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AN EXACT SEQUENCE IN DIFFERENTIAL TOPOLOGY

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1. **Introduction.** The purpose of this note is to describe an exact sequence relating three series of abelian groups: Γ^n, defined by Thom [3]; θ^n, defined by Milnor [1]; and Λ^n, defined below. The sequence is written

$$\cdots \longrightarrow \Gamma^n \xrightarrow{j} \theta^n \xrightarrow{k} \Lambda^n \xrightarrow{d} \Gamma^{n-1} \longrightarrow \cdots. \tag{1}$$

We now describe these groups briefly.

To obtain Γ^n, divide the group of diffeomorphisms of the $n-1$ sphere S^{n-1} by the normal subgroup of those diffeomorphisms that are extendable to the n-ball. See [2] for details.

The set θ^n is the set of J-equivalence classes of closed, oriented, differentiable n-manifolds that are homotopy spheres. If M is an oriented manifold, let $-M$ be the oppositely oriented manifold. Two closed oriented n-manifolds M and N are *J-equivalent* if there is an oriented $n+1$-manifold X whose boundary is the disjoint union of M and $-N$, and which admits both M and N as deformation retracts. We denote the J-equivalence class of M by $[M]$. If $[M]$ and $[N]$ are elements of θ^n, their sum is defined to be $[M \# N]$, where $[M \# N]$ is obtained by removing the interior of an n-ball from M and N and identifying the boundaries in a suitable way. Details may be found in [1].

The group Λ^n is defined analogously using combinatorial manifolds. Instead of the interior of an n-ball, the interior of an n-simplex is removed. If M is a combinatorial manifold, we write $\langle M \rangle$ for its J-equivalence class.

2. **The sequence.** To define $k: \theta^n \rightarrow \Lambda^n$, we observe that every differentiable manifold M defines a combinatorial manifold $\overline{M}$, unique up to combinatorial equivalence, by means of a smooth triangulation of M [4]. We define $k[M] = \langle \overline{M} \rangle$.

Let $g: S^{n-1} \rightarrow S^{n-1}$ represent an element γ of Γ^n. According to J. Munkres [2], there is a unique (up to diffeomorphism) differentiable manifold V_γ corresponding to γ, such that $\overline{V}_\gamma = \overline{S}^n$. To obtain V_γ, identify two copies of $R^n - 0$ by the diffeomorphism $x \rightarrow (1/|x|)g(x/|x|)$. Here R^n is Euclidean n-space and $|x|$ is the usual norm. The diffeomorphism class of V_γ depends only on γ, and

V_γ is diffeomorphic to V_δ if and only if $\gamma=\delta$. We define $j\colon \Gamma^n \to \theta^n$ by $j(\gamma)=[V_\gamma]$.

To define $d\colon \Lambda^n \to \Gamma^{n-1}$, we proceed as follows. If $\langle M\rangle$ is an element of Λ^n, let M_0 be obtained from M by removing the interior of an n-simplex. According to a recent result of A. M. Gleason, there is a differentiable manifold N such that $\overline{N}=M_0$. By [2], N is unique up to diffeomorphism, because M_0 is contractible. Since the boundary ∂N of N is combinatorially an $n-1$ sphere, there is a unique $\beta\in\Gamma^{n-1}$ such that $\partial N=V_\beta$. We define $d\langle M\rangle=\beta$. It can be shown that β depends only on $\langle M\rangle$.

THEOREM. *The sequence* (1) *is exact.*

The proof will appear in a subsequent paper.

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