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AN EXACT SEQUENCE IN DIFFERENTIAL TOPOLOGY

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1. Introduction. The purpose of this note is to describe an exact sequence relating three series of abelian groups: \( \Gamma^n \), defined by Thom \([3]\); \( \theta^n \), defined by Milnor \([1]\); and \( \Lambda^n \), defined below. The sequence is written

\[
\cdots \rightarrow \Gamma^n \xrightarrow{j} \theta^n \xrightarrow{k} \Lambda^n \xrightarrow{d} \Gamma^{n-1} \rightarrow \cdots
\]

We now describe these groups briefly.

To obtain \( \Gamma^n \), divide the group of diffeomorphisms of the \( n-1 \) sphere \( S^{n-1} \) by the normal subgroup of those diffeomorphisms that are extendable to the \( n \)-ball. See \([2]\) for details.

The set \( \theta^n \) is the set of \( J \)-equivalence classes of closed, oriented, differentiable \( n \)-manifolds that are homotopy spheres. If \( M \) is an oriented manifold, let \( -M \) be the oppositely oriented manifold. Two closed oriented \( n \)-manifolds \( M \) and \( N \) are \( J \)-equivalent if there is an oriented \( n+1 \)-manifold \( X \) whose boundary is the disjoint union of \( M \) and \( -N \), and which admits both \( M \) and \( N \) as deformation retracts. We denote the \( J \)-equivalence class of \( M \) by \([M]\). If \([M]\) and \([N]\) are elements of \( \theta^n \), their sum is defined to be \([M \# N]\), where \([M \# N]\) is obtained by removing the interior of an \( n \)-ball from \( M \) and \( N \) and identifying the boundaries in a suitable way. Details may be found in \([1]\).

The group \( \Lambda^n \) is defined analogously using combinatorial manifolds. Instead of the interior of an \( n \)-ball, the interior of an \( n \)-simplex is removed. If \( M \) is a combinatorial manifold, we write \( \langle M \rangle \) for its \( J \)-equivalence class.

2. The sequence. To define \( k: \theta^n \rightarrow \Lambda^n \), we observe that every differentiable manifold \( M \) defines a combinatorial manifold \( \overline{M} \), unique up to combinatorial equivalence, by means of a smooth triangulation of \( M \) \([4]\). We define \( k[M] = \langle \overline{M} \rangle \).

Let \( g: S^{n-1} \rightarrow S^{n-1} \) represent an element \( \gamma \) of \( \Gamma^n \). According to J. Munkres \([2]\), there is a unique (up to diffeomorphism) differentiable manifold \( V_\gamma \) corresponding to \( \gamma \), such that \( \overline{V_\gamma} = S^n \). To obtain \( V_\gamma \), identify two copies of \( R^n - 0 \) by the diffeomorphism \( x \rightarrow (1/|x|)g(x/|x|) \). Here \( R^n \) is Euclidean \( n \)-space and \( |x| \) is the usual norm. The diffeomorphism class of \( V_\gamma \) depends only on \( \gamma \), and
$V_γ$ is diffeomorphic to $V_δ$ if and only if $γ = δ$. We define $j : Γ^n → θ^n$ by $j(γ) = [V_γ]$. To define $d : Δ^n → Γ^{n-1}$, we proceed as follows. If $⟨M⟩$ is an element of $Δ^n$, let $M_0$ be obtained from $M$ by removing the interior of an $n$-simplex. According to a recent result of A. M. Gleason, there is a differentiable manifold $N$ such that $N = M_0$. By [2], $N$ is unique up to diffeomorphism, because $M_0$ is contractible. Since the boundary $∂N$ of $N$ is combinatorially an $n-1$ sphere, there is a unique $β ∈ Γ^{n-1}$ such that $∂N = V_β$. We define $d(M) = β$. It can be shown that $β$ depends only on $⟨M⟩$.

**Theorem.** The sequence (1) is exact.

The proof will appear in a subsequent paper.

**Bibliography**

1. J. Milnor, *Differentiable manifolds which are homotopy spheres*, Princeton University, 1959 (mimeographed).