the left half of Figure 1. The 24 ends of this tree correspond to 24 points $p_1, \dots, p_{24}$ on E. We label the points by roots $r_1, \dots, r_{12}$. Each of the outside vertices has two ends, we use the same label r_i for both of them.

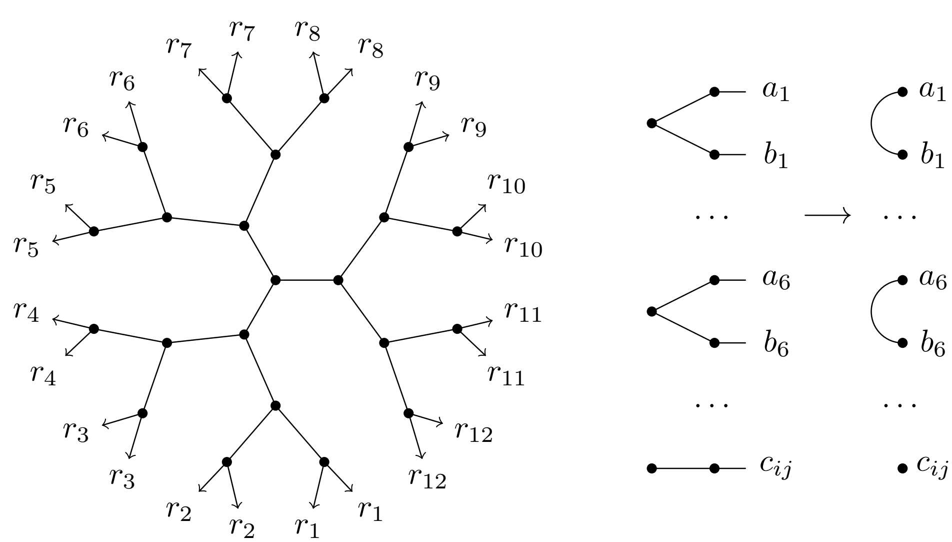


Figure 1. The tree T for the target curve E of genus 0

Definition 4.3. Let $\pi \colon C \to E$ be an admissible $27:1$ cover ramified at the point p_i with monodromy w_i for $i = 1, \dots, 24$.

For every irreducible component of E, the product of the monodromy elements equals 1; this count includes the nodes. Since we required that for every component on the boundary the two w_i's are the same, the map is unramified at the nodes. Thus, π is étale over $E \setminus \{p_1, \dots, p_{24}\}$.

4.4. Here is a concrete description of the dual graph Γ of C. It has

$$10 \times 27 + 12 \times (6 + 15) \text{ vertices and } 21 \times 27 \text{ edges}$$

Each vertex v of T in the *étale part* has 27 vertices over it. Over each of the outside 12 vertices, there are 6 vertices, where the map $\mathbb{P}^1 \to \mathbb{P}^1$ is $2:1$ and ramified at a pair of the points p_i and p_{i+12}, and 15 other vertices where the map $\mathbb{P}^1 \to \mathbb{P}^1$ is $1:1$.

All the nodes of E lie in the étale part, so for each internal edge e of the tree T there are 27 edges of Γ.

4.5. The graph Γ is homotopically equivalent to the following much simpler graph Γ'. It has:

(1) 27 vertices $\{v_s\}_{s=1}^{27}$, labeled by the lines on S. (Here, s stands for "sheets".)
(2) 12×6 edges e_{ij}. For each of the twelve roots r_i, there are 6 edges. For example, for $r = r_{\max}$, the edges are $(a_1, b_1), \dots, (a_6, b_6)$. The first edge is directed from a_1 to b_1, etc.

The graph Γ' is obtained from Γ by contracting the tree in each sheet to a point, and removing the middle vertex of degree 2 for each of the 12×6 paths corresponding to the double-sixes. The process is illustrated in the right half of Figure 1.

By Euler's formula, the genus of Γ is $12 \times 6 - 27 + 1 = 46$. Thus, the curve C has arithmetic genus 46.

4.6. Next we define a correspondence $\widetilde{D} \subset C \times C$ of bidegree $(10, 10)$, as follows. To each point $Q \in C$ over the étale part in the sheet labeled ℓ_i, associate 10 points in the same fiber of π that are labeled ℓ_{ij} by the lines that intersect ℓ_i.

This defines the curve $\widetilde{D}^0 \subset C^0 \times C^0$, where $C^0 = C \setminus \pi^{-1}\{p_1, \dots, p_{24}\}$. The correspondence $\widetilde{D} \subset C \times C$ is the closure of $\widetilde{D}^0$. Let p_i be a ramification point with monodromy w_i. Without loss of generality, we may assume $w = w_{\max}$. The points in the fiber $\pi^{-1}(p_i)$ are labeled a_1b_1, $\dots$, a_6b_6 and c_{ij} for $i \neq j$. Then the correspondence is described by:

$$a_1b_1 \mapsto \sum_{i=2}^{6} (a_ib_i + c_{1i}), \quad c_{12} \mapsto a_1b_1 + a_2b_2 + \sum_{i,j \neq 1,2} c_{ij}, \quad \text{etc.}$$

Lemma 4.7. *There exists an analytic neighborhood* $U \subset \overline{\mathcal{M}}_{0,24}$ *of the point* $[E, p_1, \dots, p_{24}]$ *and a family of covers* $\pi_t\colon C_t \to E_t$ *together with correspondences* $\widetilde{D}_t \subset C_t \times C_t$ *over* U*, which extends* $\pi\colon C \to E$ *and* $\widetilde{D}$*.*

Proof. Since the map π is étale over each node of E, the families C_t and $\widetilde{D}_t$ extend naturally. The monodromy data determine the C_t's as topological spaces. Then the finite map $C_t \to E_t$ determines a unique structure of an algebraic curve on C_t. □

Lemma 4.8. *The correspondence* $\widetilde{D} \subset C \times C$ *induces an endomorphism of the homology group* $D\colon H_1(\Gamma, \mathbb{Z}) \to H_1(\Gamma, \mathbb{Z})$ *satisfying the relation* $(D-1)(D+5) = 0$*. The* (-5)*-eigenspace* $H_1^{(-5)}$ *can be naturally identified with* $\operatorname{Ker}(\phi)$*, where*

$$\phi\colon \bigoplus_{i=1}^{12} \mathbb{Z}R_i \to E_6, \qquad R_i \mapsto r_i.$$

Here, $R_1, \dots, R_{12}$ *are standard basis vectors in* $\mathbb{Z}^{12}$*. Since the vectors* r_i *generate the lattice* E_6*, one has* $\operatorname{rk} H_1^{(-5)} = 6$*.*

Proof. We will work with the graph Γ' defined in 4.5, since the homology groups of Γ and Γ' are canonically identified. The group $C_0(\Gamma', \mathbb{Z})$ of vertices is $\bigoplus_{i=1}^{27} \mathbb{Z}v_i$. The endomorphism D^0 on it is defined in the same way as the correspondence on the 27 lines. The induced endomorphism D^1 on $C_1(\Gamma', \mathbb{Z})$ is the following. Pick one of the roots r_i. Without loss of generality, let us assume $r = \alpha_{\max}$. Then

$$D^1(a_1, b_1) = -(a_2, b_2) - \dots - (a_6, b_6).$$

By (2.6), D commutes with ∂, so defines an endomorphism on $H_1(\Gamma', \mathbb{Z})$.

The endomorphism D^1 on $C_1(\Gamma', \mathbb{Z})$ splits into 12 blocks each given by the (6×6)-matrix N such that $N_{ii} = 0$ and $N_{ij} = -1$ for $i \neq j$. It is easy to see that $(N-1)(N+5) = 0$ and that the (-5)-eigenspace of N is 1-dimensional and is generated by the vector $(a_1, b_1) + \dots + (a_6, b_6)$.

This gives an identification $C_1(\Gamma', \mathbb{Z})^{(-5)} = \bigoplus_{i=1}^{12} \mathbb{Z}R_i$. The homomorphism $\partial\colon C_1 \to C_0$ is defined by $R_i \mapsto \sum_{s=1}^{27} (r_i, e^s) v^s$, where e^s are the 27 exceptional vectors. Since the bilinear form on E_6 is nondegenerate and e^s span $E_6^\vee$, one has

$$\partial\Big(\sum_{i=1}^{12} n_i R_i\Big) = 0 \iff \left(\phi\Big(\sum_{i=1}^{12} n_i R_i\Big), e^s\right) = 0 \text{ for } s = 1, \dots, 27 \iff \phi\Big(\sum_{i=1}^{12} n_i R_i\Big) = 0.$$

Therefore, $H_1^{(-5)} = C_1^{(-5)} \cap \operatorname{Ker}(\partial) = \operatorname{Ker}(\phi)$. □

It is an elementary linear algebra exercise to pick an appropriate basis in $\operatorname{Ker}(\phi)$, which becomes especially easy if $r_1, \dots, r_6$ form a basis in E_6.

Theorem 4.9. *The limit of Prym-Tyurin varieties $P(C_t, D_t)$ as a group is the torus $(\mathbb{C}^*)^6$ with the character group $H_1^{(-5)}$. For each of the 21 internal edges e_i of the tree T, the monodromy around the divisor $\{z_i = 0\}$ in the neighborhood $U \subset \overline{\mathcal{M}}_{0,24}$ is given by the quadratic form $M_i = \sum_{s=1}^{27}((e_i^s)^*)^2$ on $H_1^{(-5)}$.*

Proof. The first statement is immediate: the limit of Jacobians as a group is a torus with the character group $H_1(\Gamma, \mathbb{Z})$, and the Prym-Tyurin varieties are obtained by taking the (-5)-eigenspace.

Every internal edge e_i of T corresponds to a node of the curve E. Over it, there are 27 nodes of the curve C. The map is étale, so the local coordinates z_i^s for the smoothings of these nodes can be identified with the local coordinate z_i. By Section 3, the matrix for the monodromy around $z_i^s = 0$ is $((e_i^s)^*)^2$. The monodromy matrix for Prym-Tyurin varieties is obtained by adding these 27 matrices together and restricting them to the (-5)-eigenspace. ☐

To compute the linear forms $(e_i^s)^*$ on $H_1(\Gamma, \mathbb{Z})$, we need to unwind the identification $H_1(\Gamma, \mathbb{Z}) = H_1(\Gamma', \mathbb{Z})$.

Lemma 4.10. *Let $p\colon \bigoplus_{i=1}^{12} \mathbb{Z}R_i \to \bigoplus_{j=1}^{21} \mathbb{Z}e_j$ be the map which associates to R_i the oriented path in the tree T of Figure 1 from the central point O to an end labeled r_i. Then, using the identification $H_1(\Gamma, \mathbb{Z})^{(-5)} = \mathrm{Ker}(\phi) \subset \bigoplus_{k=1}^{12} \mathbb{Z}R_k$, the linear functions $(e_i^s)^*$ are defined by the formula*

$$(e_i^s)^*(R_k) = \langle r_k, \ell_s\rangle \cdot \langle p(R_k), e_i^*\rangle,$$

where the first pairing is $E_6 \times E_6^* \to \mathbb{Z}$, and for the second one $\langle e_j, e_i^*\rangle = \delta_{ij}$.

Proof. Let (v_{s_1}, v_{s_2}) be an edge in Γ'. To it, we associate the path in the graph Γ going from the center of level s_1 to the center of level s_2:

$$\mathrm{path}(O_{s_1}, r_1^{s_1}) + \mathrm{path}(v_{s_1}, v_{s_2}) - \mathrm{path}(O_{s_2}, r_1^{s_2}).$$

This rule gives an identification $H_1(\Gamma', \mathbb{Z}) = H_1(\Gamma, \mathbb{Z})$.

For each of the 12 roots r_k, we have 6 edges in the graph Γ' going from the vertices s with $\langle r_k, \ell_s\rangle = 1$ to the vertices s with $\langle r_k, \ell_s\rangle = -1$. The contribution of R_k to the adjusted cycle therefore is

$$\sum_{s=1}^{27}\langle r_k, \ell_s\rangle \cdot \mathrm{path}(O_s, r_k) = \sum_{s=1}^{27}\langle r_k, \ell_s\rangle \cdot p(R_k)\big|_{e_i = e_i^s}$$

The value of the linear function e_i^s on it is therefore given by the formula in the statement. ☐

To complete the computation, we have to do the following:

(1) Choose a basis of the 6-dimensional space $H_1(\Gamma, \mathbb{Z})^{(-5)} = \mathrm{Ker}(\phi) \subset \bigoplus_{k=1}^{12} \mathbb{Z}R_k$.
(2) Compute the 21×27 linear functions $(e_i^s)^*$ on this 6-dimensional space.

(3) Compute the 21×27 quadratic functions $((e_i^s)^*)^2$, each of which is a symmetric 6×6-matrix.

(4) And finally compute the 21 monodromy matrices $M_i = \sum_{s=1}^{27}((e_i^s)^*)^2$ of Theorem 4.9.

Theorem 4.11. *There exist collections of E_6 roots $r_1, \dots, r_{12}$ generating the lattice E_6 for which the 21 symmetric (6×6)-matrices M_i of Theorem 4.9 are linearly independent.*

Proof. A concrete example is $r_1 = \alpha_{135}$, $r_2 = \alpha_{12}$, $r_3 = \alpha_{23}$, $r_4 = \alpha_{34}$, $r_5 = \alpha_{45}$, $r_6 = \alpha_{56}$, $r_7 = \alpha_{456}$, $r_8 = \alpha_{26}$, $r_9 = \alpha_{123}$, $r_{10} = \alpha_{125}$, $r_{11} = \alpha_{256}$, $r_{12} = \alpha_{15}$. An explicit computation using the the formula in Lemma 4.10, aided by a computer algebra system, shows that

(1) The monodromy matrices M_i are all divisible by 6. This corresponds to the fact that the restriction of the principal polarization from the Jacobian to the Prym-Tyurin variety is 6 times a principal polarization.

(2) For the normalized forms $M_i' = M_i/6$, the determinant of the corresponding (21×21)-matrix is $2^{12} \neq 0$.

A Mathematica notebook with an explicit computation is available at [Web15]. □

Corollary 4.12. *Theorem 0.1 holds.*

Proof. By the same argument as in the proof of Lemma 3.4, the image of the complex-analytic map $U \to \mathcal{A}_6$ has full dimension 21. Thus, the map $PT : \mathrm{Hur} \to \mathcal{A}_6$ is dominant. □

Remark 4.13. Computer experimentation shows that for a very small portion of random choices of the roots $r_1, \dots, r_{12}$, the matrices M_i are linearly independent. In most of these cases the determinant is 2^{12} but in some cases it is 2^{13}.

A necessary condition is for the roots r_1, r_2 to be non-orthogonal, and similarly for the pairs r_3, r_4, etc. Experimentation also shows that there is nothing special about the graph in Figure 1. Any other trivalent graph with 12 vertices of degree one works no worse and no better.

Remark 4.14. The theme of determining the Prym-Tyurin variety associated to an E_6-admissible cover will be picked-up again more systematically in Section 6 in the context of completely describing the map $PT : \overline{\mathrm{Hur}} \dashrightarrow \overline{\mathcal{A}}_6$ at the level of divisors.

5. The global geometry of the Hurwitz space of E_6-covers

5.1. We denote by $\mathcal{H}$ the Hurwitz space of E_6-covers $\pi : C \to \mathbb{P}^1$ together with a labeling $(p_1, \dots, p_{24})$ of its branch points. Let $\overline{\mathcal{H}}$ be the compactification of $\mathcal{H}$ by admissible $W(E_6)$-covers. By [ACV03], the stack $\overline{\mathcal{H}}$ is isomorphic to the stack of *balanced twisted stable* maps into the classifying stack $\mathcal{B}W(E_6)$ of $W(E_6)$, that is,

$$\overline{\mathcal{H}} := \overline{\mathcal{M}}_{0,24}\Big(\mathcal{B}W(E_6)\Big).$$

For details concerning the local structure of spaces of admissible coverings, we refer to [ACV03]. Note that $\overline{\mathcal{H}}$ is the normalization of the *Harris-Mumford moduli space* $\mathcal{HM}_{E_6}$ defined (in the case of covers with S_n-monodromy) in [HM82]. Points of $\mathcal{HM}_{E_6}$ are E_6-admissible coverings $[\pi : C \to R, p_1, \dots, p_{24}]$, where C and R are nodal curves of genus 46 and 0 respectively, and $p_1, \dots, p_{24} \in R_{\mathrm{reg}}$ are the branch points of π. The local monodromy of π around $p_i \in \mathbb{P}^1$ is given by a reflection $w_i \in W(E_6)$, for $i = 1, \dots, 24$. Let $\mathfrak{b} : \overline{\mathcal{H}} \to \overline{\mathcal{M}}_{0,24}$ be the *branch* morphism and $\varphi : \overline{\mathcal{H}} \to \overline{\mathcal{M}}_{46}$ be the *source* morphism. Obviously, S_{24} acts on $\overline{\mathcal{H}}$ and the projection $q : \overline{\mathcal{H}} \to \overline{\mathrm{Hur}}$ is a principal S_{24}-bundle. Passing to the S_{24}-quotient and denoting as usual $\widetilde{\mathcal{M}}_{0,n} := \overline{\mathcal{M}}_{0,n}/S_n$, we consider the induced branch and source maps

$$\mathfrak{br} : \overline{\mathrm{Hur}} \to \widetilde{\mathcal{M}}_{0,24} \quad \text{and} \quad \widetilde{\varphi} : \overline{\mathrm{Hur}} \to \overline{\mathcal{M}}_{46},$$

respectively. For each E_6-cover $[\pi : C \to \mathbb{P}^1] \in \mathrm{Hur}$, the Kanev endomorphism $D : JC \to JC$ induces, at the level of (co-)differentials, an endomorphism $D : H^0(C, K_C) \to H^0(C, K_C)$, which we denote by the same symbol, as well as a splitting

$$H^0(C, K_C) = H^0(C, K_C)^{(+1)} \oplus H^0(C, K_C)^{(-5)}$$

into $(+1)$ and (-5)-eigenspaces respectively. In turn, this induces a decomposition of the rank 46 Hodge bundle $\mathbb{E} := \widetilde{\varphi}^*(\mathbb{E})$ pulled-back from $\overline{\mathcal{M}}_{46}$ into eigenbundles

$$\mathbb{E} = \mathbb{E}^{(+1)} \oplus \mathbb{E}^{(-5)},$$

where $\mathrm{rk}(\mathbb{E}^{(+1)}) = 40$ and $\mathrm{rk}(\mathbb{E}^{(-5)}) = 6$. We set $\lambda^{(+1)} := c_1(\mathbb{E}^{(+1)})$ and $\lambda^{(-5)} := c_1(\mathbb{E}^{(-5)})$, therefore $\lambda := \widetilde{\varphi}^*(\lambda) = \lambda^{(+1)} + \lambda^{(-5)}$. We summarize the discussion in the following diagram:

(5.1)
$$\begin{tikzcd} \overline{\mathcal{H}} \arrow[r, "q"] \arrow[d, "\mathfrak{b}"'] & \overline{\mathrm{Hur}} \arrow[r, "\widetilde{\varphi}"] \arrow[d, "\mathfrak{br}"'] & \overline{\mathcal{M}}_{46} \\ \overline{\mathcal{M}}_{0,24} \arrow[r] & \widetilde{\mathcal{M}}_{0,24} & \end{tikzcd}$$

5.2. Let $\overline{\mathcal{A}}_g$ be the perfect cone compactification of $\mathcal{A}_g$. The rational Picard group of $\overline{\mathcal{A}}_g$ has rank 2 and it is generated by the first Chern class λ_1 of the Hodge bundle and the class of the *irreducible* boundary divisor $D_g := \overline{\mathcal{A}}_g - \mathcal{A}_g$. Following [Mum83], we consider the moduli space $\widetilde{\mathcal{A}}_g$ of principally polarized abelian varieties of dimension g together with their degenerations of toric rank 1. This is a partial compactification of $\mathcal{A}_g$ isomorphic to the blow-up of the open subset $\mathcal{A}_{g,\mathrm{tor.rk}\leq 1} = \mathcal{A}_g \sqcup \mathcal{A}_{g-1}$ in the Satake compactification $\overline{\mathcal{A}}_g^{\mathrm{Sat}}$.

Moreover, $\widetilde{\mathcal{A}}_g = \mathcal{A}_g \sqcup \widetilde{D}_g$, where $\widetilde{D}_g$ is an open dense subvariety of the boundary divisor D_g isomorphic to the universal Kummer variety over $\mathcal{A}_{g-1}$. Denote by b_g the proper map $\widetilde{A}_g \to \mathcal{A}_g^{\mathrm{Sat}}$.

Lemma 5.3. *Both Hodge eigenclasses $\lambda^{(+1)}$ and $\lambda^{(-5)} \in CH^1(\overline{\mathrm{Hur}})$ are nef.*

Proof. Kollár [Kol90] showed that the Hodge bundle $\mathbb{E}$ is semipositive, therefore the eigenbundles $\mathbb{E}^{(+1)}$ and $\mathbb{E}^{(-5)}$ as quotients of $\mathbb{E}$ are semipositive as well. Therefore $\det(\mathbb{E}^{(+1)})$ and $\det(\mathbb{E}^{(-5)})$ are nef line bundles. □

Theorem 5.4. *There exists an open subset $\mathrm{Hur}_{\mathrm{tor.rk}\le 1} \subset \overline{\mathrm{Hur}}$ such that PT restricts to a proper morphism $PT\colon \mathrm{Hur}_{\mathrm{tor.rk}\le 1} \to \widetilde{\mathcal{A}}_6$.*

Proof. The rational map PT extends to a *regular morphism* $PT^{\mathrm{Sat}} : \overline{\mathrm{Hur}} \to \overline{\mathcal{A}}_6^{\mathrm{Sat}}$ to the Satake compactification. At the level of points, it maps an admissible cover $\pi : C \to \mathbb{P}^1$ to the abelian part of the semiabelian Prym-Tyurin variety. In particular, the preimage

$$(PT^{\mathrm{Sat}})^{-1}(\mathcal{A}_{6,\mathrm{tor.rk}\le 1}) =: \mathrm{Hur}_{\mathrm{tor.rk}\le 1} \subseteq \overline{\mathrm{Hur}}$$

is open.

By [Ale02], the toroidal compactification $\overline{\mathcal{A}}_g^{\mathrm{vor}}$ for the 2nd Voronoi fan has a geometric meaning: its normalization is the (main irreducible component of the) moduli space of principally polarized stable semiabelic pairs $(G \curvearrowright X, \Theta)$. Here, G is a semiabelian group variety, X is a reduced projective variety with G-action, and Θ is an ample Cartier divisor not containing any G-orbits and satisfying $\Theta^g/g! = 1$. See [Ale02] for more details.

Using this fact, we obtain a rational map $PT : \overline{\mathrm{Hur}} \dashrightarrow \mathcal{A}_6^{\mathrm{vor}}$. It may not be regular. However, we claim that it is regular on $\mathrm{Hur}_{\mathrm{tor.rk}\le 1}$. The map PT is regular around a point $\pi = [\pi\colon C \to R] \in \overline{\mathrm{Hur}}$ if and only if for any 1-parameter family $[\pi_t\colon C_t \to R_t]$ with $\pi_0 = \pi$, the limit semiabelic pair $(X, \Theta) \in \overline{\mathcal{A}}_6^{\mathrm{vor}}$ depends only on π and not on the family π_t.

The semiabelian Prym-Tyurin variety G depends only on P. But for a semiabelian variety G of toric rank at most 1, there exists a unique stable pair $G \curvearrowright X \supset \Theta$. (See e.g. [Ale02] again; these are Mumford's "first order degenerations" from [Mum83]. For toric rank ≥ 2 this is no longer true). Thus, PT is a morphism over $\mathrm{Hur}_{\mathrm{tor.rk}\le 1}$ and the image is contained in $\widetilde{\mathcal{A}}_6$. Note that $\widetilde{\mathcal{A}}_g$ is an open subset that is shared by *all* toroidal compactifications of $\mathcal{A}_g$, in particular by those for perfect cones and second Voronoi fans.

Thus, we have maps $\mathrm{Hur}_{\mathrm{tor.rk}\le 1} \xrightarrow{PT} \widetilde{\mathcal{A}}_6 \xrightarrow{b_6} \mathcal{A}_{6,\mathrm{tor.rk}\le 1}$. Since both b_6 and $b_6 \circ PT$ are proper, it follows that PT is also proper. □

5.5. The conclusion of Theorem 0.1 can be restated in terms of the positivity of $\lambda^{(-5)}$. The fact that the Prym-Tyurin map $PT : \overline{\mathrm{Hur}} \dashrightarrow \overline{\mathcal{A}}_6$ is dominant can be interpreted as saying that the class $\lambda^{(-5)} \in CH^1(\overline{\mathrm{Hur}})$ is big. A similar statement can be made concerning the Kodaira dimension of $\mathcal{A}_6$, as we explain now.

Corollary 5.6. *Let D_i be the irreducible divisors supported on $\overline{\mathrm{Hur}} \setminus \mathrm{Hur}_{\mathrm{tor.rk}\leq 1}$. Then to prove that $\mathcal{A}_6$ is of general type, it suffices to show that there exist some integers a_i such that the divisor $\overline{PT^*(K_{\widetilde{\mathcal{A}}_6})} + \sum a_i D_i$ on $\overline{\mathrm{Hur}}$ is big.*

Proof. If this divisor is big on $\overline{\mathrm{Hur}}$, then its corresponding linear system has maximal Iitaka dimension. Then the linear system $|PT_* PT^*(K_{\widetilde{\mathcal{A}}_6})|$ has maximal Iitaka dimension as well. Since all boundary divisors D_i are contracted under the Prym-Tyurin map, we write $PT_*\Big(PT^*(K_{\widetilde{\mathcal{A}}_6})\Big) = PT_*\Big(PT^*(K_{\widetilde{\mathcal{A}}_6}) + \sum_i a_i D_i\Big) = \deg(PT) K_{\widetilde{\mathcal{A}}_6}$, so $K_{\widetilde{\mathcal{A}}_6}$ is big. □

It is amusing to note that one can reprove along these lines Wirtinger's classical result [Wir95] on the dominance of the Prym map in dimension 5:

Proposition 5.7. *The Prym map $P : \mathcal{R}_6 \to \mathcal{A}_5$ is generically finite if and only there exists an effective divisor $D \in \mathrm{Eff}(\overline{\mathcal{M}}_6)$ of slope $s(D) < 8$.*

Proof. We consider the rational Prym map $P : \overline{\mathcal{R}}_{g+1} \dashrightarrow \overline{\mathcal{A}}_g$ from the moduli space of stable Prym curves, the projection $\pi : \overline{\mathcal{R}}_g \to \overline{\mathcal{M}}_g$ and use the notation from [FL10] for divisors on $\overline{\mathcal{R}}_g$. In particular, δ_0^{ram} denotes the class of the ramification divisor of π and $\delta_0^{''}$ is the divisor class of Wirtinger covers. The map P is generically finite for $g = 6$ if and only if the class

$$P^*(\lambda_1) = \lambda - \frac{1}{4}\delta_0^{\mathrm{ram}} \in CH^1(\overline{\mathcal{R}}_6)$$

is big. Since $\pi^*(\delta_0) = \delta_0^{'} + \delta_0^{''} + 2\delta_0^{\mathrm{ram}}$, the class $\lambda - \frac{1}{4}\delta_0^{\mathrm{ram}} - \pi^*([D]) \in CH^1(\overline{\mathcal{R}}_6)$ is big for an effective divisor $[D] = s\lambda - \delta \in \mathrm{Eff}(\overline{\mathcal{M}}_g)$, if $s < 8$. Such a divisor D exists. For instance, one can choose D to the Petri divisor on $\overline{\mathcal{M}}_6$ consisting of curves $[C] \in \mathcal{M}_6$ having a pencil $A \in W^1_4(C)$ with $H^0(C, K_C \otimes A^{\otimes(-2)}) \neq 0$. It is known that the slope of D equals $\frac{47}{6} < 8$. □

We now turn to describing the geometry of the Hurwitz space $\overline{\mathcal{H}}$. We make the following:

Definition 5.8. For a partition $\mu = (\mu_1, \ldots, \mu_\ell) \vdash n$, we define $\mathrm{lcm}(\mu) := \mathrm{lcm}(\mu_1, \ldots, \mu_\ell)$ and $\frac{1}{\mu} := \frac{1}{\mu_1} + \cdots + \frac{1}{\mu_\ell}$. For $i = 2, \ldots, 12$, we denote by $\mathcal{P}_i$ the set of partitions $\mu \vdash 27$ describing the conjugacy class of products of i reflections in $W(E_6)$.

5.9. Each boundary divisor of $\overline{\mathcal{H}}$ corresponds to the following combinatorial data:

(1) A partition $I \sqcup J = \{1, \ldots, 24\}$, such that $|I| \geq 2$, $|J| \geq 2$.
(2) Reflections $\{w_i\}_{i\in I}$ and $\{w_j\}_{j\in J}$ in $W(E_6)$, such that $\prod_{i\in I} w_i = u$, $\prod_{j\in J} w_j = u^{-1}$, for some $u \in W(E_6)$. The sequence $w_1, \ldots, w_{24}$ is defined up to conjugation by the same element $g \in W(E_6)$.

To this data, we associate an E_6-admissible covering with a labeling of the branch points

$$t := \big[\pi : C \to R,\ p_1, \ldots, p_{24}\big] \in \overline{\mathcal{H}},$$

where $[R = R_1 \cup_q R_2, p_1, \ldots, p_{24}] \in B_{|I|} \subset \overline{\mathcal{M}}_{0,24}$ is a pointed union of two smooth rational curves meeting at the point q. The marked points lying on R_1 are precisely those labeled by the set I. Over q, the map π is ramified according to u, that is, the points in $\pi^{-1}(q)$ correspond to cycles in the permutation u considered as an element of S_{27}. Let $\mu := (\mu_1, \ldots, \mu_\ell) \vdash 27$ be the partition induced by $u \in S_{27}$ and denote by $E_{i:\mu}$ the boundary divisor on $\overline{\mathcal{H}}$ classifying E_6-admissible coverings with $\pi^{-1}(q)$ having partition type μ, and precisely i of the points $p_1, \ldots, p_{24}$ lying on R_1. Clearly only partitions from $\mathcal{P}_i$ are considered.

In Table 1 we give the list of partitions of 27 appearing as products of reflections in $W(E_6)$ (using GAP notation for the conjugacy classes). For future use, we also record the invariants $\frac{1}{\mu}$, for each $\mu \in \mathcal{P}_i$.

5.10. We recall the local structure of the morphism $\mathfrak{b} : \overline{\mathcal{H}} \to \overline{\mathcal{M}}_{0,24}$, over the point t, see also [HM82] p.62. The (non-normalized) space $\mathcal{HM}_{E_6}$ is locally described by its local ring

$$\hat{\mathcal{O}}_{t,\mathcal{HM}_{E_6}} = \mathbb{C}[[t_1, \ldots, t_{21}, s_1, \ldots, s_\ell]]/s_1^{\mu_1} = \cdots = s_\ell^{\mu_\ell} = t_1, \tag{5.2}$$

where t_1 is the local parameter on $\overline{\mathcal{M}}_{0,24}$ corresponding to smoothing the node $q \in R$. By passing to the normalization $\nu : \overline{\mathcal{H}} \to \mathcal{HM}_{E_6}$, we deduce that over each point of the fibre $\nu^{-1}(t)$ the map $\mathfrak{b} : \overline{\mathcal{H}} \to \overline{\mathcal{M}}_{0,24}$ is ramified with index $\mathrm{lcm}(\mu)$. Indeed, if $\widetilde{\pi} : \widetilde{C} \to R$ is the Galois closure of the degree 27 cover $\pi : C \to R$ corresponding to t, then each point from $\widetilde{\pi}^{-1}(q)$ has ramification index $\mathrm{lcm}(\mu)$ over q. Thus for each $i = 2, \ldots, 12$, we have a decomposition

$$\mathfrak{b}^*(B_i) = \sum_{\mu \in \mathcal{P}_i} \mathrm{lcm}(\mu) E_{i:\mu},$$

where the partitions in $\mathcal{P}_i$ can be found in Table 1.

In view of applications to the Kodaira dimension of $\overline{\mathcal{H}}$, we discuss in detail the pull-back $\mathfrak{b}^*(B_2)$. We pick a point $t = [\pi : C = C_1 \cup C_2 \to R = R_1 \cup_q R_2,\ p_1, \ldots, p_{24}] \in \mathfrak{B}_2$ like in 5.9, where $C_i = \pi^{-1}(R_i)$. Without loss of generality, we assume that $I = \{1, \ldots, 22\}$, thus $p_1, \ldots, p_{22} \in R_1$ and $p_{23}, p_{24} \in R_2$. The group $G = \langle w_1, \ldots, w_{22}\rangle$ generated by the reflections in the remaining roots $r_1, \ldots, r_{22} \in E_6$ is the Weyl group for a lattice $L = L_G \subset E_6$. Since $\prod_{i=1}^{24} w_i = 1$, it follows that $w_{23} \cdot w_{24} \in G$, hence $\mathrm{rk}(L) \geq \mathrm{rk}(E_6) - 1 = 5$.

5.11. Assume that the reflections w_{23} and w_{24} corresponding to the coalescing points p_{23} and p_{24} are equal, hence $w_{23} = w_{24}$. In this case, the corresponding partition is $\mu = (1^{27})$ and

Number of reflections	Partition μ of 27	Conjugacy class	$\frac{1}{\mu}$
0, 2, 4, 6	1^{27}	1a	27
1, 3, 5	$(2^6, 1^{15})$	2c	18
2, 4, 6	$(2^{10}, 1^7)$	2b	12
2, 4, 6	$(3^6, 1^9)$	3b	11
3, 5	$(2^{12}, 1^3)$	2d	9
3, 5	$(4^5, 2^1, 1^5)$	4d	$\frac{27}{4}$
3, 5	$(6^1, 3^4, 2^3, 1^3)$	6e	6
4, 6	$(2^{12}, 1^3)$	2a	9
4, 6	(3^9)	3c	3
4, 6	$(4^6, 1^3)$	4a	$\frac{9}{2}$
4, 6	$(4^5, 2^3, 1)$	4b	$\frac{15}{4}$
4, 6	$(5^5, 1^2)$	5a	3
4, 6	$(6^3, 2^3, 1^3)$	6b	5
4, 6	$(6^2, 3^2, 2^4, 1)$	6d	4
5	$(4^5, 2^3, 1)$	4c	$\frac{15}{4}$
5	$(6^2, 3^5)$	6f	2
5	$(6^4, 3^1)$	6g	1
5	$(8^3, 2^1, 1)$	8a	$\frac{7}{8}$
5	$(10^1, 5^3, 2^1)$	10a	$\frac{58}{5}$
5	$(12^1, 6^1, 4^2, 1)$	12b	$\frac{7}{4}$
6	$(3^9)^*$	3a	$\frac{1}{3}$
6	$(6^4, 3^1)^*$	6a	$\frac{11}{3}$
6	$(6^3, 2^3, 1^3)^*$	6c	5
6	(9^3)	9a	$\frac{1}{3}$
6	$(12^2, 3^1)$	12a	$\frac{19}{6}$

TABLE 1. Products of reflections in $W(E_6)$

we set $E_0 := E_{2:1^{27}}$. We denote by E_L the boundary divisor of admissible covers in $E_{2:(1^{27})}$ corresponding to the lattice L. The map $\mathfrak{b}$ is unramified along each divisor E_L and we have

$$E_0 = \sum_{L \subset E_6} E_L \subset \overline{\mathcal{H}}.$$

The general cover t corresponding to each divisor E_L carries no automorphism preserving all branch points $p_1, \ldots, p_{24}$, that is, $\mathrm{Aut}(t) = \{\mathrm{Id}\}$.

Suppose now that the reflections w_{23} and w_{24} are distinct. Following [Dol12], we distinguish two possibilities depending on the relative position of the two double-sixes, described in terms of a general admissible cover $t = \big[\pi : C = C_1 \cup C_2 \to R_1 \cup_q R_2,\ p_1, \ldots, p_{24}\big]$.

5.12. The reflections w_{23} and w_{24} form an *azygetic* pair, that is, the corresponding roots r_{23} and r_{24} satisfy $r_{23}\cdot r_{24}\neq 0$. In this case, $\langle w_{23}, w_{24}\rangle = W(A_2)$ and $r_{23}+r_{24}$ or $r_{23}-r_{24}$ is again a root that is azygetic to both r_{23} and r_{24}. The double-sixes associated to w_{23} and w_{24} share 6 points and the permutation $w_{23}\cdot w_{24}$ decomposes into 6 disjoint three cycles, therefore $\mu=(3^6,1^9)\vdash 27$. Accordingly, $C_2=\pi^{-1}(R_2)$ decomposes into six rational components mapping $3:1$, respectively 9 components mapping isomorphically onto R_2. If

$$E_{\rm azy} := E_{2:(3^6,1^9)} \subset \overline{\mathcal{H}}$$

is the boundary divisor parametrizing such points, then $\mathfrak{b}$ is triply ramified along $E_{\rm azy}$. The general point of $E_{\rm azy}$ has no non-trivial automorphisms preserving all the branch points.

5.13. The reflections w_{23} and w_{24} form a *syzygetic* pair, that is, $r_{23}\cdot r_{24}=0$. We have $\langle w_{23}, w_{24}\rangle = W(A_1^2)$. The two associated double-sixes share 4 points and $w_{23}\cdot w_{24}\in S_{27}$ decomposes into a product of 10 disjoint transpositions, therefore $\mu=(2^{10},1^7)$. Eight of these transpositions are parts of the double-sixes corresponding to w_{23} and w_{24} that remain disjoint respectively. Note that C_2 consists of 8 rational components mapping $2:1$ onto R_2, as well as a smooth rational component, say Z, mapping $4:1$ onto R_2. The fibers $\pi_Z^{-1}(q), \pi_Z^{-1}(p_1)$ and $\pi_Z^{-1}(p_2)$ each consist of two ramification points. We denote by

$$E_{\rm syz} := E_{2:(2^{10},1^7)} \subset \overline{\mathcal{H}}$$

the boundary divisor of admissible syzygetic covers. For a general cover $t\in E_{\rm syz}$, note that $\mathrm{Aut}(t)=\mathbb{Z}_2$, see Remark 5.16.

5.14. To summarize the discussion above, we have the following relation:

$$\mathfrak{b}^*(B_2) = E_0 + 3E_{\rm azy} + 2E_{\rm syz}. \tag{5.3}$$

In opposition to E_0, we show in Theorem 6.19 that the boundary divisors $E_{\rm azy}$ or $E_{\rm syz}$ have fewer components. Precisely, for a general element $t\in E_{\rm azy}$ or $t\in E_{\rm azy}$, we always have $G=W(L)=W(E_6)$, hence the subcurve $C_1=\pi^{-1}(R_1)$ is irreducible.

5.15. The Hurwitz formula applied to the ramified cover $\mathfrak{b}:\overline{\mathcal{H}}\to\overline{\mathcal{M}}_{0,24}$, coupled with the expression $K_{\overline{\mathcal{M}}_{0,24}}\equiv\sum_{i=2}^{12}\bigl(\frac{i(24-i)}{23}-2\bigr)B_i$ to be found e.g. in [KM13], yields

$$K_{\overline{\mathcal{H}}} = \mathfrak{b}^*K_{\overline{\mathcal{M}}_{0,24}} + \mathrm{Ram}(\mathfrak{b}) = -\frac{2}{23}[E_0] + \frac{19}{23}[E_{\rm syz}] + \frac{40}{23}[E_{\rm azy}] + N, \tag{5.4}$$

where N is the *effective* combination of the boundary divisors of $\overline{\mathcal{H}}$ disjoint from $E_0, E_{\rm syz}$ and $E_{\rm azy}$, with the coefficient of $[E_{i:\mu}]$ for $i=3,\ldots,12$ being equal to $\mathrm{lcm}(\mu)\bigl(\frac{i(24-i)}{23}-1\bigr)-1>0$.

The ramification divisor of the projection $q:\overline{\mathcal{H}}\to\overline{\mathrm{Hur}}$ is contained in the pull-back $\mathfrak{b}^*(B_2)$ (recall the commutative diagram 5.1). Note that B_2 is the ramification divisor of the map $\overline{\mathcal{M}}_{0,24} \to \widetilde{\mathcal{M}}_{0,24}$. The general point of each of the components of E_0 and E_{azy} admits an involution compatible with the involution of the rational curve R_2 preserving q and interchanging the branch points p_{23} and p_{24} respectively. No such automorphism exists for a general point of the divisor E_{syz} (see Remark 5.16), thus

$$\mathrm{Ram}(q) = E_0 + E_{\mathrm{azy}}.$$

Remark 5.16. We illustrate the above statement in the case of the divisor E_{syz}. We choose a general point $t := \big[\pi : C \to R = R_1 \cup_q R_2, p_1, \ \ldots, p_{24}\big] \in E_{\mathrm{syz}}$, and denote by $\pi_Z : Z \to R_2$, the degree 4 cover having as source a smooth rational curve Z and such that $\pi_Z^*(q) = 2u+2v$, and $\pi_Z^*(p_i) = 2x_i + 2y_i$, for $i = 23, 24$. Then $\mathrm{Aut}(t) = \mathbb{Z}_2$. Furthermore, there exist a *unique* automorphism $\tau \in \mathrm{Aut}(R_2)$ such that $\tau(q) = q$, $\tau(p_{23}) = p_{24}$ and $\tau(p_{24}) = p_{23}$, as well as a unique automorphism $\sigma \in \mathrm{Aut}(Z)$ with $\sigma(u) = u, \sigma(v) = v$, $\sigma(x_{23}) = y_{23}$ and $\sigma(x_{24}) = y_{24}$ and such that $\pi_Z \circ \sigma = \tau \circ \pi_Z$. Note that σ induces the unique non-trivial automorphism of t fixing all the branch points. In contrast, the general point of E_{azy} corresponds to an admissible cover which has no automorphisms fixing all the branch points.

Definition 5.17. On the space $\overline{\mathrm{Hur}}$ of unlabeled E_6-covers, we introduce the *reduced* boundary divisors $D_0, D_{\mathrm{syz}}, D_{\mathrm{azy}}$, as well as the boundary divisors $\big\{D_{i:\mu} : 3 \le i \le 12, \mu \in \mathcal{P}_i\big\}$ which pull-back under the map $q : \mathcal{H} \to \overline{\mathrm{Hur}}$ to the corresponding divisors indexed by E's, that is, $q^*(D_0) = 2E_0$, $q^*(D_{\mathrm{azy}}) = 2E_{\mathrm{azy}}$, $q^*(D_{\mathrm{syz}}) = E_{\mathrm{syz}}$ and $q^*(D_{i:\mu}) = E_{i:\mu}$, for $3 \le i \le 12$ and $\mu \in \mathcal{P}_i$. More generally, for each sublattice $L \subset E_6$, we denote by $D_L \subset \overline{\mathrm{Hur}}$ the reduced divisor characterized by pulling-back to E_L under the map q.

If D is an irreducible divisor on $\overline{\mathrm{Hur}}$, we denote as usual by $[D] := [D]_{\mathbb{Q}} \in CH^1(\overline{\mathrm{Hur}})_{\mathbb{Q}}$ its $\mathbb{Q}$-class, that is, the quotient of its usual class by the order of the automorphism group of a general point from D.

Theorem 5.18. *The canonical class of the Hurwitz space* $\overline{\mathrm{Hur}}$ *is given by the formula:*

$$K_{\overline{\mathrm{Hur}}} = -\frac{25}{46}[D_0] + \frac{19}{23}[D_{\mathrm{syz}}] + \frac{17}{46}[D_{\mathrm{azy}}] + \sum_{i=3}^{12}\sum_{\mu \in \mathcal{P}_i} \Big(\mathrm{lcm}(\mu)\Big(\frac{i(24-i)}{23} - 1\Big) - 1\Big)[D_{i:\mu}].$$

Proof. We apply the Riemann-Hurwitz formula to the map $q : \overline{\mathcal{H}} \to \overline{\mathrm{Hur}}$ and we find

$$q^*(K_{\overline{\mathrm{Hur}}}) = K_{\overline{\mathcal{H}}} - [E_0] - [E_{\mathrm{azy}}] = -\frac{25}{23}[E_0] + \frac{19}{23}[E_{\mathrm{syz}}] + \frac{17}{23}[E_{\mathrm{azy}}] + \cdots \in CH^1(\overline{\mathcal{H}}).$$

□

5.19. We describe the Hodge class on $\overline{\mathcal{H}}$ in terms of boundary divisors, and to that end we set notation. For $2 \le i \le 12$, let $B_i := \sum_{|T|=i} \delta_{0:T} \in \mathrm{Pic}(\overline{\mathcal{M}}_{0,24})$ be the boundary class, where the sum runs over all subsets $T \subset \{1, \ldots, 24\}$ of cardinality i. Recall that $\delta_{0:T}$

is the closure of the locus of pointed curves consisting of two rational components, such that the marked points lying on one component are precisely those labeled by T. Let $\widetilde{B}_i$ be the *reduced* boundary divisor on $\widetilde{\mathcal{M}}_{0,24}$ which pulls-back to B_i under the quotient map $\overline{\mathcal{M}}_{0,24} \to \widetilde{\mathcal{M}}_{0,24}$. Furthermore, let $\psi_1, \ldots, \psi_{24} \in \mathrm{Pic}(\overline{\mathcal{M}}_{0,24})$ be the cotangent tautological classes corresponding to the marked points. The universal curve over $\overline{\mathcal{M}}_{0,24}$ is the morphism $\pi := \pi_{25} : \overline{\mathcal{M}}_{0,25} \to \overline{\mathcal{M}}_{0,24}$, forgetting the marked point labeled by 25. The following formulas are well-known, see e.g. [FG03]:

Proposition 5.20. *The following relations hold:*

$$(1)\ c_1(\omega_\pi) = \psi_{25} - \sum_{i=1}^{24} \delta_{0:i,25} \in CH^1(\overline{\mathcal{M}}_{0,25}).$$

$$(2)\ \sum_{i=1}^{24} \psi_i = \sum_{i=2}^{12} \frac{i(24-i)}{23}[B_i] \in CH^1(\overline{\mathcal{M}}_{0,24}), \quad (3)\ \kappa_1 = \sum_{i=2}^{12} \frac{(i-1)(23-i)}{23}[B_i].$$

We now find a boundary expression for the Hodge class at the level of $\overline{\mathcal{H}}$.

Theorem 5.21. *The Hodge class at the level of $\overline{\mathcal{H}}$ is given by the following formula:*

$$\lambda = \sum_{i=2}^{12} \sum_{\mu \in \mathcal{P}_i} \frac{1}{12} \mathrm{lcm}(\mu) \Big(\frac{9i(24-i)}{23} - 27 + \frac{1}{\mu} \Big) [E_{i:\mu}].$$

Note that a boundary formula for λ in the case of S_n-covers has appeared first in [KKZ11] and confirmed later with algebraic methods in [vdGK12].

Proof. Over the Hurwitz space $\overline{\mathcal{H}}$ we consider the universal E_6-admissible cover $f : \mathcal{C} \to P$, where $P := \overline{\mathcal{H}} \times_{\overline{\mathcal{M}}_{0,24}} \overline{\mathcal{M}}_{0,25}$. Note that P is the universal *orbicurve* of genus zero over $\overline{\mathcal{H}}$. This means that over a general point $t = [C \to R, p_1, \ldots, p_{24}]$ of a boundary divisor $E_{i:\mu}$, where $\mu = (\mu_1, \ldots, \mu_\ell) \in \mathcal{P}_i$ corresponding to the local description (5.2), even though P has a singularity of type $A_{\mathrm{lcm}(\mu)-1}$, the space $\mathcal{C}$ has singularities of type $A_{\mathrm{lcm}(\mu)/\mu_i - 1}$ at the ℓ points corresponding to the inverse image of R_{sing}.

We further denote by $\phi : P \to \overline{\mathcal{H}}$ and by $\tilde{q} : P \to \overline{\mathcal{M}}_{0,25}$ the two projections and by $v := \phi \circ f : \mathcal{C} \to \overline{\mathcal{H}}$ and $\tilde{f} := \tilde{q} \circ f : \mathcal{C} \to \overline{\mathcal{M}}_{0,25}$ respectively. The ramification divisor of f decomposes into components $R = R_1 + \cdots + R_{24} \subset \mathcal{C}$, where a general point of R_i is of the form $[C \to R,\ p_1, \ldots, p_{24}, x]$, where $x \in C$ is one of the six ramification points lying over the branch point p_i. In particular $f_*([R_i]) = 6[\mathfrak{B}_i]$, where $\mathfrak{B}_i \subset P$ is the corresponding branch divisor.

We apply the Riemann-Hurwitz formula for f and write: $c_1(\omega_v) = f^* \tilde{q}^* c_1(\omega_\pi) + [R]$. We are going to push-forward via v the square of this identity and describe all the intervening terms in the process. Over $\overline{\mathcal{H}}$ we have the identity:

$$\varphi^*(\kappa_1) = \upsilon_* c_1^2(\omega_\upsilon) = \upsilon_*\Big(\big(\tilde{f}^* c_1^2(\omega_\pi) + 2\tilde{f}^* c_1(\omega_\pi)\cdot[R] + [R]\big)^2\Big).$$

We evaluate each appearing term: $\upsilon_*\Big(\tilde{f}^* c_1(\omega_\pi)\cdot[R]\Big) =$

$$\sum_{i=1}^{24}\phi_*(\tilde{q}^* c_1(\omega_\pi)\cdot 6[\mathfrak{B}_i]) = 6\sum_{i=1}^{24}\phi_*\tilde{q}^*\Big(c_1(\omega_\pi)\cdot[\Delta_{0:i,25}]\Big) = 6\mathfrak{b}^*\Big(\sum_{i=1}^{24}\psi_i\Big).$$

Furthermore, $f^*(\mathfrak{B}_i) = 2R_i + A_i$, where A_i is the *anti-ramification* divisor mapping $15:1$ onto $\mathfrak{B}_i$. Note that A_i and R_i are disjoint, hence $f^*([\mathfrak{B}_i])\cdot R_i = 2R_i^2$, therefore

$$\upsilon_*([R_i]^2) = 3\phi_*([\mathfrak{B}_i^2]) = 3\phi_*(\tilde{q}^*(\delta_{0:i,25}^2)) = -3\mathfrak{b}^*(\psi_i).$$

Using Proposition 5.20, we find that

$$\upsilon_*([R]^2) = \upsilon^*\Big(\sum_{i=1}^{24}[R_i]^2\Big) \equiv -3\sum_{i=2}^{12}\frac{i(24-i)}{23}\mathfrak{b}^*(B_i).$$

We will use Proposition 5.20, and the relation $\pi_*(\delta_{0:i,25}^2) = -\psi_i$, to write:

$$\upsilon_*\tilde{f}^* c_1^2(\omega_\pi) = \phi_*\Big(27\tilde{q}^* c_1^2(\omega_\pi)\Big) = 27\mathfrak{b}^*\pi_*\Big(\psi_{25} - \sum_{i=1}^{24}\delta_{0:i,24}\Big)^2 =$$

$$27\mathfrak{b}^*\Big(\kappa_1 - \sum_{i=1}^{24}\psi_i\Big) \equiv -27\mathfrak{b}^*\Big(\sum_{i=2}^{12}B_i\Big).$$

We find the following expression for the pull-back of the Mumford class to $\overline{\mathcal{H}}$:

$$\upsilon_* c_1^2(\omega_\upsilon) \equiv \sum_{i=2}^{12}\Big(\frac{9i(24-i)}{23} - 27\Big)\mathfrak{b}^*(B_i) \equiv \sum_{i=2}^{12}\sum_{\mu\in\mathcal{P}_i}\mathrm{lcm}(\mu)\Big(\frac{9i(24-i)}{23} - 27\Big)E_{i:\mu}. \tag{5.5}$$

Using Mumford's GRR calculation in the case of the universal genus 46 curve $\upsilon:\mathcal{C}\to\overline{\mathcal{H}}$, coupled with the local analysis of the fibres of the map $\mathfrak{b}$, we have that

$$12\varphi^*(\lambda) \equiv \upsilon_* c_1^2(\omega_\upsilon) + \sum_{i=2}^{12}\sum_{\mu\in\mathcal{P}_i}\mathrm{lcm}(\mu_1,\ldots,\mu_\ell)\Big(\frac{1}{\mu_1}+\cdots+\frac{1}{\mu_\ell}\Big)E_{j:\mu}.$$

Substituting in (5.5), we finish the proof. □

Remark 5.22. Using Definition 5.17, we spell out Theorem 5.21 at the level of $\overline{\mathrm{Hur}}$:

$$\lambda = \frac{33}{46}[D_0] + \frac{7}{46}[D_{\mathrm{azy}}] + \frac{17}{46}[D_{\mathrm{syz}}] + \cdots \in CH^1(\overline{\mathrm{Hur}}). \tag{5.6}$$

Proposition 5.23. *The morphism $\varphi:\overline{\mathcal{H}}\to\overline{\mathcal{M}}_{46}$ has ramification of order 12 along the divisor E_0. In particular, the class $\varphi^*(\delta_0) - 12[E_0] - 2[E_{\mathrm{syz}}]\in CH^1(\overline{\mathcal{H}})$ is effective.*

Proof. The morphism φ factors via $\overline{\mathrm{Hur}}$, that is, $\varphi = \widetilde{\varphi} \circ q$, where recall that $q : \overline{\mathcal{H}} \to \overline{\mathrm{Hur}}$ is the projection map and $\widetilde{\varphi} : \overline{\mathrm{Hur}} \to \overline{\mathcal{M}}_{46}$. We have observed that q is ramified along E_0. Furthermore, since the general element of $\varphi(E_0)$ is a 6-nodal curve, the local intersection number $\big(\widetilde{\varphi}(\Gamma) \cdot \delta_0\big)_{\widetilde{\varphi}(t)}$, for any curve $\Gamma \subset \overline{\mathrm{Hur}}$ passing through a point $t \in q(E_0)$, is at least equal to 6. Finally, $[E_{\mathrm{syz}}]$ appears with multiplicity 2 because, as pointed out in Remark 5.16, each point of E_{syz} has an automorphism of order 2. $\square$

5.24. To establish the bigness of the class $K_{\overline{\mathcal{H}}}$ we shall use Moriwaki's class [Mor98]

$$\mathfrak{mo} := (8g+4)\lambda - g\delta_0 - \sum_{i=1}^{\lfloor \frac{g}{2} \rfloor} 4i(g-i)\delta_i \in CH^1(\overline{\mathcal{M}}_g).$$

It is shown in [Mor98] that $\mathfrak{mo}$ non-negatively intersects all complete curves in $\overline{\mathcal{M}}_g$ whose members are stable genus g curves with at most one node. Furthermore, the rational map $\phi_{n\cdot\mathfrak{mo}} : \overline{\mathcal{M}}_g \dashrightarrow \mathbb{P}^{\nu}$ defined by a linear system $|n\cdot\mathfrak{mo}|$ with $n \gg 0$, induces a *regular* morphism on $\mathcal{M}_g$. In our situation when $g = 46$, this implies that the pull-back $\varphi^*(\mathfrak{mo})$ is an *effective* $\mathbb{Q}$-divisor class on $\overline{\mathcal{H}}$, which we shall determine. In what follows, if D_1 and D_2 are divisors on a normal variety X, we write $D_1 \geq D_2$ if $D_1 - D_2$ is effective.

Proposition 5.25. *The following divisor class on the Hurwitz space $\overline{\mathcal{H}}$*

$$-\frac{2}{23}E_0 + \frac{523}{2415}E_{\mathrm{syz}} + \frac{62}{115}E_{\mathrm{azy}} + \sum_{i=3}^{12}\sum_{\mu \in \mathcal{P}_i} \frac{93}{1610} i(24-i)\mathrm{lcm}(\mu)E_{i:\mu}$$

is effective.

Proof. We give a lower bound for the coefficient of $E_{i:\mu}$ in the expression $\varphi^*(\lambda)$ of Theorem 5.21, by observing that for a partition $(\mu_1, \ldots, \mu_\ell) \vdash 27$, the inequality $\frac{1}{\mu_1} + \cdots + \frac{1}{\mu_\ell} \leq 27$ holds. Using this estimate together with Theorem 5.21 $\varphi^*(\lambda) = \frac{33}{23}[E_0] + \frac{17}{46}[E_{\mathrm{syz}}] + \frac{7}{23}[E_{\mathrm{azy}}] + \cdots$, as well as Proposition 5.23, we write

$$0 \leq \frac{1}{210}\varphi^*(\mathfrak{mo}) \leq \frac{372}{210}\varphi^*(\lambda) - \frac{46\cdot 12}{210}[E_0] - \frac{46\cdot 2}{210}[E_{\mathrm{syz}}] =$$

$$-\frac{2}{23}[E_0] + \frac{523}{2415}[E_{\mathrm{syz}}] + \frac{62}{115}[E_{\mathrm{azy}}] + \sum_{i=3}^{12}\sum_{\mu \in \mathcal{P}_i} \frac{93}{1610} i(24-i)\mathrm{lcm}(\mu)[E_{i:\mu}].$$

The scaling has been chosen in such a way to match the negative E_0 coefficient in the canonical class $K_{\overline{\mathcal{H}}}$ of (5.4). $\square$

As a step towards determining the Kodaira dimension of $\overline{\mathrm{Hur}}$ we establish the following:

Theorem 5.26. *The canonical class of $\overline{\mathcal{H}}$ is big.*

Proof. Recalling that $\mathfrak{b} : \overline{\mathcal{H}} \to \overline{\mathcal{M}}_{0,24}$, for each $0 < \alpha < 1$, using (5.4) we write the equality

$$K_{\overline{\mathcal{H}}} = (1-\alpha)\mathfrak{b}^*(\kappa_1) + \alpha\mathfrak{b}^*(\kappa_1) - \sum_{i=2}^{12}\sum_{\mu\in\mathcal{P}_i} E_{i:\mu}.$$

Since the class $\kappa_1 \in CH^1(\overline{\mathcal{M}}_{0,24})$ is well-known to be ample, in order to establish that $K_{\overline{\mathcal{H}}}$ is big, it suffices to show that for α sufficiently close to 1, the class $\alpha\mathfrak{b}^*(\kappa_1) - \sum_{i,\mu\in\mathcal{P}_i}[E_{i:\mu}]$ is effective. After brief inspection, this turns out to be a consequence of Proposition 5.25. $\square$

6. The Prym-Tyurin map along boundary components of $\overline{\mathrm{Hur}}$

In this section we refine the analysis of the boundary divisors of $\overline{\mathcal{H}}$, in particular we identify divisors that are not contracted by the Prym-Tyurin-Kanev map.

6.1. Following 5.9, we denote by $E_{I:L_1,L_2,\mu}$ the divisor of $\overline{\mathcal{H}}$ of E_6-admissible covers

$$t := [\pi : C := C_1 \cup C_2 \to R_1 \cup_q R_2, p_1, \ldots, p_{24}],$$

where R_1 contains the branch points $\{p_i\}_{i\in I}$, with roots $\{r_i\}_{i\in I}$ generating the lattice $L_1 \subset E_6$ and the corresponding reflections generating the group $G := W(L_1) \subset W(E_6)$, whereas R_2 contains the branch points $\{p_j\}_{j\in J}$, where $I \cup J = \{1, \ldots, 24\}$, with roots $\{r_j\}_{j\in J}$ generating the lattice $L_2 \subset E_6$ and the reflections generating the group $H := W(L_2) \subset W(E_6)$ respectively. We set $u := \prod_{i\in I} w_i$, therefore $u^{-1} = \prod_{j\in J} w_j$. As before, $\mu \vdash 27$ is the partition corresponding to the cycle type of $u \in S_{27}$, and which describes the fibre $\pi^{-1}(q)$. Let O_G (respectively O_H) denote the set of orbits of G (respectively H) on the set $\overline{27} := \{1, \ldots, 27\}$. In particular, there is a bijection between O_G (respectively O_H) and the set of irreducible components of C_1 (respectively C_2).

Returning to the notation of boundary divisors used in the previous section, for $\mu \in \mathcal{P}_i$ we write $E_{i:\mu} = \sum_{|I|=i,L_1,L_2} E_{I:L_1,L_2,\mu}$, the sum being taken over all possible sublattices L_1 and L_2 of E_6 as above.

6.2. We describe the *toric rank* of the Prym-Tyurin variety associated to a general point of $E_{I:L_1,L_2,\mu}$. The following result is expressed in terms of the graph $\Gamma(u, O_G, O_H)$, to be defined below (Definition 6.6), which is the dual graph of the source of an E_6-admissible cover corresponding to a general point of $E_{I:L_1,L_2,\mu}$. This graph is endowed with an endomorphism $D : H_1(\Gamma(u, O_G, O_H), \mathbb{Q}) \to H_1(\Gamma(u, O_G, O_H), \mathbb{Q})$ induced by the action of the Kanev correspondence.

Theorem 6.3. *Let $t \in E_{I,J:L_1,L_2}$ be a general point in a boundary divisor of $\overline{\mathcal{H}}$ corresponding to the above data. Then the toric rank of the Prym-Tyurin variety $PT(C, D)$ equals the dimension of the (-5)-eigenspace $H_1\big(\Gamma(u, O_G, O_H), \mathbb{Q}\big)^{(-5)}$.*

In case both curves C_1 and C_2 are irreducible, the above result simplifies considerably.

Corollary 6.4. *Assume that $|O_G| = |O_H| = 1$, that is, both groups G and H act transitively on the set $\overline{27}$. Then the toric rank of $PT(C, D)$ equals the dimension of invariant subspace of u in the 6-dimensional representation $E_6 \otimes \mathbb{Q}$ of $W(E_6)$.*

Example 6.5. Table 2 gives the dimension of the invariant subspace $(E_6 \otimes \mathbb{Q})^u$ for the 25 conjugacy classes of $W(E_6)$, for which we use the same notation as in Table 1. The number in the name is the order of u, even elements (products of even number of reflections) are listed first, then the odd ones. We computed the dimensions of the invariant subspaces from the character table by the formula $\dim(E_6 \otimes \mathbb{Q})^u = \sum_{n=1}^{\mathrm{ord}(u)} \frac{\chi_{E_6}(u^n)}{\mathrm{ord}(u)}$.

1a	2a	2b	3a	3b	3c	4a	4b	5a	6a	6b	6c	6d	9a	12a
6	2	4	0	4	2	2	2	2	0	2	0	2	0	0

2c	2d	4c	4d	6e	6f	6g	8a	10a	12b
5	3	1	3	3	1	1	1	1	1

TABLE 2. Dimensions of $(E_6 \otimes \mathbb{Q})^u$ for $u \in W(E_6)$

The conclusions of Corollary 6.4 and Example 6.5 concerning the abelian part of $PT(C, D)$ agree with [LR08, p.236].

Definition 6.6. Let $u \in W(E_6)$ and let $A = A_1 \sqcup \ldots \sqcup A_a$ and $B = B_1 \sqcup \ldots \sqcup B_b$ be two u-invariant partitions of the set $\overline{27}$. We define the graph $\Gamma(u, A, B)$ to be the following bipartite graph:

(1) The vertices are $A_1, \ldots, A_a$ and $B_1, \ldots, B_b$ respectively.
(2) The edges correspond to cycles C_k in the cyclic representation of $u \in S_{27}$, including cycles of length 1.
(3) For each cycle C_k, there exist unique vertices A_i and B_j containing the set c_k. Then the edge C_k joins A_i and B_j.

When both partitions A and B are trivial, that is each consists of the single set $\overline{27}$, we set $\Gamma_u := \Gamma(u, \overline{27}, \overline{27})$ and $\Gamma_1 := \Gamma(1, \overline{27}, \overline{27})$ respectively.

Example 6.7. The graph Γ_1 has 2 vertices and 27 edges. One has $C_1(\Gamma_1, \mathbb{Z}) = \mathbb{Z}^{27}$, and $H_1(\Gamma_1, \mathbb{Z}) \simeq \mathbb{Z}^{26}$ consists of elements $\sum_{s=1}^{27} n_s e_s$ with $\sum_{s=1}^{27} n_s = 0$. There is a natural degree 10 homomorphism $D_1 \colon C_1(\Gamma_1, \mathbb{Q}) \to C_1(\Gamma_1, \mathbb{Q})$ with eigenvalues $10, 1, -5$, which induces a homomorphism $D \colon H_1(\Gamma_1, \mathbb{Q}) \to H_1(\Gamma_1, \mathbb{Q})$ with $(+1)$-eigenspace of dimension 20 and (-5)-eigenspace of dimension 6 respectively.

Lemma 6.8. *For $u \in W(E_6)$, the following statements hold:*

(1) $H_1(\Gamma_u, \mathbb{Q})^{(-5)} = \big(H_1(\Gamma_1, \mathbb{Q})^{(-5)}\big)^u$ *(that is, the u-invariant subspace), and*
(2) $H_1(\Gamma(u, A, B), \mathbb{Q})^{(-5)} \subset H_1(\Gamma_u, \mathbb{Z})^{(-5)}$.

Proof. Suppose we have the following cycle decomposition $u = C_1 \cdot C_2 \cdot \dots \cdot C_k \in S_{27}$ and let $n := \operatorname{ord}(u)$ and $\ell(C_i)$ denote the length of C_i. We write $C_1(\Gamma_u, \mathbb{Z}) = \bigoplus_{i=1}^k \mathbb{Z} e_{C_i}$. Then one has an orthogonal projection

$$C_1(\Gamma_1, \mathbb{Z}) \twoheadrightarrow C_1(\Gamma_u, \mathbb{Z}), \qquad \text{edge } e \mapsto \frac{1}{n}\sum_{i=0}^{n-1} u^i \cdot e,$$

identifying $C_1(\Gamma_u, \mathbb{Z})$ with a sublattice in $C_1(\Gamma_1, \mathbb{Q})$ via the injection $e_{C_i} \mapsto \frac{1}{\ell(C_i)} \sum_{j \in C_i} e_j$. This induces a surjection from $H_1(\Gamma_1, \mathbb{Z})$ to $H_1(\Gamma_u, \mathbb{Z})$, which clearly commutes with D, that is, $D(C_1(\Gamma_u, \mathbb{Z})) \subset C_1(\Gamma_u, \mathbb{Z})$. It follows that $H_1(\Gamma_u, \mathbb{Z})^{(-5)}$ is the projection of $H_1(\Gamma_1, \mathbb{Z})^{(-5)}$ to the (-5)-eigenspace in $C_1(\Gamma_u, \mathbb{Q})$ and that

$$H_1(\Gamma_u, \mathbb{Q})^{(-5)} = \big(H_1(\Gamma_1, \mathbb{Q})^{(-5)}\big)^u.$$

The graph $\Gamma(u, A, B)$ is obtained from Γ_u by splitting the two vertices into $a + b$ new vertices. This can be obtained by inserting in place of the two vertices two trees with a and b vertices – without changing H_1 – and then removing the edges of these trees. Thus, one has an inclusion $H_1(\Gamma(u, A, B), \mathbb{Z}) \subset H_1(\Gamma_u, \mathbb{Z})$, commuting with D, which gives an inclusion $H_1(\Gamma(u, A, B), \mathbb{Z})^{(-5)} \subset H_1(\Gamma_u, \mathbb{Z})^{(-5)}$. ☐

Lemma 6.9. *The (-5)-eigenspace $H_1(\Gamma(u, A, B), \mathbb{Q})^{(-5)}$ is a subspace of the u-invariant subspace $(E_6 \otimes \mathbb{Q})^u$ in the standard 6-dimensional $W(E_6)$-representation.*

Proof. Indeed, $H_1(\Gamma_1, \mathbb{Q})^{(-5)} = E_6 \otimes \mathbb{Q}$, therefore $(H_1(\Gamma_1, \mathbb{Q})^u)^{(-5)} = (E_6 \otimes \mathbb{Q})^u$. ☐

Proof of Theorem 6.3. For any E_6-admissible cover $\pi \colon C_1 \cup C_2 \to R_1 \cup R_2$ with a correspondence D, the toric part of $PT(C, D)$ is the torus with character lattice $H_1(\Gamma, \mathbb{Z})^{(-5)}$, where Γ is the dual graph of C. So we only need to show that $\Gamma = \Gamma(u, O_G, O_H)$. But that is clear, for the edges of Γ are the nodes of C, that is, the cycles in $u \in S_{27}$. ☐

6.10. In order to illustrate Theorem 6.3 in concrete situations, we classify all root sublattices of E_6. Recall that with the notation of Section 2, the standard roots in the E_6 Dynkin diagram are $r_2 = \alpha_{12}, \dots, r_6 = \alpha_{56}$, and $r_1 = \alpha_{123}$. In the extended Dynkin diagram $\widetilde{E}_6$ there is an additional root $r_0 = -\alpha_{\max}$, so that $3r_4 + 2r_1 + 2r_3 + 2r_5 + r_2 + r_6 + r_0 = 0$.

Lemma 6.11. *The following is the complete list of root sublattices $L \subset E_6$:*

(1) *If $\dim(L) = 6$, then L is either E_6, or isomorphic to A_5A_1, or A_2^3.*

(2) *If* $\dim(L) = 5$, *then* L *is isomorphic to* A_5, D_5, A_4A_1, $A_3A_1^2$, *or* $A_2^2A_1$.

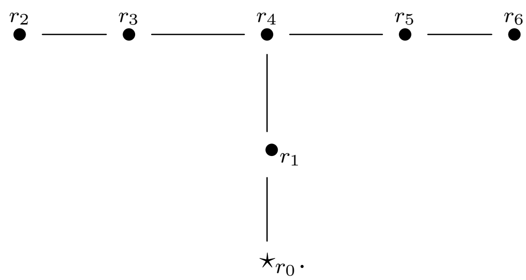
(3) *If* $\dim(L) = 4$, *then* L *is isomorphic to* A_4, D_4, A_2^2, A_3A_1, $A_2A_1^2$, *or* A_1^4.

(4) *If* $\dim(L) = 3$, *then* L *is isomorphic to* A_3, A_2A_1, *or* A_1^3.

(5) *If* $\dim(L) = 2$, *then* L *is isomorphic to* A_2, *or* A_1^2.

(6) *If* $\dim(L) = 1$, *then* L *is isomorphic to* A_1.

Furthermore, all the above sublattices (and the associated subgroups) can be obtained by removing vertices from the extended E_6 *diagram* $\widetilde{E}_6$*:*



If the root lattices L, L' *corresponding to reflections subgroups* G *and* G' *of* $W(E_6)$ *are isomorphic, then they differ by an automorphism of the* E_6 *lattice, and the corresponding subgroups* G *and* G' *are conjugate in* $W(E_6)$.

Proof. We first note that there is a natural bijection between root sublattices L of E_6 and subgroups G generated by reflections of $W(E_6)$. One has $\mathrm{Aut}(E_6) = W(E_6) \oplus \mathbb{Z}_2$, with $\mathbb{Z}_2$ acting on E_6 by multiplication by ± 1. Any automorphism of E_6 induces an automorphism of $W(E_6)$, and the kernel of $\phi\colon \mathrm{Aut}(E_6) \to \mathrm{Aut}\, W(E_6)$ is $\mathbb{Z}_2$. Finally, by [Fra01, Section 2.3]), all automorphisms of $W(E_6)$ are inner, so that $\mathrm{Aut}\, W(E_6) = W(E_6)$ and ϕ is surjective.

Thus, the proof reduces to showing that all root sublattices of E_6 are of the above types, and that if L, L' are isomorphic as abstract root lattices, then they differ by an element of $\mathrm{Aut}(E_6)$. The statement that all such root sublattices correspond to proper subdiagrams of the extended Dynkin diagram $\widetilde{E}_6$ is an *a posteriori* observation.

The standard method for finding all root sublattices of a given root lattice is described in [BdS49, Dyn52]. A modern treatment can be found in [DL11, Theorem 1]. The method is to repeatedly apply the following two procedures to Dynkin diagrams Γ, starting from $\Gamma = E_6$: (1) remove a node, and/or (2) replace one of the connected components Γ_s of Γ by an extended Dynkin diagram $\widetilde{\Gamma}_s$ and remove a node from it. Applying the above method repeatedly, we obtain all the lattices listed above. The fact that isomorphic root sublattices differ by an automorphism of E_6 is a case by case computation. This can also be found in [Osh06, Table 10.2]. $\square$

6.12. Table 3 lists the orbits for one choice of roots (the other choices being similar) for each type of lattice. The last column describes the degrees of the maps from the irreducible components of C_1 to R_1. We keep the Schläfli notation a_i, b_i, c_{ij} for the elements of the set $\overline{27}$, which is being identified with the set of lines of a cubic surface. The smooth (possibly disconnected) curve C_1 is a 27-sheeted cover of $R_1 = \mathbb{P}^1$, with branch points $\{p_i\}_{i\in I}$, the local monodromy over p_i being given by the reflection w_i, and with a branch point q, with local monodromy u^{-1}, where $u = \Pi_{i\in I} w_i$.

We apply Theorem 6.3 to compute the toric ranks associated to the divisors

$$E_L := \sum_{|I|=22} E_{I;L,A_1,(1^{27})} \subset \overline{\mathcal{H}}.$$

Since $\dim(L) \geq 5$, using Lemma 6.11, we have the following possibilities:

$$L \in \{E_6, A_5A_1, A_2^3, A_5, D_5, A_4A_1, A_3A_1^2, A_2^2A_1\}.$$

Proposition 6.13. *The toric rank of each boundary divisor E_L with $L \neq E_6$ is equal to zero. The toric rank of E_{E_6} is equal to 1.*

Proof. Note that there are $|O_H| = 21$ vertices on the right, of which 15 vertices are ends and thus can be removed without changing the homology of the graph. The remaining 6 vertices each have degree 2. Contracting unnecessary edges, we reduce the calculation to a graph with $|O_G|$ vertices and 6 edges. The 6 edges correspond to the 6 transpositions appearing in the decomposition of $w_{23} = w_{24} \in S_{27}$.

Assume $|O_G| = 1$, which, by Table 3, is the case if and only if $L = E_6$. Then $H_1(\Gamma, \mathbb{Z}) = \bigoplus_{i=1}^6 \mathbb{Z}e_i$ and $D(e_i) = -\sum_{j\neq i} e_j$. Therefore $H_1(\Gamma, \mathbb{Q})^{(-5)}$ is 1-dimensional and generated by the element $e_1 + \cdots + e_6$. The other cases follow similarly by direct calculation. □

Remark 6.14. For more details concerning the calculation of the toric rank in the case $L = D_5$, see (7.2).

6.15. Although the divisor theory of $\overline{\mathrm{Hur}}$ is quite complicated, we now show that most of these divisors are contracted under the Prym-Tyurin map. We first establish the following:

Theorem 6.16. *Assume that the image of a component B of $E_{I:L_1,L_2}$ under the map PT has codimension 1 in $\widetilde{\mathcal{A}}_6$. Then $\{|I|, |I^c|\} = \{2, 22\}$.*

6.17. Before proving the above theorem, we need a few preliminaries. Fix a general point $\left[\pi : C = C_1 \cup C_2 \to R_1 \cup R_2\right] \in B$, where C_1 and C_2 are smooth (possibly reducible) curves. As in (3.1), the generalized Jacobian JC is an extension of the Jacobian $JC_1 \times JC_2$ of the normalization $C_1 \sqcup C_2$ by a torus:

$$0 \longrightarrow H^1(\Gamma, \mathbb{C}^*) \longrightarrow JC \longrightarrow JC_1 \times JC_2 \longrightarrow 0$$

Sublattice	Roots	Orbits	Degrees
E_6	$r_1, \dots, r_6$	$\{a_i, b_i, c_{ij}\}$	27
A_5A_1	$r_0, r_2, \dots, r_6$	$\{a_i, b_i\}$, $\{c_{ij}\}$	15, 12
A_2^3	$r_i, i \neq 4$	$\{a_i, b_i, c_{ij} \mid 1 \leq i,j \leq 3\}$, $\{a_i, b_i, c_{ij} \mid 4 \leq i,j \leq 6\}$, $\{c_{ij} \mid 1 \leq i \leq 3, 4 \leq j \leq 6\}$	9^3
D_5	$r_1, \dots, r_5$	$\{a_6\}$, $\{a_i, b_6, c_{ij} \mid 1 \leq i,j \leq 5\}$, $\{b_i, c_{i6} \mid 1 \leq i \leq 5\}$	1, 10, 16
A_5	$r_2, \dots, r_6$	$\{a_i\}, \{b_i\}, \{c_{ij}\}$	$6^2, 15$
A_4A_1	$r_0, r_2, \dots, r_5$	$\{a_i, c_{ij} \mid 1 \leq i,j \leq 4\}, \{b_i, c_{56} \mid 1 \leq i \leq 4\}$ $\{a_5, a_6\}, \{b_j, c_{ij} \mid 1 \leq i \leq 4, 5 \leq j \leq 6\}$	$2, 5, 10^2$
$A_3A_1^2$	r_0, r_2, r_3, r_4, r_6	$\{a_1, b_i, c_{ij}, c_{56} \mid 2 \leq i,j \leq 4\}, \{b_1\}, \{a_i, c_{ij} \mid 2 \leq i,j \leq 4\}$ $\{a_5, a_6, c_{15}, c_{16}\}, \{b_j, c_{ij} \mid 2 \leq i \leq 4, 5 \leq j \leq 6\}$	$8^2, 6, 4, 1$
$A_2^2A_1$	$r_i, i \neq 0, 4$	$\{a_i, c_{ij} \mid 1 \leq i,j \leq 3\}, \{b_i, c_{ij} \mid 4 \leq i,j \leq 6\}, \{a_4, a_5, a_6\}$ $\{b_1, b_2, b_3\}, \{c_{ij} \mid 1 \leq i \leq 3, 4 \leq j \leq 6\}$	$9, 6^2, 3^2$
A_4	$r_2, \dots, r_5$	$\{a_i, c_{ij} \mid 1 \leq i,j \leq 4\}, \{b_i, c_{56} \mid 1 \leq i \leq 4\}, \{a_5\}, \{a_6\}$ $\{b_5, c_{i6} \mid 1 \leq i \leq 4\}, \{b_6, c_{i5} \mid 1 \leq i \leq 4\}$	$10, 5^3, 1^2$
D_4	r_1, r_3, r_4, r_5	$\{a_1, c_{ij}, b_6 \mid 2 \leq i,j \leq 5\}, \{a_i, c_{1i} \mid 2 \leq i \leq 5\}, \{a_6\}$ $\{b_1\}, \{b_i, c_{i6} \mid 2 \leq i \leq 5\}, \{c_{16}\}$	$8^3, 1^3$
A_2^2	r_2, r_3, r_5, r_6	$\{a_1, a_2, a_3\}, \{b_1, b_2, b_3\}, \{a_4, a_5, a_6\}, \{b_4, b_5, b_6\}$ $\{c_{12}, c_{13}, c_{23}\}, \{c_{45}, c_{46}, c_{56}\}, \{c_{ij} \mid 1 \leq i \leq 3, 4 \leq j \leq 6\}$	$9, 3^6$
A_3A_1	r_2, r_3, r_4, r_6	$\{c_{56}\}, \{a_5, a_6\}, \{b_5, b_6\}, \{a_1, \dots, a_4\}, \{b_1, \dots, b_4\}$ $\{c_{ij} \mid 1 \leq i,j \leq 4\}, \{c_{ij} \mid 1 \leq i \leq 4, 5 \leq j \leq 6\}$	$8, 6, 4^2, 2^2, 1$
$A_2A_1^2$	r_1, r_2, r_3, r_5	$\{a_6\}, \{b_6, c_{45}\}, \{b_1, b_2, b_3\}, \{c_{16}, c_{26}, c_{36}\}, \{b_5, b_6, c_{64}, c_{65}\}$ $\{a_4, a_5\}, \{a_i, c_{ij} \mid 1 \leq i,j \leq 3\}, \{c_{ij} \mid 1 \leq i \leq 3, 4 \leq j \leq 5\}$	$6^2, 4, 3^2, 2^2, 1$
A_1^4	r_0, r_2, r_4, r_6	$\{a_6\}, \{b_1\}, \{a_1, b_6, c_{23}, c_{45}\}, \{a_2, a_3, c_{12}, c_{13}\}, \{a_4, a_5, c_{14}, c_{15}\}$ $\{b_2, b_3, c_{26}, c_{36}\}, \{b_4, b_5, c_{46}, c_{56}\}, \{c_{24}, c_{34}, c_{25}, c_{35}\}, \{c_{16}\}$	$4^6, 1^3$
A_3	r_2, r_3, r_4	$\{c_{12}, c_{13}, c_{14}, c_{23}, c_{24}, c_{34}\}, \{c_{15}, c_{25}, c_{35}, c_{45}\}, \{c_{16}, c_{26}, c_{36}, c_{46}\}$ $\{a_1, a_2, a_3, a_4\}, \{b_1, b_2, b_3, b_4\}, \{a_i\}, \{b_i\}, 5 \leq i \leq 6, \{c_{56}\}$	$6, 4^4, 1^5$
A_2A_1	r_1, r_2, r_3	$\{b_1, b_2, b_3\}, \{c_{14}, c_{24}, c_{34}\}, \{c_{15}, c_{25}, c_{35}\}, \{a_1, a_2, a_3, c_{12}, c_{13}, c_{23}\}$ $\{c_{16}, c_{26}, c_{36}\}, \{b_j, c_{kl}\}, \{j,k,l\} = \{4,5,6\}, \{a_i\}, 4 \leq i \leq 6$	$6, 3^4, 2^3, 1^3$
A_1^3	r_2, r_4, r_5	$\{c_{13}, c_{14}, c_{23}, c_{24}\}, \{c_{15}, c_{16}, c_{25}, c_{26}\}, \{c_{35}, c_{36}, c_{45}, c_{46}\}$ $\{a_i, a_{i+1}\}, \{b_i, b_{i+1}\}, \{c_{ii+1}\}, i = 1, 3, 5$	$4^3, 2^6, 1^3$
A_2	r_2, r_3	$\{a_1, a_2, a_3\}, \{b_1, b_2, b_3\}, \{c_{12}, c_{13}, c_{23}\}, \{c_{14}, c_{24}, c_{34}\}$ $\{c_{15}, c_{25}, c_{35}\}, \{c_{16}, c_{26}, c_{36}\}, \{a_i\}, \{b_i\}, \{c_{ij}\}, 4 \leq i,j \leq 6$	$3^6, 1^9$
A_1^2	r_2, r_4	$\{c_{13}, c_{23}, c_{14}, c_{24}\}, \{c_{56}\}, \{a_i, a_{i+1}\}, \{b_i, b_{i+1}\}$, $\{c_{ij}, c_{i+1j}\}, \{a_j\}, \{b_j\}, \{c_{ii+1}\}, i = 1, 3, j = 5, 6$	$4, 2^8, 1^7$
A_1	r_0	$\{a_i, b_i\}, \{c_{ij}\}, 1 \leq i,j \leq 6$	$2^6, 1^{15}$

Table 3. Sublattices and Orbits

where Γ is the dual graph of C. In particular, $H^1(\Gamma,\mathbb{C}^*)\cong\mathbb{G}_m^{k_C}$, where k_C is the toric rank of JC. The correspondence D induces correspondences $D_i\subset C_i\times C_i$ and endomorphisms $D_i: JC_i\to JC_i$. Since D acts in the fibers of π, the induced endomorphisms $JC_i\to JC_{3-i}$ are zero, for $i=1,2$. Define $P_i:=\mathrm{Im}(D_i-1)\subset JC_i$. The above facts can be summarized as follows:

Lemma 6.18. *The abelian part of the Prym-Tyurin variety*

$$P:=PT(C,D):=\mathrm{Im}\{D-1:JC\to JC\}$$

is isogenous to $P_1\times P_2$.

Proof of Theorem 6.16. Without loss of generality, we may assume that $i:=|I|\geq 12$. If the image of B has codimension 1 inside $\widetilde{\mathcal{A}}_6$, then for a general point of B, the toric rank k_P of the corresponding Prym-Tyurin variety P is either 0 or 1.

Suppose first that $k_P=0$. Then $P\cong P_1\times P_2$. If both P_1 and P_2 have positive dimension, then P belongs to a subvariety of $\mathcal{A}_6$ parametrizing products and each such subvariety has codimension greater than 1. So one of the P_i is zero. The parameter space of P_1 has dimension at most $i-2$, that of P_2 has dimension at most $22-i$. Since the parameter space of P is 20-dimensional and $i\geq 12$, we have $P_2=0$, and $\dim(P_1)=6$. We deduce $i=22$.

Now assume $k_P=1$. In this case the image of B is the boundary divisor $\widetilde{D}_6$ of $\widetilde{\mathcal{A}}_6$ and the product $P_1\times P_2$ is a general abelian variety of dimension 5. Once again, one of the P_i is zero. The assumption $i\geq 12$ implies $P_2=0$, hence $\dim(P_1)=5$. The parameter space of P_1 is 15 dimensional, which implies $i\geq 17$. Let $\{p_1,\dots,p_\ell\}=C_1\cap C_2$ be the set of the nodes of C, which also label the edges of Γ. For each i, let p_i' and p_i'' be the points of C_1 and C_2 respectively that we identify to obtain p_i on C. Choose the orientation of Γ in such a way that each edge p_i is oriented from C_1 to C_2. The extension

$$0\longrightarrow H^1(\Gamma,\mathbb{C}^*)\longrightarrow JC\longrightarrow JC_1\times JC_2\longrightarrow 0$$

is given by the map $\phi: H_1(\Gamma,\mathbb{Z})\to JC_1\times JC_2$ sending the edge p_i to $p_i''-p_i'$. The extension

$$0\longrightarrow \mathbb{G}_m=(D-1)H^1(\Gamma,\mathbb{C}^*)\longrightarrow P\longrightarrow P_1\times P_2\longrightarrow 0$$

is therefore given by the map $\phi_P:(D-1)H_1(\Gamma,\mathbb{Z})\to P_1\times P_2$ obtained by composing the restriction of ϕ to $(D-1)H_1(\Gamma,\mathbb{Z})\subset H_1(\Gamma,\mathbb{Z})$ with $(D-1)$ and dividing by 6, for the restriction of the polarization of JC to P is 6 times a principal polarization. Since $P_2=0$, the class of the extension is therefore the projection of $\phi_P(D-1)H_1(\Gamma,\mathbb{Z})$ to P_1. Equivalently, we can project $\phi(D-1)$ first to JC_1 and then compose with (D_1-1) on JC_1.

This means that the extension class of P does not depend on C_2 (up to a finite set). Therefore the moduli of C_2 does not produce positive moduli for the extension class of P. It follows that the cover $C_1 \to R_1$ depends on 20 moduli, hence R_1 contains at least 22 branch points, therefore $i = 22$. $\square$

The following result shows that the boundary divisors have many fewer irreducible components than one would a priori expect. Recall that in 5.12 and 5.13 we have introduced the divisors $E_{\mathrm{azy}} := \sum_{|I|=22} E_{I:L1,A_2,(3^6,1^9)}$ and $E_{\mathrm{syz}} := \sum_{|I|=22} E_{I:L_1,A_1^2,(2^{10},1^7)}$ respectively.

Theorem 6.19. *Assume that $|I| = 22$ and $L_2 = A_2$ or A_1^2. Then $E_{I:L_1,L_2}$ is empty unless $L_1 = E_6$. In other words, for a general E_6-admissible cover*

$$\left[\pi : C = C_1 \cup C_2 \to R_1 \cup_q R_2,\ p_1, \dots, p_{24}\right] \in E_{\mathrm{azy}} \quad \textit{or } E_{\mathrm{syz}},$$

the curve C_1 is irreducible with monodromy $W(E_6)$ over R_1.

Proof. Consider first the azygetic case $L_2 = A_2$. Then, as we saw in 5.12, the curve C_2 has 15 components, each of which intersects C_1 in exactly one point. Therefore, no component of C_2 can connect two components of C_1 and C_1 is irreducible.

In the syzygetic case, as we saw in 5.13, the curve C_2 has 16 components. Of the components of C_2, only the 4-sheeted cover (denoted by Z in 5.13) intersects C_1 in two points. All other components of C_2 intersect C_1 in exactly one point. It follows that C_1 has at most two irreducible components. Looking now at Table 3, we see that there are only two possibilities for the lattice L_1, namely $L_1 = E_6$ or $L_1 = A_5A_1$.

We now eliminate the possibility $L_1 = A_5A_1$ in the syzygetic case. It is a consequence of Lemma 6.11 that the A_5 summand of L_1 is the orthogonal complement of the A_1 summand. Hence the lattice L_1 and the group G_1 generated by the reflections $w_1, \dots, w_{22}$ are determined by the A_1 sublattice. Since all reflections are conjugate, we can assume that the A_1 summand is generated by the reflection w_0 in the root r_0 (see 6.12). Since $\langle G_1, w_{23}\rangle = \langle G_1, w_{24}\rangle = W(E_6)$, the reflections w_{23}, w_{24} do not belong to G_1. Therefore the pairs (w_0, w_{23}), (w_0, w_{24}) are azygetic.

After a permutation of the indices $\{1, \dots, 6\}$, we can assume that w_{23} is the reflection in the root α_{123} and w_{24} is the reflection in the root α_{145} (see, e.g., [Dol12, Section 9.1]). The composition $w_{23} \cdot w_{24} \in S_{27}$, contains the double transposition $(a_1, b_6)(c_{23}, c_{45})$ which acts on the 4-sheeted cover Z. However, $w_{23} \cdot w_{24}$ also contains the transposition (a_2, c_{13}) which acts on a degree 2 component of C_2, i.e., the points corresponding to a_2 and c_{13} come together over the node. Looking at the orbits of C_1 in Table 3 in the A_5A_1 case, we see that the points a_2 and c_{13} belong to two different components of C_1 and cannot come together over the node, which is a contradiction. $\square$

We now consider the components of the divisor E_0 introduced in 5.11. Recall that

$$E_L := \sum_{|I|=22} E_{I;L,A_1,(1^{27})} \subset \overline{\mathcal{H}}.$$

Theorem 6.20. *For $L \subsetneq E_6$ the divisor E_L is contracted by PT.*

Proof. Let $[\pi : C := C_1 \cup C_2 \to R_1 \cup_q R_2]$ be a general element of a component B of E_L with $L \subsetneq E_6$. By Proposition 6.13, the toric rank of $P := PT(C, D)$ is 0. As in the proof of Theorem 6.16 and with the notation there, we have $P = P_1 \times P_2 = P_1 = PT(C_1, D_1)$ because all the components of C_2 are rational. Furthemore, since $C_1 \to R_1$ is not ramified at q, the isomorphism class of C_1 and hence also of P_1 is independent of the choice of the point q. It follows that $P = P_1$ depends on at most $19 = \dim(\mathcal{M}_{0,22})$ parameters, hence B is contracted by PT. $\square$

We summarize the results of this section as follows:

Corollary 6.21. *The only boundary divisors of $\overline{\mathcal{H}}$ that are not contracted under the Prym-Tyurin map are E_{E_6}, E_{syz} and E_{azy}. The divisor E_{E_6} maps onto the boundary divisor of $\overline{\mathcal{A}}_6$, whereas E_{syz} and E_{azy} map onto divisors not supported on the boundary of $\overline{\mathcal{A}}_6$.*

7. Ordinary Prym varieties regarded as Prym-Tyurin-Kanev varieties

The aim of this section is to illustrate how 6-dimensional Prym varieties appear as Prym-Tyurin varieties of type E_6 and thus prove Theorem 0.4. The Prym moduli space $\mathcal{R}_7$ has codimension 3 inside $\mathcal{A}_6$, where we identify $\mathcal{R}_7$ with the image of the generically injective Prym map $P : \mathcal{R}_7 \to \mathcal{A}_6$. We shall show that the boundary divisor D_{D_5} of $\overline{\mathrm{Hur}}$ is an irreducible component of $PT^{-1}(\mathcal{R}_7)$ and we shall explicitly describe the 2-dimensional fibres of the restriction $PT_{D_{D_5}} : D_{D_5} \to \overline{\mathcal{R}}_7$.

7.1. Consider an admissible cover $[\pi : C = C_1 \cup C_2 \to R_1 \cup R_2]$ in the divisor D_{D_5} of $\overline{\mathrm{Hur}}$. Such a cover can be described as follows. The cover $C_1 \to R_1$ has D_5-monodromy generated by the roots $r_1, \ldots, r_5$, and it is ramified at 22 distinct points. The local monodromy at each branch point is given by one of the reflections $w_i \in W(D_5)$ associated to r_i, choosing the ordering such that $\prod_{i=1}^{22} w_i = 1$. The cover $C_2 \to R_2$ has A_1-monodromy generated by the root r_0, and is branched at 2 points. Both covers are unramified at the point $q \in R_1 \cap R_2$. As listed in Table 3, we have the following irreducible components and orbits for C_1:

$$\begin{aligned}
F_1 :&\quad \{b_1, b_2, b_3, b_4, b_5, c_{16}, c_{26}, c_{36}, c_{46}, c_{56}\}\\
F_2 :&\quad \{a_1, a_2, a_3, a_4, a_5, c_{12}, c_{13}, c_{14}, c_{15}, c_{23}, c_{24}, c_{25}, c_{34}, c_{35}, c_{45}, b_6\}\\
F_0 :&\quad \{a_6\},
\end{aligned}$$

and the following irreducible components and orbits for C_2:

$$\begin{array}{lll} 1 \le i \le 6 & H_i: & \{a_i, b_i\} \\ 7 \le i \le 21 & H_i: & \{c_{k(i)\ell(i)}\}, \text{ for some choice of integers } k(i) < \ell(i), \text{ between 1 and 6.} \end{array}$$

One computes that the three components F_1, F_2 and F_0 of C_1 have genera $13, 29$ and 0, and map onto R_1 with degree 10, 16 and 1 respectively. The components of C_2 are all rational with $H_1, \ldots, H_6$ mapping $2:1$ to R_2 and $H_7, \ldots, H_{21}$ mapping isomorphically. The description of the orbits given above also specifies the points of intersection F_i and H_j. For instance, H_6 intersects F_2 at a point corresponding to b_6 and it intersects F_0 at a point corresponding to a_6.

7.2. In order to compute the toric rank of the Prym-Tyurin variety $P := (D-1)(JC_1)$, we apply the correspondence D to the homology group $H_1(\Gamma, \mathbb{Z})$, where Γ denotes the dual graph of the stable curve $F_1 \cup F_2$ (see Section 6 for the notation). The graph Γ consists of 2 vertices joined by 5 edges: $e_1 := (b_5, b_6), e_2 := (c_{16}, c_{15}), e_3 := (c_{26}, c_{25}), e_4 := (c_{36}, c_{35}), e_5 := (c_{46}, c_{45})$ and $H_1(\Gamma, \mathbb{Z}) = \bigoplus_{i=1}^4 \mathbb{Z}\ (e_i - e_{i+1})$ (see 3.2). One computes

$$D(\partial(e_1 - e_2)) = D(b_6 - b_5) - D(c_{15} - c_{16}) = b_6 - b_5 - (c_{15} - c_{16}) = \partial(e_1 - e_2).$$

By Remark 2.6, D commutes with ∂, hence $D(e_1 - e_2) = (e_1 - e_2)$. Similarly, one checks that $D(e_i - e_{i+1}) = (e_i - e_{i+1})$, for $i = 1, \ldots, 4$, hence $(D-1)H_1(\Gamma, \mathbb{Z}) = 0$. Therefore, the Prym-Tyurin variety $P := (D-1)(JC_1)$ has toric rank 0 and it is contained in JC_1, since $JC_2 = \{0\}$.

7.3. As is apparent from the description of the orbits, the correspondence D restricts to a fixed-point-free involution $\iota : F_1 \to F_1$, a correspondence D_2 of valence 5 on F_2, a correspondence $D_{12} : F_1 \to F_2$ and its transpose $D_{21} : F_2 \to F_1$ of degree 8 over F_1 and of degree 5 over F_2.

The variety P is the image of the following endomorphism of $JC_1 = JF_1 \times JF_2$:

$$\begin{pmatrix} \iota - 1 & D_{21} \\ D_{12} & D_2 - 1 \end{pmatrix}.$$

7.4. Let $f : F_1 \to Y$ be the induced unramified double cover on the curve $Y := F_1/\langle\iota\rangle$ of genus 7. Note that the degree 10 map $\pi_1 : F_1 \to R_1$ factors through a degree 5 map $h : Y \to R_1$. The image $Q_1 := (\iota - 1)JF_1 \subset JF_1$ is the ordinary Prym variety $P(F_1, \iota)$ associated to the double cover $[f : F_1 \to Y] \in \mathcal{R}_7$.

7.5. The relationship between the curves F_1 and F_2 (or between the tower $F_1 \xrightarrow{f} Y \xrightarrow{h} R_1$ and the map $\pi_2 : F_2 \to R_1$) is an instance of the *pentagonal construction* ([Don92, Section

5.17]). This is the $n = 5$ case of the n-gonal construction, see [Don92, Section 2] or [ILS09a, Section 1], which applies to covers $F_1 \xrightarrow{f} Y \xrightarrow{h} R_1$ whose Galois group is the Weyl group $W(D_n)$. The idea is to consider the following curve inside the symmetric product $F_1^{(n)}$:

$$h_*F_1 := \left\{ G \in F_1^{(n)} \ : \ \mathrm{Nm}_f(G) = h^{-1}(t), \text{ for some } t \in R_1 \right\}.$$

The induced map $h_*F_1 \to R_1$ is of degree $2^n = 32$ and one can check that above a branch point $t \in R_1$ there are exactly $2^{n-2} = 8$ simple ramification points in h_*F_1.

Proposition 7.6. *h_*F_1 is the union of two isomorphic components $h_*F_1 = X_0 \sqcup X_1$, with $X_0 \simeq X_1$ being smooth curves of genus $1 + 2^{n-3}(n + g(Y) - 5) = 29$.*

Proof. The splitting is explained in [Don92, Section 2.2] and [ILS09a, Section 1]. The smoothness is proved in [ILS09a, Lemma 1.1]. The genus calculation follows from the Hurwitz formula. ☐

We can distinguish the components by looking at the parity of the number of common points in two divisors in the same fiber of $h_*F_1 \to R_1$. We say that two divisors $G_1, G_2 \in h_*F_1$ are equivalent if $\mathrm{Nm}_f(G_1) = \mathrm{Nm}_f(G_2)$ and they share an *even* number of points of F_1.

We specialize to the case at hand, when $n = 5$. Let $X = X_0$ be the component of h_*F_1 whose fiber over a point $t \in R_1$ can be identified with the class of the divisor $c_{16} + \cdots + c_{56}$. The proof of the following result is immediate.

Proposition 7.7. *The map $\psi : F_2 \to X$ given by $x \mapsto D_{21}(x) \in h_*F_1$ is an isomorphism.*

Remark 7.8. Under the above identification, the restriction D_2 of the Kanev (incidence) correspondence coincides with the correspondence D defined in [ILS09a, Section 2]. Also, the restriction D_{21} of the Kanev correspondence coincides with the correspondence S defined in [IL12, Section 2]. It follows from [ILS09a, Corollary 6.2] that the image Q_2 of the ordinary Prym variety Q_1 in JF_2 by D_{21} is the eigen-abelian variety of D_2 for the eigenvalue $-n+2 = -3$. It also follows from [ILS09a, Section 6.6] and [Kan87, Theorem 3.1] that in this case Q_2 is a Prym-Tyurin variety of dimension 6 and exponent $4 = -(-3)+1$ for the correspondence D_2. The restriction ρ of the correspondence $D-1$ to $Q_1 \subset JF_1$ gives the sequence of isogenies of principally polarized abelian varieties

$$\begin{array}{ccccc} Q_1 & \xrightarrow{\rho} & P & \longrightarrow & Q_2 \\ x_1 & \longmapsto & ((\iota - 1)x_1, D_{12}x_1) & \longmapsto & D_{12}x_1. \end{array}$$

Along the boundary divisor D_{D_5}, two of the 24 branch points collide, which leads to the following:

Proposition 7.9. *The map ρ factors through multiplication by 2 to induce an isomorphism $Q_1 := P(F_1, \iota) \simeq P$ and a surjection $Q_1 \to Q_2 := P(F_2, D_2)$ whose kernel is a maximal isotropic subgroup $\mathbb{H}$ (for The Weil pairing) of the group of points of order 2 in Q_1.*

Proof. For an abelian variety A and $n \in \mathbb{Z}$, we denote by $n_A : A \to A$ the morphism given by multiplication by n. It follows from [IL12, Corollary 2.3] that $D_{21} \circ D_{12} = 8_{Q_1}$. A straightforward generalization of the proof of [IL12, Proposition 3.3] implies that $D_{12} = \varphi \circ 2_{Q_1}$ for an isogeny $\varphi : Q_1 \to Q_2$ such that $\varphi^* \Theta_{Q_2} = \Theta_{Q_1}$, where Θ_{Q_i} is the polarization of Q_i. Therefore we have $\varphi \circ \varphi^t = 2_{Q_2}$. It follows that the kernel of φ is a maximal isotropic subgroup $\mathbb{H}$ of the group of points of order 2 in Q_1. Since the restriction of $\iota - 1$ to Q_1 is -2_{Q_1}, its kernel is the subgroup of points of order 2. Therefore $\rho = \psi \circ 2_{Q_1}$, where now $\psi : Q_1 \to P$ is injective hence an isomorphism. $\square$

8. The Weyl-Petri realization of the Hodge eigenbundles

Here we show that, at least over an open dense subset of $\overline{\mathrm{Hur}}$, the Hodge eigenbundles $\mathbb{E}^{(+1)}$ and $\mathbb{E}^{(-5)}$ admit a Petri-like incarnation, which makes them amenable to intersection-theoretic calculations.

8.1. For a smooth E_6-cover $[\pi : C \to \mathbb{P}^1] \in \mathrm{Hur}$, set $L := \pi^*(\mathcal{O}_{\mathbb{P}^1}(1)) \in W^1_{27}(C)$ and let

$$\mu(L) : H^0(C, L) \otimes H^0(C, K_C \otimes L^\vee) \to H^0(C, K_C)$$

be the Petri map given by multiplication of global sections. We write $\pi_*(\mathcal{O}_C) = \mathcal{O}_{\mathbb{P}^1} \oplus V^\vee$, where V is a rank 26 vector bundle on $\mathbb{P}^1$. Then $\mu(L)$ is injective if and only if the splitting type of V is as balanced as possible, which in this case means that $V = \mathcal{O}_{\mathbb{P}^1}(2)^{\oplus 6} \oplus \mathcal{O}_{\mathbb{P}^1}(3)^{\oplus 20}$. We assume that $h^0(C, L) = 2$, hence by Riemann-Roch $h^0(C, K_C \otimes L^\vee) = 20$. If, as expected, for a general choice of the pair $[C, L] \in \mathrm{Hur}$, the map $\mu(L)$ is injective, then Im $\mu(L)$ is a codimension 6 subspace of $H^0(C, K_C)$. Remarkably enough, this turns out to be precisely the $(+1)$-eigenspace of $H^0(C, K_C)$. In Section 10, using a degeneration argument, we establish that a general covering from Hur is Petri general:

Theorem 8.2. *For a general point $[C, L] \in \mathrm{Hur}$, the multiplication map $\mu(L)$ is injective.*

We have the following description of the Hodge eigenbundles.

Theorem 8.3. *Let $[C, L] \in \overline{\mathrm{Hur}}$ be an element corresponding to a nodal curve of genus 46 and a base point free line bundle $L \in W^1_{46}(C)$, such that $h^0(C, L) = 2$ and the Petri map $\mu(L)$ is injective. One has the following canonical identifications:*

$$\text{(i)} \quad H^0(C, K_C)^{(+1)} = H^0(C, L) \otimes H^0(C, K_C \otimes L^\vee).$$

$$\text{(ii)} \qquad H^0(C, K_C)^{(-5)} = \left(\frac{H^0(C, L^{\otimes 2})}{S^2 H^0(C, L)}\right)^{\vee} \otimes \bigwedge^2 H^0(C, L).$$

Proof. Let L be an E_6-pencil on C. Consider a general divisor $\Gamma \in |L|$ and the exact sequence

$$0 \longrightarrow \mathcal{O}_C \xrightarrow{\cdot s} L \longrightarrow \mathcal{O}_\Gamma(\Gamma) \longrightarrow 0 \tag{8.1}$$

induced by a section $s \in H^0(C, L)$ with $\mathrm{div}(s) = \Gamma$, and its cohomology sequence

$$0 \longrightarrow H^0(C, \mathcal{O}_C) \longrightarrow H^0(C, L) \longrightarrow H^0(\mathcal{O}_\Gamma(\Gamma)) \xrightarrow{\alpha} H^1(C, \mathcal{O}_C) \longrightarrow H^1(C, L) \longrightarrow 0. \tag{8.2}$$

There is an action of $W(E_6)$ on $H^0(\mathcal{O}_\Gamma(\Gamma))$ compatible with the trivial action on $H^0(C, L)$, because L is the pull-back from $\mathbb{P}^1$. We identify the space $H^0(\mathcal{O}_\Gamma(\Gamma))$ with the vector space generated by the 27 lines on a smooth cubic surface; each line is represented by a point of Γ and the incidence correspondence of lines is the Kanev correspondence D. Therefore the representation of $W(E_6)$ on $H^0(\mathcal{O}_\Gamma(\Gamma))$ splits into the sum of three irreducible representations: the trivial 1-dimensional one, the 6-dimensional one which coincides with the representation on the primitive cohomology of a cubic surface and the 20-dimensional one, which coincides with the one on the space of rational equivalences on a cubic surface, see for instance [AV12].

The Kanev correspondence D induces an endomorphism on $H^0(\mathcal{O}_\Gamma(\Gamma))$ compatible with the endomorphism $D^\vee \in \mathrm{End}\big(H^1(C, \mathcal{O}_C)\big)$ via the cohomology sequence (8.2). On a cubic surface, the action of the incidence correspondence on the primitive cohomology is equal to multiplication by -5 and its action on the space of rational equivalences is the identity. Therefore this is also how the action on $H^0(\mathcal{O}_\Gamma(\Gamma))$ can be described. It follows that the image $\alpha\big(H^0(\mathcal{O}_\Gamma(\Gamma))\big)$ contains the (-5)-eigenspace, that is, we have an inclusion

$$\big(H^0(C, K_C)^{(-5)}\big)^{\vee} \subseteq \left(\frac{H^0(C, K_C)}{H^0(C, L) \otimes H^0(C, K_C \otimes L^\vee)}\right)^{\vee}.$$

When the Petri map $\mu(L)$ is injective, the two spaces appearing in this inclusion have the same dimension and the inclusion becomes an equality, which establishes the first claim.

To prove the second claim, we start by observing that the Base Point Free Pencil Trick yields the sequence $0 \to \bigwedge^2 H^0(C, L) \otimes L^\vee \to H^0(C, L) \otimes \mathcal{O}_C \to L \to 0$. After tensoring with L and taking cohomology, we arrive at the following exact sequence

$$0 \longrightarrow \frac{H^0(C, L^{\otimes 2})}{\mathrm{Sym}^2 H^0(C, L)} \longrightarrow \bigwedge^2 H^0(C, L) \otimes H^1(C, \mathcal{O}_C) \xrightarrow{u} H^0(C, L) \otimes H^1(C, L) \longrightarrow 0.$$

To describe the map u in this sequence, let us choose a basis $s_1, s_2 \in H^0(C, L)$. Then,

$$u(s_1 \wedge s_2 \otimes f) = s_1 \otimes (s_2 \cdot f) - s_2 \otimes (s_1 \cdot f) \in H^0(C, L) \otimes H^1(C, L).$$

It follows via Serre duality, that $\mathrm{Ker}(u)$ consists of all linear maps $v : H^0(C, K_C) \to \mathbb{C}$ vanishing on $H^0(C, L) \otimes H^0(C, K_C \otimes L^\vee) \subset H^0(C, K_C)$, which proves the claim. $\square$

Remark 8.4. In the case of an arbitrary E_6-admissible cover $[\pi : C \to R] \in \overline{\mathrm{Hur}}$, the proof above yields the inclusion $H^0(C, \mathcal{O}_C(\Gamma)) \otimes H^0(C, \omega_C(-\Gamma)) \subseteq H^0(C, \omega_C)^{(+1)}$, where $\Gamma = \pi^{-1}(p)$, with $p \in R_{\mathrm{reg}}$ being an arbitrary smooth point.

The identifications provided by Theorem 8.3 extend to isomorphisms of vector bundles over a partial compactification of Hur which we shall introduce now. This allows us to express to Hodge classes $\lambda^{(+1)}$ and $\lambda^{(-5)}$ in terms of certain tautological classes and define the Petri map globally at the level of the moduli stack.

8.5. Let $\widetilde{\mathcal{M}}_{46}$ be the open subvariety of $\overline{\mathcal{M}}_{46}$ parametrizing irreducible curves and denote by $\mathcal{G}^1_{27} \to \widetilde{\mathcal{M}}_{46}$ the stack parametrizing pairs $[C, L]$, where $[C] \in \widetilde{\mathcal{M}}_{46}$ (in particular C is a *stable* curve) and L is a torsion free sheaf of degree 27 on C with $h^0(C, L) \geq 2$. Note that $\mathcal{G}^1_{27}$ is a locally closed substack of the universal Picard stack of degree 27 over $\widetilde{\mathcal{M}}_{46}$. Let $\mathcal{G}_{E_6}$ be the locus of pairs $[C, L] \in \mathcal{G}^1_{27}$, where L is *locally free* and *base point free* with $h^0(C, L) = 2$ and the monodromy of the pencil $|L|$ is equal to $W(E_6)$. We denote by $\sigma : \mathcal{G}_{E_6} \to \widetilde{\mathcal{M}}_{46}$ the projection map given by $\sigma([C, L]) := [C]$.

8.6. One has a birational isomorphism $\beta : \overline{\mathrm{Hur}} \dashrightarrow \mathcal{G}_{E_6} \subset \mathcal{G}^1_{27}$ which can be extended over each boundary divisor of $\overline{\mathrm{Hur}}$ as follows. Let $t := [\pi : C = C_1 \cup C_2 \to R_1 \cup_q R_2] \in D_{i:\mu}$ be a general point, where we recall that $C_i := \pi^{-1}(R_i)$ and precisely i branch points specialize to the component R_1. We assign to t the point $\big[\mathrm{st}(C), \mathrm{st}(f^*\mathcal{O}_{R_1 \cup R_2}(1,0))\big] \in \mathcal{G}_{E_6}$, where st is the map assigning to a nodal curve X its stable model $\mathrm{st}(X)$ and to a line bundle L on X the line bundle $\mathrm{st}(L)$ on $\mathrm{st}(C)$ obtained by adding base points to each destabilizing component of X which is contracted. Geometrically, for the general point of each boundary divisor, the map $\beta : \overline{\mathrm{Hur}} \dashrightarrow \mathcal{G}_{E_6}$ contracts the curve C_2. We still denote by $D_{\mathrm{azy}}, D_{\mathrm{syz}}$ and D_{E_6} the images under β of the boundary divisors denoted by the same symbols on $\overline{\mathrm{Hur}}$.

Remark 8.7. In light of the results in Section 6, note that the partial compactification $\mathcal{G}_{E_6}$ differs from $\overline{\mathrm{Hur}}$ only along boundary divisors that are contracted under the map PT. In order to study the geometry of $\mathcal{A}_6$, it makes thus no difference whether we work with $\overline{\mathrm{Hur}}$ or we restrict to $\mathcal{G}_{E_6}$.

8.8. It is important to understand the effect of β along $\mathfrak{br}^{-1}(\widetilde{B}_2)$. If $t \in D_{\mathrm{azy}}$ is a general point, then C_1 is smooth and $L := \pi^*_{C_1}(\mathcal{O}_{R_1}(1)) \in W^1_{27}(C_1)$ has 6 triple ramification points over the branch point $q \in R_1$. Then $\beta(t) = [C_1, L] \in \mathcal{G}_{E_6}$. If, on the other hand, t is a general point of D_{syz}, then retaining the notation of Remark 5.16, C_1 is a smooth curve of genus 45, meeting the smooth rational component Z in two points $u, v \in \pi^{-1}(q)$. Then $\beta(t) = [C', L]$, where $C' := C_1/u \sim v$ is an irreducible 1-nodal curve of genus 46 and $L \in W^1_{27}(C')$ is the pencil inducing the map π. Finally, if $t \in D_{E_6}$, then $\beta(t) = [C', L]$, where C' is a 6-nodal curve obtained from C_1 by identifying the six pairs of points in $\pi^{-1}(q)$ lying also over the 6 components of C_2 which map $2:1$ onto R_2.

We record the formula $\lambda = \frac{33}{46}[D_0] + \frac{7}{46}[D_{\mathrm{azy}}] + \frac{17}{92}[D_{\mathrm{syz}}] + \cdots \in CH^1(\mathcal{G}_{E_6})$ for the Hodge class at the level of $\mathcal{G}_{E_6}$. The factor of $\frac{1}{2}$ in front of $[D_{\mathrm{syz}}]$ compared to (5.6) is explained by the fact that the general point of $D_{\mathrm{syz}} \subset \overline{\mathrm{Hur}}$ has an automorphism of order 2, whereas its image under β is automorphism free.

8.9. At the level of $\mathcal{G}_{E_6}$ one can introduce several tautological classes along the lines of [Far09]. We denote by $f : \mathcal{C}_{E_6} \to \mathcal{G}_{E_6}$ the universal genus 46 curve and choose a universal line bundle $\mathcal{L} \in \mathrm{Pic}(\mathcal{C}_{E_6})$ satisfying the property $\mathcal{L}_{|f^{-1}([C,L])} = L \in W^1_{27}(C)$, for each $[C, L] \in \mathcal{G}_{E_6}$. We then define the following tautological classes:

$$\mathfrak{A} := f_*\big(c_1^2(\mathcal{L})\big),\ \ \mathfrak{B} := f_*\big(c_1(\mathcal{L}) \cdot c_1(\omega_f)\big),\ \kappa := f_*\big(c_1^2(\omega_f)\big) \in CH^1(\mathcal{G}_{E_6}).$$

Via Grauert's theorem, we observe that $\mathcal{V} := f_*\mathcal{L}$ is a locally free sheaf of rank two on $\mathcal{G}_{E_6}$. Similarly, the sheaf

$$\mathcal{V}_2 := f_*(\mathcal{L}^{\otimes 2})$$

is a vector bundle of rank 9 over $\mathcal{G}_{E_6}$. Globalizing at the level of moduli the multiplication map of global sections $\mathrm{Sym}^2 H^0(C, L) \to H^0(C, L^{\otimes 2})$, we define the rank 6 vector bundle $\mathcal{E}_2$ over $\mathcal{G}_{E_6}$ via the following exact sequence:

$$0 \longrightarrow \mathrm{Sym}^2(\mathcal{V}) \longrightarrow \mathcal{V}_2 \longrightarrow \mathcal{E}_2 \longrightarrow 0.$$

8.10. The choice of $\mathcal{L}$ is not unique; replacing $\mathcal{L}$ by $\mathcal{L}' := \mathcal{L} \otimes f^*(\alpha)$, where $\alpha \in \mathrm{Pic}(\mathcal{G}_{E_6})$ and denoting the corresponding tautological classes by $\mathfrak{A}', \mathfrak{B}' \in CH^1(\mathcal{G}_{E_6})$ respectively, we find the relations

$$\mathfrak{A}' = \mathfrak{A} + 2d \cdot \alpha \ \ \text{and}\ \ \mathfrak{B}' = \mathfrak{B} + (2g-2) \cdot \alpha.$$

It follows that $\mathfrak{B}' - \frac{5}{3}\mathfrak{A}' = \mathfrak{B} - \frac{5}{3}\mathfrak{A}$, that is, the class

$$\gamma := \mathfrak{B} - \frac{5}{3}\mathfrak{A} \in CH^1(\mathcal{G}_{E_6}) \tag{8.3}$$

is well-defined and independent of the choice of a Poincaré bundle $\mathcal{L}$. We now describe in a series of calculations the Chern classes of the vector bundles we have just introduced.

Proposition 8.11. *The following relations hold in $CH^1(\mathcal{G}_{E_6})$:*

$$c_1(\mathcal{V}_2) = \lambda - \mathfrak{B} + 2\mathfrak{A} \quad \textit{and} \quad c_1\Big(R^1 f_*(\omega_f \otimes \mathcal{L}^\vee)\Big) = \lambda + \frac{\mathfrak{A}}{2} - \frac{\mathfrak{B}}{2} - c_1(\mathcal{V}).$$

Proof. We apply Grothendieck-Riemann-Roch to $f : \mathcal{C}_{E_6} \to \mathcal{G}_{E_6}$ and write

$$c_1(\mathcal{V}_2) = f_*\Big[\Big(1 + 2c_1(\mathcal{L}) + 2c_1^2(\mathcal{L})\Big) \cdot \Big(1 - \frac{c_1(\omega_f)}{2} + \frac{c_1^2(\Omega_f^1) + c_2(\Omega_f^1)}{12}\Big)\Big]_2.$$

Now use Mumford's formula $f_*\big(c_1^2(\Omega_f^1)+c_2(\Omega_f^1)\big) = 12\lambda$, see [HM82] p. 49, and conclude. $\square$

8.12. Theorem 8.2 (to be proved in Section 10) shows that the Petri map $\mu(L)$ is injective for a general point of $[C, L] \in \mathcal{G}_{E_6}$. However, we cannot rule out the (unlikely) possibility that $\mu(L)$ is not injective along a divisor $\mathfrak{N}$ on $\mathcal{G}_{E_6}$. We denote by $\mathfrak{n} := [\mathfrak{N}] \in CH^1(\mathcal{G}_{E_6})$. This (possibly zero) class is effective. Globalizing Theorem 8.3, we obtain isomorphisms of vector bundles over $\mathcal{G}_{E_6} - \mathfrak{N}$:

$$\mathbb{E}^{(+1)} = R^1 f_*\big(\omega_f \otimes \mathcal{L}^\vee\big) \otimes \mathcal{V} \quad \text{and} \quad \mathbb{E}^{(-5)} = \mathcal{E}_2^\vee \otimes \det(\mathcal{V}).$$

Extending this to $\mathcal{G}_{E_6}$, there exists an injection of vector bundles $R^1 f_*\big(\omega_f \otimes \mathcal{L}^\vee\big) \hookrightarrow \mathbb{E}^{(+1)}$, with the quotient being a sheaf supported on $\mathfrak{N}$ and on possibly other higher codimension cycles.

Proposition 8.13. *The following formulas hold at the level of* $\mathcal{G}_{E_6}$*:*

$$\lambda^{(+1)} = 2\lambda - \gamma + \mathfrak{n} \quad \textit{and} \ \lambda^{(-5)} = -\lambda + \gamma - \mathfrak{n}.$$

Proof. We have that $\lambda^{(+1)} = c_1\Big(R^1 f_*\big(\omega_f \otimes \mathcal{L}^\vee\big)\Big) - [\mathfrak{N}] \in CH^1(\mathcal{G}_{E_6})$ and the rest is a consequence of Theorem 8.3 coupled with Proposition 8.11. $\square$

Proposition 8.14. *We have that* $\mathfrak{A} = 27c_1(\mathcal{V}) \in CH^1(\mathcal{G}_{E_6})$.

Proof. Recall that $\mathcal{G}_{E_6}$ has been defined as a locus of pairs $[C, L]$ such that L is a base point free pencil. In particular, the image under f of the codimension 2 locus in $\mathcal{C}_{E_6}$ where the morphism of vector bundles $f^*(\mathcal{V}) \to \mathcal{L}$ is not surjective is empty, hence by Porteous' formula

$$0 = f_*\Big(c_2(f^*\mathcal{V}) - c_1(f^*\mathcal{V}) \cdot c_1(\mathcal{L}) + c_1^2(\mathcal{L})\Big) = -27c_1(\mathcal{V}) + \mathfrak{A}.$$

$\square$

Essential in all the ensuing calculations is the following result expressing the divisor D_{azy} in terms of Hodge eigenbundles and showing that its class is quite positive:

Theorem 8.15. *The following relation holds:*

$$[D_{\mathrm{azy}}] = 5\lambda + \lambda^{(-5)} - 3[D_{E_6}] - \frac{5}{6}[D_{\mathrm{syz}}] + \mathfrak{n} \in CH^1(\mathcal{G}_{E_6}).$$

Proof. The idea is to represent D_{azy} as the push-forward of the codimension two locus in the universal curve $\mathcal{C}_{E_6}$ of the locus of pairs $[C,L,p]$ such that $h^0(C,L(-3p))\geq 1$. We form the fibre product of the universal curve $\mathcal{C}_{E_6}$ together with its projections:

$$\mathcal{C}_{E_6} \xleftarrow{\pi_1} \mathcal{C}_{E_6}\times_{\mathcal{G}_{E_6}}\mathcal{C}_{E_6} \xrightarrow{\pi_2} \mathcal{C}_{E_6}.$$

For each $k\geq 1$, we consider the *locally free* jet bundle $J_k(\mathcal{L})$ defined, e.g. in [Est96], as a locally free replacement of the sheaf of principal parts $\mathcal{P}^k_f(\mathcal{L}):=(\pi_2)_*\Big(\pi_1^*(\mathcal{L})\otimes\mathcal{I}_{(k+1)\Delta}\Big)$ on $\mathcal{C}_{E_6}$. Note that $\mathcal{P}^k_f(\mathcal{L})$ is not locally free along the codimension two locus in $\mathcal{C}_{E_6}$ where f is not smooth. To remedy this problem, we consider their *wronskian* locally free replacements $J^k_f(\mathcal{L})$, which are related by the following commutative diagram for each $k\geq 1$:

$$\begin{tikzcd}
0 \arrow[r] & \Omega^k_f\otimes\mathcal{L} \arrow[r]\arrow[d] & \mathcal{P}^k_f(\mathcal{L}) \arrow[r]\arrow[d] & \mathcal{P}^{k-1}_f(\mathcal{L}) \arrow[r]\arrow[d] & 0\\
0 \arrow[r] & \omega_f^{\otimes k}\otimes\mathcal{L} \arrow[r] & J^k_f(\mathcal{L}) \arrow[r] & J^{k-1}_f(\mathcal{L}) \arrow[r] & 0
\end{tikzcd}$$

Here Ω^k_f denotes the $\mathcal{O}_{\mathcal{G}_{E_6}}$-module $\mathcal{I}_{k\Delta}/\mathcal{I}_{(k+1)\Delta}$. The first vertical row here is induced by the canonical map $\Omega^k_f\to\omega_f^{\otimes k}$, relating the sheaf of relative Kähler differentials to the relative dualizing sheaf of the family f. The sheaves $\mathcal{P}^k_f(\mathcal{L})$ and $J^k_f(\mathcal{L})$ differ only along the codimension two singular locus of f. Furthermore, for each integer $k\geq 0$ there is a vector bundle morphism $\nu_k: f^*(\mathcal{V})\to J^k_f(\mathcal{L})$, which for points $[C,L,p]\in\mathcal{G}_{E_6}$ such that $p\in C_{\mathrm{reg}}$, is just the evaluation morphism $H^0(C,L)\to H^0(L_{|(k+1)p})$. We specialize now to the case $k=2$ and consider the codimension two locus $Z\subset\mathcal{C}_{E_6}$ where $\nu_2: f^*(\mathcal{V})\to J^2_f(\mathcal{L})$ is not injective. Then, at least over the locus of smooth curves, D_{azy} is the set-theoretic image of Z. A simple local analysis shows that the morphism ν_2 is simply degenerate for each point $[C,L,p]$, where $p\in C_{\mathrm{sing}}$. Taking into account that a general point of D_{azy} corresponds to a pencil with *six* triple points aligned over one branch point, and that the stable model of a general element of the divisor D_{syz} corresponds to a curve with *one* node, whereas that of a general point of D_{E_6} to a curve with *six* nodes and so on, we obtain the formula:

$$6[D_{\mathrm{azy}}]=f_*c_2\left(\frac{J^2_f(\mathcal{L})}{f^*(\mathcal{V})}\right)-6[D_{E_6}]-3[D_{\mathrm{syz}}]\in CH^1(\mathcal{G}_{E_6}).$$

The fact that D_{syz} appears with multiplicity 3 is a result of the following local computation. We choose a family $F:X\to B$ of curves of genus 46 over a smooth 1-dimensional basis B, such that X is smooth, and there is a point $b_0\in B$ such that $X_b:=F^{-1}(b)$ is smooth for $b\in B\setminus\{b_0\}$, whereas X_{b_0} has a unique node $N\in X$. Assume also that $L\in\mathrm{Pic}(X)$ is a line bundle such that $L_b:=L_{|X_b}$ is a pencil with E_6-monodromy on X_b for each $b\in B$, and furthermore $[X_{b_0},L_{b_0}]\in D_{\mathrm{syz}}$. Choose a local parameter $t\in\mathcal{O}_{B,b_0}$ and $x,y\in\mathcal{O}_{X,N}$, such that

$xy = t$ represents the local equation of X around the point N. Then ω_F is locally generated by the meromorphic differential $\tau = \frac{dx}{x} = -\frac{dy}{y}$. We choose two sections $s_1, s_2 \in H^0(X, L)$, where s_1 does not vanish at N and s_2 vanishes with order 2 at N along both branches of X_{b_0}. Then we have the relation $s_{2,N} = (x^2 + y^2)s_{1,N}$ between the germs of the two sections s_1 and s_2 at N. We compute

$$d(s_2) = 2xdy + 2ydy = 2(x^2 - y^2)\tau, \quad \text{and} \quad d(x^2 - y^2) = 2(x^2 + y^2)\tau.$$

In local coordinates, the map $H^0\big(X_{b_0}, L_{b_0}\big) \to H^0\big(X_{b_0}, L_{b_0|3N}\big)$ is then given by the 2×2 minors of the following matrix:

$$\begin{pmatrix} 1 & 0 & 0 \\ x^2 + y^2 & x^2 - y^2 & x^2 + y^2 \end{pmatrix}.$$

This completes the proof that $[D_{\rm syz}]$ appears with multiplicity 3 in the degeneracy locus.

We compute: $c_1(J_f^2(\mathcal{L})) = 3c_1(\mathcal{L}) + 3c_1(\omega_f)$ and $c_2(J_f^2(\mathcal{L})) = 3c_1^2(\mathcal{L}) + 6c_1(\mathcal{L}) \cdot c_1(\omega_f) + 2c_1^2(\omega_f)$, hence

$$f_*c_2\left(\frac{J_f^2(\mathcal{L})}{f^*(\mathcal{V})}\right) = 3\mathfrak{A} + 6\mathfrak{B} - 3(d + 2g - 2)c_1(\mathcal{V}) + 2\kappa_1 = 6\gamma + 2\kappa_1.$$

Furthermore, $\kappa_1 = 12\lambda - 6[D_{E_6}] - [D_{\rm syz}] - \cdots$, hence after applying Proposition 8.13, we obtain the claimed formula. □

We can also express the divisors $D_{\rm syz}$ and $D_{\rm azy}$ in terms of the Hodge eigenclasses.

Proposition 8.16. *The following formulas hold in $CH^1(\mathcal{G}_{E_6})$:*

$$[D_{\rm azy}] = \frac{25}{16}\lambda + \frac{51}{16}\lambda^{(-5)} + \frac{3}{4}[D_{E_6}] + \frac{51}{16}\mathfrak{n} \quad \textit{and} \quad [D_{\rm syz}] = \frac{33}{8}\lambda - \frac{21}{8}\lambda^{(-5)} - \frac{9}{2}[D_{E_6}] - \frac{21}{8}\mathfrak{n}.$$

Proof. Combine Theorem 8.15 with the expression of the Hodge class λ in terms of the boundary divisor classes on $\mathcal{G}_{E_6}$. □

Corollary 8.17. *One has that $[D_{\rm syz}] \leq \frac{33}{8}\lambda - \frac{21}{8}\lambda^{(-5)} - \frac{9}{2}[D_{E_6}]$.*

We are now in a position to determine the class of the ramification divisor of the Prym-Tyurin map in terms of the classes already introduced. Recall that $\widetilde{D}_6 := \widetilde{\mathcal{A}}_6 \setminus \mathcal{A}_6$ is the irreducible boundary divisor of the partial compactification of the moduli space of ppav of dimension 6 and $\lambda_1 \in CH^1(\widetilde{\mathcal{A}}_6)$ denotes the Hodge class. Note that $K_{\widetilde{\mathcal{A}}_6} = 7\lambda_1 - [\widetilde{D}_6]$, see [Mum83].

Theorem 8.18. *The ramification divisor of the map $PT : \mathcal{G}_{E_6} \dashrightarrow \overline{\mathcal{A}}_6$ is given by*

$$\big[\mathrm{Ram}(PT)\big] = \frac{73}{32}\lambda - \frac{221}{32}\lambda^{(-5)} - \frac{9}{8}[D_{E_6}] + \frac{3}{32}\mathfrak{n}.$$

Proof. The general point of D_{E_6} corresponds to a semi-abelian variety of torus rank 1, whereas for all the other boundary divisors in $\mathfrak{br}^*(\widetilde{B}_2)$ the corresponding torus rank is zero. Moreover $PT^*(\widetilde{D}_6) = D_{E_6}$. Via the Hurwitz formula, we obtain that

$$\big[\mathrm{Ram}(PT)\big] = K_{\mathcal{G}_{E_6}} - PT^*\big(7\lambda_1 - [\widetilde{D}_6]\big) = K_{\mathcal{G}_{E_6}} - 7\lambda^{(-5)} + [D_{E_6}].$$

Recall that the canonical class $K_{\overline{\mathrm{Hur}}}$ has been expressed in terms of boundary divisors on $\overline{\mathrm{Hur}}$. Using Theorem 8.15, we can pass to a new basis in $CH^1(\mathcal{G}_{E_6})$ involving the Hodge eigenbundles and one boundary divisor, namely D_{E_6}. After simple manipulations we obtain

$$K_{\mathcal{G}_{E_6}} = \frac{73}{32}\lambda + \frac{3}{32}\lambda^{(-5)} - \frac{17}{8}[D_{E_6}] + \frac{3}{32}\mathfrak{n}, \tag{8.4}$$

which then leads to the claimed formula. □

We can now complete the proof of Theorem 0.2.

Theorem 8.19. *The canonical class of the partial compactification $\mathcal{G}_{E_6}$ of* Hur *is big. It follows that there exists a divisor E on $\overline{\mathrm{Hur}}$ with $PT_*(E) = 0$, such that $K_{\overline{\mathrm{Hur}}} + E$ is big.*

Proof. The varieties $\mathcal{G}_{E_6}$ and $\overline{\mathrm{Hur}}$ differ in codimension one only along boundary divisors that are collapsed under the Prym-Tyurin map. Showing that $K_{\mathcal{G}_{E_6}}$ is big implies therefore the second half of the claim, and thus Theorem 0.2. Using Theorems 5.18 (note the caveat about the already mentioned factor $1/2$ in front of the coefficient of $[D_{\mathrm{syz}}]$ when passing from $\overline{\mathrm{Hur}}$ to $\mathcal{G}_{E_6}$), coupled with Theorem 6.19, we write:

$$K_{\mathcal{G}_{E_6}} = -\frac{25}{46}[D_{E_6}] + \frac{19}{46}[D_{\mathrm{syz}}] + \frac{17}{46}[D_{\mathrm{azy}}] \geq -\frac{25}{46}[D_{E_6}] + \frac{19}{46}[D_{\mathrm{syz}}] + \frac{17}{46}\Big(\frac{25}{16}\lambda + \frac{51}{16}\lambda^{(-5)} + \frac{3}{4}[D_{E_6}]\Big)$$
$$= \frac{867}{736}\lambda^{(-5)} + \frac{425}{736}\lambda - \frac{49}{184}[D_{E_6}].$$

Putting together Proposition 5.23 together with the fact that $\lambda^{(-5)}$ is big, the conclusion follows by comparing the ratio of the λ and $[D_{E_6}]$-coefficients of the last expression. □

9. The ramification divisor of the Prym-Tyurin map

The aim of this section is to describe the differential of the Prym-Tyurin map PT and prove Theorem 0.3. As in the previous section, we fix a smooth E_6-cover $\pi : C \to \mathbb{P}^1$ with branch divisor $B := p_1 + \cdots + p_{24}$ and denote $L := \pi^*(\mathcal{O}_{\mathbb{P}^1}(1))$.

Via the étale map $\mathfrak{br} : \mathrm{Hur} \to \mathcal{M}_{0,24}/S_{24}$, we identify the cotangent space $T^\vee_{[C,\pi]}(\mathrm{Hur})$ with $H^0(\mathbb{P}^1, K_{\mathbb{P}^1}^{\otimes 2}(B))$. The cotangent space $T^\vee_{[P(C,f)]}(\mathcal{A}_6)$ is identified with $\mathrm{Sym}^2 H^0(C, K_C^{\otimes 2})^{(-5)}$.

Definition 9.1. Let R and A be the ramification and antiramification divisors of π:

$$\pi^*(B) = 2R + A, \quad K_C = \pi^*(K_{\mathbb{P}^1}) + R, \quad 2K_C + A = \pi^*(2K_{\mathbb{P}^1} + B).$$

Definition 9.2. Let $\mathrm{tr} : \pi_*\mathcal{O}_C(-A) \to \mathcal{O}_{\mathbb{P}^1}$ be the trace map on regular functions. For an open affine subset $U \subset \mathbb{P}^1$, a regular function $\varphi \in \Gamma(U, \mathcal{O}_C(-A))$, and a point $y \in U$, one has $\mathrm{tr}(\varphi)(y) = \sum_{x\in f^{-1}(y)} \varphi(x)$, counted with multiplicities. Note that tr is a surjective homomorphism of sheaves. Let $\pi_*\mathcal{O}_C(2K_C) \to \mathcal{O}_{\mathbb{P}^1}\big(2K_{\mathbb{P}^1} + B\big)$ be the induced trace map at the level of quadratic differentials. We denote the corresponding map on global sections by $\mathrm{Tr} : H^0(C, K_C^{\otimes 2}) \to H^0(\mathbb{P}^1, K_{\mathbb{P}^1}^{\otimes 2}(B))$.

Theorem 9.3. *The codifferential* $(dPT)^\vee_{[C,\pi]} : T^\vee_{[PT(C,\pi)]}\big(\mathcal{A}_6\big) \to T^\vee_{[C,\pi]}\big(\mathrm{Hur}\big)$ *is given by the following composition of maps:*

$$\mathrm{Sym}^2 H^0(K_C)^{(-5)} \hookrightarrow \mathrm{Sym}^2 H^0(K_C) \xrightarrow{\mathrm{mul}} H^0(K_C^{\otimes 2}) \xrightarrow{\mathrm{Tr}} H^0(K_{\mathbb{P}^1}^{\otimes 2}(p_1 + \cdots + p_{24})).$$

Proof. The second map is the codifferential of the Torelli map $\mathcal{M}_{46} \to \mathcal{A}_{46}$. The first map is the codifferential of the map between the moduli spaces of ppav's of dimension 46 together with an endomorphism D having eigenvalues $(+1)$ and (-5) respectively to $\mathcal{A}_6$. The third map is the codifferential of the map $\mathrm{Hur} \to \mathcal{M}_{46}$. $\square$

9.4. We now analyze the differential dPT at a point $[C, \pi] \in \mathrm{Hur}$ in detail. For each of the 24 branch points $p_i \in \mathbb{P}^1$, let $\{r_{ij}\}_{j=1}^6 \subset C$ be the ramification points lying over p_i. The formal neighborhoods of the points r_{ij} are naturally identified, so that we can choose a single local parameter x and write any quadratic differential $\gamma \in H^0(C, K_C^{\otimes 2})$ as

$$\gamma = \varphi_{ij}(x) \cdot (dx)^{\otimes 2} \ \text{ near } r_{ij} \in C.$$

Choose a local parameter y at the point p_i, so that π is given locally by the map $y = x^2$. We can use the same local parameter at the remaining 15 antiramification points $\{q_{ik}\}_{k=1}^{15}$ over p_i at which π is unramified, and write $\gamma = \psi_{ik}(y) \cdot (dy)^{\otimes 2}$ near $q_{ik} \in C$, for $k = 1, \ldots, 15$.

Lemma 9.5. *The kernel of the trace map* $\mathrm{Tr} : H^0(C, K_C^{\otimes 2}) \to H^0\big(\mathbb{P}^1, K_{\mathbb{P}^1}^{\otimes 2}(B)\big)$ *consists of quadratic differentials* γ *such that*

$$\sum_{j=1}^{6} \varphi_{ij}(r_{ij}) = 0, \quad \textit{for} \ \ i = 1, \ldots, 24.$$

Proof. From $y = x^2$, we get $dy = 2x\, dx$ and $(dx)^{\otimes 2} = (dy)^{\otimes 2}/4y$. We have that

$$\mathrm{Tr}(\gamma) = \left(\frac{1}{4y} \sum_{j=1}^{6} \big(\varphi_{ij}(x) + \varphi_{ij}(-x)\big) + \sum_{k=1}^{15} \psi_{ik}(y) \right) \cdot (dy)^{\otimes 2} \quad \text{near } p_i.$$

Suppose $\mathrm{Tr}(\gamma) = 0$. Then the leading coefficient $\frac{1}{2}\sum_{j=1}^{6} \varphi_{ij}(r_{ij})$ is zero. Conversely, assume that the 24 expressions are zero. Then $\mathrm{Tr}(\gamma) \in H^0(\mathbb{P}^1, K_{\mathbb{P}^1}^{\otimes 2}) = 0$. $\square$

In order to understand the condition in Lemma 9.5, we recall the action of the endomorphism $D\colon H^0(C,K_C)\to H^0(C,K_C)$ induced by the Kanev correspondence in local coordinates at the points $p\in C$ and $q\in\pi^{-1}(p)$ (see also Theorem 8.3 and Remark 8.4).

9.6. The unramified case. Suppose that π is unramified at p, thus $\Gamma:=\pi^{-1}(p)=\sum_{s=1}^{27}q_s$. Since π is étale, we can use the same local parameter y at p, as well as at each $q_s\in C$. Let $\alpha\in H^0(C,K_C)$. In a formal neighborhood of each point q_s, we write locally $\alpha=\alpha_s(y)dy$.

Assume $p=[0:1]\in\mathbb{P}^1$. One has $\sum_{s=1}^{27}\operatorname{Res}_{q_s}(\alpha\cdot\frac{x_0}{x_1})=0$, so $\sum_{s=1}^{27}\alpha_s(q_s)=0$. The action of the correspondence on α_s's is described by an endomorphism of a 26-dimensional space $\hat{\mathcal{O}}^{26}=\left\{\sum_{s=1}^{27}\alpha_s=0\right\}\subset\hat{\mathcal{O}}^{27}$, where $\hat{\mathcal{O}}=\hat{\mathcal{O}}_{\mathbb{P}^1,p}$. This endomorphism is given by the same integral (26×26)-matrix as the action of D on $H^0(\mathcal{O}_\Gamma(\Gamma))/H^0(C,L)$, see also the proof of Theorem 8.3. Thus, D has two eigenvalues $(+1)$ and (-5) with the eigenspaces of dimensions 20 and 6 respectively. Choose a basis $\{v_m\}_{m=1}^6$ in the (-5)-eigenspace of $\mathbb{Z}^{26}$. Then an element $\alpha^{(-5)}\in H^0(C,K_C)^{(-5)}$ can be locally written uniquely as

$$\alpha^{(-5)}=\sum_{m=1}^{6}\delta_m v_m\in\hat{\mathcal{O}}^{27},\quad\text{for some }\delta_m\in\hat{\mathcal{O}}.$$

9.7. The ramified case. Suppose π is branched at p and $\pi^{-1}(p)$ consists of ramification points $r_1,\dots,r_6$ and 15 antiramification points q_k. The points r_i correspond to the ordered pairs (a_i,b_i) of sheets coming together. On the sheets, the correspondence is defined by

$$a_i\mapsto\sum_{j\neq i}(b_j+c_{ij})\ \text{ and }\quad b_i\mapsto\sum_{j\neq i}(a_j+c_{ij}),\qquad\text{for } i=1,\dots,6.$$

As above, we use a local coordinate y for $p\in\mathbb{P}^1$ and the 15 points $q_k\in C$, and a local coordinate x for the ramification points r_i, with $y=x^2$. Thus, we write locally

$$\alpha=\alpha_{r_i}(x)dx\quad\text{near } r_i,\quad\text{and}\quad\alpha=\alpha_{q_k}(y)dy\quad\text{near } q_k.$$

The local involution $x\mapsto -x$ splits the differential form into the odd and even parts:

$$\begin{aligned}\alpha_{r_i}(x)dx={}&\quad\alpha_{r_i}^{\mathrm{odd}}(x^2)dx+\alpha_{r_i}^{\mathrm{ev}}(x^2)xdx,\\ \alpha_{r_i}(-x)d(-x)={}&-\alpha_{r_i}^{\mathrm{odd}}(x^2)dx+\alpha_{r_i}^{\mathrm{ev}}(x^2)xdx.\end{aligned}$$

The even part can be written in terms of y as $\frac{1}{2}\alpha_{r_i}^{\mathrm{ev}}(y)dy$. The odd parts have no such interpretation and we claim that they do not mix with the 15 sheets on which π is étale:

Lemma 9.8. *The correspondence D induces an endomorphism on the 6-dimensional $\hat{\mathcal{O}}_y$-module of odd parts $\langle\alpha_{r_i}^{\mathrm{odd}}dx\rangle$. It is given by a matrix which has 0 on the main diagonal and (-1) elsewhere. The (-5) eigenspace is 1-dimensional with the generator $(1,\dots,1)$, and for* *every element $\alpha \in H^0(C, K_C)^{(-5)}$ one has*

$$(\alpha^{\mathrm{odd}}_{r_1}, \dots, \alpha^{\mathrm{odd}}_{r_6}) = (\phi, \dots, \phi), \quad \textit{for some } \phi(y) \textit{ independent of } i = 1, \dots, 6.$$

Proof. This case is obtained by taking a limit of the unramified case. We work in complex-analytic topology. A ramification point $r \in C$ is a limit of two points $q_{a_i}, q_{b_i} \in C$ on the sheets a_i and b_i respectively. The local parameters x, x' at q_{a_i}, q_{b_i} are identified as $x' = -x$. One has $y = x^2 = (x')^2$ and we look at the limit x tends to 0. From $dx' = -dx$ it follows that

$$\begin{aligned} \alpha_{q_{a_i}}(x)dx = &\quad \alpha^{\mathrm{odd}}_{r_i}(x^2)dx + \alpha^{\mathrm{ev}}_{r_i}(x^2)xdx, \\ \alpha_{q_{b_i}}(x)dx = &\ -\alpha^{\mathrm{odd}}_{r_i}(x^2)dx + \alpha^{\mathrm{ev}}_{r_i}(x^2)xdx. \end{aligned}$$

Under the correspondence between the 27 sheets, the α^{odd} contributions from a_i and b_i to the 5 sheets c_{ij} $(j \neq i)$ cancel out. Conversely, the terms $\alpha_{c_{ij}}$ contribute to α^{ev} but not to α^{odd} at the point r_i.

It follows that the homomorphism D sends the $\hat{\mathcal{O}}^6$ block of the odd parts $\alpha^{\mathrm{odd}}_{r_i}$ to itself. The matrix of this linear map is the same as the matrix of an endomorphism of $\mathbb{Z}^6$ with the basis of vectors $a_i - b_i$, that is, $a_i - b_i \mapsto -\sum_{j \neq i}(a_j - b_j)$. It is easy to see that this linear map has eigenvalues $(+1)$ and (-5) and that the (-5)-eigenspace is one-dimensional and is generated by the vector $(1, \dots, 1)$. The statement now follows. $\square$

Corollary 9.9. *Let $\beta \in \mathrm{Sym}^2 H^0(C, K_C)^{(-5)}$ and let $\gamma = \mathrm{mul}(\beta)$ be its image in $H^0(C, K_C^{\otimes 2})$. Then in the notation of Lemma 9.5, one has $\varphi_{ij}(r_{ij}) = \varphi_{ij'}(r_{ij'})$, for all $i = 1, \dots, 24$ and all $1 \leq j, j' \leq 6$.*

Proof. Let $\alpha, \alpha' \in H^0(C, K_C)^{(-5)}$. Then in the notation of Lemma 9.8, one has $\mathrm{mul}(\alpha \otimes \alpha')(r_{ij}) = \alpha^{\mathrm{odd}}_{r_{ij}}(r_{ij}) \cdot (\alpha')^{\mathrm{odd}}_{r_{ij}}(r_{ij}) = \phi(0)\phi'(0)$, which is independent of $j = 1, \dots, 6$. $\square$

Lemma 9.8 also has consequences for the geometry of the Abel-Prym-Tyurin canonical curve $\varphi_{(-5)} = \varphi_{H^0(C,K_C)^{(-5)}} : C \to \mathbb{P}^5$. In stark contrast with the case of ordinary Prym-canonical curves, the map $\varphi_{(-5)}$ is far from being an embedding.

Proposition 9.10. *For an E_6-cover $\pi : C \to \mathbb{P}^1$, we have $\varphi_{(-5)}(r_{i1}) = \cdots = \varphi_{(-5)}(r_{i6})$, for each $i = 1, \dots, 24$.*

Proof. This is a consequence of Lemma 9.8: the condition that $\alpha \in H^0(C, K_C)^{(-5)}$ vanishes along the divisor $r_{i1} + \cdots + r_{i6}$ is expressed by a *single* condition $\phi(0) = 0$, therefore $\dim \left|H^0(C, K_C)^{(-5)}(-\sum_{j=1}^6 r_{ij})\right| = 4$. $\square$

Finally we are in a position to describe the ramification divisor of the map PT.

Proof of Theorem 0.3. Using (9.5) and (9.9), it follows that the map PT is ramified at a point $[C,\pi]\in \mathrm{Hur}$, if and only if there exists $0\neq\beta\in \mathrm{Sym}^2 H^0(C,K_C)^{(-5)}$ such that

$$\mathrm{mul}(\beta)\in H^0(C,2K_C-R)=H^0(C,K_C-2L)=\mathrm{Ker}(\mu(L)),$$

where the last equality follows from the Base Point Free Pencil Trick applied to the Petri map $\mu(L)$. If now $\mu(L)$ is injective, it follows that $\mathrm{mul}(\beta)=0$, which finishes the proof. □

10. Brill-Noether type divisors on $\overline{\mathrm{Hur}}$ and degenerations of E_6-covers

Determining the Kodaira dimension of $\mathcal{A}_6$ is a long-standing open question. As pointed out in Corollary 5.6, to conclude that $\mathcal{A}_6$ is of general type, it suffices to show that the class $7\lambda^{(-5)}-[D_{E_6}]$ is big on the partial compactification $\mathcal{G}_{E_6}$ of Hur introduced in Section 8. This could be achieved by constructing effective divisors which are close to being extremal in the effective cone of $\mathcal{G}_{E_6}$ and computing their classes (see [FL10] for an instance on how this is carried out for $\mathcal{R}_g$). We have computed in Proposition 8.16 the classes of the effective divisors D_{azy} and D_{syz}. In order to estimate the positivity of these classes and the sharpness of Corollary 8.17, we need an upper bound on the total Hodge class λ in terms of $\lambda^{(-5)}$ and $[D_{E_6}]$. To that end, we introduce a series of virtual effective divisors on $\mathcal{G}_{E_6}$, compute their classes, and give a sufficient condition for $\mathcal{A}_6$ to be of general type.

We start with an E_6-cover $[\pi: C\to\mathbb{P}^1]\in \mathrm{Hur}$ and set, as before, $L:=\pi^*(\mathcal{O}_{\mathbb{P}^1}(1))$. Observe that the line bundle $\eta:=3K_C-10L$ has degree zero, whereas $\deg(2K_C-5L)=45$.

Proposition 10.1. *If $D: JC\to JC$ denotes as usual the Kanev endomorphism, then*

$$D(3K_C-10L)=3K_C-10L.$$

Proof. By direct calculation, one verifies that $D(L)=10L$ and $D(K_C)=K_C+30L$. □

10.2. For each $n\in\mathbb{Z}$, we define the locus

$$\mathfrak{D}_n:=\Big\{[C,L]\in\mathcal{G}_{E_6}: H^0\big(C,(3n+2)K_C-(10n+5)L\big)\neq 0\Big\}.$$

Note that points in $\mathfrak{D}_n$ are characterized by the condition $n\cdot\eta\in\Theta_{2K_C-5L}\subset JC$. In particular, $\mathfrak{D}_n$ is a virtual divisor on $\mathcal{G}_{E_6}$.

Theorem 10.3. *For each $n\geq 0$, the virtual class of $\mathfrak{D}_n$ is given by the following formula:*

$$[\mathfrak{D}_n]=-\lambda-\binom{3n+2}{2}\kappa_1+\frac{15}{2}(2n+1)^2\cdot\gamma\in CH^1(\mathcal{G}_{E_6}).$$

Proof. We reinterpret the defining property of points in $\mathfrak{D}_n$ via the Base Point Free Pencil Trick, as saying that the multiplication map

$$\mu_n(L): H^0(C,L)\otimes H^0\big(C,(3n+2)K_C-(10n+4)L\big)\longrightarrow H^0\big(C,(3n+2)K_C-(10n+3)L\big)$$

is not bijective. Note, $h^0((3n+2)K_C-(10n+4)L)=27$ and $h^0((3n+2)K_C-(10n+3)L)=54$. The map $\mu_n(L)$ can be globalized to a morphism of vector bundles over $\mathcal{G}_{E_6}$

$$\mu_n : f_*\mathcal{L}\otimes f_*\Bigl(\omega_f^{\otimes(3n+2)}\otimes\mathcal{L}^{\otimes(-(10n+4))}\Bigr)\ \longrightarrow\ f_*\Bigl(\omega_f^{\otimes(3n+2)}\otimes\mathcal{L}^{\otimes(-(10n+3))}\Bigr),$$

where, as in the previous section, $\mathcal{L}$ is a universal pencil with $W(E_6)$-monodromy over the universal curve $f:\mathcal{C}_{E_6}\to\mathcal{G}_{E_6}$ that ultimately plays no role in the class formula for $\mathfrak{D}_n$. Clearly, $\mathfrak{D}_n$ is the degeneracy locus of μ_n.

Since $R^1f_*\mathcal{L}\otimes f_*\Bigl(\omega_f^{\otimes(3n+2)}\otimes\mathcal{L}^{\otimes(-(10n+4))}\Bigr)=0$ and $R^1f_*\Bigl(\omega_f^{\otimes(3n+2)}\otimes\mathcal{L}^{\otimes(-(10n+3))}\Bigr)=0$, the Chern classes of the sheaves appearing in the definition of the morphism μ_n can be computed via an easy GRR calculation. For instance, we obtain that

$$c_1\Bigl(f_*\bigl(\omega_f^{\otimes(3n+2)}\otimes\mathcal{L}^{\otimes(-(10n+4))}\bigr)\Bigr)=\lambda+\binom{3n+2}{2}\kappa_1+2(5n+2)^2\mathfrak{A}-2(5n+2)(6n+3)\mathfrak{B},$$

and after routine manipulations we obtained the claimed formula. $\square$

10.4. For each $n\geq 0$ such that $\mathfrak{D}_n$ is an actual divisor on $\mathcal{G}_{E_6}$, using Theorem 10.3, we obtain the following lower bound for the total Hodge class:

$$(48n^2+48n+11)\lambda\leq 15(2n+1)^2\lambda^{(-5)}+6(3n+1)(3n+2)[D_{E_6}]+(3n+1)(3n+2)[D_{\mathrm{syz}}].$$

and

$$\lambda\leq\frac{3(97n^2+97n+26)}{87n^2+87n+22}\,\lambda^{(-5)}+\frac{3(36n^2+36n+8)}{87n^2+87n+22}\,[D_{E_6}].$$

In particular, for $n=0$, Theorem 10.8 yields the following upper bound for λ:

Corollary 10.5. *On $\mathcal{G}_{E_6}$, the following divisor class is effective:*

$$\frac{39}{11}\lambda^{(-5)}+\frac{12}{11}[D_{E_6}]+\frac{39}{11}\mathfrak{n}-\lambda\geq 0.$$

A consequence of this bound is a sufficient condition for $\mathcal{A}_6$ to be of general type. Recall that $\widetilde{\varphi}:\mathcal{G}_{E_6}\to\widetilde{\mathcal{M}}_{46}$ is the map given by $\widetilde{\varphi}([C,L])=[C]$.

Proposition 10.6. *Assume that there exists an effective divisor $D\in\mathrm{Eff}(\overline{\mathcal{M}}_{46})$ having slope $s(D)<\frac{198}{41}$ and not containing the image $\widetilde{\varphi}(\mathcal{G}_{E_6})$. Then $\mathcal{A}_6$ is of general type.*

Proof. By assumption, there exists an effective $\mathbb{Q}$-divisor $D\equiv s\lambda-\delta_0\in CH^1(\widetilde{\mathcal{M}}_{46})$ with $s<\frac{198}{41}$, such that $\widetilde{\varphi}^*([D])=s\lambda-6[D_{E_6}]-[D_{\mathrm{syz}}]$ is effective on $\mathcal{G}_{E_6}$. Using Proposition 8.17 and Corollary 10.5, it follows that the class

$$\Bigl(\frac{39s}{12}-12\Bigr)\lambda^{(-5)}-\Bigl(6-\frac{12}{11}s\Bigr)[D_{E_6}]\in CH^1(\mathcal{G}_{E_6})$$

is effective. The slope of the $\lambda^{(-5)}$ and $[D_{E_6}]$-coefficients of this class is lower than $7=s(K_{\overline{\mathcal{A}}_6})$, precisely when $s<\frac{198}{41}$. Since $\lambda^{(-5)}$ is a big class, this finishes the proof. $\square$

Remark 10.7. The known effective divisors of smallest slope on $\overline{\mathcal{M}}_g$ are the Koszul divisors constructed in [Far09]. Determining the minimal slope of an effective divisor on $\overline{\mathcal{M}}_g$ is a long standing open problem.

We now address the issue whether the classes computed in Theorem 10.3 are effective. In particular, this also confirms the upper bound for λ from Corollary 10.5.

Theorem 10.8. *For a general E_6-curve $[C, L] \in \mathcal{G}_{E_6}$, we have that*

$$H^0(C, 2K_C - 5L) = 0,$$

that is, $\mathfrak{D}_0$ is an effective divisor on $\mathcal{G}_{E_6}$.

Proof. The proof of Theorem 10.8 involves a degeneration similar to the one used to establish the dominance of the map PT (in fact a simpler one). We start with a cover $\pi_t : C_t \to \mathbb{P}^1$ ramified in 24 points such that the local monodromy elements are reflections w_i in 12 pairs of roots $r_1, \dots, r_{12}$ generating the lattice E_6. We consider a degeneration in which the 12 pairs of roots with the same label come together. The degenerate cover $\pi : C \to \mathbb{P}^1$ is ramified in 12 points $q_1, \dots, q_{12} \in \mathbb{P}^1$. Over each point q_i there are 6 simple ramification points. The curve C is nodal with $12 \times 6 = 72$ ordinary double points.

Lemma 10.9. *The curve C has 27 irreducible components isomorphic to $\mathbb{P}^1$ and the restriction of π to each of them is an isomorphism.*

Proof. The dual graph of C is connected and has $|V| \le 27$ vertices and $|E| = 6 \times 12 = 72$ edges. If g_i are the genera of the normalizations of the irreducible components of C, one has $46 = p_a(C) = |E| - |V| + 1 + \sum g_i = 73 - |V|$, which implies that $|V| = 27$ (and of course all $g_i = 0$). □

Remark 10.10. Note that the cover π is not *admissible* in the sense of [ACV03]. The corresponding admissible cover is obtained by replacing each point $q_i \in \mathbb{P}^1$ by an inserted $\mathbb{P}^1$ with two additional marked points p_i, p_{i+12}, and modifying the curve C accordingly.

Notation 10.11. The 27 irreducible components $\{X_s \simeq \mathbb{P}^1\}_{s=1}^{27}$ of C are in bijection with the lines $\{\ell_s\}_{s=1}^{27}$ on a cubic surface. Let Γ be the dual graph of C. For each root r_i with $i = 1, \dots, 12$, there are 6 pairs of lines (a_{ij}, b_{ij}) such that $r_i \cdot a_{ij} = 1$, $b_{ij} = a_{ij} + r_i$, hence $r_i \cdot b_{ij} = -1$. To each pair we associate an edge (a_{ij}, b_{ij}) of Γ directed from the vertex a_{ij} to the vertex b_{ij}. We also fix 12 ramification points $q_i \in \mathbb{P}^1 \setminus \{0, \infty\} = \mathbb{C}^*$ and denote by $\{p_{si}\}_{i=1}^{n_s}$ the nodes of C lying on X_s. Clearly, $\pi(p_{si}) \in \{q_1, \dots, q_{12}\}$, for all s and i.

Lemma 10.12. *The space $H^0(C,\omega_C)$ is naturally identified with*

$$H_1(\Gamma,\mathbb{C}) = \operatorname{Ker}\Big\{\bigoplus_{i=1}^{12}\bigoplus_{j=1}^{6}\mathbb{C}(a_{ij},b_{ij}) \to \bigoplus_{s=1}^{27}\mathbb{C}\ell_s\Big\}.$$

To an edge (a_{ij},b_{ij}) over a root r_i one associates a differential form ω_{ij} equal to $\frac{dz}{z-q_i}$ on $X_{a_{ij}}$, to $-\frac{dz}{z-q_i}$ on $X_{b_{ij}}$ and 0 on X_s, for $s\neq a_{ij},b_{ij}$. Then $H^0(C,\omega_C)$ is the subspace of

$$\bigoplus_{s=1}^{27} H^0\Big(X_s, K_{X_s}\big(\sum_{i=1}^{n_s} p_{si}\big)\Big)$$

of the forms $\omega=\sum c_{ij}\omega_{ij}$, such that for $1\le s\le 27$ the sum of residues of ω on X_s is zero. Equivalently, for each $1\le s\le 27$, one considers a space of forms

$$\omega_s = \sum_{s\in\{a_{ij},b_{ij}\}} c_i\frac{dz}{z-q_i}, \quad \textit{such that } \sum_i c_i = 0.$$

Then a form $\omega\in H^0(C,\omega_C)$ is equivalent to a collection of log forms $\{\omega_s\}_{s=1}^{27}$ satisfying the 72 conditions $\operatorname{Res}_{q_i}(\omega_{a_{ij}})+\operatorname{Res}_{q_i}(\omega_{b_{ij}})=0$, for each edge (a_{ij},b_{ij}) of Γ.

Proof. This follows by putting together two well known facts:
(1) Let C be a nodal curve with normalization $\nu\colon \widetilde{C}\to C$ and nodes $p_i\in C$ such that $\nu^{-1}(p_i)=\{p_i^+,p_i^-\}$. Then $H^0(C,\omega_C)$ is identified with the space of sections

$$\tilde\omega\in H^0\Big(\widetilde{C}, K_{\widetilde{C}}\big(\sum(p_i^+ + p_i^-)\big)\Big) \text{ satisfying } \operatorname{Res}_{p_i^+}(\tilde\omega)+\operatorname{Res}_{p_i^-}(\tilde\omega)=0.$$

(2) A section of $H^0\big(\mathbb{P}^1, K_{\mathbb{P}^1}(\sum_i q_i)\big)$ is a linear combination $\sum_i c_i\frac{dz}{z-q_i}$, with $\sum_i c_i=0$. ☐

In practice, assume that $H^0(X_s,\omega_{C|X_s})$ is identified with the space of fractions

$$\frac{P_s(x)}{\prod_{i=1}^{n_s}(x-\pi(p_{si}))}\,dx,$$

where $P_s(x)$ is a polynomial of degree n_s-2. Then $H^0(C,\omega_C)\subseteq\bigoplus_{s=1}^{27}H^0(X_s,\omega_{C|X_s})$ is characterized by the condition that for every node of $p_{sj}=p_{s'j'}\in C$ joining components X_s and $X_{s'}$, the *sum* of the residues

$$\operatorname{Res}_s := \frac{P_s(\pi(p_{sj}))}{\prod_{i\neq j}\big(\pi(p_{sj})-\pi(p_{si})\big)}$$

and $\operatorname{Res}_{s'}$ respectively is 0.

Similarly, if $H^0(X_s,\omega_{C|X_s}^{\otimes 2})$ is identified with the space of fractions

$$\frac{P_s(x)}{\prod_{i=1}^{n_s}(x-\pi(p_{si}))^2}\,(dx)^{\otimes 2},$$

where deg $P_s(x) = 2(n_s - 2)$, then $H^0(C, \omega_C^{\otimes 2}) \subseteq \bigoplus_{s=1}^{27} H^0(X_s, \omega_{C|X_s}^{\otimes 2})$ is the subspace characterized by the condition that for every node $p_{sj} = p_{s'j'} \in C$ joining components X_s and $X_{s'}$, the *difference* of the residues

$$\mathrm{Res}_s := \frac{P_s(q_j)}{\prod_{i \neq j} \big(\pi(p_{sj}) - \pi(p_{si})\big)^2}$$

and $\mathrm{Res}_{s'}$ respectively is 0.

To describe $H^0(C, 2\omega_C - 5L)$, we take the subspace in the above space of sections that vanish at ∞ with multiplicity at least 5. This means that instead of polynomials of degree $2(n_s-2)$ one should consider polynomials of degree $2(n_s-2)-5$. Determining h^0 reduces to a linear algebra computation. The unknowns are the coefficients of the polynomials $\{P_s(x)\}_{s=1}^{27}$ and the equations are the residue anti-matching, respectively matching, conditions. The initial input is:

(1) 12 points $q_i \in \mathbb{P}^1 \setminus \{0, \infty\} = \mathbb{C}^*$, and
(2) 12 roots r_i generating the lattice E_6.

For $H^0(C, \omega_C)$, there are $\sum_{s=1}^{27}(n_s - 1) = 117$ unknowns and 72 equations. The matrix has rank 71, and the space of solutions has dimension $46 = g$. For $H^0(C, \omega_C^{\otimes 2})$ there are $\sum_{s=1}^{27}(2n_s - 3) = 207$ unknowns and 72 equations. The matrix has full rank 72, and the space of solutions has dimension $135 = 3g - 3$.

In order to have $H^0(C, 2\omega_C - 5L) = 0$, at least $n_s \geq 4$ nodes must lie on each irreducible component X_s. Otherwise, $\deg(P_s) = 2(n_s-2)-5 \leq -2$ and the system has more unknowns than equations. It is allowed to have $\deg(P_s) = -1$; this case gives $\deg(P_s)+1 = 2n_s-8 = 0$ unknowns, which fits the formula. Assuming each $n_s \geq 4$ one has $\sum_{s=1}^{27}(2n_s - 8) = 72$ unknowns and 72 equations, so the expected dimension is 0. This situation is achieved for the following choice of ramification points and roots:

(1) $q_i = i \in \mathbb{C}$, for $i = 1, \ldots, 12$.
(2) $r_1 = \alpha_{135}$, $r_2 = \alpha_{12}$, $r_3 = \alpha_{23}$, $r_4 = \alpha_{34}$, $r_5 = \alpha_{45}$, $r_6 = \alpha_{56}$, $r_7 = \alpha_{16}$, $r_8 = \alpha_{456}$, $r_9 = \alpha_{123}$, $r_{10} = \alpha_{346}$, $r_{11} = \alpha_{234}$, $r_{12} = \alpha_{156}$, using the standard notation for roots in $\mathbb{Z}^{1,6}$, that is, $\alpha_{ij} = f_i - f_j$ and $\alpha_{ijk} = f_0 - f_i - f_j - f_k$.

A Mathematica notebook with an explicit computation can be found at [Web15]. Note that $r_1, \ldots, r_6$ generate E_6, so the cover $\pi : C \to \mathbb{P}^1$ is connected. This completes the proof of Theorem 10.8. $\square$

10.13. A Petri theorem at the level of the E_6-Hurwitz space. We now move on to prove Theorem 8.2 and show that a Petri-like theorem holds generically on Hur. Using the same framework and notation, we study the Petri map

$$\mu(L) : H^0(C, L) \otimes H^0(C, \omega_C \otimes L^\vee) \to H^0(C, \omega_C)$$

for a 72-nodal curve C corresponding to a cover $\pi : C \to \mathbb{P}^1$ as above.

Lemma 10.14. *Let $H^0(\mathbb{P}^1, \mathcal{O}(1)) = \langle x_0, x_1 \rangle \subset H^0(C, L)$. Then the subspace*

$$H^0(C, \omega_C \otimes L^\vee) \otimes \langle x_0 \rangle \subset H^0(C, \omega_C \otimes L^\vee) \otimes H^0(C, L) \subset H^0(C, \omega_C)$$

consists of elements $\{\omega_s\}_{s=1}^{27}$ as above, satisfying for each $1 \le s \le 27$ the additional condition $\sum c_i q_i = 0$. Similarly, the subspace $H^0(C, \omega_C \otimes L^\vee) \otimes \langle x_1 \rangle \subseteq H^0(C, \omega_C)$ consists of elements $\{\omega_s\}_{s=1}^{27}$ as above, satisfying for $1 \le s \le 27$ an additional condition $\sum \frac{c_i}{q_i} = 0$.

Proof. We identify $H^0(C, \omega_C \otimes L^\vee)$ with $H^0(C, \omega_C(-\pi^*\infty))$, hence $H^0(C, \omega_C \otimes L^\vee) \otimes \langle x_0 \rangle$ is the space of forms $\{\omega_s\}_{s=1}^{27}$ such that for each $s = 1, \ldots, 27$, they satisfy the equality

$$0 = \mathrm{Res}_\infty\big(\omega_s \cdot \frac{x_1}{x_0}\big) = -\sum \mathrm{Res}_{q_i}\big(\omega_s \cdot \frac{x_1}{x_0}\big) = -\sum c_i q_i.$$

□

To prove Theorem 8.2, it is sufficient to find one degeneration $\pi : C \to \mathbb{P}^1$ such that

(1) $h^0(C, L) = 2$.
(2) The linear subspaces $H^0(C, \omega_C \otimes L^\vee) \otimes \langle x_0 \rangle$, $H^0(C, \omega_C \otimes L^\vee) \otimes \langle x_1 \rangle$ and $H^0(C, \omega_C)^{(-5)}$ generate the vector space $H^0(C, \omega_C)$.

We obtain a system of linear equations in the 72 variables

$$x_{ij} = \mathrm{Res}_{q_i}(\omega_{a_{ij}}) = -\mathrm{Res}_{q_i}(\omega_{b_{ij}}), \quad \text{for} \quad i = 1, \ldots, 12 \text{ and } \; j = 1, \ldots, 6.$$

For each of the spaces $H^0(C_0, \omega_C \otimes L^\vee) \otimes \langle x_0 \rangle$, respectively $H^0(C, \omega_C \otimes L^\vee) \otimes \langle x_1 \rangle$, we get 2×27 equations. By Lemma 9.8, $H^0(C, \omega_C)^{(-5)}$ is the subspace of $H^0(C, \omega_C)$ of forms $\{\omega_s\}_{s=1}^{27}$ satisfying for each $1 \le i \le 12$ the condition $x_{ij} = x_{ij'}$ for all $1 \le j, j' \le 6$. This gives a system of $27 + 12 \times 5$ equations.

Lemma 10.15. *The above conditions are satisfied for the following choices of roots and ramification points:*

(1) $r_1 = \alpha_{135}$, $r_2 = \alpha_{12}$, $r_3 = \alpha_{23}$, $r_4 = \alpha_{34}$, $r_5 = \alpha_{45}$, $r_6 = \alpha_{56}$, $r_7 = \alpha_{\max}$, $r_8 = \alpha_{124}$, $r_9 = \alpha_{234}$, $r_{10} = \alpha_{35}$, $r_{11} = \alpha_{13}$, $r_{12} = \alpha_{36}$.
(2) $q_i = i$, *for* $i = 1, \ldots, 12$.

Proof. This is now a straightforward linear algebra computation, which we performed in Mathematica. It can be found at [Web15]. □

References

[ACV03] Dan Abramovich, Alessio Corti and Angelo Vistoli, *Twisted bundles and admissible covers*, Comm. Algebra **31** (2003), 3547–3618.

[Ale02] Valery Alexeev, *Complete moduli in the presence of semiabelian group action*, Annals of Math. **155** (2002), 611–708.

[Ale04] Valery Alexeev, *Compactified Jacobians and Torelli map*, Publ. Res. Inst. Math. Sci. **40** (2004), 1241–1265.

[AB12] Valery Alexeev and Adrian Brunyate, *Extending the Torelli map to toroidal compactifications of Siegel space*, Invent. Math. **188** (2012), 175–196.

[ABH02] Valery Alexeev, Christina Birkenhake, and Klaus Hulek, *Degenerations of Prym varieties*, J. Reine Angew. Math. **553** (2002), 73–116.

[ACGH85] Enrico Arbarello, Maurizio Cornalba, Philip Griffiths and Joseph Harris, *Geometry of Algebraic Curves, Volume I*, Springer, Berlin, 1985.

[AV12] Maksym Arap and Robert Varley, *On algebraic equivalences among the 27 Abel-Prym curves on a generic abelian* 5*-fold*, arXiv:1206.6517, to appear in International Math. Research Notices.

[Bea77] Arnaud Beauville, *Prym varieties and the Schottky problem*, Invent. Math. **41** (1977), 149–196.

[BdS49] Armand Borel and J. de Siebethal, *Les sous-groupes fermés de rang maximum des groupes de Lie clos*, Comment. Math. Helv. 23, (1949), 200–221.

[Cle83] Herbert Clemens, *Double Solids*, Advances in Math. **47** (1983), 107–230.

[CG72] Herbert Clemens and Phillip Griffiths, *The intermediate Jacobian of the cubic threefold*, Annals of Math. **95** (1972), 281–356.

[Dol12] Igor V. Dolgachev, *Classical algebraic geometry: A modern view*, Cambridge University Press, Cambridge, 2012.

[DS81] Ron Donagi and Roy Smith, *The structure of the Prym map*, Acta Math. **146** (1981), 25–102.

[Don84] Ron Donagi, *The unirationality of $\mathcal{A}_5$*, Annals of Math. **119** (1984), 269–307.

[Don92] Ron Donagi, *The fibers of the Prym map,* Curves, Jacobians, and abelian varieties (Amherst, MA, 1990) Contemp. Math., vol. 136, Amer. Math. Soc., Providence, RI, 1992, 55–125.

[DL11] M. J. Dyer and G. I. Lehrer, *Reflection subgroups of finite and affine Weyl groups*, Transactions American Math. Soc. **363** (2011), 5971–6005.

[Dyn52] E. Dynkin, *Semisimple subalgebras of semisimple Lie algebras*, Mat. Sbornik N.S. **30** (1952), 349–462.

[Est96] Eduardo Esteves, *Wronski algebra systems on families of singular curves*, Ann. Scient. Éc. Norm. Sup. **29** (1996), 107–134.

[Far09] Gavril Farkas, *Koszul divisors on moduli spaces of curves*, American J. Math. **131** (2009), 819–869.

[FG03] Gavril Farkas and Angela Gibney, *The Mori cones of moduli spaces of pointed curves of small genus*, Transactions American Math. Soc. **355** (2003), 1183–1199.

[FL10] Gavril Farkas and Katharina Ludwig, *The Kodaira dimension of the moduli space of Prym varieties*, J. European Math. Society **12** (2010), 755–795.

[FV14] Gavril Farkas and Alessandro Verra, *The universal abelian variety over $\mathcal{A}_5$*, arXiv:1402.0539, to appear in Ann. Scient. Éc. Norm. Sup.

[Fra01] William N. Franzsen, *Automorphisms of Coxeter groups*, PhD thesis, University of Sydney, 2001. http://www.maths.usyd.edu.au/u/PG/Theses/franzsen.pdf

[vdGK12] Gerard van der Geer and Alexis Kouvidakis, *The Hodge bundle on Hurwitz spaces*, Pure Appl. Math. Quarterly **7** (2011), 1297–1308.

[HM82] Joe Harris and David Mumford, *On the Kodaira dimension of* $\overline{\mathcal{M}}_g$, Invent. Math. **67** (1982), 23–88.

[IL12] Elham Izadi and Herbert Lange, *Counter-examples of high Clifford index to Prym-Torelli*, J. Algebraic Geom. **21** (2012), 769–787.

[ILS09a] Elham Izadi, Herbert Lange, and Volker Strehl, *Correspondences with split polynomial equations*, J. Reine Angew. Math. **627** (2009), 183–212.

[ILS09b] Elham Izadi, Marco LoGiudice and Gregory Sankaran, *The moduli space of étale double covers of genus 5 curves is unirational*, Pacific J. Math. **239** (2009), 39–52.

[Kan87] Vassil Kanev, *Principal polarizations on Prym-Tyurin varieties*, Compositio Math. **64** (1987), 243–270.

[Kan89a] Vassil Kanev, *Intermediate Jacobians and Chow groups of threefolds with a pencil of del Pezzo surfaces*, Annali Mat. Pura ed Applicata **154** (1989), 13–48.

[Kan89b] Vassil Kanev, *Spectral curves, simple Lie algebras, and Prym-Tjurin varieties*, Theta functions—Bowdoin 1987, Proc. Sympos. Pure Math., vol. 49, Amer. Math. Soc. 1989, 627–645.

[Kan06] Vassil Kanev, *Hurwitz spaces of Galois coverings of* $\mathbb{P}^1$*, whose Galois groups are Weyl groups*, J. Algebra **305** (2006), 442–456.

[KM13] Sean Keel and James McKernan, *Contractible extremal rays on* $\overline{\mathcal{M}}_{0,n}$, in Handbook of Moduli Vol. 2, 115–130, International Press 2013.

[KKZ11] Alexey Kokotov, Dmitry Korotkin and Peter Zograf, *Isomonodromic tau function on the space of admissible covers*, Advances Math. **227** (2011), 586–600.

[Kol90] János Kollár, *Projectivity of complete moduli*, J. Differential Geometry **32** (1990), 235–268.

[LR08] Herbert Lange and Anita M. Rojas, *A Galois-theoretic approach to Kanev's correspondence*, Manuscripta Math. **125** (2008), 225–240.

[Mor98] Atsushi Moriwaki, *Relative Bogomolov inequality and the cone of positive divisors on the moduli space of stable curves*, J. Amer. Math. Soc. **11** (1998), 569–600.

[MM83] Shigeyumi Mori and Shigeru Mukai, *The uniruledness of the moduli space of curves of genus* 11, Springer Lecture Notes in Mathematics **1016** (1983), 334–353.

[Mum83] David Mumford, *On the Kodaira dimension of the Siegel modular variety*, Algebraic geometry—open problems (Ravello, 1982), Lecture Notes in Math., vol. 997, Springer, 1983, 348–375.

[Nam73] Yukihiko Namikawa, *On the canonical holomorphic map from the moduli space of stable curves to the Igusa monoidal transform*, Nagoya Math. J. **52** (1973), 197–259.

[Nam76] Yukihiko Namikawa, *A new compactification of the Siegel space and degeneration of Abelian varieties II*, Math. Ann. **221** (1976), no. 3, 201–241.

[Osh06] Toshio Oshima, *A classification of subsystems of a root system*, arXiv:math/0611904 (2006).

[Tai82] Y. Tai, *On the Kodaira dimension of the moduli space of abelian varieties*, Invent. Math. **68** (1982), 425–439.

[Ver84] Alessandro Verra, *A short proof of the unirationality of* $\mathcal{A}_5$, Indagationes Math. **46** (1984), 339–355.

[Web15] *Website with supporting computer computations*, http://alpha.math.uga.edu/~valery/a6.

[Wir95] Wilhelm Wirtinger, *Untersuchungen über Thetafunktionen*, Teubner, Leipzig, 1895.

Valery Alexeev: Department of Mathematics, University of Georgia
Athens GA 30602, USA
E-mail address: valery@math.uga.edu

Ron Donagi: Department of Mathematics, University of Pennsylvania
209 South 33rd Street, Philadelphia, PA 19104-6395, USA
E-mail address: donagi@math.upenn.edu

Gavril Farkas: Institut für Mathematik, Humboldt-Universität zu Berlin
Unter den Linden 6, 10099 Berlin, Germany
E-mail address: farkas@math.hu-berlin.de

Elham Izadi: Department of Mathematics, University of California, San Diego
La Jolla, CA 92093-0112, USA
E-mail address: eizadi@math.ucsd.edu

Angela Ortega: Institut für Mathematik, Humboldt-Universität zu Berlin
Unter den Linden 6, 10099 Berlin, Germany
E-mail address: ortega@math.hu-berlin.de