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by

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CHAPTER 1. INTRODUCTION

It is generally agreed that numerical methods for the solution of initial boundary value problems of continuum mechanics should be consistent, in some defined sense, with the governing field equations, if they are to be successful according to the criteria of numerical analysis. This minimal requirement is lacking in many of the numerical formulations for finite deformation inelasticity reported in the computational literature. No amount of numerical sophistication will compensate for this shortcoming. The research reported herein is essentially a search for such consistency for a limited problem class.

Attention is restricted to materials characterized by constitutive equations which are insensitive to the rate of deformation but which are appropriate for deformations of arbitrary magnitude. Such rate-independent characterization includes hyperelasticity, hypoelasticity, and elasto-plasticity. These material classes are the main constitutive focus of the following developments.

Discussion of numerical consistency presupposes knowledge of the governing nonlinear field equations, many aspects of which, for finite deformations, are still subject to conjecture. Consequently, in this development of a numerical formulation for the finite deformation problem of rate-independent solids, two fundamental and distinct aspects are given attention:

(a) The mathematical description of the physical theory.

(b) The development of numerical solution procedures consistent with (a).

These two aspects are discussed briefly below.

Mathematical Description of Physical Theory

Emphasis is placed on the geometric interpretation of the concepts of continuum mechanics. This is accomplished using the results of the calculus on manifolds and follows the work of Marsden and Hughes [3]. The coordinate free notation removes the opacity given to many results by layers of complicated coordinate formulas, revealing the heart of the theory. Such an approach not only makes standard results transparent but has also led to new results in the area of constitutive theory as well as lending itself admirably to the development of consistent numerical solution algorithms.

As an example of new results in constitutive theory, the findings in Chapter 3, which considers a spatial setting for the constitutive theory of hyperelasticity, may be mentioned. Although a theory of finite elasticity is necessary for elasto-plasticity, the study of this subject proves to be revealing in its own right. It is shown that the spatial (Cauchy) stress tensor is derivable from a thermodynamic potential which is a function of the point values of the convected metric tensor. This in turn leads to a specific rate form of the hyperelastic constitutive equations involving the Lie derivative of the spatial stress field taken with respect to the spatial velocity field. Furthermore, using results from geometry, other objective stress rates are interpreted in terms of the Lie derivative and conclusions drawn about the thermodynamic admissibility for such stress rates as they appear in specific, but frequently used, constitutive equations. It is noted that stress rates used in constitutive equations in the computational literature appear to be selected only on the grounds of making the boundary value problem self-adjoint. This practice may have serious consequences.

Constitutive theory for rate-independent elasto-plasticity employing an internal variable formalism and developed within the framework of irreversible continuum thermodynamics is considered in Chapter 4. The use of thermodynamic potentials which depend on the internal variables leads to the additive decomposition of the rate of deformation tensor into elastic and plastic parts, independently of any kinematic arguments or approximations. The consistent thermodynamic approach leads to evolution equations for the spatial stress field and the spatial internal variables in terms of their Lie derivatives taken with respect to the spatial velocity field.

A Consistent Numerical Solution Procedure

There is now a considerable body of literature describing a great diversity of numerical formulations for the finite deformation problem of rate-independent elasto-plasticity. Many of these formulations [5-10] have been based on the boundary value problem for rate of momentum balance, motivated by the rate nature of the constitutive equations. For certain material classes, the rate problem may be characterized by a variational principle due to Hill [11]. Conditions for the existence of such a principle demand that the constitutive relations have a symmetry such that the stress rates are derivable from potential functions of the strain rate [4]. Some objective stress rates do not admit the existence of such potentials and, accordingly, the rate problem will not be self-adjoint [5,8]. Finite element formulations based on the rate problem yield the nodal velocities as the solution, and these must be integrated if the displacement field is required.

An equivalent incremental solution procedure, where the rate problem is numerically integrated over a time step using a predictor/corrector technique, has been developed by a number of workers [12-17].

The rate of momentum balance formulation is not employed in the present study. Instead, an approach based on consistent linearization of the weak form of a variational equation equivalent to the boundary value problem of momentum balance is used.

There is an extensive literature in solid and structural mechanics dealing with concepts of linearization. However, many of the proposed methods lack a sound analytical basis; consequently, the relationship between the solutions of the linearized theory and its nonlinear progenitor is often obscure. The material of Chapter 5 represents an application and extension of the analytically consistent linearization theory presented by Marsden and Hughes [3].

In the present work, equations expressing the dynamic equilibrium of loaded bodies employ the motion as an independent variable. The functional dependence on the motion of all tensors, including the stress tensor, appearing in the equilibrium equations provides the basis for consistent linearization. However, for the rate-independent material classes under consideration, some objective rate of a spatial stress tensor will be expressed as a homogeneous linear function of the spatial rate of deformation tensor. Since these rate constitutive equations are not directly integrable, in general, the functional dependence of the stress tensor on the motion is not available and the usual methodology of linearization is not effective. Using geometric concepts involving the Lie derivative of the stress field taken with respect to the incremental motion, the theory of linearization is extended in Chapter 5 to cover the case of rate-independent constitutive equations.

The linearized weak form, consistent with the rate constitutive equations, provides a basis for defining a finite element, Newton-Raphson solution procedure in which the resulting variables are the nodal incremental motions and which has so-called Total Lagrangian and Updated Lagrangian interpretations. The relationship between the symmetry property of the finite element tangent operator and the structure of the rate constitutive equations is considered. Since the rate constitutive equations are not directly integrable, the stress update in the Newton-Raphson procedure must be accomplished by numerical integration of the constitutive equations, taking proper account of the finite deformation effects occuring over the time steps.

Chapter 6 considers the evolution of the stress tensor subject to the constitutive equations of hyperelasticity, hypoelasticity, and rate-independent elasto-plasticity. Taking advantage of the geometric interpretation of the mechanics underlying the evolution of the stress tensor, an algorithmic treatment of the integration problem is developed. The resulting incrementally objective family of algorithms is intimately connected with the structure of the constitutive equations and, in this respect, marks a departure from the work of others who have considered the integration problem [13,16,17,47,51].

Examples of finite homogeneous deformation are considered in Chapter 8, where analytical solutions for a number of problems are developed. These problems serve two purposes:

(a) They illustrate that certain thermodynamically inconsistent (but frequently used) constitutive equations lead to "non-physical" instabilities in the solution of boundary value problems. These instabilities are absent for thermodynamically consistent constitutive equations.

(b) They demonstrate the consistency and accuracy of the numerical solution procedure proposed in this work.

CHAPTER 2. CONTINUUM BASIS

2.1 KINEMATICS AND DEFORMATION RATES

A kinematic description of finite deformations, appropriate for the mechanics of solid continua, is considered in this section. Definitions of tensors required for later chapters are given and connections among certain of these are established.

Coordinate Systems and Motion

A deformable body is assumed to be a smooth Euclidean manifold M which deforms in a smooth Euclidean manifold N. A configuration of M is described by the mapping:

$$\mathbf{\Phi} : M \to N$$

The set of all configurations of M is denoted C(M) and called the configuration space. One configuration in C(M), not necessarily corresponding to an actual configuration occupied by the body, will be designated as the reference configuration and specified by $\bar{\mathbf{\Phi}} \in C(M)$.

With no loss of generality and for notational simplicity, it will be assumed that $\bar{\mathbf{\Phi}}(M)$ is the identity mapping. Accordingly, the reference configuration, to be denoted B, is defined by:

$$B = \mathrm{I}(M).$$

Three coordinate systems, associated with the reference configuration, the ambient space and a deformed configuration, are introduced along with the definition of motion.

(a) Material Coordinate System

The reference configuration B is endowed with a fixed material coordinate system $\{X^A\}$ so that points in B are denoted $\mathbf{X} \in B$ and have coordinates X^A. The covariant base vectors, $\mathbf{G}_A$, associated with this coordinate system, generate a covariant metric tensor $\mathbf{G}$, with the property:

$$\frac{d}{dt}(\mathbf{G}) = 0$$

where t denotes time.

(b) Spatial Coordinate System

The ambient deformation space N is endowed with a fixed spatial coordinate system $\{x^a\}$, so that points in N are denoted $\mathbf{x} \in N$ and have coordinates x^a. The covariant base vectors, $\mathbf{g}_a$, associated with this coordinate system, generate a covariant spatial metric tensor denoted $\mathbf{g}$.

Before defining the final coordinate system, which depends on the deformation, it is necessary to introduce the description of motion.

Definition 2.1

A motion of the body in the deformation space N, relative to the reference configuration B, is a curve in the configuration space C(M) given by:

$$\mathbf{x} = \boldsymbol{\phi}(\mathbf{X},t) : B \times R \rightarrow N$$

where $\boldsymbol{\phi}$ is assumed to be a C^1 regular mapping.

The motion $\boldsymbol{\phi}(\mathbf{X},t)$ evaluated for some fixed $\mathbf{X} \in B$ is written as $\boldsymbol{\phi}_t(\mathbf{X})$. The configuration at time t, referred to as the current configuration, is denoted $\boldsymbol{\phi}_t(B)$.

A third coordinate system associated with the current configuration is defined as follows:

(c) Convected Coordinate System

The material coordinate system $\{X^A\}$ defined on B is supposed to be mapped with the body through the motion $\boldsymbol{\phi}_t$ into the current configuration $\boldsymbol{\phi}_t(B)$. The convected coordinate system so defined is denoted $\{\bar{x}^\alpha\}$, with points in $\boldsymbol{\phi}_t(B)$ given by $\bar{\mathbf{x}} \in \boldsymbol{\phi}_t(B)$ and having coordinates $\bar{x}^\alpha$. Note that the material and convected coordinates of a given material point will be identical. The covariant base vectors, $\bar{\mathbf{g}}_\alpha$, associated with $\{\bar{x}^\alpha\}$, generate a covariant metric tensor denoted $\bar{\mathbf{g}}$ [1].

Figure 1 depicts the three coordinate systems described above.

Associated contravariant metric tensors can be constructed from the covariant metric tensors $\mathbf{G}$, $\mathbf{g}$, and $\bar{\mathbf{g}}$, and these are denoted $\mathbf{G}^*$, $\mathbf{g}^*$, and $\bar{\mathbf{g}}^*$, respectively.

The deformation gradient is defined by

$$\mathbf{F} = \frac{\partial \boldsymbol{\phi}}{\partial \mathbf{X}}$$

with components, $F^a{}_A = \dfrac{\partial x^a}{\partial X^A}$. It is assumed that $\boldsymbol{\phi}$ satisfies:

$$J = \det\left(\frac{\partial \boldsymbol{\phi}(\mathbf{X},t)}{\partial \mathbf{X}}\right) > 0 \qquad \text{for all} \quad t \in [0,\infty)$$

which ensures that $\boldsymbol{\phi}^{-1}$ exists and allows the polar decomposition of $\mathbf{F}$, such that:

$$\mathbf{F} = \mathbf{R}\mathbf{U} = \mathbf{V}\mathbf{R}.$$

In the above expression, $\mathbf{R}$ is an orthogonal rotation tensor and $\mathbf{U}$ and $\mathbf{V}$ are respectively right and left positive definite and symmetric stretch tensors.

Pull Back, Push Forward and the Piola Transformation

Definition 2.2

If $\boldsymbol{\gamma}$ is a vector field defined on $\boldsymbol{\phi}_t(B)$, the pull back of $\boldsymbol{\gamma}$ through the motion $\boldsymbol{\phi}_t$ defines a vector field $\boldsymbol{\Gamma}$ on B given by:

$$\boldsymbol{\Gamma} = \boldsymbol{\phi}_t^*(\boldsymbol{\gamma})$$

In coordinates:

$$\Gamma^A = (F^{-1})^A{}_a \gamma^a$$

or

$$\Gamma_A = F^a{}_A \gamma_a.$$

Again, this definition may be immediately generalized for tensors of higher order. For example:

$$\Gamma^{AB} = (F^{-1})^A{}_a (F^{-1})^B{}_b \gamma^{ab}.$$

or

$$\Gamma_{AB} = F^a{}_A F^b{}_B \gamma_{ab}.$$

Definition 2.3

If $\mathbf{\Gamma}$ is a vector field defined on B, the push forward of $\mathbf{\Gamma}$ through the motion $\boldsymbol{\phi}_t$ defines a vector field $\boldsymbol{\gamma}$ on $\boldsymbol{\phi}_t(B)$ given by:

$$\gamma = \boldsymbol{\phi}_{t*}(\Gamma)$$

In coordinates:

$$\gamma^a = F^a{}_A \Gamma^A$$

or

$$\gamma_a = (\mathbf{F}^{-1})^A{}_a \Gamma_A.$$

This definition may be immediately generalized for tensors of higher order. For example:

$$\gamma^{ab} = F^a{}_A F^b{}_B \Gamma^{AB}$$

or

$$\gamma_{ab} = (\mathbf{F}^{-1})^A{}_a (\mathbf{F}^{-1})^B{}_b \Gamma_{AB}.$$

Definition 2.4

If $\boldsymbol{\gamma}$ is a tensor field defined on $\boldsymbol{\phi}_t(B)$, the Piola transformation of $\boldsymbol{\gamma}$ with respect to the motion $\boldsymbol{\phi}_t$ defines a tensor field $\mathbf{\Gamma}$ on B given by:

$$\Gamma = J \boldsymbol{\phi}_t^{*}(\gamma)$$

where $J = \det \left| \dfrac{\partial \boldsymbol{\phi}_t}{\partial \mathbf{X}} \right|$.

Strain Measures

Definition 2.5

The right Cauchy-Green tensor $\mathbf{C}$ and the right spatial (Eulerian) deformation tensor $\mathbf{c}$ are defined by:

$$\mathbf{C} = \boldsymbol{\phi}_t^{*}(\mathbf{g})$$
$$\mathbf{c} = \boldsymbol{\phi}_{t*}(\mathbf{G}).$$

In coordinates:

$$C_{AB} = F^a{}_A F^b{}_B g_{ab}$$
$$c_{ab} = (F^{-1})^A{}_a (F^{-1})^B{}_b G_{AB}.$$

Definition 2.6

The left Cauchy-Green tensor **b** and the left material deformation tensor **B** are defined by:

$$\mathbf{b} = \phi_{t*}(\mathbf{G}^*)$$

$$\mathbf{B} = \phi_t^*(\mathbf{g}^*).$$

In coordinates:

$$b^{ab} = F^a{}_A F^b{}_B G^{AB}$$
$$B^{AB} = (F^{-1})^A{}_a (F^{-1})^B{}_b g^{ab}.$$

In the literature, the left Cauchy-Green deformation tensor is usually denoted **B**. Here **b** is used for notational consistency.

Velocity Field and Flow

The material velocity of a motion ϕ is defined as a vector field **V** over the deformed configuration, such that:

$$\mathbf{V}(\mathbf{X},t) = \frac{d}{dt}\phi(\mathbf{X},t)$$

The spatial velocity field **v** is defined by:

$$\mathbf{v} = \mathbf{V} \circ \phi_t^{-1}.$$

The spatial velocity gradient tensor **l** is given by:

$$\mathbf{l} = \frac{\partial \mathbf{v}}{\partial \mathbf{x}} = \nabla \mathbf{v}$$

in coordinates

$$l^a{}_b = v^a|_b = v^a{}_{,b} + \Gamma^a_{mb} v^m$$

where the vertical bar and comma denote covariant and partial differentiation respectively and Γ^a_{mb} is the Christoffel symbol of the second kind.

The spatial deformation rate tensor **d** is defined by:

$$\mathbf{d} = \nabla^S \mathbf{v} = \tfrac{1}{2}(\mathbf{l} + \mathbf{l}^T)$$

and the spatial spin rate or vorticity tensor $\boldsymbol{\omega}$ by:

$$\boldsymbol{\omega} = \nabla^A \mathbf{v} = \tfrac{1}{2}(\mathbf{l} - \mathbf{l}^T).$$

Definition 2.7

Let $\mathbf{v}$ be the spatial velocity field on N and $\boldsymbol{\phi}_t$ the corresponding motion. Then the collection

$$\left\{ \boldsymbol{\phi}_{s,t} \,\middle|\, \boldsymbol{\phi}_{s,t} = \boldsymbol{\phi}_s \circ \boldsymbol{\phi}_t^{-1} : \boldsymbol{\phi}_t(B) \rightarrow \boldsymbol{\phi}_s(B) \right\}$$

is defined to be the flow of $\mathbf{v}$.

The Lie Derivative and Strain Rates

The Lie derivative [2,3], which gives a measure of the rate of change of a tensor with respect to a vector field, is an extremely useful concept. In this section, the Lie derivative is defined and simple applications to strain rate definitions are considered. In subsequent chapters, the following definitions are drawn upon frequently as further applications of the Lie derivative are considered.

Definition 2.8

Let $\mathbf{v}$ be the spatial velocity field on N and $\boldsymbol{\phi}_{s,t}$ denote its flow. If $\boldsymbol{\gamma}$ is a possibly time dependent C^1 tensor field on N, then the Lie derivative of $\boldsymbol{\gamma}$ with respect to $\mathbf{v}$ is defined by:

$$L_{\mathbf{v}}(\boldsymbol{\gamma}) = \left[\frac{d}{ds} \boldsymbol{\phi}_{s,t}^{*}(\boldsymbol{\gamma}_s) \right]_{s=t}.$$

Proposition 2.1

The Lie derivative has an alternative definition given by:

$$L_{\mathbf{v}}(\boldsymbol{\gamma}) = \boldsymbol{\phi}_{t*} \left(\frac{d}{dt} \boldsymbol{\phi}_t^{*}(\boldsymbol{\gamma}) \right)$$

for the motion ϕ_t corresponding to $\mathbf{v}$.

Proof

By Definition 2.7, the flow of $\mathbf{v}$ is given by $\phi_{s,t} = \phi_s \circ \phi_t^{-1}$. Accordingly,

$$\begin{aligned} L_{\mathbf{v}}(\gamma) &= \left[\frac{d}{ds} (\phi_s \circ \phi_t^{-1})^* (\gamma_s) \right]_{s=t} \\ &= \phi_{t*} \left(\frac{d}{ds} \phi_s^* (\gamma_s) \right)_{s=t} \\ &= \phi_{t*} \left(\frac{d}{dt} \phi_t^* (\gamma) \right) \quad \blacksquare \end{aligned}$$

The alternative definition given by Proposition 2.1 is often the most useful in applications.

In view of the assumed invertibility of $\mathbf{F}$, the following result is immediate:

$$\phi_t^* (L_{\mathbf{v}}(\gamma)) = \frac{d}{dt} \phi_t^* (\gamma).$$

Coordinate formulas for the Lie derivative with respect to the spatial velocity field of second order spatial tensors, expressed in terms of the covariant derivative, are given as follows [3] (generalization to other orders is immediate):

$$\begin{aligned} [L_{\mathbf{v}}(\gamma)]^{ab} &= \dot{\gamma}^{ab} - v^a|_m \gamma^{mb} - v^b|_m \gamma^{am} \\ [L_{\mathbf{v}}(\gamma)]^a{}_b &= \dot{\gamma}^a{}_b - v^a|_m \gamma^m{}_b + v^m|_b \gamma^a{}_m \\ [L_{\mathbf{v}}(\gamma)]_{ab} &= \dot{\gamma}_{ab} + v^m|_a \gamma_{mb} + v^m|_b \gamma_{am} \end{aligned}$$

where the superimposed dot implies material time differentiation given by:

$$\dot{\gamma} = \frac{\partial \gamma}{\partial t} + \nabla \gamma \cdot \mathbf{v}.$$

Proposition 2.2

Let $\mathbf{v}$ be the spatial velocity field on N corresponding to the motion ϕ_t, then

$$\begin{aligned} (a) \quad L_{\mathbf{v}}(\mathbf{g}) &= 2\mathbf{d} \\ (b) \quad L_{\mathbf{v}}(\mathbf{g}^*) &= -2\mathbf{d} \end{aligned}$$

where $\mathbf{d}$ is the spatial rate of deformation tensor. This important result is trivial but included for completeness.

Proof

(a) Using the coordinate formulas:

$$[L_{\mathbf{v}}(\mathbf{g})]_{ab} = \dot{g}_{ab} + v^m|_a g_{mb} + v^m|_b g_{am}$$

$$\text{But } \dot{g}_{ab} = \frac{\partial g_{ab}}{\partial t} + g_{ab}|_c v^c = 0$$

since $\mathbf{g}$ does not depend parametrically on time and $g_{ab}|_c = 0$ by Ricci's Theorem. It follows that $L_{\mathbf{v}}(\mathbf{g}) = 2\,\mathbf{d}$.

(b) The proof is similar to (a) ■

The following results on strain rate definitions are employed in subsequent chapters.

Proposition 2.3

Let $\mathbf{v}$ be the spatial velocity field corresponding to the motion $\boldsymbol{\phi}_t$, then

$$(a)\quad \dot{\mathbf{C}} = 2\boldsymbol{\phi}_t^{*}(\mathbf{d})$$

$$(b)\quad L_{\mathbf{v}}(\mathbf{c}) = 0$$

$$(c)\quad \dot{\mathbf{B}} = 2\boldsymbol{\phi}_t^{*}(-\mathbf{d})$$

$$(d)\quad L_{\mathbf{v}}(\mathbf{b}) = 0$$

Proof

(a) Using Definition 2.5 and Propositions 2.1 and 2.2 it follows that:

$$\dot{\mathbf{C}} = \frac{d}{dt}\boldsymbol{\phi}_t^{*}(\mathbf{g}) = \boldsymbol{\phi}_t^{*}(L_{\mathbf{v}}(\mathbf{g})) = 2\boldsymbol{\phi}_t^{*}(\mathbf{d}).$$

(b) Using definition 2.5 and Proposition 2.1 it follows that:

$$L_{\mathbf{v}}(\mathbf{c}) = \boldsymbol{\phi}_{t*}\left(\frac{d}{dt}\mathbf{G}\right) = 0$$

Which is equivalent to:

$$\dot{\mathbf{c}} = -\mathbf{c}\mathbf{l} - \mathbf{l}^T\mathbf{c} \quad \text{or} \quad \dot{\mathbf{e}} = \mathbf{d} - \mathbf{e}\mathbf{l} - \mathbf{l}^T\mathbf{e}$$

Where $\mathbf{e} = \frac{1}{2}(\mathbf{g} - \mathbf{c})$ is the Eulerian strain tensor.

(c) and (d) follow similarly ■

2.2 STRESS TENSORS

Definition 2.9

The Cauchy stress vector $\bar{\mathbf{t}}(\mathbf{x},t,\mathbf{n})$ is a vector field on $\phi_t(B)$ depending on the spatial point $\mathbf{x}$, time t and a unit vector $\mathbf{n}$ such that $\bar{\mathbf{t}}$ represents the force per unit area exerted on a surface element oriented with normal $\mathbf{n}$. The symmetric Cauchy stress tensor $\boldsymbol{\sigma}$ is defined by the Cauchy stress principle which states that:

$$\bar{\mathbf{t}} = \boldsymbol{\sigma} \cdot \mathbf{n}$$

in terms of contravariant components $\bar{t}^a = \sigma^{ab} n_b$.

Definition 2.10

Starting with the contravariant components of the Cauchy stress tensor $\boldsymbol{\sigma}$, the first Piola-Kirchhoff stress tensor $\mathbf{P}$ is defined as the Piola transform on the first index of $\boldsymbol{\sigma}$, that is:

$$P^{Aa} = J(\mathbf{F}^{-1})^A{}_b \sigma^{ba}$$

Definition 2.11

The symmetric second Piola-Kirchhoff stress tensor S is defined by the pull back of the second index of $\mathbf{P}$ by ϕ_t, that is:

$$S^{AB} = (F^{-1})^B{}_a P^{Aa} = J(F^{-1})^A{}_a (F^{-1})^B{}_b \sigma^{ab}.$$

Definition 2.12

The Kirchhoff stress tensor $\boldsymbol{\tau}$ is defined as the tensor density of $\boldsymbol{\sigma}$, that is,

$$\tau = J\sigma$$

From the above definitions it follows that:

$$\mathbf{S} = J\boldsymbol{\phi}_t^*(\boldsymbol{\sigma}) = \boldsymbol{\phi}_t^*(\boldsymbol{\tau}).$$

2.3 OBJECTIVE STRESS RATES

The principle of material frame indifference requires that the intrinsic physical properties of a body be independent of the body's location or orientation in space. This principle is embodied in constitutive theory by the requirement that constitutive equations must contain only objective tensor fields.

Definition 2.13

Let γ be a time-dependent tensor (or tensor density) field on N and let $\xi : N \rightarrow N$ be a mapping which defines a superimposed rigid body translation and rotation of $\phi_t(B)$. Then γ is objective if it transforms under the mapping ξ according to:

$$\gamma^+ = \xi_*(\gamma).$$

It is noted that if Γ is a material tensor, then Γ, $\dot{\Gamma}$, $\phi_{t*}(\Gamma)$, and $\phi_{t*}(\dot{\Gamma})$ are all objective tensors.

Proposition 2.4

Objective tensors (or tensor densities) have objective Lie derivatives taken with respect to the spatial velocity field $\mathbf{v}$.

Proof

The proof (which is intuitively clear from the above discussion) is straight-forward and may be found in [3].

In subsequent chapters, it will be shown that a number of objective stress rate definitions arise naturally from thermodynamic arguments. The definition of these stress rates and their component representations are presented here for future reference.

Definition 2.14

Let τ and σ be defined on $\phi_t(B)$ with spatial velocity field $\mathbf{v}$ corresponding to the motion ϕ_t. Then:

(a) The Lie derivative of τ with respect to $\mathbf{v}$ is defined by

$$L_{\mathbf{v}}(\tau) = \phi_{t*}\left[\frac{d}{dt}\phi_t^*(\tau)\right]$$

for contravariant components of τ:

$$L_{\mathbf{v}}(\tau) = \dot{\tau} - \mathbf{l}\cdot\tau - \tau\cdot\mathbf{l}^T$$

or

$$\left[L_{\mathbf{v}}(\tau)\right]^{ab} = \dot{\tau}^{ab} - v^a|_m \tau^{mb} - v^b|_m \tau^{am}$$

(b) The Lie derivative of σ with respect to $\mathbf{v}$ is defined by

$$L_{\mathbf{v}}(\sigma) = \boldsymbol{\phi}_{t*}\left(\frac{d}{dt}\boldsymbol{\phi}_t^*(\sigma)\right)$$

the contravariant component form is similar to that given in (a).

(c) The Truesdell rate of σ with respect to $\mathbf{v}$ is defined by

$$\overset{\circ}{\sigma} = J^{-1}\boldsymbol{\phi}_{t*}\left(\frac{d}{dt}\boldsymbol{\phi}_t^*(J\sigma)\right) = J^{-1}L_{\mathbf{v}}(\tau)$$

for contravariant components of σ:

$$\overset{\circ}{\sigma} = \dot{\sigma} - \mathbf{l}\cdot\sigma - \sigma\cdot\mathbf{l}^T + \sigma\, tr(\mathbf{d})$$

or

$$(\overset{\circ}{\sigma})^{ab} = \dot{\sigma}^{ab} - v^a|_m \sigma^{mb} - v^b|_m \sigma^{am} + \sigma^{ab} v^m|_m$$

Since τ, σ, and J are objective tensors and since the Lie derivatives of objective tensors are also objective, it follows that all the stress rates given in Definition 2.14 are objective.

Definition 2.15

The Jaumann (or co-rotational) rate of σ, denoted $\overset{\nabla}{\sigma}$, is defined for contravariant components by:

$$\overset{\nabla}{\sigma} = \dot{\sigma} - \boldsymbol{\omega}\cdot\sigma + \sigma\cdot\boldsymbol{\omega}$$

or

$$(\overset{\nabla}{\sigma})^{ab} = \dot{\sigma}^{ab} - \omega^a{}_m \sigma^{mb} - \omega^b{}_m \sigma^{am}$$

The form of this objective rate for τ is similar. The Jaumann rate and Lie derivative coincide under the assumption that the spatial deformation rate tensor $\mathbf{d}$ is instantaneously zero.

Stated alternatively, if the pull back and push forward operations in the definition of the Lie derivative are performed with respect to only the rotational part of the motion $\boldsymbol{\phi}_t$, the Jaumann rate results. These alternative views of the Jaumann rate prove to be useful in the construction of algorithms for the integration of certain rate constitutive equations and are considered in detail later.

Finally, it is noted that if the reference configuration coincides instantaneously with the current configuration, then the Lie derivative is Hill's convected derivative [4].

2.4 MOMENTUM BALANCE

Spatial Form of Linear Momentum Balance

Assuming the conservation of mass in $\boldsymbol{\phi}_t(B)$, the localization of global linear momentum balance leads to the field equations of momentum balance. Appending the boundary conditions defines the boundary value problem of momentum balance:

$$\begin{aligned} \operatorname{div}\boldsymbol{\sigma} + \rho\mathbf{b} &= \rho\dot{\mathbf{v}} && \mathbf{x} \in \boldsymbol{\phi}_t(B) \\ \boldsymbol{\sigma}\cdot\mathbf{n} &= \bar{\mathbf{t}} && \mathbf{x} \in \partial(\boldsymbol{\phi}_t(B))_{\bar{\mathbf{t}}} \\ \mathbf{x} &= \bar{\mathbf{x}} && \mathbf{x} \in \partial(\boldsymbol{\phi}_t(B))_{\bar{\mathbf{x}}} \end{aligned}$$

where $(\operatorname{div}\boldsymbol{\sigma})^a = \sigma^{ab}|_b$, $\mathbf{b}$ is a body force field on $\boldsymbol{\phi}_t(B)$, and ρ is the mass density in $\boldsymbol{\phi}_t(B)$. Tractions are specified on $\partial(\boldsymbol{\phi}_t(B))_{\bar{\mathbf{t}}}$, which has a unit outward normal $\mathbf{n}$, and the motion is specified on $\partial(\boldsymbol{\phi}_t(B))_{\bar{\mathbf{x}}}$. Furthermore,

$$\begin{aligned} \partial(\boldsymbol{\phi}_t(B))_{\bar{\mathbf{t}}} \bigcup \partial(\boldsymbol{\phi}_t(B))_{\bar{\mathbf{x}}} &= \partial(\boldsymbol{\phi}_t(B)) \\ \partial(\boldsymbol{\phi}_t(B))_{\bar{\mathbf{t}}} \bigcap \partial(\boldsymbol{\phi}_t(B))_{\bar{\mathbf{x}}} &= \varnothing. \end{aligned}$$

Assuming the motion $\boldsymbol{\phi}_t$ to be the independent variable, then, by the fundamental lemma of the calculus of variations and the assumption of any necessary differentiability, the following weak form of a variational equation is equivalent to the boundary value problem of momentum balance:

$$G(\mathbf{x},\boldsymbol{\eta}) = \int_{\boldsymbol{\phi}_t(B)} tr(\boldsymbol{\sigma}\cdot\nabla\boldsymbol{\eta})\, dv + \int_{\boldsymbol{\phi}_t(B)} \rho(\dot{\mathbf{v}}-\mathbf{b})\cdot\boldsymbol{\eta}\, dv - \int_{\partial(\boldsymbol{\phi}_t(B))_{\bar{\mathbf{t}}}} \bar{\mathbf{t}}\cdot\boldsymbol{\eta}\, da = 0 \qquad (2.1)$$

$$\text{for all} \quad \boldsymbol{\eta} \in \mathbf{E}_t = \left\{ \boldsymbol{\eta} \mid \boldsymbol{\eta} : \boldsymbol{\phi}_t(B)\times\mathbf{R} \to \mathbf{R}^3 \text{ such that } \boldsymbol{\eta} = 0 \text{ on } \partial(\boldsymbol{\phi}_t(B))_{\bar{\mathbf{x}}} \right\}$$

and $\quad \mathbf{x} \in \mathbf{F}_t = \left\{ \mathbf{y} \,|\, \mathbf{y} : B \times R \rightarrow N \text{ such that } \mathbf{y} = \bar{\mathbf{x}} \text{ on } \partial(\boldsymbol{\phi}_t(B))_{\bar{\mathbf{x}}} \right\}.$

Material Form of Linear Momentum Balance

Assuming the conservation of mass in $\boldsymbol{\phi}_t(B)$ and the localization of global momentum balance for $\boldsymbol{\phi}_t(B)$ referred to B, leads to the field equations of momentum balance. Appending the boundary conditions defines the boundary value problem of momentum balance:

$$
\begin{aligned}
\mathrm{DIV}\,\mathbf{P} + \rho_B \mathbf{B} &= \rho_B \dot{\mathbf{V}} & \mathbf{X} &\in B \\
\mathbf{P}^T \cdot \mathbf{N} &= \bar{\tau} & \mathbf{X} &\in \partial B_{\bar{\tau}} \\
\boldsymbol{\phi}_t(\mathbf{X}) &= \bar{\mathbf{x}}(t) & \mathbf{X} &\in \partial B_{\bar{\mathbf{x}}}
\end{aligned}
$$

where $(\mathrm{DIV}\,\mathbf{P})^a = P^{Aa}|_A$, $\mathbf{B} = \mathbf{b} \circ \boldsymbol{\phi}_t$, and $\rho_B = J\rho$. Tractions are specified on $\partial B_{\bar{\tau}}$ which has a unit outward normal $\mathbf{N}$, and the motion is specified on $\partial B_{\bar{\mathbf{x}}}$. Furthermore,

$$
\begin{aligned}
\partial B_{\bar{\tau}} \bigcup \partial B_{\bar{\mathbf{x}}} &= \partial B \\
\partial B_{\bar{\tau}} \bigcap \partial B_{\bar{\mathbf{x}}} &= \varnothing
\end{aligned}
$$

Assuming the motion $\boldsymbol{\phi}_t$ to be the independent variable, the weak form equivalent to the boundary value problem of momentum balance has the form:

$$
G(\mathbf{x},\boldsymbol{\eta}) = \int_B tr(\mathbf{P} \cdot \mathbf{D}\boldsymbol{\eta})\, dv + \int_B \rho_B (\dot{\mathbf{V}} - \mathbf{B}) \cdot \boldsymbol{\eta}\, dv - \int_{\partial B_{\tau}} \bar{\tau} \cdot \boldsymbol{\eta}\, dA = 0 \qquad (2.2)
$$

for all $\quad \boldsymbol{\eta} \in E_t = \left\{ \boldsymbol{\eta} \,|\, \boldsymbol{\eta} : B \times R \rightarrow R^3 \text{ such that } \boldsymbol{\eta} = 0 \text{ on } \partial B_{\bar{\mathbf{x}}} \right\}$

and $\quad \mathbf{x} \in F_t = \left\{ \mathbf{y} \,|\, \mathbf{y} : B \times R \rightarrow N \text{ such that } \mathbf{y} = \bar{\mathbf{x}} \text{ on } \partial B_{\bar{\mathbf{x}}} \right\}$

and where $\mathbf{D}\boldsymbol{\eta} = \dfrac{\partial \boldsymbol{\eta}}{\partial \mathbf{X}}$.

Obviously, (2.2) can be obtained from (2.1) directly by means of a transformation from spatial to material coordinates. In order for (2.1) or (2.2) to be formally well-posed, it is necessary to specify how the stress tensor, $\boldsymbol{\sigma}$ or $\mathbf{P}$, depends on the motion. For certain material classes (including hypoelasticity and rate-independent elasto-plasticity) the constitutive equations are expressed in a rate form from which the explicit dependence of the stress tensor on the motion is not available. A method of dealing with this problem class, based on a theory of linearization, is developed in the following chapters. The material version of the weak form, with domain independent of the motion, will be useful in this respect.

2.5 RATE OF MOMENTUM BALANCE

There is now a considerable body of literature describing a great diversity of numerical formulations for the finite deformation problem of rate-independent elasto-plasticity. Many of these formulations [5-10] have been based on the boundary value problem for rate of momentum balance, motivated by the rate nature of the constitutive equations. For certain material classes, the rate problem may be characterized by a variational principle due to Hill [11]. Conditions for the existence of such a principle demand that the constitutive relations have a symmetry such that the stress rates are derivable from potential functions of the strain rate [4]. Some objective stress rates do not admit the existence of such potentials and, accordingly, the rate problem will not be self-adjoint [5,8]. Finite element formulations based on the rate problem yield the nodal velocities as the solution, and these must be integrated if the displacement field is required.

An equivalent incremental solution procedure, where the rate problem is numerically integrated over a time step using a predictor/corrector technique, has been developed by a number of workers [12-17].

The rate of momentum balance formulation is not employed in the present study. However, it is useful to summarize the basic statements of the rate formulation for comparison with an alternative formulation to be developed subsequently.

Weak Form For Rate Of Momentum Balance

Considering a material setting and assuming, for simplicity, that the body force field **B** and applied boundary tractions $\bar{\tau}$ have vanishing material derivatives, the rate problem is given by:

$$\begin{aligned} \mathrm{DIV}\,\dot{\mathbf{P}} &= \rho_B \ddot{\mathbf{V}} && \mathbf{X} \in B \\ \dot{\mathbf{P}}^T \mathbf{N} &= 0 && \mathbf{X} \in \partial B_{\bar{\tau}} \\ \dot{\boldsymbol{\phi}}_t(\mathbf{X}) &= \dot{\bar{\mathbf{x}}} && \mathbf{X} \in \partial B_{\bar{\mathbf{x}}} \end{aligned}$$

A weak form equivalent to the rate problem is given by:

$$\int_B tr\,(\dot{\mathbf{P}} \cdot \mathbf{D}\boldsymbol{\eta})\, dV + \int_B \rho_B \ddot{\mathbf{V}} \cdot \boldsymbol{\eta}\, dV \;=\; 0 \tag{2.3}$$

for all

$$\boldsymbol{\eta} \in \hat{\mathbf{E}} = \left\{ \boldsymbol{\eta} \mid \boldsymbol{\eta} : B \times \mathbf{R} \rightarrow \mathbf{R}^3 \text{ such that } \boldsymbol{\eta} = 0 \text{ on } \partial B_{\bar{\mathbf{x}}} \right\}$$

and

$$\boldsymbol{\phi}_t \in \hat{\mathbf{F}} = \left\{ \mathbf{y} \mid \mathbf{y} : B \times \mathbf{R} \rightarrow N \text{ such that } \mathbf{y} = \bar{\mathbf{x}} \text{ on } \partial B_{\bar{\mathbf{x}}} \right\}$$

As before, $\mathbf{D}\boldsymbol{\eta} = \dfrac{\partial \boldsymbol{\eta}}{\partial \mathbf{X}}$.

Proposition 2.5

Considering *contravariant* components of stress, then:

$$\begin{aligned}
&(a) \quad \mathbf{F} \cdot \dot{\mathbf{P}} &&= L_{\mathbf{v}}(\boldsymbol{\tau}) + \boldsymbol{\tau} \cdot \mathbf{l}^T \\
&(b) &&= \overset{\triangledown}{\boldsymbol{\tau}} - 2(\boldsymbol{\tau} \cdot \mathbf{d})^S + \boldsymbol{\tau} \cdot \mathbf{l}^T \\
&(c) &&= J[\overset{\circ}{\boldsymbol{\sigma}} + \boldsymbol{\sigma} \cdot \mathbf{l}^T] \\
&(d) &&= J[L_{\mathbf{v}}(\boldsymbol{\sigma}) + \boldsymbol{\sigma} \cdot \mathbf{l}^T + \boldsymbol{\sigma}\, tr(\mathbf{d})] \\
&(e) &&= J[\overset{\triangledown}{\boldsymbol{\sigma}} - 2(\boldsymbol{\sigma} \cdot \mathbf{d})^S + \boldsymbol{\sigma} \cdot \mathbf{l}^T + \boldsymbol{\sigma}\, tr(\mathbf{d})]
\end{aligned}$$

Proof

The proof follows from Definitions 2.10, 2.12 and 2.14 and is omitted here.

Proposition 2.6

The weak form of the material version of the rate of momentum balance (2.3) has the alternative representations.

(i)

$$\int_{\boldsymbol{\phi}_t(B)} J^{-1}[L_{\mathbf{v}}(\boldsymbol{\tau}) : \nabla^S \boldsymbol{\eta} + (\boldsymbol{\tau} \cdot \mathbf{l}^T) : \nabla \boldsymbol{\eta}^T]\, dv + \int_{\boldsymbol{\phi}_t(B)} \rho \ddot{\mathbf{v}} \cdot \boldsymbol{\eta}\, dv \;=\; 0$$

(ii)

$$\int_{\phi_t(B)} J^{-1}[\overset{\nabla}{\tau} : \nabla^S \eta - 2(\tau \cdot \mathbf{d})^S : \nabla^S \eta + (\tau \cdot \mathbf{l}^T) : \nabla \eta^T]\, dv + \int_{\phi_t(b)} \rho \dot{\mathbf{v}} \cdot \eta \, dv = 0$$

(iii)

$$\int_{\phi_t(B)} [\overset{\circ}{\sigma} : \nabla^S \eta + (\sigma \cdot \mathbf{l}^T) : \nabla \eta^T]\, dv + \int_{\phi_t(B)} \rho \dot{\mathbf{v}} \cdot \eta \, dv = 0$$

(iv)

$$\int_{\phi_t(B)} [L_v(\sigma) : \nabla^S \eta + (\sigma \cdot \mathbf{l}^T) : \nabla \eta^T + tr(\mathbf{d}) \sigma : \nabla^S \eta]\, dv + \int_{\phi_t(B)} \rho \dot{\mathbf{v}} \cdot \eta \, dv = 0$$

(v)

$$\int_{\phi_t(B)} [\overset{\nabla}{\sigma} : \nabla^S \eta - 2(\tau \cdot \mathbf{d})^S : \nabla^S \eta + (\sigma \cdot \mathbf{l}^T) : \nabla \eta^T + tr(\mathbf{d}) \sigma : \nabla^S \eta]\, dv + \int_{\phi_t(B)} \rho \dot{\mathbf{v}} \cdot \eta \, dv = 0$$

Proof

Substitution of (a) - (e) from Proposition 2.5 into (2.3), use of the properties of the trace operator and transformation of the weak form (2.3) from material to spatial coordinates results in (i) - (v).

If the stress rates appearing in (i) - (v) above are characterized by constitutive equations of the form:

$$\text{stress rate} = L{:}\mathbf{d}$$

where L is a fully symmetric fourth order tensor, then weak forms (i) - (iii) are symmetric with respect to interchange of indices between $\nabla \eta$ and $\nabla \mathbf{v}$, whereas (iv) and (v) are not. Accordingly, finite element formulations based on (i) - (iii) have symmetric tangent operators. This is not the case for (iv) - (v). These conclusions correspond to the conditions for the existence of Hill's rate potentials [11] (stress rates lead to the self-adjointness of the rate problem if they are derivable from potentials which are Legendre transformations of the potential for $\dot{\mathbf{P}}$).

Finally, in (i) - (v), it should be noted that terms in $\ddot{\mathbf{v}}$ occur. If the corresponding displacement field is required, the rate formulation will require the integration of a third order system. This presents a significant challenge to the accuracy of numerical integration techniques.

CHAPTER 3. CONSTITUTIVE THEORY FOR FINITE ELASTICITY

3.1 THERMODYNAMICS AND HYPERELASTICITY

The development here is restricted to isothermal elasticity, but generalization to thermoelasticity does not change the results in any fundamental way. Defining $\mathbf{D} = \boldsymbol{\phi}^*(\mathbf{d}) = \frac{1}{2}\dot{\mathbf{C}}$, the integral

$$P_d(t) = \int_B tr(\mathbf{P} \cdot \dot{\mathbf{F}})\, dv = \int_B tr(\mathbf{S} \cdot \mathbf{D})\, dv$$

represents the deformation power of the continuum at time t, referred to the reference configuration B.

Definition 3.1

A body will be called hyperelastic if it satisfies the thermodynamic hypothesis that:

$$\int_{t_1}^{t_2} P_d(t)\, dt = 0$$

whenever $\boldsymbol{\phi}_{t_1}(B)$ and $\boldsymbol{\phi}_{t_2}(B)$ differ by at most a rigid body displacement.

Using the axioms of local action and material frame indifference, it may be shown that the necessary and sufficient conditions for the hypothesis to hold are:

(i) $tr(\mathbf{P} \cdot \dot{\mathbf{F}})$ and $tr(\mathbf{S} \cdot \mathbf{D})$ are at each material point perfect differentials,

(ii) there exists a functional $\Psi : B \rightarrow R$, called the material free energy density, which depends only on the point values of $\mathbf{F}$ or $\mathbf{C}$ and is a potential for the stress such that:

$$\mathbf{P} = \rho_B \frac{\partial \hat{\Psi}(\mathbf{F})}{\partial \mathbf{F}} \cdot \mathbf{g}^* \quad \text{or} \quad P^{Aa} = \rho_B \frac{\partial \hat{\Psi}(\mathbf{F})}{\partial F^b{}_A} g^{ab} \tag{3.1}$$

and

$$\mathbf{S} = 2\rho_B \frac{\partial \hat{\Psi}(\mathbf{C})}{\partial \mathbf{C}} \quad \text{or} \quad S^{AB} = 2\rho_B \frac{\partial \hat{\Psi}(\mathbf{C})}{C_{AB}}$$

It may also be shown [19] that Ψ is an isotropic function whenever the material is isotropic.

The constitutive equations of hyperelasticity (3.1) may be written in the alternative form:

$$\rho_B \dot{\Psi} - tr(\mathbf{P} \cdot \dot{\mathbf{F}}) = 0 \tag{3.2}$$

or

$$\rho_B \dot{\Psi} - tr(\mathbf{S} \cdot \mathbf{D}) = 0$$

which are expressions of the balance of energy equation (first law of thermodynamics) for isothermal processes. By definition, (3.2) also has the representations:

$$\rho_B \dot{\Psi} - tr(\boldsymbol{\tau} \cdot \mathbf{d}) = 0 \tag{3.3}$$

or

$$\rho \dot{\Psi} - tr(\boldsymbol{\sigma} \cdot \mathbf{d}) = 0$$

Under the present hypothesis for elasticity, all processes are non-dissipative, consequently the second law of thermodynamics is satisfied trivially and does not require further elaboration. In contrast, materials exhibiting inelastic dissipation must be considered within the framework of irreversible continuum thermodynamics, c.f. Chapter 4.

Proposition 3.1

(i) Assuming that $\Psi : B \to R$ is a C^2 function of the point values of $\mathbf{C}$, then

$$\frac{d}{dt}\mathbf{S} = \mathbf{A}:\mathbf{D} \tag{3.4}$$

is equivalent to $(3.1)_2$, that is, $\mathbf{S} = 2\rho_B \dfrac{\partial \Psi}{\partial \mathbf{C}}$

where $\mathbf{A}$ is the (second) material elasticity tensor defined by:

$$\mathbf{A} = 4\rho_B \frac{\partial^2 \Psi}{\partial \mathbf{C}^2}$$

The first material elasticity tensor will be defined in a subsequent chapter.

(ii) Equation (3.4) is objective.

Proof

Taking the material derivative of $(3.1)_2$ and noting that the material derivative of a material tensor is objective, supplies the result.

Notice that the material elasticity tensor $\mathbf{A}$ is fully symmetric by virtue of the continuity of Ψ and symmetry of $\mathbf{C}$. In Section 3.2, spatial forms of (3.1) and (3.4) are developed. The development of spatial forms of (3.4) is motivated by the observation that plasticity results from kinetic processes at the microscale and it is natural that the constitutive hypothesis of rate-independent plasticity lead to constitutive equations of *spatial* form in which components of an objective stress rate are given as homogeneous linear functions of the components of the rate of deformation. For the plasticity theory discussed in Chapter 4, solutions in the neighborhood of points interior to the yield hypersurface are considered to be governed by the thermodynamic hypothesis of hyperelasticity. Thus, for the convenience of solution procedures to be described later, it is desirable to obtain spatial versions of (3.4), to which the plasticity equations will reduce under the necessary conditions.

3.2 CONSTITUTIVE EQUATIONS FOR ISOTHERMAL HYPERELASTICITY

In this section, the spatial version of the constitutive equation of hyperelasticity $(3.1)_2$, that is,

$$\mathbf{S} = 2\rho_B \frac{\partial \Psi}{\partial \mathbf{C}}$$

and its equivalent form (3.4), that is,

$$\frac{d}{dt}\mathbf{S} = \mathbf{A} : \mathbf{D}$$

are sought.

Referring to Figure 1, it is recalled that three coordinate systems have been introduced:

(1) Material coordinate system $\{X^A\}$ embedded on the reference configuration B with basis vectors $\mathbf{G}_A$ and covariant metric tensor $\mathbf{G}$.

(2) Convected coordinate system $\{\bar{x}^\alpha\}$ deforming with the continuum and having basis vectors $\bar{\mathbf{g}}_\alpha$ and covariant metric tensor $\bar{\mathbf{g}}$.

(3) Coordinate system on the ambient space, having basis vectors $\mathbf{g}_a$ and covariant metric tensor $\mathbf{g}$.

Proposition 3.2

The convected coordinate basis vectors $\bar{\mathbf{g}}_\alpha$ and convected metric tensor $\bar{\mathbf{g}}$, have the properties:

$$(i) \quad \dot{\bar{\mathbf{g}}}_\alpha = v^\beta|_\alpha \bar{\mathbf{g}}_\beta \tag{3.5}$$
$$(ii) \quad \dot{\bar{\mathbf{g}}} = 2\mathbf{d}$$

Proof

(i) Let $\mathbf{r}$ be the position vector of a material point in the current configuration (see Figure 1), such that [1]:

$$\bar{\mathbf{g}}_\alpha = \frac{\partial \mathbf{r}}{\partial \bar{x}^\alpha}$$

Taking the material time derivative ($\bar{x}^\alpha = \text{constant}$) of this expression and noting that $\dot{\mathbf{r}} = \mathbf{v}$, leads to:

$$\dot{\bar{\mathbf{g}}}_\alpha = \frac{\partial \mathbf{v}}{\partial \bar{x}^\alpha} = v^\beta|_\alpha \mathbf{g}_\beta$$

(ii) By definition,

$$\bar{g}_{\alpha\beta} = \bar{\mathbf{g}}_\alpha \cdot \bar{\mathbf{g}}_\beta$$

Taking the material time derivative of the above expression and using the results of (i), leads to:

$$\dot{\bar{g}}_{\alpha\beta} = v^\gamma|_\alpha \bar{\mathbf{g}}_\gamma \cdot \bar{\mathbf{g}}_\beta + v^\gamma|_\beta \bar{\mathbf{g}}_\gamma \cdot \bar{\mathbf{g}}_\alpha = 2d_{\alpha\beta}$$

that is, $\dot{\bar{\mathbf{g}}} = 2\mathbf{d}$ ■.

The spatial form of $(3.1)_2$ is given by the following theorem.

Theorem 3.1

For a hyperelastic material under isothermal conditions, there exists a scalar function $\bar{\psi} : \boldsymbol{\phi}_t(B) \to R$, called the spatial free energy density, which depends only on the point values of $\bar{\mathbf{g}}$ in $\boldsymbol{\phi}_t(B)$ and is a potential for stress such that:

$$\sigma = 2\rho \frac{\partial \bar{\psi}}{\partial \bar{\mathbf{g}}} \tag{3.6}$$

or

$$\tau = 2\rho_B \frac{\partial \bar{\psi}}{\partial \bar{\mathbf{g}}}$$

Proof

The deformation power of the continuum at time t is given by:

$$P_d(t) = \int_B tr\,(\mathbf{S} \cdot \mathbf{D})\, dv = \int_{\phi_t(B)} tr\,(\sigma \cdot \mathbf{d})\, dv$$

Using the results of the previous proposition, it follows that:

$$tr\,(\sigma \cdot \mathbf{d}) = \tfrac{1}{2} tr\,(\sigma \cdot \dot{\bar{\mathbf{g}}})$$

and

$$P_d(t) = \int_{\phi_t(B)} \tfrac{1}{2}\, tr\,(\bar{\sigma} \cdot \dot{\bar{\mathbf{g}}})\, dv$$

Referring to the definition of hyperelasticity in Section 3.1, a necessary and sufficient condition for the thermodynamic hypothesis to hold is that there exists a scalar potential $\bar{\psi}'$ per unit volume of $\phi_t(B)$ which depends only on the point values of $\bar{\mathbf{g}}$ in $\phi_t(B)$ such that

$$\sigma = 2 \frac{\partial \bar{\psi}'}{\partial \bar{\mathbf{g}}}$$

in which case

$$\tfrac{1}{2}\, tr\,(\sigma \cdot \dot{\bar{\mathbf{g}}}) = \frac{d}{dt} \bar{\psi}' = tr\,(\sigma \cdot \mathbf{d})$$

is a perfect differential.

Defining $\bar{\psi} = \frac{1}{\rho} \bar{\psi}'$ as the current free energy density per unit mass of $\phi_t(B)$, it follows that :

$$\sigma = 2 \frac{\partial \bar{\psi}'}{\partial \bar{\mathbf{g}}} = 2\rho \frac{\partial \bar{\psi}}{\partial \bar{\mathbf{g}}}$$

or

$$\tau = 2J \frac{\partial \bar{\psi}'}{\partial \bar{\mathbf{g}}} = 2\rho_\beta \frac{\partial \bar{\psi}}{\partial \bar{\mathbf{g}}} \quad \blacksquare$$

An objective rate form of (3.6) is developed in the next proposition.

Proposition 3.3

Assuming that $\bar{\psi} : \boldsymbol{\phi}_t(B) \rightarrow R$ is a C^2 function of the point values of $\bar{\mathbf{g}}$, then

(i)

$$\overset{\circ}{\sigma} = \mathbf{a} : \mathbf{d} \tag{3.7}$$

is equivalent to $(3.6)_1$, that is, $\sigma = 2\rho \dfrac{\partial \bar{\psi}}{\partial \bar{\mathbf{g}}}$

where $\mathbf{a}$ is the spatial elasticity tensor defined by:

$$\mathbf{a} = 4\rho \frac{\partial^2 \bar{\psi}}{\partial \bar{\mathbf{g}}^2}$$

and

$$\overset{\circ}{\sigma} = J^{-1} \boldsymbol{\phi}_{t*} \left(\frac{d}{dt} \boldsymbol{\phi}_t^{*} (J\sigma) \right)$$

(Truesdell rate of Cauchy stress, see Proposition 2.5).

(ii) Equation (3.7) is objective.

Proof

(i) From $(3.6)_1$, for contravariant components of σ,

$$\sigma = 2\rho \frac{\partial \bar{\psi}}{\partial \bar{\mathbf{g}}} = 2\rho \frac{\partial \bar{\psi}}{\partial \bar{g}_{\alpha\beta}} \bar{\mathbf{g}}_\alpha \times \bar{\mathbf{g}}_\beta$$

Taking the material time derivative of this equation and using the results of Proposition 3.2, it follows that:

$$\dot{\sigma} = 4\rho \frac{\partial^2 \bar{\psi}}{\partial \bar{\mathbf{g}}^2} : \mathbf{d} + \mathbf{l} \cdot \sigma + \sigma \cdot \mathbf{l}^T - \sigma \, tr(\mathbf{d})$$

The last term in the above equation results from the material time derivative of ρ. Thus,

$$\dot{\sigma} - \mathbf{l} \cdot \sigma - \sigma \cdot \mathbf{l}^T + \sigma \, tr(\mathbf{d}) = \mathbf{a} : \mathbf{d}$$

But

$$\overset{\circ}{\sigma} \;=\; J^{-1}\phi_{t*}\left[\frac{d}{dt}\phi_t^{*}(J\sigma)\right] \;=\; J^{-1}L_v(J\sigma) \;=\; L_v(\sigma) + J^{-1}\sigma\, L_v(J)$$

For the contravariant components of σ under consideration, it follows that:

$$\overset{\circ}{\sigma} \;=\; \dot{\sigma} - \mathbf{l}\cdot\sigma - \sigma\cdot\mathbf{l}^T + \sigma\, tr(\mathbf{d})$$

By comparison with the equation above,

$$\overset{\circ}{\sigma} \;=\; \mathbf{a}:\mathbf{d}$$

(ii) Since J and $J\sigma$ are objective, it follows that $L_v(J\sigma)$ is also objective, and hence, equation (3.7) is objective ■

It is noted that the above result in the form of equation (3.7) is, by the invariance properties of tensors, valid in any coordinate system. Before generalizing (3.7) to include other stress rates, it is useful to have the following result.

Proposition 3.4

The (second) material and spatial elasticity tensors, A and **a**, respectively, are related through the motion ϕ_t by:

$$\mathbf{a} \;=\; J^{-1}\phi_{t*}(\mathrm{A}) \tag{3.8}$$

Proof

By definition and using $(3.1)_2$:

$$\begin{aligned} L_v(\tau) \;&=\; \phi_{t*}\left[\frac{d}{dt}\phi_t^{*}(\tau)\right] \\ &=\; \phi_{t*}\left[\frac{d}{dt}\mathrm{S}\right] \\ &=\; \phi_{t*}\left[2\rho_\beta\frac{\partial^2\Psi}{\partial \mathrm{C}^2} : \frac{d}{dt}\mathrm{C}\right] \end{aligned}$$

But, by Proposition 2.3, $\frac{d}{dt}\mathrm{C} = 2\phi_t^{*}(\mathbf{d})$ and, by definition (see Proposition 3.1),

$$2\rho_\beta\frac{\partial^2\Psi}{\partial \mathrm{C}^2} \;=\; \frac{1}{2}\mathrm{A}$$

Thus,

$$L_v(\tau) = \phi_{t*}(\mathrm{A} : \phi_t^{*}(\mathbf{d})) = \phi_{t*}(\mathrm{A}) : \mathbf{d}$$

But, by Proposition 3.3, $L_v(\tau) = J\overset{\circ}{\sigma} = J\mathbf{a} : \mathbf{d}$, from which it follows that:

$$\mathbf{a} = J^{-1}\phi_{t*}(\mathrm{A})$$

in components:

$$a^{abmn} = J^{-1} F^a{}_A F^b{}_B F^m{}_M F^n{}_N A^{ABMN} \quad \blacksquare$$

The above proof has a straight-forward development in components but is omitted here.

Using the framework of the propositions above, the relationships of a number of stress rates to gradients of the free energy potential can be developed as follows:

Proposition 3.5

The constitutive equation given by:

$$L_v(\tau) = J\,\mathbf{a} : \mathbf{d} \qquad (3.9)$$

is equivalent to (3.7), that is, $\overset{\circ}{\sigma} = \mathbf{a} : \mathbf{d}$

Proof

From (3.7), $\overset{\circ}{\sigma} = J^{-1} L_v(\tau) = \mathbf{a} : \mathbf{d}$, thus, $L_v(\tau) = J\,\mathbf{a} : \mathbf{d} = 4\rho_\beta \dfrac{\partial^2 \overline{\psi}}{\partial \overline{g}^2} : \mathbf{d} \quad \blacksquare$

Proposition 3.6

Assuming that $\overline{\psi}' : \phi_t(B) \rightarrow R$ is a C^2 function of the point values of $\overline{\mathbf{g}}$ and is defined per unit volume of $\phi_t(B)$, then:

(i)

$$L_v(\sigma) = \mathbf{a}' : \mathbf{d} \qquad (3.10)$$

is equivalent to $(3.6)_1$, that is, $\sigma = 2\dfrac{\partial \overline{\psi}'}{\partial \overline{\mathbf{g}}}$

where $\mathbf{a}'$ is the spatial elasticity tensor defined per unit *volume* of $\boldsymbol{\phi}_t(B)$ by:

$$\mathbf{a}' = 4\,\frac{\partial^2\bar{\psi}'}{\partial\bar{\mathbf{g}}^2}$$

(ii) Equation (3.10) is objective.

Proof

The proof follows the lines given for the proof of equation (3.7) in proposition (3.3), that is, material time differentiation of:

$$\boldsymbol{\sigma} = 2\,\frac{\partial\bar{\psi}'}{\partial\bar{g}_{\alpha\beta}}\,\bar{\mathbf{g}}_\alpha \times \bar{\mathbf{g}}_\beta$$

and use of the properties of the convected metric tensor and base vectors (Proposition 3.2).

3.3 CONSTITUTIVE EQUATIONS FOR ISOTHERMAL HYPOELASTICITY

The rate form of the spatial constitutive equation of hyperelasticity (3.7) expresses the linear dependence of the Truesdell rate of Cauchy stress $\overset{\circ}{\boldsymbol{\sigma}}$ on the spatial rate of deformation tensor $\mathbf{d}$ through the spatial elasticity tensor $\mathbf{a}$. By hypothesis, $\mathbf{a}$ is a function of the deformation through its dependence on the point values of $\bar{\mathbf{g}}$. It is observed that (3.7) may be expressed in terms of other stress rates if the difference between these rates and the Truesdell rate of Cauchy stress is absorbed in the definition of the compliance tensor. In this case, the compliance tensor will, in general, be a function of the stress tensor and the deformation from the reference state (if $\boldsymbol{\tau}$ is involved). For example, if:

$$\overset{\triangledown}{\boldsymbol{\tau}} = \bar{\mathbf{a}}:\mathbf{d} \tag{3.11}$$

Then (3.11) is equivalent to (3.7) if:

$$\bar{a}^{abcd} = J\,[a^{abcd} + \sigma^{bd}g^{ac} + \sigma^{ac}g^{bd}] \tag{3.12}$$

This follows from the definitions of $\overset{\circ}{\boldsymbol{\sigma}}$ and $\overset{\triangledown}{\boldsymbol{\tau}}$.

In constitutive equations of this special kind, the difference between various definitions of the stress rate is *not essential.* In this section, rate constitutive equations which are *not* equivalent to (3.7) are considered.

A hypoelastic material is defined [20,53] by the minimal requirement that a spatial stress rate is a homogeneous linear function of the spatial rate of deformation tensor. For example:

$$\overset{\nabla}{\tau} = \mathbf{C}:\mathbf{d} \tag{3.13}$$

$$\overset{\nabla}{\sigma} = \mathbf{C}:\mathbf{d} \tag{3.14}$$

where $\mathbf{C}$ is an isotropic tensor which depends on J and σ in general. Usually, the dependence of $\mathbf{C}$ on J and σ is not sufficient to render (3.13) or (3.14) equivalent to (3.7) but, by specialization of $\mathbf{C}$, the dependence may be sufficient, in which case it is seen that hypoelasticity contains hyperelasticity, as they are defined here.

In the present study, the effect of choosing $\mathbf{C}$ in (3.13) and (3.14) to be a constant tensor is investigated. Such a constitutive assumption has been used frequently in the computational literature. Clearly, this assumption does not render (3.13) or (3.14) equivalent to (3.7). In fact, the differences in these constitutive equations leads to marked variations in the solution boundary value problems employing these definitions. It may be shown [20,53] that, in general, hypoelasticity is equivalent to elasticity only for infinitesimal deformations from an arbitrary reference configuration. The following result will be of some use when the integration of (3.13) is contemplated in Chapter 6.

Proposition 3.7

The co-rotational rate of Kirchhoff stress $\overset{\nabla}{\tau}$ coincides with the Lie derivative of τ with respect to the spatial velocity field $\mathbf{v}$ under the assumption that the rate of deformation tensor $\mathbf{d}$ vanishes. That is,

$$\overset{\nabla}{\tau} = L_{\mathbf{v}}(\tau)\big|_{\mathbf{d}=0} \tag{3.15}$$

Proof

By Definition 2.14(a) and the Polar Decomposition Theorem:

$$L_{\mathbf{v}}(\tau) = (\phi_t^R \circ \phi_t^U)_* \left[\frac{d}{dt} (\phi_t^R \circ \phi_t^U)^* (\tau) \right]$$

where ϕ_t^R and ϕ_t^U are associated with pure rotation and stretching of material neighborhoods, respectively.

It follows that:

$$L_v(\tau) = \phi_{t\,*}^{U}\left(\phi_{t\,*}^{R}\left(\frac{d}{dt}\phi_t^{U^*}\left(\phi_t^{R^*}(\tau)\right)\right)\right)$$

Suppose that local material motion is characterized by pure spin such that $\phi_{t\,*}^{U}(\cdot)$ is the identity mapping, corresponding to $\mathbf{d} = 0$. Then:

$$L_v(\tau)\big|_{\mathbf{d}=0} = \phi_{t\,*}^{R}\left(\frac{d}{dt}\phi_t^{R^*}(\tau)\right)$$

Noting that $\dot{\mathbf{F}} = \boldsymbol{\omega}\cdot\mathbf{F}$, where $\boldsymbol{\omega}$ is the spin rate tensor, it follows that:

$$L_v(\tau)\big|_{\mathbf{d}=0} = \dot{\tau} - \boldsymbol{\omega}\cdot\tau - \tau\cdot\boldsymbol{\omega}^T = \overset{\triangledown}{\tau}$$

for contravariant components of τ ■

It is clear from the above proposition that the arguments of Proposition 3.3 cannot be applied to find constitutive equations of the form (3.13) or (3.14) from thermodynamics. Nevertheless, these equations are objective.

Homogeneous deformation problems governed by (3.13) or (3.14) display a yield-like instability phenomenon in which the material response softens at certain critical deformation states (for example, one can demonstrate the necessity of a zero force to obtain an infinite deformation in one case, see Chapter 8). This phenomenon is referred to as hypoelastic yield [20] and a number of examples are given in Chapter 8. For some problems the instability may occur at small strains (although the effects are associated with terms arising from large deformation analysis). Thus, for some problems, constitutive equations of the form (3.13) or (3.14) may be quite inappropriate. This would appear to include the constitutive equations of elasto-plasticity. The difference between $\overset{\circ}{\sigma}$ and $\overset{\triangledown}{\tau}$ being responsible for an instability in the solution of problems, quite independent of physical processes. In Chater 4, a plasticity theory employing $\overset{\circ}{\sigma}$ is developed.

Finally, it is interesting to note that a rational theory of rate-independent plasticity can be developed as a generalization of the theory of hypoelasticity [38,39]. A hypoelastic material has a nonfading memory and behavior independent of time scale [20]. When the dependence of $\mathbf{C}$ in (3.13) or (3.14) on τ and J is adequate, a hypoelastic yield results. With respect to these characteristics, rate-independent plasticity lies within the scope of hypoelasticity. However, internal variables must be introduced into the theory to obtain correct loading-unloading and hardening characteristics [39]. Such an approach to rate-independent plasticity, the subject of the next chapter, is not employed here.

CHAPTER 4. CONSTITUTIVE THEORY FOR ELASTO-PLASTICITY

4.1 THERMODYNAMICS AND PLASTICITY

This section deals with the application of thermodynamics to materials which exhibit rate-independent plastic deformations. The development is appropriate for finite strains but is restricted to isothermal conditions. Attention is focused on the definition of the thermodynamic state and on the expression of the second law of thermodynamics. The formulation presented uses an internal variable formalism.

Internal Variables and the Thermodynamic State

In the case of metals, plastic behavior arises as a consequence of slip rearrangements of crystallographic planes through the motion of dislocations. The resulting plastic flow will be modified by other microstructural phenomena such as twinning in crystals, grain-boundary sliding and stress induced phase transformation. Point defects in the crystal lattice structure, such as vacancies and solute atoms of different species, may introduce a strong viscous component into the plastic deformation behavior.

It is well known that two substantially different dislocation arrangements, both of which have resulted in the identical extension of a loaded specimen, lead, under the application of subsequent loading, to quite different responses. Accordingly, the thermodynamic state is not determined by the current state of stress or strain alone. The current extent of the local structural rearrangements produced by the operative microstructural mechanisms is included in the definition of the thermodynamic state by supplementing the stress or strain tensors with a finite set of internal variables.

The objective of the present work is to arrive at a macroscopic constitutive theory of plasticity which will be characterized by as few variables as possible. Such a task is not without its difficulties. As suggested above, materials deforming plastically will exhibit intense local inhomogeneities in the material distribution and deformation at the microscale. In contrast, the concept of local homogeneity is central to the framework of the continuum approximation. In this dichotomy of view lies the challenge of developing macroscopic models from microscopic considerations. It is currently believed that internal variables, which may be identified with specific microstructural mechanisms, offer a viable thermodynamic approach to this problem.

The internal variables are objective tensors and are denoted collectively by the n-vector $\{\mathbf{q}\}$ or simply $\mathbf{q}$. Each component of $\mathbf{q}$, denoted $\mathbf{q}_a$, will represent an internal variable. The internal variables, being tensors, will have the usual material or spatial forms. However, in these introductory remarks, no distinction will be made between material and spatial descriptions of the tensors $\mathbf{q}$ and so a slight abuse of notation is committed for the sake of economy. This is rectified when a detailed development is undertaken.

The state variables, given by the stress or strain tensor associated with the current state (and excluding temperature for the present isothermal development), are denoted $\boldsymbol{\Lambda}$. Thus, as suggested above, the thermodynamic state is fully defined by the state space vector denoted $(\boldsymbol{\Lambda}, \mathbf{q})$. Again, $\boldsymbol{\Lambda}$ may assume a material or spatial description consistent with $\mathbf{q}$.

In a formal sense, the number of internal variables, n, is chosen in view of the principle of determinism [20]. This principle requires that the history and present values of the state space variables, $(\boldsymbol{\Lambda},\mathbf{q})$, be sufficient to uniquely determine the thermodynamic potential functions, such as the free energy, etc.

The internal variables are assumed to evolve according to local rate equations [21]. Thus, some objective time derivative of each internal variable is determined by the present thermodynamic state. The internal variable hypothesis [22] assumes that the dependence of the thermodynamic potentials on the history of the state (principle of determinism) is achieved by their dependence on the current values of the internal variables. More simply stated, the thermodynamic potentials depend on the history of the state through their dependence on what the state has produced, namely, the current values of the internal variables. The relevance of such considerations to "path-dependent" plasticity described above should be clear.

Characterization of Rate-Independent Plasticity

A theory of plasticity which includes rate-dependent and rate-independent effects may be based on internal variables which are governed by rate equations of the form [21-23]:

$$\dot{\mathbf{q}} = \Pi\,(\mathbf{\Lambda}, \mathbf{q}, \dot{\mathbf{\Lambda}})$$

where the superimposed dot implies some suitable time derivative to satisfy objectivity requirements. The material will be rate-dependent if Π does not depend on $\dot{\mathbf{\Lambda}}$ and viscoplastic if there is a region E in $(\mathbf{\Lambda}, \mathbf{q})$ space such that $(\mathbf{\Lambda}, \mathbf{q}) \in E$ implies $\Pi\,(\mathbf{\Lambda}, \mathbf{q}) = 0$. On the other hand, a material is rate-independent if Π is homogeneous of the first degree in $\dot{\mathbf{\Lambda}}$ [23]. Although viscous effects may be important under certain loading conditions and despite the fact that rate-independence may be viewed as a limiting case of rate-dependent response, the present work is, nevertheless, restricted to the special case of rate-independent behavior.

Rate-independent plasticity is characterized by the existence of a region R in state space $(\mathbf{\Lambda}, \mathbf{q})$ such that:

(a) R defines an admissible region of state space, which the state cannot leave, and is characterized by a yield function $f(\mathbf{\Lambda}, \mathbf{q})$, such that:

$$(\mathbf{\Lambda}, \mathbf{q}) \in \text{int(R)} \;\rightarrow\; f < 0$$
$$(\mathbf{\Lambda}, \mathbf{q}) \in \partial\text{R} \;\rightarrow\; f = 0$$

all other states are inadmissible.

(b) if $(\mathbf{\Lambda}, \mathbf{q}) \in \text{int(R)}$ then $\dot{\mathbf{q}} = 0$.

The interior of R, denoted E, is called the elastic region. All internal variable rates vanish at points in E or during unloading when the state lies on the yield surface (f = 0) but moves towards points in E. During loading, when the state lies on the yield surface and remains on it, the internal variable rates are non-vanishing. This situation is summarized by:

$$\begin{aligned} f &< 0 && \rightarrow \dot{\mathbf{q}} = 0 \\ f &= 0 \;\; (\textit{unloading}) && \rightarrow \dot{\mathbf{q}} = 0 \\ f &= 0 \;\; (\textit{loading}) && \rightarrow \dot{\mathbf{q}} = \lambda\,\Gamma\,(\mathbf{\Lambda}, \mathbf{q}) \end{aligned} \tag{4.1}$$

where λ is a scalar to be determined by the consistency condition during loading, that is,

$\dot{f}(\mathbf{\Lambda},\mathbf{q}) = 0$, and where $\mathbf{\Gamma}$ is a tensor valued constitutive function.

The loading index $\overset{\Delta}{f}$ is introduced as:

$$\overset{\Delta}{f}(\mathbf{\Lambda},\mathbf{q},\dot{\mathbf{\Lambda}}) = \frac{\partial f}{\partial \mathbf{\Lambda}} : \dot{\mathbf{\Lambda}} \tag{4.2}$$

in which case it may be shown that (4.1) is equivalent to:

$$\dot{\mathbf{q}} = \mathbf{r} < \overset{\Delta}{f} > \tag{4.3}$$

where

$$\mathbf{r}(\mathbf{\Lambda},\mathbf{q}) = -\left[tr\left(\frac{\partial f}{\partial \mathbf{q}} \cdot \mathbf{\Gamma} \right) \right]^{-1} \mathbf{\Gamma}$$

and $<\cdot>$ is the Macauley bracket.

It is noted that rate-independence is preserved in (4.3), which is homogeneous of the first degree in $\dot{\mathbf{\Lambda}}$ by virtue of (4.2). The product $\left(\frac{\partial f}{\partial \mathbf{q}} \cdot \mathbf{\Gamma} \right)$ in the last equation should be interpreted appropriately for the order of $\mathbf{q}$ (here unspecified).

Second Law of Thermodynamics

A Lagrangian formulation in terms of material tensors is considered here, although the spatial form of the resulting expressions will be derived. The state variable $\mathbf{\Lambda}$ is identified as the Lagrangian strain tensor $\mathbf{E}$ defined by $\mathbf{E} = ½(\mathbf{C} - \mathbf{G})$. The internal variable vector is denoted $\mathbf{Q}$ with components $\mathbf{Q}_a$, a = 1, ..., n. The internal variables are assumed to be material tensors. The thermodynamic state is described by the state space vector $(\mathbf{E},\mathbf{Q})$ in this case.

The existence of the free energy per unit mass of B, denoted $\Psi(\mathbf{E},\mathbf{Q})$, is assumed on the grounds that at any point in state space there are neighboring points that can be reached by unloading. That is, an elastic process in which the internal variables have vanishing rates. Such a state is one of constrained equilibrium [23].

The second law of thermodynamics, restricted for isothermal conditions, is expressed by the local form of the Clausius-Duhem inequality [25]:

$$-\rho_B \dot{\Psi} + tr\,(\mathbf{S} \cdot \dot{\mathbf{E}}) \;\geqslant\; 0 \tag{4.4}$$

The material version of the internal variable rate equation (4.3) is taken to be:

$$\dot{\mathbf{Q}} \;=\; \mathbf{R} < \overset{\Delta}{f} > \tag{4.5}$$

Since $\dot{\mathbf{Q}}$ is homogeneous of the first degree in $\dot{\mathbf{E}}$, it is notable that the Clausius-Duhem inequality (4.4) is not sufficient to determine the constitutive equations for stress [23]. The usual arguments regarding the satisfaction of (4.4) for arbitrary process must be supplemented by the additional assumption that unloading processes ($f < 0$) are elastic (non-dissipative). In this case, (4.4) is satisfied if and only if:

$$\mathbf{S} \;=\; \rho_B \frac{\partial \Psi}{\partial \mathbf{E}} \tag{4.6}$$

provided that the dissipation inequality:

$$-\frac{\partial \Psi}{\partial \mathbf{Q}} \cdot \mathbf{R} < \overset{\Delta}{f} > \;\geqslant\; 0 \tag{4.7}$$

is satisfied.

The dissipation inequality (4.7) implies that internal dissipative processes cannot increase the free energy. This inequality acts as a restriction on the admissible forms of (4.5).

Introducing the complementary free energy, χ, per unit mass of B, by the Legendre transformation:

$$\chi \;=\; \frac{1}{\rho_B}\, tr\,(\mathbf{S} \cdot \mathbf{E}) - \Psi$$

it follows that (4.6) and (4.7) have the complementary forms:

$$\mathbf{E} \;=\; \rho_B \frac{\partial \chi}{\partial \mathbf{S}} \tag{4.8}$$

$$\frac{\partial \chi}{\partial \mathbf{Q}} \cdot \mathbf{R} < \overset{\Delta}{f} > \;\geqslant\; 0 \tag{4.9}$$

Proposition 4.1

Assuming χ is a C^2 function of the point values of $\mathbf{S}$ and $\mathbf{Q}$, the constitutive equation (4.8) is equivalent to:

$$\dot{\mathbf{E}} \;=\; \mathbf{M} : \dot{\mathbf{S}} + \mathbf{N} \cdot \dot{\mathbf{Q}} \tag{4.10}$$

where $\mathbf{M}$ is the fully symmetric elasticity compliance tensor defined by:

$$\mathbf{M} = \rho_B \frac{\partial^2 \chi}{\partial \mathbf{S}^2}$$

and $\mathbf{N}$ is an inelastic compliance tensor defined by:

$$\mathbf{N} = \rho_B \frac{\partial^2 \chi}{\partial \mathbf{Q} \partial \mathbf{S}}$$

(As above, the product $\mathbf{N} \cdot \dot{\mathbf{Q}}$ in (4.10) is to be interpreted appropriately for the order of $\mathbf{Q}$).

Proof

The result follows from the material time derivative of (4.8).

The detailed plasticity model to be presented in the next section and used for computational purposes later will be expressed in terms of spatial tensors defined on $\boldsymbol{\phi}_t(B)$. Accordingly, the spatial version of (4.10) is now considered.

Proposition 4.2

For all regular motions $\boldsymbol{\phi} : B \times R \rightarrow N$,

(i) Equation (4.10) is equivalent to

$$\mathbf{d} = \mathbf{m} : \overset{\circ}{\boldsymbol{\sigma}} + \mathbf{n} \cdot \overset{\circ}{\mathbf{q}} \tag{4.11}$$

or

$$\mathbf{d} = J^{-1} m : L_v(\boldsymbol{\tau}) + J^{-1} \mathbf{n} \cdot L_v(J\mathbf{q}) \tag{4.12}$$

where the spatial elastic compliance tensor $\mathbf{m}$ is given by:

$$\mathbf{m} = J \boldsymbol{\phi}_{t*}(\mathbf{M})$$

and the spatial inelastic compliance tensor $\mathbf{n}$ by:

$$\mathbf{n} = J \boldsymbol{\phi}_{t*}(\mathbf{N})$$

The spatial internal variables $\mathbf{q}$ satisfy

$$\mathbf{q} = J^{-1} \boldsymbol{\phi}_{t*}(\mathbf{Q})$$

and

$$\overset{\circ}{\sigma} = J^{-1}\boldsymbol{\phi}_{t*}\left(\frac{d}{dt}\overset{*}{\boldsymbol{\phi}_t}(J\sigma)\right)$$

$$\overset{\circ}{\mathbf{q}} = J^{-1}\boldsymbol{\phi}_{t*}\left(\frac{d}{dt}\overset{*}{\boldsymbol{\phi}_t}(J\mathbf{q})\right)$$

(ii) Equations (4.11) and (4.12) are objective.

Proof

(i) Recalling $\dot{\mathbf{E}} = \overset{*}{\boldsymbol{\phi}_t}(\mathbf{d})$, $\mathbf{S} = \overset{*}{\boldsymbol{\phi}_t}(J\sigma)$ and using $\mathbf{Q} = \overset{*}{\boldsymbol{\phi}_t}(J\mathbf{q})$, then (4.10) has the form:

$$\begin{aligned}\overset{*}{\boldsymbol{\phi}_t}(\mathbf{d}) &= \mathbf{M} : \frac{d}{dt}\overset{*}{\boldsymbol{\phi}_t}(J\sigma) + \mathbf{N}\cdot\frac{d}{dt}\overset{*}{\boldsymbol{\phi}_t}(J\mathbf{q}) \\ &= \mathbf{M} : \overset{*}{\boldsymbol{\phi}_t}(L_{\mathbf{v}}(J\sigma)) + \mathbf{N}\cdot\overset{*}{\boldsymbol{\phi}_t}(L_{\mathbf{v}}(J\mathbf{q}))\end{aligned}$$

from which it follows that

$$\mathbf{d} = \boldsymbol{\phi}_{t*}(\mathbf{M}) : L_{\mathbf{v}}(J\sigma) + \boldsymbol{\phi}_{t*}(\mathbf{N})\cdot L_{\mathbf{v}}(J\mathbf{q})$$

using (4.13) and (4.14),

$$\begin{aligned}\mathbf{d} &= J^{-1}\mathbf{m} : L_{\mathbf{v}}(\tau) + J^{-1}\mathbf{n}\cdot L_{\mathbf{v}}(J\mathbf{q}) \\ &= \mathbf{m} : \overset{\circ}{\sigma} + \mathbf{n}\cdot\overset{\circ}{\mathbf{q}}\end{aligned}$$

(ii) Objectivity follows from the properties of the Lie derivative (see Section 2.1) ■

It is recalled from Definition 2.14 that $\overset{\circ}{\sigma}$ denotes the Truesdell rate of Cauchy stress.

Hypothesis

In accord with experimental evidence, it is postulated that processes which occur at fixed values of the internal variables are governed by the equations of hyperelasticity, which are independent of the history of deformation. Thus, **m** is independent of **q** and will usually be assumed to be a constant isotropic tensor.

Uncoupled instantaneous elastic response corresponds to an additive decomposition of the complementary free energy into elastic and inelastic parts [21].

Proposition 4.3

The spatial elastic compliance tensor **m** is the inverse of the spatial elasticity tensor **a** (see Proposition 3.3), such that:

$$\mathbf{m} : \mathbf{a} = \mathbf{I}$$

Proof

From (4.8),

$$\rho_B \frac{\partial^2 \chi}{\partial \mathbf{S}^2} : \frac{\partial \mathbf{S}}{\partial \mathbf{E}} = \mathbf{I}$$

Using (4.6), it follows that $\mathbf{M} : \mathbf{A} = \mathbf{I}$ where $\mathbf{M} = \rho_b \dfrac{\partial^2 \chi}{\partial \mathbf{S}^2}$ and $\mathbf{A} = \rho_B \dfrac{\partial^2 \Psi}{\partial \mathbf{E}^2}$.

Since $\mathbf{m} = J\, \phi_{t*}(\mathbf{M})$, from Proposition 4.2, and $\mathbf{a} = J^{-1}\, \phi_{t*}(\mathbf{A})$, from Proposition 3.4,

$$\text{then} \quad \phi_t^*(\mathbf{m}) : \phi_t^*(\mathbf{a}) = \mathbf{I} \quad \text{which implies} \quad \mathbf{m} : \mathbf{a} = \mathbf{I} \quad \blacksquare$$

In the event that $\overset{\circ}{\mathbf{q}} = 0$ in (4.11) (that is, during loading in the elastic range or unloading from the yield hypersurface), the above proposition demonstrates that (4.11) represents a hyperelastic material model as discussed in Chapter 3.

Note that the material form of the internal variable rate equation (4.5) has a spatial representation as follows:

$$\dot{\mathbf{Q}} = \mathbf{R} < \overset{\Delta}{f} >$$

implies

$$\frac{d}{dt} \phi_t^*(J\mathbf{q}) = \mathbf{R} < \overset{\Delta}{f} >$$

and

$$J^{-1} L_v(J\mathbf{q}) = J^{-1} \phi_{t*}(\mathbf{R}) < \overset{\Delta}{f} >$$

or

$$\overset{\circ}{\mathbf{q}} = \mathbf{r} < \overset{\Delta}{f} > \tag{4.13}$$

where

$$\mathbf{r} = J^{-1} \phi_{t*}(\mathbf{R})$$

Using (4.13), it is now seen that (4.11) and (4.12) have the identical form:

$$\mathbf{d} = \mathbf{m} : \overset{\circ}{\sigma} + \mathbf{n} \cdot \mathbf{r} < \overset{\Delta}{f} > \tag{4.14}$$

Kinematic Considerations

Equation (4.14) has the interpretation that the total deformation rate $\mathbf{d}$ has an additive decomposition into an elastic part and an inelastic or plastic part. This decomposition has been obtained independently of any kinematic arguments or approximations. Introducing the notation:

$$\mathbf{d}^e = \mathbf{m} : \overset{\circ}{\sigma} = J^{-1}\mathbf{m} : L_v(\tau) \tag{4.15}$$

and

$$\mathbf{d}^p = \mathbf{n} \cdot \overset{\circ}{\mathbf{q}} = \mathbf{n} \cdot \mathbf{r} < \overset{\Delta}{f} > \tag{4.16}$$

where $\mathbf{d}^e$ is associated with the rate of total deformation when $\mathbf{d}^p = 0$, that is, elastic deformation and $\mathbf{d}^p$ is associated with the rate of plastic deformation. Thus,

$$\mathbf{d} = \mathbf{d}^e + \mathbf{d}^p \tag{4.17}$$

A number of plasticity theories have been based on assumptions that are kinematic in nature, and it is worthwhile to briefly contrast such an approach to the essentially thermodynamic result of (4.17). One kinematic approach, introduced by Lee and Liu [26], is based on the concept of the "intermediate configuration." This approach is based on the assumption that infinitesimal neighborhoods of plastically deformed material can be unloaded elastically without any additional plastic flow. This assumption leads to the multiplicative decomposition of the deformation gradient $\mathbf{F}$, such that [26]:

$$\mathbf{F} = \mathbf{F}^e \mathbf{F}^p \tag{4.18}$$

The intermediate configuration Ω is defined by the (generally non-differentiable) map:

$$(\mathbf{F}^e)^{-1} : \phi_t(B) \rightarrow \Omega$$

which describes the elastic unloading. The remainder of the motion is supposed to be plastic and is described by the (generally non-differentiable) map $\mathbf{F}^p$, to within a rotation. Other decompositions of the motion are possible [27,28]. The intermediate configuration Ω has been utilized in a number of internal variable plasticity theories [22,29,30,31,35]. Also, important attempts aimed at a finite deformation theory for single crystals based on an internal variable description of microdynamic crystal defects have used the intermediate configuration [32,33]. A continuum model based on the above kinematical decomposition has also been developed by Rice [34]. The incremental or rate form of (4.18) is [36]:

$$\mathbf{d} \;=\; D^e + D^p \tag{4.19}$$

where

$$D^e \;=\; [\dot{\mathbf{F}}^e\,(\mathbf{F}^e)^{-1}]^S$$

and

$$D^p \;=\; [\mathbf{F}^e\,\dot{\mathbf{F}}^p\,(\mathbf{F}^p)^{-1}\,(\mathbf{F}^e)^{-1}]^S$$

Often the elastic coupling terms in D^p are ignored by order of magnitude arguments [36] or otherwise incorporated into the theory [29,37].

It should be emphasized that (4.19) results from kinematical arguments whereas (4.17), the basis of the present work, results from a thermodynamic development without kinematical assumptions. While it is possible to compare (4.17) and (4.19) to obtain:

$$D^e + D^p \;=\; \mathbf{m}:\overset{\circ}{\sigma} + \mathbf{n}\cdot\mathbf{r}<\overset{\Delta}{f}>$$

there appears to be no compelling reasons to identify:

$$D^e \;\approx\; \mathbf{m}:\overset{\circ}{\sigma}$$

$$D^p \;\approx\; \mathbf{n}\cdot\mathbf{r}<\overset{\Delta}{f}>$$

4.2 A MODEL FOR RATE-INDEPENDENT PLASTICITY

A model for rate-independent plastic behavior will be developed by:

(a) Selecting a set of internal variables.

(b) Specifying the form of the internal variable rate equations (4.13), that is,

$$\overset{\circ}{\mathbf{q}} \;=\; \mathbf{r}<\overset{\Delta}{f}>$$

(c) Specifying the form of the plasticity constitutive equation (4.16), that is,

$$\mathbf{d}^p = \mathbf{n} \cdot \overset{\circ}{\mathbf{q}}$$

Plastic Flow Potential

A normality structure for plastic deformation rates may be demonstrated in two ways:

(1) By the assumption of a material stability postulate such as Drucker's (suitably generalized for finite strain).

(2) By the assumption of certain restrictions on the internal variable rate equations.

In general, the first method is not a thermodynamic result but the second method, adopted here, has a clear thermodynamic basis.

Letting $\overline{\chi}(\sigma,\mathbf{q})$ be the specific complementary free energy in $\phi_t(B)$, the thermodynamic force ξ conjugate to the internal variable fluxes $\overset{\circ}{\mathbf{q}}$ is defined by:

$$\xi\,(\sigma,\mathbf{q}) = \frac{\partial\overline{\chi}}{\partial\mathbf{q}}$$

Provided the following two restrictions on the internal variable rate equations are satisfied, a plastic potential may be shown to exist:

(a) Local dependence hypothesis [34,40], the dependence of $\overset{\circ}{\mathbf{q}}$ on σ is of the form:

$$\overset{\circ}{\mathbf{q}} = \hat{\mathbf{g}}(\xi\,(\sigma,\mathbf{q}),\mathbf{q})$$

that is, the internal variable rate equation depends on the stress σ only through its dependence on ξ.

(b) The internal variable rates and their conjugate forces satisfy a generalized Onsager reciprocity condition [41].

Under these conditions, there exists a flow potential [34,40,41] $\Omega\,(\sigma,\mathbf{q})$ such that:

$$\mathbf{d}^p = \lambda\,\frac{\partial\Omega}{\partial\sigma}$$

where λ is a scalar to be determined from the plastic consistency condition. The satisfaction of the above conditions has been investigated on the basis of dislocation mechanics for the case of single crystals [32,33]. In the sequel, the flow potential Ω will be taken as the yield function f, such that:

$$\mathbf{d}^p = \lambda \frac{\partial f}{\partial \sigma}(\sigma,\mathbf{q}) \tag{4.20}$$

Hardening Internal Variables

Two internal variables are introduced:

(1) A scalar α associated with the expansion of the yield surface for isotropic hardening, and

(2) A symmetric second order tensor $\boldsymbol{\beta}$ with units of stress associated with the translation of the yield surface for kinematic hardening.

No relation to kinetic processes at the microscale will be pursued here, although such an approach is desirable.

Following Prager [42], the translated stress space defined by:

$$\boldsymbol{\delta} = \boldsymbol{\sigma} - \boldsymbol{\beta}$$

is used with α to define the state space vector, which is taken to be $(\boldsymbol{\delta},\alpha)$.

It is convenient in introducing the rate equations for α and $\boldsymbol{\beta}$ to work with the inverse of (4.16), that is:

$$\overset{\circ}{\mathbf{q}} = \mathbf{n}^{-1}:\mathbf{d}^p$$

In fact, this equation will be generalized so that $\overset{\circ}{\mathbf{q}}$ is not linear in $\mathbf{d}^p$ but merely homogeneous of the first degree. The rate equations are assumed to be of the form:

$$\overset{\circ}{\alpha} = \mathbf{r}_1(\boldsymbol{\delta},\alpha):\mathbf{d}^p + r_2(\boldsymbol{\delta},\alpha)\,[\mathbf{d}^p:\mathbf{d}^p]^{1/2} \tag{4.21}$$

$$\overset{\circ}{\boldsymbol{\beta}} = r_3(\boldsymbol{\delta},\alpha)\mathbf{d}^p + \mathbf{r}_4(\boldsymbol{\delta},\alpha)\,[\mathbf{d}^p:\mathbf{d}^p]^{1/2} \tag{4.22}$$

where the order of the respective $\mathbf{r}_i$ should be apparent. Constitutive assumptions for the form of the $\mathbf{r}_i$ will be made subsequently. The terms involving $\mathbf{r}_1$ and r_3 are associated with work hardening rules in classical plasticity and the terms involving r_2 and $\mathbf{r}_4$ with effective-strain hardening rules.

Yield Function

The yield function $f(\boldsymbol{\delta},\alpha)$ is assumed to be an analytic function of the second and third invariants of the deviatoric part of $\boldsymbol{\delta}$, denoted $\boldsymbol{\delta}'$, where $\boldsymbol{\delta}' = \boldsymbol{\delta} - \frac{1}{3} tr(\boldsymbol{\delta})\mathbf{I}$. In this way, f will be pressure-insensitive. For simplicity, a von Mises type criterion is taken, having the form:

$$f(\boldsymbol{\delta},\alpha) = \boldsymbol{\delta}' : \boldsymbol{\delta}' + k(\alpha) \qquad (4.23)$$

where k is a scalar function of α.

The consistency condition during plastic loading on the yield surface requires $\dot{f} = 0$ during a plastic deformation process. An expression for $\dot{f}$ is given by the following lemma.

Lemma 4.1

Suppose f is a scalar function of $\boldsymbol{\sigma}$ and $\mathbf{q}_\alpha$, $\alpha = 1,\dots,n$, then:

$$\dot{f} = \frac{\partial f}{\partial \boldsymbol{\sigma}} : \overset{\circ}{\boldsymbol{\delta}} + \frac{\partial f}{\partial \mathbf{q}_\alpha} \cdot \overset{\circ}{\mathbf{q}}_\alpha \qquad \text{(sum on alpha)}$$

where the last contraction is to be interpreted for the order of $\mathbf{q}_\alpha$.

Proof

The second Piola-Kirchhoff stress tensor and material internal variable tensors are defined by:

$$\mathbf{S} = \boldsymbol{\phi}_t^*(J\boldsymbol{\sigma}), \qquad \mathbf{Q}_\alpha = \boldsymbol{\phi}_t^*(J\mathbf{q}_\alpha)$$

and let $f(\boldsymbol{\sigma},\mathbf{q}_\alpha) = \bar{f}(\mathbf{S},\mathbf{Q}_\alpha)$, then:

$$\dot{f} = \frac{\partial \bar{f}}{\partial \mathbf{S}} : \dot{\mathbf{S}} + \frac{\partial \bar{f}}{\partial \mathbf{Q}_\alpha} \cdot \dot{\mathbf{Q}}_\alpha$$

But, by definition,

$$\dot{\mathbf{S}} = \frac{d}{dt}\boldsymbol{\phi}_t^*(J\boldsymbol{\sigma}) = \boldsymbol{\phi}_t^*(L_v(J\boldsymbol{\sigma})) = J\boldsymbol{\phi}_t^*(\overset{\circ}{\boldsymbol{\sigma}})$$

similarly,

$$\dot{\mathbf{Q}}_\alpha = J\boldsymbol{\phi}_t^*(\overset{\circ}{\mathbf{q}}_\alpha)$$

Using these results and the property of the contraction, it is easy to show that:

$$\dot{f} = J\phi_{t*}(\frac{\partial \bar{f}}{\partial \mathbf{S}}) : \overset{\circ}{\sigma} + J\phi_{t*}(\frac{\partial \bar{f}}{\partial \mathbf{Q}_\alpha}) \cdot \overset{\circ}{\mathbf{q}}_\alpha$$

Using the chain rule, it may also be shown that:

$$J\phi_{t*}(\frac{\partial \bar{f}}{\partial \mathbf{S}}) = \frac{\partial f}{\partial \sigma} \quad \text{and} \quad J\phi_{t*}(\frac{\partial \bar{f}}{\partial \mathbf{Q}_\alpha}) = \frac{\partial f}{\partial \mathbf{q}_\alpha}$$

from which it follows that:

$$\dot{f} = \frac{\partial f}{\partial \sigma} : \overset{\circ}{\sigma} + \frac{\partial f}{\partial \mathbf{q}_\alpha} \cdot \overset{\circ}{\mathbf{q}}_\alpha \quad \blacksquare$$

Applying this lemma to (4.23), the consistency condition during plastic loading is given by:

$$\dot{f} = \frac{\partial f}{\partial \boldsymbol{\delta}} : \overset{\circ}{\boldsymbol{\delta}} + \frac{\partial f}{\partial \alpha} \overset{\circ}{\alpha} = 0 \tag{4.24}$$

where it is noted that, by definition:

$$\overset{\circ}{\alpha} = J^{-1}\phi_{t*}(\frac{d}{dt}\phi_t^{*}(J\alpha)) = J^{-1}\frac{d}{dt}(J\alpha) = \alpha \, tr(\mathbf{d}) + \dot{\alpha}$$

Before summarizing the model so far developed, observe that (4.11) has the alternative form:

$$\overset{\circ}{\sigma} = \mathbf{m}^{-1} : (\mathbf{d} - \mathbf{d}^P) = \mathbf{a} : (\mathbf{d} - \mathbf{d}^P) \tag{4.25}$$

where, by hypothesis, $\mathbf{a}$ is independent of σ, $\boldsymbol{\beta}$, and α.

Summary of Model

A general theory of isotropic and kinematic hardening can be constructed from (4.20) - (4.25). However, in the sequel only isotropic hardening is considered, in which case the model is summarized by:

$$\begin{aligned} \overset{\circ}{\sigma} &= \mathbf{a} : (\mathbf{d} - \mathbf{d}^P) \\ \mathbf{d}^P &= \lambda \frac{\partial f}{\partial \sigma}(\sigma, \mathbf{q}) \\ f &= \sigma' : \sigma' + k(\alpha) \\ \dot{f} &= \frac{\partial f}{\partial \sigma} : \overset{\circ}{\sigma} + \frac{\partial f}{\partial \alpha} \overset{\circ}{\alpha} = 0 \\ \overset{\circ}{\alpha} &= \mathbf{r}_1 : \mathbf{d}^P + r_2 [\mathbf{d}^P : \mathbf{d}^P]^{1/2} \end{aligned} \tag{4.26}$$

Kinematic hardening is easily incorporated into the theory and is neglected here only for simplicity.

Elasticity Tensor

The spatial elasticity tensor **a** is assumed to be a constant isotropic tensor:

$$a^{ijkl} = \lambda g^{ij} g^{kl} + \mu (g^{ik} g^{jl} + g^{il} g^{jk}) \tag{4.27}$$

where λ and μ are material constants and g^{ij} are components of the spatial metric tensor.

From $(4.26)_3$, it is noted that $\overset{\Delta}{f} = \frac{\partial f}{\partial \boldsymbol{\sigma}} : \overset{\circ}{\boldsymbol{\sigma}} = 2\boldsymbol{\sigma}' : \overset{\circ}{\boldsymbol{\sigma}}$

Introduction of (4.27) into $(4.26)_1$ and defining $\xi = \boldsymbol{\sigma}' : \overset{\circ}{\boldsymbol{\sigma}}$ results, after some manipulations, in:

$$\begin{aligned} \mathbf{d}^P &= H(\xi)\, \frac{8\mu}{h}\, (\boldsymbol{\sigma}' : \mathbf{d})\, \boldsymbol{\sigma}' \\ \overset{\circ}{\boldsymbol{\sigma}} &= \mathbf{a} : \mathbf{d} - H(\xi)\, \frac{16\mu^2}{h}\, (\boldsymbol{\sigma}' : \mathbf{d}) \boldsymbol{\sigma}' \end{aligned} \tag{4.28}$$

where $H(\cdot)$ is the Heaviside function and the hardening parameter $\hat{h}(\alpha)$ is given by:

$$h = 4\boldsymbol{\sigma}' : (\mathbf{a} : \boldsymbol{\sigma}') - 2\frac{\partial f}{\partial \alpha}\, [\mathbf{r}_1 : \boldsymbol{\sigma}' + r_2 (\boldsymbol{\sigma}' : \boldsymbol{\sigma}')^{1/2}] \tag{4.29}$$

Rate Equation for α

The following constitutive assumptions are adopted:

(i) $\mathbf{r}_1$ and r_2 are pressure insensitive and depend on $\boldsymbol{\sigma}$ through $\boldsymbol{\sigma}'$,

(ii) $(\boldsymbol{\sigma}', \alpha, \mathbf{d}^P)$ and $(-\boldsymbol{\sigma}', \alpha, -\mathbf{d}^P)$ give the same rates of α.

The simplest form of $(4.26)_5$ satisfying these conditions is given by:

$$\begin{aligned} \mathbf{r}_1(\boldsymbol{\sigma}', \alpha) &= r_1(\alpha) \boldsymbol{\sigma}' \\ r_2(\boldsymbol{\sigma}', \alpha) &= r_2(\alpha) \end{aligned} \tag{4.30}$$

Using these assumptions in $(4.26)_5$ together with $(4.28)_1$ and $(4.26)_3$ leads to:

$$\overset{\circ}{\alpha} = H(\xi)\, \frac{8\mu}{h}\, (\boldsymbol{\sigma}' : \mathbf{d}) \gamma \tag{4.31}$$

where

$$\gamma(\alpha) = -r_1(\alpha) k(\alpha) + r_2(\alpha)\, [\boldsymbol{\sigma}' : \boldsymbol{\sigma}']^{1/2} \tag{4.32}$$

Using the above assumptions in (4.29) together with (4.27) and $(4.26)_3$ leads to:

$$h(\alpha) = -2\frac{\partial k(\alpha)}{\partial \alpha}\gamma - 8\mu k(\alpha) \tag{4.33}$$

where γ is given by (4.32).

The model may be summarized by:

$$\overset{\circ}{\sigma} = \mathbf{a}:\mathbf{d} - H(\xi)\,\frac{16\mu^2}{h}\,(\sigma':\mathbf{d})\sigma' \tag{4.34}$$

$$\overset{\circ}{\alpha} = H(\xi)\,\frac{8\mu\gamma}{h}\,\sigma':\mathbf{d} \tag{4.35}$$

where γ is given by (4.32) and h by (4.33).

4.3 ISOTROPIC HARDENING MODEL FOR COMPUTATIONAL PURPOSES

In this section a simple constitutive model for isotropic hardening plasticity corresponding to a Prandtl-Reuss type approximation is developed by further specialization of the model discussed in Section 4.2. This simple model will, nevertheless, serve adequately as a constitutive basis for illustrating features of the algorithmic treatment of the boundary value problem of momentum balance to be developed subsequently. The model will also allow comparison with other studies which have used material models somewhat similar in form [5,7,8,9,13,14,16,17]. The resulting model is discussed in detail in [43]. It must be emphasized that the present model and those referenced above are similar only in general form. The thermodynamic arguments of Section 4.1 have lead to the use of the Truesdell rate of Cauchy stress, σ, and the Lie derivative of the Kirchhoff stress, $L_v(\tau)$, in the constitutive equations. Most of the models referenced above employ the co-rotational rate of Kirchhoff stress. These differences have been alluded to in Section 3.2 and will be discussed further in subsequent chapters.

The constitutive assumptions given by (4.30) are specialized to:

$$\begin{aligned} r_1(\alpha) &= 0 \\ r_2(\alpha) &= \left(\frac{2}{3}\right)^{1/2} \end{aligned} \tag{4.36}$$

For convenience $k(\alpha)$ in $(4.26)_3$ is replaced by the function $K(\alpha)$ such that:

$$k(\alpha) = -\frac{2}{3}K^2 \tag{4.37}$$

and by constitutive hypothesis it is supposed that:

$$\frac{\partial K}{\partial \alpha} = H = \frac{E\,E_t}{E - E_t} \tag{4.38}$$

where H is the hardening modulus given in terms of the elastic modulus E and a strain hardening modulus E_t.

In this case, substituting (4.36) into (4.33) and using the result in (4.34) gives, in components:

$$\overset{\circ}{\sigma}{}^{ij} = \left[a^{ijkl} - 2H(\xi)\,n^{ij}\,n^{kl}\right] d_{kl} \tag{4.39}$$

where a^{ijkl} is given by (4.27) and n^{ij} by:

$$n^{ij} = \frac{\sigma'^{ij}}{K\left[\frac{2}{3}\left(\frac{h}{3\mu} + 1\right)\right]^{1/2}}$$

Noting that $\overset{\circ}{K} = H\overset{\circ}{\alpha}$ and using (4.36) in (4.35) together with (4.32) and (4.33) leads to:

$$\overset{\circ}{K} = \frac{1}{K\left[\frac{1}{3\mu} + \frac{1}{H}\right]}\,(\sigma'^{ij}\,d_{ij}) \tag{4.40}$$

which is subject to the initial condition:

$$K(0) = \bar{\sigma}_y = \left(\frac{3}{2}\boldsymbol{\sigma}'_{yP} : \boldsymbol{\sigma}'_{yP}\right)^{1/2} \tag{4.41}$$

Finally, the yield condition $(4.26)_3$ is given by:

$$\frac{3}{2}\sigma'^{ij}\sigma'^{ij} - K^2 = 0 \tag{4.42}$$

Equations (4.38) - (4.42) represent the final model for isotropic hardening. Direct dependence on α has been eliminated essentially by the introduction of (4.40) and the final form allows comparison with many models in current use according to the comments given in the introduction to this section.

CHAPTER 5

LINEARIZATION IN THE MECHANICS OF SOLIDS

5.1 DEFINITION OF LINEARIZATION

Introduction

Linearization is one of the most useful areas of mathematical analysis, not only for carrying out the linearization of the field equations of thermomechamics but also because of its utility in defining numerical procedures for the solution of nonlinear problems. Many properties of nonlinear theories are linearization stable [3], which means that the linearized theory may be used to deduce significant results about the nonlinear theory.

There is an extensive literature in solid and structural mechanics dealing with concepts of linearization. However, many of the proposed methods are associated with the dropping of higher order terms and lack any sound analytical basis. As a result, the relationship of the linearized theories to their nonlinear progenitors is often obscure.

A consistent theory of linearization with special reference to finite elasticity is presented in [3]. The same theory with applications to nonlinear elastic plates is given in [44]. References [3,44] provide a consistent method of linearization with a sound analytical basis. The material in this chapter represents an application and extension of the results in [3,44].

In mechanics of solids and structures, the sources of nonlinearity lie in the constitutive equations and the kinematics of motion. The terms "material" and "geometric" nonlinearity are often employed in this context. The method of linearization discussed below will lead to consistent definitions of linearized operators associated with "material" and "geometric" effects without recourse to any ad hoc assumptions regarding approximations.

In the present work, equations expressing the dynamic equilibrium of loaded bodies employ the motion as the independent variable. The functional dependence on the motion of all tensors, including the stress tensor, appearing in the equilibrium equations provides the basis for consistent linearization [3,44]. However, for certain material classes, such as hypoelasticity (including rate forms of hyperelasticity) and rate-independent elasto-plasticity, some objective rate of a spatial stress tensor will be expressed as a homogeneous linear function of the spatial rate of deformation tensor. The functional dependence of the spatial stress tensor on the motion is not available.

The objectives of this chapter are three-fold:

(1) To consider an extension of the linearization theory presented in [3,44] to the constitutive equations of hypoelasticity and rate-independent elasto-plasticity.

(2) To consider the consequences of (1) in defining the linearization of the linear momentum balance equation as expressed through the weak form of a variational equation.

(3) To consider a numerical solution procedure for the boundary value problem of momentum balance for finite deformation hypoelasticity and rate-independent elasto-plasticity.

General Theory

Formally, it is found that a consistent linearization procedure may be based on Taylor's formula for C^1 functions, in which case an estimate of the difference in solutions of the nonlinear and linearized equations may be obtained by using the implicit function theorem together with Taylor's formula [3].

Definition 5.1

Let X and Y be Banach spaces and $\mathbf{f}: Q = X \to Y$ be a C^1 mapping. Let $\mathbf{x}' \in Q$, then the linearization of $\mathbf{f}$ about $\mathbf{x}'$ is given by:

$$L[\mathbf{f},\mathbf{u}]_{\mathbf{x}'} = \mathbf{f}(\mathbf{x}') + D\mathbf{f}(\mathbf{x}') \cdot \mathbf{u} \tag{5.1}$$

for $\mathbf{u} \in Q$.

In (5.1), $D\mathbf{f}(\mathbf{x}')$ is the Frechet derivative of $\mathbf{f}$ at $\mathbf{x}'$. In Euclidean spaces, $D\mathbf{f}$ is the linear map whose matrix in the standard bases is the Jacobian matrix of $\mathbf{f}$.

Note on Notation

Equation (5.1) will often be written simply as:

$$L[\mathbf{f},\mathbf{u}]_{\mathbf{x}'} = \mathbf{f} + D\mathbf{f}\cdot\mathbf{u}$$

where it is understood that the terms on the right-hand side are to be evaluated at $\mathbf{x}' \in Q$.

Assuming $\mathbf{f}$ to be Gateaux differentiable, $D\mathbf{f}\cdot\mathbf{u}$ can be related to the directional derivative of $\mathbf{f}$ in the direction $\mathbf{u}$, according to:

$$D\mathbf{f}(\mathbf{x}')\cdot\mathbf{u} = \frac{d}{d\epsilon}\,[\mathbf{f}(\mathbf{x}' + \epsilon\mathbf{u})]_{\epsilon=0} \tag{5.2}$$

It is noted that the directional derivative is a linear operator and follows rules similar to ordinary differentiation when applied to maps defined by composition [3,44]. Higher derivatives are defined by induction.

Motion Relative to a Deformed Configuration

The interpretation of (5.1) and (5.2) for the mechanics of a deformable medium are now considered. As suggested above, the motion will play the role of an independent variable in the field equations of thermomechanics.

Referring to definition 5.1, the point $\mathbf{x}' \in Q$ will be interpreted as a "reference state" defined by the motion $\psi_t : B \rightarrow N$, such that:

$$\mathbf{x}' = \psi_t(\mathbf{X})$$

and about which the field equations will be linearized. The reference state $\psi_t(\mathbf{X})$ is not necessarily, and in general will not be, an equilibrium configuration (that is to say, ψ is not necessarily an actual motion of the continuum).

Referring again to definition 5.1, the vector $\mathbf{u} \in Q$ will be interpreted as an infinitesimal deformation superimposed on the reference state $\psi_t(B)$, such that $\mathbf{u} : \psi_t(B) \rightarrow N$. A material form of $\mathbf{u}$ is defined by $\mathbf{U} = \mathbf{u} \circ \psi_t : B \rightarrow N$. The difference between these two forms is not emphasized in what follows and (with an abuse of notation) $\mathbf{u}$ is used for both cases (see Fig. 2).

Finally, the current configuration is defined by the regular motion $\boldsymbol{\phi}_t$ where:

$$\boldsymbol{\phi}_t = (\boldsymbol{\psi}_t + \mathbf{u}) : B \rightarrow N$$

such that

$$\mathbf{x} = \mathbf{x}' + \mathbf{u} \tag{5.3}$$

5.2 LINEARIZATION OF THE STRESS TENSOR FOR NONLINEAR ELASTICITY

It is demonstrated in Section 5.4 that linearization of the stress tensor is necessary in order to facilitate linearization of the boundary value problem of momentum balance. In the case of nonlinear elasticity, linearization of the stress tensor proceeds without difficulty, since the constitutive equations supply the functional dependence of the stress tensor on the motion. This is not the case for hypoelasticity (including the rate form of hyperelasticity) where an extension of the theory of linearization is needed and is considered in Section 5.3. The present section provides motivation for the ideas in Section 5.3 as well as illustrating the basic methodology of linearization.

The first Piola-Kirchhoff stress tensor, $\mathbf{P}$, has a constitutive equation for nonlinear elasticity given by $(3.1)_1$. That is,

$$\mathbf{P} = \rho_B \frac{\partial \Psi(\mathbf{F})}{\partial \mathbf{F}} \cdot \mathbf{g}^* \quad \text{in components} \quad P^{Aa} = \rho_B \frac{\partial \Psi}{\partial F^b{}_A} g^{ba} \tag{5.4}$$

Proposition 5.1

The linearization of the first Piola-Kirchhoff stress tensor $\mathbf{P}$ with nonlinear elastic constitutive equation (5.4), about the reference state $\mathbf{x}'$, is given by:

$$L[\mathbf{P},\mathbf{u}]_{\mathbf{x}'} = \mathbf{P} + \bar{\mathbf{A}} : \mathbf{Du} \tag{5.5}$$

where $\bar{\mathbf{A}}$ is the (first) material elasticity tensor defined by:

$$\bar{\mathbf{A}} = \rho_B \frac{\partial^2 \Psi}{\partial \mathbf{F}^2} \cdot \mathbf{g}^* \quad \text{with components} \quad \bar{A}^{aA}{}_b{}^B = \rho_B \frac{\partial^2 \Psi}{\partial F^c{}_A \partial F^b{}_B} g^{ca}$$

$$\text{and} \quad \mathbf{Du} = \frac{\partial \mathbf{u}}{\partial \mathbf{X}} \quad \text{or} \quad (\mathbf{Du})^a{}_A = u^a|_A$$

(Note: Referring to the "note on notation" following definition 5.1, the right-hand side of (5.5) is considered to be evaluated at the reference state $\mathbf{x}'$).

Proof

Using (5.2), it is noted that:

$$D\mathbf{F}\cdot\mathbf{u} = \frac{d}{d\epsilon}\left[\frac{\partial}{\partial \mathbf{x}}(\mathbf{x}' + \epsilon\mathbf{u})\right]_{\epsilon=0} = \mathbf{Du} \tag{5.6}$$

Applying (5.2) to (5.4) and using the properties of the directional derivative, it follows that:

$$D\mathbf{P}\cdot\mathbf{u} = \rho_B \frac{\partial}{\partial \mathbf{F}}\left(\frac{\partial \Psi}{\partial \mathbf{F}}\cdot\mathbf{g}^{*}\right):\left(D\mathbf{F}\cdot\mathbf{u}\right) = \bar{\mathbf{A}}:\mathbf{Du}$$

Referring to (5.1), the linearization of $\mathbf{P}$ is given by:

$$L\,[\mathbf{P},\mathbf{u}]_{\mathbf{x}'} = \mathbf{P} + \bar{\mathbf{A}}:\mathbf{Du} \quad \blacksquare$$

Recalling the definition of the fully symmetric (second) material elasticity tensor $\mathbf{A}$ given in Proposition 3.1, it is shown in [44] (for the case of rectangular cartesian coordinates - here generalized) that $\mathbf{A}$ and $\bar{\mathbf{A}}$ are related by:

$$\bar{A}^{aA}{}_{b}{}^{B} = A^{CAEB} F^{a}{}_{C} F^{m}{}_{E}\, g_{mb} + S^{AB} g^{a}{}_{b}$$

which emphasizes the dependence of $\bar{\mathbf{A}}$ on the stress tensor defined in the reference state $\psi_t(B)$.

Proposition 5.2

The linearization of the Cauchy stress tensor $\boldsymbol{\sigma}$, consistent with the constitutive equation (5.4), about the reference state $\mathbf{x}'$, is given by:

$$L\,[\boldsymbol{\sigma},\mathbf{u}]_{\mathbf{x}'} = \boldsymbol{\sigma}\,[1 - tr(\nabla\mathbf{u})] + 2(\boldsymbol{\sigma}\cdot\nabla\mathbf{u}^{T})^{S} + \mathbf{a}:\nabla^{S}\mathbf{u} \tag{5.7}$$

where $\mathbf{a}$ is the spatial elasticity tensor defined in Proposition 3.3 and

$$\nabla\mathbf{u} = \frac{\partial \mathbf{u}}{\partial \mathbf{x}} \quad \text{in components} \quad (\nabla\mathbf{u})^{a}{}_{b} = u^{a}|_{b}$$

Proof

For contravariant components of stress, by Definition 2.11, $\boldsymbol{\sigma} = J^{-1}\mathbf{F}\cdot\mathbf{S}\cdot\mathbf{F}^{T}$. Using (5.2), and the properties of the directional derivative, it follows that:

$$D\sigma \cdot \mathbf{u} \; = \; -\sigma \, tr(\nabla \mathbf{u}) + 2(\sigma \cdot \nabla \mathbf{u}^T)^S + J^{-1}\mathbf{F} \cdot (D\mathbf{S} \cdot \mathbf{u}) \cdot \mathbf{F}^T$$

where $D(J) \cdot \mathbf{u} = J tr(\nabla \mathbf{u})$ has been used. Using the constitutive equation for $\mathbf{S}$ $(3.1)_2$, the definition of $\mathbf{A}$ (Proposition 3.1) and the relationship between $\mathbf{A}$ and $\mathbf{a}$ (Proposition 3.4), it is easy to show that:

$$J^{-1}\mathbf{F}\,(D\mathbf{S} \cdot \mathbf{u})\, F^T \; = \; \mathbf{a} : \nabla^S \mathbf{u}$$

Referring to (5.1), the linearization of σ is given by:

$$L[\sigma, \mathbf{u}]_{\mathbf{x}'} \; = \; \sigma[1 - tr(\nabla \mathbf{u})] + 2(\sigma \cdot \nabla \mathbf{u}^T)^S + \mathbf{a} : \nabla^S \mathbf{u} \qquad \blacksquare$$

In the next section, which deals with linearization of hypoelastic material models, a theorem connecting the linearization of material and spatial tensors is developed, from which the above result is easily deduced.

5.3 LINEARIZATION OF THE STRESS TENSOR FOR HYPOELASTICITY

For hypoelastic constitutive equations (including the rate form of hyperelasticity c.f. (3.7) and rate-independent elasto-plasticity c.f. (4.31)), the methodology of the previous section is not effective. For such material models, the functional dependence of the stress tensor on the motion is not available since the rate constitutive equations are not directly integrable, in general. The objective of this section is to extend the methodology of linearization to cover the case of hypoelasticity.

Some preliminary results are required. It is useful to provide an alternative interpretation of the directional derivative of a material tensor as given by (5.2).

Definition 5.2

Let $\psi_\epsilon = (\psi_t + \epsilon \mathbf{u}) : B \rightarrow N$ and let Λ be a material tensor defined on B. The directional derivative of Λ in the direction of an incremental motion $\mathbf{u}$ superimposed on the reference state $\psi_t(B)$ is defined to be:

$$D\Lambda(\mathbf{X}, t) \cdot \mathbf{u} \; = \; \left[\frac{d}{d\epsilon} \Lambda\,(\psi_\epsilon^{-1}, t) \right]_{\epsilon = 0} \qquad (5.8)$$

Figure 2 illustrates the configurations utilized in this definition.

Theorem 5.1

The directional derivative with respect to the incremental motion $\mathbf{u}$ of the Kirchhoff stress tensor τ, defined on the reference state $\psi_t(B)$, and the associated second Piola-Kirchhoff stress tensor S, defined on B, are related by:

$$DS \cdot \mathbf{u} = \Psi_t^{*}[L_{\mathbf{u}}(\tau)] \tag{5.9}$$

where

$$L_{\mathbf{u}}(\tau) = \left[\frac{d}{d\epsilon}(\psi_\epsilon \circ \psi_t^{-1})^{*}(\tau_\epsilon)\right]_{\epsilon=0} \tag{5.10}$$

and

$$\psi_\epsilon = \psi_t + \epsilon\,\mathbf{u} \tag{5.11}$$

$$\tau_\epsilon = \hat{\tau}(\psi_\epsilon, t) \tag{5.12}$$

(See Figure 2.)

Proof

Using (5.10), it follows that:

$$\begin{aligned}
\psi_t^{*}[L_{\mathbf{u}}(\tau)] &= \psi_t^{*}\left\{\left[\frac{d}{d\epsilon}(\psi_\epsilon \circ \psi_t^{-1})^{*}(\tau_\epsilon)\right]_{\epsilon=0}\right\} \\
&= \left[\frac{d}{d\epsilon}\psi_t^{*}\left\{\left(\psi_\epsilon \circ \psi_t^{-1}\right)^{*}\left(\tau_\epsilon\right)\right\}\right]_{\epsilon=0} \\
&= \left[\frac{d}{d\epsilon}\psi_\epsilon^{*}\left(\tau_\epsilon\right)\right]_{\epsilon=0} \\
&= \left[\frac{d}{d\epsilon}\mathrm{S}\left(\psi_\epsilon^{-1}, t\right)\right]_{\epsilon=0} \\
&= DS \cdot \mathbf{u}
\end{aligned}$$

where the results of definition 5.2 have been used ■

Coordinate formulas may be developed for $L_{\mathbf{u}}(\tau)$ as follows. From (5.6), it is noted that:

$$DF \cdot \mathbf{u} = \frac{\partial \mathbf{u}}{\partial \mathbf{X}} = \mathbf{Du}$$

which implies,

$$D\mathbf{F}^{-1}\cdot\mathbf{u} \;=\; -\mathbf{F}^{-1}\cdot\left(\frac{\partial\mathbf{u}}{\partial\mathbf{x}}\right) \;=\; -\mathbf{F}^{-1}\cdot\nabla\mathbf{u} \tag{5.13}$$

Considering contravariant components of τ (and referring to Figure 2 for the geometric view), it follows that:

$$(\psi_\epsilon \circ \psi_t^{-1})^*(\tau_\epsilon) \;=\; \mathbf{F}\cdot\mathbf{F}_\epsilon^{-1}\cdot\tau_\epsilon\cdot\mathbf{F}_\epsilon^{-T}\cdot\mathbf{F}^T$$

where $\mathbf{F} \;=\; \dfrac{\partial\psi_t}{\partial\mathbf{X}}$ and $\mathbf{F}_\epsilon \;=\; \dfrac{\partial\psi_\epsilon}{\partial\mathbf{X}}$.

Substituting this result into (5.10) and noting that by definition,

$$\left[\frac{d}{d\epsilon}\tau_\epsilon\right]_{\epsilon=0} \;=\; D\tau\cdot\mathbf{u}$$

also, from (5.13),

$$\left[\frac{d}{d\epsilon}\mathbf{F}_\epsilon^{-1}\right]_{\epsilon=0} \;=\; -\mathbf{F}^{-1}\cdot\nabla\mathbf{u}$$

and observing that $[\mathbf{F}_\epsilon]_{\epsilon=0} = \mathbf{F}$ and $[\tau_\epsilon]_{\epsilon=0} = \tau$, it follows that:

$$L_\mathbf{u}(\tau) \;=\; D\tau\cdot\mathbf{u} - \nabla\mathbf{u}\cdot\tau - \tau\cdot\nabla\mathbf{u}^T$$

in components:

$$[L_\mathbf{u}(\tau)]^{ab} \;=\; (D\tau\cdot\mathbf{u})^{ab} - \tau^{cb}u^a|_c - \tau^{ac}u^b|_c \tag{5.14}$$

Similarly, it may be shown that:

$$[L_\mathbf{u}(\tau)]_{ab} \;=\; (D\tau\cdot\mathbf{u})_{ab} + \tau_{cb}u^c|_a + \tau_{ac}u^c|_b \tag{5.15}$$

and

$$[L_\mathbf{u}(\tau)]^a{}_b \;=\; (D\tau\cdot\mathbf{u})^a{}_b - \tau^c{}_b\,u^a|_c + \tau^a{}_c\,u^c|_b \tag{5.16}$$

It may also be shown that:

$$L_\mathbf{u}(\tau) \;=\; L_\mathbf{u}(J\sigma) \;=\; J\sigma\,tr(\nabla\mathbf{u}) + J\,L_\mathbf{u}(\sigma) \tag{5.17}$$

where $\boldsymbol{\sigma}$ is the Cauchy stress tensor and $J = \det(\mathbf{F})$. The proof of (5.17) follows from the definition of $L_\mathbf{u}(\cdot)$ given by (5.10) and is omitted here.

Example 5.1

This example considers the linearization of the Cauchy stress tensor $\boldsymbol{\sigma}$ for nonlinear elasticity (c.f. Proposition 5.2). By theorem 5.1 and (5.17), it follows that:

$$L_{\mathbf{u}}(\sigma) + \sigma\, tr(\nabla \mathbf{u}) = J^{-1}\psi_{t*}(D\mathbf{S}\cdot\mathbf{u})$$

Considering contravariant components of stress and noting (as in Proposition 5.2) that:

$$J^{-1}\psi_{t*}(D\mathbf{S}\cdot\mathbf{u}) = \mathbf{a} : \nabla^S\mathbf{u}$$

where $\mathbf{a}$ is the spatial elasticity tensor evaluated on $\psi_t(B)$, then:

$$D\sigma\cdot\mathbf{u} - \nabla\mathbf{u}\cdot\sigma - \sigma\cdot\nabla\mathbf{u}^T + \sigma\, tr(\nabla\mathbf{u}) = \mathbf{a} : \nabla^S\mathbf{u}$$

Solving for $D\sigma\cdot\mathbf{u}$, the final result (5.7) follows immediately.

Proposition 5.3

The directional derivatives of the contravariant components of **S** and **P** are related by:

$$D\mathbf{S}\cdot\mathbf{u} = \left[D\mathbf{P}\cdot\mathbf{u} - \mathbf{P}\cdot\nabla\mathbf{u}^T\right]\cdot\mathbf{F}^{-T} \tag{5.18}$$

Proof

From the relationship $\mathbf{S} = \mathbf{P}\cdot\mathbf{F}^{-T}$ and (5.13), the result follows immediately.

Using (5.18) in (5.9), it follows that Theorem 5.1 has the alternative form:

$$D\mathbf{P}\cdot\mathbf{u} = \mathbf{P}\cdot\nabla\mathbf{u}^T + \psi_t^{*}[L_{\mathbf{u}}(\tau)]\cdot\mathbf{F}^T \tag{5.19}$$

when contravariant components of stress are considered. The preliminary results are now in hand. However, before proceeding to the final conclusions of this section, a unification of notation for constitutive equations will be helpful.

Unification of Notation for Constitutive Equations

Hypoelasticity (including the rate forms of hyperelasticity and rate-independent elasto-plasticity) is characterized in this section by the universal constitutive equation:

$$L_{\mathbf{v}}(\tau) = L(J,\tau) : \mathbf{d} \tag{5.20}$$

where L is a fourth order isotropic tensor. All the constitutive equations considered in Chapters 3 and 4 may be expressed in the form (5.20). The dependence of L on J and τ will reflect the difference between the various forms of constitutive equations, examples are given below.

Linearization of Rate Constitutive Equations

It is demonstrated in Section 5.4 that $D\mathbf{P}\cdot\mathbf{u}$ is required for the linearization of the boundary value problem of momentum balance. Thus, the issue of concern here is the evaluation of (5.19) and of $L_{\mathbf{u}}(\tau)$ in particular.

Recalling Definition 2.8, the Lie derivative of τ with respect to $\mathbf{v}$ is given by:

$$L_{\mathbf{v}}(\tau) = \left[\frac{d}{ds}(\phi_s \circ \phi_t^{-1})^*(\tau_s)\right]_{s=t} \tag{5.21}$$

where it is noted that $\left[\frac{d\phi_s}{ds}\right]_{s=t} = \mathbf{v}$ and τ_s means τ evaluated a time s.

Consider the definition of $L_{\mathbf{u}}(\tau)$ given by (5.10); that is,

$$L_{\mathbf{u}}(\tau) = \left[\frac{d}{d\epsilon}(\psi_\epsilon \circ \psi_\epsilon^{-1})^*(\tau_\epsilon)\right]_{\epsilon=0} \tag{5.22}$$

where it is noted that $\left[\frac{d\psi_\epsilon}{d\epsilon}\right]_{\epsilon=0} = \mathbf{u}$, from (5.11), and τ_ϵ means τ evaluated on configuration $\psi_\epsilon + \epsilon\mathbf{u}$.

Clearly, (5.21) and (5.22) define Lie differentiation of τ with respect to $\mathbf{v}$ and $\mathbf{u}$, respectively.

Introducing the universal constitutive equation (5.20), it follows that:

$$\left[\frac{d}{ds}\left(\phi_s \circ \phi_t^{-1}\right)^*\left(\tau_s\right)\right]_{s=t} = L:\nabla^S\mathbf{v} \tag{5.23}$$

Since (5.23) is assumed by constitutive hypothesis to be valid for any vector $\mathbf{v}$ emanating from the current configuration, then the approximation:

$$L_{\mathbf{u}}(\tau) \approx L:\nabla^S\mathbf{u} \tag{5.24}$$

is immediately suggested. Furthermore,

$$\lim_{\psi_t \to \phi_t} L_{\mathbf{u}}(\tau) = L : \nabla^S \mathbf{u} \tag{5.25}$$

It is shown in the next section that practical solution schemes for the boundary value problem of momentum balance, based on the above theory, do satisfy the limit condition of (5.25).

Returning to (5.19) and using the approximation (5.24) results in:

$$D\mathbf{P} \cdot \mathbf{u} = \mathbf{P} \cdot \nabla \mathbf{u}^T + \psi_t^* \left[L : \nabla^S \mathbf{u} \right] \cdot \mathbf{F}^T \tag{5.26}$$

Expanding (5.26) for contravariant components of stress in (5.20) (see pull back, Definition 2.2) and noting that $\mathbf{P} = \mathbf{F}^{-1} \cdot \boldsymbol{\tau}$ leads to:

$$D\mathbf{P} \cdot \mathbf{u} = \mathbf{F}^{-1} \cdot \left[\boldsymbol{\tau} \cdot \nabla \mathbf{u}^T + L : \nabla^S \mathbf{u} \right] \tag{5.27}$$

Some examples of the application of (5.27) follow the next proposition.

Proposition 5.4

Considering *contravariant* components of stress, then:

$$\begin{aligned}
(a) \quad L_{\mathbf{v}}(\boldsymbol{\tau}) &= J\overset{\circ}{\boldsymbol{\sigma}} \\
(b) \quad &= J[L_{\mathbf{v}}(\boldsymbol{\sigma}) + \boldsymbol{\sigma}\, tr(\mathbf{d})] \\
(c) \quad &= \overset{\triangledown}{\boldsymbol{\tau}} - 2(\boldsymbol{\tau} \cdot \mathbf{d})^S \\
(d) \quad &= J[\overset{\triangledown}{\boldsymbol{\sigma}} - 2(\boldsymbol{\sigma} \cdot \mathbf{d})^S + \boldsymbol{\sigma}\, tr(\mathbf{d})]
\end{aligned}$$

Proof

The proof, which is straight-forward, follows from Definitions 2.10, 2.12, and 2.14 and is omitted here.

Proposition 5.5

If constitutive equations are given in the form:

$$\begin{aligned}
(a) \quad \bar{\mathbf{a}} : \mathbf{d} &= \overset{\circ}{\boldsymbol{\sigma}} \\
(b) \quad &= L_{\mathbf{v}}(\boldsymbol{\sigma}) \\
(c) \quad &= \overset{\triangledown}{\boldsymbol{\tau}}
\end{aligned}$$

$$(d) \qquad = \overset{\nabla}{\sigma}$$

then the corresponding forms of the directional derivative of **P** are given by:

$$
\begin{aligned}
(a')\quad D\mathbf{P}\cdot\mathbf{u} &= \mathbf{F}^{-1}\cdot[\tau\cdot\nabla\mathbf{u}^T + J\bar{\mathbf{a}}:\nabla^S\mathbf{u}] \\
(b') \qquad &= \mathbf{F}^{-1}\cdot[\tau\cdot\nabla\mathbf{u}^T + J\bar{\mathbf{a}}:\nabla^S\mathbf{u} + \tau\, tr(\nabla^S\mathbf{u})] \\
(c') \qquad &= \mathbf{F}^{-1}\cdot[\tau\cdot\nabla\mathbf{u}^T + \bar{\mathbf{a}}:\nabla^S\mathbf{u} - 2(\tau\cdot\nabla^S\mathbf{u})^S] \\
(d') \qquad &= \mathbf{F}^{-1}\cdot[\tau\cdot\nabla\mathbf{u}^T + J\bar{\mathbf{a}}:\nabla^S\mathbf{u} - 2(\tau\cdot\nabla^S\mathbf{u})^S + \tau\, tr(\nabla^S\mathbf{u})]
\end{aligned}
$$

Proof

Consider Case (c) as being representative. By Proposition 5.4(c) and using (c) above, it follows that:

$$L_v(\tau) = \bar{\mathbf{a}}:\mathbf{d} - 2(\tau\cdot\mathbf{d})^S = L:\mathbf{d}$$

It follows immediately that $L:\nabla^S\mathbf{u}$ occurring in (5.27) may now be replaced by:

$$L:\nabla^S\mathbf{u} = \bar{\mathbf{a}}:\nabla^S\mathbf{u} - 2(\tau\cdot\nabla^S\mathbf{u})^S$$

which, when substituted into (5.27), produces the required result.

The other results of this proposition follow similarly ∎

NOTE: No restrictions on the form of $\bar{\mathbf{a}}$ are implied. In general, $\bar{\mathbf{a}}$ may depend on the stress tensor and the results of the above proposition apply to all the constitutive equations considered in Chapters 3 and 4 by suitable specialization of $\bar{\mathbf{a}}$. Furthermore, component forms of L corresponding to (a) - (d) may easily be determined if $\bar{\mathbf{a}}$ is specified. However, the component form of L (which was introduced only for convenience of presentation) is not required for this development. The above results play a central role in the subject matter of the next section.

5.4 LINEARIZATION OF THE BOUNDARY VALUE PROBLEM OF MOMENTUM BALANCE

A weak form equivalent to the boundary value problem of momentum balance for the current configuration $\boldsymbol{\phi}_t(B)$, is defined in Section 2.4 by (2.1) or (2.2), such that:

$$G(\mathbf{x},\eta) = 0 \tag{5.28}$$

In general, this form is nonlinear with respect to the motion $\mathbf{x}$ but has a locally linear approximation about the reference state $\mathbf{x}' = \boldsymbol{\psi}_t$ given by (5.1):

$$L[G,\mathbf{u}]_{x'} = 0 \tag{5.29}$$

where

$$L[G,\mathbf{u}]_{\mathbf{x}'} = G(\mathbf{x}',\boldsymbol{\eta}) + DG(\mathbf{x}',\boldsymbol{\eta}) \cdot \mathbf{u} \tag{5.30}$$

In evaluating $DG \cdot \mathbf{u}$, it is convenient to work with the material form given by (2.2) since, in this case, the domains of the integrals are independent of the motion. It is assumed for simplicity that $\mathbf{B}$ and $\bar{\tau}$ occurring in (2.2) are independent of the motion (this assumption is not essential to the development). Then, referring to (2.2), it follows that:

$$DG \cdot \mathbf{u} = \int_B D[tr(\mathbf{P} \cdot \mathbf{D}\boldsymbol{\eta})] \cdot \mathbf{u}\, dV + \int_B \rho_B \ddot{\mathbf{u}} \cdot \boldsymbol{\eta}\, dV \tag{5.31}$$

where $D(\ddot{\mathbf{x}}) \cdot \mathbf{u} = \ddot{\mathbf{u}}$ has been used.

In general, $\boldsymbol{\eta}$ will depend on the motion $\mathbf{x}$ and, accordingly, will make a contribution to $D[tr(\mathbf{P} \cdot \mathbf{D}\boldsymbol{\eta})] \cdot \mathbf{u}$. For example, (5.29) may be used as the basis for formulating problems of reduced dimensionality, such as beam or plate theory [44], in which the motion $\mathbf{x}$ is restricted to reflect the characteristics of the particular structural element under consideration. In this case and for Galerkin type approximations, $\boldsymbol{\eta}$ may depend nonlinearly on the parameters characterizing the motion [45,46]. In the present study, a continuum type approximation is used which places no restrictions on the motion $\mathbf{x}$ and considers $\boldsymbol{\eta}$ independent of the motion. Accordingly,

$$\int_B D[tr(\mathbf{P} \cdot \mathbf{D}\boldsymbol{\eta})] \cdot \mathbf{u}\, dV = \int_B tr([D\mathbf{P} \cdot \mathbf{u}] \cdot \mathbf{D}\boldsymbol{\eta})\, dV \tag{5.32}$$

Combining (5.29) - (5.32), the locally linear approximation to the weak form about the motion $\mathbf{x}' = \boldsymbol{\psi}_t$, is given by:

$$\int_B tr([D\mathbf{P} \cdot \mathbf{u}] \cdot \mathbf{D}\boldsymbol{\eta})\, dV + \int_B \rho_B \ddot{\mathbf{u}} \cdot \boldsymbol{\eta}\, dV = -G(\mathbf{x}',\boldsymbol{\eta}) \tag{5.33}$$

Two forms of (5.33) are of particular interest:

(i) Material form. Equation (2.2) is used for the evaluation of $-G(\mathbf{x}',\boldsymbol{\eta})$,

$$\int_B tr([D\mathbf{P}\cdot\mathbf{u}]\cdot\mathbf{D}\boldsymbol{\eta})\,dV + \int_B \rho_B\,\ddot{\mathbf{u}}\cdot\boldsymbol{\eta}\,dV =$$
$$\int_{\partial B_\tau^-} \bar{\boldsymbol{\tau}}\cdot\boldsymbol{\eta}\,dA - \int_B tr(\mathbf{P}\cdot\mathbf{D}\boldsymbol{\eta})\,dV - \int_B \rho_B(\dot{\mathbf{V}}-\mathbf{B})\cdot\boldsymbol{\eta}\,dV \tag{5.34}$$

(ii) Spatial form. Transforming the left-hand side of (5.33) from material to spatial coordinates (associated with the reference state $\boldsymbol{\psi}_t(B)$), using $dV = J^{-1}dv$ and (2.1) for the evaluation of $-G(\mathbf{x}',\boldsymbol{\eta})$,

$$\int_{\boldsymbol{\psi}_t(B)} J^{-1}\,tr([D\mathbf{P}\cdot\mathbf{u}]\cdot\mathbf{D}\boldsymbol{\eta})\,dv + \int_{\boldsymbol{\psi}_t(B)} \rho\,\ddot{\mathbf{u}}\cdot\boldsymbol{\eta}\,dv =$$
$$\int_{\partial(\boldsymbol{\psi}_t(B))_t^-} \bar{\mathbf{t}}\cdot\boldsymbol{\eta}\,da - \int_{\boldsymbol{\psi}_t(B)} tr\,(\boldsymbol{\sigma}\cdot\nabla\boldsymbol{\eta})\,dv - \int_{\boldsymbol{\psi}_t(B)} \rho(\dot{\mathbf{v}}-\mathbf{b})\cdot\boldsymbol{\eta}\,dv \tag{5.35}$$

where $\rho = J^{-1}\rho_B$ is the density in $\boldsymbol{\psi}_t(B)$.

The work of Sections 5.2 and 5.3 is now brought to bear on $D\mathbf{P}\cdot\mathbf{u}$ occurring in (5.34) and (5.35). For example, for nonlinear elasticity, Proposition 5.1 demonstrates that $D\mathbf{P}\cdot\mathbf{u} = \bar{\mathbf{A}}:\mathbf{Du}$, where $\bar{\mathbf{A}}$ is the (first) material elasticity tensor.

Applications to hypoelasticity (including rate forms of hyperelasticity and rate-independent elasto-plasticity) are accomplished by the following discussion.

Proposition 5.6

If constitutive equations are given in the form:

$$\begin{array}{lrl}
(a) & \bar{\mathbf{a}}:\mathbf{d} = & \overset{\circ}{\boldsymbol{\sigma}} \\
(b) & = & L_{\mathbf{v}}(\boldsymbol{\tau}) \\
(c) & = & L_{\mathbf{v}}(\boldsymbol{\sigma}) \\
(d) & = & \overset{\nabla}{\boldsymbol{\tau}} \\
(e) & = & \overset{\nabla}{\boldsymbol{\sigma}}
\end{array}$$

then the kernel $J^{-1}tr([D\mathbf{P}\cdot\mathbf{u}]\cdot\mathbf{D}\boldsymbol{\eta})$ occurring in (5.35) has the corresponding forms:

$$\begin{array}{lrl}
(a') & J^{-1}tr([D\mathbf{P}\cdot\mathbf{u}]\cdot D\boldsymbol{\eta}) = & tr[(\boldsymbol{\sigma}\cdot\nabla\mathbf{u}^T + \bar{\mathbf{a}}:\nabla^S\mathbf{u})\cdot\nabla\boldsymbol{\eta}] \\
(b') & = & tr[(\boldsymbol{\sigma}\cdot\nabla\mathbf{u}^T + J^{-1}\bar{\mathbf{a}}:\nabla^S\mathbf{u})\cdot\nabla\boldsymbol{\eta}] \\
(c') & = & tr[(\boldsymbol{\sigma}\cdot\nabla\mathbf{u}^T + \bar{\mathbf{a}}:\nabla^S\mathbf{u} + \boldsymbol{\sigma}\,tr(\nabla^S\mathbf{u}))\cdot\nabla\boldsymbol{\eta}] \\
(d') & = & tr[(\boldsymbol{\sigma}\cdot\nabla\mathbf{u}^T + J^{-1}\bar{\mathbf{a}}:\nabla^S\mathbf{u} - 2(\boldsymbol{\sigma}\cdot\nabla^S\mathbf{u})^S)\cdot\nabla\boldsymbol{\eta}]
\end{array}$$

$$(e') \qquad = tr[(\sigma \cdot \nabla \mathbf{u}^T + \bar{\mathbf{a}} : \nabla^S \mathbf{u} - 2(\sigma \cdot \nabla^S \mathbf{u})^S + \sigma\, tr(\nabla^S \mathbf{u})) \cdot \nabla \boldsymbol{\eta}]$$

Proof

Proposition 5.5 is used to evaluate $D\mathbf{P} \cdot \mathbf{u}$. Noting that $\sigma = J^{-1}\tau$ and $tr(\mathbf{F}^{-1}[\cdot] \cdot \mathbf{D}\boldsymbol{\eta}) = tr([\cdot] \cdot \nabla \boldsymbol{\eta})$ supplies the required results ∎

Since all the tensors in (a') - (e') are spatial tensors defined on the reference state $\psi_t(B)$, it is convenient to use the spatial version of the linearized weak form given by (5.35). The results (a') - (e') may be substituted directly. For example, if by constitutive hypothesis $\overset{\nabla}{\tau} = \bar{\mathbf{a}} : \mathbf{d}$, then by Proposition 5.6 (d prime) and (5.35), the linearized weak form is given by:

$$\int_{\psi_t(B)} tr[(\sigma \cdot \nabla \mathbf{u}^T + J^{-1}\bar{\mathbf{a}} : \nabla^S u - 2(\sigma \cdot \nabla^S \mathbf{u})^S) \cdot \nabla \boldsymbol{\eta}]dv + \int_{\psi_t(B)} \rho \ddot{\mathbf{u}} \cdot \boldsymbol{\eta}\, dv =$$
$$\int_{\partial(\psi_t(B))_t^-} \bar{\mathbf{t}} \cdot \boldsymbol{\eta}\, da - \int_{\psi_t(B)} tr(\sigma \cdot \nabla \boldsymbol{\eta})dv - \int_{\psi_t(B)} \rho(\dot{\mathbf{v}} - \mathbf{b}) \cdot \boldsymbol{\eta}\, dv$$

which is linear in the incremental motion $\mathbf{u}$.

Note on Symmetry

Linearized weak forms associated with (a), (b) and (d) in Proposition 5.5 are symmetric with respect to interchange of indices between $\nabla \mathbf{u}$ and $\nabla \boldsymbol{\eta}$ (providing $\bar{\mathbf{a}}$ is fully symmetric) whereas (c) and (e) are not.

It is interesting to note the correspondence between these conclusions and those reported for the symmetry of the rate problem discussed in Section 2.5.

5.5 A NUMERICAL SOLUTION PROCEDURE

The results of Section 5.4 provide a basis for defining a Newton-Raphson iteration scheme for the solution of the boundary value problem of momentum balance.

It has been shown that the weak form (2.1) or (2.2) has a locally linear approximation given by (5.29), such that:

$$G(\mathbf{x}',\boldsymbol{\eta}) + DG(\mathbf{x}',\boldsymbol{\eta}) \cdot \mathbf{u} = 0 \tag{5.36}$$

However, if (5.36) is solved for $\mathbf{u}$, then, in general, $G(\mathbf{x}' + \mathbf{u},\boldsymbol{\eta}) \neq 0$. That is, $\mathbf{x}' + \mathbf{u} \neq \mathbf{x}$, where $\mathbf{x}$ is the solution corresponding to the actual motion $\boldsymbol{\phi}_t$.

A Newton-Raphson iteration procedure based on (5.36) and schematically represented by:

$$
\begin{aligned}
&(i) \quad G^i = G(\mathbf{x}^i, \boldsymbol{\eta}), \qquad DG^i \cdot \mathbf{u}^{i+1} = \mathbf{K}^i \cdot \mathbf{u}^{i+1} \\
&(ii) \quad \mathbf{u}^{i+1} = -(\mathbf{K}^i)^{-1} \cdot G^i \\
&(iii) \quad \mathbf{x}^{i+1} = \mathbf{x}^i + \mathbf{u}^{i+1} \\
&(iv) \quad i \leftarrow i+1, \ go\ (i)
\end{aligned}
$$

will converge to the solution, provided that G is well-behaved between $\mathbf{x}^o$ and $\mathbf{x}$.

The linearized weak form (5.35) may be projected into a finite dimensional setting by using a finite element spatial discretization in which the resulting variables are the nodal incremental motions. Some implementation details are given in Chapter 7. In this finite dimensional case, the updating of the nodal motion proceeds according to (iii) in the scheme above. However, the spatial stress field must also be updated at the finite element quadrature points. Since the rate constitutive equations are not integrable, in general, this stress update cannot be based directly on the nodal motion update. It is thus necessary to introduce a numerical algorithm for the time integration of the rate constitutive equations in order to effect the update of the spatial stress field. Such a numerical integration algorithm, which takes proper account of the effects of finite deformation occuring over the time steps, is the subject of the next chapter.

CHAPTER 6

INTEGRATION OF RATE CONSTITUTIVE EQUATIONS

6.1 INTRODUCTION

This chapter is concerned with the evolution of the spatial stress *tensor*. That is to say, it is concerned with the evolution of the components of the tensor *and* of the coordinate system basis vectors to which the components are referred.

In the previous chapter, a Newton-Raphson solution procedure for the momentum balance equation, based on a finite element spatial discretization of the linearized weak form, was proposed. This finite dimensional setting of the problem yields the nodal incremental motion as the independent variable. Updating of the nodal motion is direct. However, the spatial stress tensor and internal variables must also be updated at the spatial integration points by temporal integration of their respective evolution equations. It is to the temporal integration of the rate constitutive equations for the spatial stress field that the present chapter is addressed.

6.2 MATHEMATICAL BASIS

The objective of this section is to clarify the mathematical description of the mechanics underlying the evolution of the spatial stress field. In so doing, an algorithmic treatment of the integration problem is suggested. However, discussion of algorithmic issues, such as incremental objectivity, stability and accuracy, are deferred until Section 6.3.

The constitutive equations which govern the evolution of the stress tensor are here restricted to be those of hypoelasticity (including rate forms of hyperelasticity) as described in Chapter 3. Generalization of the following results for rate-independent elasto-plasticity is considered in Section 6.4.

In order to motivate the methodology of the present section, it is useful to anticipate the algorithmic treatment of the problem. The loading process applied to the continuum is considered to be discretized in time. In particular, it is assumed that at time t_n the configuration

$\phi_n(B)$, denoted Ω_n, and the stress tensor σ_n at each material point in Ω_n is known. At time t_{n+1}, the continuum occupies the known configuration $\phi_{n+1}(B)$, denoted Ω_{n+1}. The problem is to determine the corresponding σ_{n+1} at each material (integration) point in Ω_{n+1}. Some implicit numerical integration schemes [47,48] applied to spatial rate constitutive equations have employed difference operations on the stress *components* of the form $[\sigma^{ij}_{n+1} - \sigma^{ij}_n]$. However, such quantities are of limited value. Consider the following argument.

Convected, spatial and material coordinate systems have been defined in Section 2.1. The covariant basis vectors corresponding to these coordinate systems are denoted $\bar{\mathbf{g}}_\alpha$, $\mathbf{g}_a$, and $\mathbf{G}_A$, respectively. In Proposition 3.2 it has been noted that $\bar{\mathbf{g}}_\alpha$ evolves according to:

$$\frac{d}{dt}(\bar{\mathbf{g}}_\alpha) = v^\beta|_\alpha \bar{\mathbf{g}}_\beta \tag{6.1}$$

Furthermore, for a fixed material particle ($\mathbf{X}$ = constant), the basis vectors associated with the spatial coordinate system $\mathbf{g}_a$ evolve according to:

$$\frac{d}{dt}(\mathbf{g}_a) = \frac{\partial}{\partial t}(\mathbf{g}_a) + \frac{\partial \mathbf{g}_a}{\partial x^b} v^b = v^b \Gamma^m_{ab} \mathbf{g}_m \tag{6.2}$$

(although it is clear by Proposition 2.2 that the spatial metric tensor $\mathbf{g}$ satisfies $\dot{\mathbf{g}} = 0$).

Noting that:

$$\sigma = \bar{\sigma}^{\alpha\beta} \bar{\mathbf{g}}_\alpha \otimes \bar{\mathbf{g}}_\beta = \sigma^{ab} \mathbf{g}_a \otimes \mathbf{g}_b$$

it is seen that by virtue of (6.1) and (6.2) σ_n and σ_{n+1} will, in general, have components referred to different basis vectors whether the spatial or convected coordinate system is employed. That is to say, $[\bar{\sigma}^{\alpha\beta}_{n+1} - \bar{\sigma}^{\alpha\beta}_n]$ or $[\sigma^{ab}_{n+1} - \sigma^{ab}_n]$ do not represent the components of a tensorial quantity as a result of the evolution of the basis vectors over the time step.

On the other hand, the basis vectors $\mathbf{G}_A$ associated with the material coordinate system have a stationary evolution in time since

$$\frac{d}{dt}(\mathbf{G}_A) = \frac{\partial}{\partial t}(\mathbf{G}_A) + \frac{\partial \mathbf{G}_a}{\partial X^B} \frac{\partial X^B}{\partial t} = 0 \tag{6.3}$$

The second Piola-Kirchhoff stress tensor S defined by:

$$\mathbf{S} = \phi_t^*(J\sigma) \tag{6.4}$$

is a material tensor on B and is in a one-to-one relation with σ through $\boldsymbol{\phi}_t$. Since the $\mathbf{G}_A$ are stationary in time, the quantity

$$[\mathbf{S}_{n+1} - \mathbf{S}_n] = [S_{n+1}^{AB} - S_n^{AB}]\, \mathbf{G}_A \otimes \mathbf{G}_B \tag{6.5}$$

might usefully be employed in a numerical algorithm since only the evolution of the components S^{AB} need be addressed.

The following development employs the material setting based on an arbitrary reference configuration B, which is not necessarily and, in general, will not be the same reference configuration employed in the momentum balance equation, discussed in Chapter 5. In fact, advantage is taken of the freedom allowed in choosing B. A spatial development, equivalent to the following material development, which considers the evolution of the stress tensor components and basis vectors, is possible. However, the material setting, utilizing the concept of Lie differentiation, leads to a very direct and appealing development.

By Definition 2.14(c), for a fixed material point $\mathbf{X} \in B$ and a regular motion $\boldsymbol{\phi}_t : R \to N$, the Truesdell rate of σ is given by:

$$\overset{\circ}{\sigma} = J^{-1} \boldsymbol{\phi}_{t*} \left[\frac{d}{dt} \boldsymbol{\phi}_t^{*} (J\sigma) \right] = J^{-1} \boldsymbol{\phi}_{t*} \left[\frac{d}{dt} \mathbf{S} \right]$$

which, in view of the assumed invertibility of $\mathbf{F}$, implies:

$$\frac{d}{dt} \mathbf{S} = \boldsymbol{\phi}_t^{*} (J \overset{\circ}{\sigma}) = \boldsymbol{\phi}_t^{*} (L_{\mathbf{v}}(\tau)) \tag{6.6}$$

The last result in (6.6) follows from $\overset{\circ}{\sigma} = J^{-1} L_{\mathbf{v}}(\tau)$. The body is assumed to occupy the known configurations $\Omega_n = \boldsymbol{\phi}_n(B)$ and $\Omega_{n+1} = \boldsymbol{\phi}_{n+1}(B)$. An intermediate configuration $\Omega_{n+\alpha} = \boldsymbol{\phi}_{n+\alpha}(B)$ is introduced, where

$$\boldsymbol{\phi}_{n+\alpha} = \alpha \boldsymbol{\phi}_{n+1} + (1 - \alpha)\, \boldsymbol{\phi}_n \qquad 0 \leqslant \alpha \leqslant 1 \tag{6.7}$$

Clearly, $\Omega_{n+\alpha}$ is not, in general, an actual configuration occupied by the body. Figure 3 illustrates the configurations in use. It is also assumed that σ_n is known at each material (integration) point in Ω_n. The generalized midpoint rule [49,50] is used to numerically integrate the material rate equation (6.6), such that:

$$\mathbf{S}_{n+1} - \mathbf{S}_n = \Delta t \left(\frac{d}{dt} \mathbf{S} \right)_{n+\alpha} \qquad 0 \leqslant \alpha \leqslant 1 \tag{6.8}$$

where $\Delta t = t_{n+1} - t_n$ and $(\cdot)_{n+\alpha}$ will be evaluated on the intermediate configuration $\Omega_{n+\alpha}$. Using (6.6) and noting that $\mathbf{S} = \overset{*}{\boldsymbol{\phi}_t}(J\boldsymbol{\sigma}) = \overset{*}{\boldsymbol{\phi}_t}(\boldsymbol{\tau})$, (6.8) has the alternative representations:

$$\overset{*}{\boldsymbol{\phi}}_{n+1}(J\boldsymbol{\sigma}) - \overset{*}{\boldsymbol{\phi}}_{n}(J\boldsymbol{\sigma}) = \Delta t\, \overset{*}{\boldsymbol{\phi}}_{n+\alpha}(\overset{\circ}{J\boldsymbol{\sigma}}) \tag{6.9}$$

or

$$\overset{*}{\boldsymbol{\phi}}_{n+1}(\boldsymbol{\tau}) - \overset{*}{\boldsymbol{\phi}}_{n}(\boldsymbol{\tau}) = \Delta t\, \overset{*}{\boldsymbol{\phi}}_{n+\alpha}(L_{\mathbf{v}}(\boldsymbol{\tau})) \tag{6.10}$$

In reference to the comments above, it is noted that (6.8) - (6.10) have components referred to the stationary material basis vectors. Accordingly, *they may be expressed in terms of their components alone.*

To simplify (6.9) or (6.10), it is advantageous to select the reference configuration B to coincide instantaneously with Ω_{n+1}. That is, $\boldsymbol{\phi}_{n+1} = \mathbf{I}$, in which case (6.9) and (6.10) reduce to:

$$\boldsymbol{\sigma}_{n+1} - \overset{*}{\boldsymbol{\phi}}_{n}(J\boldsymbol{\sigma}) = \Delta t\, \overset{*}{\boldsymbol{\phi}}_{n+\alpha}(\overset{\circ}{J\boldsymbol{\sigma}}) \tag{6.11}$$

$$\boldsymbol{\tau}_{n+1} - \overset{*}{\boldsymbol{\phi}}_{n}(\boldsymbol{\tau}) = \Delta t\, \overset{*}{\boldsymbol{\phi}}_{n+\alpha}(L_{\mathbf{v}}(\boldsymbol{\tau})) \tag{6.12}$$

Defining the deformation gradients:

$$\boldsymbol{\Lambda}_{n+\alpha} = \frac{\partial \mathbf{x}_{n+1}}{\partial \mathbf{x}_{n+\alpha}} \qquad 0 \leqslant \alpha \leqslant 1 \tag{6.13}$$

where $\mathbf{x}_t = \boldsymbol{\phi}_t$, and Jacobian determinants:

$$J_{n+\alpha} = \det(\boldsymbol{\Lambda}_{n+\alpha}) \qquad 0 \leqslant \alpha \leqslant 1 \tag{6.14}$$

then, for contravariant components of stress, (6.11) and (6.12) have the form:

$$\boldsymbol{\sigma}_{n+1} - J_n \boldsymbol{\Lambda}_n \cdot \boldsymbol{\sigma}_n \cdot \boldsymbol{\Lambda}_n^T = \Delta t\, J_{n+\alpha} \boldsymbol{\Lambda}_{n+\alpha} \cdot \overset{\circ}{\boldsymbol{\sigma}}_{n+\alpha} \cdot \boldsymbol{\Lambda}_{n+\alpha}^T \tag{6.15}$$

$$\boldsymbol{\tau}_{n+1} - \boldsymbol{\Lambda}_n \cdot \boldsymbol{\tau}_n \cdot \boldsymbol{\Lambda}_n^T = \Delta t\, \boldsymbol{\Lambda}_{n+\alpha} \cdot L_{\mathbf{v}}(\boldsymbol{\tau})_{n+\alpha} \cdot \boldsymbol{\Lambda}_{n+\alpha}^T \tag{6.16}$$

These equations become well-posed by introducing the rate constitutive equations for $\overset{\circ}{\boldsymbol{\sigma}}_{n+\alpha}$ or $L_{\mathbf{v}}(\boldsymbol{\tau})_{n+\alpha}$. This is considered in Section 6.3.

Co-rotational Rate of Stress

Once again, the Truesdell rate of $\boldsymbol{\sigma}$ and the Lie derivative of $\boldsymbol{\tau}$ with respect to $\mathbf{v}$ have appeared naturally in the development. It is, nevertheless, interesting to see how algorithms (6.15) and (6.16) might be modified for the co-rotational rate of $\boldsymbol{\tau}$, denoted $\overset{\triangledown}{\boldsymbol{\tau}}$, which is a stress rate often employed in the computational literature. The following discussion is easily extended to $\overset{\triangledown}{\boldsymbol{\sigma}}$.

Recalling Section 3.3, the motion $\boldsymbol{\phi}_t$ is again assumed to be given by the composition mapping:

$$\boldsymbol{\phi}_t = \boldsymbol{\phi}_t^R \circ \boldsymbol{\phi}_t^u \tag{6.17}$$

where $\boldsymbol{\phi}_t^R$ and $\boldsymbol{\phi}_t^u$ correspond to rotation and stretching of material neighborhoods, respectively. It is shown in Section 3.3 that:

$$\overset{\triangledown}{\boldsymbol{\tau}} = L_{\mathbf{v}}(\boldsymbol{\tau})\big|_{\boldsymbol{\phi}_t^u = \mathbf{I}} = (\boldsymbol{\phi}_t^R)_* \left[\frac{d}{dt} (\boldsymbol{\phi}_t^R)^* (\boldsymbol{\tau}) \right] \tag{6.18}$$

From (6.18) it follows that:

$$\frac{d}{dt} (\boldsymbol{\phi}_t^R)^* (\boldsymbol{\tau}) = (\boldsymbol{\phi}_t^R)^* (\overset{\triangledown}{\boldsymbol{\tau}}) \tag{6.19}$$

which is analogous to (6.6). The tensors in (6.19) will be material only in the event that the actual motion of material neighborhoods is purely rotational, such that $\mathbf{d} = 0$. If it is assumed that such an approximation may be made over the time step, the generalized midpoint rule applied to (6.19) yields, after choosing B to coincide instantaneously with Ω_{n+1}:

$$\boldsymbol{\tau}_{n+1} - \mathbf{R}_n \cdot \boldsymbol{\tau}_n \cdot \mathbf{R}_n^T = \Delta t\, \mathbf{R}_{n+\alpha} \cdot \overset{\triangledown}{\boldsymbol{\tau}}_{n+\alpha} \cdot \mathbf{R}_{n+\alpha}^T \tag{6.20}$$

where the orthogonal rotation tensor $\mathbf{R}_{n+\alpha}$ is defined by:

$$\mathbf{R}_{n+\alpha} = \frac{\partial \boldsymbol{\phi}_{n+1}^R}{\partial \mathbf{x}_{n+\alpha}} \tag{6.21}$$

It is noted that the constitutive equation for $\overset{\triangledown}{\boldsymbol{\tau}}$ is usually of the form $\overset{\triangledown}{\boldsymbol{\tau}} = \bar{\mathbf{a}} : \mathbf{d}$. This is interesting, since it has been assumed that $\mathbf{d}$ is instantaneously zero for the development leading to (6.20). However, (6.20) is similar to results obtained in [16,51]. The algorithm (6.20) is unsatisfactory for two reasons:

(i) The approximation $\mathbf{d} = 0$ instantaneously is not appropriate.

(ii) Evaluation of $\mathbf{R}_{n+\alpha}$ from $\boldsymbol{\phi}_{n+\alpha}$ (or from $\boldsymbol{\Lambda}_{n+\alpha}$ by polar decomposition) is not readily accomplished.

The algorithms (6.15) and (6.16), in contrast, use the deformation gradient which is readily computed.

It is shown in the following section how integration of $\overset{\nabla}{\boldsymbol{\tau}}$ may be accomplished consistently using (6.16).

6.3 NUMERICAL ALGORITHM

Equations (6.15) or (6.16) provide a basis for a numerical algorithm for the integration of the rate constitutive equations of hypoelasticity (including rate forms of hyperelasticity). It remains only to find some approximation for $\overset{\circ}{\boldsymbol{\sigma}}_{n+\alpha}$ or $L_v(\boldsymbol{\tau})_{n+\alpha}$ occurring in (6.15) and (6.16), respectively.

The following development is based on (6.16) although, since (6.15) and (6.16) are entirely equivalent, either equation could be used. For convenience of presentation, the universal constitutive equation (5.20) is reintroduced:

$$L_v(\boldsymbol{\tau}) = \hat{L}(J,\boldsymbol{\tau}):\mathbf{d} \tag{6.22}$$

where it is again noted that all the constitutive equations of Chapters 3 and 4 are contained in (6.22) if the dependence of L on J and $\boldsymbol{\tau}$ is adequate.

Example 6.1

If constitutive equations are given in the form:

$$\begin{aligned} (a) \quad \bar{\mathbf{a}}:\mathbf{d} &= \overset{\circ}{\boldsymbol{\sigma}} \\ (b) \quad\quad\quad &= L_v(\boldsymbol{\tau}) \\ (c) \quad\quad\quad &= \overset{\nabla}{\boldsymbol{\tau}} \end{aligned}$$

then the corresponding L in (6.22) has the form:

$$
\begin{aligned}
(a') \quad L^{abmn} &= J\,\bar{a}^{abmn} \\
(b') \quad &= \bar{a}^{abmn} \\
(c') \quad &= a^{abmn} - \tau^{nb} g^{ma} - \tau^{am} g^{nb}
\end{aligned}
$$

These results follow from definitions 2.14 and 2.15. Similar results may be constructed for $L_v(\sigma) = \mathbf{a}:\mathbf{d}$, $\overset{\circ}{\sigma} = \mathbf{a}:\mathbf{d}$, etc.

Example 6.1 demonstrates that even when $\bar{\mathbf{a}}$ is a constant tensor, L will, in general, depend on J and τ. Of course, $\bar{\mathbf{a}}$ may itself be a function of J and τ as in the case of rate-independent plasticity.

Substituting (6.22) into (6.16) gives:

$$\boldsymbol{\tau}_{n+1} - \boldsymbol{\Lambda}_n \cdot \boldsymbol{\tau}_n \cdot \boldsymbol{\Lambda}_n^T = \Delta t\, \boldsymbol{\Lambda}_{n+\alpha} \cdot \left[\hat{L}\,(J_{n+\alpha}, \boldsymbol{\tau}_{n+\alpha}):\mathbf{d}_{n+\alpha}\right] \cdot \boldsymbol{\Lambda}_{n+\alpha}^T \tag{6.23}$$

Assuming that $\boldsymbol{\Lambda}_n$ is known, it remains to evaluate $\boldsymbol{\Lambda}_{n+\alpha}$ and to obtain approximations for $\mathbf{d}_{n+\alpha}$ and $\boldsymbol{\tau}_{n+\alpha}$.

Evaluation of $\Lambda_{n+\alpha}$

From (6.7) and (6.13) it follows that:

$$\boldsymbol{\Lambda}_{n+\alpha} = [\alpha\,\mathbf{I} + (1-\alpha)\,\boldsymbol{\Lambda}_n^{-1}]^{-1} \tag{6.24}$$

Treating the components of these tensors as matrices, and using the following result from linear algebra:

$$(\mathbf{A}^{-1} + \mathbf{B}^{-1})^{-1} = \mathbf{A}\,(\mathbf{A} + \mathbf{B})^{-1}\mathbf{B} = \mathbf{B}\,(\mathbf{A} + \mathbf{B})^{-1}\mathbf{A} \tag{6.25}$$

where $\mathbf{A}$, $\mathbf{B}$, and $\mathbf{A} + \mathbf{B}$ are square nonsingular matrices, then (6.24) may be simplified to:

$$\boldsymbol{\Lambda}_{n+\alpha} = [(1-\alpha)\,\mathbf{I} + \alpha\,\boldsymbol{\Lambda}_n]^{-1} \cdot \boldsymbol{\Lambda}_n \tag{6.26}$$

Approximation for $d_{n+\alpha}$

The displacement increment $\boldsymbol{\delta}$ over the time step Δt is:

$$\boldsymbol{\delta} = \mathbf{x}_{n+1} - \mathbf{x}_n \tag{6.27}$$

The spatial rate of deformation tensor at time $t_{n+\alpha}$ is approximated by:

$$\mathbf{d}_{n+\alpha} = \frac{1}{\Delta t}\left[\frac{\partial \delta}{\partial \mathbf{x}_{n+\alpha}}\right]^S \tag{6.28a}$$

and the spin rate $\boldsymbol{\omega}_{n+\alpha}$ by:

$$\boldsymbol{\omega}_{n+\alpha} = \frac{1}{\Delta t}\left[\frac{\partial \delta}{\partial x_{n+\alpha}}\right]^A \tag{6.28b}$$

From (6.27), (6.7), and (6.13), it follows that:

$$\frac{\partial \delta}{\partial x_{n+\alpha}} = [\alpha \mathbf{I} + (1-\alpha)\boldsymbol{\Lambda}_n^{-1}]^{-1} - [\alpha \boldsymbol{\Lambda}_n + (1-\alpha)\mathbf{I}]^{-1} \tag{6.29}$$

Using (6.25) to simplify (6.29), it follows from (6.28a) that $\mathbf{d}_{n+\alpha}$ is given by:

$$\mathbf{d}_{n+\alpha} = \frac{1}{\Delta t}\left[\left\{(1-\alpha)\mathbf{I} + \alpha\boldsymbol{\Lambda}_n\right\}^{-1}\left\{\boldsymbol{\Lambda}_n - \mathbf{I}\right\}\right]^S \tag{6.30}$$

This approximation for $\mathbf{d}_{n+\alpha}$ serves to restrict the admissible choices for α if the algorithm is to remain incrementally objective with respect to superimposed rigid body rotations. Before discussing this issue, the final approximation, for $\boldsymbol{\tau}_{n+\alpha}$, is presented.

Approximation for $\tau_{n+\alpha}$

The stress $\boldsymbol{\tau}_{n+\alpha}$ defined on the intermediate configuration $\Omega_{n+\alpha}$ is approximated by:

$$\boldsymbol{\tau}_{n+\alpha} = \alpha(\boldsymbol{\phi}_{n+1} \circ \boldsymbol{\phi}_{n+\alpha}^{-1})^{*}(\boldsymbol{\tau}_{n+1}) + (1-\alpha)(\boldsymbol{\phi}_n \circ \boldsymbol{\phi}_{n+\alpha}^{-1})^{*}(\boldsymbol{\tau}_n) \tag{6.31}$$

which, by the chain rule, has the representation:

$$\boldsymbol{\tau}_{n+\alpha} = \alpha\boldsymbol{\Lambda}_{n+\alpha}^{-1}\cdot\boldsymbol{\tau}_{n+1}\cdot\boldsymbol{\Lambda}_{n+\alpha}^{-T} + (1-\alpha)\boldsymbol{\Lambda}_{n+\alpha}^{-1}\cdot\boldsymbol{\Lambda}_n\cdot\boldsymbol{\tau}_n\cdot\boldsymbol{\Lambda}_n^T\cdot\boldsymbol{\Lambda}_{n+\alpha}^{-T} \tag{6.32}$$

(6.32) may be simplified by noting from (6.26) that:

$$\boldsymbol{\Lambda}_{n+\alpha}^{-1} = (1-\alpha)\boldsymbol{\Lambda}_n^{-1} + \alpha\mathbf{I} \tag{6.33}$$

and

$$\boldsymbol{\Lambda}_{n+\alpha}^{-1}\cdot\boldsymbol{\Lambda}_n = (1-\alpha)\mathbf{I} + \alpha\boldsymbol{\Lambda}_n \tag{6.34}$$

Since $\boldsymbol{\tau}_{n+\alpha}$ is expressed in terms of $\boldsymbol{\tau}_n$ and $\boldsymbol{\tau}_{n+1}$ by virtue of (6.31), it follows that the algorithm (6.23) is implicit in $\boldsymbol{\tau}_{n+1}$, in general. The solution of this implicit algorithm for $\boldsymbol{\tau}_{n+1}$ may be accomplished by use of an iteration scheme.

It is recalled that the integration algorithm is required for updating the stress tensor corresponding to Newton-Raphson iterations on the linearized weak form of the momentum balance equation. The following subiteration scheme for the solution of $\boldsymbol{\tau}_{n+1}$ in (6.23) has proven to significantly improve the convergence rate of the Newton-Raphson scheme:

1. Set j = 1.

2. Initialize $\boldsymbol{\tau}^{o}_{n+1} = \overline{\boldsymbol{\tau}}_{n+1}$.

3. Solve:

$$\boldsymbol{\tau}^{j}_{n+1} = \Lambda_n \cdot \boldsymbol{\tau}_n \cdot \Lambda^T_n + \Delta t\, \boldsymbol{\Lambda}_{n+\alpha} \cdot \left[\hat{L}(J_{n+\alpha}, \boldsymbol{\tau}^{j-1}_{n+\alpha}) : \mathbf{d}_{n+\alpha} \right] \cdot \boldsymbol{\Lambda}^T_{n+\alpha}$$

4. If

$$\frac{||\boldsymbol{\tau}^{j}_{n+1}|| - ||\boldsymbol{\tau}^{j-1}_{n+1}||}{||\boldsymbol{\tau}^{j}_{n+1}||} \leqslant tol,$$

   stop.

5. $j \leftarrow j+1, \qquad \text{if } j > j_{\max},$

   stop.

6. Go to 3.

In Step 2, $\overline{\boldsymbol{\tau}}_{n+1}$ is the value of $\boldsymbol{\tau}_{n+1}$ determined from the previous Newton-Raphson iteration. In Step 3, $\boldsymbol{\tau}^{j-1}_{n+\alpha}$ is meant to imply the evaluation of $\boldsymbol{\tau}_{n+\alpha}$ according to (6.32) but with $\boldsymbol{\tau}_{n+1}$ replaced by $\boldsymbol{\tau}^{j-1}_{n+1}$.

In practice, the above scheme converged very rapidly in all cases (only 1 or 2 iterations generally being required).

6.4 INCREMENTAL OBJECTIVITY

It is essential that the algorithm (6.23) with $\mathbf{d}_{n+\alpha}$ given by (6.30) and $\boldsymbol{\tau}_{n+\alpha}$ by (6.32) be incrementally objective with respect to rotation of material neighborhoods over the time step. It is seen in this section that such a requirement leads to restrictions on α.

Definition 6.1

Let R be the group of all orthogonal second order tensors and M the group of all positive definite symmetric second order tensors. The algorithm defined by (6.23), (6.30), and (6.32) is *incrementally objective* if:

$$(a)\quad \mathbf{\Lambda}_n \in R \quad <=> \quad \mathbf{d}_{n+\alpha} = 0 \tag{6.35}$$

$$(b)\quad \mathbf{\Lambda}_n \in M \quad <=> \quad \boldsymbol{\omega}_{n+\alpha} = 0 \tag{6.36}$$

where $\boldsymbol{\omega}_{n+\alpha}$ is the spin rate tensor determined by the approximation (6.28b).

Condition (a) ensures that the integration algorithm reduces to:

$$\boldsymbol{\tau}_{n+1} = \mathbf{\Lambda}_n \cdot \boldsymbol{\tau}_n \cdot \mathbf{\Lambda}_n^T$$

in the event that $\mathbf{\Lambda}_n \in R$. Condition (b) ensures that no arbitrary rotations are introduced under the stated conditions.

Proposition 6.1

The algorithm defined by (6.23), (6.30), and (6.32) is incrementally objective if and only if $\alpha = 0.5$.

Proof

(a) Consider $\mathbf{\Lambda}_n \in R$, such that $\mathbf{\Lambda}_n^T \cdot \mathbf{\Lambda}_n = \mathbf{I}$. From (6.30):

$$\begin{aligned} 2\Delta t\, \mathbf{d}_{n+\alpha} &= [(1-\alpha)\,\mathbf{I} + \alpha\mathbf{\Lambda}_n]^{-1}[\mathbf{\Lambda}_n - \mathbf{I}] + [\mathbf{\Lambda}_n - \mathbf{I}]^T[(1-\alpha)\,\mathbf{I} + \alpha\mathbf{\Lambda}_n]^{-T} \\ &= (\beta\mathbf{I} + \alpha\mathbf{\Lambda}_n)^{-1}\mathbf{\Lambda}_n + \mathbf{\Lambda}_n^T(\beta\mathbf{I} + \alpha\mathbf{\Lambda}_n)^{-T} - (\beta\mathbf{I} + \alpha\mathbf{\Lambda}_n)^{-1} \\ &\quad - (\beta\mathbf{I} + \alpha\mathbf{\Lambda}_n)^{-T} \end{aligned} \tag{6.37}$$

where $\beta = 1 - \alpha$.

But, using (6.25) and the condition $\mathbf{\Lambda}_n \in R$ gives:

$$\begin{aligned} (\beta\mathbf{I} + \alpha\mathbf{\Lambda}_n)^{-1} &= [(\beta\mathbf{I} + \alpha\mathbf{\Lambda}_n^T)^{-1}]^T \\ &= [(\beta\mathbf{I} + \alpha\mathbf{\Lambda}_n^{-1})^{-1}]^T \\ &= [\frac{1}{\alpha\beta}(\frac{1}{\beta}\mathbf{I} + \frac{1}{\alpha}\mathbf{\Lambda}_n)^{-1}\mathbf{\Lambda}_n]^T \\ &= \frac{1}{\alpha\beta}\mathbf{\Lambda}_n^T(\frac{1}{\beta}\mathbf{I} + \frac{1}{\alpha}\mathbf{\Lambda}_n)^{-T} \end{aligned} \tag{6.38}$$

Similarly,

$$(\beta\mathbf{I} + \alpha\mathbf{\Lambda}_n)^{-T} = \frac{1}{\alpha\beta}(\frac{1}{\beta}\mathbf{I} + \frac{1}{\alpha}\mathbf{\Lambda}_n)^{-1}\mathbf{\Lambda}_n \tag{6.39}$$

Substituting (6.38) and (6.39) into (6.37) yields:

$$2\Delta t\,\mathbf{d}_{n+\alpha} = \left[(\beta\mathbf{I} + \alpha\mathbf{\Lambda}_n)^{-1} - \frac{1}{\alpha\beta}(\frac{1}{\beta}\mathbf{I} + \frac{1}{\alpha}\mathbf{\Lambda}_n)^{-1}\right]\mathbf{\Lambda}_n + \mathbf{\Lambda}_n^T\left[(\beta\mathbf{I} + \alpha\mathbf{\Lambda}_n)^{-T} - \frac{1}{\alpha\beta}(\frac{1}{\beta}\mathbf{I} + \frac{1}{\alpha}\mathbf{\Lambda}_n)^{-T}\right] \tag{6.40}$$

From (6.40), it follows that:

$$\mathbf{d}_{n+\alpha} = 0 \iff \alpha = \beta = 0.5$$

(b) Consider $\mathbf{\Lambda}_n \in M$, such that $\mathbf{\Lambda}_n = \mathbf{\Lambda}_n^T$. Using (6.25) to simplify (6.29) to:

$$\frac{\partial\boldsymbol{\delta}}{\partial x_{n+\alpha}} = (\beta\mathbf{I} + \alpha\mathbf{\Lambda}_n)^{-1}(\mathbf{\Lambda}_n - \mathbf{I})$$

where $\beta = 1-\alpha$, it follows from (6.28b) that:

$$2\Delta t\,\boldsymbol{\omega}_{n+\alpha} = (\beta\mathbf{I} + \alpha\mathbf{\Lambda}_n)^{-1}(\mathbf{\Lambda}_n - \mathbf{I}) - (\mathbf{\Lambda}_n^T - \mathbf{I})(\beta\mathbf{I} + \alpha\mathbf{\Lambda}_n)^{-T} \tag{6.41}$$

But, since $\mathbf{\Lambda}_n \in M$,

$$(\beta\mathbf{I} + \alpha\mathbf{\Lambda}_n)^{-1} = (\beta\mathbf{I} + \alpha\mathbf{\Lambda}_n)^{-T}$$

and noting that $\mathbf{\Lambda}_n$ is a diagonal tensor, it follows from (6.41) that:

$$2\Delta t\,\boldsymbol{\omega}_{n+\alpha} = 0$$

Thus, $\boldsymbol{\omega}_{n+\alpha} = 0$ for all $\alpha \in [0,1]$ ■

It is clear from part (a) of the above proof that the approximation for $\mathbf{d}_{n+\alpha}$ given by (6.28a) results in a restriction of α to 0.5. The parameter α was initially introduced into (6.8) to define the generalized midpoint rule. Fortunately, this integration rule is unconditionally stable and second order accurate at the value of $\alpha = 0.5$ [49,50].

The algorithm described above is generalized for the case of rate-independent elasto-plasticity as described in the next section.

6.5 INTEGRATION OF THE CONSTITUTIVE EQUATIONS OF RATE-INDEPENDENT ELASTO-PLASTICITY

The integration algorithm developed in Sections 6.2 - 6.4 is directly applicable to the constitutive equations of hypoelasticity (including the rate forms of hyperelasticity). Extension of the algorithm for the case of rate-independent elasto-plasticity is considered here.

The procedure is based on the additive decomposition of the spatial rate of deformation tensor **d** into elastic and plastic parts:

$$\mathbf{d} = \mathbf{d}^e + \mathbf{d}^P \tag{6.42}$$

according to the discussion of Section 4.1. The evolutionary equations for the problem comprise the linear momentum balance equation, the rate constitutive equations for the spatial stress field, and the rate equations for the internal variables. Due to (6.42), the evolutionary equations admit an additive decomposition (operator split) into elastic and plastic parts corresponding to the nonlinear elasticity wave operator and the plasticity flow rule operator [52]. The product formula techniques of nonlinear semigroup theory may be applied to this decomposition, resulting in a step-by-step integration procedure in which a nonlinear elasto-dynamic problem is first solved, followed by the application of algorithms that bring in the plastic part of the evolutionary equations. It is shown in [52] that these latter algorithms result in a projection of the stresses corresponding to the solution of the elasto-dynamic problem, onto the convex yield hypersurface. Thus, at every step, one first solves the incremental elasticity problem for the elastic "trial" stress, which is then projected onto the convex yield hypersurface. This composition mapping of elastic solution $\mathbf{S}_e$ and return mapping to the yield hypersurface $\mathbf{P}_c$ is depicted in Figure 4.

In order to obtain the "trial" stresses ($\mathbf{S}_e$), the nonlinear elasto-dynamic problem is solved by means of the Newton-Raphson procedure discussed in Section 5.5. The elastic rate constitutive equations are integrated using the algorithm defined by (6.23), (6.30), and (6.32).

The convergence of the product formulae requires consistency and stability of the algorithms used for the elastic and plastic parts of the evolutionary equations [52]. It is shown in

[52] that this places general restrictions on the form of the return mapping $\mathbf{P}_c$ of the elastic trial stress. These restrictions are:

(a) Stability

The return mapping $\mathbf{P}_c$ should be a contraction mapping such that if $\boldsymbol{\sigma}_1$ and $\boldsymbol{\sigma}_2$ are two elastic trial solutions, $\mathbf{P}_c$ must satisfy:

$$||\mathbf{P}_c(\sigma_1) - \mathbf{P}_c(\sigma_2)|| \leqslant ||\sigma_1 - \sigma_2||$$

(b) Consistency

The return mapping $\mathbf{P}_c$ should be a normal projection of the stress point very close to the yield hypersurface.

The above formalism does not depend on any notion of smoothness of the yield hypersurface and is applicable to arbitrary convex elastic regions with or without corners [52]. For hardening plasticity, the yield surface will evolve according to its dependence on the internal variables. In this case, the return mapping will project the trial stress onto the updated yield surface.

A number of return maps satisfying the above conditions have been proposed [55-57]. Error analyses of the resulting procedures have been considered for the case of infinitesimal perfect plasticity [56,57] and, in one case, finite deformation hardening plasticity [55]. The plasticity model used in the present study is restricted to isotropic hardening. Two algorithms have been implemented and are based on return maps, $\mathbf{P}_c$, which are essentially finite deformation/hardening generalizations of the projections reported in [56,57]. The algorithms are:

(i) tangent predictor-radial return

(ii) closest point projection (radial return).

These projections are illustrated in Figure 5.

The essential features are:

(a) In both algorithms, the elastic trial stress is obtained by integration of the equations of hypoelasticity according to the discussion of the previous section.

(b) The so-called plastic contact stress $\boldsymbol{\sigma}_c$ (corresponding to the stress state at first contact with the yield surface during the time step) is used to update the yield surface radius K by integrating (c.f. eq. (4.40)):

$$\dot{K} = \frac{1}{K\left(\frac{1}{3\mu} + \frac{1}{H}\right)} \boldsymbol{\sigma}'_c : \mathbf{d}_{n+\alpha}$$

(c) In method (i), the tangent predictor stress point must be tangent to the updated yield surface to maintain the contractive property of the plastic projection.

(d) All the above operations are in deviatoric stress space.

Pressure Calculations

For the plasticity model under consideration, (4.38)-(4.42), the trace of the Truesdell rate of Cauchy stress is determined purely by the volumetric deformation rate and the elastic constants. However, the pressure rate must be evaluated correctly. The pressure p is defined p by:

$$p = \frac{1}{3} tr(\boldsymbol{\sigma})$$

Constitutive equation (4.39) may be written as:

$$\overset{\circ}{\boldsymbol{\sigma}} = \mathbf{a} : \mathbf{d} - 2H(\xi)(\mathbf{n} : \mathbf{d})\,\mathbf{n}$$

where the spatial elasticity tensor $\mathbf{a}$ is defined by (4.27) and the $\mathbf{n}$ terms by $(4.39)_2$.

It may be shown that under this hypothesis, the pressure rate is given by:

$$\dot{p} = J\left(\lambda + \frac{2}{3} mu\right) tr(\mathbf{d}) + \frac{2}{3} tr(J\,\boldsymbol{\sigma} \cdot \mathbf{d})$$

This equation may be integrated using the generalized mid-point rule, resulting in:

$$p_{n+1} = p_n + \Delta t\,(\dot{p})_{n+\alpha}$$

where $(\dot{p})_{n+\alpha}$ implies evaluation of $\dot{p}$ at time $n+\alpha$.

If the Truesdell rate of Cauchy stress in the constitutive equation above is replaced by other objective stress rates, different expressions for the pressure rate will result. If the pressure is to be calculated independently, it is important that the correct definition of the pressure rate be used.

CHAPTER 7. FINITE ELEMENT IMPLEMENTATION

7.1 INTRODUCTION

A Newton-Raphson solution procedure for the momentum balance equation, based on a finite element spatial discretization of the linearized weak form, was proposed and schematically outlined in Section 5.5. The variables resulting in this formulation are the nodal incremental motions which are directly updated to define the nodal motion. The corresponding spatial stress field at the finite element quadrature points may be evaluated by integration of the spatial rate constitutive equations using the algorithm developed in Chapter 6. Some details of the finite element implementation of the Newton-Raphson solution procedure and integration algorithm are considered in this chapter.

7.2 ITERATIVE SOLUTION PROCEDURE

A locally linear approximation for the weak form of a variational equation, equivalent to the boundary value problem of momentum balance, is given by (5.33). This locally linear approximation is based on a Lagrangian formulation referred to a reference configuration B having an expression in material coordinates given by (5.34) and in spatial coordinates by (5.35).

It is seen from Proposition 5.6 that for spatial rate constitutive equations the kernel $tr([D\mathbf{P}\cdot\mathbf{u}]\cdot\mathbf{D}\boldsymbol{\eta})$ appearing in (5.34) and (5.35) is expressed most directly in terms of spatial tensors (transformation of the results of Proposition 5.6 to material coordinates is possible but would introduce additional complications into the expressions). For this reason alone, the form (5.35), in which the Lagrangian formulation is expressed in terms of spatial tensors, is adopted here as the basis for a finite dimension approximation using spatial discretization of the reference state $\psi_t(B)$ by the finite element method.

The reference configuration B must be selected and two possibilities are of interest:

(a) **Total Lagrangian Formulation**

The reference configuration B is selected to coincide with the initial configuration of the continuum, such that:

$$\phi_o = \mathbf{I} \tag{7.1}$$

It is supposed that ρ_o and $\bar{\tau}$ on $\partial B_{\bar{\tau}}$ are specified. Using the result that:

$$\int_{\partial(\psi_t(B))_{\bar{t}}} \bar{\mathbf{t}} \cdot \boldsymbol{\eta}\, da = \int_{\partial B_{\bar{\tau}}} \bar{\tau} \cdot \boldsymbol{\eta}\, dA$$

then (5.35) has the form:

$$\int_{\psi_t(B)} J^{-1} tr([D\mathbf{P} \cdot \mathbf{u}] \cdot \mathbf{D}\boldsymbol{\eta}) dv + \int_{\psi_t(B)} J^{-1} \rho_B \ddot{\mathbf{u}} \cdot \boldsymbol{\eta}\, dv =$$
$$\int_{\partial B_{\bar{\tau}}} \bar{\tau} \cdot \boldsymbol{\eta}\, dA - \int_{\psi_t(B)} tr(\sigma \cdot \nabla\boldsymbol{\eta}) dv - \int_{\psi_t(B)} J^{-1} \rho_B (\dot{\mathbf{v}} - \mathbf{b}) \cdot \boldsymbol{\eta}\, dv \tag{7.2}$$

(b) **Updated Lagrangian Formulation**

The reference configuration B is selected to instantaneously coincide with the reference state $\psi_t(B)$, such that:

$$\psi_t = \mathbf{I} \tag{7.3}$$

It is supposed that ρ and $\bar{\mathbf{t}}$ on $\partial(\psi_t(B))_{\bar{t}}$ are specified. From (7.3) it follows that J = 1, and (5.35) has the form:

$$\int_{\psi_t(B)} tr([D\mathbf{P} \cdot \mathbf{u}] \cdot \mathbf{D}\boldsymbol{\eta})\, dv + \int_{\psi_t(B)} \rho \ddot{\mathbf{u}} \cdot \boldsymbol{\eta}\, dv =$$
$$\int_{\partial(\psi_t(B))_{\bar{t}}} \bar{\mathbf{t}} \cdot \boldsymbol{\eta}\, da - \int_{\psi_t(B)} tr(\sigma \cdot \nabla\boldsymbol{\eta})\, dv - \int_{\psi_t(B)} \rho(\dot{\mathbf{v}} - \mathbf{b}) \cdot \boldsymbol{\eta}\, dv \tag{7.4}$$

For both (7.2) and (7.3), Proposition 5.6 is used to evaluate the kernel of the first integral (for the updated Lagrangian formulation, J = 1 would be used in Proposition 5.6).

Following Section 5.5, notation for the Newton-Raphson procedure is introduced by letting $(\cdot)^i_{n+1}$ denote $(\cdot)$ at time $n+1$ and iteration i. For example, if the constitutive hypothesis is of the form:

$$\overset{\nabla}{\tau} = \bar{\mathbf{a}} : \mathbf{d} \tag{7.5}$$

then, using Proposition 5.6(d') and (7.4), the updated Lagrangian formulation is defined by:

$$\int_{\psi^i_{n+1}(B)} tr[(\sigma\cdot\nabla\mathbf{u}^T + \bar{\mathbf{a}}:\nabla^S\mathbf{u} - 2(\sigma\cdot\nabla^S\mathbf{u})^S)^i_{n+1}\cdot\nabla\boldsymbol{\eta}]dv^i_{n+1} + \int_{\psi^i_{n+1}(B)} \rho^i_{n+1}\ddot{\mathbf{u}}^i_{n+1}\cdot\boldsymbol{\eta}dv^i_{n+1} =$$

$$\int_{\partial\psi^i_{n+1}(B)_t} \bar{\mathbf{t}}_{n+1}\cdot\boldsymbol{\eta}\, da^i_{n+1} - \int_{\psi^i_{n+1}(B)} tr(\sigma^i_{n+1}\cdot\nabla\boldsymbol{\eta})\, dv^i_{n+1} - \int_{\psi^i_{n+1}(B)} \rho^i_{n+1}(\mathbf{b}n+1-\dot{\mathbf{v}}^i_{n+1})\cdot\boldsymbol{\eta}\, dv^i_{n+1} \quad (7.6)$$

where $J^i_{n+1} = 1$ has been used.

Updating the motion is achieved by setting:

$$\mathbf{x}^{i+1}_{n+1} = \mathbf{x}^i_{n+1} + \mathbf{u}^i_{n+1}$$

The finite element discretization of (7.2) or (7.4) follows standard procedures [58]. It is noted that (7.2) and (7.4) involve integration over the reference state and spatial gradients. The calculations involving these terms are conveniently accomplished using the concept of isoparametric mapping, standard to all finite element codes. Integration over the reference state will employ the Jacobian of the mapping from the current coordinates to the local finite element coordinates. Spatial gradients are obtained by updating the finite element spatial coordinates and using these quantities in the shape function routines that calculate gradients.

In the numerical examples given in Chapter 8, the total Lagrangian formulation (7.2) was employed for two dimensional analysis using constitutive equations appropriate for conditions of plane strain.

CHAPTER 8. EXAMPLES

8.1 INTRODUCTION

If the finite deformation of a continuum is characterized by a spatially uniform spatial velocity gradient, such that:

$$\nabla \mathbf{v} = \mathbf{g}(t) \tag{8.1}$$

then the corresponding deformations are referred to as being homogeneous. For hypoelasticity, the time dependent spatial stress field corresponding to (8.1) will also be spatially uniform with a vanishing spatial divergence. Consequently, the linear momentum balance equation will be satisfied identically for any velocity field corresponding to homogeneous deformation.

It follows that complete analytical solutions to problems of homogeneous deformation may be obtained if the constitutive equations can be integrated. In which case, the velocity field may also be integrated to define the deformed configuration (i.e. the motion). As noted in previous chapters, the constitutive equations of hypoelasticity and rate-independent elasto-plasticity are not, in general, directly integrable (which fact motivated the numerical integration procedure of Chapter 6). However, for a number of simple cases, analytical solutions may be obtained and are considered in the following section.

Despite the simplicity of the homogeneous deformation states to be considered, they retain considerable nonlinearity and provide a reasonable basis for evaluation of the performance of the numerical solution procedure.

The solutions of the homogeneous deformation problems considered below are useful for three reasons:

(1) They provide a clear illustration of the non-physical instability of response due to certain stress rate definitions used in the constitutive equations (hypoelastic yield phenomenon).

(2) Analytical solutions to homogeneous finite deformation problems are valuable for assessing the performance of numerical solution algorithms.

(3) Solution algorithms must be effective on problems of homogeneous deformation if they are to be useful in the inhomogeneous case.

Three problems are considered:

(a) Finite extension.

(b) Finite simple shear.

(c) Finite simple extension (with restrained Poisson effect) and simultaneous rigid rotation.

In all cases, dimension changes of a full order of magnitude are considered and in (b) rotation through 360 degrees. The problem (b) provides an effective means of checking the incremental objectivity of the algorithm for integrating the constitutive equations. Finally, the problem of an elastic-perfectly plastic cylinder subjected to internal pressure is considered.

8.2 HOMOGENEOUS FINITE EXTENSION

Kinematic Description

Consider a rectangular block whose edges coincide with the directions of the axes of a Cartesian coordinate system $\{X_A\}$ $A = 1,2,3$, in which X_1 represents the direction of loading. A material point **X** in the undeformed block is mapped to **x** at time t, where:

$$\lambda_i = \frac{x_i}{X_i} \tag{8.2}$$

defines the coordinate stretch ratios.

Restricting attention to *plane strain,* the spatial velocity field defined by:

$$v_1 = \alpha x_1, \quad v_2 = k\alpha x_2, \quad v_3 = 0 \tag{8.3}$$

where α and k are constants, corresponds to finite extension of the block in the X_1 direction. Since the spatial gradients of this velocity field are spatially uniform (independent of **x**), (8.3) characterizes a state of homogeneous deformation.

Constitutive Equations

An isotropic constant tensor **a** is defined (for Cartesian spatial coordinates) by:

$$a_{ijkl} = \lambda\, \delta_{ij}\, \delta_{kl} + \mu\, (\delta_{ik}\, \delta_{jl} + \delta_{il}\, \delta_{jk}) \tag{8.4}$$

where λ and μ are the Lame constants.

Three constitutive equations are defined by:

$$\overset{\circ}{\sigma} = \mathbf{a} : \mathbf{d} \tag{8.5}$$

$$L_v(\tau) = \mathbf{a} : \mathbf{d} \tag{8.6}$$

$$\overset{\triangledown}{\tau} = \mathbf{a} : \mathbf{d} \tag{8.7}$$

where $\overset{\circ}{\sigma}$ is the Truesdell rate of Cauchy stress, $L_v(\tau)$ is the Lie derivative of the Kirchhoff stress, and $\overset{\triangledown}{\tau}$ is the co-rotational rate of Kirchhoff stress (see Definitions 2.14 and 2.15).

The direct integration of these constitutive equations is possible under the present deformation hypothesis (8.3). Appendix I contains details of the direct integration of (8.5) - (8.7) subject to (8.3).

Analytical Results

The following is a summary of results developed in Appendix I. For the prescribed velocity field (8.3) and constitutive equations (8.5) - (8.7), the stretch ratios are related by:

$$\ln \lambda_2 = -\left(\frac{\nu}{1-\nu}\right) \ln \lambda_1, \quad \ln \lambda_3 = 0 \tag{8.8}$$

where $\nu = \dfrac{\lambda}{2(\lambda + \mu)}$.

For constitutive equation (8.5), the stress components and axial force associated with an axial stretch ratio of λ_1 are given by:

$$\sigma_{11} = \left(\frac{E}{1+\nu}\right)\left[(\lambda_1)^{\frac{1}{1-\nu}} - 1\right] \tag{8.9}$$

$$\sigma_{33} = \frac{E\nu}{(1+\nu)(1-2\nu)}\left[1 - (\lambda_1)^{\frac{1-2\nu}{\nu-1}}\right] \tag{8.10}$$

$$P_1 = Y_2 Y_3 \left(\frac{E}{1+\nu}\right) (\lambda_1) \left[1 - (\lambda_1)^{\frac{1}{\nu-1}}\right] \tag{8.11}$$

where Y_2 and Y_3 are the initial dimensions of the loaded cross section.

For constitutive equation (8.6), these quantities become:

$$\sigma_{11} = \frac{1}{2}\left(\frac{E}{1-\nu^2}\right)\left(\lambda_1\right)^{\frac{2\nu-1}{1-\nu}}\left[\left(\lambda_1\right)^2 - 1\right] \tag{8.12}$$

$$\sigma_{33} = \left(\frac{E}{1-\nu^2}\right)\left(\lambda_1\right)^{\frac{2\nu-1}{1-\nu}} \ln\lambda_1 \tag{8.13}$$

$$P_1 = Y_2 Y_3 \frac{1}{2}\left(\frac{E}{1-\nu^2}\right)\left(\lambda_1\right)^{-1}\left[\left(\lambda_1\right)^2 - 1\right] \tag{8.14}$$

For constitutive equation (8.7), these quantities become:

$$\sigma_{11} = \left(\frac{E}{1-\nu^2}\right)\left(\lambda_1\right)^{\frac{2\nu-1}{1-\nu}} \ln\lambda_1 \tag{8.15}$$

$$\sigma_{33} = \left(\frac{E\nu}{1-\nu^2}\right)\left(\lambda_1\right)^{\frac{2\nu-1}{1-\nu}} \ln\lambda_1 \tag{8.16}$$

$$P_1 = Y_2 Y_3\left(\frac{E}{1-\nu^2}\right)\left(\lambda_1\right)^{-1} \ln\lambda_1 \tag{8.17}$$

In (8.9) - (8.17), $E = \dfrac{\mu(3\lambda + 2\mu)}{\lambda + \mu}$.

A plot of axial force P_1 versus stretch ratio λ_1 for constitutive equations (8.5) - (8.7) is given by the solid curves in Fig. 6. The axial force P_1 increases uniformly with λ_1 for constitutive equations (8.5) and (8.6), the difference between these curves being due only to J. However, the axial force (8.17) associated with constitutive equation (8.7) exhibits a maximum load effect corresponding to the hypoelastic yield phenomenon discussed in Section 3.3. The critical axial load P_c occurs at a stretch ratio given by $\ln\lambda_c = 1$.

For this model, all stretch ratio values of $\lambda_1 > \lambda_c$ require an axial force P_1 *less* than P_c. In fact,

$$\lim_{\lambda_1 \to \infty} P_1 = 0$$

This instability arises from the stress rate definition used in the constitutive equation. However, it is clear from Fig. 5 that even for axial stretch ratios less than λ_c, the axial force resulting from (8.7) is considerably different to that resulting from (8.5) or (8.6). (The axial forces only coincide in the infinitesimal limit, which also has $\dfrac{dP_1}{d\lambda_1}$ equal for all constitutive equations.

This is as it should be since the phenomenon under discussion is a finite deformation effect.) The discussion above suggests that even for elasto-plastic constitutive equations expressed in terms of $\overset{\nabla}{\tau}$ the *non-physical* instability associated with this stress rate may be significant. It is interesting to note that the thermodynamic development of elasto-plasticity in Chapter 4 favored the Truesdell rate of Cauchy stress over the co-rotational rate of Kirchhoff stress for the constitutive equations.

Numerical Results

The finite extension problem was solved by discretization of the block into four quadrilateral plane strain finite elements. Figure 6 shows the parameters employed in this analysis. The constitutive models (8.5) - (8.7) were incorporated into a Newton-Raphson solution procedure according to the methods of Chapter 5. Updating of the spatial stress field at the element quadrature points was accomplished by use of the time integration algorithm developed in Chapter 6. Subiteration on the constitutive equations was used when appropriate; that is, for (8.7). Axial load versus stretch results are shown as dots in Figure 6 for the three constitutive models. The finite element procedure converged for a minimum of ten load increments over the range corresponding to $1 \leqslant \lambda_1 \leqslant 2$.

The results of the finite element analysis for the stress components and axial load were within 0.1 percent of the analytical results presented above. Such results, for the finite element mesh employed, may be interpreted as satisfaction of a generalized patch test. The Newton-Raphson procedure generally required 4 to 5 iterations for convergence within a time step.

Finally, the numerical results for the coordinate stretch ratios satisfied conditions (8.8) very closely over the range $1 \leqslant \lambda_1 \leqslant 2$.

8.3 HOMOGENEOUS FINITE SIMPLE SHEAR

Kinematic Description

Consider a rectangular block deforming with a spatial velocity field defined by:

$$v_1 = 2\alpha x_2, \quad v_2 = v_3 = 0 \tag{8.18}$$

where α is a parameter. Since the spatial gradients of this velocity field, which corresponds to finite simple shear of the block in the $X_1 - X_2$ plane, are spatially uniform, (8.18) characterizes a state of homogeneous deformation.

Constitutive Equations

The constitutive equations are again those given by (8.4) - (8.7) in the previous example. The direct integration of the constitutive equations (8.5) - (8.7) subject to (8.18) is considered in detail in Appendix I.

Analytical Results

The following is a summary of results developed in Appendix I. It is shown in Appendix I that (8.18) represents isochoric deformation with J = 1. In this case, constitutive equations (8.5) and (8.6) are equivalent. For these constitutive equations, the non-zero stress components are given by hyperbolic functions:

$$\sigma_{12} = \mu \sinh(2\alpha t) \tag{8.19}$$

$$\sigma_{11} = \mu[1 - \cosh(2\alpha t)] \tag{8.20}$$

$$\sigma_{22} = \mu[\cosh(2\alpha t) - 1] \tag{8.21}$$

where t is time.

For constitutive equation (8.7), these stress components are given by trigonometric functions:

$$\sigma_{12} = \mu \sin(2\alpha t) \tag{8.22}$$

$$\sigma_{11} = \mu\,[1 - \cos(2\alpha t)] \tag{8.23}$$

$$\sigma_{22} = \mu\,[\cos(2\alpha t) - 1] \tag{8.24}$$

It is noteworthy that once again, constitutive equations (8.5) and (8.6) result in stable and reasonable behavior, whereas (8.7) leads to unstable response at relatively small deformations. It is clear that (8.22) - (8.24) do *not* constitute reasonable behavior. Again, the phenomenon of hypoelastic yield is seen to be associated with (8.7), which has a *non-physical* instability due to the stress rate definition. It is recalled also that (8.5) and (8.6) are thermodynamically consistent, whereas (8.7) is not.

In Fig. 7, (8.22) and (8.23) are depicted by solid curves where $2\alpha t$ in these equations is replaced by x_1 for the unit cube under consideration (see Appendix I). Curves corresponding to (8.19) - (8.21) are not shown.

Numerical Results

The finite shear of a unit cube corresponding to (8.18) was solved by discretization of the block into four quadrilateral plane strain finite elements. The finite element mesh and values of the material constants are shown in Fig. 7. The solution procedure follows that described for the finite extension problem discussed above. Displacement control of the boundary nodes was used to define the homogeneous deformation state of simple shear.

The finite element results corresponding to constitutive equation (8.7) are shown as dots in Fig. 7. Convergence was excellent, even for the maximum step size corresponding to x_1 increments of 0.2. Accuracy was within 0.1% of the analytical results presented above for the full range of deformation considered.

Further Example

It is seen that (8.22) - (8.24) become unstable at $2\alpha t = \frac{\pi}{2}$. Clearly, (8.19) - (8.21) experience no such instability. An example which further illustrates the unreasonable behavior associated with (8.7) is depicted in Fig. 8. Here the torsion of an elastic annulus was numerically modeled for the constitutive equations (8.5) - (8.7). The deformed configuration of the Truesdell rate model (8.5) is shown with a 50 degree angle of twist. The co-rotational rate model (8.7) became unstable at a 28.5 degree angle of twist, corresponding to a principal shear strain (c.f. x_1) of 1.87, close to $\frac{\pi}{2}$.

8.4 HOMOGENEOUS FINITE SIMPLE EXTENSION AND ROTATION

Kinematic Description

This section considers the problem of a rectangular block which is subject to simple axial extension (that is, with restrained transverse Poisson effect) and simultaneous rigid rotation. A solution for the stress components relative to the fixed spatial coordinate system is required. This is accomplished in Appendix I by first solving the simple extension problem for the stress components relative to a coordinate system rotating rigidly with the block. The resulting components are then transformed to components referred to the fixed spatial coordinate system.

The spatial velocity field under consideration here and detailed in Appendix I is not the same as in Section 8.2 due to the restraint on transverse deformation.

The purpose of this example is principally to check the incremental objectivity of the integration algorithm.

Constitutive Equation

Only constitutive equation (8.7) is considered, although (8.5) or (8.6) would be equally appropriate for testing the efficacy of the solution procedure.

Analytical Results

The analytical solution to this problem for constitutive equation (8.7) is detailed in Appendix I. Referring to Fig. 9 and noting that $\theta(t)$ represents the rigid rotation of the block at time t and $\lambda(t)$ the axial stretch ratio (in the rotated coordinated frame), the analytical solution for two of the stress components (relative to the fixed spatial coordinate system) is given in Appendix I, for a unit cube, by:

$$\begin{Bmatrix} \tau_{11} \\ \tau_{22} \end{Bmatrix} = \begin{bmatrix} \cos^2\theta & \sin^2\theta \\ \sin^2\theta & \cos^2\theta \end{bmatrix} \begin{Bmatrix} 1 \\ \dfrac{\nu}{1-\nu} \end{Bmatrix} \frac{E(1-\nu)}{(1+\nu)(1-2\nu)} \ln\lambda \tag{8.25}$$

The particular problem to be solved is defined by:

$$\theta(t) = 2\pi t \quad \text{and} \quad \lambda(t) = 1+t$$
$$\text{for} \quad 0 \leqslant t \leqslant 1$$

Thus, the cube is stretched to double its length and simultaneously rotated through 360 degrees. The solid curves of Fig. 9 depict the analytical solutions (8.25) for this combination of θ and λ.

Numerical Results

The numerical solution was obtained (as in the previous examples) by discretization of the block into four quadrilateral plane strain finite elements. The mesh is shown in Fig. 9. The analysis is performed in the fixed spatial coordinate system with the θ and λ deformation states imposed by suitable prescription of the boundary node displacements. The numerical integration algorithm for constitutive equation (8.7) used subiteration within the Newton-Raphson equilibrium iterations. The stress components resulting from the analysis are shown as dots in Fig. 9. They display accuracy to within 0.1% of the analytical solution over the full range of deformation. Convergence was obtained for the problem using as few as ten equal increments of (θ,λ).

This analysis tends to confirm the incremental objectivity of the numerical solution procedure.

8.5 ELASTIC-PERFECTLY PLASTIC INFINITE CYLINDER SUBJECTED TO TO INTERNAL PRESSURE

In this elasto-plastic example, the hardening model represented by (4.38) - (4.42) is specialized to perfect plasticity by setting $E_t = 0$ in (4.38). The parameters employed are:

$$E = 2.07\times 10^{11}, \quad \nu = 0.3, \quad \bar{\sigma}_Y = 3.1\times 10^{8}.$$

The problem was analyzed by considering a quadrant of the tube modeled with a finite element mesh of 64 plane strain quadrilateral elements and employed displacement control of the boundary nodes. The results, Fig. 10, compare well with the analytical solution [54].

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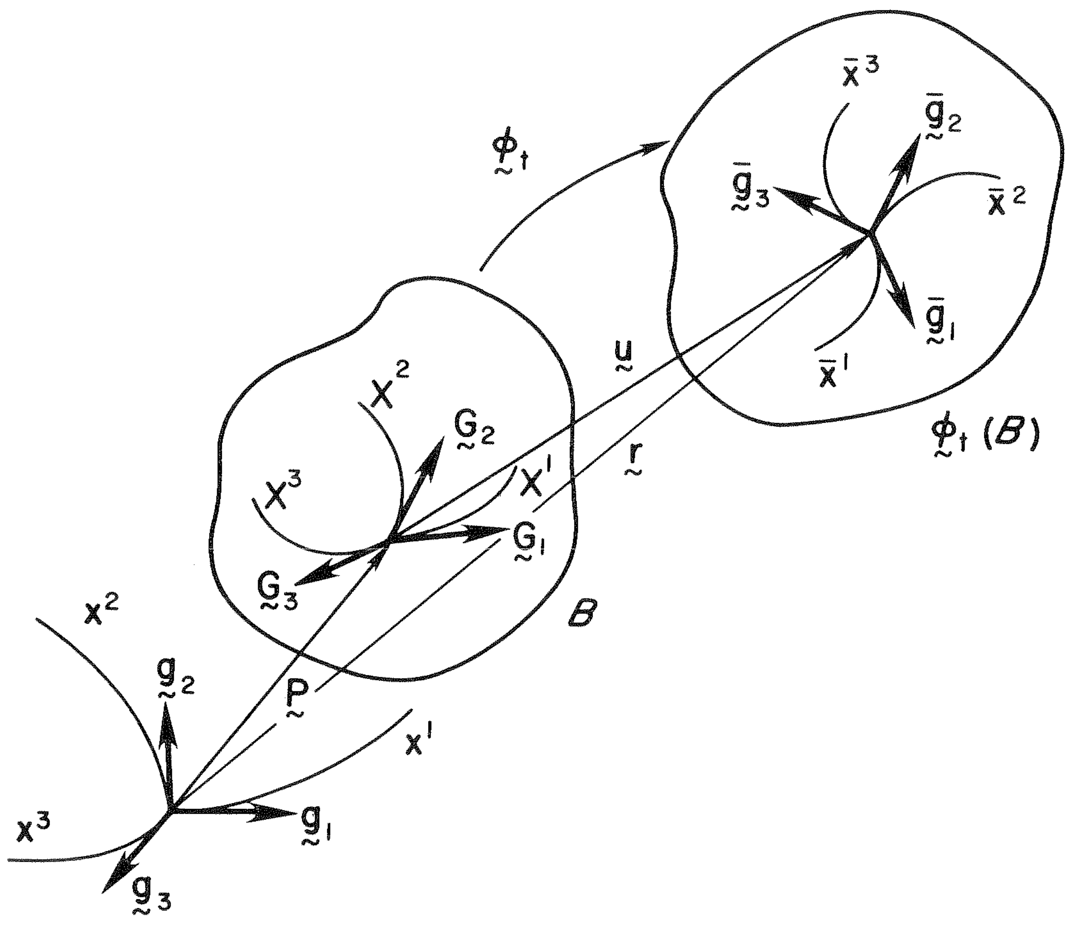


FIGURE 1. COORDINATE SYSTEMS

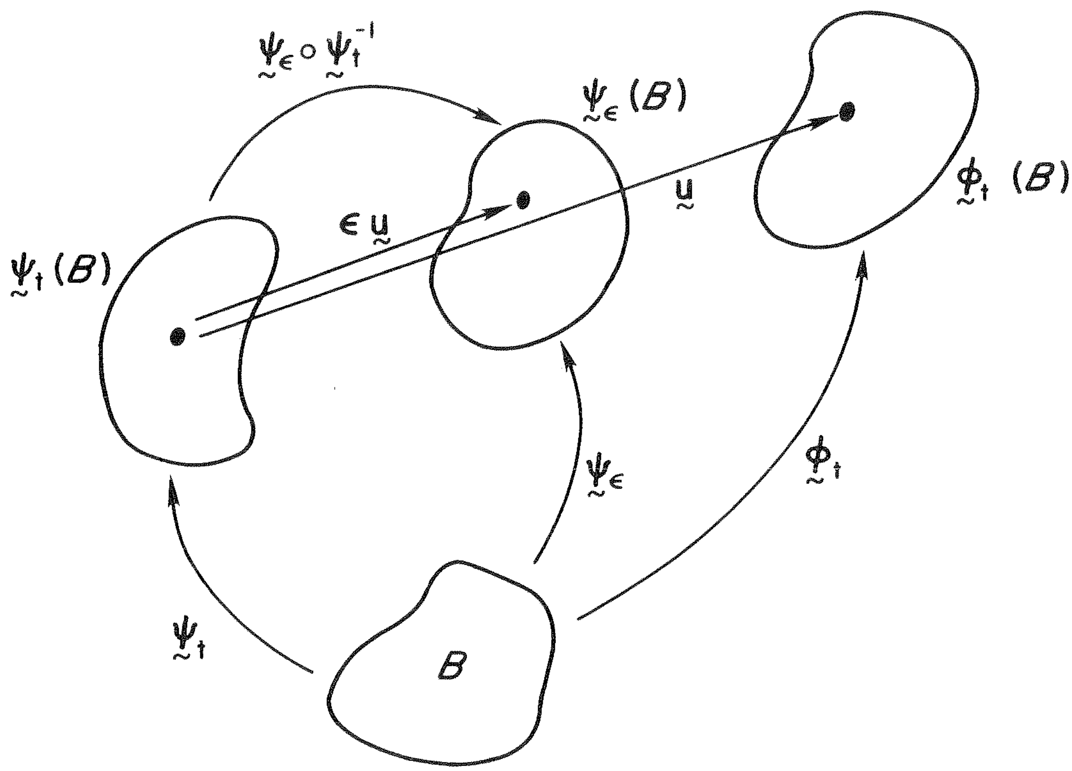


FIGURE 2. CONFIGURATIONS EMPLOYED IN THE DEFINITION OF LINEARIZATION

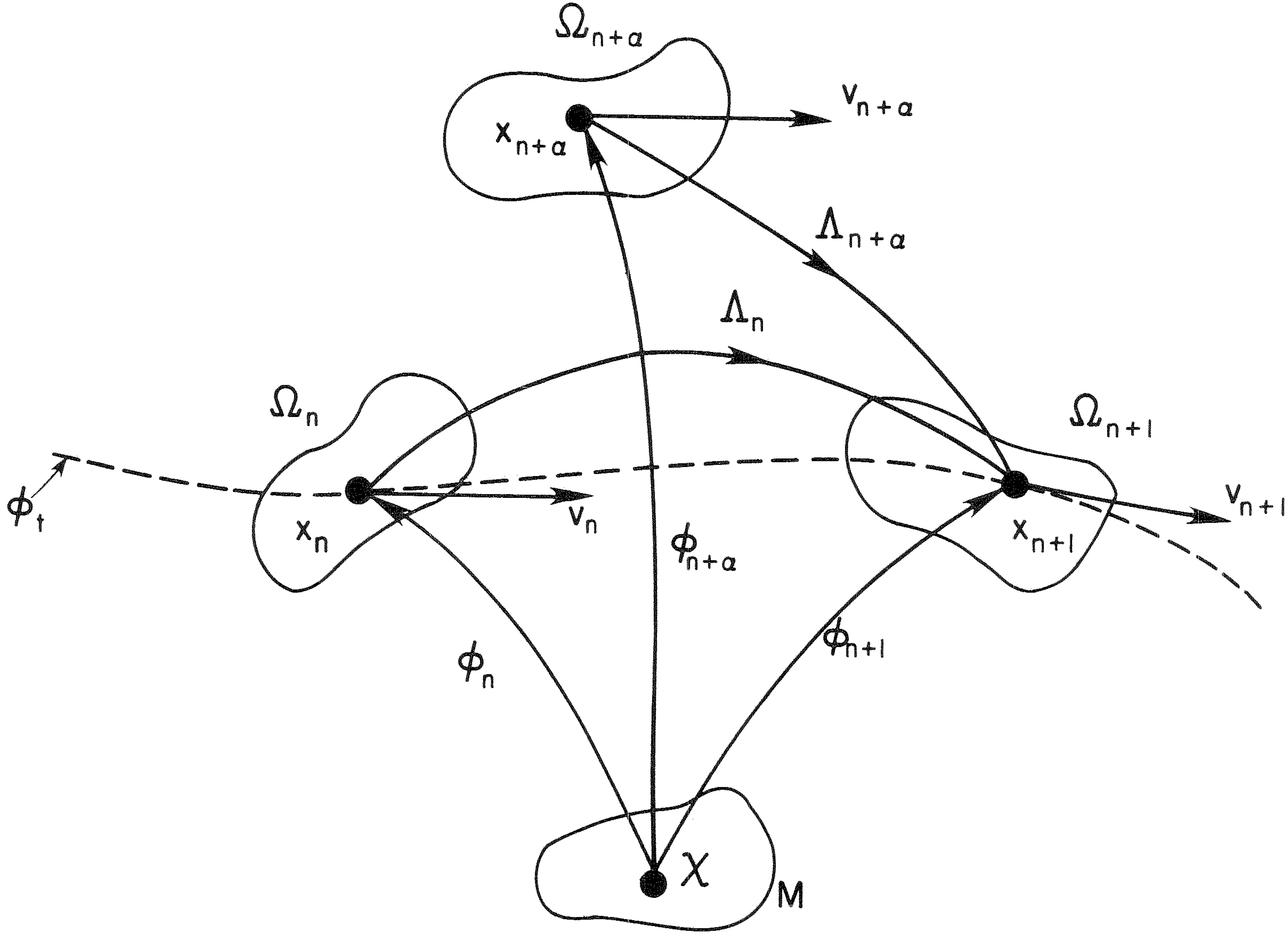


FIGURE 3. CONFIGURATIONS EMPLOYED IN THE NUMERICAL INTEGRATION OF RATE CONSTITUTIVE EQUATIONS BETWEEN TIMES t_n AND t_{n+1}

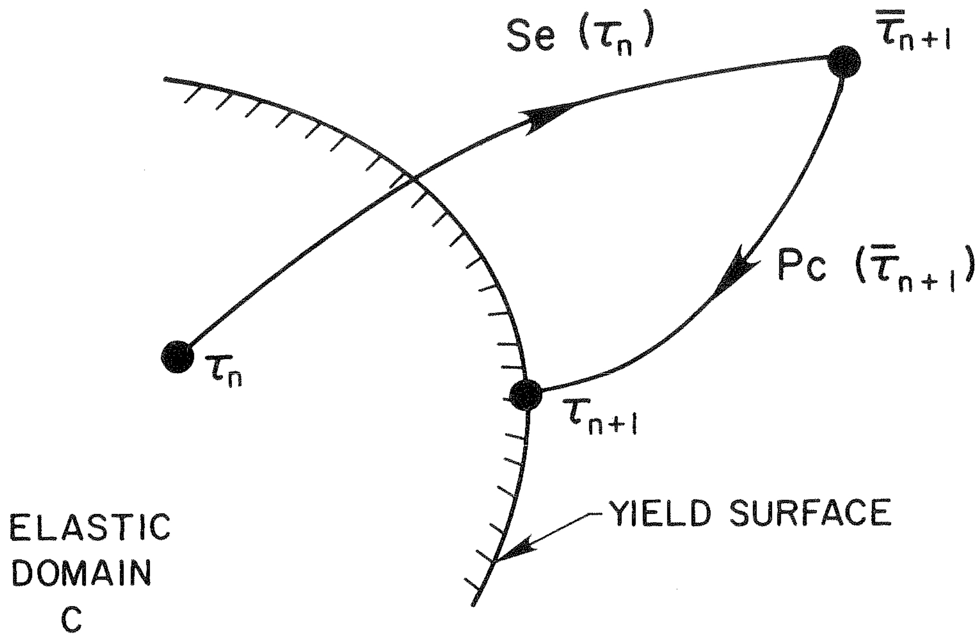
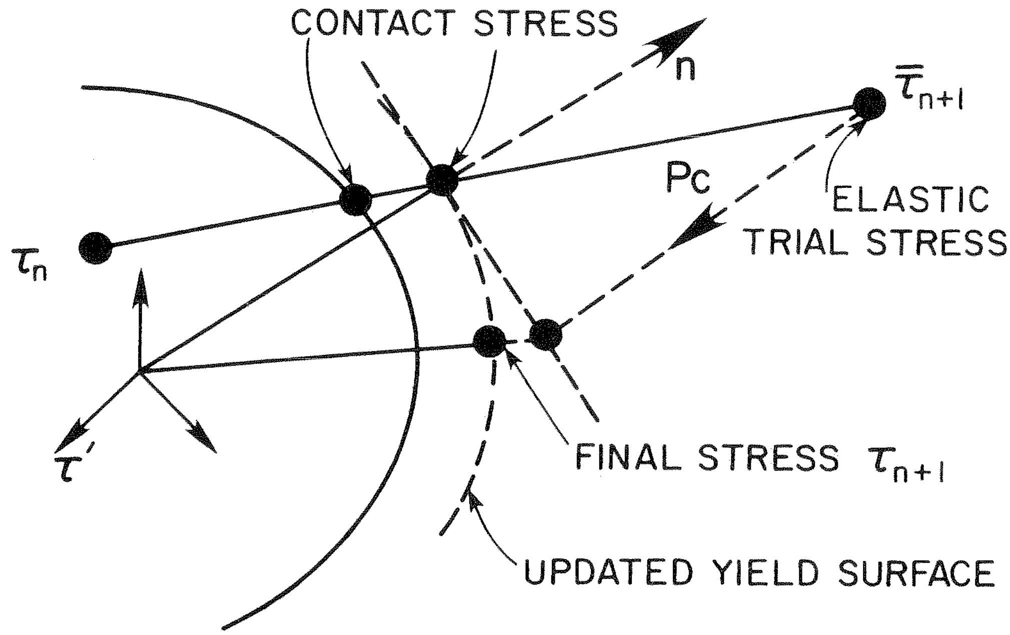


FIGURE 4. ELASTIC SOLUTION AND RETURN MAPPING OF STRESS POINT TO THE YIELD SURFACE



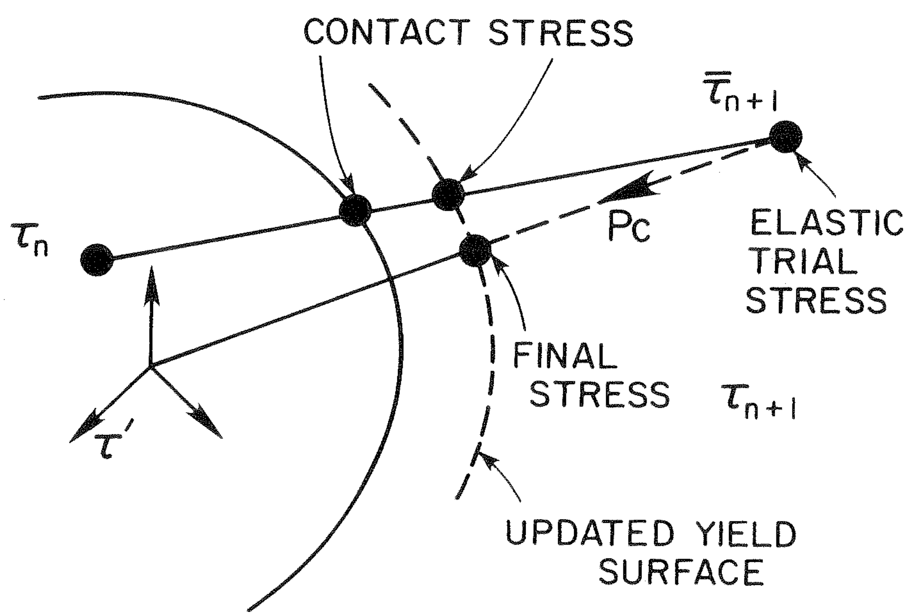


FIGURE 5. (A) TANGENT PREDICTOR, RADIAL RETURN

(B) RADIAL RETURN

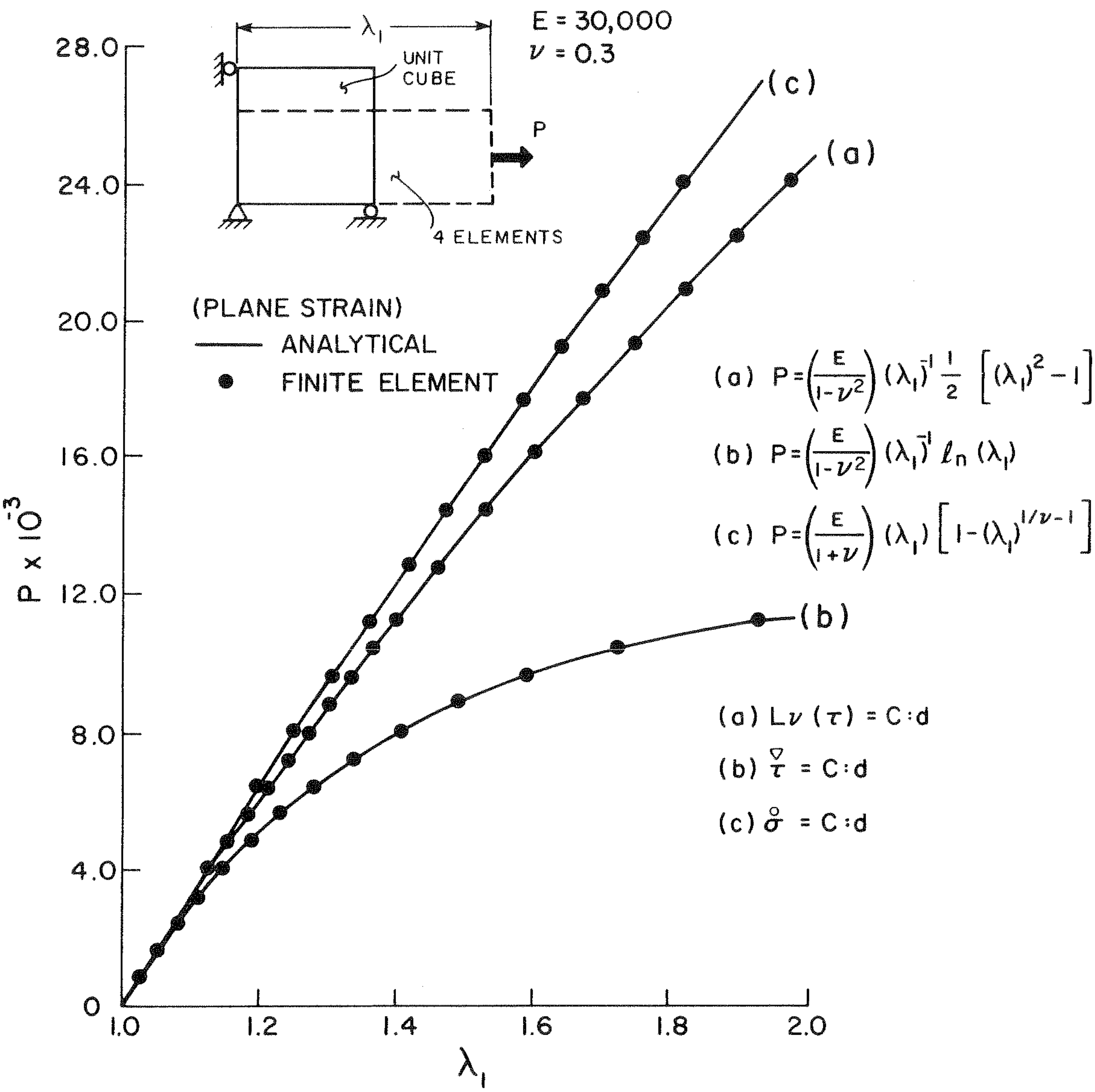


FIGURE 6. HOMOGENEOUS FINITE EXTENSION

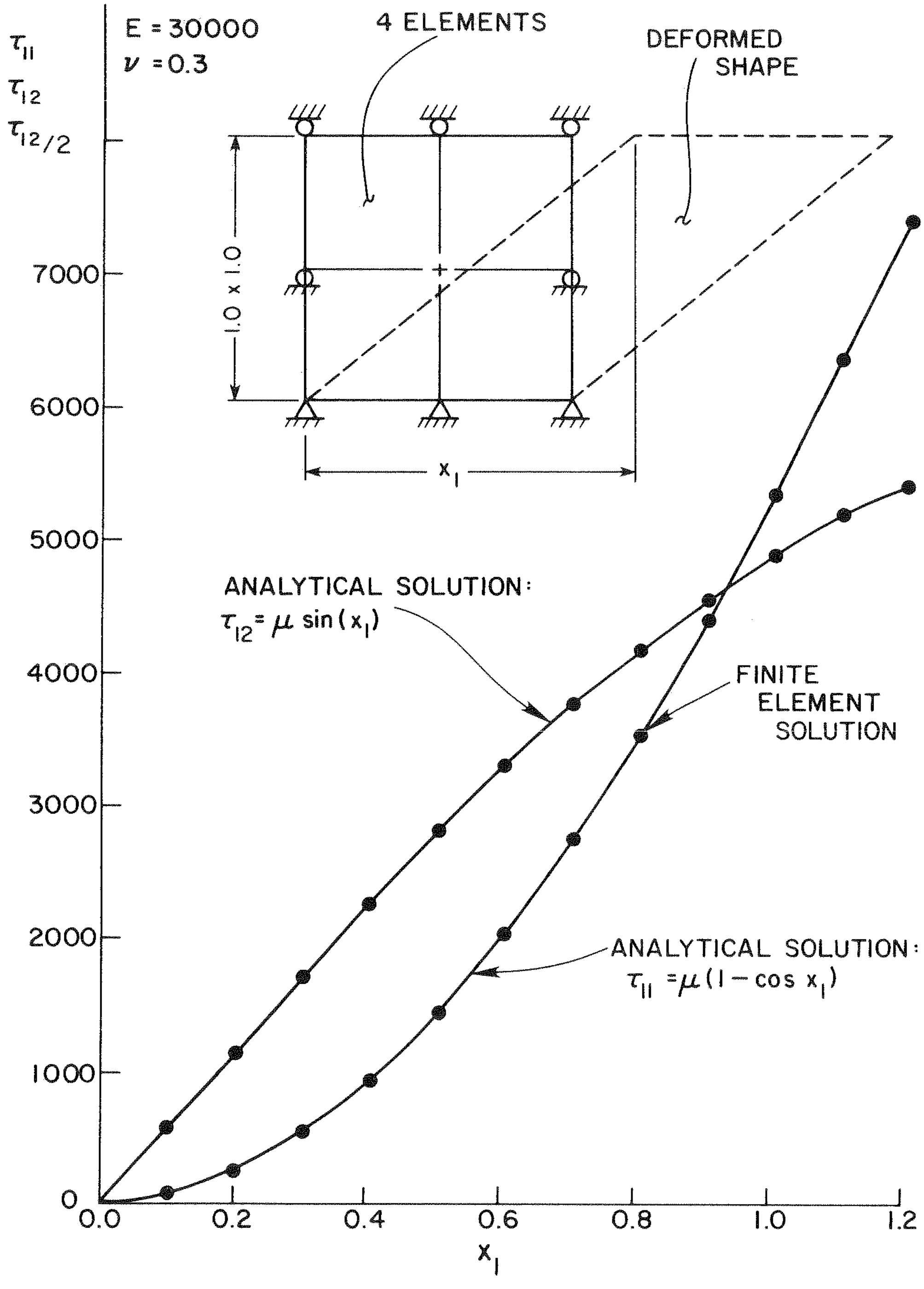


FIGURE 7. HOMOGENEOUS FINITE SIMPLE SHEAR

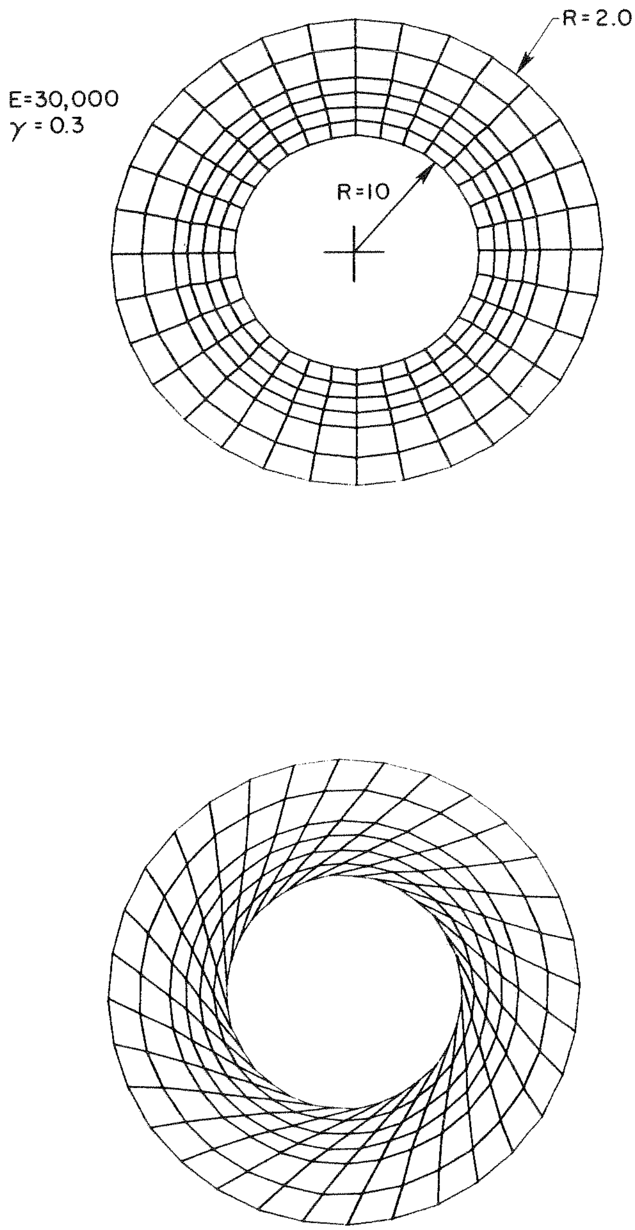


FIGURE 8. TORSION OF AN ELASTIC ANNULUS FOR MATERIAL MODEL CORRESPONDING TO (8.5)

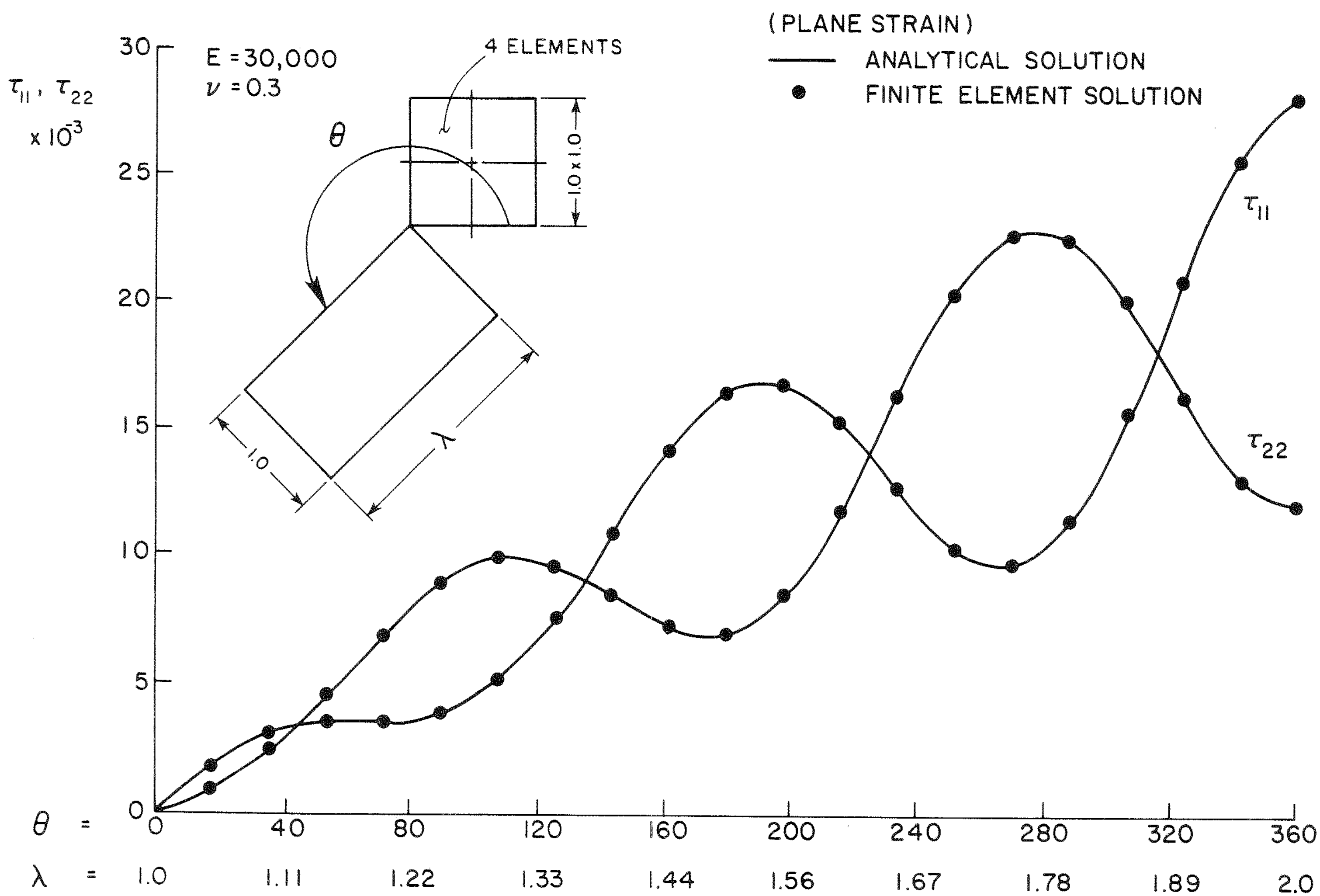
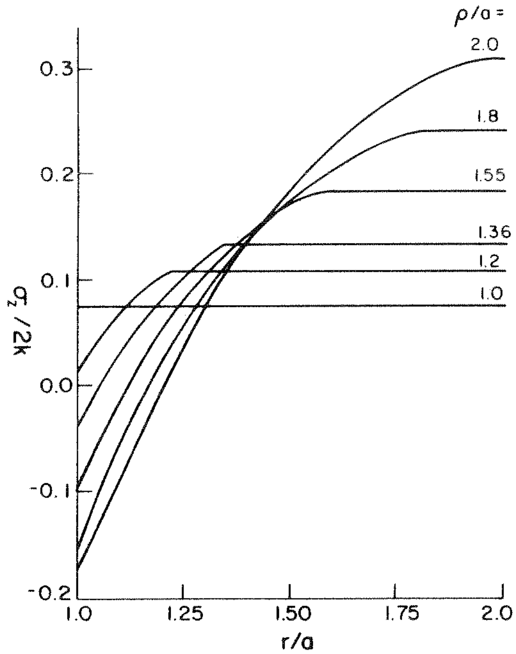
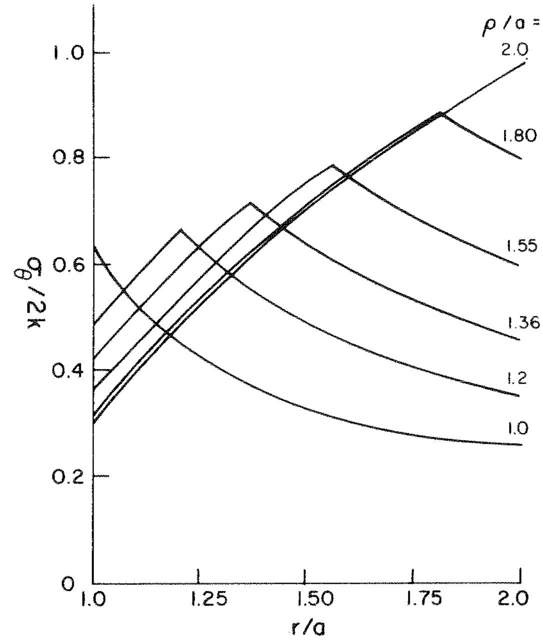


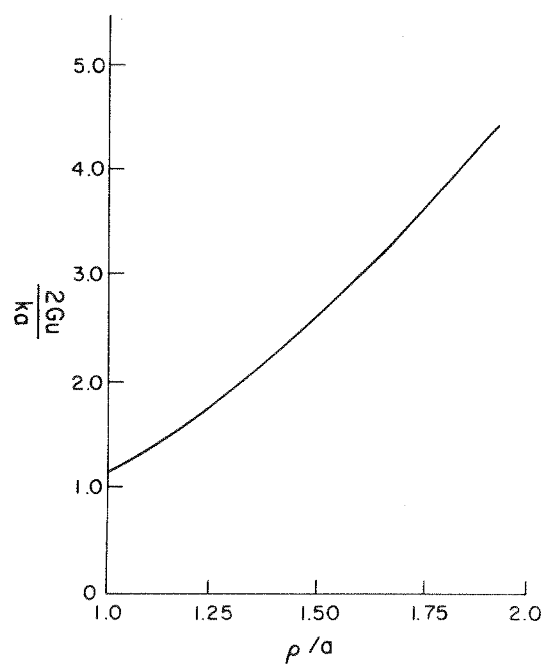
FIGURE 9. HOMOGENEOUS FINITE SIMPLE EXTENSION AND SIMULTANEOUS RIGID ROTATION



Distribution of axial stress.



Distribution of circumferential stress.



Inner radial displacement versus radius of elastic-plastic boundary.

FIGURE 10. INTERNALLY PRESSURIZED INFINITE CYLINDER

APPENDIX I

The direct integration of constitutive equations (8.5) - (8.7) subject to homogeneous deformation hypotheses corresponding to:

(i) Finite extension.

(ii) Finite simple shear.

(iii) Finite simple extension and rotation.

is considered.

(i) FINITE EXTENSION

See Section 8.2 for the problem description. The spatial velocity field is defined by (8.3):

$$v_1 = \alpha x_1, \quad v_2 = k\alpha x_2, \quad v_3 = 0 \tag{I.1}$$

where α and k are constants (k is problem dependent). From (I.1) it follows that:

$$d_{11} = \alpha, \quad d_{22} = k\alpha \tag{I.2}$$

with all other components of the rate of deformation tensor $\mathbf{d}$ being zero. The spin rate tensor $\boldsymbol{\omega}$ associated with (I.1) is identically zero.

Equation (I.1a) may be rewritten as:

$$\frac{d\left(\dfrac{x_1}{X_1}\right)}{dt} = \alpha\left(\frac{x_1}{X_1}\right) \tag{I.3}$$

Integrating (I.3) results in:

$$\ln \lambda_1 = \alpha t \tag{I.4}$$

where λ_1 is defined by (8.2). Similarly, it follows from (I.1) that:

$$\ln \lambda_2 = k\alpha t, \quad \ln \lambda_3 = 0 \tag{I.5}$$

Consequently,

$$x_1 = X_1 e^{\alpha t}, \quad x_2 = X_2 e^{k\alpha t}, \quad x_3 = X_3 \tag{I.6}$$

If the initial dimensions of the loaded cross section are given as Y_2 and Y_3, the cross section at time t has area:

$$A_1 = Y_2 Y_3 e^{k\alpha t} = Y_2 Y_3 (\lambda_1)^k \tag{I.7}$$

The constitutive equations to be integrated are (8.5), (8.6), and (8.7) and are considered in turn.

(a) Truesdell Rate of Cauchy Stress

From Definition 2.14(c):

$$\overset{\circ}{\sigma} = \dot{\sigma} - \mathbf{l}\cdot\sigma - \sigma\cdot\mathbf{l}^T + \sigma\, tr(\mathbf{d}) \tag{I.8}$$

Using (I.2), (8.4) and (I.8) in (8.5) results in:

$$\begin{aligned} \dot{\sigma}_{11} - 2\alpha\sigma_{11} + \sigma_{11}(1+k)\alpha &= (\lambda+2\mu) + \lambda k\alpha \\ \dot{\sigma}_{22} - 2k\alpha\sigma_{22} + \sigma_{22}(1+k)\alpha &= (\lambda+2\mu)k\alpha + \lambda\alpha \\ \dot{\sigma}_{33} + \sigma_{33}(1+k)\alpha &= \lambda\alpha(1+k) \end{aligned} \tag{I.9}$$

From the condition $\dot{\sigma}_{22} = \sigma_{22} = 0$ is obtained:

$$k = \frac{-\lambda}{\lambda+2\mu} = \frac{\nu}{\nu-1} \tag{I.10}$$

Integrating (I.9a) and (I.9c) subject to the initial conditions $\sigma_{11}(0) = \sigma_{33}(0) = 0$ and using (I.4) and (I.10) leads to:

$$\sigma_{11} = \left(\frac{E}{1+\nu}\right)\left[\left(\lambda_1\right)^{\frac{1}{1-\nu}} - 1\right] \tag{I.11}$$

$$\sigma_{33} = -\frac{E\nu}{(1+\nu)(1-2\nu)}\left[\left(\lambda_1\right)^{\frac{1-2\nu}{\nu-1}} - 1\right] \tag{I.12}$$

The axial load required to produce an extension λ_1 is given by $P_1 = \sigma_{11}A_1$. Using the value of k given by (I.10) in (I.7), it follows that:

$$A_1 = Y_2 Y_3 (\lambda_1)^{\frac{\nu}{\nu-1}} \tag{I.13}$$

and

$$P_1 = Y_2 Y_3\left(\frac{E}{1+\nu}\right)\left(\lambda_1\right)\left[1 - (\lambda_1)^{\frac{1}{\nu-1}}\right] \tag{I.14}$$

(b) Lie Derivative of Kirchhoff Stress

From Definition 2.14(a), it is noted that:

$$L_v(\tau) = \dot{\tau} - \mathbf{l}\cdot\tau - \tau\cdot\mathbf{l}^T \tag{I.15}$$

Observing that $\mathbf{l}=\mathbf{d}$ according to (I.1), using (I.2), (8.4) and (I.15) in (8.6) results in:

$$\begin{aligned}
\dot{\tau}_{11} - 2\alpha\tau_{11} &= (\lambda+2\mu)\,\alpha + k\lambda\alpha \\
\dot{\tau}_{22} - 2k\alpha\tau_{22} &= (\lambda+2\mu)\,k\alpha + \lambda\alpha \\
\dot{\tau}_{33} &= \lambda\alpha(1+k)
\end{aligned} \tag{I.16}$$

From the condition $\dot{\tau}_{22} = \tau_{22} = 0$, (I.16b) yields:

$$k = \frac{\nu}{\nu - 1} \tag{I.17}$$

Integration of (I.16a) and (I.16c) and use of (I.4) and (I.17) yields:

$$\tau_{11} = \left(\frac{E}{1-\nu^2}\right)\frac{1}{2}\left[\left(\lambda_1\right)^2 - 1\right] \tag{I.18}$$

$$\tau_{33} = \left(\frac{E\nu}{1-\nu^2}\right)\ln\lambda_1 \tag{I.19}$$

Recall that $\sigma = J^{-1}\tau$, where $J = \det(\mathbf{F})$. From (I.6) and (I.4) it follows that:

$$\mathbf{F} = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & (\lambda_1)^{\frac{\nu}{\nu-1}} & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{I.20}$$

from which $J = (\lambda_1)^{\frac{2\nu-1}{\nu-1}}$.

From (I.7) and (I.17),

$$A_1 = Y_2 Y_3 (\lambda_1)^{\frac{\nu}{\nu-1}}$$

The axial force required to produce an extension λ_1 is again given by $P_1 = \sigma_{11}A_1 = J^{-1}\tau_{11}A_1$.

From the above, it follows that:

$$P_1 = Y_2 Y_3\left(\frac{E}{1-\nu^2}\right)\frac{1}{2}(\lambda_1)^{-1}\left[(\lambda_1)^2 - 1\right] \tag{I.21}$$

(c) Co-Rotational Rate of Kirchhoff Stress

Following an analysis similar to (b) above, it may be shown that for constitutive equation (8.7):

$$\sigma_{11} = \left(\frac{E}{1-\nu^2}\right)\left(\lambda_1\right)^{\frac{2\nu-1}{1-\nu}} \ln\lambda_1 \tag{I.22}$$

$$\sigma_{33} = \left(\frac{E\,\nu}{1-\nu^2}\right)\left(\lambda_1\right)^{\frac{2\nu-1}{1-\nu}} \ln\lambda_1 \tag{I.23}$$

The axial force P_1 required to produce an extension λ_1 is in this case:

$$P_1 = Y_2 Y_3 \left(\frac{E}{1-\nu^2}\right)\left(\lambda_1\right)^{-1} \ln\lambda_1 \tag{I.24}$$

(ii) FINITE SIMPLE SHEAR

See Section 8.3 for the problem description. The spatial velocity field is defined by (8.18):

$$v_1 = 2\alpha X_2, \quad v_2 = v_3 = 0 \tag{I.25}$$

where α is a parameter. From (I.25) it follows that:

$$d_{12} = d_{21} = \alpha \tag{I.26}$$

$$\omega_{12} = -\omega_{21} = \alpha \tag{I.27}$$

all other components of $\mathbf{d}$ and $\boldsymbol{\omega}$ are zero.

From (I.25), $x_1 = X_1 + 2\alpha t X_2$, $x_2 = X_2$, and $x_3 = X_3$. Thus, $J = \det(\mathbf{F}) = 1$ and the motion is isochoric. In this case, constitutive equations (8.5) and (8.6) are equivalent. The constitutive equations to be integrated are thus (8.5) and (8.7) and are considered in turn.

(a) Truesdell Rate of Cauchy Stress

From Definition 2.14(c):

$$\overset{\circ}{\sigma} = \dot{\sigma} - \mathbf{l}\cdot\sigma - \sigma\cdot\mathbf{l}^T + \sigma\, tr(\mathbf{d}) \tag{I.28}$$

Using (I.26), (I.27), (8.4) and (I.28) in (8.5) results in:

$$\begin{aligned} \dot{\sigma}_{11} &= 2\alpha\sigma_{12} \\ \dot{\sigma}_{22} &= 2\alpha\sigma_{12} \\ \dot{\sigma}_{12} &= 2\mu\alpha + \alpha(\sigma_{11} + \sigma_{22}) \end{aligned} \tag{I.29}$$

Integration of (I.29) subject to the initial conditions $\sigma_{11}(0) = \sigma_{22}(0) = \sigma_{12}(0) = 0$ yields:

$$\begin{aligned} \sigma_{12} &= \mu \sinh(2\alpha t) \\ \sigma_{11} &= \mu\,[1-\cosh(2\alpha t)] \\ \sigma_{22} &= \mu\,[\cosh(2\alpha t)-1] \end{aligned} \tag{I.30}$$

(b) Co-Rotational Rate of Kirchhoff Stress

From Definition 2.15:

$$\overset{\nabla}{\tau} = \dot{\tau} - \omega \cdot \tau + \tau \cdot \omega \tag{I.31}$$

Using (I.26), (I.27), (8.4) and (I.31) in (8.7) results in:

$$\begin{aligned} \dot{\tau}_{11} &= 2\alpha\tau_{12} \\ \dot{\tau}_{22} &= -2\alpha\tau_{12} \\ \dot{\tau}_{12} &= 2\mu\alpha + \alpha(\tau_{22}-\tau_{12}) \end{aligned} \tag{I.32}$$

Integration of (I.32) subject to the initial conditions $\tau_{11}(0) = \tau_{22}(0) = \tau_{12}(0) = 0$ yields:

$$\begin{aligned} \tau_{12} &= \mu \sin(2\alpha t) \\ \tau_{11} &= \mu\,[1-\cos(2\alpha t)] \\ \tau_{22} &= \mu\,[\cos(2\alpha t)-1] \end{aligned} \tag{I.33}$$

Since $J = 1$, $\tau = \sigma$. Referring to Fig. 7, $X_1 = 0$, $X_2 = 1 \rightarrow x_1 = 2\alpha t$.

(iii) FINITE SIMPLE EXTENSION AND ROTATION

See Section 8.4 for the problem description. The problem is solved in two parts: first a solution to the simple extension problem relative to a coordinate system rotating rigidly with the block is obtained and then the results are transformed to components referred to the fixed spatial coordinate system.

Simple Extension Problem

The spatial velocity field:

$$v_1 = \alpha x_1, \quad v_2 = v_3 = 0 \tag{I.34}$$

defines a state of homogeneous simple extension. From (I.34):

$$d_{11} = \alpha \tag{I.35}$$

all other components of $\mathbf{d}$ and $\boldsymbol{\omega}$ are zero. The constitutive equation to be integrated is (8.7). Using (I.35) and (8.4) in (8.7) results in:

$$
\begin{aligned}
\dot{\bar{\tau}}_{11} &= \alpha(\lambda + 2\mu) \\
\dot{\bar{\tau}}_{22} &= \alpha\lambda \\
\dot{\bar{\tau}}_{33} &= \alpha\lambda
\end{aligned}
\tag{I.36}
$$

Defining $\lambda_1 = \dfrac{x_1}{X_1}$ as the axial stretch ratio, the integration of (I.36) subject to the initial conditions $\bar{\tau}_{11}(0) = \bar{\tau}_{22}(0) = \bar{\tau}_{33}(0) = 0$, yields:

$$
\bar{\tau}_{11} = (\lambda + 2\mu)\ln\lambda_1 = \frac{E(1-\nu)}{(1+\nu)(1-2\nu)}\ln\lambda_1 \tag{I.37}
$$

$$
\bar{\tau}_{22} = \bar{\tau}_{33} = \lambda\ln\lambda_1 = \frac{E\nu}{(1+\nu)(1-2\nu)}\ln\lambda_1 \tag{I.38}
$$

where $\ln\lambda_1 = \alpha t$ has been used and which results from the integration of (I.34a).

Assuming (I.37) and (I.38) to be referred to a coordinate system which has rotated counter-clockwise through an angle θ from the fixed spatial coordinate system, they may be transformed to components referred to the fixed spatial coordinate system by the operation:

$$
\begin{Bmatrix} \tau_{11} \\ \tau_{22} \end{Bmatrix} = \begin{bmatrix} \cos^2\theta & \sin^2\theta \\ \sin^2\theta & \cos^2\theta \end{bmatrix} \begin{Bmatrix} 1 \\ \dfrac{\nu}{1-\nu} \end{Bmatrix} \frac{E(1-\nu)}{(1+\nu)(1-2\nu)}\ln\lambda_1 \tag{I.39}
$$