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SANGAKU: SACRED MATHEMATICS IN JAPAN

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Table of Contents

List of Figures	iv
Abstract	v
1 Introduction	1
1.1 Conventions	1
1.2 Japanese history related to mathematics	1
1.3 Sangaku and mathematical documents	3
2 Sangaku Problems	4
2.1 Diophantine problem	5
2.2 Osculating circles problem	6
2.3 Ellipses problem	9
2.4 Dense packing problem	11
3 Mathematical Education and Communication	13
3.1 Mathematical education	13
3.2 Mathematical communication	15
4 Conclusion	16
Figures	17
Bibliography	22

List of Figures

Figure 1: Find N, m , and n	17
Figure 2: Find r_4 in terms of r_1, r_2 , and r_3	17
Figure 3: Drop a perpendicular k through the center of r_3	18
Figure 4: Find the semi-major axis in terms of the radius.	18
Figure 5: Choose the coordinate axes.	19
Figure 6: Find the radius of the small circles, r , in terms of S	19
Figure 7: Similar triangles.	20
Figure 8: Nine circles can also be packed into the same circle.	20
Figure 9: Malfatti's solution.	21
Figure 10: The combined area of circles on the left is larger.	21
Figure 11: One on the right wins if $\sin(A/2) < \tan(B/4)$	21

Abstract

SANGAKU: SACRED MATHEMATICS IN JAPAN

Katsuhito Sugano

Japan in the Edo period (1603 - 1867) closed virtually all direct contacts with the Western countries and subsequently developed a unique style of mathematics independently of the West. We present a historical account of various mathematical problems described in "sangaku" (mathematical tablets displayed in shrines and temples in Japan) that were hung by many mathematics aficionados from different socioeconomic classes. These tablets contain interesting mathematical problems and answers, and the contributors considered the tablets sacred because they spent years to solve their problems. Sometimes, interesting mathematical problems and theorems were discovered many years before the Western counterparts. We describe some of the problems and prove them, using a modern mathematical notation. We also mention the history of mathematics related to sangaku problems and discuss the mechanisms of mathematical education and communication in Japan at that time period.

1 Introduction

1.1 Conventions

We study the history of mathematics in Japan during the Edo period (1603 - 1867) mainly based on the book entitled, *Sacred Mathematics: Japanese Temple Geometry*, by Fukagawa Hidetoshi and Tony Rothman.[1] We follow the naming and pronunciation convention of Japanese people in this book: The surname is first, the given name at birth is next, and the middle name sometimes adopted after adulthood is omitted; the pronunciation of given name is based on “kun yomi” (the traditional Japanese pronunciation of kanji), instead of “on doku” (the Chinese style pronunciation); and the spelling of Japanese word is the Hepburn romanization system for translation into English. For example, one of the most famous Japanese mathematicians, known as Kowa Seki in the West, becomes Seki Takakazu, and the name spelled “Ōno” with a bar on top of a vowel “O” is pronounced like “Ohno” instead of “Ono” without a bar.

1.2 Japanese history related to mathematics

In 718 A.D., the Japanese government under the empress Genshō in the Nara period (710 - 794) passed a series of new laws called “Yōrō Ritsuryō” (the Laws in the years of Yōrō), by which they officially remeasured the size of fields, levied taxes, and created the office of “San Hakase” that means literally “Mathematics Doctorate” but more appropriately “Arithmetic Intelligence” in this context. The core members of the department consisted of about seventy midlevel functionaries who were proficient in arithmetic operations. They learned the skills in arithmetics from the Chinese books, using a precursor to the abacus that consisted of a set of small bamboo sticks known as “sauzi” in Chinese and “sangi” in Japanese. This device was a very powerful computational tool that was capable of performing some high-level mathematical operations such as square root if it was properly mastered. The most important Chinese mathematical text to them would have been the *Jiu zhang Suanshu* (Nine Chapters on the Mathematical Art), the date and author of which is unknown although most experts seem to believe it was finished by the late second or early third century A.D. The total number of people involved in each tax-assessment team under the direction of a San Hakase was very large; for

example, it is estimated to be 1,330 persons for the Suruga province, which had a relatively small size in 738, based on the inspection of extensive collections of historical documents about the expense reports and maps.

Around the year 750, Buddhism arrived from China and has coexisted with the native Shintoism in relative peace. Most Japanese people have been practicing both religions socially and individually without serious conflicts since then. Buddhist temples in particular became repositories of learning, while Shinto shrines remained as places for conducting traditional Japanese festivals and rituals. Hence, both temples and shrines became the centers of gathering for the Japanese people as well as the places for worship. This fact will play an important role later in the Edo period, when many mathematics aficionados from various socioeconomic classes hung “sangaku” (mathematical tablets) dedicated to shrines and temples. These tablets contain interesting mathematical problems and their solutions; the contributors felt that the tablets were sacred since they sometimes had to spend years to solve their problems until some inspirations arrived at their minds at last. Moreover, by displaying mathematical problems in sangaku in public places like temples and shrines, mathematicians at that time communicated with one another for interesting problems and solutions.

In 1603, Tokugawa Ieyasu became the shogun of Japan, establishing the Edo period, after he defeated his rival warlords at the battle of Sekigahara three years before. He consolidated power by confiscating the lands of other warlords and carried on substantial trade with foreign countries, both Eastern and Western. Consequently, information about mathematics was available from all over the world mainly through books, especially about Euclidean geometry and basic algebra.

In the 1630s, however, the third shogunate Tokugawa Iemitsu, like other predecessors, issued series of decrees to expel Christian missionaries mainly based on the real threat of the European colonizations of Asia, Africa, and America, which had already been happening in the South and Southeast Asia by that time. In fact, some of the missionaries secretly gathered information on the military might of Japan such as the number of guns each region had, and sent it back to their countries for their feasibility studies of military invasion into Japan, should their trade or religion negotiations for their favors fall apart. Also for a more cultural reason, the Japanese government feared that the religion of Christianity would divide the nation because

it might not coexist with Shintoism and Buddhism peacefully in Japan, especially in each individual's faith. By 1641, Japan closed all direct contacts with the Western countries except Holland; the representatives of the Dutch East India Company were forced to move onto a small man-made island called "Deshima" in Nagasaki harbor that had been created for the Portuguese previously. This foreign policy known as "sakoku" (closed country) lasted for the next two hundred years; consequently, Japan was virtually isolated from the West. In terms of mathematics, it is extremely unlikely that anyone in Japan during this period learned about the creation of modern calculus by Newton and Leibnitz later in the seventeenth century.[1]

Approximately from 1640 to 1860, the Tokugawa shogunate provided political and economic stability in Japan by eliminating foreign conflicts due to the "sakoku" policy; hence the 250 years of the Edo period became known as the "Great Peace." Moreover, Japanese culture experienced a brilliant flowering during the years of "Genroku" in the late seventeenth century, which is commonly referred to as "Genroku Renaissance": The renowned "Ukiyo-e" attained its highest achievements, for example by Katsushika Hokusai and Kitagawa Utamaro; "Haiku" achieved its literary significance in the work of Matsuo Basho; "Noh" and "Kabuki" flourished; Drama was popular; and Tea ceremonies, flower arrangement, and garden architecture were on the ascendant. As to mathematics, people began hanging sangaku in shrines and temples during this Genroku period. In those sangaku, they created and solved many interesting problems in Euclidean geometry, calculus, and number theory. Moreover, they developed a unique style of calculus called "enri" (circle principle), which amounts to definite integration for calculating areas but does not mention the Fundamental Theorem of Calculus.

1.3 Sangaku and mathematical documents

The oldest surviving sangaku has been found in Tochigi prefecture and dates from 1683 although we do not know exactly when the tradition of hanging sangaku began. Yamaguchi Kanzan, a nineteenth century mathematician, mentions an even older sangaku dating from 1668 in his travel diary, but this sangaku is now lost. Over the next two centuries, these tablets appeared all over Japan: about two-thirds in Shinto shrines and one-third in Buddhist temples. We do not know how many of them are originally produced but are certain that at least 1,738 have

been lost according to contemporary mathematics texts that mentioned sangaku. For instance, only two percent of the tablets recorded in Yamaguchi Kanzan's diary survive. We estimate that there were originally thousands more than the 900 tablets extant today. The practice of hanging sangaku gradually died out after the fall of the Tokugawa shogunate, but some devotees continued to post them as late as 2000, and sangaku continue to be discovered even now. In 2005, five tablets were found in Toyama prefecture alone. The most recent one discovered at the Ubara shrine dates from 1870.[1]

The size of sangaku varies considerably: from a small one of size 45 cm by 30 cm to a large one of 620 cm by 140 cm. Most of them contain multiple problems by different proposers and show only the final answer to each problem, rarely a detailed solution, mainly because of space limitation and partly because of the proposer's challenge for others to try the problem themselves. It seems that the presenters of sangaku also took the spiritual and even religious aspect of the practice seriously. Moreover, they show precise geometrical figures on the tablets that are often beautifully colored for easy recognition; sometimes the frame of a tablet is also decorated with art-works such as engraved metal plates or figures of wood-curved dragons. Today, sangaku is inspiring not only mathematicians for its mathematical gems but also artists for its artistically colored geometric figures.

The majority of the presenters of sangaku seem to have been members of the samurai class. During the Edo period, most samurai were not engaging in battles in full armor around the countryside, but worked as government functionaries; many became mathematicians. Nonetheless, the inscriptions on the tablets make clear that whole class of students, children, and occasionally women dedicated sangaku.[1][6]

2 Sangaku Problems

To show the flavor of the diversity of sangaku problems[1][2], we select several examples that represent different types of mathematics. At the beginning of each problem we introduce the name of the proposer, his/her brief background, and the year when the problem was proposed. We also mention the location, the size, and the specific history of the sangaku that contained the problem, or the reference to the problem if the sangaku is lost. Then, we describe the

problem in a mathematically concise way and give a solution/proof of each problem in detail, using a modern mathematical notation because the process of derivation is usually omitted on a sangaku due to its space limitation. Finally, we discuss the historical significance of some problems.

2.1 Diophantine problem

The sangaku containing the problem below was hung by Nagayoshi Nobuhiro of the Itō Yasusada school in 1858 in the Kumano shrine of Senhoku city in Mie prefecture. It measures 89 cm wide and 38 cm high. For decades the tablet had gone unrecognized as a sangaku and was on the verge of being discarded when Fukagawa Hidetoshi visited the shrine and recognized its value in 2005; subsequently it has been restored in the shrine.[1] This problem is about the integer solutions of equations (Diophantine problem) and number theory in mathematics, which is rare among the sangaku problems, but it seems to have been inspired by geometry in the sense that the “ball” mentioned in the problem can be interpreted as a barrel of rice, which was traditionally shaped like an approximate cylinder, and these barrels were often stacked in the shape of trapezoid for storage.

Problem: We have N balls. First we stack them with 19 balls on the top and m balls on the bottom, as on the left side of figure 1 in the figures section. Then, we can re-stack them with 6 balls on top and n on the bottom, as on the right side of the figure 1. Find N , m , and n .

Solution: By the summation formula, we have

$$N = \sum_{i=19}^m i = \sum_{i=1}^m i - \sum_{i=1}^{18} i = \frac{m(m+1)}{2} - \frac{18(18+1)}{2} = \frac{m^2 + m - 342}{2}.$$

Similarly,

$$N = \sum_{i=6}^n i = \frac{n^2 + n - 30}{2}.$$

Hence,

$$m^2 + m - 342 = n^2 + n - 30.$$

$$m^2 - n^2 + m - n = 312.$$

$$(m - n)(m + n + 1) = 312.$$

Since m and n are positive integers, we notice that if $m - n$ is even, then $m + n + 1$ is odd, whereas if $m - n$ is odd, then $m + n + 1$ is even. For if $m - n$ is even, then m and n are either both even or both odd, which implies $m + n + 1$ is odd; similarly if $m - n$ is odd, then only one of m and n is odd and the other is even, which implies $m + n + 1$ is even. Therefore, we can narrow down the search of $m - n$ and $m + n + 1$ among the possible positive integer factors of 312 for $0 < m - n < m + n + 1$:

$$312 = 1 \cdot 312, 2 \cdot 156, 3 \cdot 104, 4 \cdot 78, 6 \cdot 52, 8 \cdot 39, 12 \cdot 26, \text{ and } 13 \cdot 24$$

The only candidate for the *even · odd* solutions above is $8 \cdot 39$, significantly reducing the possibilities. By setting $m - n = 8$ and $m + n + 1 = 39$, we get $m = 23, n = 15$, and $N = 105$.

In the case of *odd · even* solutions above, there is no reduction of possibilities because of the fact that 312 is already an even number, *i.e.*, if one factor is odd, then the other factor must be even. So, we have to examine all three candidates above, which rejects $3 \cdot 104$ and $13 \cdot 24$ because they lead to inconsistent N values that are calculated separately from m and n . Hence, the only candidate for the *odd · even* solutions above is $1 \cdot 312$. In this case we get $m = 156, n = 155$, and $N = 12075$. However, this solution is not written on the tablet.

Note: The method of this solution is different from the one in the book of Fukagawa and Rothman[1] in terms of narrowing down the possibilities of factors. In the book they write, “Notice that $8 \cdot 39$ can also be factored into, for example, $6 \cdot 52$ or $24 \cdot 13$, but these do not give a solution in positive integers. ... Convince yourself that $8 \cdot 39$ and $1 \cdot 312$ are the only correct possibilities.”

2.2 Osculating circles problem

This problem was proposed by Sugano Teizou and one of the twenty-one problems in the sangaku that was hung by the pupils of Sakuma Yōken in 1877 in the Abe no Monjuin temple in Fukushima prefecture. The tablet measures 620 cm in width and 140 cm in height and is the largest known sangaku. However, weather damage has made many of the problems indistinct.

Problem: A chain of four circles of radii r_1, r_2, r_3 , and r_4 are touched on one side by the

line l and on the other side by a circular arc of radius r . Find r_4 in terms of r_1, r_2 , and r_3 . (See figure 2 in the figures section.)

Solution:[1] In the following subsections, we adopt a convention to abbreviate the name of a circle by its length of radius: “circle r ” instead of “circle with radius r that is centered at a certain location.” First, we prove the following lemma quoted in the 1810 book, *Sanpō Tenshōhō Shinan* (Guidebook to Algebra and Geometry) by Aida Yasuaki (1747 - 1817):

Lemma: Three circles r_1, r_2 , and r_3 mutually touch. (See figure 3.) Two of the circles, r_1 and r_2 , are tangent to the line l , while the third circle, r_3 , is tangent to a parallel line f . Then,

$$r_3 = \frac{[h(r_1 + r_2) - 2r_1r_2]^2}{8hr_1r_2},$$

where h is the distance between l and f .

Proof: We drop a perpendicular line k from f through the center of r_3 to l . The distance between the center of r_1 and k is s ; the distance between the center of r_2 and k is t . Then the horizontal distance between the center of r_1 and r_2 is $s + t$. Moreover, the square of this horizontal distance can be calculated by the Pythagorean theorem: $(r_1 + r_2)^2 - (r_2 - r_1)^2 = 4r_1r_2$. Hence,

$$s + t = 2\sqrt{r_1r_2}. \quad (1)$$

On the other hand, s and t can be calculated by the Pythagorean theorem, too:

$$s^2 = (r_1 + r_3)^2 - (h - r_1 - r_3)^2. \quad (2)$$

$$t^2 = (r_2 + r_3)^2 - (h - r_2 - r_3)^2. \quad (3)$$

Combining the equations (1), (2), and (3), we get the following equation after some tedious algebra:

$$8hr_1r_2r_3 = 4r_1^2r_2^2 - 4hr_1r_2(r_1 + r_2) + h^2(r_1 + r_2)^2,$$

which is equivalent to the equation in the lemma.

Q.E.D.

Now, we solve the problem. Applying the above lemma to r_1 , r_2 , and r in the figure 2, we have

$$r = \frac{[h(r_1 + r_2) - 2r_1r_2]^2}{8hr_1r_2}.$$

Similarly from r_2 , r_3 , and r , we have

$$r = \frac{[h(r_2 + r_3) - 2r_2r_3]^2}{8hr_2r_3}.$$

Hence, by equating the above two expressions, we have h in terms of r_1 , r_2 , and r_3

$$h = \frac{2r_2\sqrt{r_1r_3}(\sqrt{r_1} - \sqrt{r_3})}{(r_1 + r_2)\sqrt{r_3} - (r_2 + r_3)\sqrt{r_1}} = \frac{2r_2\sqrt{r_1r_3}}{\sqrt{r_1r_3} - r_2}.$$

Repeating the above process for r_2 , r_3 , and r_4 , we have

$$h = \frac{2r_3\sqrt{r_2r_4}}{\sqrt{r_2r_4} - r_3}.$$

Thus, eliminating h from the above two expressions, we have an equation in terms of r_1 , r_2 , r_3 , and r_4 :

$$\frac{\sqrt{r_1r_2}}{\sqrt{r_1r_3} - r_2} = \frac{\sqrt{r_3r_4}}{\sqrt{r_2r_4} - r_3}.$$

Solving for $\sqrt{r_4}$,

$$\sqrt{r_4} = \frac{r_3\sqrt{r_1r_2}}{(\sqrt{r_1} + \sqrt{r_3})r_2 - r_3\sqrt{r_1}}.$$

Therefore,

$$r_4 = \frac{r_2r_3^2}{[(\sqrt{r_3/r_1} + 1)r_2 - r_3]^2},$$

which is written in the tablet.

During the later Edo period it became popular to consider problems that could be drawn

upon the shapes of ordinary objects such as the shape of a folding fan (a sector of an annulus). This problem may have been inspired by the shape of a “gunpai” (umpire’s fan) in sumo wrestling. (See figure 2.)

Note: We assumed that the configuration of the problem was possible. However, this configuration that four circles r_1, r_2, r_3 , and r_4 are tangent to both l and r may not be possible for all values of r . For instance, if the circle r is too far away from the line l , then the circle r_4 becomes too large; consequently, there is not enough space for the remaining three circles r_3, r_2 , and r_1 . Some necessary conditions for the circle r are such that $h \leq 2r$, where h is the height of four circles r_1, r_2, r_3 , and r_4 combined, the circle r is tangent to both l and the horizontal bottom line in figure 2, and so on.

2.3 Ellipses problem

This problem was proposed by Kobayashi Tadayoshi (1796 - 1871) on a sangaku hung in 1824 at Konpira shrine in Komoro city in Nagano prefecture. He was a student of the Takeuchi school and published a collection of sangaku problems called, *Sanpō Koren* (Mathematical Gems) in 1836. The collection records only five tablets, including this one and dating from 1824 to 1834. The problems in the collection, however, are all extremely difficult to solve by using only the mathematical knowledge at that time; for example, one problem asks for the area of ellipsoids, requiring the techniques of “enri” (definite integral) described previously. All sangaku mentioned in the *Sanpō Koren* have been lost.

Problem: As shown in figure 4, three identical ellipses of major axis $2a$ are inscribed in a large circle of radius r . Find a in terms of r when the area S of the ellipse is a maximum.

Solution:[1] Since this problem is difficult to solve by a traditional method, we present a modern solution. We choose the coordinate axes to lie along the major and minor axes of the bottom horizontal ellipse, $2a$ and $2b$, as in figure 5.

Then, the equation of the horizontal ellipse is

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1. \tag{1}$$

The equation of the large circle with center at $(0, p)$ is

$$x^2 + (y - p)^2 = r^2. \quad (2)$$

By considering the dashed equilateral triangle shown in figure 5, we see that the equation of the dotted tangent line to the ellipses is

$$y = \frac{1}{\sqrt{3}}x + p. \quad (3)$$

Inserting equation (3) for y into equation (1) gives

$$\frac{x^2}{a^2} + \frac{[\frac{1}{\sqrt{3}}x + p]^2}{b^2} = 1. \quad (4)$$

Solving for x in equation (4), we get

$$x = \frac{\sqrt{3}a^2}{a^2 + 3b^2} [-p \pm \sqrt{p^2 - (p^2 - b^2)\frac{a^2 + 3b^2}{a^2}}].$$

However, since the dotted tangent line intersects the ellipse at only one point, the roots must be the same, implying vanishing of the discriminant (*i.e.*, the expression inside the radical). This gives a relationship between a , b , and p .

$$p^2 = \frac{a^2 + 3b^2}{3}. \quad (5)$$

From equations (1) and (2) for the intersection of the circle and the ellipse, we also have

$$\frac{r^2 - (y - p)^2}{a^2} + \frac{y^2}{b^2} = 1.$$

As before, the discriminant of this equation must vanish. With expression (5) for p^2 , we get

$$4a^4 = 3r^2(a^2 - b^2).$$

Hence, the area of each ellipse is

$$A = \pi ab = \pi\sqrt{a^2b^2} = \pi\sqrt{a^4 - \frac{4a^6}{3r^2}}.$$

The area will be maximized when the quantity under the radical is maximized. Taking the derivative with respect to a , and setting it to zero gives $4a^3 - 8a^5/r^2 = 0$.

Therefore, the maximum is attained when $a = r/\sqrt{2}$ and $S = \pi r^2/2\sqrt{3}$.

2.4 Dense packing problem

This problem proposed by Kawano Michimuku, a student of the Fujita school, was written on a sangaku hung in 1804 at the Udo shrine in Miyazaki prefecture. We know of it from Fujita Kagen's 1807 version of the *Zoku Shinpeki Sanpō* (Sacred Mathematics II).

Problem: As shown in figure 6, ten circles of radius r touch each other externally and touch the larger circle internally. If S is the area of the big circle minus the area of the ten little circles, find r in terms of S .

Solution:[1] We offer a traditional solution from the manuscript, *Zoku Shinpeki Sanpō Kai* (Solutions to the Sacred Mathematics II) by Okayu Yasumoto (1794 - 1862).

Referring to figure 7, let R be the radius of the largest outer circle, $p = R - r$, and $q = AB$.

The marked angles in this figure allow us to use similar triangles, such that

$$\frac{n}{q} = \frac{r}{p}; \quad \frac{2n}{p} = \frac{q/2}{2r}.$$

Thus, $pq = 8nr = 8qr^2/p$, which gives $(R - r)^2 = p^2 = 8r^2$, or $R = (1 + \sqrt{8})r = (1 + 2\sqrt{2})r$.

Since S is defined as the area of the large circle minus the area of the ten small circles, $S = \pi R^2 - 10\pi r^2 = \pi(4\sqrt{2} - 1)r^2$, we have the following result:

$$r = \sqrt{\frac{S}{\pi(4\sqrt{2} - 1)}}.$$

This formula agrees with the specific numerical answer in the tablet.

This problem is interesting because we can also pack nine circles of the radius r into a circle of the same radius $R = (1 + 2\sqrt{2})r$. (See figure 8.)

Proof: Given the radius of small nine circles r , we prove $R = (1 + 2\sqrt{2})r$ for the radius of the large circle R . We notice that the diagonal line through the centers of three small diagonal circles coincides with the radius of the large circle. Moreover, the radius R on this diagonal

line is the sum of r and the hypotenuse of a right isosceles triangle with the equal sides $2r$.

$$\text{Thus, } R = r + 2r\sqrt{2} = (1 + 2\sqrt{2})r. \quad \text{Q.E.D.}$$

Moreover, Fukagawa and Rothman provide additional historical detail: *“In 1717, Minami Koushin hung a tablet in a small shrine in the samurai Egawa’s garden, which is located in Shizuoka prefecture. On the tablet were two figures, the same as figure 6 and 8, which asked for R for nine and ten circles when r is given. The problem was recorded in the unpublished 1830 manuscript of Nakamura Tokikazu (? - ?), Saishi Shinzan (The Mathematics of Shrines).”*[1]

Furthermore, this problem is similar to the problem known in the West as “Malfatti problem,” which has a long and tortuous history. In 1803, Gian Francesca Malfatti (1731 - 1807), an Italian mathematician, asked the following practical question: *“Given a right triangular prism of any sort of material, such as marble, how shall three circular cylinders of the same height as the prism and of the greatest possible volume of material be related to one another in the prism and leave over the least possible amount of material?”*[5]

His question can be extended to a general triangle and simply paraphrased as “How can we place three circles in a given triangle such that the area of the circles is maximized?”

Malfatti intuitively assumed that the maximum was obtained by three circles that are mutually tangent as in figure 9 and provided a solution to how to inscribe these three circles.

A number of later geometers in the West, such as Jakob Steiner, devised other proofs in 1826, verifying Malfatti’s solution. Meanwhile, in Japan, Ajima Naonobu (1732 - 1798) gave a solution to this problem in a manuscript approximately thirty years before Malfatti proposed it:

$$r_1 = \frac{q + t}{2(s - c)},$$

$$r_2 = \frac{q - t}{2(s - b)},$$

$$r_3 = \frac{r_1[b - (t/r + c)]^2}{(2r_1 - p)^2},$$

where a , b , and c are the lengths of corresponding three sides, respectively; $s = (a+b+c)/2$ is the

semiperimeter; $r = \sqrt{(s-a)(s-b)(s-c)}/s$ is the radius of the inscribed circle of the triangle ABC ; and $p := r - (s-a) + \sqrt{r^2 + (s-a)^2}$, $q := r(a-p)$, and $t := \sqrt{q^2 - p^2(s-b)(s-c)}$.

However, in 1930, H. Lob and H.W. Richmond pointed out that the Malfatti configuration was not always the solution to the problem as originally posed.[4] For example, in figure 10, the configuration of circles on the left, which are not mutually tangent, has a greater area than the mutually tangent circles on the right.

In 1967, Michael Goldberg showed that the Malfatti configuration is *never* the solution to the original problem.[3]

Regarding what exactly is the best configuration, this question was only laid to rest fairly recently, in 1992, by V.A. Zalgaller and G.A. Los'.[7] Although their proof is quite involved, there are only two different configurations, each of which could take on the maximum value depending on the angles of the triangle as shown in figure 11.

Note: It is fascinating that our intuition for dense packing is not the solution in these cases: a circular container or a triangular container with circular contents packed inside them. In other words, a highly symmetric solution may not be the optimal configuration for dense packing. Since symmetry breaking happens quite often in physics and in nature, there may be real-life examples in which nature takes advantage of the best solution for dense packing. In mathematics, the dense packing problem for other shapes of containers and contents, as well as for higher dimensions than three, is still an open issue.

3 Mathematical Education and Communication

3.1 Mathematical education

Basic mathematical education for ordinary people during the Edo period was provided mainly by small private schools called “juku,” which also taught reading and writing. Those schools emphasized teaching arithmetic skills, especially the skills to efficiently use “soroban” (Japanese style abacus) that replaced “sangi” in everyday household and business calculations. Merchants in particular cherished the education of soroban skills for young people because quick calculations with a soroban were necessary in business transactions. In fact, the juku that specialized

in teaching only the soroban skills flourished and lasted in Japan until the advent of inexpensive electronic pocket calculators in the late twentieth century.

Most of the mathematics teachers in juku were samurai who had already made transitions from battle field warriors to ordinary civil servants due to the enduring peace of the Edo era. They were highly educated and tried to maintain the high standards of both academic excellence and physical strength. They went to the provincial castle to work for three or four days a week, but their salaries were so meager that they had to take on side jobs. Since there were no universities or high-schools in Japan during the Edo period, many samurai moonlighted as teachers in juku; however, their juku salaries were also meager because they often despised making money. As a matter of fact, the hierarchy of social orders during the Edo period was such that the samurai class was at the top and the merchant class was at the bottom.

On the other hand, the students of juku came from various socioeconomic classes: they were samurai, farmers, technicians, and merchants; the rich and the poor; the young and the old; and men and women. The low attendance fee made it possible for many motivated people to attend the juku. For instance, the Yōken juku had the total of 2,144 students including many adults over the course of fifty years, according to its roster. Mathematician Sakuma Yōken (1819 - 1896) was their teacher, and the small wooden school room has been well preserved. By the late Edo period, about 80,000 juku existed throughout Japan. Some juku students, who were inspired by mathematics, went on to learn high level mathematics from their juku teachers like Sakuma Yōken or from famous mathematicians who individually formed exclusive schools like the Seki school.

The most influential book for mathematical education in Japan was *Jinkō-ki* (Large and Small Numbers) by Yoshida Mitsuyoshi (1598 - 1672), which included the arithmetic skills with soroban as well as higher mathematics such as the approximate value of π . Not only did this book in many revisions serve as a mathematics textbook for ordinary people, but it also motivated many high level mathematicians including Seki Takakazu (1640? - 1710) to calculate the value of π accurate to forty digits, which was well ahead of their Western counter parts.

3.2 Mathematical communication

Ordinary people at the juku, who could not afford to publish their own books, took up the ancient custom of bringing votive tablets to shrines and temples, and began to hang sangaku. In this way, they were both making a religious offering and advertising their results. Since ordinary people frequently visited both shrines and temples, sangaku was an effective medium for disseminating new mathematical results and also for communicating the result with other mathematicians at a leisurely pace.

During the Edo period, traveling required official permission from the government for some national security reasons. Yet, most people loved traveling for sight seeing and pilgrimage to famous shrines and temples. Sometimes they had hidden motives for their travels such as relaxing in legendary hot springs like Dōgo Onsen in Ehime prefecture or enjoying famous local cuisines like Kaiseki Ryōri in Kyoto. Since the Tokugawa shogunate, having been founded on an austere samurai culture, discouraged such luxuries, it was easier to obtain a travel permit if the travel was for academic reasons like recording sangaku problems for juku and schools. In fact, mathematicians such as Yamaguchi Kanzan (1781? - 1850), Sakuma Yōken (1819 - 1896), and Hōdōji Zen (1820 - 1868) made extensive travels all over Japan to teach mathematics, encourage young students, record sangaku problems, and directly communicate with other mathematicians. Especially, Yamaguchi Kanzan made six walking tours all over Japan during his lifetime. In his longest journey from the spring of 1816 to the winter of 1821, he discussed new technical methods of solving mathematical problems with many distant mathematicians. Also, in his relatively short travel between 1820 and 1822, he recorded more sangaku problems than in his five other trips combined.

Yamaguchi's total travel diary is substantial, comprising about seven hundred pages in total. In it he describes the sights, speaks of meetings with friends and other mathematicians, and also records problems from eighty-seven sangaku, only two of which survive to the present. The original diary has survived and currently resides in the city of Agano in Niigata prefecture where he was born. The diary has been declared as a cultural asset.

Another method of mathematical communication during the Edo period was through books and manuscripts. However, it was a slow process because most mathematical books were

produced by woodblocks on Japanese paper although some rudimentary printing machines from Korea existed at that time. The authors of mathematical books and manuscripts preferred the method of woodblocks for it was capable of producing fine lines like those in ukiyo-e. Nevertheless, many mathematicians published books and manuscripts, which became a reliable method of mathematical communication and also a permanent repository of sangaku problems. For example, Yoshida Tameyuki (1819 - 1892), who was a samurai in the Owari clan of Nagoya, left numerous manuscripts containing solutions to sangaku geometry problems. His solutions are noteworthy for their simplicity, clarity, and beauty.[1]

4 Conclusion

The problems written on sangaku demonstrate a strikingly high standard in mathematics in Japan during the Edo period, especially in geometry and calculus. It is even more amazing that such a high level was accomplished independently of the West due to the strict isolation policy of the Tokugawa shogunate. Some interesting problems such as a dense packing problem were discovered almost simultaneously in Japan and in the Western countries, which may indicate that mathematics is universal in space and time. Furthermore, it is remarkable that the sangaku problems were proposed by many people from various socioeconomic backgrounds: not only by samurai but also by farmers, technicians, and merchants, and sometimes by women and young students. Again, this fact may indicate that mathematics is attractive to many people across the boundaries of class, economy, ethnicity, age, and gender.

Fukagawa Hidetoshi, a high-school math teacher by occupation, almost single-handedly founded, developed, and popularized the field of sangaku study. Although he initially faced some indifference and many obstacles in promoting the field, he has finally succeeded in establishing the mathematical, historical, and cultural importance of sangaku in Japan; hence, sangaku is considered as a national cultural asset. Subsequently, Fukagawa introduced his study of sangaku to the West through his books with the help of Tony Rothman, Dan Pedoe, and Freeman Dyson among others. Now, there are many people from all over the world who appreciate the mathematical beauty of sangaku problems as well as their artistic displays. We feel very fortunate to be able to witness this fascinating development in the sangaku history.

Figures [1]

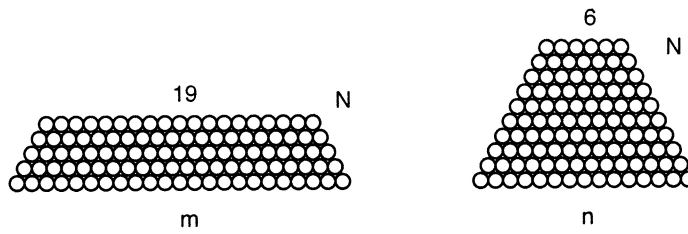


Figure 1. Find N , m , and n .

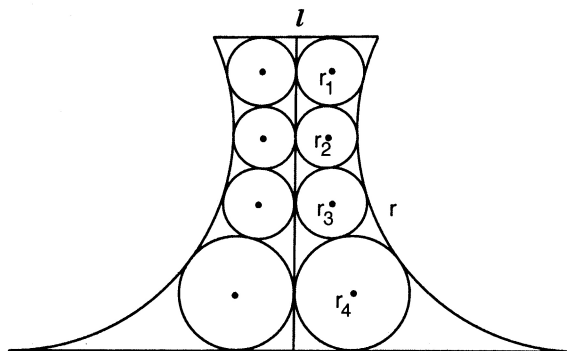


Figure 2. Find r_4 in terms of r_1 , r_2 , and r_3 .

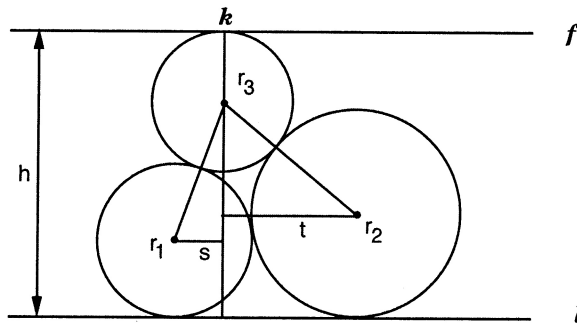


Figure 3. Drop a perpendicular k through the center of r_3 .

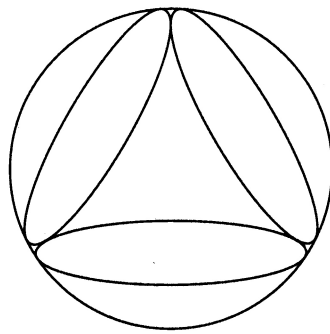


Figure 4. Find the semi-major axis of an ellipse in terms of the radius of the circle when the area of each ellipse is maximized.

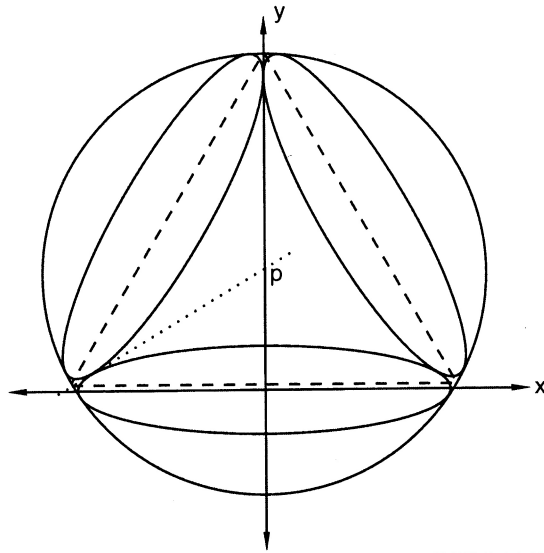


Figure 5. Choose these coordinate axes.

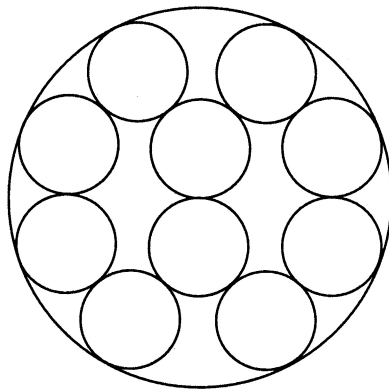


Figure 6. Find the radius of the small circles, r , in terms of S , the area of the larger circle minus the area of the ten small circles.

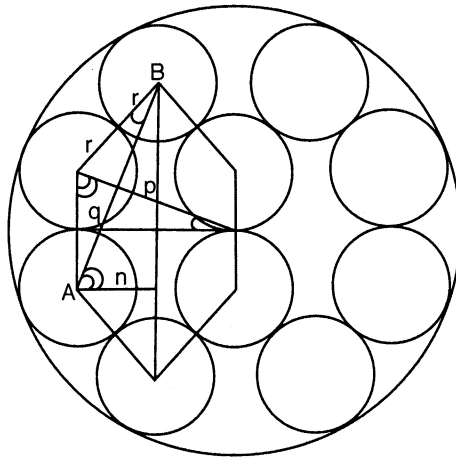


Figure 7. Similar triangles.

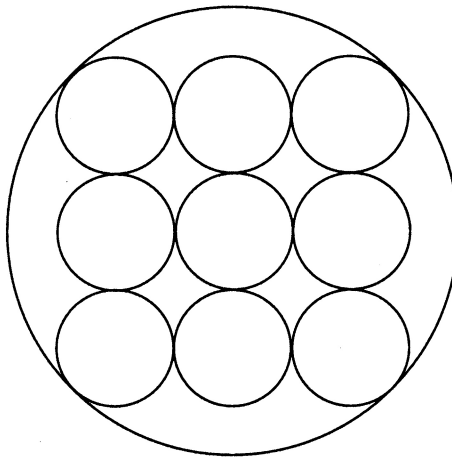


Figure 8. Nine circles of radius r can also be packed into the same circle of radius R .

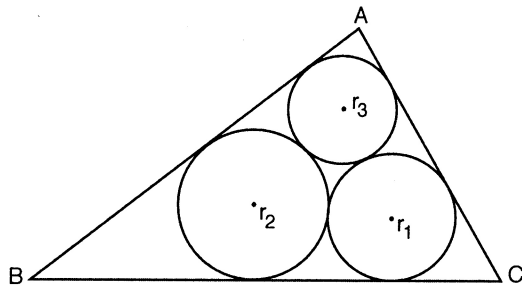


Figure 9. Malfatti's solution of three mutually tangent circles.

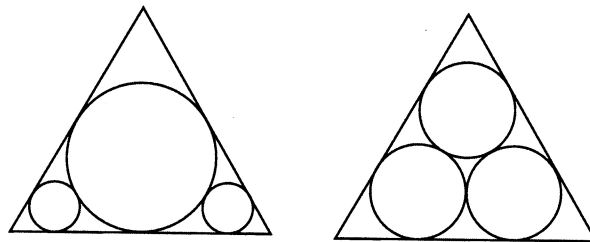


Figure 10. The combined area of the circles on the left is larger than that on the right.

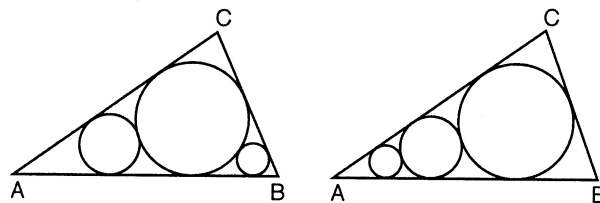


Figure 11. The winning configurations. The one on the right wins if $\sin(A/2) < \tan(B/4)$.

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