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By

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Abstract

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In this dissertation we prove results relating to Martin's Conjecture, a foundational conjecture in Recursion Theory, as well as results in Inner Model Theory, a part of Set Theory which plays a central role in the meta-mathematics of Set Theory. While the different parts of the thesis are not closely related, our interest in the work presented here stems from an interest in the foundations of mathematics.

In chapter 1, we prove results relating to Martin's Conjecture which are joint work with Patrick Lutz. Among other things, we show that part of the conjecture holds for a natural class of functions, the order-preserving functions. In chapter 2, we prove uniqueness theorems about the core model, a fundamental object of study in Inner Model Theory. Our theorems identify the core model in elementary set-theoretic terms, whereas the usual definitions of the core model require deep knowledge of Inner Model Theory. Finally, in chapter 3, we develop the theory meta-iteration trees, a framework for studying the kind of iteration tree combinatorics which has become central to the study of mouse pairs and is relevant for applications of Inner Model Theory to Descriptive Set Theory.

For Theo, my cat

Contents

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This dissertation is dedicated to my cat Theo, who passed away very suddenly shortly before the filling deadline. He was a constant distraction at home, where I wrote most of this thesis. I've missed him dearly in the short time spent editing this without him.

Chapter 1

Progress on Martin's Conjecture

There seems to be significant global structure to the functions on the Turing degrees which come up naturally in Recursion Theory. In particular, the only *natural* functions seem to be constant functions, the identity, and iterates of the Turing jump, or minor variations of these functions. As one might expect, we can use a well-order of the reals to build pathological functions, so that this observed structure does not hold for *all* functions on the Turing degrees, but such functions are not thought to be *natural* from the point of view of Recursion Theory. Martin's Conjecture asserts, roughly, that the observed structure on natural functions on the Turing degrees holds for all functions under the Axiom of Determinacy, AD. Assuming the existence of large cardinals, $L(\mathbb{R})$ satisfies AD, so that results under AD apply to functions which are constructible from the reals. While the relationship between the "natural" functions and those in $L(\mathbb{R})$ is not clear, establishing the truth of Martin's Conjecure would vindicate the intuition that there is something real to the observed structure on the natural functions.

In this chapter, we show that part of Martin's Conjecture holds for order-preserving functions on the reals, along with some other results relating to Martin's Conjecture. All of this work is joint with Patrick Lutz and also appears in his PhD thesis [12].

1.1 Preliminaries

In this section we'll state Martin's Conjecture precisely, along with some relevant well-known theorems of ZF+AD. We'll also discuss previous work on Martin's Conjecture due to Ted Slaman and John Steel. First, we establish some notation and review basic definitions.

We'll use the lowercase Latin alphabet to denote reals, either elements of Cantor space, 2^ω, Baire space, ω^ω, or some other standard structure. We let $\mathcal{D}_T$ denote the quotient of 2^ω under Turing-equivalence, $\equiv_T$, and use boldface lowercase Latin letters to denote elements of $\mathcal{D}_T$.

Definition 1.1.1. A function $f : \mathbb{R} \to \mathbb{R}$ is *Turing-invariant* iff for all $x, y \in \mathbb{R}$, $x \equiv_T y$ implies $f(x) \equiv_T f(y)$.

A Turing-invariant function f induces a function on the Turing degrees F by letting $F([x]_T) = [f(x)]_T$, where $[z]_T$ denotes the degree of the real z, i.e. its equivalence class under Turing-equivalence.

Under AC, every function on the Turing degrees arises in this way, though it is not known whether this holds under ZF+AD. As we will mention later on, it does hold under Woodin's AD^+, and so holds in $L(\mathbb{R})$ under AD, in particular.

Going forward, we take ZF as our base theory and state any additional hypothesis. We'll need the following well-known consequences of determinacy, which can be found in Moschovakis [14], for example.

Theorem 1.1.2 (Mycielski). *Assume AD. Then countable choice for reals, $\mathsf{CC}_{\mathbb{R}}$, holds, i.e. for every countable family of sets of reals, $\{A_n \mid n \in \omega\}$, there is a function $f : \omega \to \mathbb{R}$ such that $f(n) \in A_n$ for all $n \in \omega$.*

Theorem 1.1.3 (Davis). *Assume AD. Then every set of reals A has the perfect set property, i.e. for every set of reals A, either A is countable or there is a perfect tree S such that $[S] \subseteq A$.*

The most essential result is the following theorem of Martin.

Theorem 1.1.4 (Martin). *Assume AD. Then for every set of reals A which is $\leq_T$-cofinal, there is a pointed perfect tree S such that $[S] \subseteq A$.*

Here, a tree $S \subseteq 2^{<\omega}$ is *pointed perfect* if it is perfect and every $x \in [S]$ (the infinite paths of S), $x \geq_T S$.

A *cone of Turing degrees* is a set of Turing degrees of the form $\{\mathbf{x} \in \mathcal{D}_T \mid \mathbf{x} \geq_T \mathbf{b}\}$ for some degree $\mathbf{b}$.

It's easy to see that if a tree S is pointed perfect, the degrees of infinite paths of S, i.e. $\{[x]_T \mid x \in [S]\}$, is a cone.

For $A \subseteq \mathcal{D}_T$, let's put $A \in \mu$ iff A contains a cone. μ is called the *Martin measure*. The following result, Martin's Cone Theorem, is an immediate corollary of the previous theorem.

Theorem 1.1.5 (Martin). *Assume AD.*

1. *For every $A \subseteq \mathcal{D}_T$, either A contains a cone or A is disjoint from a cone,*
2. *μ is a countably complete ultrafilter on $\mathcal{D}_T$.*

(1) is sometimes called *Turing Determinacy*, TD. Note that (2) easily implies (1) over ZF. (1) clearly implies (2) over $\mathrm{CC}_{\mathbb{R}}$. Recently, Peng and Yu [15] showed that (1) implies $\mathrm{CC}_{\mathbb{R}}$, so (1) and (2) are equivalent. These statements are also equivalent to the hypothesis that every game with Turing-invariant pay-off set is determined.

We make the following definition.

Definition 1.1.6. For $F, G : \mathcal{D} \to \mathcal{D}$,

$$F \leq_M G \Leftrightarrow \text{ for } \mu\text{-a.e. } \mathbf{x},\ F(\mathbf{x}) \leq_T G(\mathbf{x})$$
$$F \equiv_M G \Leftrightarrow \text{ for } \mu\text{-a.e. } \mathbf{x},\ F(\mathbf{x}) = G(\mathbf{x})$$

Of course, these relations also make sense between degree-invariant functions by comparing the induced functions on degrees. We can now state Martin's Conjecture.

Definition 1.1.7. Martin's Conjecture is the assertion that the following are provable from ZF+AD+DC.

1. For any Turing-invariant $f : \mathbb{R} \to \mathbb{R}$, if f is not $\equiv_M$-equivalent to a constant function, then $f \geq_M id_{\mathbb{R}}$,

2. $\leq_M$ is a pre-well-order of the Turing-invariant functions $f : \mathbb{R} \to \mathbb{R}$ such that $id_{\mathbb{R}} \leq_M f$ and the $\leq_M$-successor of any f is $J \circ f$.

Here $J : \mathbb{R} \to \mathbb{R}$ is the *Turing jump*, i.e. $J(z) = z'$ for any real z. Of course, this conjecture is due to Tony Martin, in the late 70's.

Both parts of Martin's Conjecture are still open, even for Borel functions, but there have been partial results.

Definition 1.1.8. A function $f : \mathbb{R} \to \mathbb{R}$ is *uniformly invariant* iff there is a function $u : \omega \times \omega \to \omega \times \omega$ such that if $\Phi_i(x) = y$ and $\Phi_j(y) = x$, then $\Phi_{u(i,j)_0}(f(x)) = f(y)$ and $\Phi_{u(i,j)_1}(f(y)) = f(x)$.

Theorem 1.1.9 (Slaman-Steel, [18])**.** *Assume AD. Then Part 1 of Martin's Conjecture holds for the class of uniformly invariant functions.*

Theorem 1.1.10 (Steel, [21])**.** *Assume AD. Then Part 2 of Martin's Conjecture holds for the class of uniformly invariant functions.*

In [18], Slaman and Steel also proved the following result, which has no uniformity hypothesis.

Theorem 1.1.11 (Slaman-Steel)**.** *Assume AD. If $f <_M id_{\mathbb{R}}$, then f is $\equiv_M$-equivalent to a constant function.*

It follows that the full Martin's Conjecture holds for continuous functions. Both parts of Martin's Conjecture remain open for $\mathbf{\Sigma}^0_2$ functions.

Finally, Slaman and Steel also proved a result about Borel, order-preserving functions.

Definition 1.1.12. A function $f : \mathbb{R} \to \mathbb{R}$ is *order-preserving* iff $x \leq_T y$ implies $f(x) \leq_T f(y)$.

Of course, order-preserving functions are Turing-invariant.

Theorem 1.1.13 (Slaman-Steel)**.** *Part 2 of Martin's Conjecture holds for Borel, order-preserving functions.*

While these early results point to a positive answer for Martin's Conjecture, there hadn't been much progress in this direction in the last 30 years. Going the other way, Alexander Kechris has conjectured that $\equiv_T$ is a universal countable Borel equivalence relation, which would imply that there is a Borel f witnessing that Part 1 fails.

We end the preliminary section with a refinement of Martin's Theorem 1.1.4, which we will need. See [13] for a proof.

Theorem 1.1.14. *Assume AD. Let A be $\leq_T$-cofinal and $\pi : A \to \omega$. Then there is a pointed perfect S such that $[S] \subseteq A$ and $\pi \restriction [S]$ is constant.*

1.2 Measure-preserving functions

In this section, we introduce a natural class of functions and show, basically, that Part 1 of Martin's Conjecture holds for these functions.

Definition 1.2.1. A function $f : 2^\omega \to 2^\omega$ is *measure-preserving* iff for any $x \in 2^\omega$, there is a $y \in 2^\omega$ such that for all $z \geq_T y$, $f(z) \geq_T x$ (equivalently, $f[C_y] \subseteq C_x$).

It is easy to see that increasing functions, i.e. f such that $f(x) \geq_T x$, are measure-preserving (we can just take $y = x$). There are many natural equivalences to being measure-preserving; our next proposition includes some that we'll use.

First, we make the following bit of notation. For $\pi : X \to Y$ and $S \subseteq P(X)$, we let $\pi_*(S)$ be the subset of $P(Y)$ given by $A \in \pi_*(S)$ iff $\pi^{-1}(A) \in S$.

Proposition 1.2.2. *Assume AD. Let $f : 2^\omega \to 2^\omega$ be Turing-invariant. The following are equivalent.*

1. *f is measure-preserving,*
2. *for every $x \in 2^\omega$, $f \geq_M c_x$, the constantly x function,*
3. *$F_*(U_M) = U_M$, where F is the function on the Turing degrees induced by f,*

Proof. Fix f a Turing-invariant function. Then for any x, $f \geq_M c_x$ iff there is a y such that for all $z \geq_T y$, $f(z) \geq_T x$. So (1) is equivalent to (2).

Now let F be the function on the Turing degrees induced by f. We show (1) implies (3). Let $A \subseteq \mathcal{D}_T$ and B the set of reals with degree in A. First suppose $A \in U_M$. Then there is an x such that $C_x \subseteq B$. Since f is measure preserving, there is a y such that $f[C_y] \subseteq C_x$. So $C_y \subseteq f^{-1}[C_x] \subseteq f^{-1}[B]$. It follows that $F^{-1}[A] \in U_M$, so $A \in F_*(U_M)$. Now suppose $A \notin U_M$. By Martin's Cone Theorem, B is disjoint from a cone, say $C_x \cap B = \emptyset$. Letting y be such that $f[C_y] \subseteq C_x$, we have that $f^{-1}[B] \cap C_y = \emptyset$. It follows that $A \notin F_*(U_M)$.

Finally we show (3) implies (1). Suppose that $F_*(U_M) = U_M$. Let $x \in 2^\omega$. Let $C = \{\mathbf{y} \in \mathcal{D}_T \mid \mathbf{y} \geq_T [x]_T\}$. Then $C \in U_M$, so $F^{-1}[C] \in U_M$, too. Since C_x is the set of reals with degree in C, it follows that $f^{-1}[C_x]$ contains a cone, i.e. there is a y such that $f[C_y] \subseteq C_x$. So f is measure-preserving. □

We already observed that increasing functions are trivially measure-preserving. One of our main results is that functions which are order-preserving with respect to $\leq_T$ are either constant on a cone or measure-preserving.

Definition 1.2.3. A function $f : 2^\omega \to 2^\omega$ is *order-preserving* iff whenever $x \leq_T y$, $f(x) \leq_T f(y)$.

Notice that order-preserving functions must be Turing-invariant.

Theorem 1.2.4. *Assume AD. Let $f : 2^\omega \to 2^\omega$ be order-preserving. Then either f is constant on a cone or f is measure-preserving.*

This is an easy consequence of an interesting criterion for a set to be $\leq_T$-cofinal.

Definition 1.2.5. A set $A \subseteq 2^\omega$ is *countably directed* iff for every countable $X \subseteq A$, there is a $z \in A$ such that $z \geq_T x$ for all $x \in X$.

Notice that A is countably directed exactly when $\langle A, \leq_T \restriction A \rangle$ is a countably directed quasi-order.

Theorem 1.2.6. *Let $A \subseteq 2^\omega$. Suppose that A contains a perfect set and is countably directed. Then A is $\leq_T$-cofinal.*

We defer the proof of this theorem to §1.4

Proof of Theorem 1.2.4 Let f be order-preserving. Since every set has the perfect set property, either $\mathrm{ran}(f)$ is countable or $\mathrm{ran}(f)$ contains a perfect set. In the former case, we'll show f is constant on a cone; in the latter case, we'll show f is measure-preserving.
Case 1. $\mathrm{ran}(f)$ is countable.

Let $\mathrm{ran}(f) = \{z_i \mid i \in \omega\}$. Let $A_i = \{x \in 2^\omega \mid f(x) \equiv_T z_i\}$. Then the A_i are Turing-invariant since f is order-preserving. Since $\bigcup_i A_i = 2^\omega$, the countable completeness of the Martin measure implies one of the A_i's must contain a cone. So for some i, $f(x) \equiv_T z_i$ for a cone of x, i.e. f is constant on a cone.
Case 2. $\mathrm{ran}(f)$ contains a perfect set.

We want to show that $\mathrm{ran}(f)$ is $\leq_T$-cofinal, using Theorem 1.2.6. So let $X = \{x_i \mid i \in \omega\}$ be a countable subset of $\mathrm{ran}(f)$. By $\mathrm{CC}_\mathbb{R}$, let $\{a_i \mid i \in \omega\}$ such that $f(a_i) = x_i$. Since f is order-preserving, $f(\bigoplus_i a_i) \geq_T x_i$ for all $i \in \omega$. So $\mathrm{ran}(f)$ is countably directed. Theorem 1.2.6 implies it is $\leq_T$-cofinal.

Now fix x. Since $\mathrm{ran}(f)$ is $\leq_T$-cofinal, we can find some y such that $f(y) \geq_T x$. Since f is order-preserving, for any $z \geq_T y$, $f(z) \geq_T x$. So f is measure-preserving, as desired.

□

Our main theorem on measure-preserving functions is the following.

Theorem 1.2.7. *Assume AD+DC$_\mathbb{R}$. Let $f : 2^\omega \to 2^\omega$ be Turing-invariant and measure-preserving. Then $f(x) \geq_T x$ on a cone.*

Combining this with Theorem 1.2.4, we immediately get that Part 1 of Martin's Conjecture holds for order-preserving functions, assuming AD+DC$_\mathbb{R}$. We'll give another proof of this just from AD in the next section.

As a warm up to Theorem 1.2.7, we'll prove a weaker version. The proof is an elaboration of Martin's proof of the following theorem.

Theorem 1.2.8 (Martin, [18])**.** *Assume AD. Suppose that $f : 2^\omega \to 2^\omega$ is Turing-invariant, measure-preserving, and $f(x) \leq_T x$ on a cone. Then $f(x) \equiv_T x$ on a cone.*

This easily follows from the following lemmas.

Lemma 1.2.9. *Let T be a pointed perfect tree and Φ a Turing functional which is total on $[T]$. Then there is a pointed perfect subtree $S \subseteq T$ such that either*

1. *Φ is constant on $[S]$ or*

2. Φ *is one-to-one on* $[S]$ *and* $\Phi(x) \oplus S \equiv_T x$ *for all* $x \in [S]$.

A proof of this lemma can be found in [18].

Lemma 1.2.10 (Martin). *Assume AD. Let* A $\leq_T$*-cofinal and* $f : A \to 2^\omega$ *be such that for all* $x \in A$, $f(x) \leq_T x$. *Then either* f *is constant on a* $\leq_T$*-cofinal set or else there is a* $\leq_T$*-cofinal* $B \subseteq A$ *and a real* z *such that for all* $x \in B$, $f(x) \oplus z \equiv_T x$.

Note that we do not assume f is Turing-invariant in the lemma statement.

Proof. Let $\pi : A \to \omega$ given by $\pi(x)$ is the least e such that $\Phi_e(x) \downarrow= f(x)$. By Theorem 1.1.14, there is a pointed perfect tree T such that $[T] \subseteq A$ and π is constant on $[T]$. Let $\Phi = \Phi_e$ for e this constant value. Then, by moving to a subtree of T if necessary, Lemma 1.2.9 gives that either Φ is constant on $[T]$ or $\Phi(x) \oplus T \equiv_T x$ for all $x \in [T]$. But $f \upharpoonright [T] = \Phi \upharpoonright [T]$, so we're done. $\square$

Corollary 1.2.11. *Assume AD. Let* $g : 2^\omega \to 2^\omega$ *be such that* $g(x) \geq_T x$ *on a* $\leq_T$*-cofinal set* A. *Then there is a function* h, *a real* z, *and* $\leq_T$*-cofinal set* B *such that for all* $x \in B$,

1. $g(h(x)) = x$ *and*
2. $h(x) \oplus z \equiv_T x$.

Proof. Since g is increasing on a $\leq_T$-cofinal set, $\mathrm{ran}(g)$ is $\leq_T$-cofinal. So we can define h on $\mathrm{ran}(g)$ by $h(x) = \Phi_e(x)$ for e is the least n such that $\Phi_n(x) \downarrow$ and $g(\Phi_n(x)) = x$. Then clearly $g(h(x)) = x$ for any $x \in \mathrm{ran}(g)$. So applying Lemma 1.2.10 to h gives a $\leq_T$-cofinal $B \subseteq \mathrm{ran}(g)$ and a real z as desired. $\square$

To state the weaker version of Theorem 1.2.7, we need one more definition.

Definition 1.2.12. Let $f, g : 2^\omega \to 2^\omega$. g is a *modulus for* f iff for any x, $f[C_{g(x)}] \subseteq C_x$.

It is immediate from the definitions that if f has a modulus, it is actually measure-preserving. If (the graph of) f is measure-preserving and Suslin, the closure properties of the Suslin sets guarantee that f has a modulus (we just uniformize the relation $R(x, y) \Leftrightarrow f[C_y] \subseteq C_x$). Since increasing functions easily have a modulus, Theorem 1.2.7 easily implies that every measure-preserving function has a modulus, under AD+$\mathrm{DC}_{\mathbb{R}}$. We do not know how to prove this in just AD.

We'll need the following easy observation.

Proposition 1.2.13. *Let* $f : 2^\omega \to 2^\omega$ *and suppose that* f *has a modulus. Then* f *has an increasing modulus, i.e. a modulus* g *such that* $g(x) \geq_T x$ *for all* x.

Proof. If g is a modulus for f, then $x \mapsto g(x) \oplus x$ is an increasing modulus for f. $\square$

Lemma 1.2.14. *Assume AD. Let* $f : 2^\omega \to 2^\omega$ *measure-preserving. Suppose that* f *has a modulus. Then there is a* $\leq_T$*-cofinal set* A *such that for all* $x \in A$, $f(x) \geq_T x$.

Proof. Fix g an increasing modulus for f. Let B, h and z be as in Corollary 1.2.11 for g. Now, for any $x \in B$ with $x \geq_T g(z)$, $f(x) \geq_T z$ since g is a modulus for f. So as $g(h(x)) = x$ for any $x \in B$, whenever $x \in B$ and $x \geq_T g(z)$,

$$\begin{aligned} f(x) &= f(g(h(x))) \\ &\geq_T h(x) \oplus z \\ &\geq_T x, \end{aligned}$$

using for the middle inequality that $f(g(y)) \geq_T y$ for any y, since g is a modulus for f (we apply this for $y = h(x)$). Since there are $\leq_T$-cofinally many $x \in B$ with $x \geq_T g(z)$, we're done. □

In the case that f is Turing-invariant, we immediately get our weaker version of Theorem 1.2.7.

Theorem 1.2.15. *Assume AD. Assume f is Turing-invariant and has a modulus. Then $f(x) \geq_T x$ for a cone of x.*

As an aside, note that this version is actually enough to get Theorem 1.2.7 under the (possibly) stronger hypothesis of AD^+.

Corollary 1.2.16. *Assume AD^+. Suppose f is Turing-invariant and measure-preserving. Then $f(x) \geq_T x$ for a cone of x.*

Proof. Suppose that there is a counter example, i.e. there is a measure-preserving, Turing-invariant function f such that $f(x) \not\geq_T x$ on a cone. This statement is projective in f, so by the Σ^2_1-Reflection theorem, there is a Suslin-co-Suslin counter example, f. But then we can uniformize the relation $R(x, y)$ given by

$$R(x, y) \Leftrightarrow f[C_x] \subseteq C_y,$$

since this is projective in f and the Suslin sets are closed under real quantification. So let g be a function uniformizing R. Then g is a modulus for f. So Theorem 1.2.15 implies $f(x) \geq_T x$ on a cone, a contradiction. □

For a while, this was our best result on measure-preserving functions. Of course, it is made irrelevant by Theorem 1.2.7, which we turn to now. The challenge here is how to make due without the existence of a modulus. The first idea is to work with ω-sequences which approximate iteratively applying an increasing modulus.

Definition 1.2.17. Let $f : 2^\omega \to 2^\omega$. Say a sequence $\vec{x} = \langle x_i \mid i < \omega \rangle$ is a *modulus sequence for f* if $x_i \leq_T x_{i+1}$ and for all $y \geq_T x_{i+1}$, $f(y) \geq_T x_i$.

Lemma 1.2.18. *Assume ZF+AD+$DC_\mathbb{R}$. Let $f : 2^\omega \to 2^\omega$ be measure-preserving. Then for any $x \in 2^\omega$ there is a modulus sequence $\vec{x}$ for f with $x_0 = x$.*

Proof. This is immediate from the definitions and $\mathrm{DC}_\mathbb{R}$. □

We can now prove the main theorem.

Proof of Theorem 1.2.7. We want to end up in a situation where we can apply Lemma 1.2.10 and then argue in a way similar to the proofs of Corollary 1.2.11 and Lemma 1.2.14. The idea is to map some point x to a real $y \leq_T x, f(x)$ but in such a way that the map $x \mapsto y$ cannot be constant on a $\leq_T$-cofinal set. To do this, we will arrange that y is sufficiently close to x in the sense that some nicely behaved function $\pi : 2^\omega \to \omega_1$ has $\pi(x) = \pi(y)$. This will guarantee that $x \mapsto y$ cannot be constant on an $\leq_T$-cofinal set. Lemma 1.2.7 will then imply that $f(x) \geq_T x$ on a cone.

We start by introducing this ordinal-valued function. For any $x \in 2^\omega$, we let $\pi(x)$ be the least ordinal α such that there is a modulus sequence $\vec{x}$ for f with $x_0 = x$ and $\sup\{\omega_1^{x_i} \mid i \in \omega\} = \alpha$. Note that there is such a sequence by Lemma 1.2.18. For every x, $\omega_1^x \leq \pi(x) < \omega_1$ (using $\mathrm{CC}_\mathbb{R}$). Moreover, π is Turing invariant: if $x \equiv_T y$, then $\pi(x) = \pi(y)$ (since swapping x for y in a modulus sequence for f that starts with x produces a modulus sequence which starts with y).

The following is the key observation about π.

Claim 1. *For any y there is an x such that $y \leq_T x, f(x)$ and $\pi(x) = \pi(y)$.*[1]

Proof. Fix y. Let $\vec{x}$ be a modulus sequence for f starting with $x_0 = y$ such that $\sup\{\omega_1^{x_i} \mid i \in \omega\} = \pi(y)$. Let $x = x_1$. Then, by definition, $y \leq_T x, f(x)$. So we just need to see that $\pi(x) = \pi(f(x)) = \pi(y)$. First we check $\pi(x) = \pi(y)$. But $\langle x_{i+1} \mid i \leq \omega \rangle$ is a modulus for x with $\sup\{\omega_1^{x_i} \mid i \in \omega\} = \pi(y)$. So $\pi(x) \leq \pi(y)$. But for $\vec{z}$ any modulus sequence for f starting with x, $\langle y \rangle^\frown \vec{z}$ is a modulus sequence for f starting with y; moreover, $\omega_1^y \leq \omega_1^x$, so the corresponding supremum doesn't change. So we also have $\pi(y) \leq \pi(x)$. This finishes the claim. $\square$

By the claim, there is a $\leq_T$-cofinal set A such that for every $x \in A$, there is an x and a $y \leq_T x, f(x)$ such that $\pi(y) = \pi(x) = \pi(z)$. By replacing A with a $\leq_T$-cofinal subset of A if necessary, the proof of Lemma 1.2.10 gives a function $h : A \to 2^\omega$ such that for all $x \in A$, $h(x)$ is such a y. In particular $h(x) \leq_T x, f(x)$ for all $x \in A$ and $\pi \circ h = \pi \restriction A$.

But then h cannot be constant on a $\leq_T$-cofinal $B \subseteq A$. If it were, $\pi \circ h \restriction B = \pi \restriction B$ would also be constant. But for all x, $\pi(x) \geq \omega_1^x$ and ω_1^x takes on arbitrarily large values on any $\leq_T$-cofinal set, a contradiction. So, by Lemma 1.2.10, there is a $\leq_T$-cofinal $B \subseteq A$ and a real z such that $h(x) \oplus z \equiv_T x$ for all $x \in B$. Now, since f is measure-preserving, there is a y such that for all $x \geq_T y$, $f(x) \geq_T z$. So for any $x \in B$ with $x \geq_T y$, $f(x) \geq_T h(x) \oplus z \equiv_T x$.

Since f is Turing-invariant, it follows that $f(x) \geq_T x$ on a cone, as desired. $\square$

We'll discuss more consequences of this theorem in other sections, but let us point out a particularly simple but interesting one.

Corollary 1.2.19. *Assume ZF+ AD+DC$_\mathbb{R}$. Then $\mathrm{id}_\mathbb{R}$ is the $\leq_M$-least upper bound of the constant functions.*

Proof. By Proposition 1.2.2, we have that a Turing-invariant function f is an upper-bound of the constant functions under $\leq_M$ iff it is measure-preserving. So any Turing-invariant $\leq_M$-upper-bound for the constant functions is $\geq_M \mathrm{id}_\mathbb{R}$ by Theorem 1.2.7. $\square$

[1] We actually only need that $\pi(y) \geq \pi(x)$, but since we can prove they're equal, we do.

This perspective on Theorem 1.2.7 reveals that it is a genuine structural result about $\leq_M$, much like the Slaman-Steel theorem on regressive functions, not merely a partial result on Martin's Conjecture.

1.3 Order-preserving functions

In this section we'll show that under just AD, Part 1 of Martin's Conjecture holds for the class of order-preserving functions.

Theorem 1.3.1. *Assume AD. Let $f : 2^\omega \to 2^\omega$. Then f is either constant on a cone or $f(x) \geq_T x$ on a cone.*

Again, with the additional hypothesis of $\mathrm{DC}_{\mathbb{R}}$, this is a consequence of Theorems 1.2.7 and 1.2.4, but we seem to need a modified proof that only works for order-preserving functions if we want to drop $\mathrm{DC}_{\mathbb{R}}$.

We'll need some simple observations about measure-preserving functions which are easy consequences of Theorem 1.2.7, but provable under just AD.

Lemma 1.3.2. *Suppose $f, g : 2^\omega \to 2^\omega$ are measure-preserving. Then $f \circ g$ is measure-preserving.*

Proof. Fix x. Since f is measure-preserving, let y be such that $f(z) \geq_T x$ for all $z \geq_T y$. Since g is measure-preserving, let u be such that $g(z) \geq_T y$ for all $z \geq_T u$. Then for any $z \geq_T u$, $f \circ g(z) \geq_T x$. So $f \circ g$ is measure-preserving. □

Lemma 1.3.3. *Assume AD. Let $f : 2^\omega \to 2^\omega$ be measure-preserving and $\pi : 2^\omega \to Ord$ be Turing-invariant. Then $\pi \circ f(x) \geq \pi(x)$ on a cone.*

Proof. Towards a contradiction, assume $\pi \circ f(x) < \pi(x)$ on a cone, say above x_0. By our previous lemma, the finite iterates of f are measure-preserving. By $\mathrm{CC}_{\mathbb{R}}$, let choose a sequence $\langle x_n \mid n \in \omega \rangle$ such that for every n, for all $z \geq_T x_n$, $f^n(z) \geq_T x_0$. Let $y = \bigoplus_n x_n$ and $\alpha_n = \pi(f^n(y))$. Then for every n, $f^n(y) \geq_T x_0$, and so as $\pi \circ f(x) < \pi(x)$ for all $x \geq_T x_0$,

$$\begin{aligned} \alpha_{n+1} &= \pi(f^{n+1}(y)) \\ &= \pi \circ f(f^n(y)) \\ &< \pi(f^n(y)) = \alpha_n. \end{aligned}$$

So $\langle \alpha_n \mid n \in \omega \rangle$ is an infinite decreasing sequence of ordinals, a contradiction. □

Proof of Theorem 1.3.1 Assume that f is not constant on a cone. Then Theorem 1.2.4 implies that f is measure-preserving. The rest of the proof will be exactly like the proof of Theorem 1.2.7, except we use a different ordinal-valued function (since the ordinal-valued function in that proof needed $\mathrm{DC}_{\mathbb{R}}$ to work).

For each x, define $C(x)$ to be the smallest set such that

- $x \in C(x)$;

- if $y \in C(x)$ and $z \leq_T y$, then $z \in C(x)$;
- if $y, z \in C(x)$, then $y \oplus z \in C(x)$; and
- if $y \in C(x)$, then $f(y) \in C(x)$.

Note that for every x, $C(x)$ is a countable set. We now define our ordinal valued function π by $\pi(x) = \sup\{\omega_1^y \mid y \in C(x)\}$. As in the proof of Theorem 1.2.7, we have that $\pi : 2^\omega \to \omega_1$, π is Turing-invariant, and $\pi(x) \geq \omega_1^x$.

To finish, it is enough to show that the for $\leq_T$-cofinally many x there is a $y \leq_T x, f(x)$ such that $\pi(y) = \pi(x)$. Once we have this, we proceed exactly as in the proof of Theorem 1.2.7 to conclude that $f(x) \geq_T x$ on a cone.

By Lemma 1.3.3, $\pi \circ f(x) \geq \pi(x)$ on a cone. Fix z in this cone. Let $x = z \oplus f(z)$ and $y = f(z)$. We immediately have $z \leq_T x$ and $y = f(z) \leq_T x$. Since f order-preserving, we also have we have $y = f(z) \leq_T f(x)$ ($z \leq_T x$ implies $f(z) \leq_T f(x)$). So we just need to see that $\pi(x) = \pi(y)$.[2]

By our choice of x, y and the definition of the C operation,

$$C(y) \subseteq C(z) = C(x).$$

So by our choice of π,

$$\pi(y) \leq \pi(z) = \pi(x).$$

But we chose z to have $\pi(f(z)) \geq \pi(z) = \pi(x)$, so $\pi(y) \geq \pi(x)$, too (using again that $y = f(z)$). So $\pi(x) = \pi(y)$. □

The above proof is fairly local, for example when f has $\mathbf{\Pi}^1_1$ graph, one only needs $\mathbf{\Pi}^1_1$-Determinacy to get that $f(x) \geq_T x$ on a cone. On the other hand, if f is Borel, we still seem to be using $\mathbf{\Pi}^1_1$-Determinacy. In unpublished work, Takayuki Kihara showed that this is provable in just ZF via a different argument which uses Theorem 1.2.4 together with a version of the Solecki Dichotomy.

1.4 Proof of Theorem 1.2.6

In this last subsection, we finally prove Theorem 1.2.6, our local criterion for a set of reals to be $\leq_T$ cofinal.

This will be an easy consequence of a variant of a theorem of Groszek and Slaman [7].[3] In a sense, this variant is an improvement of the Groszek-Slaman result: they showed, more or less, that if $\{x_i \mid i \in \omega\}$ contains a countable dense subset of $[T]$, any real is computable from $\bigoplus_i x_i$ together with finitely many branches of T. Our result says that we can replace $\bigoplus_i x_i$ with any real z such that $\{y \in 2^\omega \mid y \leq_T z\}$ contains a countable dense subset of $[T]$. However, Groszek-Slaman only needed two branches of T for their result, whereas we use four.

[2] As in that proof, we only actually need $\pi(y) \geq \pi(x)$, but again end up with equality.

[3] Interestingly, the technique used by Groszek and Slaman has its roots Martin's Conjecture: it builds on the methods used in Slaman and Steel's proof that there are no regressive functions on the Turing degrees, under AD, which needed to accomplish something similar.

Theorem 1.4.1. *Assume ZF. Let $T \subseteq 2^{<\omega}$ be a perfect tree. Let z be a real such that for every $\sigma \in T$ there is $b \leq_T z$ such that $\sigma \trianglelefteq b$ and $b \in [T]$. Then for any real x there are reals $y_0, y_1, y_2, y_3 \in [T]$ such that*

$$z \oplus y_0 \oplus y_1 \oplus y_2 \oplus y_3 \geq_T x$$

Proof. Fix z and the real $x \in 2^\omega$ we're trying to compute. Also fix X a countable dense subset of $[T]$ such that $z \geq_T b$ for every $b \in X$ (we can find such an X by choosing for any $\sigma \in T$ the left-most branch b of T through σ such that $b \leq_T z$). We'll describe how to build the reals $y_0, y_1, y_2, y_3 \in [T]$ so that $z \oplus y_0 \oplus y_1 \oplus y_2 \oplus y_3 \geq_T x$.

We'll build the y_i via finite approximations y_i^n (we'll actually have that the sequence $\langle (y_0^n, y_1^n, y_2^n, y_3^n) \mid n \in \omega \rangle$ can be enumerated recursively in $z \oplus y_0 \oplus y_1 \oplus y_2 \oplus y_3$). To guarantee that $y_i \in [T]$, we'll also maintain that there are infinite paths b_i^n of $[T]$ such that $y_i^n \trianglelefteq b_i^n \in X$.

The bits of x will be coded into how the b_i^n deviate from the y_i, as follows. We will make sure we can enumerate a sequence $\langle (e_0^n, e_1^n, e_2^n, e_3^n) \mid n \in \omega \rangle$ recursively in $z \oplus y_0 \oplus y_1 \oplus y_2 \oplus y_3$ such that for all $n \in \omega$,

1. for all i, $\Phi_{e_i^n}(z)$ is total and $\Phi_{e_i^n} \neq y_i$;
2. if n is even, then $\Phi_{e_0^n} = b_0^n$, $\Phi_{e_1^n} = b_1^n$, and $x(n) = 0$ iff $e_0^n < e_1^n$ (so $x(n) = 1$ iff $e_0^n \geq e_1^n$);
3. if n is odd, then $\Phi_{e_2^n} = b_2^n$, $\Phi_{e_3^n} = b_3^n$, and $x(n) = 0$ iff $e_2^n < e_3^n$.

If we can enumerate such a sequence, then clearly we can compute x by checking the order on the appropriate e_i^n, e_{i+1}^n.

We'll now describe how we intend to enumerate this sequence and then show that we can build the y_i's so that this intended method succeeds.

Assume we have e_0^n, e_1^n, e_2^n, and e_3^n.

Case 1. n is even.

First, we look for the least l and t such that $\Phi_{e_0^n}$ disagrees with y_0 at l and $\Phi_{e_1^n}$ disagrees with y_1 at t. Now let e_2^{n+1} be the least index e such that $\Phi_e(z)$ converges on all inputs $< l$ in at most t steps and $\Phi_e(z) \restriction l = y_2 \restriction l$. Similarly, let e_3^{n+1} be the least index e such that $\Phi_e(z)$ converges on all inputs $< l$ in at most t steps and $\Phi_e(z) \restriction l_0 = y_3 \restriction l_0$. Finally, we let $e_0^{n+1} = e_0^n$ and $e_1^{n+1} = e_1^n$.

Case 2. n is odd.

This case is basically the same; we just swap the roles of $i = 0, 1$ and $i = 2, 3$. Let l, t be the first places where $\Phi_{e_2^n}$ disagrees with y_2 and $\Phi_{e_3^n}$ disagrees with y_3, respectively. Let e_0^{n+1} be the least index e such that $\Phi_e(z)$ converges on all inputs $< l$ in at most t steps and $\Phi_e(z) \restriction l = y_0 \restriction l$. Similarly, let e_3^{n+1} be the least index e such that $\Phi_e(z)$ converges on all inputs $< l$ in at most t steps and $\Phi_e(z) \restriction l_0 = y_1 \restriction l$. Also let $e_2^{n+1} = e_2^n$ and $e_3^{n+1} = e_3^n$.

This finishes the description of how we will enumerate the e_i^n using $z \oplus y_0 \oplus y_1 \oplus y_2 \oplus y_3$, and so how we will compute x from $z \oplus y_0 \oplus y_1 \oplus y_2 \oplus y_3$, assuming that the e_i^n are as desired.

All that remains is to show how to build the y_i^n and b_i^n so that following the above procedure for picking the e_i^n actually terminates and also has the desired properties. In addition to properties (1)-(3) listed above, we will also need to ensure that we know where the disagreements l, t occur at stage n. To do this, we will also maintain the following:

4. if n is even, then letting $l = \mathit{length}(y_0^n)$ and $t = \mathit{length}(y_1^n)$, $y_0(l) \neq b_0^n(l)$ and $y_1(t) \neq b_1^n(t)$;

5. if n is odd, then letting $l = \mathit{length}(y_2^n)$ and $t = \mathit{length}(y_3^n)$, $y_2(l) \neq b_2^n(l)$ and $y_3(t) \neq b_3^n(t)$.

Note that since $y_i^n \trianglelefteq y_i, b_i^n$, it follows that these l, t are the same l, t from the procedure for enumerating the e_i^n, i.e. the least disagreements between the appropriate y_i and b_i^n. It follows that we can enumerate up through $e_0^n, \ldots, e_3^n$ just using z together with $y_0^n, \ldots, y_3^n$ and $b_0^n, \ldots, b_3^n$.

Now suppose we've built up through y_i^n and b_i^n, maintaining that (1)-(5) hold for the e_i^k for $k \leq n$. We'll describe how to pick the new branches $b_i^{n+1} \in X \subseteq [T]$ and extend the y_i^n to y_i^{n+1} so that (1)-(5) hold at $n + 1$.

We again split into an even and odd case.

Case 1. n is odd.

First, we let $b_2^{n+1} = b_2^n$. Now let j be the least index such that $\Phi_j(z) = b_2^n = b_2^{n+1}$ (recall, such an index exists because $b_2^n \in X$ and so $b_n^2 \leq_T z$). Now let k be the least index greater than j such that $\Phi_k(z)$ converges, $\Phi_k(z) \in X$, and $y_3^n \trianglelefteq \Phi_k(z)$ and set $b_3^{n+1} = \Phi_k(z)$. Note that such a k exists by the density of X. We now need to extend the y_i^n's to the y_i^{n+1}'s to force $e_2^{n+2} = j$ and $e_3^{n+2} = k$, recalling the procedure for picking these indices.

Let l, t sufficiently large such that

- for any $k < i$ and $m < l$, $\Phi_k(z)$ either doesn't converge on input r in t steps or converges but disagrees with $b_2^{n+1}(m)$

- for any $k < j$ and $m < l$, $\Phi_l(z)$ either doesn't converge on input l in t steps or converges but disagrees with $b_3^{n+1}(m)$,

- $b_0^n \upharpoonright l$ and $b_1^n \upharpoonright t$ are splitting nodes of T.

Of course, we are using that T is perfect to guarantee such l, t exist.

We then let b_0^{n+1} be the least element of X extending $b_0^n \upharpoonright l$ which disagrees with b_0^n at l_0 (this exists by density of X and since $b_0^n \upharpoonright l$ is a splitting node of T) and b_1^{n+1} is the least element of X extending $b_1^n \upharpoonright t$ which disagrees with b_1^n at t (using that $b_1^n \upharpoonright t$ is a splitting node). The rest of our assignments are:

- $y_0^{n+1} = b_0^{n+1} \upharpoonright l$,

- $y_1^{n+1} = b_1^{n+1} \upharpoonright t$,

- $y_2^{n+1} = b_2^{n+1} \upharpoonright l$,

- $y_3^{n+1} = b_3^{n+1} \upharpoonright l$.

This maintains (1)-(5), by construction.

Case 2. n is even.

This case is entirely symmetric, swapping $i = 0, 1$ and $i = 2, 3$.

This finishes the construction of the y_i^n and b_i^n. As promised, we set the $y_i = \bigcup_n y_i^n$. Our construction guaranteed that $z \oplus y_0 \oplus y_1 \oplus y_2 \oplus y_3 \geq_T x$ and, since $[T]$ is closed, we have $y_i \in [T]$ (because we maintained that for every n, $y_i^n \trianglelefteq b_i^n \in X \subseteq [T]$). $\square$

Proof of Theorem 1.2.6. Let A be a countably directed set of reals which contains a perfect set. We want to show that A is $\leq_T$-cofinal. So fix x. We want to find a $y \in A$ such that $y \geq_T x$.

Fix $T \subseteq 2^{<\omega}$ a perfect tree such that $[T] \subseteq A$. Let X be a countable dense subset of $[T]$ (as in the previous proof, we can define such an X: take the set of left-most paths through any $\sigma \in T$). By countable directedness of A, we can find $z \in A$ such that $z \geq_T x$ for every $y \in X$. The density of X guarantees that for any $\sigma \in T$, z computes some infinite path b through σ. So applying Theorem 1.4.1 to T, z, and x gives that there are reals y_0, y_1,y_2, $y_3 \in [T] \subseteq A$ such that

$$z \oplus y_0 \oplus y_1 \oplus y_2 \oplus y_3 \geq_T x.$$

Using the countable directedness of A again, we get that there is a real $y \in A$ with $y \geq_T z \oplus y_0 \oplus y_1 \oplus y_2 \oplus y_3$. So $y \geq_T x$, as desired. $\square$

1.5 Ultrafilters on $\mathcal{D}_T$ under AD^+

Martin's Conjecture was stated by Martin under ZF+AD+DC, before AD^+ had been isolated by Woodin. It seems likely that if it were first stated today, Martin's Conjecture would be stated under the hypothesis of AD^+ instead, as the consensus nowadays is that this is the right generalization of the theory of $L(\mathbb{R})$ under determinacy.

The main advantage to working with AD^+ in the context of Martin's Conjecture is Woodin's Countable Section Uniformization Theorem.

Theorem 1.5.1 (Woodin)**.** *Assume AD^+. Suppose $R(x, y)$ is a relation on $\mathbb{R}$ with countable sections (i.e. for every x, $\{y \mid R(x, y)\}$ is countable). Then R admits a uniformization.*

This theorem immediately implies that, like in the context of AC, Turing-invariant functions and functions on the Turing degrees are really the same. That is, we have the following.

Corollary 1.5.2. *Assume AD^+. Let $F : \mathcal{D}_T \to \mathcal{D}_T$. Then there is a Turing-invariant function $f : 2^\omega \to 2^\omega$ such that $F([x]_T) = [f(x)]_T$ for all $x \in 2^\omega$.*

Proof. Let $R(x, y)$ be the relation on $2^\omega \times 2^\omega$ given by $F([x]_T) = [y]_T$. This relation has countable sections, so it has a uniformization f by Woodin's theorem. It is easy to see this f is as desired. $\square$

The relevance (and existence) of the Countable Section Uniformization Theorem was pointed out to us by William Chan. See his paper [2] for a proof of the theorem.

This means that both parts of Martin's Conjecture are equivalent, under AD^+, to corresponding statements about functions on $\mathcal{D}_T$.

Proposition 1.5.3. *Assume AD^+.*

1. *The following are equivalent.*

   (a) *Part 1 of Martin's Conjecture*

   (b) *For any $F : \mathcal{D}_T \to \mathcal{D}_T$, either F is (literally) constant on a cone or $F \geq_M id_{\mathcal{D}_T}$.*

2. *The following are equivalent.*

   (a) *Part 2 of Martin's Conjecture*

   (b) *$\leq_M$ is a pre-well-order of the functions $F : \mathcal{D}_T \to \mathcal{D}_T$ such that $F \geq_M id_{\mathcal{D}_T}$ and for any $F : \mathcal{D}_T \to \mathcal{D}_T$, $J \circ F$ is the $\leq_M$-successor on F.*

Here, J is the Turing jump as a function on $\mathcal{D}_T$, i.e. $J([x]_T) = [x']_T$ (contrary to our earlier notation).

Combining this proposition with Theorem 1.2.7 (or the easier Corollary 1.2.16), we get the following characterization of Part 1 of Martin's Conjecture in terms of the Rudin-Keisler order, where for U, W ultrafilters on X and Y, respectively, $W \leq_{\mathrm{RK}} U$ iff there is a $\pi : X \to Y$ such that $\pi_*(U) = W$.

Theorem 1.5.4. *Assume AD^+. The following are equivalent.*

1. *Part 1 of Martin's Conjecture*

2. *For every ultrafilter W on $\mathcal{D}_T$, $W \leq_{RK} U_M$ iff W is principal or $W = U_M$.*

Proof. Assume (1) holds. Let W on $\mathcal{D}_T$ with $W \leq_{RK} U_M$. Let $F : \mathcal{D}_T \to \mathcal{D}_T$ be such that $W = F_*(U_M)$. By the previous proposition and (1), F is either constant on a set in U_M or $F \geq_M id_{\mathcal{D}_T}$. If F is constant on a set in U_M, W is principal. If $F \geq_M id_{\mathcal{D}_T}$, then F is measure-preserving, so $F_*(U_M) = U_M$ by Proposition 1.2.2. So (1), Part 1 of Martin's Conjecture, immediately implies (2).

For the converse, suppose (2) holds. Let $F : \mathcal{D}_T \to \mathcal{D}_T$. If $F_*(U_M)$ is principal, then F is constant on a set in U_M, i.e. constant on a cone. Otherwise, (2) gives $F_*(U_M) = U_M$, so F is measure-preserving (using Proposition 1.2.2). So Theorem 1.2.7 gives that $F \geq_M id_{\mathcal{D}_T}$. By the previous proposition, we have (1). □

In light of (2), natural test questions are whether we can rule out that known ultrafilters on $\mathcal{D}_T$ are Rudin-Keisler below the Martin measure. For example, Lebesgue measure or the comeager filter, which are ultrafilters on $\mathcal{D}_T$ because Turing degrees are tailsets; see [1]. While this is open, we can say something about the relationship between these ultrafilters and U_M under $\leq_{\mathrm{RK}}$.

The following result is due to Andrew Marks and Adam Day, in unpublished work. We proved it independently, but later. We give a particularly simple proof here, pointed out to us by Gabriel Goldberg.

Definition 1.5.5. For W a filter on a set X, say that W is *commutative* if for any $R \subseteq X \times X$,

$$\{x \mid \{y \mid R(x, y)\} \in W\} \in W \Leftrightarrow \{y \mid \{x \mid R(x, y)\} \in W\} \in W.$$

Lemma 1.5.6. *Suppose W is commutative and $U \leq_{RK} W$. Then U is commutative.*

Proof. Say W on Y and U is on X. Let $f : Y \to X$ such that $U = f_*(W)$. Fix $R \subseteq X \times X$. But then

$$\begin{aligned}\{x \in X \mid \{y \in X \mid R(x,y)\} \in U\} \in U &\Leftrightarrow f^{-1}[\{x \in X \mid \{y \mid R(x,y)\} \in U\}] \in W \\ &\Leftrightarrow \{a \in Y \mid f^{-1}[\{y \in X \mid R(f(a),y)\}] \in W\} \in W \\ &\Leftrightarrow \{a \in Y \mid \{b \in Y \mid R(f(a),f(b))\} \in W\} \in W.\end{aligned}$$

By applying this to R^{-1}, we of course get:

$$\{y \in X \mid \{x \in X \mid R(x,y)\} \in U\} \in U \Leftrightarrow \{b \in Y \mid \{a \in Y \mid R(f(a),f(b))\} \in W\} \in W.$$

That W is commutative (applied to the relation on $Y \times Y$ given by $R(f(a), f(b))$) gives these conditions on membership in W are equivalent, so that the conditions on membership in U are equivalent as well, i.e.

$$\{x \in X \mid \{y \in X \mid R(x,y)\} \in U\} \in U \Leftrightarrow \{y \in X \mid \{x \in X \mid R(x,y)\} \in U\} \in U$$

So U is commutative, as desired. $\square$

Corollary 1.5.7. *Assume AD. Suppose that W is a commutative ultrafilter on $\mathcal{D}_T$. Then $U_M \not\leq_{RK} W$. In particular, Lebesgue measure and the Comeager filter are not Rudin-Keisler above the Martin measure.*

Proof. By the previous lemma, it is enough to check that U_M is *not* commutative. But this is easy. Let $R(x,y)$ iff $x \notin C_y$. So for any x, $\{y \mid R(x,y)\}$ contains a cone (the cone above x', say). But for any y, $\{x \mid R(x,y)\} = \bar{C}_y$. So we have

$$\{x \mid \{y \mid R(x,y)\} \in U_M\} = \mathcal{D}_T \in U_M$$

but

$$\{y \mid \{x \mid R(x,y)\} \in U\} = \emptyset \notin U_M.$$

So R witnesses that U_M is not commutative. The rest of the corollary follows from the Fubini theorem and the Kuratowski-Ulam theorem along with the fact that Turing degrees are tailsets (as mentioned above, see [1]). $\square$

Using Theorem 1.4.1, we can prove one last easy result.

Theorem 1.5.8. *Assume AD^+. Suppose W is a countably complete ultrafilter on $\mathcal{D}_T$. Then one of the following holds.*

1. *W is principal,*
2. *$\{x \mid \bar{C}_x \in W\} \in W$, or*
3. *$W = U_M$*

Proof. Consider $A = \{x \mid C_x \in W\}$. If $A \notin W$, then its complement, $\{x \mid C_x \notin W\} = \{x \mid \bar{C}_x \in W\} \in W$, giving (2). So suppose $A \in W$. If A is countable, then since W is countably complete, W must be principal. In the remaining case, we have $A \in W$ and A is not countable. We want to show that $W = U_M$.

Since A is not countable, let A^* be the set of reals with degree in A, that is, $A^* = \{x \in 2^\omega \mid [x]_T \in A\}$. Then A^* is uncountable, so it contains a perfect set. But for any countable subset $\{x_i \mid i \in \omega\} \subseteq A^*$, we have $C_{[x_i]} \in W$ so $\bigcap_{i\in\omega} C_{[x_i]} \in W$, by countable completeness of W. But then we must have $A \cap \bigcap_{i\in\omega} C_{[x_i]} \in W$, too. In particular, $A \cap \bigcap_{i\in\omega} C_{[x_i]}$ is non-empty. So there is a $z \in A^*$ such that $z \geq_T x_i$ for all i. So A^* is countably directed. Theorem 1.2.6 implies that A^* is $\leq_T$-cofinal. But this means for every x, there is a $y \geq_T x$ such that $C_y \in W$. But if $y \geq_T x$ and $C_y \in W$, then $C_x \in W$ since $C_x \supseteq C_y$ and W is a filter. This easily implies $W = U_M$. □

Determinacy in $L(\mathbb{R})$ allows for a complete analysis of the countably complete ultrafilters on the projective ordinals. The results of this section show that a complete analysis of the countably complete ultrafiters on $\mathcal{D}_T$ would decide Part 1 of Martin's Conjecture in $L(\mathbb{R})$. While a complete analysis may be very difficult, we suspect that this may be a site of more tractable problems relating to Martin's Conjecture, perhaps ones involving more sophisticated Descriptive Set Theory.

1.6 Additional results

In this final section we use results from the previous sections to identify a criterion for when $g \leq_M f$ and also refute a natural variant of Kechris's conjecture.

Theorem 1.6.1. *Assume AD+DC$_\mathbb{R}$. For any $f, g : 2^\omega \to 2^\omega$ with $f, g \geq_M id_\mathbb{R}$, the following are equivalent.*

1. *there is a z such that for all $x, y \geq_T z$,*
$$g(x) \equiv_T g(y) \Rightarrow f(x) \equiv_T f(y),$$
2. *there is a Turing-invariant, measure-preserving function $h : 2^\omega \to 2^\omega$ such that $f(x) \equiv_T h \circ g(x)$ on a cone,*
3. *there is a Turing-invariant function $h : 2^\omega \to 2^\omega$ such that $h \geq_M id_\mathbb{R}$ and $f(x) \equiv_T h \circ g(x)$ on a cone,*
4. *there is a Turing-invariant function $h : 2^\omega \to 2^\omega$ such that $f(x) \equiv_T h \circ g(x)$ on a cone.*

Moreover, these all imply $g \leq_M f$.

Proof. (1) implies (2) is the most work, so we'll save it for last. (2) implies (3) by Theorem 1.2.7. (3) trivially implies (4). Now we'll show that (4) implies (1). Let z be such that

$f(x) \equiv_T h \circ g(x)$ for all $x \geq_T z$. Let $x, y \geq_T z$ and suppose $g(x) \equiv_T g(y)$. Then since $x, y \geq_T z$ and h is Turing-invariant, we have

$$\begin{aligned} f(x) &\equiv_T h(g(x)) \\ &\equiv_T h(g(y)) \\ &\equiv_T f(y). \end{aligned}$$

This gives (1).

Finally, we show (1) implies (2). The idea here is simple: we'll just let h be an extension of $f \circ g^{-1}$, defined on the $\leq_T$-cofinal set $\mathrm{ran}(g)$, and check that h is Turing-invariant and measure-preserving. Of course, g is not one-to-one, so how to make sense of $f \circ g^{-1}$ requires some care.

Fix z such that for all $x, y \geq_T z$, $g(x) \equiv_T g(y) \Rightarrow f(x) \equiv_T f(y)$. Since $g \geq_M id_{\mathbb{R}}$, we can also fix u such that for all $x \geq_T u$, there is some $y \geq_T z$ such that $g(y) \equiv_T x$ (because $g[C_z]$ is $\leq_T$-cofinal, there is a cone of such x). For $x \geq_T u$, let $x^* = \Phi_e(x)$, for e the least index such that $\Phi_e(x)$ is total, $\Phi_e(x) \geq_T z$, and $g(\Phi_e(x)) = x$. By our choice of u, x^* is defined for $x \geq_T u$. We now define our h as follows.

$$h(x) = \begin{cases} f(x^*) & \text{if } x \geq_T u, \\ x & \text{otherwise.} \end{cases}$$

We need to check that h is Turing-invariant. So suppose $x \equiv_T y$. Clearly either both of x, y are defined via the first case or both are defined via the second case in our definition of h. In the second case, we have $h(x) = x \equiv_T y = h(y)$. So suppose we're in the first case, i.e. $x, y \geq_T u$. By the definition of the $*$-operation, we have that $x^*, y^* \geq_T z$ and $g(x^*) \equiv_T x \equiv_T y \equiv_T g(y^*)$. So by our choice of z, $h(x) = f(x^*) \equiv_T f(y^*) = h(y)$. So h is Turing-invariant.

All that remains is to check that h is measure-preserving. So fix x. We need to find a y so that $h[C_y] \subseteq C_x$. First, since $f \geq_M id_{\mathbb{R}}$, we may let a be such that $f(c) \geq_T x$ for all $c \geq_T a$. Since $g[C_{x \oplus z \oplus a}]$ is $\leq_T$-cofinal, we can let $y \geq_T u, x, a$ such that for every $c \geq_T y$, $c \equiv_T g(b)$ for some $b \geq_T x, z, a$. We just need to check that $h(c) \geq_T x$ for every $c \geq_T y$.

So suppose $c \geq_T y$. Then there is $b \geq_T x, z, a$ such that $c \equiv_T g(b)$. Since $c \geq_T u$, c^* is defined and $g(c^*) \equiv_T c \equiv_T g(b)$. But then by the definition of h and since $c^*, b \geq_T z$, $h(c) = f(c^*) \equiv_T f(b)$. But $b \geq_T x, a$, so $f(b) \geq_T x$ (by our choice of a). So $h(c) \geq_T x$. This shows that h is measure-preserving, so (1) implies (2). $\square$

Note that these (equivalent) criteria are *not* equivalent to $g \leq_M f$. For example, letting g be the hyperjump, i.e. $x \mapsto \mathcal{O}(x)$, and f be the function $x \mapsto \mathcal{O}(x)^{(\omega_1^x)}$, we have that $g <_M f$, but (1) must fail because we can have $\mathcal{O}(x) \equiv_T \mathcal{O}(y)$ but $\omega_1^x < \omega_1^y$ so that $g(x) \equiv_T g(y)$ but $f(x) <_T f(y)$. On the other hand, Steel's Theorem 1.1.10 implies that these criteria are actually equivalent to $g \leq_M f$ when f, g are Borel uniformly invariant functions. It is natural to ask how the natural order on Turing invariant functions determined by the equivalent criteria (1)-(4) differs from the Martin order on the uniformly invariant functions, in general, but we do not know the answer.

In §1.1, we mentioned that Kechris has conjectured $\equiv_T$ is a universal countable Borel equivalence relation. We end this chapter by showing a natural variant of this conjecture has a negative answer.

Definition 1.6.2. A quasi-order on $\leq$ on $\mathbb{R}$ is *locally countable* if for every $x \in \mathbb{R}$, $\{y \mid y \leq x\}$ is countable.

A locally countable quasi-order $\leq$ on $\mathbb{R}$ is a *universal Borel locally countable quasi-order* if $\leq$ is Borel and for any Borel locally countable quasi-order $\preceq$ on $\mathbb{R}$, there is a Borel function $f : \mathbb{R} \to \mathbb{R}$ such that $x \preceq y \Leftrightarrow f(x) \leq f(y)$.

Of course, $\leq_T$ is a Borel locally countable quasi-order. Whether it is a universal such quasi-order seems to us to be a very natural variant of Kechris's conjecture, though, as far as we know, it was not asked anywhere. In any case, it has a negative answer.

Theorem 1.6.3. *$\leq_T$ is not a universal Borel locally countable quasi-order.*

Proof. Fix A_0, A_1 disjoint, $\leq_T$-cofinal, Borel sets of reals such that $A_0 \cup A_1 = \mathbb{R}$ (e.g. reals that start with 0 and reals that start with 1).

Let $\preceq$ be the countable Borel quasi-order $(\leq_T \restriction A_0) \cup (\leq_T \restriction A_1)$. We claim that there is no Borel f such that $x \preceq y$ iff $f(x) \leq_T f(y)$.

Suppose not and fix such an f. Then $f \restriction A_0$ and $f \restriction A_1$ easily extend to Borel order-preserving maps $f_0, f_1 : \mathbb{R} \to \mathbb{R}$. Since f_0, f_1 are Borel, $\mathrm{ran}(f_0)$ and $\mathrm{ran}(f_1)$ are $\mathbf{\Sigma}^1_1$ and so must either be countable or contain a perfect set, by the Perfect Set Theorem. But $\mathrm{ran}(f_0)$ and $\mathrm{ran}(f_1)$ cannot be countable since A_0 and A_1 are not countable. It follows that $\mathrm{ran}(f_0)$ and $\mathrm{ran}(f_1)$ contain a perfect set and are countably directed (using here that f_0, f_1 are order-preserving). So Theorem 1.2.6 gives that both of their ranges are $\leq_T$-cofinal. So we can find $x \in A_0$ and $y \in A_1$ such that $f_0(x) \leq_T f_1(y)$. So $f(x) \leq_T f(y)$ but $x \not\preceq y$, a contradiction. $\square$

Chapter 2

The Uniqueness of the Core Model

A major achievement of Inner Model Theory is Jensen and Steel's identification of the core model K under the hypothesis that there is no inner model with a Woodin cardinal in [11]. Under this hypothesis, K is a canonical inner model which is close to V, generalizing Jensen's seminal result that L is close to V under the more restrictive hypothesis that $0^{\#}$ doesn't exist. K plays a central role in the meta-mathematics of Set Theory: it is an essential tool in establishing strong consistency strength lower bounds, for example in Steel's result that PFA implies $\mathrm{AD}^{L(\mathbb{R})}$ [22]

Jensen and Steel identify K as an inner model whose levels are certain *premice*, fine-structural models of set theory which have a complicated definition involving somewhat arbitrary decisions. In particular, Jensen and Steel use what are known as ms-indexed (pure-extender) premice to build K. Other varieties of premice have been studied, for example Jensen-indexed premice or the recent pfs-premice and least branch strategy mice of Steel's [25], and could give rise to ostensibly different versions of K. However, it is expected that all these ostensibly different versions are the same. One reason for this expectation is that one should be able to translate premice of one variety into premice of another variety. This has been realized in some cases; for example, Fuchs [4] and [5] showed that one can translate ms-indexed premice into a modified Jensen-indexed premice, and vice-versa. These translation methods are carefully tailored to the varieties one is translating between. So, such methods don't seem like they can yield the kind of general result one would really like to show: *any* successful notion of premouse must give rise to the same core model. In this chapter we take a new approach to establishing sufficiently general results along these lines: we show that in some contexts, abstract properties of the core model uniquely determine it, i.e. there is at most one inner model with these properties. So any notion of premouse for which the associated core model enjoys the abstract properties of K must actually give rise to the same core model.

This chapter is organized as follows. In §2, we'll prove a result for the core model below $0^{\P}$, as various things are easier in this context. Moreover, the properties we'll use about the core model have actually been verified for both the Jensen-indexed and ms-indexed core model below $0^{\P}$, so our result gives that these two versions of the core model are the same under this hypothesis. In §3, we characterize the core model under the additional hypothesis that there is a proper class of measurable cardinals. We start with some preliminary definitions and observations, then discuss some new and folklore facts about K.

2.1 Preliminaries

We will consider transitive models of ZFC^-, that is ZFC, stated with the Replacement Schema and the Well-Ordering Theorem,[1] but without the Power Set Axiom. We introduce the following bits of notation.

Definition 2.1.1. For M a transitive model of ZFC^-, $o(M) = \mathrm{Ord} \cap M$. For M, N transitive models of ZFC^-, and $\pi : M \to N$ an elementary embedding, we let $\pi(o(M)) = o(N)$.

We also introduce the following nonstandard notation, for convenience.

Definition 2.1.2. For μ a limit cardinal we let

$$H_\mu = \bigcup \{H_\kappa \mid \kappa < \mu \text{ a regular cardinal}\}.$$

For M a transitive model of ZFC^-, we also set $H^M_{o(M)} = M$.

This notation is useful because this hierarchy comes up naturally in Inner Model Theory: for M a premouse and μ a limit cardinal of M, H_μ is the universe of $M|\mu$.

We also make the following definition.

Definition 2.1.3. "V=HOD" be the sentence in the language of set theory expressing that every set is Σ_3-definable in ordinal parameters.

Note that satisfying "V=HOD" sufficiently locally implies GCH: for M a transitive model of ZFC and $\mu < o(M)$ an M-cardinal, if $H^M_{(\mu^+)^M} \models$"V=HOD", then $M \models$ "$2^\mu = \mu^+$", since the order-type of the resulting well-ordering of $H^M_{(\mu^+)^M} \supseteq P(\mu)$ definable over $H^M_{(\mu^+)^M}$ must have M-cardinality $(\mu^+)^M$.

We'll also need to look at directed systems of elementary embeddings between transitive models of ZFC^-, which we'll just call "directed systems of models of set theory".

Definition 2.1.4. A *directed system of models of set theory* is a system $\mathcal{D} = \{M_i, \pi_{i,j} \mid i, j \in D \text{ and } i \leq j\}$ such that

1. $\leq$ is a directed partial order on D,
2. for every $i, j, k \in D$,
   (a) M_i is a transitive model of ZFC^-,
   (b) if $i \leq j$, then $\pi_{i,j}$ is an elementary embedding from M_i into M_j,
   (c) $\pi_{i,i}$ is the identity on M_i, and
   (d) if $i \leq j \leq k$, then $\pi_{i,k} = \pi_{j,k} \circ \pi_{i,j}$.

Definition 2.1.5. For $\mathcal{D} = \{M_i, \pi_{i,j} \mid i, j \in D \text{ and } i \leq j\}$ a directed system of models of set theory $\mathcal{D}$ is *well-founded* if the direct limit (M_∞, E) is well-founded, in which case we take M_∞ to be transitive and $E = \in\restriction M_\infty$.

[1] The Well-Ordering Theorem is the statement that every set admits a well-ordering.

We will use this terminology and the results to follow even when the directed system $\mathcal{D}$ is a definable family of transitive proper class models of ZFC (and elementary embeddings between them). Of course, as ZFC theorems, any results proven about such a system is schematic.

Our first lemma is implicit in the computations of HOD in models of determinacy, isolated in this general form by Gabriel Goldberg.

Lemma 2.1.6. *Let $\mathcal{D} = \{M_i, \pi_{i,j} \mid i, j \in D$ and $i \leq j\}$ be a well-founded directed system of models of set theory. Let M_∞ be its direct limit, $\pi_{i,\infty} : M_i \to M_\infty$ the direct limit maps, and $X \subseteq M_\infty$.*

The following are equivalent.

1. *$X \in M_\infty$,*
2. *there is an $i \in D$ such that*
   (a) *$\pi_{i,\infty}^{-1}[X] \in M_i$ and*
   (b) *for any $j \geq i$, $\pi_{i,j}(\pi_{i,\infty}^{-1}[X]) = \pi_{j,\infty}^{-1}[X]$.*

Proof. First we show (1) $\Rightarrow$ (2). Suppose $X \in M_\infty$. Then X is the image of an element of some model in our system, i.e. we can find an $i \in D$ and $\bar{X} \in M_i$ such that $\pi_{i,\infty}(\bar{X}) = X$. We check i witnesses (2) holds. For (2)(a), it's enough to see that $\bar{X} = \pi_{i,\infty}^{-1}[X]$. But this is trivial by elementarity: for any $x \in M_i$,

$$\begin{aligned} x \in \bar{X} &\iff \pi_{i,\infty}(x) \in \pi_{i,\infty}(\bar{X}) = X \\ &\iff x \in \pi_{i,\infty}^{-1}[X]. \end{aligned}$$

Since for any $j \geq i$, $\pi_{j,\infty}(\pi_{i,j}(\bar{X})) = X$, the corresponding calculation at j also gives (2)(b).

Now we show (2) $\Rightarrow$ (1). So let i witness that (2) holds. For $j \geq i$, let $\bar{X}_j = \pi_{j,\infty}^{-1}[X]$. (2)(a) says $\bar{X}_i \in M_i$. Since $X \subseteq M_\infty$, (2)(b) gives that $\pi_{i,\infty}(\bar{X}_i) \subseteq M_\infty$, too. So, it's enough to show that $\pi_{i,\infty}(\bar{X}_i)$ and X have the same elements of M_∞. So fix $x \in M_\infty$. Then x is the image of an element of some point in our system, so we can find a $j \geq i$ and $\bar{x} \in M_j$ such that $\pi_{j,\infty}(\bar{x}) = x$. By (2)(b), $\pi_{i,j}(\bar{X}_i) = \bar{X}_j$ (in particular, $\bar{X}_j \in M_j$). So since $\bar{X}_j = \pi_{j,\infty}^{-1}[X]$,

$$\bar{x} \in \bar{X}_j \iff x \in X.$$

Since $\pi_{j,\infty}(\bar{X}_j) = \pi_{i,\infty}(\bar{X}_i)$, applying $\pi_{j,\infty}$ to the left-hand side, gives

$$x \in \pi_{i,\infty}(\bar{X}_i) \iff x \in X.$$

□

We'll identify a definability criterion which is sufficient for (2) and typical in applications of the lemma.

Definition 2.1.7. Let $\mathcal{D} = \{M_i, \pi_{i,j} \mid i, j \in D$ and $i \leq j\}$ be a directed system of models of set theory. For $A \subseteq D$, an A-indexed family of n-ary relations $\{R_i \mid i \in A\}$ is *uniformly definable over $\mathcal{D}$* if there is an $i \in A$, $a \in M_i$, and formula in the language of set theory $\varphi(v_1, \ldots, v_n, u)$ such that for all $j \geq i$,

1. $j \in A$,
2. $R_j \subseteq M_j^n$ and
3. for all $x_1, \ldots, x_n \in M_j$, $R_j(x_1, \ldots, x_n) \iff M_j \models \varphi(x_1, \ldots, x_n, \pi_{i,j}(a))$.

A single relation R is *uniformly definable over* $\mathcal{D}$ if the constantly R D-indexed family is uniformly definable over $\mathcal{D}$.

Definition 2.1.8. For M a transitive set, a set X is a *bounded subset of* M if there is a $y \in M$ such that $X \subseteq y$.

Lemma 2.1.9. *Let $\mathcal{D} = \{M_i, \pi_{i,j} \mid i, j \in D$ and $i \leq j\}$ be a well-founded directed system of models of set theory. Let M_∞ be its direct limit, $\pi_{i,\infty} : M_i \to M_\infty$ the direct limit maps. Let X be a bounded subset of M_∞.*

Suppose that $\{\pi_{i,\infty}^{-1}[X] \mid i \in D\}$ is uniformly definable over $\mathcal{D}$. Then $X \in M_\infty$.

Proof. Since X is a bounded subset of M_∞, we can fix $y \in M_\infty$ such that $X \subseteq y$. Let $i \in D$ and $\bar{y} \in M_i$ such that $\pi_{i,\infty}(\bar{y}) = y$. Then for all $j \geq i$, $\pi_{j,\infty}[X] \subseteq \pi_{i,j}(\bar{y})$. Since $\{\pi_{i,\infty}^{-1}[X] \mid i \in D\}$ is uniformly definable over $\mathcal{D}$, by increasing i if necessary, we have that $\pi_{j,\infty}[X]$ is a bounded subset of M_j which is definable over M_j, and so $\pi_{j,\infty}[X] \in M_j$ by Replacement in M_j. But then the elementarity of $\pi_{i,j}$ and the uniform definability of the $\pi_{i,\infty}^{-1}[X]$ immediately gives $\pi_{i,j}(\pi_{i,\infty}^{-1}[X]) = \pi_{j,\infty}^{-1}[X]$, so (2)(b) holds as well. So $X \in M_\infty$ by Lemma 2.1.6. □

We can use this lemma to show that appropriately intertwined directed systems of models of set theory have the same direct limit when the points in the systems are models of "V=HOD" and the direct limit models and maps are uniformly definable over both systems. We state the result as a condition for when an initial segment of the direct limit of one system is a subset of another.

Theorem 2.1.10. *Let $\mathcal{C}$ and $\mathcal{D}$ be well-founded directed systems of models of set theory with the same underlying partial order, $\mathcal{C} = \{N_i, \sigma_{i,j} \mid i, j \in D$ and $i \leq j\}$ and $\mathcal{D} = \{M_i, \pi_{i,j} \mid i, j \in D$ and $i \leq j\}$. Let N_∞, M_∞ be the direct limit models and $\sigma_{i,\infty}$, $\pi_{i,\infty}$ be the direct limit maps.*

Suppose that

1. *for all $i \in D$, $N_i \subseteq M_j$,*
2. *for all $i \in D$, $N_i \models$ "V=HOD", and*
3. *N_∞, M_∞, $\{\sigma_{i,\infty}\}_{i \in D}$, $\{\pi_{i,\infty}\}_{i \in D}$ are uniformly definable over $\mathcal{D}$.*

Then $N_\infty \subseteq M_\infty$.

Proof. Suppose that $N_\infty \not\subseteq M_\infty$. Since N_i satisfies "V=HOD", N_∞ satisfies "V=HOD", too. So, we can look at the least set of ordinals $A \in N_\infty$ such that $A \notin M_\infty$ under the definable well-order of N_∞. Since N_∞ and M_∞ are uniformly definable over $\mathcal{D}$, so is A. Further, since A is a member of M_i for all sufficiently large i, by the uniform definability of the N_∞, so A is a bounded subset of M_∞. Since $\{\sigma_{i,\infty}\}_{i \in D}$, $\{\pi_{i,\infty}\}_{i \in D}$, and A are uniformly definable over $\mathcal{D}$, we easily get $\{\pi_{i,\infty}^{-1}[A] \mid i \in D\}$ is also uniformly definable over $\mathcal{D}$. So Lemma 2.1.9 gives $A \in M_\infty$, a contradiction. □

Next, we'll look at inner models which are definable via special kinds of Σ_2-formulas, where by *inner models* we mean transitive proper class models of ZF, as is standard. We will look at formulas which provably define inner models over some base theory, T/ In the subsequent sections, we will consider just two base theories: ZFC+"$0^\P$ does not exist" and ZFC+"there is no inner model of ZFC with a Woodin cardinal", the latter of which we'll shorten to "there is no inner model with a Woodin cardinal". When considered formally, we mean that this is expressed in the language of set theory in the standard way. Note that the hypotheses that there is no inner model with a Woodin cardinal and that $0^\P$ does not exist are both absolute to inner models: that is, if they hold, they hold in any inner model of ZFC. We introduce the following definition to capture this phenomenon.

Definition 2.1.11. A theory T is a *nice* extension ZFC iff it has the form ZFC+φ for a Π_2-sentence φ of the form "for every set of ordinals A, $L[A] \models \theta$" for some Π_2-sentence θ.

We leave the following easy proposition to the reader.

Proposition 2.1.12. *Let T be a nice extension of ZFC. Assume T. Then for any transitive proper class model W of ZFC, $W \models T$.*

It is straightforward to see that ZFC+"$0^\P$ does not exist" and ZFC+"there is no inner model with a Woodin cardinal" are nice extensions of ZFC.

We'll use the following standard notation.

Definition 2.1.13. For $\varphi(\vec{u}, \vec{v})$ a formula with free variables $\vec{u}, \vec{v}$, M a transitive class and $\vec{x} \in M$, we let

$$\varphi(\vec{x}, \vec{v})^M = \{\vec{y} \in M \mid M \models \varphi(\vec{x}, \vec{y})\}.$$

Also, if $\varphi(u)$ has just one free variable u, we'll write φ^M instead of $\varphi(u)^M$.

Definition 2.1.14. Let T be a nice extension of ZFC. A Σ_2-formula $\varphi(v)$ *locally defines an inner model over T* iff $\varphi(v)$ has the form

$$\exists\mu\,\big(\mu \text{ is a strong limit cardinal } \wedge v \in H_\mu \wedge H_\mu \models \theta(v)\big).$$

for some formula $\theta(v)$ and, letting $M = \varphi^V$, the following is provable over T:

- M is an inner model of ZFC,[2]
- for every strong limit cardinal μ, $H_\mu^M = \theta^{H_\mu}$.

Note that if φ locally defines an inner model over T, then φ is a Σ_2-formula. Also note that we can always take the μ in the displayed formula above to be the least strong limit cardinal such that $v \in H_\mu$. We'll see that these formulas are more nicely behaved than arbitrary Σ_2-formulas which provably define inner models.

We need one more bit of notation.

[2]Recall that this is expressible in the language of set theory by asserting that φ^V is a transitive class which is almost universal, closed under Gödel operations, and satisfies AC; see [8]. Note that since we assumed T is a nice extension of ZFC, it follows that $M \models T$, as well.

Definition 2.1.15. Let $\rho = \rho(u_1, \ldots, u_n)$ and $\chi = \chi(\vec{v}, w)$ be formulas. The formula ρ^χ is

$$\rho^{\chi-} \wedge \chi(\vec{v}, u_1) \wedge \cdots \wedge \chi(\vec{v}, u_n),$$

where $\rho^{\chi-}$ defined recursively on the complexity of ρ as follows:

- for ρ an atomic formula, $\rho^{\chi-} = \rho$,
- $(\rho \wedge \xi)^{\chi-} = \rho^{\chi-} \wedge \xi^{\chi-}$,
- $(\rho \vee \xi)^{\chi-} = \rho^{\chi-} \vee \xi^{\chi-}$,
- $(\neg\rho)^{\chi-} = \neg(\rho^{\chi-})$,
- $(\exists u\, \rho)^{\chi-} = \exists u\, (\chi(\vec{v}, u) \wedge \rho^{\chi-})$, and
- $(\forall u\, \rho)^{\chi-} = \forall u\, \big(\chi(\vec{v}, u) \to \rho^{\chi-}\big)$.

The point of this is just that if χ^V is a transitive class, say $\chi^V = M$, then $(\rho^\chi)^V = \rho^M$.

Lemma 2.1.16. *Let T be a nice extension of ZFC. Let φ, ψ be formulas which locally define inner models over T. Then there is a formula τ which locally defines an inner model over T such that τ equivalent to ψ^φ, provably over T.*

Proof. Let θ, ρ be formulas witnessing that φ and ψ locally define inner models over T, that is, such that φ is $\exists\mu\, \big(\mu \text{ is a strong limit cardinal } \wedge v \in H_\mu \wedge H_\mu \models \theta(v)\big)$ and ψ is $\exists\mu\, \big(\mu \text{ is a strong limit cardinal } \wedge v \in H_\mu \wedge H_\mu \models \rho(v)\big)$.

Let τ be $\exists\mu\, \big(\mu \text{ is a strong limit cardinal } \wedge v \in H_\mu \wedge H_\mu \models \rho^\theta(v)\big)$. We'll show that τ is our desired formula. Work in T. Let $M = \varphi^V$ and $N = \psi^M$. Then M and N are both inner models of T. We need to show that $N = \tau^V$ and that for any strong limit μ, $H_\mu^N = (\rho^\theta)^{H_\mu}$. This latter claim immediately implies the former, so we just need to verify it.

Let μ be a strong limit cardinal. Then μ is a strong limit cardinal of M, so

$$\begin{aligned} H^N_{(\mu)^N} &= \rho^{H^M_\mu} \\ &= \rho^{(\theta^{H_\mu})} \\ &= (\rho^\theta)^{H_\mu}, \end{aligned}$$

using that φ and ψ locally define inner models over T (as witnessed by θ and ρ) for the second and first equivalences, respectively. □

For arbitrary Σ_2-formulas ψ and φ which provably define inner models, it seems that ψ^φ should not be provably equivalent to a Σ_2-formula, but we do not have an example.

Our next goal is to shows that for M, N inner models defined via local formulas over T, the ω-sequence of inner models $\langle M, N^M, M^{N^M}, N^{M^{N^M}}, \ldots\rangle$ is definable. The problem is that this we may have no bound on the quantifier complexity of the formulas $\varphi, \varphi^\psi, \psi^{\varphi^\psi}, \varphi^{\psi^{\varphi^\psi}}, \ldots$ (where φ, ψ are some witnessing formulas to the definability of M, N). We get around this by using our previous proposition.

Fix $\langle \varphi_i \mid i \in \omega\rangle$ a primitive recursive enumeration of formulas of the language of set theory in one free variable. For γ such a formula, let $\ulcorner\gamma\urcorner$ be the i such that $\gamma = \varphi_i$. For φ, ψ formulas, let $F_{\varphi,\psi}$ be the primitive recursive function outputting the Gödel numbers of the sequence $\langle \varphi, \psi^\varphi, \psi^{\varphi^\psi}, \ldots\rangle$. That is, $F_{\varphi,\psi}$ is the function F defined by

- $F(0) = \ulcorner \varphi \urcorner$ and
- $F(k+1) = \begin{cases} \ulcorner \psi^{\varphi_{F(k)}} \urcorner & \text{if } k \text{ is even,} \\ \ulcorner \varphi^{\varphi_{F(k)}} \urcorner & \text{if } k \text{ is odd.} \end{cases}$

Let $\mathrm{Sat}_0(w, v, u)$ be the usual definition of the Δ_0-satisfaction predicate.[3]

For φ, ψ formulas which locally define inner models over T, say φ is $\exists\mu\,(\mu \text{ is a strong limit cardinal } \wedge v \in H_\mu \wedge H_\mu \models \theta(v))$ and ψ is $\exists\mu\,(\mu \text{ is a strong limit cardinal } \wedge v \in H_\mu \wedge H_\mu \models \rho(v))$, we also let $\xi_{\varphi,\psi}(u, v)$ be the formula

$$\exists\mu\big(\mu \text{ is a strong limit cardinal } \wedge v \in H_\mu \wedge \mathrm{Sat}_0(H_\mu, F_{\theta,\rho}(u), v)\big).^{4}$$

Proposition 2.1.17. *Let T be a nice extension of ZFC. Suppose that φ, ψ are formulas which locally define inner models over T. Let $\xi = \xi_{\varphi,\psi}$. Then the following is provable in T.*

For every $k \in \omega$,

1. *$\xi(k, v)^V$ is an inner model of T*[5]
2. *$\xi(0, v)^V = \varphi^V$,*
3. *$\xi(k+1, v)^V = \psi^{\xi(k,v)^V}$ if k is even, and*
4. *$\xi(k+1, v)^V = \varphi^{\xi(k,v)^V}$ if k is odd.*

Proof. Assume T. We prove (1)-(4) by induction on $k \in \omega$.

Since $F_{\theta,\rho}(0) = \ulcorner \theta \urcorner$, we immediately get $\xi(0, v)^V = \varphi^V$, giving (2). Since φ^V is an inner model of T, by hypothesis, (1) holds for $k = 0$.

Now suppose (1) holds at k, i.e. $\xi(k, v)^V$ is an inner model of T. Assume k is even. We'll verify $\xi(k+1, v)^V = \psi^{\xi(k,v)^V}$. Then, since $\xi(k, v)^V$ is an inner model of ZFC, so is $\xi(k+1, v)^V = \psi^{\xi(k,v)^V}$ by our hypothesis about ψ. We have that $\xi(k, v)$ is

$$\exists\mu\big(\mu \text{ is a strong limit cardinal} \wedge v \in H_\mu \wedge \mathrm{Sat}_0(H_\mu, F_{\theta,\rho}(k), v)\big),$$

which is equivalent to

$$\exists\mu\big(\mu \text{ is a strong limit cardinal} \wedge v \in H_\mu \wedge H_\mu \models \varphi_{F_{\theta,\rho}(k)}(v)\big).$$

Now, since k is even, $F_{\theta,\rho}(k+1) = \ulcorner \rho^{\varphi_{F_{\theta,\rho}}} \urcorner$. So, $\xi(k+1, v)$ is equivalent to

$$\exists\mu\big(\mu \text{ is a strong limit cardinal} \wedge v \in H_\mu \wedge H_\mu \models \rho^{\varphi_{F_{\theta,\rho}(k)}}(v)\big).$$

By the proof of Lemma 2.1.16, we get that $\xi(k+1, v)^V = \psi^{\xi(k,v)^V}$, as desired. The case that k is odd is basically the same (just replace ρ with θ). $\square$

[3]So for any transitive x, $y \in x$, and $n \in \omega$, $\mathrm{Sat}_0(x, n, y) \Leftrightarrow x \models \varphi_n(y)$.

[4]Here we really mean that we've replaced $F_{\theta,\rho}$ with a formula defining defining it over ZFC.

[5]Here we mean $\xi(k, v)^V$ is almost universal, etc., and satisfies the additional sentence witness that T is nice.

Again, although the precise statement is technical, we think of this proposition as saying that for M, N locally definable inner models (over some nice T), the ω-sequence of inner models $\langle M, N^M, M^{N^M}, N^{M^{N^M}}, \ldots\rangle$ is actually definable (over T). This proposition is a major reason why we've focused on inner models which are locally definable—it is not clear that the sequence $\langle M, N^M, M^{N^M}, \ldots\rangle$ is definable at all for two arbitrary Σ_2-formulas which provably define inner models (over some T), as the quantifier complexity of the resulting definitions gets arbitrarily large.

Unsurprisingly, one of the key properties we will use in identifying the core model is covering. Informally, the covering properties of an inner model W are thought of asserting that W is "close" to V. The specific covering property will make use of in most of our results is captured in the following definition.

Definition 2.1.18. For inner models $M \subseteq N$ of ZFC, we say that M is *close* to N if for all measurable or singular strong limit cardinals μ of N,

1. μ is measurable or singular in M and
2. $(\mu^+)^M = (\mu^+)^N$.

We'll say that M is *close* if M is close to V.

The ms-indexed core model K of [11] is close, provably in ZFC+"there is no inner model with a Woodin cardinal". This follows by combining the covering theorems of from Jensen-Steel [11] and Mitchell-Schimmerling [**?**]. That is, we have the following.

Theorem 2.1.19. *Assume there is no inner model with a Woodin cardinal. Then K is a close inner model.*

As far as we can tell, ordinary weak covering, i.e. that $\mathrm{cof}(\lambda) \geq |\lambda|$ whenever $\lambda \geq \omega_2$ is a successor cardinal of M, may not be transitive whereas the property just introduced is transitive, that is we have the following.[6]

Proposition 2.1.20. *Suppose $M \subseteq N \subseteq P$ are inner models of ZFC, M is close to N, and N is close to P. Then M is close to P.*

Proof. This is immediate from the definition. $\square$

We make another definition which is just a strengthening of locally defining an inner model over T.

Definition 2.1.21. A formula φ *locally defines a close inner model over T* iff φ provably defines a close inner model over T and T proves that φ^V is close.

In the rest of this preliminary section, we dip into Inner Model Theory proper, reviewing some well-known facts about K as well as proving some new ones. First, we will state a folklore theorem about the absoluteness of iterability when there is no inner model with a Woodin cardinal.

[6] There is a consequence of weak covering which is transitive: if $\mu \geq \omega_2$ is a regular cardinal, then $cof((\mu^+)^M) \geq \mu$. This actually works fine for our purposes below $0^\P$ but does not seem to work below a Woodin cardinal, in general.

Theorem 2.1.22. *Assume there is no inner model with a Woodin cardinal. Let W be an inner model of ZFC. Then K^W is iterable.*

This is a corollary of the following result, due to Steel (this is easy to obtain from iterability absoluteness results in [27] and standard facts about the existence of Q-structures for normal iteration trees on 1-small premice).

Theorem 2.1.23. *Let W be an inner model of ZFC, κ be an uncountable cardinal of W, and P a 1-small premouse with $P \in H_\kappa^W$. Then P is iterable iff $H_\kappa^W \models$ "P is iterable".*

Proof of Theorem 2.1.22. Since there is no inner model with a Woodin cardinal, K^W is defined and also has no Woodin cardinals. It follows that for any successor K^W-cardinal, η, $K^W|\eta$ is 1-small. But if there is a bad normal iteration tree $\mathcal{T}$ on K^W in V, then there is a bad tree on $K^W|\eta$ for some such η. But $K^W|\eta$ is iterable in W and so iterable in V, by Theorem 2.1.23, a contradiction. □

Next, we'll confirm that there is a definition of K which locally defines an inner model over ZFC+"there is no inner model with a Woodin cardinal". This follows from Steel's inductive definition of K from [20]. One could also use Schindler's result from [6] that, above ω_2, levels of K are just obtained by stacking, however this has not been checked in context without the measurable cardinal.

Definition 2.1.24. For α a K-cardinal, a countably iterable, 1-small premouse N is α-strong iff $K|\alpha \trianglelefteq N$ and for all premice M such that M is β-strong for all K-cardinals $\beta < \alpha$, the phalanx (N, M, α) is iterable.

Arguments from [20] give

Theorem 2.1.25. *Assume there is no inner model with a Woodin cardinal. Let α be a cardinal of K. Then*

1. *N is α-strong iff $K|\alpha \trianglelefteq N$ and for all premice M of size $\leq |N|$ such that M is β-strong for all K-cardinals $\beta < \alpha$, the phalanx (M, N, α) is ω_1-iterable;*
2. *$K|(\alpha^+)^K = \bigcup\{N|(\alpha^+)^N \mid N$ is α-strong and $|N| = |\alpha|\}$.*

This immediately gives the following result.

Theorem 2.1.26. *Assume there is no inner model with a Woodin cardinal. Let μ be a strong limit cardinal. Then $K|\mu$, i.e. H_μ^K together with the extender sequence of $K|\mu$, is definable without parameters over H_μ, uniformly in μ.*

Proof. Fix μ. The idea here is to define when some premouse is an initial segment of $K|\mu$ by asserting there are sufficiently long sequences of sets S_α, ordinals κ_α, and premice P_α such that the κ_α are the K cardinals, S_α is the set of $< \kappa_\alpha$-strong premice of some *fixed* cardinality, and $P_\alpha = K|\kappa_\alpha$, using the inductive definition of K (i.e. the previous theorem). We use that μ is a strong limit to guarantee that the set of all $< \kappa_\alpha$-strong premice of our fixed cardinality $< \mu$ is a member of H_μ, since premice of size θ are essentially subsets of θ.

This is routine, but we include it here for completeness. A premouse Q of size $< \mu$ is a proper initial segment of $K|\mu$ iff there sequences $\langle S_\alpha \mid \alpha \leq \zeta\rangle$, $\langle P_\alpha \mid \alpha \leq \zeta\rangle$, $\langle \kappa_\alpha \mid \alpha \leq \zeta\rangle$, for some $\zeta < \mu$, such that

1.
   - S_0 is the set of countably iterable, 1-small premice of size $\leq |Q|$,
   - $P_0 = \langle V_\omega, \emptyset \rangle$,and
   - $\kappa_0 = \omega$;

2. for $\alpha + 1 \leq \zeta$,
   - $S_{\alpha+1}$ is the set of all $N \in S_\alpha$ such that $P_\alpha \trianglelefteq N$ and for all $M \in S_\alpha$, (M, N, α) is ω_1-iterable,
   - $P_{\alpha+1} = \bigcup\{N|(\kappa_\alpha^+)^N \mid N \in S_{\alpha+1}\}$, and
   - $\kappa_{\alpha+1} = o(P_{\alpha+1})$;

3. for $\lambda \leq \zeta$ a limit ordinal,
   - $S_\lambda = \bigcup\{S_\alpha \mid \alpha < \lambda\}$,
   - $P_\lambda = \bigcup\{P_\alpha \mid \alpha < \lambda\}$, and
   - $\kappa_\lambda = \sup\{\kappa_\alpha \mid \alpha < \lambda\}$; and

4. $Q \trianglelefteq P_\zeta$.

By Theorem 2.1.25 and our above comments, this gives a definition for $K|\mu$ over H_μ and is clearly uniform in μ. □

We fix $\varphi_K(v)$ be the formula which locally defines an inner model over over ZFC+"there is no inner model with a Woodin cardinal" given by the above inductive definition of K. We also let "V=K" be the sentence $\forall v\, \varphi_K(v)$.

Theorem 2.1.25 also gives the following.

Theorem 2.1.27. *Assume there is no inner model with a Woodin cardinal. Then $K \models$ "$V = K$". In particular, $K \models$ "$V = HOD$".*

Proof sketch. The point is that, by induction, we'll have that for all K-cardinals $\beta < \alpha$, $K|\alpha$ is β-strong inside of K. For 1-small N which is α-strong in K, the iterability of $(K|\alpha, N, \alpha)$ inside K implies that this phalanx is actually iterable in V, which suffices for showing that N is actually α-strong.

This implies $K \models$ "$V = HOD$" because K has a global well-order definable over V, by Theorem 2.1.26. □

One can also prove Theorem 2.1.27 using Theorem 2.1.41, below.

While very general local definability results can likely be obtained from the methods of [6], we easily get the following additional definability result.

Proposition 2.1.28. *Assume there is no inner model with a Woodin cardinal. Let μ be an inaccessible cardinal such that $(\mu^+)^K = \mu^+$. Then $K|\mu^+$ is definable without parameters over H_{μ^+}, uniformly in μ.*

Proof. This follows from Theorem 2.1.26 together with the fact that, under the hypotheses of the proposition, $K|\mu^+ = S(K|\mu)$, the stack of countably iterable sound premice extending $K|\mu$ which project to μ. This is sufficiently definable by Theorem 2.1.23, since the 1-small premice which are levels of this stack are cofinal. □

Note that there may be no μ as in the hypothesis of the proposition.

The following is an immediate corollary to the previous proposition, Theorem 2.1.26, and Theorem 2.1.27.

Corollary 2.1.29. *Assume there is no inner model with a Woodin cardinal. Let α be a limit cardinal of K or the K-successor of an inaccessible cardinal of K. Let P be a transitive model of ZFC$^-$. Then*

1. *if $\pi : H^K_\alpha \to P$ is elementary (in the language of set theory), there is a unique premouse $\hat{P}$ with universe P such that $\pi : K|\alpha \to \hat{P}$ is elementary (in the language of premice),*

2. *if $\pi : P \to H^K_\alpha$ is elementary (in the language of set theory), there is a unique premouse $\hat{P}$ of P such that is $\pi : \hat{P} \to K|\alpha$ is elementary (in the language of premice).*

Finally, we'll work towards showing that, in certain situations, elementary embeddings from initial segments of K are uniquely determined by their target model. These results are new, as far as we know, but just require known techniques.

We'll use the following easy criterion for being a fixed point of an embedding $\pi : M \to N$.

Lemma 2.1.30. *Let M, N be transitive models of ZFC$^-$ and $\pi : M \to N$ elementary. Suppose that $\sup \pi``\alpha = \alpha$ and π is continuous at $cof^M(\alpha)$. Then $\pi(\alpha) = \alpha$.*

Proof. Let $\gamma = \mathrm{cof}^M(\alpha)$. Fix $\langle \beta_\xi \mid \xi < \gamma \rangle \in M$ a cofinal increasing sequence in α. Then $\pi``\{\beta_\xi \mid \xi < \gamma\}$ is cofinal in $\pi(\{\beta_\xi \mid \xi < \gamma\})$ since $\sup \pi``\gamma = \pi(\gamma)$. So we have

$$\begin{aligned} \pi(\alpha) &= \sup \pi(\{\beta_\xi \mid \xi < \gamma\}) \\ &= \sup \pi``\{\beta_\xi \mid \xi < \gamma\} \\ &= \alpha. \end{aligned}$$

□

Lemma 2.1.31. *Let μ be a regular cardinal and M transitive models of ZFC$^-$ such that $o(M) = \mu^+$ and μ is the largest cardinal of M. Then for any $\alpha < \mu^+$,*

$$cof^M(\alpha) = \mu \iff cof(\alpha) = \mu.$$

Proof. If an ordinal $\alpha < \mu^+$ has $\mathrm{cof}(\alpha) = \mu$, then $\mathrm{cof}^M(\alpha) = \mu$ since $\mathrm{cof}(\mu) \leq \mathrm{cof}^M(\alpha) \leq \mu$, as $|\alpha|^M \leq \mu$, since μ is the largest cardinal of M. Conversely, if $\mathrm{cof}^M(\alpha) = \mu$, then $\mathrm{cof}(\alpha) = \mu$ since μ is a regular cardinal. □

Definition 2.1.32. Let μ be regular cardinal. We let $\mathcal{C}_{\mu,\mu^+}$ be the μ-club filter on μ^+, that is the filter generated by the cofinal subsets of μ^+ which are closed under increasing μ-sequences.

Proposition 2.1.33. *Let μ be a regular cardinal and M, N be transitive models of ZFC$^-$ such that $o(M) = o(N) = \mu^+$ and μ is the largest cardinal of M. Suppose that $\pi : M \to N$ is elementary and π is continuous at μ. Then fix$(\pi) \in \mathcal{C}_{\mu,\mu^+}$.*

Proof. If α is a limit of fixed points of π which has cofinality μ, then Lemma 2.1.31 gives $\mathrm{cof}^M(\alpha) = \mu$, so that $\pi(\alpha) = \alpha$ by Lemma 2.1.30 (since $\sup \pi"\alpha = \alpha$, as it is a limit of fixed points). So we just need to see π has arbitrarily large fixed points.

Fix $\beta < \mu^+$. Above β, we can build a μ-sequence $\langle \alpha_\xi \mid \xi < \mu \rangle$ in μ^+ such for all $\eta < \xi < \mu$, that $\pi(\alpha_\eta) < \alpha_\xi$. Let $\alpha = \sup\{\alpha_\xi \mid \xi < \mu\}$. Then $\alpha = \sup \pi"\alpha$ and, by Lemma 2.1.31, $\mathrm{cof}^M(\alpha) = \mathrm{cof}(\alpha) = \mu$. So by Lemma 2.1.30, $\pi(\alpha) = \alpha$. □

We'll typically use this in the following situation.

Corollary 2.1.34. *Let μ be a regular cardinal and M, N be transitive models of ZFC$^-$ such that $o(M) = o(N) = \mu^+$ and μ is the largest cardinal of M, N. Suppose that $\pi : M \to N$ is elementary. Then fix$(\pi) \in \mathcal{C}_{\mu,\mu^+}$.*

Proof. Since μ is definable in the same way in M, N (as the largest cardinal), $\pi(\mu) = \mu$. In particular, π is continuous at μ. So the previous proposition applies. □

Definition 2.1.35. Let μ be a regular cardinal. An iterable premouse P is *μ-universal* if $o(P) = \mu^+$ and P has largest cardinal μ.

Note that, in general, there may be no premouse P which is μ-universal, according to this definition. However, if there is no inner model with a Woodin cardinal and there is a regular cardinal μ such that $(\mu^+)^K = \mu^+$ (e.g. for a measurable cardinal μ), then $K|\mu^+$ is μ-universal.

Theorem 2.1.36. *Assume there is no inner model with a Woodin cardinal. Suppose μ is a regular cardinal such that $(\mu^+)^K = \mu^+$. Let P be μ-universal. Then there is a unique elementary embedding $\pi : K|\mu^+ \to P$. Moreover, π is definable over H_{μ^+} in parameters $K|\mu$ and $P|\mu$, uniformly in $P|\mu$.*

First we need to see that there is an embedding $\pi : K|\mu^+ \to P$ at all. For this, we extend the definition of $\tilde{K}(\tau, \Omega)$ from [11] to the case $\tau = \mu$ and $\Omega = \mu^+$.

Definition 2.1.37. Suppose that P μ-universal. $\mathrm{Def}^P = \bigcap\{Hull^P(\Gamma) \mid \Gamma \in \mathcal{C}_{\mu,\mu^+}\}$.

Standard arguments, as in [11], give

Proposition 2.1.38. *Suppose that P and Q are μ-universal. Then Def$^P \cong$ Def^Q.*

Definition 2.1.39. If there is a μ-universal P, then $\tilde{K}(\mu, \mu^+)$ is the common transitive collapse of Def^P for any μ-universal P.

Now, the collapsing weasel case of the proof of Lemma 4.31 from [11] gives

Proposition 2.1.40. *Suppose that there is a μ-universal P. Then $\tilde{K}(\mu, \mu^+)$ is μ-universal and there is a $\Gamma \in \mathcal{C}_{\mu,\mu^+}$ such that Def^P = Hull$^P(\Gamma)$.*

Theorem 2.1.41. *Assume there is no inner model with a Woodin cardinal. Suppose μ is a regular cardinal such that $(\mu^+)^K = \mu^+$. Then $K|\mu^+ = \tilde{K}(\mu, \mu^+)$.*

Proof. We have that $K|\mu = \tilde{K}(\mu, \mu^+)|\mu$, since $\tilde{K}(\mu, \mu^+)|\mu$ also satisfies the inductive definition of K, as in the proof of Lemma 6.1 of [11]. Since μ is regular, the stack over $K|\mu = \tilde{K}(\mu, \mu^+)|\mu$ is well-defined, i.e. the sound, iterable premice extending $K|\mu = \tilde{K}(\mu, \mu^+)|\mu$ and projecting to μ are totally ordered by the initial segment relation. It follows that $K|\mu^+ \trianglelefteq \tilde{K}(\mu, \mu^+)$ or $\tilde{K}(\mu, \mu^+) \trianglelefteq K|\mu^+$. But both have height μ^+, so they must be equal. $\square$

This easily gives the following.

Proposition 2.1.42. *Assume there is no inner model with a Woodin cardinal. Suppose μ is a regular cardinal such that $(\mu^+)^K = \mu^+$. Then for any $\Gamma \in \mathcal{C}_{\mu,\mu^+}$, $K|\mu^+ = Hull^{K|\mu^+}(\Gamma)$.*

Proof. It is enough to show that $K|\mu^+ = \mathrm{Hull}^P(\Gamma)$ for *some* $\Gamma \in \mathcal{C}_{\mu,\mu^+}$. Let P be μ-universal. Using Proposition 2.1.40 and Theorem 2.1.41, we let Γ be such that $K|\mu^+$ is the transitive collapse of $\mathrm{Hull}^P(\Gamma)$. Let $\pi : K|\mu^+ \to P$ be the uncollapse map. By Proposition 2.1.34, we can assume that Γ is a set of fixed points of π. For any $\Lambda \subseteq \Gamma$, since $\pi``\Lambda = \Lambda$, $\pi``\mathrm{Hull}^{K|\mu^+}(\Lambda) = \mathrm{Hull}^P(\Lambda) = \mathrm{Def}^P$, and so $\mathrm{ran}(\pi) \subseteq \pi``\mathrm{Hull}^{K|\mu^+}(\Lambda)$. It follows that $K|\mu^+ = \mathrm{Hull}^{K|\mu^+}(\Lambda)$. $\square$

Proof of Theorem 2.1.36. By Theorem 2.1.41, there is an embedding from $K|\mu^+$ into P, and by Proposition 2.1.40, we actually have that $K|\mu^+$ is the transitive collapse of $\mathrm{Def}^P = \mathrm{Hull}^P(\Gamma)$ for some $\Gamma \in \mathcal{C}_{\mu,\mu^+}$. So suppose $\pi : K|\mu^+ \to P$ is elementary. Then by Proposition 2.1.34, $\mathrm{fix}(\pi) \in \mathcal{C}_{\mu,\mu^+}$, so we can find some $\Lambda \in \mathcal{C}_{\mu,\mu^+}$ which is a set of fixed points of π such that $\mathrm{Def}^P = \mathrm{Hull}^P(\Lambda)$. We also get $K|\mu^+ = \mathrm{Hull}^{K|\mu^+}(\Lambda)$, by Proposition 2.1.42. It follows that

$$\begin{aligned} \pi``K|\mu^+ &= \pi``\mathrm{Hull}^{K|\mu^+}(\Lambda) \\ &= \mathrm{Hull}^P(\pi``\Lambda) \\ &= \mathrm{Hull}^P(\Lambda) \\ &= \mathrm{Def}^P. \end{aligned}$$

Since π was arbitrary, Def^P is the range of any elementary embedding from $K|\mu^+$ into P. So there is at most one such embedding.

For the definability of π, since μ is regular, standard arguments give that $K|\mu$ and $P|\mu$ have a common iterate, Q, and letting $i : K|\mu \to Q$ and $j : P|\mu \to Q$ be the iteration maps of the comparison, and E, F the length μ extenders of these iteration maps,

$$S(Q) = \mathrm{Ult}(K|\mu^+, E) = \mathrm{Ult}(P, F),$$

where $S(Q)$ is the stack over Q. Let $\hat{\imath} : K|\mu^+ \to S(Q)$ and $\hat{\jmath} : P \to S(Q)$ be the ultrapower maps. Then we also have that $\hat{\imath}``K|\mu^+ = \hat{\jmath}``\mathrm{Def}^P$. It follows that $\pi = \hat{\jmath}^{-1} \circ \hat{\imath}$. Since $K|\mu$, $P|\mu$, Q, and E, F are all in H_{μ^+} and since $K|\mu^+ = S(K|\mu)$ and $P = S(P|\mu)$, we get the required definability of π (using for uniformity that E, F came from the comparison). $\square$

To finish this preliminary section, we review some results which hold below $0^\P$. In this context, the theory of the Jensen-indexed core model, which we denote J, has been developed by Schindler, see [16] or [28].

The covering theorems of Schindler [16] and Cox [3] immediately give

Theorem 2.1.43. *Assume $0^\P$ does not exist. Then J is close to V.*

J also has an inductive definition—we let φ_J be the resulting formula. We have that φ_J locally defines a close inner model over ZFC+"$0^\P$ does not exist". We also let "$V = J$" be the formula $\forall v\, \varphi_J(v)$.

One important feature of Inner Model Theory below $0^\P$ is that the theory of (definable) proper class premice is well-behaved. This is because definable iteration trees on iterable premice have definable well-founded branches, even when they are proper class sized. This fails below a Woodin cardinal in general.

We have the following below $0^\P$.

Theorem 2.1.44. *Assume $0^\P$ does not exist. Suppose that φ is a Σ_n-formula which defines a close inner model W. Then*

1. *there is an elementary embedding $k : K \to K^W$, definable uniformly in φ,*
2. *there is an elementary embedding $j : J \to J^W$, definable uniformly in φ.*

Proof sketch. We'll just talk about the ms-indexed core model K, as it is symmetric. K^W is iterable in W and so it is actually iterable, by Theorem 2.1.22.[7] Moreover, K^W is close to M by Theorem 2.1.19 in W. So K^M is close to V by Proposition 2.1.20. In particular, $(\mu^+)^{K^M} = \mu^+$ for all singular strong limit cardinals μ, so K^M is *universal* in the sense that it is maximal in the ms-indexed mouse-order, by standard arguments (cf. Lemma 6.3.1 of Zeman [28]). In particular, K and K^M have a common, non-dropping iterate, obtained by comparing the two inner models. By standard arguments (which can be found in [28] or [20]), K^W doesn't move in this comparison and so there is an elementary embedding $k : K \to K^W$. Since k was obtained as the iteration map of the (definable) comparison, it is definable, uniformly in the definition of W. $\square$

We suspect that Theorem 2.1.44 may fail below a Woodin cardinal, but we do not have a counterexample.

Finally, we also have that K and J are *rigid*. Because rigidity of an inner model is not expressible in the language of set theory, in general, we make the following definition.

Definition 2.1.45. An inner model M is Σ_n*-rigid* if there is no Σ_n-definable, non-trivial elementary embedding $j : M \to M$.

By standard techniques (cf [28] or [20]), we have the following, for any n.

Theorem 2.1.46. *Assume $0^\P$ does not exist. Then*

1. *K is Σ_n-rigid,*

[7] Below $0^\P$, this absoluteness fact is actually easier and holds for Jensen-indexed premice as well.

2. J is Σ_n-rigid.

This easily implies the following analogue of Theorem 2.1.27.

Theorem 2.1.47. *Assume $0^\P$ does not exist. Then $J \models$ "$V = J$". In particular, $J \models$ "$V = HOD$".*

Proof sketch. By Theorem 2.1.44 in V, there is a definable elementary embedding $j_0 : J \to J^J$. But also since J is a universal Jensen-indexed proper class premouse (i.e. Jensen-indexed mouse-order maximal), the proof of 2.1.44[8] gives there is also an elementary embedding $j_1 : J^J \to J$ which is still definable in V. But then $j_1 \circ j_0 : J \to J$ and so must be the identity, by Theorem 2.1.46. It follows that $J = J^J$, as desired. We get $J \models$ "$V = HOD$" just as in the proof of Theorem 2.1.27. □

Of course, this also gives another proof of Theorem 2.1.27 under the additional hypothesis that $0^\P$ does not exist. This same argument will come up again in our uniqueness results.

In [20], Steel proves that if there is no inner model with a Woodin cardinal and Ω is a measurable cardinal, then $K|\Omega$ is rigid. Surprisingly, it appears to be open whether K is (Σ_n-)rigid just under the hypothesis that there is no inner model with a Woodin cardinal. The difficulty in adapting arguments from [20] to this context is that these arguments rely on the existence of very soundness witnesses for initial segments of K which are definable over K. We do not see how to get such witnesses in the general context.

2.2 Below $0^\P$

In this section, we will prove our simplest uniqueness result about the core model under the hypothesis that $0^\P$ does not exist. We prove, basically, that the core model is the unique inner model which "resembles the core model", which we define shortly. The proof that there is at most one such inner model doesn't make use of the existence of the core model at all and doesn't use our hypothesis that $0^\P$ does not exist So there is a corresponding uniqueness theorem which holds under just ZFC, although it is possibly trivial.

Definition 2.2.1. Let T be a nice extension of ZFC. For M an inner model, $\varphi(v)$ a formula which locally defines a close inner model over T, and $\psi(u, v, w)$ a formula, M *resembles the core model via* (φ, ψ) iff $M = \varphi^V$ and the following is provable in T:

1. $\varphi^V \models$"V=HOD",
2. $\varphi^V \models \forall x\, \varphi(x)$,
3. for any Σ_2-formula γ, if $W = \gamma^V$ is a close inner model of ZFC, then $\psi(\ulcorner\gamma\urcorner, v, w)$ defines an elementary embedding from φ^V into φ^W.[9]

[8] We mean the proof of the Jensen-indexed case, which was omitted but is symmetric to the ms-indexed case, which we sketched.

[9] Recall that an elementary embedding between definable inner models of ZFC is expressible in the language of set theory by asserting just Σ_1-elementarity.

We'll say that M *Σ_n-resembles the core model over T* if we can take ψ to be a Σ_n-formula. We'll just say that M *resembles the core model* if it Σ_{10000}-resembles the core model over ZFC+"$0^\P$ does not exist". Of course, this number is overkill: we just need n sufficiently large so that the actual core model, K, Σ_n-resembles the core model over ZFC+"$0^\P$ does not exist".

Our main result is the following schema, for $n \geq 1$.

Theorem 2.2.2. *Assume T. Suppose M and N resemble the core model over T via Σ_n-formulas and M is Σ_n-rigid. Then $M = N$.*

Proof. Fix φ, ρ such that M resembles the core model via (φ, ρ) and let ψ, χ such that N resembles the core model via (ψ, χ). So φ and ψ are formulas which locally define close inner models over T. For W an inner model, we'll write M^W instead of φ^W and N^W instead of ψ^W.

We first show $N^M = M$ and $M^N = N$. Since it's symmetric, we'll just show the former. To get this, we'll show that M elementarily embeds into some M_∞ such that $M_\infty \models \forall x\, \psi(x)$, so that $M \models \forall x\, \psi(x)$, i.e. $N^M = M$.

Let $\xi = \xi_{\varphi,\psi}$ be the formula from Proposition 2.1.17 (defined in the discussion preceeding it). Let $M_k = \xi(2k, v)^V$ and $N_k = \xi(2k+1, v)^V$. Then by Proposition 2.1.17, $M_0 = \varphi^V = M$, $N_k = N^{M_k}$, and $M_{k+1} = M^{N_k}$. Let $\pi_i = \rho(\ulcorner\psi\urcorner, u, v)^{M_i}$ and $\sigma_i = \chi(\ulcorner\varphi\urcorner, u, v)^{N_i}$. So π_i is an elementary embedding from M_i into M_{i+1} and σ_i is an elementary embedding from N_i into N_{i+1}. For $i \leq j$, let $\pi_{i,j} : M_i \to M_j$ and $\sigma_{i,j} : N_i \to N_j$ the natural maps obtained from composing the π_k and σ_k, respectively. Let $\mathcal{C} = \{N_i, \sigma_{i,j} \mid i, j \in \omega \text{ and } i \leq j\}$ and $\mathcal{D} = \{M_i, \pi_{i,j} \mid i, j \in \omega \text{ and } i \leq j\}$. Let N_∞ be the direct limit of $\mathcal{C}$, M_∞ the direct limit of $\mathcal{D}$, and $\sigma_{i,\infty}$ and $\pi_{i,\infty}$ the direct limit maps.

First we'll show M_∞, N_∞ are well-founded. Of course, the argument is the same for M_∞ and N_∞; moreover, it is just Gaifman's argument that the ω^{th}-iterate of V by a countably complete ultrafilter is well-founded.

Suppose M_∞ is not well-founded. Let α least such M_∞ is ill-founded below $\pi_{0,\infty}(\alpha)$. By Proposition 2.1.17 and how we chose the $\pi_{i,j}$, $\{\langle \pi_{i,j} \mid j \geq i\rangle \mid i \in \omega\}$ is uniformly definable over $\mathcal{D}$. But then for α least such that M_∞ is ill-founded below $\pi_{0,\infty}(\alpha)$, for any i, $\pi_{0,i}(\alpha)$ is the least β such that M_∞ is ill-founded below $\pi_{i,\infty}(\beta)$, so M_∞ cannot be ill-founded below $\pi_{0,\infty}(\alpha)$ after all, a contradiction.

As we mentioned in the preceding argument, $\{\langle \pi_{i,j} \mid j \geq i\rangle \mid i \in \omega\}$ is uniformly definable over $\mathcal{D}$. So by Replacement, the (transitivized) M_∞ and a tail of the direct limit maps $\pi_{i,\infty}$ are also uniformly definable over $\mathcal{D}$. Since N_i is definable in M_i we also get the (transitivized) N_∞ and the direct limit maps $\sigma_{i,\infty}$ are also uniformly definable over $\mathcal{D}$. Since we also have $N_i \subseteq M_i$ and $N_i \models$ "V=HOD" (by clause 1 of the definition of resembles the core model), Theorem 2.1.10 gives $N_\infty \subseteq M_\infty$. A symmetric argument shows $M_\infty \subseteq N_\infty$. But by elementarity, $N_\infty \models \forall x\, \psi(x)$ so M_∞ does too, as desired.

To finish, let $\pi = \rho(\ulcorner\psi\urcorner, u, v)^V$ and $\sigma = \chi(\ulcorner\varphi\urcorner, u, v)^V$. So $\pi : M \to M^N = N$ and $\sigma : N \to N^M = M$ are elementary. If $M \neq N$ then at least one of π, σ is not the identity on the ordinals. But then as σ, π are definable by the Σ_n-formulas ψ and τ, $\sigma \circ \pi : M \to M$ is a

Σ_n-definable elementary embedding[10] which is not the identity, contradicting the Σ_n-rigidity of M. So $M = N$ after all. $\square$

As in the preliminary section, we let K be the ms-indexed core model and J the Jensen-indexed core model (defined below $0^\P$).

Theorem 2.2.3. *Assume $0^\P$ doesn't exist. Then $K = J$ is the unique inner model which resembles the core model.*

Proof. Recall that the inductive definitions of K and J give formulas φ_K and φ_J which locally define close inner models over ZFC+"$0^\P$ doesn't exist" (see the preliminaries section). We want to see that there are formulas ψ_K and ψ_J such that K and J resemble the core model via (φ_K, ψ_K) and (φ_J, ψ_J), respectively. But (1) and (2), which only mention φ, are immediate from Theorems 2.1.27 and 2.1.47 and Theorem 2.1.44 immediately gives us our desired formulas ψ_K and ψ_J witnessing (3) for φ_K and φ_J. Finally we actually have that both J and K are both sufficiently rigid, by Theorem 2.1.46, so Theorem 2.2.2 gives $K = J$ is the unique inner model which resembles the core model. $\square$

2.3 Below a Woodin cardinal

In this section we prove a uniqueness theorem about the core model under the less restrictive hypothesis that there is no inner model with a Woodin cardinal, assuming that there is a proper class of measurable cardinals. We proceed similarly to before: we'll (re-)define "resembles the core model", show there is at most one such inner model, and then prove that the ms-indexed K is that model. We do not know whether the Jensen-indexed core model satisfies the definition given in this section. While one can show that the ms-indexed K is a modified Jensen-indexed proper class premouse, using the translation theorems of Fuchs [4], [5], and so identify a (modified) Jensen-indexed core model in some sense, the theory developed in [20] or [11] has not been worked out for Jensen-indexed premice. This is almost certainly possible and will likely appear in Jensen's in-progress manuscript [10]. In any case, our theorem indicates that the resulting core model will likely just be the ms-indexed core model, K.

Under the hypothesis that there is inner model with a Woodin cardinal but there is a proper class of measurable cardinals, the ms-indexed K is just $\bigcup\{K_\mu \mid \mu \text{ measurable}\}$, where K_μ is the core model from Steel's [20] (at the measurable cardinal μ). In this context, we can identify K while avoiding almost all the technicalities around definability from the previous section. For μ a measurable cardinal, we'll identify $H^K_{\mu^+}$ as the unique $H^M_{\mu^+}$ for M which "resembles the core model at (μ, λ)", for λ any inaccessible cardinal above μ (assuming there is no inner model with a Woodin cardinal, of course).

Definition 2.3.1. Let $\mu < \lambda$ with μ measurable and λ inaccessible. A transitive model P is *μ-full at λ* iff $P = V^W_\lambda$ for an inner model W of ZFC such that μ is measurable in W and $(\mu^+)^W = \mu^+$.

[10] Using here that $n \geq 1$.

It is not immediately obvious that being μ-full at λ is expressible in the language of set theory, since we quantified over the proper class W in the above definition. We leave it to the reader to check the following easy proposition.

Proposition 2.3.2. *P is μ-full at λ iff P is a transitive model of ZFC, $o(P) = \lambda$, $P \models \mu$ is measurable, $(\mu^+)^P = \mu^+$, and there is a well-order $\leq$ of P such that every bounded subset of λ constructible from P and $\leq$ is in P (i.e. $P = V_\lambda^{L(P,\leq)}$).*

We let $\mathrm{Full}_{\mu,\lambda}$ be the set of all P which are μ-full at λ. We can now state the main definition of the section.

Definition 2.3.3. Let μ be a measurable cardinal and $\lambda > \mu$ inaccessible. A transitive model M *resembles the core model at* (μ, λ) if there is a function from $\mathrm{Full}_{\mu,\lambda}$ into $\mathrm{Full}_{\mu,\lambda}$, $P \mapsto M^P$, such that

1. for all $P \in \mathrm{Full}_{\mu,\lambda}$, $M^P \subseteq P$,
2. $M = M^{V_\lambda}$,
3. for all $P \in \mathrm{Full}_{\mu,\lambda}$, $M^{M^P} = M^P$.
4. for any $P, Q \in \mathrm{Full}_{\mu,\lambda}$, if $\pi : H^P_{\mu^+} \to H^Q_{\mu^+}$ is elementary, then $\pi \restriction H^{M^P}_{\mu^+}$ is elementary from $H^{M^P}_{\mu^+}$ into $H^{M^Q}_{\mu^+}$,
5. for any $P, Q \in \mathrm{Full}_{\mu,\lambda}$ such that $Q \subseteq P$, there is an elementary embedding $\pi : H^{M^P}_{\mu^+} \to H^{M^Q}_{\mu^+}$ such that
   (a) $\pi \in P$,
   (b) $P \models$ "π is the unique elementary embedding from $H^{M^P}_{\mu^+}$ into $H^{M^Q}_{\mu^+}$."

Let us briefly discuss this definition. First, the function $P \mapsto M^P$ is really just proxy for M having something like a local definition which provably defines a close inner model. This is why (4) is at all plausible. Still, it is convenient to abstract away from definability to the extent we can. Also note that (2) and (5) give that $H^M_{\mu^+}$ elementarily embeds into $H^{M^P}_{\mu^+}$ for any $P \in \mathrm{Full}_{\mu,\lambda}$. Finally note that (5) for $P = Q = V_\lambda$ implies that there is no non-trivial elementary embedding $\pi : H^M_{\mu^+} \to H^M_{\mu^+}$, since the identity must be the unique such embedding (all such embeddings are in V_λ).

Under the hypothesis that there is no inner model with a Woodin cardinal, we will show that levels of K resemble the core model, as witnessed by the function $P \mapsto (\varphi_K)^P$, and that the maps π witnessing (5) for levels of K are actually uniformly definable, which will be important for the uniqueness proof. We make the following definition capturing the additional properties of the way K resembles the core model.

Definition 2.3.4. Let μ be a measurable cardinal and $\lambda > \mu$ inaccessible. A transitive model M *strongly resembles the core model at* (μ, λ) if there is a function $P \mapsto M^P$ such that (1)-(5) hold, $H^{M^P}_{\mu^+}$ is uniformly definable over $H^P_{\mu^+}$, and the maps $\pi : H^{M^P}_{\mu^+} \to H^{M^Q}_{\mu^+}$ witnessing (5) are definable over $H^P_{\mu^+}$, uniformly in parameter H^Q_μ.

We now prove the uniqueness result.

Theorem 2.3.5. *Suppose that μ is a measurable cardinal and $\lambda > \mu$ is inaccessible. Suppose that N resembles the core model at (μ, λ) and M strongly resembles the core model at (μ, λ). Then $H^M_{\mu^+} = H^N_{\mu^+}$.*

Proof. Fix a function $P \mapsto N^P$ witnessing that N resembles the core model at (μ, λ) and a function $P \mapsto M^P$ witnessing that M *strongly* resembles the core model at (μ, λ). We also fix φ a formula witnessing that $H^{M^P}_{\mu^+}$ is uniformly definable over $H^P_{\mu^+}$ for $P \in \mathrm{Full}_{\mu,\lambda}$, i.e. such that $H^{M^P}_{\mu^+} = \varphi^{H^P_{\mu^+}}$ for all $P \in \mathrm{Full}_{\mu,\lambda}$ (such a φ is guaranteed by the definition of strongly resembling the core model).

First we'll verify the following.

Claim 1. $H^{N^M}_{\mu^+} = H^M_{\mu^+}$

Proof. Let $M_0 = M$, $N_0 = N^M$, $M_1 = M^{N_0}$, $N_1 = N^{M_1}$, and $M_2 = M^{N_1}$. Then, using (1) for N, $M_2 \subseteq M_1 \subseteq M_0$, so we can fix elementary embeddings $\pi : H^{M_0}_{\mu^+} \to H^{M_1}_{\mu^+}$ and $\sigma : H^{M_1}_{\mu^+} \to H^{M_2}_{\mu^+}$ witnessing (5) for M. Similarly, fix an elementary embedding $\tau : H^{N_0}_{\mu^+} \to H^{N_1}_{\mu^+}$ witnessing (5) for M. (4) for N gives $\pi \restriction H^{N_0}_{\mu^+} : H^{N_0}_{\mu^+} \to H^{N_1}_{\mu^+}$, so since $\pi \in M_0$ by (5)(a) for M, (5)(b) for N gives that $\pi \restriction H^{N_0}_{\mu^+} = \tau$. A symmetric argument gives that $\tau \restriction H^{M_1}_{\mu^+} = \sigma$. So we have that $\pi \restriction H^{M_1}_{\mu^+} = \sigma$.

Now suppose that π is not the identity and let $\kappa = \mathrm{crit}(\pi)$. Then κ is definable over $H^{M_0}_{\mu^+}$ in parameter $H^{N_0}_{\mu}$, since π is, using (3) for M together with our assumption that M *strongly* resembles the core model at (μ, λ). These assumptions together with (4) for N give that $\mathrm{crit}(\sigma)$ is defined in the same way over $H^{M_1}_{\mu^+}$ in parameter $H^{N_1}_{\mu} = \pi(H^{N_0}_{\mu})$ as $\kappa = \mathrm{crit}(\pi)$ is over $H^{M_0}_{\mu^+}$ in parameter $H^{N_0}_{\mu}$. Since π is elementary, it follows that $\pi(\kappa) = \mathrm{crit}(\sigma)$. But σ and π agree on the ordinals, so $\mathrm{crit}(\sigma) = \mathrm{crit}(\pi) = \kappa$. So $\pi(\kappa) = \kappa$, contradicting that κ is the critical point of π. So π is the identity and $H^{N^M}_{\mu^+} = H^M_{\mu^+}$, as claimed □

Next we show

Claim 2. $H^{M^N}_{\mu^+} = H^N_{\mu^+}$

Proof. Since our hypotheses on M and N are not symmetric, this doesn't follow immediately from the proof of the previous claim. What that proof does give that $H^{N^{M^N}}_{\mu^+} = H^{M^N}_{\mu^+}$. So, by (5) for N, we get an elementary embedding $\pi : H^N_{\mu^+} \to H^{M^N}_{\mu^+}$. By (3) for M, we have that $H^{M^{M^N}}_{\mu^+} = H^{M^N}_{\mu^+}$. In particular, $H^{M^N}_{\mu^+} \models \forall v\, \varphi(v)$, by our choice of φ. Since π is elementary, $H^N_{\mu^+} \models \forall v\, \varphi(v)$ as well. So $H^{M^N}_{\mu^+} = H^N_{\mu^+}$, as claimed. □

By these claims and (5) for M and N, there are elementary embeddings $\pi : H^M_{\mu^+} \to H^{M^N}_{\mu^+} = H^N_{\mu^+}$ and $\sigma : H^N_{\mu^+} \to H^{N^M}_{\mu^+} = H^M_{\mu^+}$. So $\sigma \circ \pi : H^M_{\mu^+} \to H^M_{\mu^+}$ is elementary and so must be the identity, by (5) for M (see discussion following the definition). It follows that π and σ are the identity as well and so $H^M_{\mu^+} = H^N_{\mu^+}$, as desired. □

We now show that levels of K strongly resemble the core model under the hypothesis that there is no inner model with a Woodin cardinal.

Lemma 2.3.6. *Assume there is no inner model with a Woodin cardinal. Suppose that μ is measurable and $\lambda > \mu$ is inaccessible. Then the function on $Full_{\mu,\lambda}$ given by $P \mapsto K^P = (\varphi_K)^P$ witnesses that V_λ^K strongly resembles the core model.*

Proof. (1) is immediate since K is provably close. (2) follows by how we chose our function. (3) follows from Theorem 2.1.27. (4), (5), and the additional definability requirement on the witnessing maps follow from Theorem 2.1.36 together with Theorem 2.1.26. Finally, the fact that $H_{\mu^+}^{K^P}$ is definable over $H_{\mu^+}^P$ follows from the fact that, working inside P, $K|\mu$ is definable over H_μ, since μ is a strong limit, by Theorem 2.1.26, together with the fact that $K|\mu^+ = S(K|\mu)$, the stack over $K|\mu$, as μ is regular and $(\mu^+)^K = \mu^+$. $\square$

This lemma and the previous theorem immediately imply the following.

Theorem 2.3.7. *Assume there is no inner model with a Woodin cardinal. Suppose there is a proper class of measurable cardinals. Then K is the unique inner model such that for all measurable cardinals $\mu < \lambda$, V_λ^K resembles the core model at (μ, λ).*

Chapter 3

Meta-Iteration Trees

In [25], John Steel isolates the notion of a *mouse pair*, a premouse P together with a sufficiently well-behaved iteration strategy Σ for P. Steel proves a comparison theorem for mouse pairs and shows that many of the basic results from lower-level inner model theory can be stated in their proper general form by considering mouse pairs instead of just premice. For example, we have the full Dodd-Jensen property for mouse pairs, and thus a well-founded mouse pair order, whereas these both fail if we consider iterable premice in isolation. These and other results seem to indicate that mouse pairs, not just their mouse components, are the fundamental objects of study in inner model theory. In this chapter, we contribute to this study by developing a useful framework for examining some of the nice properties of iteration strategies which come up in work of Steel and others.

Given a stack of normal iteration trees $\vec{\mathcal{S}}$ on a premouse P, Steel and Schlutzenberg identified a procedure for rearranging the extenders of $\vec{\mathcal{S}}$ so that they generate a single normal iteration tree $W(\vec{\mathcal{S}})$, the *embedding normalization of* $\vec{\mathcal{S}}$. The embedding normalization process is best viewed as a kind tree *of* normal iteration trees, which we'll call a *meta-iteration tree*, or *meta-tree*. The meta-tree notion evolved from the work of Jensen, Schlutzenberg, and Steel, but was first explicitly isolated by Schlutzenberg (see [9] and [19], which use somewhat different terminology for meta-trees and their associated apparatus.) We use the meta-iteration tree framework to prove some new results, for example, that some nice properties of iteration strategies (versions of normalizing well and strong hull condensation) pass to tail strategies. We also use this framework to give what we think is a more perspicuous proof of Schlutzenberg's theorem on extending iteration strategies to infinite stacks. Outside of this thesis, variants of this framework will be used to give a proof of full normalization for mouse pairs in joint, in-progress work with Steel. Full normalization is used in Steel's work on optimal Suslin representations, which he has used to give descriptive-set-theoretic characterizations of Woodin cardinals of the HOD of models of determinacy, see [23]

This chapter assumes familiarity with Jensen-indexed premice and their basic theory—we refer the reader to Steel's [25], §2.

3.1 Tree embeddings and meta-iteration trees

In this section we introduce the basic objects we'll study in this chapter: tree embeddings, the embedding normalization, and meta-iteration trees. We'll review some results of Steel and Schlutzenberg and give most of a proof of Schlutzenberg's theorem on extending iteration strategies to infinite stacks. We postpone the proof of one relevant result until near the end of the next section.

3.1.1 Tree embeddings

Tree embeddings were isolated by Steel in [24]. They arise naturally in the context of embedding normalization and will feature prominently in the rest of the chapter. In this subsection, we introduce tree embeddings and look at directed systems of normal iteration trees under tree embeddings, identifying the natural direct limit normal iteration tree, when it exists.

Definition 3.1.1. Let P be a premouse and $\mathcal{S}, \mathcal{T}$ be normal iteration trees on P. A *tree embedding* $\Phi : \mathcal{S} \to \mathcal{T}$ is a system $\langle u^\Phi, v^\Phi, \{s^\Phi_\xi\}_{\xi<\mathrm{lh}(\mathcal{S})}, \{t^\Phi_\zeta\}_{\zeta+1<\mathrm{lh}(\mathcal{S})}\rangle$ such that

1. $v^\Phi : \mathrm{lh}(\mathcal{S}) \to \mathrm{lh}(\mathcal{T})$ is tree-order preserving, $u^\Phi : \{\eta \mid \eta + 1 < \mathrm{lh}(\mathcal{S})\} \to \mathrm{lh}(\mathcal{T})$, $v^\Phi(\xi) = \sup\{u^\Phi(\eta) + 1 \mid \eta < \xi\}$, and $v^\Phi(\xi) \leq_\mathcal{T} u^\Phi(\xi)$;
2. for all $\xi < \mathrm{lh}(\mathcal{S})$ and $\eta \leq_\mathcal{S} \xi$,
   (a) $s^\Phi_\xi : M^\mathcal{S}_\xi \to M^\mathcal{T}_{v^\Phi(\xi)}$ is elementary and $s^\Phi_0 = id_{M^\mathcal{S}_0}$;
   (b) $\hat{\imath}^\mathcal{T}_{v^\Phi(\eta),v^\Phi(\xi)} \circ s^\Phi_\eta = s^\Phi_\xi \circ \hat{\imath}^\mathcal{S}_{\eta,\xi}$,[1]
   (c) if $\xi + 1 < \mathrm{lh}(\mathcal{S})$, then $t^\Phi_\xi = \hat{\imath}^\mathcal{T}_{v(\xi),u(\xi)} \circ s^\Phi_\xi$ with $E^\mathcal{S}_\xi \in \mathrm{dom}(t^\Phi_\xi)$;[2]
3. for all $\xi + 1 < \mathrm{lh}(\mathcal{S})$, letting $\eta = \mathcal{S}\text{-pred}(\xi + 1)$, and $\eta^* = \mathcal{T}\text{-pred}(u(\xi) + 1)$,
   (a) $E^\mathcal{T}_{u(\xi)} = t^\Phi_\xi(E^\mathcal{S}_\xi)$,
   (b) $\eta^* \in [v(\eta), u(\eta)]_\mathcal{T}$,
   (c) $s^\Phi_{\xi+1} \restriction \mathrm{lh}(E^\mathcal{S}_\xi) = t^\Phi_\xi \restriction \mathrm{lh}(E^\mathcal{S}_\xi)$.

Going forward, we'll use the following notation for applications of the Shift Lemma.

Definition 3.1.2. For maps $\pi : \bar{N} \to N$, $\sigma : \bar{M} \to M$, and an extender E on the $\bar{N}$-sequence, we'll say *the Shift Lemma applies to* (π, σ, E) iff

1. $\bar{N}|\mathrm{dom}(E) = \bar{M}|\mathrm{dom}(E)$ and
2. $\pi \restriction \mathrm{dom}(E) = \sigma \restriction \mathrm{dom}(E)$

[1] We are using $\hat{\imath}^\mathcal{T}$ to denote the possibly partial branch embeddings of $\mathcal{T}$, following Steel [25].

[2] Note that t^Φ_ξ is only a partial elementary map from $M^\mathcal{S}_\xi$ into $M^\mathcal{T}_{u(\xi)}$ in general, since $[v(\xi), u(\xi)]_\mathcal{T}$ may drop. The demand that $E^\mathcal{S}_\xi \in \mathrm{dom}(t^\Phi_\xi)$ just means that we don't drop below the image of $E^\mathcal{S}_\xi$ along $[v(\xi), u(\xi)]_\mathcal{T}$.

In this case, *the copy map associated to* (π,σ,E) is the unique map $\tau:\mathrm{Ult}(\bar P,E)\to \mathrm{Ult}(P,\pi(E))$, for $\bar P$ the least initial segment of $\bar M$ beyond $\mathrm{dom}(E)$ with $\rho(\bar P)\le \mathrm{crit}(E)$ or $\bar P=\bar M$ and P the least initial segment of M beyond $\mathrm{dom}(\pi(E))$ with $\rho(P)\le \mathrm{crit}(\pi(E))$ or $P=M$, such that

1. $\tau\circ i_E^{\bar P}=i^P_{\pi(E)}\circ\sigma$ and
2. $\tau\restriction \mathrm{lh}(E)=\pi\restriction\mathrm{lh}(E)$.

Note that in (3) of the definition of tree embedding above, we get that

$$i^{M^{\mathcal T}_{\eta^*}}_{E^{\mathcal T}_{u(\xi)}}\circ \hat\imath^{\mathcal T}_{v(\eta),\eta^*}\circ s_\eta = s_{\xi+1}\circ i^{M^{\mathcal S}_\eta}_{E^{\mathcal S}_\xi},$$

which together with (3)(c) guarantees that $s_{\xi+1}$ is the copy map associated to $(t_\xi,\hat\imath^{\mathcal T}_{v(\eta),\eta^*}\circ s_\eta,E^{\mathcal S}_\xi)$. Also note that while we didn't explicitly mention how the s^Φ_λ are defined at limit λ in the definition of tree embedding, they are uniquely determined by the commutativity conditions (because we take direct limits at limit ordinals in iteration trees). Together, these observations reveal that there is some redundant information in the definition of a tree embedding: a tree embedding $\Phi:\mathcal S\to\mathcal T$ is actually totally determined by $\mathcal S,\mathcal T$, and u^Φ. In fact, it is totally determined by $\mathcal S, u^\Phi$, the extenders $E^{\mathcal T}_\xi$ with $\xi\notin\mathrm{ran}(u^\Phi)$, and the branch choices $[0,\lambda)_{\mathcal T}$ for λ such that $\mathrm{ran}(u^\Phi)\cap\lambda$ is bounded below λ.

Definition 3.1.3. Let $\mathcal S$ be a normal tree of successor length $\gamma+1$ and $\mathcal T$ a normal tree of has successor length $\delta+1$. For $\Phi:\mathcal S\to\mathcal T$ a tree embedding such that $v^\Phi(\gamma)\le_{\mathcal T}\delta$, we let $u^\Phi(\gamma)=\delta$ and $t^\Phi_\gamma=\hat\imath^{\mathcal T}_{v^\Phi(\gamma),\delta}\circ s^\Phi_\gamma$ and call the resulting system an *extended tree embedding.* An extended tree embedding is *non-dropping* if $(v^\Phi(\gamma),\delta]_{\mathcal T}$ doesn't drop.

Remark 3.1.4. Note that if $\Phi:\mathcal S\to\mathcal T$ is a tree embedding and $\mathcal S$ has successor length $\delta+1$, then we can always view Φ as a non-dropping extended tree embedding $\Phi:\mathcal S\to\mathcal T\restriction v(\delta)+1$. On the other hand, if $\mathcal S$ has limit length, b is a cofinal branch of $\mathcal S$, and $\Phi:\mathcal S\to\mathcal T$ is a tree embedding, there may be no extension of Φ to an extended tree embedding from $\mathcal S^\frown b$ into any extension of $\mathcal T$, even when $\mathcal S$ and $\mathcal T$ are by the same nice iteration strategy.

Here is an example due to Steel. Assume $M_1^\#$ exists and let Λ be the iteration strategy for M_1. Toward a contradiction, suppose for every $\mathcal S,\mathcal T$ of limit lengths by Λ such that there is $\Phi:\mathcal S\to\mathcal T$ with $\mathrm{ran}(v^\Phi)$ cofinal in $\mathrm{lh}(\mathcal T)$, $v^\Phi[\Lambda(\mathcal S)]\subseteq\Lambda(\mathcal T)$.

Now let $\mathcal T\in M_1$ be a normal tree by Λ of height δ^{+M_1} which has no branch in M_1, where δ is the Woodin cardinal of M_1 (that such a tree exists is due to Woodin, see Lemma 1.1 of [17]). Let g be $Col(\omega,\delta)$ generic over M_1 and h generic for the Namba forcing over $M_1[g]$. The restriction of Λ to countable trees which are in $M_1[g][h]$ is in $M_1[g][h]$ since Namba forcing adds no reals and M_1 contains the restriction of Λ to trees of length δ which are in M_1. Now, in $M_1[g][h]$, $\mathrm{lh}(\mathcal T)$ has countable cofinality. We can take a Skolem hull to get $\mathcal S$ countable in $M_1[g][h]$ and $\Phi:\mathcal S\to\mathcal T$ with $\mathrm{ran}(v^\Phi)$ cofinal in $\mathrm{lh}(\mathcal T)$. Since $\mathcal S$ is countable and by Λ, $\Lambda(\mathcal S)\in M_1[g][h]$. So, by assumption, $v^\Phi[\Lambda(\mathcal S)]\subseteq\Lambda(\mathcal T)$; so $\Lambda(\mathcal T)$ is just the downwards closure of $v^\Phi[\Lambda(\mathcal S)]$ (in $M_1[g][h]$). This identification of $\Lambda(\mathcal T)$ was independent of our choice of $g,h,\mathcal S$, so we get $\Lambda(\mathcal T)\in M_1$, a contradiction.

Despite this, we will be able to view the tree embeddings that appear naturally in embedding normalization as extended tree embeddings and deal almost exclusively with extended tree embeddings in the rest of the paper.

Definition 3.1.5. For tree embeddings or extended tree embeddings $\Phi : \mathcal{S} \to \mathcal{T}, \Psi : \mathcal{U} \to \mathcal{V}$, we put

$$\Phi \restriction \xi + 1 \approx \Psi \restriction \xi + 1$$

iff $\mathcal{S} \restriction \xi + 1 = \mathcal{U} \restriction \xi + 1$, $v^\Phi \restriction \xi + 1 = v^\Psi \restriction \xi + 1$, and $\mathcal{T} \restriction v^\Phi(\xi) + 1 = \mathcal{V} \restriction v^\Psi(\xi) + 1$. It follows that $s^\Phi_\eta = s^\Psi_\eta$ for $\eta \leq \xi$, $u^\Phi \restriction \xi = u^\Psi \restriction \xi$ and $t^\Phi_\eta = t^\Psi_\eta$ for $\eta < \xi$. Notice that we do *not* demand that $u^\Phi(\xi) = u^\Psi(\xi)$, even when Φ, Ψ are extended tree embeddings.

We introduce a few more useful bits of notation about tree embeddings. First, we will sometimes write u^Φ_α for $u^\Phi(\alpha)$ and v^Φ_α for $v^\Phi(\alpha)$ and for $\alpha \in \mathrm{dom}(u^\Phi)$ and $\xi \in [v^\Phi(\alpha), u^\Phi(\alpha)]_\mathcal{T}$,

$$s^\Phi_{\alpha,\xi} = \hat{\imath}^\mathcal{T}_{v^\Phi(\alpha),\xi} \circ s^\Phi_\alpha.$$

So $s^\Phi_{\alpha,\xi}$ is a partial elementary embedding from $M^\mathcal{S}_\alpha$ into $M^\mathcal{T}_\xi$. Also note that $s^\Phi_\alpha = s^\Phi_{\alpha,v^\Psi(\bar{\alpha})}$ and $t^\Phi_\alpha = s^\Phi_{\alpha,u^\Psi(\alpha)}$.

Definition 3.1.6. For a premouse P a *directed system of normal iteration trees (on P)* is a system $\mathcal{D} = \langle \{\mathcal{T}_a\}_{a \in A}, \{\Psi_{a,b}\}_{a \preceq b} \rangle$, where $\preceq$ is a directed partial order on A and

(a) for any $a \in A$, $\mathcal{T}_a$ is a normal iteration tree of successor length on P

(b) for any $a, b \in A$ with $a \preceq b$, $\Psi_{a,b} : \mathcal{T}_a \to \mathcal{T}_b$ is an extended tree embedding,

(c) for any $a, b, c \in A$ such that $a \preceq b \preceq c$, $\Psi_{a,c} = \Psi_{b,c} \circ \Psi_{a,b}$.[3]

We'll sometimes use variant notation for directed systems, e.g. $\langle \mathcal{T}_a, \Psi_{a,b} \mid a, b \in A \wedge a \preceq b \rangle$.

Let $\mathcal{D} = \langle \{\mathcal{T}_a\}_{a \in A}, \{\Psi_{a,b}\}_{a \preceq b} \rangle$ be a directed system of normal iteration trees. We'll define an object $\lim \mathcal{D}$ which will be the direct limit of $\mathcal{D}$ in the category of trees (of successor lengths) and extended tree embeddings, if this direct limit exists.
We'll define

$$\lim \mathcal{D} = \langle D, \leq, \leq^*, \{M_x\}_{x \in D}, \{E_x\}_{x \in D}, \{\Gamma_a\}_{a \in A} \rangle$$

as follows.

We'll need names for the components of $\Psi_{a,b}$, so let

$$\Psi_{a,b} = \langle u_{a,b}, v_{a,b}, \{s^{a,b}_\gamma\}_{\gamma < \mathrm{lh}(\mathcal{T}_a)}, \{t^{a,b}_\gamma\}_{\gamma < \mathrm{lh}(\mathcal{T}_a)} \rangle.$$

A *u-thread* is a partial function $x : A \rightharpoonup Ord$ such that

(i) for all $a, b \in \mathrm{dom}(x)$, there is $c \in \mathrm{dom}(x)$ with $a, b \preceq c$ and $u_{a,c} \circ x(a) = u_{b,c} \circ x(b)$,

(ii) $\mathrm{dom}(x)$ is a maximal subset of A with property (i).

[3]It follows from the definition that $\Psi_{a,a}$ must be the identity extended tree embedding on $\mathcal{T}_a$, since for any increasing ordinal-valued function u, $u \circ u = u$ implies that u is the identity on it's domain.

Notice that the domains of u-threads have non-empty intersection since, by (ii), if $a \in \mathrm{dom}(x)$ then $b \in \mathrm{dom}(x)$ for any $b \succeq a$, so this just follows from the directedness of $\preceq$. By (ii) again, for any $a \in A$ and $\gamma < \mathrm{lh}(\mathcal{T}_a)$, there is exactly one u-thread x such that $x(a) = \gamma$, we write $x = [a, \gamma]_{\mathcal{D}}$ in this case. So u-threads are just a useful way of describing the equivalence classes of pairs (a, γ) under applying u-maps $u_{a,b}$.

D is the set of all u-threads and $\leq$, $\leq^*$ will be certain partial orders with field D. Going forward, we'll write $\forall^* a\, \varphi(a)$ to abbreviate $\exists b \forall a \succeq b\, \varphi(a)$.

For u-threads x, y we put

$$x \leq y \Leftrightarrow \forall^* a \left(x(a) \leq y(a)\right),$$
$$x \leq^* y \Leftrightarrow \forall^* a \left(x(a) \leq_{\mathcal{T}_a} y(a)\right).$$

These definitions makes sense since the domains of u-threads have non-empty intersection. Since $\leq_{\mathcal{T}_a}$ is a refinement of the order $\leq$ on ordinals, we get that $\leq^*$ is a refinement of $\leq$.

It's easy to see that $\leq$ is a linear order on D, but it could fail to be a well-order. If it is a well-order, we identify it with its order-type δ. In any case, we will think of $\langle D, \leq \rangle$ as the length of the direct limit. In the case that the direct limit produces a normal iteration tree, δ will really be its length.

We now define $\Gamma_a = \langle u_a, v_a, \{s^a_\gamma\}_{\gamma < \mathrm{lh}(\mathcal{T})_a} \{t^a_\gamma\}_{\gamma < \mathrm{lh}(\mathcal{T}_a)} \rangle$ along with M_x and E_x for x such that $a \in \mathrm{dom}(x)$. We will actually only define the u-map and t-maps of the Γ_a; these determine the whole tree embedding, in the case that the target is actually a normal iteration tree. Fix a and $\gamma < \mathrm{lh}(\mathcal{T}_a)$. Let $x = [a, \gamma]_{\mathcal{D}}$ and set $u_a(\gamma) = [a, \gamma]_{\mathcal{D}}$. We'll actually leave M_x, E_x, and t^a_γ undefined unless the t-maps along x are total.

So, suppose we're in this case, i.e. $\forall^* b \forall c \geq b\, (t^{b,c}_{x(b)}$ is total). We define

$$M_x = \lim \langle M^{\mathcal{T}_b}_{x(b)}, t^{b,c}_{x(b)} \mid b \preceq c \text{ and for all } d \succeq b,\ t^{b,d}_{x(b)} \text{ is total} \rangle.$$

For any b such that for all $d \succeq b$, $t^{b,d}_{x(b)}$ is total, we let $t^b_{x(b)}$ be the direct limit map and we put $t^a_\gamma = t^b_{x(b)} \circ t^{a,b}_\gamma$ for any such b (this is independent of the choice of b).

We also let
$$E_x = t^b_{x(b)}(E^{\mathcal{T}_b}_{x(b)}),$$
for any such b (again, this is independent of the choice of b). We say that $\lim \mathcal{D}$ is *well-founded* iff

1. for all $x \in D$, the model M_x is defined and well-founded,
2. $\leq$ is well-founded,
3. $\mathcal{U} = \langle M_x, E_x, \leq^* \rangle$ is a normal iteration tree (i.e. with models M_x, exit extenders E_x, and tree-order $\leq^*$).

If $\lim \mathcal{D}$ is well-founded, one can show that letting $v_a(x) = \sup\{u_a(y) + 1 \mid y < x\}$, we can define s^a_γ to be the required copy maps so that $\Gamma_a = \langle u_a, v_a, \{s^a_\gamma\}_{\gamma < \mathrm{lh}(\mathcal{T})_a} \{t^a_\gamma\}_{\gamma < \mathrm{lh}(\mathcal{T}_a)} \rangle$ is an extended tree embeddins from $\mathcal{T}_a$ into $\mathcal{U}$ and $\Gamma_b \circ \Psi_{a,b} = \Gamma_a$ for every $a \preceq b$. Part of this is the analysis of successors in the $\leq^*$-order, below.

Perhaps surprisingly, we can drop conditions (2) and (3) in the definition of the well-foundedness of the direct limit.

Proposition 3.1.7. *Let $\mathcal{D}$ be a directed system of normal iteration trees. Then $\lim \mathcal{D}$ is well-founded iff for every u-thread x, the models M_x are defined and well-founded.*

Before we give a proof, we need the following observations about iterated applications of the Shift Lemma. The first observation is especially easy, so we omit the proof.

Lemma 3.1.8. *Let $\pi_0 : M_0 \to M_1$, $\pi_1 : M_1 \to M_2$ and $\sigma_0 : N_0 \to N_1$, $\sigma_1 : N_1 \to N_2$ are elementary and let E be on the M_0-sequence.*

Suppose that the Shift Lemma applies to (π_0, σ_0, E) and to (π_1, σ_1, $\pi_0(E)$) and let τ_0, τ_1 be the respective associated copy maps.

Then the Shift Lemma applies to ($\pi_1 \circ \pi_0$, $\sigma_1 \circ \sigma_0$, E) and $\tau_1 \circ \tau_0$ is the associated copy map.

Next we see how the Shift Lemma interacts with direct limits. This is implicit in [24].

Definition 3.1.9. A *directed system of premice* is a system $\mathcal{D} = \langle \{M_a\}_{a\in A}, \{\pi_{a,b}\}_{a\preceq b}\rangle$, where $\preceq$ is a directed partial order on A and

(a) for any $a \in A$, M_a is a premouse,

(b) for any $a, b \in A$ with $a \preceq b$, $\pi_{a,b} : M_a \to M_b$ is elementary, and

(c) for any $a, b, c \in A$ such that $a \preceq b \preceq c$, $\pi_{a,c} = \pi_{b,c} \circ \pi_{a,b}$.

Lemma 3.1.10. *Let $\mathcal{M} = \langle \{M_a\}_{a\in A}, \{\pi_{a,b}\}_{a\preceq b}\rangle$ and $\mathcal{N} = \langle \{N_a\}_{a\in A}, \{\sigma_{a,b}\}_{a\preceq b}\rangle$ be directed systems of premice and $\{E_a\}_{a\in A}$ extenders such that*

(a) E_a is on the M_a-sequence,

(b) for all $a, b \in A$ such that $a \preceq b$, $E_b = \pi_{a,b}(E_a)$, and

(c) for all $a, b \in A$ such that $a \preceq b$, the Shift Lemma applies to ($\pi_{a,b}$, $\sigma_{a,b}$, E_a).

For $a, b \in A$ such that $a \preceq b$, let P_a be the least initial segment of N_a beyond dom(E_a) such that $\rho(P_a) \leq crit(E_a)$ or $P_a = N_a$ and let $\tau_{a,b}$ be the copy map associated to ($\pi_{a,b}$, $\sigma_{a,b}$, E_a). Let $M_\infty = \lim \mathcal{M}$, $N_\infty = \lim \mathcal{N}$, $\pi_a : M_a \to M_\infty$ and $\sigma_a : N_a \to N_\infty$ be the direct limit maps, and E_∞ the common value of $\pi_a(E_a)$. Suppose that M_∞ and N_∞ are well-founded. Let P_∞ be the least initial segment of N_∞ beyond dom(E_∞) such that $\rho(P_\infty) \leq crit(E_\infty)$.

Let $\mathcal{Q} = \langle \{Ult(P_a, E_a)\}_{a\in A}, \{\tau_{a,b}\}_{a\preceq b}\rangle$. Let $Q_\infty = \lim \mathcal{Q}$, $\tau_a : Ult(P_a, E_a) \to Q_\infty$ the direct limit maps, and $j : N_\infty \to Q_\infty$ the unique map such that for every $a \in A$, $\tau_a \circ i^{P_a}_{E_a} = j \circ \sigma_a$.

Then $Q_\infty = Ult(P_\infty, E_\infty)$, $j = i^{P_\infty}_{E_\infty}$, and for all $a \in A$, τ_a is the copy map associated to (π_a, σ_a, E_a).

Proof. By replacing N_a with P_a if necessary, we may assume that $P_a = N_a$. The following diagram illustrates the situation discribed in the lemma, in the this case, along a chain of $\preceq$.

$$
\begin{tikzcd}
M_a \arrow[r, "\pi_{a,b}"] & M_b \arrow[r] \arrow[rr, bend left, "\pi_b"] & \cdots & M_\infty \\
\rotatebox{90}{$\in$} & & & \rotatebox{90}{$\in$} \\
E_a \arrow[r, phantom, "\mapsto"] & E_b \arrow[r, phantom, "\mapsto"] \arrow[rr, mapsto, bend left] & \cdots & E_\infty \\
N_a \arrow[r, "\sigma_{a,b}"] \arrow[d, "E_a"] & N_b \arrow[r] \arrow[rr, bend left, "\sigma_b"] \arrow[d, "E_b"] & \cdots & N_\infty \arrow[d, "j"] \\
Ult(N_a, E_a) \arrow[r, "\tau_{a,b}"'] & Ult(N_b, E_b) \arrow[r] \arrow[rr, bend left, "\tau_b"] & \cdots & Q_\infty
\end{tikzcd}
$$

We want to show that j is just the ultrapower embedding by the image of E under the direct limit map. Let $\kappa = \text{crit}(E_\infty)$ and $\lambda = \lambda(E_\infty)$. Let E_j be the (κ, λ)-extender derived from j. We show that $j = i^{N_\infty}_{E_j}$ and that E_∞ is the trivial completion of E_j. We just do the case that the we're taking zero ultrapowers everywhere; the general case is basically the same.

For $a \in A$, let $\kappa_a = \text{crit}(E_a)$ and $\lambda_a = \lambda_{E_a}$. To see $j = i^N_{E_j}$, it's enough to see that

$$Q_\infty = \{j(f)(s) \mid s \in [\lambda]^{<\omega},\ f : [\kappa_\gamma]^{|s|} \to N,\ f \in N\}.$$

This is easy. For $x \in Q_\infty$, $x = \tau_a(\bar{x})$ for some $a \in A$ and $\bar{x}$. Since $Ult(N_a, E_a)$ is the zero ultrapower of N_a by E_a, $\bar{x} = i^{N_a}_{E_a}(g)(t)$ for some $t \in [\lambda_a]^{<\omega}$ and $g : [\kappa_a]^{|t|} \to N_a$, $g \in N_a$. But then taking $f = \sigma_a(g)$ and $s = \pi_a(t) = \tau_a(t)$, we have

$$\begin{aligned}
j(f)(s) &= j \circ \sigma_a(g)(\pi_a(t)) \\
&= \tau_a \circ i^{N_a}_{E_a}(g)(\tau_a(t)) \\
&= \tau_a(i^{N_a}_{E_a}(g)(t)) \\
&= \tau_a(\bar{x}) \\
&= x.
\end{aligned}$$

Checking that E_∞ is the trivial completion of E_j is similar. Let $s \in [\lambda]^{<\omega}$ and $X \subseteq [\kappa]^{|s|}$, $X \in M_\infty$. Letting $a \in A$, $\bar{s}$, and $\bar{X}$ be such that $\tau_a(\bar{s}) = \pi_a(\bar{s}) = s$, $\sigma_a(\bar{X}) = \pi_a(\bar{X}) = X$, we have

$$\begin{aligned}
X \in (E_a)_s &\Leftrightarrow \bar{X} \in (E_a)_{\bar{s}} \\
&\Leftrightarrow \bar{s} \in i^{N_a}_{E_a}(\bar{X}) \\
&\Leftrightarrow \tau_a(\bar{s}) \in \tau_a \circ i^{N_a}_{E_a}(\bar{X}) \\
&\Leftrightarrow \tau_a(\bar{s}) \in j \circ \sigma_a(\bar{X}) \\
&\Leftrightarrow s \in j(X) \\
&\Leftrightarrow X \in (E_j)_s.
\end{aligned}$$

That the τ_a are just the appropriate copy follows from the various direct limit maps having enough agreement. We need to check

$$\sigma_a \restriction \text{dom}(E_a) = \pi_a \restriction \text{dom}(E_a)$$

and

$$\tau_a \restriction \mathrm{lh}(E_a) = \pi_a \restriction \mathrm{lh}(E_a).$$

These just follow from the fact that the relevant restrictions of these maps are the direct limit maps for the systems of restricted models and maps, for example $\tau_a \restriction \mathrm{lh}(E_a) = \pi_a \restriction \mathrm{lh}(E_a)$ is the direct limit map for the system

$$\langle \{Ult(N_a, E_a)|\mathrm{lh}(E_a)\}_{a\in A}, \{\tau_{a,b} \restriction \mathrm{lh}(E_a)\}_{a\preceq b}\rangle = \langle \{M_a|(\mathrm{lh}(E_a), -1)\}_{a\in A}, \{\pi_{a,b} \restriction \mathrm{lh}(E_a)\}_{a\preceq b}\rangle.$$

□ Lemma 3.1.10

Proof of Proposition 3.1.7. Again, this proposition amounts to saying (1) implies (2) and (3) in the above definition of the well-foundedness of the direct limit. We first show (1) implies (2).

Claim 1. *Let $x \in D$. Suppose M_x is defined and that $\leq$ is ill-founded below x. Then M_x is ill-founded.*

Proof. We define an order preserving embedding f from $\leq\restriction x$ into the ordinals of M_x.

Let $x = [a, \alpha]_{\mathcal{D}}$ and fix $y = [b, \beta]_{\mathcal{D}} < x$. Without loss of generality, we may assume $b \preceq a$ and $\beta + 1 < \mathrm{lh}(\mathcal{T}_a)$ (this is just because we can move to $c \geq a, b$ where we have $[c, u_{a,c}(\alpha)]_{\mathcal{D}} = [a, \alpha]_{\mathcal{D}}$ and $[c, u_{b,c}(\beta)]_{\mathcal{D}} = [b, \beta]_{\mathcal{D}}$, so that $u_{b,c}(\beta) + 1 < u_{a,c}(\alpha) + 1 \leq \mathrm{lh}(\mathcal{T}_c)$, since $[b, \beta]_{\mathcal{D}} < [a, \alpha]_{\mathcal{D}}$). We let

$$f(y) = t^b_{u_{a,b}(\alpha)}(\mathrm{lh}(E^{\mathcal{T}_a}_\beta)).$$

Clearly f maps y to an ordinal of M_x. It's easy to check that it is (strictly) order preserving, so M_x is ill-founded. □ Claim 1

So now suppose that $(D, \leq)$ is a well-order and that the models M_x exist and are well-founded (i.e. (1) and (2)). We show (3) by induction on $(D, \leq)$.

More specifically, for $x \in D$, we let

$$\mathcal{D}_x = \langle \mathcal{T}_a \restriction (x(a) + 1), \Psi_{a,b} \restriction (\mathcal{T}_a \restriction (x(a) + 1)) \mid a \preceq b \wedge a, b \in \mathrm{dom}(x)\rangle.$$

It's easy to see that $\lim \mathcal{D}_x = (\lim \mathcal{D}) \restriction x + 1$, where for the Γ systems, we just mean that for any $a \in \mathrm{dom}(x)$, $(u_a)^{\mathcal{D}_x} = u_a \restriction x(a) + 1$ and the corresponding t-maps are the equal.

We show that the u-maps of the Γ systems preserve tree-predecessors of successor u-threads on a tail, in the following sense.

Claim 2. *For any $x \in D$ which has a $\leq$-successor z in D, there is $y \in D$ such that*

$$y = \leq^* \text{-}pred(z).$$

Moreover,

$$y = \leq^* \text{-}pred(z) \Leftrightarrow \forall^* a \left(y(a) = \mathcal{T}_a\text{-}pred(x(a) + 1)\right).$$

Proof. Fix x and z it's $\leq$-successor. First note that for any $a \in \mathrm{dom}(z) \cap \mathrm{dom}(x)$, $z(a) = x(a)+1$, since $z(a) \leq x(a)+1$ (as $[a, x(a)+1]_{\mathcal{D}}$ is a u-thread $> x$) but $z(a) \neq x(a)$ (since $z \neq x$). We'll show that there is a u-thread y such that $\forall^* a \left(y(a) = \mathcal{T}_a\text{-pred}(x(a)+1)\right)$. Suppose that there is no immediate $\leq^*$-predecessor of z. We'll show that $\leq$ is ill-founded, a contradiction.

We define sequences $\langle a_n \mid n \in \omega\rangle$, $\langle \beta_n \mid n \in \omega\rangle$ such that $a_n \prec a_{n+1}$ and $\beta_n = \mathcal{T}_{a_n}\text{-pred} x(a_n)+1$ but $\beta_{n+1} < u_{a_n,a_{n+1}}(\beta_n)$. Then, taking $y_n = [a_n, \beta_n]_{\mathcal{D}}$ gives a witness to the ill-foundedness of $\leq$.

We start with any $a_0 \in \mathrm{dom}(z)$ and take $\beta_0 = \mathcal{T}_{a_0}\text{-pred}(x(a_0)+1)$, as we must.

Given a_n and $\beta_n = \mathcal{T}_{a_n}\text{-pred}(x(a_n)+1)$, let

$$a_{n+1} > \xi_n \text{ least such that } u_{a_n,a_{n+1}}(\beta_n) \neq \mathcal{T}_{a_{n+1}}\text{-pred}(x(a_{n+1})+1),$$

We have that such an a_{n+1} exists, since otherwise $y_n = [a_n, b_n]_{\mathcal{D}}$ is the immediate predecessor of z in $\leq^*$. Now let $\beta_{n+1} = \mathcal{T}_{a_{n+1}}\text{-pred}(x(a_{n+1})+1)$. Since $\Psi_{a_n,a_{n+1}}$ is a tree embedding, we must have that $\beta_{n+1} \in [v_{a_n,a_{n+1}}(\beta_n), u_{a_n,a_{n+1}}(\beta_n)]_{\mathcal{T}_{a_{n+1}}}$. So since $\beta_{n+1} \neq u_{a_n,a_{n+1}}(\beta_n)$, we have $\beta_{n+1} < u_{a_n,a_{n+1}}(\beta_n)$, as desired. □ Claim 2

Let z be a u-thread of successor rank, say z is the $\leq$-successor of x (i.e. the rank of z is the rank of x plus one). The observation made at the start of the previous proof shows $z(a) = x(a)+1$ for all most all a. Fix such an a, so for all $b \succeq a$ $z(b) = x(b)+1$. It follows that for all $b \succeq a$,

$$\begin{aligned} z(b) &= x(b)+1 \\ &= u_{a,b}(x(a))+1 \\ &= v_{a,b}(x(a)+1) \\ &= v_{a,b}(z(a)). \end{aligned}$$

This shows that all successor u-threads are actually v-threads (defined in the obvious way) when $\leq$ is well-founded. Even when $\leq$ is well-founded, there may be u-threads which are not v-threads, so it was important to use u-threads in defining the direct limit.

Going forward, if x is a u-thread which is not the $\leq$-largest u-thread, we'll let $x+1$ be the $\leq$-successor of x.

Claim 3. *For all u-threads $x \in D$,*

(i) $\lim \mathcal{D}_x$ *is well-founded.*

(ii) for all $a \in dom(x)$, $\Gamma_a \restriction (\mathcal{T}_a \restriction x(a)+1)$ is an extended tree embedding from $\mathcal{T}_a \restriction x(a)+1$ into $\lim \mathcal{D} \restriction x+1$ and

(iii) for all $a \in dom(x)$, all $b \succeq a$, and

$$(\Gamma_b \circ \Phi_{a,b}) \restriction x(a)+1 \approx \Gamma_a \restriction x(a)+1.$$

Proof. We proceed by induction. We already know all the M_y are defined and well-founded and $\leq$ is well-founded, so to show (i) we just need to see that $\langle M_y, E_y, \leq^* \restriction x\rangle$ is a normal iteration tree.

In the base case, where x is the minimum u-thread, this is unique normal iteration tree of length one on the base model and (ii) and (iii) hold trivially. For the successor case, suppose we have (i)-(iii) for all $z \leq x$ and suppose that x is not the last u-thread. So x has a $\leq$-successor, $x+1$ and, appealing to Claim 2, we can take $y = \leq^*$ -pred$(x+1)$. To show (i) we need to see that we're applying E_x following the rules of normality, i.e.

Subclaim 3.1. *y is $\leq$-least such that $crit(E_x) < \lambda(E_y)$ and for $P \trianglelefteq M_y$ least such that $M_y|lh(E_y) \trianglelefteq P$ and $\rho(P) \leq crit(E_x)$,*

$$M_{x+1} = Ult(P, E_y).$$

Proof. By Claim 2, we may take a such that for all $b \succeq a$, $y(b) = \mathcal{T}_b$-pred$(x(b)+1)$. For the appropriate choice of maps and models, we are now exactly in the situation of Lemma 3.1.10. The rest of the subclaim follows from the normality of each of the trees $\mathcal{T}_b$. We leave the details to the reader. $\square$

Since (ii) and (iii) hold at x and y, it is easy to see that they hold at $x+1$

Suppose now x has limit rank and (i)-(iii) hold for all $z < x$. We first need to see

Subclaim 3.2. *For $b = [0, x)_{\leq^*} =_{def} \{y \mid y <^* x\}$, b is $<$-cofinal in x, there are finitely many drops along b, and M_x is the direct limit along b.*

Proof. To see that b is cofinal, let $y < x$. Since x has limit rank, $y+1 < x$. Let a sufficiently large such that $y+1 = [a, y(a)+1]_{\mathcal{D}}$. Let $\gamma+1$ least such that $y(a)+1 \leq \gamma+1 \leq_{\mathcal{T}_a} x(a)$. We have that for all $b \succeq a$,

$$y(b) < u_{a,b}(\gamma) + 1 = v_{a,b}(\gamma+1) \leq_{\mathcal{T}_b} v_{a,b}(x(a)) \leq_{\mathcal{T}_b} x(b),$$

using here that $\Phi_{a,b}$ is a tree embeddings (and the v-maps of tree embeddings are tree-order preserving). So letting $z = [a, \gamma]_{\mathcal{D}} + 1$, we have that $y < z \leq^* x$. Since x is not a successor, we actually have $y < z <^* x$, as desired.

Since the model M_x is defined, there is an a such that for all $b \succeq a$, $t^{a,b}_{x(a)}$ is total. Suppose first that there is some successor $\eta <_{\mathcal{T}_a} x(a)$ such that $(\eta, x(a)]_{\mathcal{T}_a}$ doesn't drop. Then for all $b \succeq a$, we have that $[v_{a,b}(\eta), x(b))_{\mathcal{T}_b}$ doesn't drop. Now, there is some u-thread z such that $z = [b, v_{a,b}(\eta)]_{\mathcal{D}}$ for all sufficiently large b. But any drops from z to x in the direct limit corresponds to a drop in $[z(b), x(b))_{\mathcal{T}_b}$, for all sufficiently large b, so there are no such drops.

In the remaining case, $x(a)$ is a successor ordinal and a drop in $\mathcal{T}_a$. Let $\beta = \mathcal{T}_a$-pred$(x(a))$. Letting z the u-thread such that $z(b) = v_{a,b}(x(a))$ for all sufficiently large b, we have $z <^* x$ and there can be no drops between z and x in the direct limit tree, just as before (since for all sufficiently large b, there are no drops in $(z(b), x(b)]_{\mathcal{T}_b}$, as $t^{a,b}_{x(a)}$ is total). By induction, this means there are only finitely many drops.

Using our induction hypotheses (ii) and (iii) for $z < x$, it is straightforward to check that M_x is the direct limit along b, so we leave it to the reader. $\square$

The subclaim immediately gives us (i) at x and it is straightforward to verify (ii) and (iii). $\square$

□

Notice that the proof of Proposition 3.1.7 implies that when the direct limit $\lim \mathcal{D}$ is well-founded, then $\Gamma_a = \langle u_a, v_a, \{s^a_\xi\}, \{t^a_\xi\}\rangle$ is an extended tree embedding from $\mathcal{T}_\xi$ into $\mathcal{U}$, the direct limit tree, and $\Gamma_b \circ \Psi_{a,b} = \Gamma_a$ when $a \preceq b$, as promised.

Going forward, we when $\lim \mathcal{D}$ is well-founded, we will often identify the direct limit normal trees with $\lim \mathcal{D}$.

We end this subsection by verifying that direct limit we've defined really is the direct limit in the category of normal iteration trees (of successor lengths) and extended tree embeddings. That is, we have the following.

Proposition 3.1.11. *Let $\mathcal{D} = \langle \{\mathcal{T}_a\}_{a\in A}, \{\Psi_{a,b}\}_{a\preceq b}\rangle$ be a directed system of normal iteration trees.*

Suppose there is a normal tree $\mathcal{S}$ and for all $a \in A$ extended tree embeddings $\Pi_a : \mathcal{T}_a \to \mathcal{S}$ such that whenever $a \preceq b$, $\Pi_b = \Psi_{a,b} \circ \Pi_a$.

Then the direct limit $\lim \mathcal{D}$ is well-founded and there is a unique tree embedding $\Pi : \lim \mathcal{D} \to \mathcal{S}$ such that $\Pi_a = \Pi \circ \Gamma_a$ for all $a \in A$.

Proof. The proof is easy using Proposition 3.1.7. Let $\Pi_a = \langle u^*_a, v^*_a, \{(s^a_\alpha)^*\}_{a\in A}, \{(t^a_\alpha)^*\}_{a\in A}\rangle$. We define $\Pi = \langle u^*, v^*, \{s^*_x\}_{x\in D}, \{t^*_x\}_{x\in D}\rangle$ from $\lim \mathcal{D}$ into $\mathcal{S}$, verifying that it is as desired.

We let $u^*([a,\alpha]_{\mathcal{D}}) = u^*_a(\alpha)$. Notice that $u^* \circ u_a = u^*_a$ (here we're just recalling our definition of u_a, i.e. $u_a(\alpha) = [a,\alpha]_{\mathcal{D}}$). Since $\mathcal{S}$ is a normal iteration tree, for any u-thread x, there is an a such that for all $b \succeq a$, $(t^b_{x(b)})^*$ is total. If $u^*_a(\alpha)$ is a successor in $\mathcal{S}$, we have that the predecessor stabilizes for all sufficiently large b, as in Claim 2, but then if for all sufficiently large b, $(t^b_{x(b)})^*$ are *not* total, we must drop progressively further as we move to $c \succeq b$, so that the model which is the predecessor of $u^*_a(\alpha)$ in $\mathcal{S}$ is ill-founded, a contradiction. If $u^*_a(\alpha)$ is a limit ordinal, we get that we must drop infinitely often in $[0, u^*_a(\alpha))_{\mathcal{S}}$, a contradiction.

Since for $a \preceq b$, $(t^a_{x(a)})^* = (t^b_{x(a)})^* \circ t^{a,b}_{x(a)}$, this tells us that for all sufficiently large a, the $t^{a,b}_{x(a)}$ are total so that M_x is defined, and there is a unique map $t^*_x : M_x \to M^{\mathcal{S}}_{u^*(x)}$ such that $(t^b_{x(b)})^* = t^*_x \circ t^b_{x(b)}$ for all $b \succeq a$ (recalling that $t^b_{x(b)}$ was the direct limit map from $M^b_{x(b)}$ to M_x). In particular, M_x is well-founded. By Proposition 3.1.7, we get the direct limit is well-founded and we've already verified $\Pi_a = \Pi \circ \Gamma_a$, as desired. □ Proposition 3.1.11

We can define the direct limit of a commuting system of normal trees under ordinary tree embeddings in the obvious way and verify versions of Propositions 3.1.7 and 3.1.11. It's easy to see that the direct limit of a system of normal trees under extended tree embeddings is either the same as the corresponding direct limit of under ordinary tree embeddings, or else is some tree $\mathcal{U}$ with length $\gamma+1$ for a limit ordinal γ, and the corresponding direct limit under ordinary tree embeddings is just $\mathcal{U}\restriction\gamma$.

3.1.2 Embedding normalization

[24] isolates the process of embedding normalization, a procedure of "normalizing" a (finite) stack of normal iteration trees, i.e. given a stack of normal trees $\vec{\mathcal{S}}$, we produce a single

normal tree $W(\vec{\mathcal{S}})$ which embeds the last model of $\vec{\mathcal{S}}$. For example, if $\vec{\mathcal{S}} = \langle \mathcal{T}, \mathcal{U}\rangle$, embedding normalization produces a normal tree $W(\mathcal{T}, \mathcal{U})$ whose last model embeds the last model of $\mathcal{U}$. Roughly, the procedure works by doing the one-step embedding normalization at successors and taking direct limits of a resulting directed system of normal iteration trees at limits.

In this subsection, we'll review the one-step normalization in some detail. We'll then use the one-step embedding normalization to analyze *inflationary* tree embeddings, a particularly nice class of tree embeddings identified by Schlutzenberg. We then discuss the embedding normalization in general and review the Schlutzenberg-Steel theorem on extending iteration strategies for single normal trees to strategies for finite stacks of normal trees.

Let $\mathcal{S}, \mathcal{T}$ be normal iteration trees of successor length and F be on the sequence of last model of $\mathcal{T}$. Let $\alpha = \alpha(\mathcal{T}, F) < \mathrm{lh}(\mathcal{T})$ be least such that F is on the sequence of $M^{\mathcal{T}}_\alpha$ and let $\beta = \beta(\mathcal{T}, F)$ be least such that $\beta = \alpha$ or $\lambda(E^{\mathcal{T}}_\beta) > \mathrm{crit}(F)$.[4] Suppose that $\mathcal{S} \restriction \beta + 1 = \mathcal{T} \restriction \beta + 1$ and $\mathrm{dom}(F) \trianglelefteq M^{\mathcal{S}}_\beta | \mathrm{lh}(E^{\mathcal{S}}_\beta)$, if $\beta + 1 < \mathrm{lh}(\mathcal{S})$. In this case, we define $\mathcal{W} = W(\mathcal{S}, \mathcal{T}, F)$, $\Phi = \Phi^{\mathcal{S},\mathcal{T},F}$ a partial extended tree embedding from $\mathcal{S}$ into $\mathcal{W}$, and $\sigma = \sigma^{\mathcal{S},\mathcal{T},F}$ a weakly elementary map from $Ult(P, F)$ into the last model of $\mathcal{W}$, where P is the largest initial segment of the last model of $\mathcal{S}$ to which we can apply F. In general, we may reach ill-founded models in forming $\mathcal{W}$ and stop when we do. We say that $W(\mathcal{S}, \mathcal{T}, F)$ is *well-founded* if we never reach ill-founded models. If $\mathcal{S}$ and $\mathcal{T}$ are by a strategy Σ which has SHC^{-}, a variant of Steel's strong hull condensation which we define at the end of this section, then $\mathcal{W}$ will be well-founded.

First, we let $\mathcal{W} \restriction \alpha + 1 = \mathcal{T} \restriction \alpha + 1$ and $E^{\mathcal{W}}_\alpha = F$. For the rest of $\mathcal{W}$, we consider cases.

The dropping case.

Suppose F is applied to a proper initial segment $P \triangleleft M^{\mathcal{S}}_\beta | \mathrm{lh}(E^{\mathcal{S}}_\beta)$, if $\beta + 1 < \mathrm{lh}(\mathcal{S})$, or $P \triangleleft M^{\mathcal{S}}_\beta$ if $\beta + 1 = \mathrm{lh}(\mathcal{S})$.

In this case we've described all of $\mathcal{W}$ already:

$$\mathcal{W} = \mathcal{T} \restriction \alpha + 1^\frown \langle F \rangle$$

and Φ is just the identity on $\mathcal{S} \restriction \beta + 1$ except we set $u(\beta) = \alpha + 1$. Letting P the largest initial segment of the last model of $\mathcal{S}$ to which we can apply F, we have $P \trianglelefteq M^{\mathcal{S}}_\beta$ and $t_\beta = i^P_F$. So $Ult(P, F)$ is the last model of $\mathcal{W}$ and we take $\sigma = id$.

Note that in this case, Φ is total exactly when $\beta + 1 = \mathrm{lh}(\mathcal{S})$ and $\mathrm{Ult}(P, F)$ is well-founded.

The non-dropping case.

Suppose F is applied to an initial segment $P \trianglelefteq M^{\mathcal{S}}_\beta$ with $M^{\mathcal{S}}_\beta | \mathrm{lh}(E^{\mathcal{S}}_\beta) \trianglelefteq P$, if $\beta + 1 < \mathrm{lh}(\mathcal{S})$, or no proper initial segment of $M^{\mathcal{S}}_\beta$ projects across $\mathrm{dom}(F)$, if $\beta + 1 = \mathrm{lh}(\mathcal{S})$. We define Φ and $\mathcal{W}$ as follows.

We can say the u map of Φ at the outset:

$$u(\xi) = \begin{cases} \xi & \text{if } \xi < \beta, \\ \alpha + 1 + (\xi - \beta) & \text{if } \xi \geq \beta. \end{cases}$$

We also will have $\mathrm{lh}(\mathcal{W}) = \alpha + 1 + (\mathrm{lh}(\mathcal{S}) - \beta)$, so $\mathrm{ran}(u) = [0, \beta) \cup [\alpha + 1, \mathrm{lh}(\mathcal{W}))$.

[4]Note that these definitions of $\alpha(\mathcal{T}, F)$ and $\beta(\mathcal{T}, F)$ also make sense when F is on the sequence of *some* model of $\mathcal{T}$, not just the last model; we may use the notation in this more general context.

As mentioned above, tree embeddings are actually uniquely determined by their domain tree, their u-maps, extenders of the target tree with index not in the range of u, and branch choices where $\mathrm{ran}(u)$ is bounded. So the rest of Φ is already determined by the assignment $\mathcal{W} \restriction \alpha+1 = \mathcal{T} \restriction \alpha+1$ and $E^{\mathcal{W}}_\alpha = F$, although we need to verify that there is such a tree embedding. This is done in [24]. In this case, Φ is a total, non-dropping extended tree embedding, as long as $W(\mathcal{S}, \mathcal{T}, F)$ is well-founded.

Since $t_\beta = i^P_F$ and all the t_ξ for $\xi \geq \beta$ agree with t_β beyond $\mathrm{dom}(F)$, we have that F is an initial factor of the extender of these t_ξ. We let σ be the unique map such that the last t-map factors as $\sigma \circ i^N_F$, where N is the last model of $\mathcal{S}$ (using here that F is total on the last model by our case hypothesis). Steel shows that σ is weakly elementary in [24].

The following lemma shows that the one-step embedding normalization comes up naturally in the context of tree embeddings.

Lemma 3.1.12 (Factor Lemma)**.** *Let $\Psi : \mathcal{S} \to \mathcal{T}$ be an extended tree embedding such that $\Psi \neq Id$. Let $\beta = crit(u^\Psi)$ and $\alpha+1$ be the successor of $\beta = v^\Psi(\beta)$ in $(\beta, u^\Psi(\beta)]_{\mathcal{T}}$.*

Suppose that $\beta+1 = lh(\mathcal{S})$ or $\beta+1 < lh(\mathcal{S})$ and $dom(E^{\mathcal{T}}_\alpha) \trianglelefteq M^{\mathcal{S}}_\beta | lh(E^{\mathcal{S}}_\beta)$. Then $W(\mathcal{S}, \mathcal{T} \restriction \alpha+1, E^{\mathcal{T}}_\alpha)$ is defined and well-founded and there is a unique extended tree embedding $\Gamma : W(\mathcal{S}, \mathcal{T} \restriction \alpha+1, E^{\mathcal{T}}_\alpha) \to \mathcal{T}$ such that $u^\Gamma \restriction \alpha+1 = id$ and $\Psi = \Gamma \circ \Phi^{\mathcal{S}, \mathcal{T} \restriction \alpha+1, E^{\mathcal{T}}_\alpha}$.

Proof. First notice our hypotheses guarantee that $W(\mathcal{S}, \mathcal{T} \restriction \alpha+1, E^{\mathcal{T}}_\alpha)$ is defined. We will get that it is well-founded inductively, as we build our tree embedding Γ.

Let $\mathcal{W} = W(\mathcal{S}, \mathcal{T} \restriction \alpha+1, E^{\mathcal{T}}_\alpha)$ and $\Phi = \Phi^{\mathcal{S}, \mathcal{T} \restriction \alpha+1, E^{\mathcal{T}}_\alpha}$. Note that if $\beta+1 < \mathrm{lh}(\mathcal{S})$, then $W(\mathcal{S}, \mathcal{T} \restriction \alpha+1, E^{\mathcal{T}}_\alpha)$ is not in the dropping case, as if $\mathrm{dom}(E^{\mathcal{T}}_\alpha) < \mathrm{lh}(E^{\mathcal{S}}_\beta)$, then $\mathcal{T}$ would drop below $\mathrm{lh}(E^{\mathcal{T}}_\beta)$ so that $E^{\mathcal{T}}_\beta$ is not in the domain of t^Ψ_β, contradicting that Ψ is a tree embedding. So $\Phi : \mathcal{S} \to \mathcal{W}$ is a total, extended tree embedding (since either we are not in the dropping case or $\beta+1 = \mathrm{lh}(\mathcal{S})$ and this is trivial). The commutativity condition totally determines the u-map of Γ:

$$u^\Gamma(\xi) = \begin{cases} \xi & \text{if } \xi < \alpha+1 \\ u^\Psi \circ (u^\Phi)^{-1}(\xi) & \text{if } \xi \geq \alpha+1, \end{cases}$$

using in the second case that $[\alpha+1, \mathrm{lh}(\mathcal{W})) \subseteq \mathrm{ran}(u^\Phi)$. We just need to check by induction on ξ that $u^\Gamma \restriction (\xi+1)$ is the u-map of a tree embedding from $\mathcal{W} \restriction (\xi+1)$ into $\mathcal{T}$. This shows, in particular, that $\mathcal{W} \restriction \xi+1$ is well-founded.

We have that $\Gamma \restriction \beta+1 \approx \Psi \restriction \beta+1 \approx Id$, so we just need to see by induction on $\gamma \geq \beta$ that $u^\Gamma \restriction u^\Phi(\gamma)+1$ is the u-map of a tree embedding from $\mathcal{W} \restriction u^\Phi(\gamma)+1$ into $\mathcal{T}$. This easily passes through limits, so we just handle the successor case. The key thing here is verifying that we have the required relationship between tree predecessors: for $\eta = \mathcal{S}\text{-pred}(\gamma+1)$, $\eta^* = \mathcal{T}\text{-pred}(u^\Psi(\gamma)+1)$, and $\zeta = \mathcal{W}\text{-pred}(u^\Phi(\gamma)+1)$, $\eta^* \in [v^\Gamma(\zeta), u^\Gamma(\zeta)]_{\mathcal{T}}$. Then we can define our s-maps using the Shift Lemma and define the t-maps in the required way. Since Ψ is a tree embedding, we have $\eta^* \in [v^\Psi(\eta), u^\Psi(\eta)]_{\mathcal{T}}$. We split into cases.

Case 1. $\eta \neq \beta$.

Then $v^\Phi(\eta) = u^\Phi(\eta) = \zeta$, so we have

$$v^\Gamma \circ u^\Phi(\eta) = v^\Psi(\eta) \leq_{\mathcal{T}} \eta^* \leq_{\mathcal{T}} u^\Psi(\eta) = u^\Gamma \circ u^\Phi(\eta),$$

as desired.

We have that $s^{\Phi}_{\gamma+1}$ is the copy map associated to $(t^{\Phi}_{\gamma},\ t^{\Phi}_{\eta},\ E^{\mathcal{S}}_{\gamma})$. We let $s^{\Gamma}_{u^{\Gamma}(\gamma)+1}$ be the copy map associated to $(t^{\Gamma}_{u^{\Phi}(\gamma)},\ \hat{\imath}^{\mathcal{T}}_{v^{\Psi}(\eta),\eta^*} \circ s^{\Gamma}_{v^{\Phi}(\eta)},\ E^{\mathcal{W}}_{u^{\Phi}(\gamma)})$. By our induction hypothesis, we have that $t^{\Psi}_{\gamma} = t^{\Gamma}_{u^{\Phi}(\gamma)} \circ t^{\Phi}_{\gamma}$ and $\hat{\imath}^{\mathcal{T}}_{v^{\Psi}(\eta),\eta^*} \circ s^{\Psi}_{\eta} = \hat{\imath}^{\mathcal{T}}_{v^{\Psi}(\eta),\eta^*} \circ s^{\Gamma}_{v^{\Phi}(\eta)} \circ s^{\Phi}_{\eta}$.

So, since $s^{\Psi}_{\gamma+1}$ is the copy map associated to $(t^{\Psi}_{\gamma},\ \hat{\imath}^{\mathcal{T}}_{v^{\Psi}(\eta),\eta^*} \circ s^{\Psi}_{\eta},\ E^{\mathcal{S}}_{\gamma})$, we have $s^{\Psi}_{\gamma+1} = s^{\Gamma}_{u^{\Phi}(\gamma)+1} \circ s^{\Phi}_{\gamma+1}$ by Lemma 3.1.8. We then let $t^{\Gamma}_{u^{\Phi}(\gamma)+1} = \hat{\imath}^{\mathcal{T}}_{u^{\Psi}(\gamma)+1,u^{\Psi}(\gamma+1)} \circ s^{\Gamma}_{u^{\Phi}(\gamma)+1}$, as we must, which clearly maintains the commutativity condition.

Case 2. $\eta = \eta^* = \beta$.

In this case, $\mathrm{crit}(E^{\mathcal{S}}_{\gamma}) = \mathrm{crit}(E^{\mathcal{T}}_{u^{\Psi}(\gamma)})$. It follows that these are equal to $\mathrm{crit}(E^{\mathcal{W}}_{u^{\Phi}(\gamma)})$ and $\zeta = \beta$ as well. So we trivially have $\eta^* \in [v^{\Gamma}(\zeta), u^{\Gamma}(\zeta)]_{\mathcal{T}}$ (since $\eta^* = v^{\Gamma}(\zeta) = \beta$). We now continue as in Case 1.

Case 3. $\eta = \beta$ and $\eta^* > \beta$.

In this case we have, by our choice of α, that $\eta^* \geq_{\mathcal{T}} \alpha + 1$. We want to see that $\zeta = \alpha + 1$, since then $\eta^* \in [v^{\Gamma}(\alpha+1), u^{\Gamma}(\alpha+1)]_{\mathcal{T}}$, using here that $v^{\Gamma}(\alpha+1) = \alpha+1$ and $u^{\Gamma}(\alpha+1) = u^{\Psi}(\beta)$.

Since ζ is either β or $\alpha+1$, we just need to see β can't be $\mathcal{W}$-pred$(u^{\Phi}(\gamma)+1)$ because $\lambda(E^{\mathcal{T}}_{\beta}) \leq \mathrm{crit}(E^{\mathcal{W}}_{u^{\Phi}(\gamma)})$, using here that $E^{\mathcal{T}}_{\beta} = E^{\mathcal{W}}_{\beta}$.

Since $\mathrm{crit}(E^{\mathcal{S}}_{\gamma}) < \lambda(E^{\mathcal{S}}_{\beta})$, using that Ψ is a tree embedding and our induction hypothesis, we have

$$\begin{aligned}
\mathrm{crit}(E^{\mathcal{T}}_{u^{\Psi}(\gamma)}) &= t^{\Psi}_{\beta}(\mathrm{crit}(E^{\mathcal{S}}_{\gamma}))\\
&= t^{\Gamma}_{\alpha+1} \circ t^{\Phi}_{\beta}(\mathrm{crit}(E^{\mathcal{S}}_{\gamma}))\\
&= \hat{\imath}^{\mathcal{T}}_{\alpha+1,u^{\Psi}(\beta)} \circ t^{\Phi}_{\beta}(\mathrm{crit}(E^{\mathcal{S}}_{\gamma}))\\
&= \hat{\imath}^{\mathcal{T}}_{\alpha+1,u^{\Psi}(\beta)}(\mathrm{crit}(E^{\mathcal{W}}_{u^{\Phi}(\gamma)})).
\end{aligned}$$

So since $\mathrm{crit}(\hat{\imath}^{\mathcal{T}}_{\alpha+1,u(\beta)}) \geq \lambda(E^{\mathcal{T}}_{\alpha}) \geq \lambda(E^{\mathcal{T}}_{\beta})$, if $\mathrm{crit}(E^{\mathcal{W}}_{u^{\Phi}(\gamma)}) < \lambda(E^{\mathcal{T}}_{\beta})$, then $\mathrm{crit}(E^{\mathcal{T}}_{u^{\Psi}(\gamma)}) = \mathrm{crit}(E^{\mathcal{W}}_{u^{\Phi}(\gamma)}) < \lambda(E^{\mathcal{T}}_{\beta})$. But then $\beta \geq \eta^*$, contradicting our case hypothesis. So $\zeta = \alpha+1$, as desired. We now continue as in Case 1.

This finishes the successor case and the proof. □ Lemma 3.1.12

Definition 3.1.13. An extended tree embedding $\Psi : \mathcal{S} \to \mathcal{T}$ is *inflationary* if for any $\xi + 1 < \mathrm{lh}(\mathcal{S})$ and $\gamma + 1 \in (v^{\Psi}(\xi), u^{\Psi}(\xi)]_{\mathcal{T}}$, letting $\eta = \mathcal{T}$-pred$(\gamma+1)$,

$$\mathrm{dom}(E^{\mathcal{T}}_{\gamma}) \trianglelefteq M^{\mathcal{T}}_{\eta} | s^{\Psi}_{\xi,\eta}(\mathrm{lh}(E^{\mathcal{S}}_{\xi})).$$

Not all extended tree embeddings are inflationary, but it is easy to see that $\Phi^{\mathcal{S},\mathcal{T},F}$ is inflationary. If Ψ is inflationary and not the identity, then Ψ satisfies the hypothesis of the Factor Lemma. Letting $\Psi = \Gamma \circ \Phi$ be the resulting factorization, we'll see that Γ is also inflationary so that if Γ is not the identity it also satisfies the hypothesis of the Factor Lemma. This observation will allow us to use iterated applications of the Factor Lemma to completely factor a non-identity inflationary extended tree embedding.

Before returning to the factorization, we establish some basic facts about inflationary tree embeddings. We start with the following easy proposition.

Proposition 3.1.14. *Let $\Phi : \mathcal{S} \to \mathcal{T}$ and $\Psi : \mathcal{T} \to \mathcal{U}$ be extended tree embeddings. Suppose that $\Psi \circ \Phi$ is inflationary. Let $\xi + 1 < lh(\mathcal{T})$ with $\xi \in ran(u^\Phi)$. Then for any $\gamma + 1 \in (v^\Psi(\xi), u^\Psi(\xi)]_\mathcal{U}$, letting $\eta = \mathcal{U}$-pred$(\gamma + 1)$,*

$$dom(E^\mathcal{U}_\gamma) \trianglelefteq M^\mathcal{U}_\eta | s^\Psi_{\xi,\eta}(lh(E^\mathcal{T}_\xi)).$$

Proof. Let $\xi + 1 < \mathrm{lh}(\mathcal{T})$ with $\xi \in \mathrm{ran}(u^\Phi)$, say $\xi = u^\Phi(\bar{\xi})$. We have that $(v^\Psi(\xi), u^\Psi(\xi)]_\mathcal{U} \subseteq (v^{\Psi\circ\Phi}(\bar{\xi}), u^{\Psi\circ\Phi}(\bar{\xi})]_\mathcal{U}$. So for any $\gamma+1 \in (v^\Psi(\xi), u^\Psi(\xi)]_\mathcal{U}$, letting $\rho = \mathcal{U}$-pred$(\gamma+1)$, $\mathrm{dom}(E^\mathcal{U}_\gamma) \trianglelefteq M^\mathcal{U}_\eta | s^{\Psi\circ\Phi}_{\bar{\xi},\eta}(\mathrm{lh}(E^\mathcal{S}_{\bar{\xi}}))$, since $\Psi \circ \Phi$ is inflationary. But

$$\begin{aligned} s^{\Psi\circ\Phi}_{\bar{\xi},\eta} &= \hat{\imath}^\mathcal{U}_{v^{\Psi\circ\Phi}(\bar{\xi}),\eta} \circ s^{\Psi\circ\Phi}_{\bar{\xi}} \\ &= \hat{\imath}^\mathcal{U}_{v^\Psi(\xi),\eta} \circ s^\Psi_\xi \circ t^\Phi_{\bar{\xi}} \\ &= s^\Psi_{\xi,\eta} \circ t^\Phi_{\bar{\xi}}. \end{aligned}$$

So since $E^\mathcal{T}_\xi = t^\Phi_{\bar{\xi}}(E^\mathcal{S}_{\bar{\xi}})$, $\mathrm{dom}(E^\mathcal{U}_\gamma) \trianglelefteq M^\mathcal{U}_\eta | s^\Psi_{\xi,\eta}(\mathrm{lh}(E^\mathcal{T}_\xi))$. □

In general, it seems that it should be possible for $\Psi \circ \Phi$ to be inflationary even when Ψ is not inflationary, though we have not tried to come up with an example. However, if $\Psi \circ \Phi$ and Φ are inflationary and $[\mathrm{crit}(u^\Psi), \mathrm{lh}(\mathcal{T})) \subseteq \mathrm{ran}(u^\Phi)$, then the previous proposition immediately gives that Ψ is inflationary.

The situation with right factors of inflationary tree embeddings is simpler: they are always inflationary.

Proposition 3.1.15. *Let $\Phi : \mathcal{S} \to \mathcal{T}$ and $\Psi : \mathcal{T} \to \mathcal{U}$ be extended tree embeddings. Suppose that $\Psi \circ \Phi$ is inflationary. Then Φ is inflationary.*

Proof. Let $\xi + 1 < \mathrm{lh}(\mathcal{S})$, $\gamma + 1 \in (v^\Phi(\xi), u^\Phi(\xi)]_\mathcal{T}$ and $\eta = \mathcal{T}$-pred$(\gamma + 1)$. Let $\eta^* = \mathcal{U}$-pred$(u^\Psi(\gamma)+1)$. We have that $u^\Psi(\gamma)+1 = v^\Psi(\gamma+1)$ and so $u^\Psi(\gamma)+1 \in (v^{\Psi\circ\Phi}(\xi), u^{\Psi\circ\Phi}(\xi)]_\mathcal{U}$. So since $\Psi \circ \Phi$ is inflationary, $\mathrm{dom}(E^\mathcal{U}_{u^\Psi(\gamma)}) \trianglelefteq M^\mathcal{U}_{u^\Psi(\gamma)} | s^{\Psi\circ\Phi}_{\xi,\eta^*}(\mathrm{lh}(E^\mathcal{S}_\xi))$. But

$$\begin{aligned} s^{\Psi\circ\Phi}_{\xi,\eta^*} &= \hat{\imath}^\mathcal{U}_{v^{\Psi\circ\Phi}(\xi),\eta^*} \circ s^{\Psi\circ\Phi}_\xi \\ &= \hat{\imath}^\mathcal{U}_{v^\Psi(\eta),\eta^*} \circ s^\Psi_\eta \circ \hat{\imath}^\mathcal{T}_{v^\Phi(\xi),\eta} \circ s^\Phi_\xi \\ &= s^\Psi_{\eta,\eta^*} \circ s^\Phi_{\xi,\eta} \end{aligned}$$

and $\mathrm{dom}(E^\mathcal{T}_{u^\Psi(\gamma)}) = s^\Psi_{\eta,\eta^*}(\mathrm{dom}(E^\mathcal{T}_\gamma))$, so we must have $\mathrm{dom}(E^\mathcal{T}_\gamma) \trianglelefteq M^\mathcal{T}_\eta | s^\Phi_{\xi,\eta}(\mathrm{lh}(E^\mathcal{S}_\xi))$, as desired. □

Going the other way, we have that the inflationary tree embeddings are closed under composition.

Proposition 3.1.16. *Let $\Phi : \mathcal{S} \to \mathcal{T}$ and $\Psi : \mathcal{T} \to \mathcal{U}$ be inflationary extended tree embeddings. Then $\Psi \circ \Phi$ is an inflationary extended tree embedding.*

Proof. This follows, more or less, directly from the definitions, but it still fairly involved. We just need to verify that for $\xi+1<\mathrm{lh}(\mathcal{S})$, $\tau+1\in(v^\Gamma(\xi),u^\Gamma(\xi)]_{\mathcal{U}}$, and $\rho=\mathcal{U}\text{-pred}(\tau+1)$, we need to show that $\mathrm{dom}(E^{\mathcal{U}}_\tau)\trianglelefteq M^{\mathcal{U}}_\rho|s^\Gamma_{\xi,\rho}(\mathrm{lh}(E^{\mathcal{S}}_\xi))$.

Notice that

$$(v^\Gamma(\xi),u^\Gamma(\xi)]_{\mathcal{U}}=(v^\Gamma(\xi),v^\Psi(u^\Phi(\xi))]_{\mathcal{U}}\cup(v^\Psi(u^\Phi(\xi)),u^\Gamma(\xi)]_{\mathcal{U}}.$$

We'll handle these two regions separately, in order. Our first goal is to show the following.

Claim 1. *For any $\xi+1<lh(\mathcal{S})$ and any $\gamma+1\in(v^\Phi(\xi),u^\Phi(\xi)]_{\mathcal{T}}$, letting $\eta=\mathcal{T}$-pred$(\xi+1)$ and $\eta^*=\mathcal{U}$-pred$(u^\Psi(\gamma)+1)$, for any $\tau+1\in(v^\Psi(\eta),\eta^*]_{\mathcal{U}}$, letting $\rho=\mathcal{U}$-pred$(\tau+1)$, $dom(E^{\mathcal{U}}_\tau)\trianglelefteq M^{\mathcal{U}}_\rho|s^\Gamma_{\xi,\rho}(lh(E^{\mathcal{S}}_\xi))$.*

Proof. Since Φ is inflationary, we have $\mathrm{dom}(E^{\mathcal{T}}_\gamma)\trianglelefteq s^\Phi_{\xi,\eta}(\mathrm{lh}(E^{\mathcal{S}}_\xi))$. It follows that $s^\Psi_{\eta,\rho}(\mathrm{dom}(E^{\mathcal{T}}_\gamma))\trianglelefteq s^\Gamma_{\xi,\rho}(\mathrm{lh}(E^{\mathcal{S}}_\xi))$, since

$$\begin{aligned}s^\Gamma_{\xi,\rho}&=\hat{\imath}^{\mathcal{U}}_{v^\Gamma(\xi),\rho}\circ s^\Gamma_\xi\\&=\hat{\imath}^{\mathcal{U}}_{v^\Psi(\eta),\rho}\circ s^\Psi_\eta\circ\hat{\imath}^{\mathcal{T}}_{v^\Phi(\xi),\eta}\circ s^\Phi_\xi\\&=s^\Psi_{\eta,\rho}\circ s^\Phi_{\xi,\eta}.\end{aligned}$$

Now η^* is the first place along $[v^\Psi(\eta),u^\Psi(\eta)]_{\mathcal{U}}$ where we've finished moving up $\mathrm{crit}(E^{\mathcal{T}}_\gamma)$ to $\mathrm{crit}(E^{\mathcal{U}}_{u(\gamma)})$; that is, η^* is the least $\zeta\in[v^\Psi(\eta),u^\Psi(\eta)]_{\mathcal{U}}$ such that $\zeta=u^\Psi(\eta)$ or else $\mathrm{crit}(\hat{\imath}^{\mathcal{U}}_{\zeta,u^\Psi(\eta)})>s^\Psi_{\eta,\zeta}(\mathrm{crit}(E^{\mathcal{T}}_\gamma))$. So since $\rho<\eta^*$, $\mathrm{crit}(E^{\mathcal{U}}_\tau)\leq s^\Psi_{\eta,\rho}(\mathrm{crit}(E^{\mathcal{T}}_\gamma))$. Since Ψ is inflationary, we have that $\mathrm{dom}(E^{\mathcal{U}}_\tau)\trianglelefteq s^\Psi_{\eta,\rho}(\mathrm{lh}(E^{\mathcal{T}}_\eta))$. It follows that $\mathrm{dom}(E^{\mathcal{U}}_\tau)=(\mathrm{crit}(E^{\mathcal{U}}_\tau)^+)^{M^{\mathcal{U}}_\rho|s^\Psi_{\eta,\rho}(\mathrm{lh}(E^{\mathcal{T}}_\eta))}$, as we cannot drop below the image of $s^\Psi_\eta(\mathrm{lh}(E^{\mathcal{T}}_\eta))$ along $[v^\Psi(\eta),u^\Psi(\eta)]_{\mathcal{U}}$. But also, $\mathrm{dom}(E^{\mathcal{U}}_{u(\gamma)})=(\mathrm{crit}(E^{\mathcal{U}}_{u(\gamma)})^+)^{M^{\mathcal{U}}_{u(\eta)}|\mathrm{lh}(E^{\mathcal{U}}_{u(\eta)})}$, so we must have $s^\Psi_{\eta,\rho}(\mathrm{dom}(E^{\mathcal{T}}_\gamma))=(s^\Psi_{\eta,\rho}(\mathrm{crit}(E_\gamma)^{\mathcal{T}})^+)^{M^{\mathcal{U}}_\rho|s^\Psi_{\eta,\rho}(\mathrm{lh}(E^{\mathcal{T}}_\eta))}$, by elementarity. So actually $\mathrm{dom}(E^{\mathcal{U}}_\tau)\leq s^\Psi_{\eta,\rho}(\mathrm{dom}(E^{\mathcal{T}}_\gamma))$, since $\mathrm{crit}(E^{\mathcal{U}}_\tau)\leq s^\Psi_{\eta,\rho}(\mathrm{crit}(E^{\mathcal{T}}_\gamma))$ and the respective domains are just the successors of these cardinals in $M^{\mathcal{U}}_\rho|s^\Psi_{\eta,\rho}(\mathrm{lh}(E^{\mathcal{T}}_\eta))$. So $\mathrm{dom}(E^{\mathcal{U}}_\tau)\trianglelefteq M^{\mathcal{U}}_\rho|s^\Gamma_{\xi,\rho}(\mathrm{lh}(E^{\mathcal{S}}_\xi))$, as desired. □

Using this claim, we can now prove the following by induction on η.

Claim 2. *For any $\xi+1<lh(\mathcal{S})$ and any $\eta\in[v^\Phi(\xi),u^\Phi(\xi)]_{\mathcal{T}}$, for all $\tau+1\in(v^\Gamma(\xi),v^\Psi(\eta)]_{\mathcal{U}}$, letting $\rho=\mathcal{U}$-pred$(\tau+1)$, $dom(E^{\mathcal{U}}_\tau)\trianglelefteq s^\Gamma_{\xi,\rho}(lh(E^{\mathcal{S}}_\xi))$.*

Proof. The base case $\eta=v^\Phi(\xi)$ is trivial (there are no such τ) and since v-maps are tree-order preserving and continuous at limits, the limit case is immediate from the induction hypothesis. So we just need to handle the successor case.

So suppose the claim holds at $\eta<u^\Phi(\xi)$ and let $\gamma+1$ be the successor of η in $[v^\Phi(\xi),u^\Phi(\xi)]_{\mathcal{T}}$. Let $\tau+1\in(v^\Gamma(\xi),v^\Gamma(\gamma)]_{\mathcal{U}}$ and $\rho=\mathcal{U}$-pred$(\tau+1)$. Also let $\eta^*=\mathcal{U}$-pred$(u^\Psi(\gamma)+1)$. Now,

$$(v^\Gamma(\xi),v^\Psi(\gamma)]_{\mathcal{U}}=(v^\Gamma(\xi),v^\Psi(\eta)]_{\mathcal{U}}\cup(v^\Psi(\eta),\eta^*]_{\mathcal{U}}\cup\{u^\Psi(\eta)+1\}.$$

By our induction hypothesis, we have $\mathrm{dom}(E^{\mathcal{U}}_\tau)\trianglelefteq M^{\mathcal{U}}_\rho|s^\Gamma_{\xi,\rho}(\mathrm{lh}(E^{\mathcal{S}}_\xi))$ when $\tau+1\in(v^\Gamma(\xi),v^\Psi(\eta)]_{\mathcal{U}}$. By Claim 1, we also have $\mathrm{dom}(E^{\mathcal{U}}_\tau)\trianglelefteq M^{\mathcal{U}}_\rho|s^\Gamma_{\xi,\rho}(\mathrm{lh}(E^{\mathcal{S}}_\xi))$ when $\tau+1\in(v^\Psi(\eta),\eta^*]_{\mathcal{U}}$. So we just need to consider the case that $\tau=u^\Psi(\eta)$ and $\rho=\eta^*$.

Since Φ is inflationary, $\mathrm{dom}(E^{\mathcal{T}}_\gamma) \trianglelefteq s^\Phi_{\xi,\eta}(\mathrm{lh}(E^{\mathcal{S}}_\xi))$ so that $s^\Psi_{\eta,\eta^*}(\mathrm{dom}(E^{\mathcal{T}}_\gamma)) \trianglelefteq s^\Gamma_{\xi,\eta^*}(\mathrm{lh}(E^{\mathcal{S}}_\xi))$. η^* is the least place along $[v^\Psi(\eta), u^\Psi(\eta)]_{\mathcal{U}}$ where we finish moving up $\mathrm{dom}(E^{\mathcal{T}}_\gamma)$, i.e. $\mathrm{dom}(E^{\mathcal{U}}_{u^\Psi(\gamma)}) = s^\Psi_{\eta,\eta^*}(\mathrm{dom}(E^{\mathcal{T}}_\gamma))$. So $\mathrm{dom}(E^{\mathcal{U}}_{u^\Psi(\gamma)}) \trianglelefteq s^\Gamma_{\xi,\eta^*}(\mathrm{lh}(E^{\mathcal{S}}_\xi))$, as desired. This finishes the successor step. $\square$

The case $\eta = u^\Phi(\xi)$ just says that for $\tau+1 \in (v^\Gamma(\xi), v^\Psi(u^\Phi(\xi))]_{\mathcal{U}}$, $\mathrm{dom}(E^{\mathcal{U}}_\tau) \trianglelefteq M^{\mathcal{U}}_\rho | s^\Gamma_{\xi,\rho}(\mathrm{lh}(E^{\mathcal{S}}_\xi))$. So we've handled the first region, $(v^\Gamma(\xi), v^\Psi(u^\Phi(\xi))]_{\mathcal{U}}$.

To finish the proof, we just need to deal with the second region, which is basically immediate from the fact that Ψ is inflationary. Let $\tau+1 \in (v^\Psi(u^\Phi(\xi)), u^\Gamma(\xi)]_{\mathcal{U}}$ and $\rho = \mathcal{U}\text{-pred}(\tau+1)$. Since Ψ is inflationary, $\mathrm{dom}(E^{\mathcal{U}}_\tau) \trianglelefteq M^{\mathcal{U}}_\rho | s^\Psi_{u^\Phi(\xi),\rho}(\mathrm{lh}(E^{\mathcal{T}}_{u^\Phi(\xi)}))$. But $E^{\mathcal{T}}_{u^\Phi(\xi)} = t^\Phi_\xi(E^{\mathcal{S}}_\xi)$ and $s^\Psi_{u^\Phi(\xi),\rho} \circ t^\Phi_\xi = s^\Gamma_{\xi,\rho}$, so $\mathrm{dom}(E^{\mathcal{U}}_\tau) \trianglelefteq M^{\mathcal{U}}_\rho | s^\Gamma_{\xi,\rho}(\mathrm{lh}(E^{\mathcal{S}}_\xi))$, as desired. $\square$

Combining these propositions immediately gives us the following.

Proposition 3.1.17. *Let $\Phi : \mathcal{S} \to \mathcal{T}$ and $\Psi : \mathcal{T} \to \mathcal{U}$ be extended tree embeddings with $[crit(u^\Psi), lh(\mathcal{T})) \subseteq ran(u^\Phi)$. Then $\Psi \circ \Phi$ is inflationary iff both Φ and Ψ are inflationary.*

We also have inflationary tree embeddings are closed under (well-founded) direct limits, in the following sense.

Proposition 3.1.18. *Let $\mathcal{D} = \langle \{\mathcal{T}_a\}_{a\in A}, \{\Psi_{a,b}\}_{a\preceq b}\rangle$ be a directed system of normal trees such that for all $a, b \in A$ with $a \preceq b$, $\Psi_{a,b}$ is inflationary. Suppose that $\lim \mathcal{D}$ is well-founded and let $\Gamma_a : \mathcal{T}_a \to \lim \mathcal{D}$ be the direct limit extended tree embeddings. Then for all $a \in A$, Γ_a is inflationary.*

Proof. Fix $a \in A$, $\xi+1 < \mathrm{lh}(\mathcal{T}_a)$, $\gamma+1 \in (v^{\Gamma_a}(\xi), u^{\Gamma_a}(\xi)]_{\mathcal{T}}$, and $\eta = \mathcal{T}_a\text{-pred}(\gamma+1)$. Fix b with $a \preceq b$ such that $[b, \bar\eta]_{\mathcal{D}} = \eta$ and $[b, \bar\gamma+1]_{\mathcal{D}} = \gamma+1$. So we must have that $\bar\eta = \mathcal{T}_b\text{-pred}(\bar\gamma+1)$. Since $\Psi_{a,b}$ is inflationary, we have

$$\mathrm{dom}(E^{\mathcal{T}_b}_{\bar\gamma}) \trianglelefteq M^{\mathcal{T}_b}_{\bar\eta} | s^{\Psi_{a,b}}_{\xi,\bar\eta}(\mathrm{lh}(E^{\mathcal{T}_a}_\xi)).$$

Applying $t^{\Gamma_b}_{\bar\eta}$ gives

$$\begin{aligned} \mathrm{dom}(E^{\lim \mathcal{D}}_\gamma) &\trianglelefteq M^{\lim \mathcal{D}}_\eta | t^{\Gamma_b}_{\bar\eta} \circ s^{\Psi_{a,b}}_{\xi,\bar\eta}(\mathrm{lh}(E^{\mathcal{T}_a}_\xi)) \\ &= M^{\lim \mathcal{D}}_\eta | \hat\imath^{\lim \mathcal{D}}_{v^{\Gamma_b}(\bar\eta),\eta} \circ s^{\Gamma_b}_{\bar\eta} \circ \hat\imath^{\mathcal{T}_b}_{v^{\Psi_{a,b}}(\xi),\bar\eta} \circ s^{\Psi_{a,b}}_\xi(\mathrm{lh}(E^{\mathcal{T}_a}_\xi)) \\ &= M^{\lim \mathcal{D}}_\eta | s^{\Gamma_a}_{\xi,\eta}(\mathrm{lh}(E^{\mathcal{T}_a}_\xi)). \end{aligned}$$

$\square$

Now we'll return to iteratively factoring inflationary tree embeddings.

Theorem 3.1.19. *Let $\Psi : \mathcal{S} \to \mathcal{T}$ be an inflationary extended tree embedding. Then there is a unique sequence of extenders $\langle F_\xi \mid \xi < \lambda\rangle$ such that there is a directed system of inflationary extended tree embeddings $\mathcal{D} = \langle \{\mathcal{S}_\xi\}_{\xi\le\lambda}, \{\Psi_{\eta,\xi}\}_{\eta\le\xi\le\lambda}\rangle$ satisfying:*

1. *$\mathcal{S}_0 = \mathcal{S}$, $\mathcal{S}_\lambda = \mathcal{T}$, and $\Psi_{0,\lambda} = \Psi$;*

2. *for $\xi+1 \le \lambda$, letting $\beta_\xi = crit(u^{\Psi_{\xi,\lambda}})$ and $\alpha_\xi + 1$ the successor of β_ξ in $(\beta_\xi, u^{\Psi_{\xi,\lambda}}(\beta_\xi)]_{\mathcal{T}}$,*

   (a) *$F_\xi = E^{\mathcal{T}}_{\alpha_\xi}$,*

   (b) *$\mathcal{S}_{\xi+1} = W(\mathcal{S}_\xi, \mathcal{T}\restriction\alpha_\xi+1, F_\xi)$, and*

   (c) *$\Psi_{\xi,\xi+1} = \Phi^{\mathcal{S}_\xi, \mathcal{T}\restriction\alpha_\xi+1, F_\xi}$,*

3. *for $\gamma \le \lambda$ a limit ordinal,*

   (a) *$\mathcal{S}_\gamma = \lim\langle\{\mathcal{S}_\xi\}_{\xi<\gamma}, \{\Psi_{\eta,\xi}\}_{\eta\le\xi<\gamma}\rangle$, and*

   (b) *for $\xi < \gamma$, $\Psi_{\xi,\gamma}$ is the direct limit extended tree embedding.*

4. *for $\xi \le \lambda$ and $\eta < \xi$, $u^{\Psi_{\xi,\lambda}}\restriction\alpha_\eta + 1 = id$.*

Proof. The proof is by induction, using the Factor Lemma at successor steps. The uniqueness is guaranteed at successors by (2), (4), and the uniqueness of the tree embedding produced by the Factor Lemma; it follows at limits by (3) since direct limit tree embeddings are unique.

We'll keep track of some auxiliary objects not mentioned in the theorem statement and will maintain some additional hypotheses about them in our induction. We'll define $\langle F_\xi \mid \xi < \lambda\rangle$ and $\mathcal{D} = \langle\{\mathcal{S}_\xi\}_{\xi\le\lambda}, \{\Psi_{\eta,\xi}\}_{\eta\le\xi\le\lambda}\rangle$ along with $\langle\Gamma_\xi \mid \xi \le \lambda\rangle$, $\langle\beta_\xi \mid \xi < \lambda\rangle$, $\langle\alpha_\xi \mid \xi < \lambda\rangle$, and $\langle\delta_\xi \mid \xi \le \lambda\rangle$ by transfinite recursion, maintaining that for all $\xi \le \lambda$,

1. $\mathcal{D}\restriction\xi = \langle\{\mathcal{S}_\zeta\}_{\zeta<\xi}, \{\Psi_{\eta,\zeta}\}_{\eta\le\zeta<\xi}\rangle$ is a directed system of inflationary extended tree embeddings with $\mathcal{S}_0 = \mathcal{S}$,

2. $\Gamma_\xi : \mathcal{S}_\xi \to \mathcal{T}$ is an inflationary extended tree embedding such that

   (a) $\Gamma_0 = \Psi$ and

   (b) $\Gamma_\xi \circ \Psi_{\zeta,\xi} = \Gamma_\zeta$

3. if $\Gamma_\xi = Id_{\mathcal{T}}$, then $\xi = \lambda$,

4. if $\Gamma_\xi \neq Id_{\mathcal{S}_\xi}$, then $\xi < \lambda$, $\beta_\xi = \mathrm{crit}(u^{\Gamma_\xi})$, $\alpha_\xi + 1$ is the successor of β_ξ in $(\beta_\xi, u^{\Gamma_\xi}(\beta_\xi)]_{\mathcal{T}}$, and

   (a) $F_\xi = E^{\mathcal{T}}_{\alpha_\xi}$,

   (b) $\mathcal{S}_{\xi+1} = W(\mathcal{S}_\xi, \mathcal{T}\restriction\alpha_\xi+1, F_\xi)$,

   (c) $\Psi_{\xi,\xi+1} = \Phi^{\mathcal{S}_\xi, \mathcal{T}\restriction\alpha_\xi+1, F_\xi}$;

5. $\delta_\xi = \sup\{\alpha_\eta + 1 \mid \eta < \xi\}$ and

   (a) $u^{\Gamma_\xi}\restriction\delta_\xi = id$ and

   (b) $[\delta_\xi, \mathrm{lh}(\mathcal{S}_\xi)) \subseteq \mathrm{ran}(u^{\Psi_{0,\xi}})$

6. if ξ is a limit ordinal,

   (a) $\mathcal{S}_\xi = \lim\mathcal{D}\restriction\xi$, and

   (b) for $\eta < \xi$, $\Psi_{\eta,\xi}$ is the direct limit extended tree embedding.

Notice that, if we succeed in our induction, then for some stage $\xi \leq \mathrm{lh}(\mathcal{T})$ we must have Γ_ξ is the identity and so have $\mathcal{S}_\xi = \mathcal{T}$ and set $\lambda = \xi$, by (3). This is because, by (4), the sequence $\{F_\eta \mid \eta < \xi\}$ is a subset of the exit extenders of $\mathcal{T}$ and (4) demands we add in another F_ξ from further out in $\mathcal{T}$ as long as Γ_ξ is not the identity. Here we are using that $\alpha_\eta < \alpha_\xi$ whenever $\eta < \xi$, since $\alpha_\eta < \mathrm{crit}(u^{\Gamma_\xi}) < \alpha_\xi$, by (5), and that the α_ξ is the index of F_ξ as an exit extender of $\mathcal{T}$, by (4). Using these observations, it is easy to see that claims (1)-(4) of our theorem statement are subsumed by our inductive hypotheses (1)-(6) in the case $\xi = \lambda$.

To start, we let $\mathcal{S}_0 = \mathcal{S}$ and $\Gamma_0 = \Psi$. In this case, (1)-(6) are all trivial or hold by hypothesis, except for (4)(b) and (c), which we verify at $\xi = 1$.

Now suppose we've defined our objects for all $\eta < \xi$, maintaining (1)-(6), except for (4)(b) and (c) in the case $\eta + 1 = \xi$.[5]

First suppose ξ is a successor, say $\xi = \eta + 1$. If $u^{\Gamma_\eta} = id$, then we must have $\Gamma_\eta = Id_{\mathcal{T}}$ and so we have $\mathcal{S}_\eta = \mathcal{T}$, set $\lambda = \eta$ and stop our construction. So suppose u^{Γ_η} is not the identity. We need to continue our construction one more step.

We have that $\beta_\eta = \mathrm{crit}(u^\Gamma_\eta)$ and $\alpha_\eta + 1$ is the successor of β_η in $(\beta_\eta, u^{\Gamma_\eta}(\beta_\eta)]_{\mathcal{T}}$. By (2) at η, we have that Γ_η is inflationary, so that the Factor Lemma applies to it. So $W(\mathcal{S}_\eta, \mathcal{T}\restriction \alpha_\eta + 1, E^{\mathcal{T}}_{\alpha_\xi})$ is defined and well-founded and we set $\mathcal{S}_\xi = W(\mathcal{S}_\eta, \mathcal{T}\restriction\alpha_\eta + 1, E^{\mathcal{T}}_{\alpha_\xi})$, maintaining (4)(b). We also let $\Psi_{\eta,\xi} = \Phi^{\mathcal{S}_\eta, \mathcal{T}\restriction\alpha_\eta+1, E^{\mathcal{T}}_{\alpha_\xi}}$, maintaining (4)(c). For all $\zeta < \eta$, set $\Psi_{\zeta,\xi} = \Psi_{\eta,\xi} \circ \Psi_{\zeta,\xi}$ and $\Psi_{\xi,\xi} = Id_{\mathcal{S}_\xi}$. The $\Psi_{\zeta,\xi}$ are inflationary by Proposition 3.1.16 (or Proposition 3.1.15) and our induction hypothesis. This maintains (1).

Now we let Γ_ξ be the result of the Factor Lemma applied to Γ_η, so that $\Gamma_\xi : \mathcal{S}_\xi \to \mathcal{T}$, $u^{\Gamma_\xi} \restriction \alpha_\eta + 1 = id$, and $\Gamma_\eta = \Gamma \circ \Psi_{\eta,\xi}$. We let $\delta_\xi = \sup\{\alpha_\zeta + 1 \mid \zeta < \xi\}$. Then $\delta_\xi = \alpha_\eta + 1$ since $\alpha_\zeta < \mathrm{crit}(u^{\Gamma_\zeta}) < \alpha_\eta$, by (5) at η. So we immediately get (5)(a). Towards (5)(b), note that we have $[\alpha_\eta + 1, \mathrm{lh}(\mathcal{S}_\xi)) \subseteq \mathrm{ran}(u^{\Psi_{\eta,\xi}})$ since $\Psi_{\eta,\xi} = \Phi^{\mathcal{S}_\eta, \mathcal{T}\restriction\alpha_\eta+1, E^{\mathcal{T}}_{\alpha_\xi}}$. Moreover, $u^{\Psi_{\eta,\xi}}$ maps $[\beta_\eta, \mathrm{lh}(\mathcal{S}_\eta))$ onto $[\alpha_\eta + 1, \mathrm{lh}(\mathcal{S}_\xi))$. We have that $\Psi_{0,\xi} = \Psi_{\eta,\xi} \circ \Psi_{0,\eta}$ and our induction hypothesis (5)(b) at η gives $[\delta_\eta, \mathrm{lh}(\mathcal{S}_\eta)) \subseteq \mathrm{ran}(u^{\Psi_{0,\eta}})$. So since $\mathrm{crit}(u^{\Psi_{\eta,\xi}}) = \beta_\eta \geq \delta_\eta$ (by (5)(a) at η), $[\beta_\eta, \mathrm{lh}(\mathcal{S}_\eta)) \subseteq \mathrm{ran}(u^{\Psi_{0,\eta}})$. This gives (5)(b) since $[\alpha_\eta + 1, \mathrm{lh}(\mathcal{S}_\xi)) = u^{\Psi_{\eta,\xi}}``[\beta_\eta, \mathrm{lh}(\mathcal{S}_\eta)) \subseteq \mathrm{ran}(u^{\Psi_{\eta,\xi}\circ\Psi_{0,\eta}}) = \mathrm{ran}(u^{\Psi_{0,\xi}})$.

The last thing we need to check in the successor case is that Γ_ξ is actually inflationary. We have that $\Gamma_\eta = \Gamma_\xi \circ \Psi_{\eta,\xi}$ is inflationary, $\Psi_{\eta,\xi}$ is inflationary, and $\mathrm{crit}(u^{\Gamma_\xi}) \geq \delta_\eta = \alpha_\eta + 1$ so that $[\mathrm{crit}(u^{\Gamma_\xi}), \mathrm{lh}(\mathcal{S}_\xi)) \subseteq \mathrm{ran}(u^{\Gamma_\eta})$. So Proposition 3.1.17 implies Γ_ξ must be inflationary, too. This finishes the successor case.

Now suppose that ξ is a limit ordinal. (1) is trivial since it holds at all $\eta < \xi$. Proposition 3.1.11 gives that $\lim \mathcal{D}\restriction\xi$ is well-founded, since $\mathcal{T}$ and the Γ_η's witness that the hypothesis of that proposition obtains, by hypothesis (2). So we set $\mathcal{S}_\xi = \lim \mathcal{D}\restriction\xi$ and let $\Psi_{\eta,\xi}$ be the associated direct limit extended tree embeddings. We let Γ_ξ be the unique tree embedding from $\mathcal{S}_\xi$ into $\mathcal{T}$ such that $\Gamma_\eta = \Gamma_\xi \circ \Phi_{\eta,\xi}$ for all $\eta < \xi$, which exists by Proposition 3.1.11. Since the $\Phi_{\eta,\xi}$ are right factors of the inflationary tree embeddings Γ_η, we have that $\Phi_{\eta,\xi}$ is inflationary by Proposition 3.1.15. Of the remaining clauses, (5) is the only one which is not immediate. To verify this, we will go through the representation of $\lim \mathcal{D}\restriction\xi$ in terms of u-threads, as in §3.1.

[5] As stated, (4)(b) and (c) are hypotheses about $\xi + 1$ and will be verified at stage $\xi + 1$.

Let $\tau < \delta_\xi = \sup\{\delta_\eta \mid \eta < \xi\}$. Then as $\delta_\eta \leq \mathrm{crit}(u^{\Gamma_\eta}) = \beta_\eta$ for $\eta < \xi$, by (5) at η, we can find some η so that $\tau < \beta_\eta$. But $\beta_\eta = \mathrm{crit}(u^{\Psi_{\eta,\zeta}})$ for all $\zeta < \xi$ with $\eta < \zeta$, since $\beta_\eta = \mathrm{crit}(u^{\Psi_{\eta,\eta+1}})$ by (4)(c) and so the commutativity of the Ψ's gives $\beta_\eta \leq \mathrm{crit}(u^{\Psi_{\eta,\zeta}})$ but also $\Gamma_\zeta \circ \Psi_{\eta,\zeta} = \Gamma_\eta$, by (2)(b) below ξ, and so $\beta_\eta = \mathrm{crit}(u^{\Gamma_\eta}) = \min\{\mathrm{crit}(u^{\Psi_{\eta,\zeta}}), \mathrm{crit}(\Gamma_\zeta)\} \leq \mathrm{crit}(u^{\Psi_{\eta,\zeta}})$, too. It follows that the u-thread $[\eta,\tau]_{\mathcal{D}}$ is fixed by all $u^{\Psi_{\eta,\zeta}}$ so that $[\eta,\tau]_{\mathcal{D}}$ actually has rank τ in the order $\leq$ on u-threads, i.e. $[\eta,\tau]_{\mathcal{D}} = \tau$. So $u^{\Psi_{\eta,\xi}}(\tau) = \tau$, by the characterization of the u-map of the direct limit tree embedding $\Psi_{\eta,\xi}$ in §3.1. Now since $\Gamma_\eta = \Gamma_\xi \circ \Psi_{\eta,\xi}$ and $u^{\Gamma_\eta}(\tau) = \Psi_{\eta,\xi}(\tau) = \tau$, we must also have $u^{\Gamma_\xi}(\tau) = \tau$. Since τ was arbitary, this shows $u^{\Gamma_\xi} \restriction \delta_\xi = id$, which is (5)(a) at ξ.

Since (5)(a) holds at all $\eta \leq \xi$ and $\Gamma_\xi \circ \Psi_{\eta,\xi} = \Gamma_\eta$, we must have that $\mathrm{crit}(u^{\Psi_{\eta,\xi}}) = \beta_\eta \geq \delta_\eta$.[6] First suppose that for cofinally many $\eta < \xi$, $\beta_\eta > \delta_\eta$. Then for cofinally many η and any $\zeta \geq \eta$, $\delta_\eta = [\eta,\delta_\eta]_{\mathcal{D}} = [\zeta,\delta_\eta]_{\mathcal{D}}$. Fix such an η and $\tau < \mathrm{lh}(\mathcal{S}_\eta)$ such that $[\eta,\tau]_{\mathcal{D}} \geq \delta_\xi = \sup\{\delta_\zeta \mid \zeta < \xi\}$. Then $\tau > \delta_\eta$ so that $\tau \in \mathrm{ran}(u^{\Psi_{0,\eta}})$ by (5)(b) at η. But then $[\eta,\tau]_{\mathcal{D}} \in \mathrm{ran}(u^{\Psi_{0,\xi}})$ since, letting $\bar{\tau}$ such that $u^{\Psi_{0,\eta}}(\bar{\tau}) = \tau$,

$$\begin{aligned}[\eta,\tau]_{\mathcal{D}} &= [0,\bar{\tau}]_{\mathcal{D}} \\ &= u^{\Psi_{0,\xi}}(\bar{\tau}).\end{aligned}$$

Now suppose that for all sufficiently large η, $\beta_\eta = \delta_\eta$. Then $[\eta,\beta_\eta]_{\mathcal{D}} = [\zeta,\beta_\zeta]_{\mathcal{D}}$ for all sufficiently large η and $\zeta \geq \eta$. Fix such an η. Then for all $\zeta > \eta$, $\delta_\eta = [\zeta,\delta_\eta]_{\mathcal{D}}$ and $\delta_\zeta > \delta_\eta$, so we have that $[\eta,\beta_\eta]_{\mathcal{D}} \geq \sup\{\delta_\zeta \mid \zeta < \xi\} = \delta_\xi$. But then $\delta_\xi = [\eta,\beta_\eta]_{\mathcal{D}}$ since if $\zeta \geq \eta$ and $\tau < \mathrm{lh}(\mathcal{S}_\zeta)$ is such that $[\zeta,\tau]_{\mathcal{D}} < [\eta,\beta_\eta]_{\mathcal{D}}$, then as $[\eta,\beta_\eta]_{\mathcal{D}} = [\zeta,\beta_\zeta]_{\mathcal{D}}$, $\tau < \beta_\zeta = \mathrm{crit}(u^{\Psi_{\zeta,\xi}})$. So $[\zeta,\tau]_{\mathcal{D}} = \tau < \beta_\zeta = \delta_\zeta < \delta_\xi$. Our induction hypothesis then gives that for any $\zeta \geq \eta$, any $[\zeta,\tau]_{\mathcal{D}} \geq [\zeta,\delta_\zeta]_{\mathcal{D}} = \delta_\xi$ is in $\mathrm{ran}(u^{\Phi_{0,\gamma}})$, as in the previous case. This finishes our verification of (5) and the limit step of the construction. □

Definition 3.1.20. For an inflationary tree embedding Ψ, the *factorization of* Ψ is the sequence $\langle F_\xi \mid \xi < \lambda\rangle$ as in the previous theorem.

In the course of the proof we showed the following, which we isolate for later.

Proposition 3.1.21. *Let $\Psi : \mathcal{S} \to \mathcal{T}$ be an inflationary extended tree embedding and $\langle F_\xi \mid \xi < \lambda\rangle$ its factorization. Let $\delta(\Psi) = \sup\{\alpha(\mathcal{T}, F_\xi) + 1 \mid \xi < \lambda\}$.*

Then $[\delta(\Psi), lh(\mathcal{T})) \subseteq ran(u^\Psi)$.

Suppose $\bar{\mathcal{T}}$ and $\mathcal{T}$ are two trees on M (perhaps via different strategies) with some common extender F on the sequence of their last models and that $\mathcal{S}$ is a tree such that both $W(\mathcal{S},\bar{\mathcal{T}},F)$ and $W(\mathcal{S},\mathcal{T},F)$ are defined. We won't have that $W(\mathcal{S},\bar{\mathcal{T}},F) = W(\mathcal{S},\mathcal{T},F)$ in general, but these trees will have many models in common and the one-step normalization tree embeddings into these trees will be nearly the same, in the sense expressed in the following definition of similar tree embeddings, defined below.

First, recall the following definition from [24].

Definition 3.1.22. For $\mathcal{T}$ a normal tree and $\eta \leq_{\mathcal{T}} \xi$, we let $e^{\mathcal{T}}_{\eta,\xi}$ be the sequence of extenders used along $(\eta,\xi)_{\mathcal{T}}$, i.e. $e^{\mathcal{T}}_{\eta,\xi}$ is the length-increasing enumeration of $\{E^{\mathcal{T}}_\alpha \mid \alpha + 1 \in (\eta,\xi]_{\mathcal{T}}\}$.

[6] This was implicit in our verification of (5)(a) at ξ, but is easy to see granting it: $\beta_\eta = \mathrm{crit}(u^{\Gamma_\eta}) = \min\{\mathrm{crit}(u^{\Gamma_\xi}), \mathrm{crit}(u^{\Psi_{\eta,\xi}})\}$ and $\mathrm{crit}(u^{\Gamma_\xi}) \geq \alpha_\eta + 1 > \beta_\eta$.

Definition 3.1.23. Let $\Phi : \mathcal{S} \to \mathcal{T}$ and $\Psi : \mathcal{S} \to \mathcal{U}$ be extended tree embeddings. We say Φ and Ψ are *similar*, which we denote $\Phi \equiv \Psi$, iff for all $\xi < \mathrm{lh}(\mathcal{S})$, $e^{\mathcal{T}}_{0,u^\Phi(\xi)} = e^{\mathcal{U}}_{0,u^\Psi(\xi)}$.

We can completely characterize this notion for inflationary Φ, Ψ using the Factor Lemma: $\Phi \equiv \Psi$ iff the factorizations of Φ and Ψ are equal.

We'll need the following fact which says that if two trees $\mathcal{T}, \mathcal{U}$ have two common models M, N which are tree-related in both trees and have the same branch embeddings between them in both trees, then the extenders used to get from M to N are the same in both trees.

Proposition 3.1.24. *Suppose $\mathcal{T}$, $\mathcal{U}$ are normal trees, $\eta \leq_{\mathcal{T}} \xi$, $\eta^* \leq_{\mathcal{U}} \xi^*$, $M^{\mathcal{T}}_\xi = M^{\mathcal{U}}_{\xi^*}$, and either*

1. *η-to-ξ and η^*-to-ξ^* don't drop, $M^{\mathcal{T}}_\eta = M^{\mathcal{U}}_{\eta^*}$, and $i^{\mathcal{T}}_{\eta,\xi} = i^{\mathcal{U}}_{\eta^*,\xi^*}$, or*

2. *η^* and η are the locations of the last drop along η-to-ξ and η^*-to-ξ^*.*

Then $e^{\mathcal{T}}_{\eta,\xi} = e^{\mathcal{U}}_{\eta^*,\xi^*}$.

Proof. Of course, in case (2), we get that $\hat{\imath}^{\mathcal{T}}_{\eta,\xi}$ and $\hat{\imath}^{\mathcal{U}}_{\eta^*,\xi^*}$ are the uncoring map into $M^{\mathcal{T}}_\xi = M^{\mathcal{U}}_{\xi^*}$. So in either case, $\hat{\imath}^{\mathcal{T}}_{\eta,\xi} = \hat{\imath}^{\mathcal{U}}_{\eta^*,\xi^*}$ and are elementary on their common domain.

We can easily verify $e^{\mathcal{T}}_{\eta,\xi} = e^{\mathcal{U}}_{\eta^*,\xi^*}$ by induction using normality and the initial segment condition. Suppose $\gamma \in [\eta, \xi)_{\mathcal{T}}$ and $\gamma^* \in [\eta^*, \xi^*)_{\mathcal{U}}$ and $e^{\mathcal{T}}_{\eta,\gamma} = e^{\mathcal{U}}_{\eta^*,\gamma^*}$. Let $\zeta + 1, \zeta^* + 1$ be the successors of γ, γ^* in $(\eta, \xi]_{\mathcal{T}}$ and $(\eta^*, \xi^*]_{\mathcal{U}}$, respectively. We just need to show that $E^{\mathcal{T}}_\zeta = E^{\mathcal{U}}_{\zeta^*}$. Suppose not. Now since $e^{\mathcal{T}}_{\eta,\gamma} = e^{\mathcal{U}}_{\eta^*,\gamma^*}$ and $\hat{\imath}^{\mathcal{T}}_{\eta,\xi} = \hat{\imath}^{\mathcal{U}}_{\eta^*,\xi^*}$, the tail embeddings $\hat{\imath}^{\mathcal{T}}_{\gamma,\xi}$ and $\hat{\imath}^{\mathcal{U}}_{\gamma^*,\xi^*}$ must be equal and elementary as well (since we've just factored out a common initial segment of the extender of this common elementary map). So $E^{\mathcal{T}}_\zeta$ and $E^{\mathcal{U}}_{\zeta^*}$ are both (trivial completions of) initial segments of the extender of $\hat{\imath}^{\mathcal{T}}_{\gamma,\xi} = \hat{\imath}^{\mathcal{U}}_{\gamma^*,\xi^*}$. If they have the same length, we're done. So without loss of generality, suppose $\mathrm{lh}(E^{\mathcal{T}}_\zeta) < \mathrm{lh}(E^{\mathcal{U}}_{\zeta^*})$. Then $E^{\mathcal{T}}_\zeta$ is a whole initial segment of $E^{\mathcal{U}}_{\zeta^*}$ and so on the $M^{\mathcal{U}}_{\zeta^*}$-sequence. By coherence, $E^{\mathcal{T}}_\zeta$ is on the sequence of $M^{\mathcal{U}}_{\xi^*} = M^{\mathcal{T}}_\xi$, a contradiction (since $\zeta < \xi$, $E^{\mathcal{T}}_\zeta$ is not on the $M^{\mathcal{T}}_\xi$-sequence). Of course, we can't have that one of $e^{\mathcal{T}}_{\eta,\xi}$ and $e^{\mathcal{U}}_{\eta^*,\xi^*}$ is a proper initial segment of the other since the embeddings are the same, so the above argument shows they must be equal.

$\square$

Lemma 3.1.25. *Suppose $\Phi : \mathcal{S} \to \mathcal{T}$ and $\Psi : \mathcal{S} \to \mathcal{U}$ are inflationary extended tree embeddings and $\Phi \equiv \Psi$. Then for all $\xi < lh(\mathcal{S})$, $M^{\mathcal{T}}_{v^\Phi(\xi)} = M^{\mathcal{U}}_{v^\Psi(\xi)}$, $M^{\mathcal{T}}_{u^\Phi(\xi)} = M^{\mathcal{U}}_{u^\Psi(\xi)}$, $s^\Phi_\xi = s^\Psi_\xi$, and $t^\Phi_\xi = t^\Psi_\xi$.*

Proof. Our hypothesis that $\Phi \equiv \Psi$ immediately gives that $M^{\mathcal{T}}_{u^\Phi(\xi)} = M^{\mathcal{U}}_{u^\Psi(\xi)}$ for all $\xi < \mathrm{lh}(\mathcal{S})$. So we just need verify $M^{\mathcal{T}}_{v^\Phi(\xi)} = M^{\mathcal{U}}_{v^\Psi(\xi)}$, $s^\Phi_\xi = s^\Psi_\xi$, and $t^\Phi_\xi = t^\Psi_\xi$ by induction on ξ.

To start, $M^{\mathcal{T}}_{v^\Phi(0)} = M^{\mathcal{U}}_{v^\Psi(0)} = M^{\mathcal{S}}_0$, $s^\Phi_0 = s^\Psi_0 = id$. Also, $t^\Phi_0 = \hat{\imath}^{\mathcal{T}}_{0,u^\Phi(0)}$ and $t^\Psi_0 = \hat{\imath}^{\mathcal{U}}_{0,u^\Psi(0)}$, and so these are equal since $e^{\mathcal{T}}_{0,u^\Phi(0)} = e^{\mathcal{U}}_{0,u^\Psi(0)}$.

So suppose we've verified the equalities for all $\eta \leq \xi$. Since $t^\Phi_\xi = t^\Psi_\xi$, we have that $E^{\mathcal{T}}_{u^\Phi(\xi)} = E^{\mathcal{U}}_{u^\Psi(\xi)}$. Let $\eta = \mathcal{S}$-pred$(\xi+1)$, $\zeta = \mathcal{T}$-pred$(u^\Phi(\xi)+1)$, and $\zeta^* = \mathcal{U}$-pred$(u^\Psi(\xi)+1)$. Since Φ and Ψ are tree embeddings, we have $\zeta \in [v^\Phi(\eta), u^\Phi(\eta)]_{\mathcal{T}}$ and $\zeta^* \in [v^\Psi(\eta), u^\Psi(\eta)]_{\mathcal{U}}$. By hypothesis, $e^{\mathcal{T}}_{0,u^\Phi(\eta)} = e^{\mathcal{U}}_{0,u^\Psi(\eta)}$ and by our induction hypothesis, $M^{\mathcal{T}}_{v^\Phi(\eta)} = M^{\mathcal{U}}_{v^\Psi(\eta)}$. So we must have the extenders leading up to this model are the same, i.e. $e^{\mathcal{T}}_{0,v^\Phi(\eta)} = e^{\mathcal{U}}_{0,v^\Psi(\eta)}$, and so the tail of the extenders used are the same, too, i.e. $e^{\mathcal{T}}_{v^\Phi(\eta),u^\Phi(\eta)} = e^{\mathcal{U}}_{v^\Phi(\eta),u^\Psi(\eta)}$. But the sequence of models along $[v^\Phi(\eta), u^\Phi(\eta)]_{\mathcal{T}}$ and $[v^\Psi(\eta), u^\Psi(\eta)]_{\mathcal{U}}$ are the same and so $M^{\mathcal{T}}_\zeta = M^{\mathcal{U}}_{\zeta^*}$ since this is the first model along these branches containing the domain of $E^{\mathcal{T}}_{u^\Phi(\xi)} = E^{\mathcal{U}}_{u^\Psi(\xi)}$. Letting P be the initial segment of this model to which we apply this extender in both $\mathcal{T}$ and $\mathcal{U}$, we have that $M^{\mathcal{T}}_{v^\Phi(\xi+1)} = Ult(P, E^{\mathcal{T}}_{u^\Phi(\xi)}) = Ult(P, E^{\mathcal{U}}_{u^\Psi(\xi)}) = M^{\mathcal{U}}_{v^\Psi(\xi+1)}$. We also get $s^\Phi_{\xi+1} = s^\Psi_{\xi+1}$ since they are just the copy map associated to the same objects. Since $v^\Phi(\xi+1) \leq_{\mathcal{T}} u^\Phi(\xi+1)$, $v^\Psi(\xi+1) \leq_{\mathcal{U}} u^\Psi(\xi+1)$, and $e^{\mathcal{T}}_{0,u^\Phi(\xi+1)} = e^{\mathcal{U}}_{0,u^\Psi(\xi+1)}$, we also get that $e^{\mathcal{T}}_{0,v^\Phi(\xi+1)} = e^{\mathcal{U}}_{0,v^\Psi(\xi+1)}$ and $e^{\mathcal{T}}_{v^\Phi(\xi+1),u^\Phi(\xi+1)} = e^{\mathcal{U}}_{v^\Psi(\xi+1),u^\Psi(\xi+1)}$ (like before, using here that $M^{\mathcal{T}}_{v^\Phi(\xi+1)} = M^{\mathcal{U}}_{v^\Psi(\xi+1)}$). But then $\hat{\imath}^{\mathcal{T}}_{v^\Phi(\xi+1),u^\Phi(\xi+1)} = \hat{\imath}^{\mathcal{U}}_{v^\Psi(\xi+1),u^\Psi(\xi+1)}$ and so $t^\Phi_{\xi+1} = t^\Psi_{\xi+1}$, too. This finishes the successor case.

At limit λ, since the v-maps are continuous and tree-order preserving and the s-maps commute with branch embeddings, we get $M^{\mathcal{T}}_{v^\Phi(\lambda)} = M^{\mathcal{U}}_{v^\Psi(\lambda)}$ and $s^\Phi_\lambda = s^\Psi_\lambda$ for free. We then get $t^\Phi_\lambda = t^\Psi_\lambda$ just as in the successor case. □

Lemma 3.1.26. *Suppose $\bar{\Psi} = \mathcal{S} \to \bar{\mathcal{T}}$ and $\Psi : \mathcal{S} \to \mathcal{T}$ are inflationary extended tree embeddings such that $\bar{\Psi} \equiv \Psi$. Suppose F is on the sequence of the last models of $\bar{\mathcal{U}}$ and $\mathcal{U}$. Let $\bar{\beta} = \beta(\bar{\mathcal{U}}, F)$ and $\beta = \beta(\mathcal{U}, F)$. Suppose that $W(\bar{\mathcal{T}}, \bar{\mathcal{U}}, F)$ and $W(\mathcal{T}, \mathcal{U}, F)$ are defined and well-founded, $[\bar{\beta}, lh(\bar{\mathcal{T}})) \subseteq ran(u^{\bar{\Psi}})$, and $[\beta, lh(\mathcal{T})) \subseteq ran(u^\Psi)$.*

Then $\Phi^{\bar{\mathcal{T}},\bar{\mathcal{U}},F} \circ \bar{\Psi} \equiv \Phi^{\mathcal{T},\mathcal{U},F} \circ \Psi$.

Proof. Let $\bar{\mathcal{W}} = W(\bar{\mathcal{T}}, \bar{\mathcal{U}}, F)$, $\mathcal{W} = W(\mathcal{T}, \mathcal{U}, F)$, $\bar{\Phi} = \Phi^{\bar{\mathcal{T}},\bar{\mathcal{U}},F}$, and $\Phi = \Phi^{\mathcal{T},\mathcal{U},F}$.

By our hypotheses that $\bar{\beta}$ and β are in the ranges of $u^{\bar{\Psi}}$ and u^Ψ, respectively, let $\bar{\xi}$ such that $\bar{\beta} = u^{\bar{\Psi}}(\bar{\xi})$ and ξ such that $\beta = u^\Psi(\xi)$. We actually have that $\bar{\xi} = \xi$, since for any ζ, $E^{\bar{\mathcal{T}}}_{u^{\bar{\Psi}}(\zeta)} = E^{\mathcal{T}}_{u^\Psi(\zeta)}$, so $\mathrm{crit}(F) < \lambda(E^{\bar{\mathcal{T}}}_{u^{\bar{\Psi}}(\zeta)})$ iff $\mathrm{crit}(F) < \lambda(E^{\mathcal{T}}_{u^\Psi(\zeta)})$. Since $\bar{\Psi} \equiv \Psi$ and $\bar{\Phi} \restriction \bar{\beta} + 1$ and $\Phi \restriction \beta + 1$ are the identity, we get that for $\zeta < \xi$, $e^{\bar{\mathcal{W}}}_{0,u^{\bar{\Phi}\circ\bar{\Psi}}(\zeta)} = e^{\mathcal{W}}_{0,u^{\Phi\circ\Psi}(\zeta)}$, since they're equal to $e^{\bar{\mathcal{T}}}_{0,u^{\bar{\Psi}}(\zeta)} = e^{\mathcal{T}}_{0,u^\Psi(\zeta)}$.

Now since $\bar{\beta} = u^{\bar{\Psi}}(\bar{\xi}) = u^\Psi(\xi)$, we have $M^{\bar{\mathcal{T}}}_{\bar{\beta}} = M^{\bar{\mathcal{T}}}_{\bar{\beta}}$ and $e^{\bar{\mathcal{T}}}_{0,\bar{\beta}} = e^{\mathcal{T}}_{0,\beta}$. So let P be the level of this model to which we apply F in both $\bar{\mathcal{W}}$ and $\mathcal{W}$. Then $M^{\bar{\mathcal{W}}}_{\alpha+1} = M^{\mathcal{W}}_{\alpha+1} = Ult(P, F)$ and $u^{\bar{\Phi}\circ\bar{\Psi}}(\xi) = \alpha + 1$, $u^{\Phi\circ\Psi}(\xi) = \alpha^* + 1$. So $e^{\bar{\mathcal{W}}}_{0,u^{\bar{\Phi}\circ\bar{\Psi}}(\xi)} = e^{\mathcal{W}}_{0,u^{\Phi\circ\Psi}(\xi)}$, since both are just $e^{\bar{\mathcal{T}}}_{0,\bar{\beta}} = e^{\mathcal{T}}_{0,\beta}$ followed by F. If either of $W(\bar{\mathcal{T}}, \bar{\mathcal{U}}, F)$ or $W(\mathcal{T}, \mathcal{U}, F)$ is in the dropping case, both are and we're done. So suppose that neither is in the dropping case, so $\bar{\Phi} \circ \bar{\Psi}$ and $\Phi \circ \Psi$ are total extended tree embeddings. We prove $\bar{\Phi} \circ \bar{\Psi} \restriction \zeta + 1 \equiv \Phi \circ \Psi \restriction \zeta + 1$ by induction on $\zeta < \mathrm{lh}(\bar{\mathcal{S}})$. Note that we've already established this for $\zeta \leq \xi$. Notice that the hypotheses $[\bar{\beta}, \mathrm{lh}(\bar{\mathcal{T}})) \subseteq \mathrm{ran}(u^{\bar{\Psi}})$ and $[\beta, \mathrm{lh}(\mathcal{T})) \subseteq \mathrm{ran}(u^\Psi)$ imply that $v^{\bar{\Psi}}(\zeta) = u^{\bar{\Psi}}(\zeta)$ and $v^\Psi(\zeta) = u^\Psi(\zeta)$ for every $\zeta > \xi$. So we just need to show that $e^{\bar{\mathcal{W}}}_{0,v^{\bar{\Phi}\circ\bar{\Psi}}(\zeta)} = e^{\mathcal{W}}_{0,v^{\Phi\circ\Psi}(\zeta)}$ for $\zeta > \xi$. These hypothesis easily carry through limit ordinals, so we just need to handle successors.

Suppose we're at a successor $\zeta + 1 > \xi$ and let $\chi = \bar{\mathcal{S}}\text{-pred}(\zeta + 1)$, $\bar{\eta} = \bar{\mathcal{T}}\text{-pred}(u^{\bar{\Psi}}(\zeta) + 1)$, and $\eta = \bar{\mathcal{T}}\text{-pred}(u^\Psi(\zeta) + 1)$. We have $\bar{\eta} \in [v^{\bar{\Psi}}(\chi), u^{\bar{\Psi}}(\chi)]_{\bar{\mathcal{T}}}$ and $\eta \in [v^\Psi(\chi), u^\Psi(\chi)]_{\mathcal{T}}$, so we get $M^{\bar{\mathcal{T}}}_{\bar{\eta}} = M^{\mathcal{T}}_\eta$ and $\hat{\imath}^{\bar{\mathcal{T}}}_{v^{\bar{\Psi}}(\chi),\bar{\eta}} = \hat{\imath}^{\mathcal{T}}_{v^\Psi(\chi),\eta}$ (since $e^{\bar{\mathcal{T}}}_{v^{\bar{\Psi}}(\chi),u^{\bar{\Psi}}(\chi)} = e^{\mathcal{T}}_{v^\Psi(\chi),u^\Psi(\chi)}$). By our induction hypothesis at ζ, we have (by Lemma 3.1.25) that $t^{\bar{\Phi}\circ\bar{\Psi}}_\zeta = t^{\Phi\circ\Psi}_\zeta$, and so $E^{\bar{\mathcal{W}}}_{u^{\bar{\Phi}\circ\bar{\Psi}}(\zeta)} = E^{\mathcal{W}}_{u^{\Phi\circ\Psi}(\zeta)}$. Letting $\bar{\gamma} = \bar{\mathcal{W}}\text{-pred}(u^{\bar{\Phi}\circ\bar{\Psi}}(\zeta) + 1)$ and $\gamma = \mathcal{W}\text{-pred}(u^{\Phi\circ\Psi}(\zeta) + 1)$, we have that $\bar{\gamma} \in [v^{\bar{\Phi}\circ\bar{\Psi}}(\chi), u^{\bar{\Phi}\circ\bar{\Psi}}(\chi)]_{\bar{\mathcal{W}}}$ and $\gamma \in [v^{\Phi\circ\Psi}(\chi), u^{\Phi\circ\Psi}(\chi)]_{\mathcal{W}}$, using our induction hypothesis (i.e. that $e^{\bar{\mathcal{W}}}_{0,u^{\bar{\Phi}\circ\bar{\Psi}}(\chi)} = e^{\mathcal{W}}_{0,u^{\Phi\circ\Psi}(\chi)}$)

and normality, we have $M^{\bar{\mathcal{W}}}_{\bar{\gamma}} = M^{\mathcal{W}}_{\gamma}$. Letting $P \trianglelefteq M^{\bar{\mathcal{W}}}_{\bar{\gamma}} = M^{\mathcal{W}}_{\gamma}$ be the initial segment to which we apply $E^{\bar{\mathcal{W}}}_{u^{\bar{\Phi}\circ\bar{\Psi}}(\zeta)} = E^{\mathcal{W}}_{u^{\Phi\circ\Psi}(\zeta)}$ in both $\bar{\mathcal{W}}$ and $\mathcal{W}$, we get that both $M^{\bar{\mathcal{W}}}_{v^{\bar{\Phi}\circ\bar{\Psi}}(\zeta+1)}$ and $M^{\mathcal{W}}_{v^{\Phi\circ\Psi}(\zeta+1)}$ are both just the ultrapower of P by $E^{\bar{\mathcal{W}}}_{u^{\bar{\Phi}\circ\bar{\Psi}}(\zeta)}$ and so $e^{\bar{\mathcal{W}}}_{0,v^{\bar{\Phi}\circ\bar{\Psi}}(\zeta+1)}$ and $e^{\mathcal{W}}_{0,v^{\Phi\circ\Psi}(\zeta+1)}$ are both just $e^{\bar{\mathcal{W}}}_{0,\bar{\gamma}} = e^{\mathcal{W}}_{0,\gamma}$ followed by $E^{\mathcal{W}}_{u^{\Phi\circ\Psi}(\zeta)}$. This finishes the successor case and the proof. □

We can now verify our characterization of the relation $\Phi \equiv \Psi$ for inflationary extendend tree embeddings Φ, Ψ.

Proposition 3.1.27. *Suppose $\Phi : \mathcal{S} \to \mathcal{T}$ and $\Psi : \mathcal{S} \to \mathcal{U}$ are inflationary extended tree embeddings. Then $\Phi \equiv \Psi$ iff Φ and Ψ have the same factorizations.*

Proof. Let $\langle F_\xi \mid \xi < \lambda\rangle$ be the factorization of Φ and $\langle G_\xi \mid \xi < \gamma\rangle$ the factorization of Ψ. If $\langle F_\xi \mid \xi < \lambda\rangle = \langle G_\xi \mid \xi < \gamma\rangle$, then applying Lemma 3.1.26 to each of the factor tree embeddings easily gives $\Psi \equiv \Phi$ (the lemma explicitly handles the successor step, but limit stages are straightforward).

For the converse, suppose $\Phi \equiv \Psi$. Towards a contradiction, suppose $\langle F_\xi \mid \xi < \lambda\rangle \neq \langle G_\xi \mid \xi < \gamma\rangle$. First we observe that we can't have that one of these factorizations is an initial segment of the other. Without loss of generality, suppose $\gamma < \lambda$ and $G_\xi = F_\xi$ for all $\xi < \gamma$. Let Φ^* and Γ be the inflationary extended tree embeddings such that $\Phi = \Gamma \circ \Phi^*$ and Φ^* has factorization $\langle F_\xi \mid \xi < \gamma\rangle$. By the previous direction, we have $\Psi \equiv \Phi^*$. Let $\beta = \beta(\mathcal{T}, F_\gamma)$, so $\beta = \mathrm{crit}(u^\Gamma)$. By Proposition 3.1.21 (and since $\mathrm{crit}(u^\Gamma) \geq \delta(\Phi^*)$ by the proof of Proposition 3.1.19, as $\Gamma = \Gamma_\gamma$, using the notation of that proof), $\beta \in \mathrm{ran}(u^{\Phi^*})$, say $\beta = u^{\Phi^*}(\xi)$. Then $t^{\Phi^*}_\xi \neq t^{\Phi}_\xi$, since $t^{\Gamma}_{u^{\Phi^*}(\xi)} \neq id$. But $t^{\Phi^*}_\xi = t^{\Psi}_\xi = t^{\Phi}_\xi$, using Lemma 3.1.25, a contradiction.

Now let ξ least such that $F_\xi \neq G_\xi$ (we just showed that there is such a $\xi < \gamma, \lambda$). Let $\beta = \beta(\mathcal{T}, F_\xi)$ and $\beta' = \beta(\mathcal{U}, G_\xi)$. As in the preceeding argument, we decompose Φ as $\Gamma \circ \Phi^*$ and Ψ as $\Delta \circ \Psi^*$, where Φ^* and Ψ^* have factorization $\langle F_\eta \mid \eta < \xi\rangle = \langle G_\eta \mid \eta < \xi\rangle$. By the previous direction of the proposition, we have $\Phi^* \equiv \Psi^*$. We also have that $\beta \in \mathrm{ran}(u^{\Phi^*})$ and $\beta' \in \mathrm{ran}(u^{\Psi^*})$, so let η such that $u^{\Phi^*}(\eta) = \beta$ and ζ such that $u^{\Psi^*}(\zeta) = \beta'$. First notice that we must have that $\eta = \zeta$ since if, for example, $\eta < \zeta$, then $u^{\Psi^*}(\eta) < \beta'$ and so $t^{\Psi}_\eta = t^{\Psi^*}_\eta = t^{\Phi^*}_\eta$ (using that $\Phi^* \equiv \Psi^*$ and Lemma 3.1.25). But since $\Phi \equiv \Psi$, using Lemma 3.1.25, gives $t^{\Psi}_\eta = t^{\Phi}_\eta \neq t^{\Phi^*}_\eta$, since $u^{\Phi^*}(\eta)$ is the critical point of u^Γ, a contradiction. Now, since $u^{\Phi^*}(\eta) = \mathrm{crit}(u^\Gamma)$, and $u^{\Psi}(\eta) = \mathrm{crit}(u^\Delta)$, we also have that F_ξ is the first extender used in $(u^{\Phi^*}(\eta), u^{\Phi}(\eta)]_{\mathcal{T}}$ and G_ξ is the first extender used in $(u^{\Psi^*}(\eta), u^{\Psi}(\eta)]_{\mathcal{U}}$, by the definition of the factorization. But then since $\Phi \equiv \Psi$ and $\Phi^* \equiv \Psi^*$, we have that $F_\xi = G_\xi$, a contradiction. □

We now briefly describe the embedding normalization of a stack of normal trees $\langle \mathcal{T}, \mathcal{U}\rangle$, $W(\mathcal{T}, \mathcal{U})$, which we get as the last normal tree in a system

$$\langle \mathcal{W}_\xi, \sigma_\xi, F_\zeta, \Phi^{\eta,\xi} \mid \eta, \xi, \zeta + 1 < \mathrm{lh}(\mathcal{U}),\, \eta \leq_{\mathcal{U}} \xi\rangle,$$

which we define by induction on $\mathrm{lh}(\mathcal{U})$, where

1. $\mathcal{W}_\xi = W(\mathcal{T}, \mathcal{U}\restriction \xi + 1)$;

2. $\sigma_\xi : M^{\mathcal{U}}_\xi \to M^{\mathcal{W}_\xi}_\infty$ is weakly elementary and if $\xi + 1 < \mathrm{lh}(\mathcal{U})$, $F_\xi = \sigma_\xi(E^{\mathcal{U}}_\xi)$;

3. For $\zeta \leq_{\mathcal{U}} \eta \leq_{\mathcal{U}} \xi$,
   (a) $\Phi^{\eta,\xi} : \mathcal{W}_\eta \to \mathcal{W}_\xi$ is a partial extended tree embedding,
   (b) $\Phi^{\zeta,\xi} = \Phi^{\eta,\xi} \circ \Phi^{\zeta,\eta}$, and
   (c) $\sigma_\xi \circ \hat{\imath}^{\mathcal{U}}_{\eta,\xi} = t^{\Phi^{\eta,\xi}}_\infty \circ \sigma_\eta$;

4. For $\eta = \mathcal{U}\text{-pred}(\xi + 1)$,
   (a) $\mathcal{W}_{\xi+1} = W(\mathcal{W}_\eta, \mathcal{W}_\xi, F_\xi)$,
   (b) $\Phi^{\eta,\xi+1} = \Phi^{\mathcal{W}_\eta,\mathcal{W}_\xi,F_\xi}$ and $\sigma_{\xi+1} = \sigma^{\mathcal{W}_\eta,\mathcal{W}_\xi,F_\xi}$;

5. For $\lambda < \mathrm{lh}(\mathcal{U})$ a limit and $b = [0, \lambda)_{\mathcal{U}}$, $\mathcal{W}_\lambda = \lim\langle \mathcal{W}_\xi, \Phi^{\eta,\xi} \mid \eta \leq_{\mathcal{U}} \xi \in b\rangle$ and $\Phi^{\xi,\lambda}$ is the direct limit tree embedding.

If $\mathcal{U}$ has successor length, we let $\sigma^{\mathcal{T},\mathcal{U}}$ be the last of the σ_ξ.

Remark 3.1.28. Let us point out one subtlety here: we may have $\deg(M^{\mathcal{U}}_\xi) < \deg(M^{\mathcal{W}_\xi}_\infty)$ and so the sense in which σ_ξ is weakly elementary is really as a map from $M^{\mathcal{U}}_\xi$ into $M^{\mathcal{W}_\xi}_\infty|\langle o(M^{\mathcal{W}_\xi}_\infty), \deg(M^{\mathcal{U}}_\xi)\rangle$. This same phenomenon occurs in copying across a weakly elementary map, but let us give a simple example in this context, due to Steel.

Let M a premouse, $k = \deg(M)$, $\mathcal{T} = \langle E\rangle$ with $\mathrm{crit}(E) > \rho_k(M)$ and let $N = \mathrm{Ult}_{k-1}(M, E) = M^{\mathcal{T}}_1$. We'll explain how we can have a tree $\mathcal{U} = \langle F, G\rangle$ on N such that $\deg(M^{\mathcal{U}}_2) < \deg(M^{\mathcal{W}_2}_\infty)$, so that $\sigma_2 : M^{\mathcal{U}}_2 \to M^{\mathcal{W}_2}_\infty$ is only weakly elementary in this extended sense.

Suppose that $\mathrm{crit}(F) < \rho_k(M)$, $i^{M^-}_E$ is continuous at $\rho_{k-1}(M)$, and $\mathrm{crit}(F) = \eta^M_{k-1} = \eta^N_{k-1}$, so $i^N_F : N \to \mathrm{Ult}(N, F)$ is discontinuous at $\rho_{k-1}(N)$. So

$$\rho_{k-1}(M^U_1) < i^N_F \circ i^{M^-}_E(\rho_{k-1}(M)),$$

hence

$$\sigma_1(\rho_{k-1}(M^{\mathcal{U}}_1)) < \sigma_1 \circ i^N_F \circ i^{M^-}_E(\rho_{k-1}(M)),$$

so

$$\sigma_1(\rho_{k-1}(M^{\mathcal{U}}_1)) < i^{M^{\mathcal{U}}_1}_{i^M_F(E)} \circ i^M_F(\rho_{k-1}(M)).$$

But i^M_F preserves $\rho_{k-1}(M)$, even though it is discontinuous there, because $\mathcal{W}_1$ took the full k-ultrapower of M. So $i^{M^{\mathcal{U}}_1}_{i^M_F(E)} \circ i^M_F(\rho_{k-1}(M)) = \rho_{k-1}(M^{\mathcal{W}_1}_\infty)$ and

$$\sigma_1(\rho_{k-1}(M^{\mathcal{U}}_1)) < \rho_{k-1}(M^{\mathcal{W}_1}_\infty).$$

Now suppose G, our next extender used in $\mathcal{U}$, is such that $\rho_{k-1}(M^{\mathcal{U}}_1) \leq \mathrm{crit}(G)$, but $\sigma_1(\mathrm{crit}(G)) < \rho_{k-1}(M^{\mathcal{W}_1}_\infty)$.

Then in $\mathcal{U}$, we drop to degree $k - 2$ when applying G to $M^{\mathcal{U}}_1$, i.e. $\deg(M^{\mathcal{U}}_2) = k - 2$, but applying $\sigma_1(G)$ to $M^{\mathcal{W}_1}_\infty$ doesn't result in a drop, so $\deg(M^{\mathcal{W}_2}_\infty) = \deg(M^{\mathcal{W}_1}_\infty) = k - 1$. So we have $\deg(M^{\mathcal{U}}_2) < \deg(M^{\mathcal{W}_2}_\infty)$, as claimed.

In general, we define $W(\vec{\mathcal{S}})$ for a stack of normal iteration trees $\vec{\mathcal{S}}$ of length $n+1$ with a last model and $\sigma_{\vec{\mathcal{S}}}$ from the last model of $\vec{\mathcal{S}}$ to the last model of $W(\vec{\mathcal{S}})$ by induction on length n. For $\vec{\mathcal{S}} = \langle \mathcal{S}_0, \dots \mathcal{S}_{n-1}\rangle$ we put

$$W(\vec{\mathcal{S}}) = W(W(\vec{\mathcal{S}} \restriction (n-1)), \sigma_{\vec{\mathcal{S}}\restriction(n-1)}\mathcal{S}_{n-1}),$$

and, if $\mathcal{S}_{n-1}$ has successor length, we let

$$\sigma_{\vec{\mathcal{S}}} = \sigma^{W(\vec{\mathcal{S}}\restriction(n-1)), \sigma_{\vec{\mathcal{S}}\restriction(n-1)}\mathcal{S}_{n-1}} \circ \sigma^*,$$

where σ^* is the last copy map from the last model of $\mathcal{S}_{n-1}$ to the last model of $\sigma_{\vec{\mathcal{S}}\restriction(n-1)}\mathcal{S}_{n-1}$. Here an else where, when we copy a normal tree across a map which is only weakly elementary (understood in the extended sense of the previous remark), we take the normal copied tree, not the weakly normal one, that is, if $\sigma : M \to N$ is weakly elementary and $\mathcal{S}$ is a normal tree on M, $\sigma\mathcal{S}$ is a normal tree on N (if we never reach ill-founded models). Such a copying construction is carried out in [24].

Note that this procedure can be continued for infinite stacks by taking direct limits of the resulting directed system of normal trees at limits, and then continuing as above if the direct limit is well-founded. We'll go through the details in the proof of Theorem 3.1.48, below.

Definition 3.1.29. Let M be a premouse and Σ be an (ω, θ)-iteration strategy for M. Σ *bottom-up normalizes well* iff for any $\vec{\mathcal{S}}$ by Σ with a last model P,

1. $W(\vec{\mathcal{S}})$ is by Σ and
2. letting Q be the last model of $W(\vec{\mathcal{S}})$, $\Sigma_{\vec{\mathcal{S}},P} = (\Sigma_{W(\vec{\mathcal{S}}),Q})^{\sigma_{\vec{\mathcal{S}}}}$.

Bottom-up normalizing well makes sense for strategies for infinite stacks as well, once we have defined $W(\vec{\mathcal{S}})$ and $\sigma_{\vec{\mathcal{S}}}$ for an infinite stack (as we do in the proof of Theorem 3.1.48, below).

Definition 3.1.30. Let M be a premouse and Σ a (κ, θ)-iteration strategy for M.[7] iff whenever $\mathcal{T}$ is a normal tree by Σ and $\mathcal{S}$ is a normal tree such that there is a tree embedding $\Phi : \mathcal{S} \to \mathcal{T}$, then $\mathcal{S}$ is by Σ.

Remark 3.1.31. Bottom-up normalizes well and SHC^- are variants of normalizing well and strong hull condensation from [24]. We discuss this a bit more at the end of the chapter, but one difference between our notions is that Steel demands that all the different possible ways of normalizing a stack are by Σ (for example, we must have both $W(W(\mathcal{S}, \mathcal{T}), \sigma\mathcal{U})$ and $W(\mathcal{S}, W(\mathcal{T}, \mathcal{U}))$ are by Σ) and that all tails of an (ω, θ)-strategy Σ have SHC^-. We will prove that these properties follow from our definitions.

The following lemma is implicit in [24] §2.7.

[7] A (κ, θ)-iteration strategy for a premouse M is a strategy for building stacks of length $< \kappa$ consisting of normal trees of length $< \theta$, except we allow the possibility that the last tree in the stack has length θ. A θ-iteration strategy is a strategy for building single normal trees of length $\leq \theta$.

Lemma 3.1.32. *Let M be a premouse, θ a regular cardinal, and Σ a θ-iteration strategy for M with SHC$^-$. Let $\mathcal{T}$ be a normal tree on M by Σ of successor length. Suppose that $\mathcal{U}$ is a normal tree on $M_\infty^{\mathcal{T}}$ with lh$(\mathcal{U}) < \theta$ a limit ordinal such that for all $\xi <$ lh$(\mathcal{U})$, $\mathcal{W}_\xi = W(\mathcal{T}, \mathcal{U}\restriction\xi+1)$ is by Σ.*

Then there is a unique branch b of $\mathcal{U}$ such that $\lim\langle \mathcal{W}_\xi, \Phi^{\eta,\xi} \mid \eta \leq_{\mathcal{U}} \xi \in b\rangle$ is by Σ.

Theorem 3.1.33 (Schlutzenberg, Steel)**.** *Let M be a premouse, θ a regular cardinal, and Σ a θ-iteration strategy for M with SHC$^-$. Then there is a unique extension of Σ to an (ω,θ)-iteration strategy for M which bottom-up normalizes well.*

Proof sketch. We define the extension of Σ to stacks by induction: suppose we have $\vec{\mathcal{S}}$ is a finite stack with last model P and $\mathcal{U}$ a tree of limit length on P such that $\vec{\mathcal{S}}^\frown\langle\mathcal{U}\rangle$ is by Σ. By the previous lemma, we let $\Sigma(\vec{\mathcal{S}}^\frown\langle\mathcal{U}\rangle)$ be the unique branch b of $\mathcal{U}$ such that $W(W(\vec{\mathcal{S}}), \sigma_{\vec{\mathcal{S}}}\mathcal{U}^\frown b)$ is by Σ. By construction, this strategy bottom-up normalizes well and clearly is the unique such strategy.

□

In [19], Schlutzenberg shows that we can actually extend Σ to a $(\theta, \theta+1)$-strategy if we start with Σ a $(\theta+1)$-strategy (with SHC$^-$). We'll prove this result in the next section.

We mention one more useful result of Steel from [24]: embedding normalization commutes with copying.

Theorem 3.1.34 (Steel)**.** *Let $\pi : M \to N$ be elementary and $\vec{\mathcal{S}}$ be a finite stack of normal trees on M such that $\pi\vec{\mathcal{S}}$ is well-founded. Suppose that $W(\pi\vec{\mathcal{S}})$ is well-founded.*

Then $W(\vec{\mathcal{S}})$ is well-founded and $\pi W(\vec{\mathcal{S}}) = W(\pi\vec{\mathcal{S}})$.

3.1.3 Meta-iteration trees

The embedding normalization process produces a kind of tree of iteration trees with tree embeddings between tree-order related nodes. This perspective is used in [24] and abstracted in [19], [9] to their notions of "factor trees of inflations" and "insertion iterations", respectively. Here, we also isolate an abstraction of this kind of tree of iteration trees which we'll call *meta-iteration trees*, or *meta-trees*.

Definition 3.1.35. A *meta-iteration tree* (or *meta-tree*) is a system

$$\mathbb{S} = \langle \{\mathcal{S}_\xi\}_{\xi<\mathrm{lh}(\mathbb{S})}, \{F_\xi\}_{\xi+1<\mathrm{lh}(\mathbb{S})}, \{\Phi^{\eta,\xi}\}_{\eta\leq_{\mathbb{S}}\xi}\rangle$$

such that

1. lh$(\mathbb{S})$ is an ordinal and $\leq_{\mathbb{S}}$ is a tree-order on lh$(\mathbb{S})$;
2. for all $\zeta \leq_{\mathbb{S}} \eta \leq_{\mathbb{S}} \xi <$ lh$(\mathbb{S})$,
   (a) $\mathcal{S}_\xi$ has a last model,
   (b) if $\xi + 1 <$ lh$(\mathbb{S})$, F_ξ is an extender on the last model of $\mathcal{S}_\xi$,
   (c) $\Phi^{\eta,\xi}$ is a partial extended tree embedding from $\mathcal{S}_\eta$ into $\mathcal{S}_\xi$

(d) $\Phi^{\zeta,\xi} = \Phi^{\eta,\xi} \circ \Phi^{\zeta,\eta}$;

3. (Normality) for $\xi + 1 < \mathrm{lh}(\mathbb{S})$, letting $\alpha_\xi = \alpha(\mathcal{S}_\xi, F_\xi)$,

   (a) for all $\eta < \xi$, $\mathrm{lh}(F_\eta) < \mathrm{lh}(F_\xi)$,

   (b) for $\eta = \mathbb{S}\text{-pred}(\xi + 1)$,

      i. η is least such that $\mathrm{crit}(F_\xi) < \lambda(F_\eta)$,

      ii. $\mathcal{S}_{\xi+1} = W(\mathcal{S}_\eta, \mathcal{S}_\xi, F_\xi)$,

      iii. $\Phi^{\eta,\xi+1} = \Phi^{\mathcal{S}_\eta,\mathcal{S}_\xi,F_\xi}$ (as a partial extended tree embedding from $\mathcal{S}_\eta$ into $\mathcal{S}_{\xi+1}$);

4. for $\lambda < \mathrm{lh}(\mathbb{S})$, $b = [0, \lambda)_\mathbb{S}$ is a cofinal subset of λ and there is a tail $c \subseteq b$ such that

   (a) for all $\eta, \xi \in c$ with $\eta \leq \xi$, $\Phi^{\eta,\xi}$ is total,

   (b) $\mathcal{S}_\lambda = \lim\langle \mathcal{S}_\xi, \Phi^{\eta,\xi} \mid \xi \leq_\mathbb{S} \xi \in c\rangle$

   (c) for all $\eta \in c$, $\Phi^{\eta,\lambda}$ is the direct limit extended tree embedding and for $\xi \in b \setminus c$, $\Phi^{\xi,\lambda} = \Phi^{\eta,\lambda} \circ \Phi^{\xi,\eta}$ where $\eta = \min c$.

For a meta-tree $\mathbb{S}$ and a branch b of $\mathbb{S}\restriction\gamma$, we let

$$\lim_b(\mathbb{S}\restriction\gamma) = \lim\langle \mathcal{S}_\xi, \Phi^{\eta,\xi} \mid \eta \leq_\mathbb{S} \xi \in c\rangle,$$

where c is any tail of b where the $\Phi^{\eta,\xi}$ are total (if there is such a c).

A meta-tree $\mathbb{S}$ *drops along* $(\eta, \xi]_\mathbb{S}$ iff there is some successor $\gamma + 1 \in (\eta, \xi]_\mathbb{T}$ such that for $\zeta = \mathbb{S}\text{-pred}(\gamma+1)$ and $W(\mathcal{S}_\zeta, \mathcal{S}_\gamma, F_\gamma)$ is in the dropping case. We'll use other natural variants of this terminology (e.g. "η-to-ξ drops in $\mathbb{S}$") with their obvious meaning.

When $\mathbb{S}$ has successor length $\xi + 1$, we write $\Phi^\mathbb{S}$ for the main branch partial extended tree embedding $\Phi^{0,\xi}$.

Using results from the previous subsection, it is easy to see that the tree embeddings $\Phi^{\eta,\xi}$ of a meta-tree are inflationary and that the length-increasing enumeration of $\langle F_\zeta \mid \zeta + 1 \in (\eta, \xi]_\mathbb{S}\rangle$ is the factorization of $\Phi^{\eta,\xi}$.

We now give a couple examples of meta-trees. We start with a very familiar example: ordinary normal iteration trees can be viewed as meta-trees with the same tree-order and exit extenders.

Example 3.1.36. Let $\mathcal{T}$ be an iteration tree. Let $\mathcal{T}_\xi = \mathcal{T} \restriction \xi + 1$ and $F_\xi = E^\mathcal{T}_\xi$ and for $\eta = \mathcal{T}\text{-pred}(\xi + 1)$. Then

$$\mathbb{T} = \langle \mathcal{T}_\xi, \Phi_{\xi,\eta}, F_\zeta \mid \xi, \eta, \zeta + 1 < \mathrm{lh}(\mathcal{T}),\ \xi \leq_T \eta\rangle$$

is a meta-tree with underlying tree structure $\mathcal{T}$ (i.e. $\mathrm{lh}(\mathbb{T}) = \mathrm{lh}(\mathcal{T})$ and $\leq_\mathbb{T}=\leq_\mathcal{T}$).

This is a meta-tree since for $\eta = \mathcal{T}\text{-pred}(\xi + 1)$, we have

$$\begin{aligned}\mathcal{T}_{\xi+1} &= \mathcal{T}_\xi{}^\frown\langle E^\mathcal{T}_\xi\rangle \\ &= W(\mathcal{T}_\xi, \mathcal{T}_\eta, F_\xi).\end{aligned}$$

Notice that we haven't explicitly defined the tree embeddings of $\mathbb{T}$, As remarked above, they are uniquely determined by the extenders F_ζ and the tree order $\leq_{\mathcal{T}}$. One can check that for $\eta = \mathcal{T}\text{-pred}(\xi+1)$, the (extended) tree embedding $\Phi^{\eta,\xi+1}$ is just the unique extended tree embedding with associated u-map given by

$$u(\zeta) = \begin{cases} \zeta & \text{if } \zeta < \eta, \\ \xi+1 & \text{if } \zeta = \eta. \end{cases}$$

The more important example of meta-trees comes from embedding normalization.

Example 3.1.37. Using our notation above for the embedding normalization of a stack $\langle \mathcal{T}, \mathcal{U} \rangle$,

$$\mathbb{W}(\mathcal{T}, \mathcal{U}) = \langle \mathcal{W}_\xi, \Phi^{\xi,\eta}, F_\zeta \mid \xi, \eta, \zeta + 1 < \text{lh}(\mathcal{U}),\ \xi \leq_{\mathcal{U}} \eta \rangle$$

is a meta-tree with underlying tree structure $\mathcal{U}$.

Definition 3.1.38. A *stack of meta-trees* is a sequence $\langle \mathbb{S}^\xi \mid \xi < \gamma \rangle$ of meta-trees such that for all $\xi + 1 < \gamma$, $\mathbb{S}^\xi$ has successor length and the first tree of $\mathbb{S}^{\xi+1}$ is the last tree of $\mathbb{S}^\xi$ and at limit $\lambda < \gamma$, the first tree of $\mathbb{S}^\lambda$ is the direct limit of the directed system of normal trees generated by the trees of $\mathbb{S}^\xi$ for $\xi < \lambda$.[8]

Definition 3.1.39. Let $\mathcal{S}$ be a normal iteration tree of successor length. A stack of meta-trees $\vec{\mathbb{S}}$ is *on* $\mathcal{S}$ when $\mathcal{S}$ is the first tree of the first meta-tree in the stack. A (κ, θ)*-meta-iteration strategy for* $\mathcal{S}$ is a strategy for building stacks of length $< \kappa$ of meta-trees of length $< \theta$ on $\mathcal{S}$, allowing the possibility that there is a final meta-tree in the stack of length θ.[9]

Remark 3.1.40. If $\mathcal{S}$ is the unique tree of length 1 on M (i.e. $\mathcal{S} = \langle M \rangle$), then meta-trees on $\mathcal{S}$ are just normal trees on M. In particular, a meta-iteration strategy for $\mathcal{S}$ is just a normal iteration strategy for M.

The following is the main theorem on the existence of strategies for stacks of meta-trees. This theorem is a generalization, due to Jensen, of Schlutzenberg's extension of Theorem 3.1.33 to infinite stacks. The arguments are due to Steel and Schlutzenberg.

Theorem 3.1.41. *Let M be a premouse, θ a regular cardinal, and Σ a $\theta + 1$ strategy for M that has SHC$^-$. Then for every normal tree $\mathcal{S}$ by Σ of successor length $< \theta$, there is a unique $(\theta, \theta + 1)$-meta-iteration strategy $\Sigma^*_{\mathcal{S}}$ for $\mathcal{S}$ such that*

$$\vec{\mathbb{S}} \textit{ is by } \Sigma^* \iff \textit{ every tree in every meta-tree of } \vec{\mathbb{S}} \textit{ is by } \Sigma.$$

We'll give a proof of this theorem in this chapter. To start, we generalize Lemma 3.1.32.

[8] Here we mean the directed system of normal trees whose trees are the trees of the $\mathbb{S}^\xi$ and tree embeddings are generated from the tree embeddings of the $\mathbb{S}^\xi$ by closing under composition and direct limits, for $\xi < \lambda$. Also, since we may drop along some of these meta-trees, we mean that there is tail of $\xi < \lambda$ such that the main branch of $\mathbb{S}^\xi$ does not drop, and we only use the meta-trees in this tail to form the system.

[9] So a strategy picks cofinal well-founded branches at limit stages in building a meta-tree in a stack and we never must stop we building a stack by the strategy because we reach an ill-founded model or drop infinitely often along an infinite stack.

Lemma 3.1.42. *Let M be a premouse, θ a regular cardinal, and Σ a $\theta+1$-iteration strategy for M with SHC$^-$. Let $\mathcal{S}$ be a normal tree on M by Σ with successor length lh$(\mathcal{S}) < \theta$ and $\mathbb{S} = \langle \mathcal{S}_\xi, F_\xi, \ldots \rangle$ be a meta-tree on $\mathcal{S}$ of limit length $\leq \theta$ such that for all $\xi <$ lh$(\mathbb{S})$, $\mathcal{S}_\xi$ is by Σ. Let $\delta = \sup\{\alpha_\xi^{\mathbb{S}} + 1 \mid \xi <$ lh$(\mathbb{S})\}$.*

Then there is a unique branch b of $\mathbb{S}$ such that $(\lim_b \mathbb{S}) \restriction \delta + 1$ is by Σ. Moreover, for this b, $\lim_b \mathbb{S}$ is well-founded and is by Σ if lh$(\mathbb{S}) < \theta$.

Proof. The existence and uniqueness of the branch b is just the same argument from [24] which gave us b in the case $\mathbb{S} = \mathbb{W}(\mathcal{S}, \mathcal{U})$ for some tree $\mathcal{U}$ in Lemma 3.1.32,[10] so we just need to show the "moreover" part of the lemma.

In the case that $\mathrm{lh}(\mathbb{S}) < \theta$, $\lim_b \mathbb{S}$ is too, and one can use SHC$^-$ to show this well-founded and by Σ (this is basically part of the proof of Lemma 3.1.32). So, we only need to deal with the case that $\mathrm{lh}(\mathbb{S}) = \theta$. We do this via an easy reflection argument.

First notice that since $\mathrm{lh}(\mathbb{S}) = \theta$, we get that $\delta = \theta$ as well. Let $\mu >> \theta$ and H transitive with $\pi : H \to V_\mu$ elementary, $|H| = \theta$, $M, \Sigma, \mathcal{S}, \mathbb{S}, b \in \mathrm{ran}(\pi)$ and for $\alpha = \mathrm{crit}(\pi)$, $\pi(\alpha) = \theta$, and $\mathrm{lh}(\mathcal{S}) < \alpha$. Let $\bar{M}$, $\bar{\Sigma}$, etc., denote the relevant preimages under π.

Let $\mathcal{T}$ be a normal tree on $\bar{M}$ in H by $\bar{\Sigma}$ of length $\leq \alpha$. Then $\pi(\mathcal{T})$ is by Σ and the copy tree $\pi\mathcal{T}$ is well-founded, the copy maps π_ξ for $\xi < \alpha$ are just restrictions of π, and $\pi\mathcal{T} = \pi(\mathcal{T}) \restriction \mathrm{lh}(\mathcal{T})$. In particular, for all $\xi < \alpha$, $\mathrm{lh}(\bar{\mathcal{S}}_\xi) < \alpha$, by elementarity, so $\mathrm{lh}(\mathcal{S}_\xi) = \mathrm{lh}(\bar{\mathcal{S}}_\xi)$ and $\pi\bar{\mathcal{S}}_\xi = \mathcal{S}_\xi$. So, letting $\pi\bar{\mathbb{S}} = \langle \pi\bar{\mathcal{S}}_\xi, \pi(\bar{F}_\xi), \ldots \rangle$, $\pi\bar{\mathbb{S}} = \mathbb{S} \restriction \alpha$. Let $\bar{b} = \pi^{-1}(b)$. Then since b was chosen so that $(\lim_b \mathbb{S}) \restriction \theta + 1$ is by Σ, we have that $(\lim_{\bar{b}} \bar{\mathbb{S}}) \restriction \alpha + 1$ is by $\bar{\Sigma}$. It follows that the copied tree $\pi[\lim_{\bar{b}}(\bar{\mathbb{S}}) \restriction \alpha + 1] = \lim_{\bar{b}} \pi\bar{\mathbb{S}} \restriction \alpha + 1 = \lim_{\bar{b}}(\mathbb{S} \restriction \alpha + 1) \restriction \alpha + 1$ is by Σ. Now, in V (by the lemma in the case $\mathrm{lh}(\mathbb{S}) < \theta$), letting $c = [0, \alpha)_{\mathbb{S}}$, we have that c is the unique branch of $\mathbb{S} \restriction \alpha$ such that $\lim_c(\mathbb{S} \restriction \alpha) \restriction \alpha + 1$ is by Σ, so $\bar{b} = c$. So the full $\lim_{\bar{b}}(\mathbb{S} \restriction \alpha)$ is by Σ, and so well-founded. But then $\lim_{\bar{b}} \bar{\mathbb{S}}$ is well-founded and so well-founded in H, too, since this is absolute. By elementarity, $\lim_b \mathbb{S}$ is well-founded, as desired. □

Notice that if we started with Σ a θ-strategy and $\mathbb{S}$ of length $< \theta$ with the properties in the hypothesis of the lemma, the conclusion still holds.

As a corollary, we get that Σ generates an $(\omega, \theta+1)$-meta-iteration strategy for $\mathcal{S}$ for normal trees $\mathcal{S}$ on M by Σ of successor length $< \theta$.

Lemma 3.1.43. *Let M be a premouse, θ a regular cardinal, and Σ a $\theta+1$-strategy for M with SHC$^-$. Then for any $\mathcal{S}$ by Σ of successor length $< \theta$, there is a unique $(\omega, \theta+1)$-meta-strategy for $\mathcal{S}$ such that*

$$\vec{\mathbb{S}} \textit{ is by } \Sigma^*_{\mathcal{S}} \Leftrightarrow \textit{ for every } i < \mathit{lh}(\mathbb{S}) \textit{ and } \xi < \mathit{lh}(\mathbb{S}^i),\ \mathcal{S}^i_\xi \restriction \theta + 1 \textit{ is by } \Sigma.$$ [11]

Proof. This is an easy induction using the previous lemma at limit stages and SHC$^-$ at successor stages. □

Note that if we started with Σ a θ-strategy for M, we would still get an (ω, θ) strategy $\Sigma^*_{\mathcal{S}}$ for normal trees $\mathcal{S}$ by Σ of successor length. For M, Σ, $\mathcal{S}$ as in the hypothesis of the previous lemma, we let Σ^* be the union of the $\Sigma^*_{\mathcal{S}}$ from the conclusion.

[10]See [9] for an explicit proof.

[11]If $\mathrm{lh}(\mathbb{S}^i) < \theta$, $\mathrm{lh}(\mathcal{S}^i_\xi) < \theta$, too, so this just means that $\mathcal{S}^i_\xi$ is by Σ.

We need a few more results about meta-trees. The most involved is the following result due to Schlutzenberg and the author, whose proof we postpone till the last section.[12]

Theorem 3.1.44 (Schlutzenberg, Siskind)**.** *Let M be a premouse, θ a regular cardinal, and Σ a $\theta+1$-iteration strategy for M with SHC$^-$. Let $\mathcal{S}$ be a tree by Σ of successor length $<\theta$ and $\langle \mathbb{S}, \mathbb{T}\rangle$ is a stack of meta-trees on $\mathcal{S}$ by $\Sigma^*_{\mathcal{S}}$ with last tree $\mathcal{U}$. Then there is a meta-tree $\mathbb{U}$ on $\mathcal{S}$ by $\Sigma^*_{\mathcal{S}}$ with last tree $\mathcal{U}$ and $\Phi^{\mathbb{U}} = \Phi^{\mathbb{T}} \circ \Phi^{\mathbb{S}}$. Moreover, $\mathcal{S}$-to-$\mathcal{U}$ doesn't drop in $\mathbb{U}$ iff $\mathcal{S}$-to-$\mathcal{U}$ doesn't drop in $\langle \mathbb{S}, \mathbb{T}\rangle$.*

This theorem looks like full-normalization for stacks of meta-trees. We'll come back to this perspective in §3.2, where we prove the theorem.

Lemma 3.1.45. *Let M be a premouse, θ a regular cardinal, and Σ a θ-iteration strategy for M with SHC$^-$. Let $\mathcal{S}$ be a normal tree by Σ of successor length and $\mathbb{S}$, $\mathbb{T}$ meta-trees on $\mathcal{S}$ by Σ with the same last tree $\mathcal{T}$. Then $\mathbb{S} = \mathbb{T}$.*

Proof. The proof is really the same as the proof that there is a unique normal tree by Σ giving rise to any Σ-iterate of M.

Without loss of generality, suppose $\mathrm{lh}(\mathbb{S}) \leq \mathrm{lh}(\mathbb{T})$. We show that $\mathbb{S}\restriction\xi+1 = \mathbb{T}\restriction\xi+1$ by induction on $\mathrm{lh}(\mathbb{S})$. Since both meta-trees have last tree $\mathcal{T}$, it follows that $\mathrm{lh}(\mathbb{S}) = \mathrm{lh}(\mathbb{T})$ and $\mathbb{S} = \mathbb{T}$.

Let $\mathbb{S} = \langle \mathcal{S}_\xi, F_\xi, \ldots\rangle$ and $\mathbb{T} = \langle \mathcal{T}_\xi, G_\xi, \ldots\rangle$ (so $\mathcal{T}_0 = \mathcal{S}_0 = \mathcal{S}$ and $\mathcal{T}_\infty = \mathcal{S}_\infty = \mathcal{T}$). Suppose $\mathcal{S}_\xi = \mathcal{T}_\xi$. Suppose $\mathrm{lh}(F_\xi) < \mathrm{lh}(G_\xi)$. It follows that F_ξ is on the sequence of last model of $\mathcal{T}$. But F_ξ is used in $\mathcal{T}$ since it is used in $\mathcal{S}_\eta$ for every $\eta > \xi$, a contradiction.

The same argument shows that we can't have $\mathrm{lh}(G_\xi) < \mathrm{lh}(F_\xi)$, either. So $F_\xi = G_\xi$. So we get $\mathcal{S}_{\xi+1} = \mathcal{T}_{\xi+1}$.

At limit stages we just use that both meta-trees are by the same meta-strategy. □

The following is a comparison theorem for normal trees. It is basically a variation of a theorem of Schlutzenberg (see [19]), but discovered later and independently by the author. The proof is a fairly straightforward modification of the ordinary premouse comparison by least extender disagreement.

Theorem 3.1.46 (Tree comparison)**.** *Let M be a premouse, θ a regular cardinal, and Σ a $\theta+1$-strategy for M with SHC$^-$. Let $\kappa < \theta$ and $\{\mathcal{S}_\xi \mid \xi \in \kappa\}$ a set of normal trees by Σ of successor lengths $<\theta$. Then there is a normal tree $\mathcal{T}$ by Σ with $lh(\mathcal{T}) < \theta$ and meta-trees $\mathbb{S}^\xi$ on $\mathcal{S}_\xi$ by Σ^* with $lh(\mathbb{S}^\xi) < \theta$ each with last tree $\mathcal{T}$. Moreover, for some $\mathcal{S}_\xi$, the main branch $\mathcal{S}_\xi$-to-$\mathcal{T}$ of $\mathbb{S}^\xi$ doesn't drop.*

To prove this, we need an easy lemma about the effect of drops in a meta-tree.

Lemma 3.1.47. *Let $\mathbb{S} = \langle \mathcal{S}_\xi, F_\xi, \Phi_{\eta,\xi}\rangle$ be a meta-tree and let $\delta_\xi + 1 = lh(\mathcal{S}_\xi)$. Suppose $\eta = \mathbb{S}$-pred$(\xi+1)$ is such that $(\eta, \xi+1]_{\mathbb{S}}$ is a drop and let $\gamma \geq_{\mathbb{S}} \xi+1$. Then*

1. *$\Phi_{\eta,\gamma}$ is a total extended tree embedding from $\mathcal{S}_\eta\restriction\beta_\xi+1$ into $\mathcal{S}_\gamma$[13];*

[12] This is version of the community of inflation, which is due Schlutzenberg (see [19]). As stated, it was discovered later but independently by the author.

[13] Recall that $\Phi_{\eta,\gamma}$ is a partial extended tree embedding, in general. Here we are just saying that it has domain $\mathcal{S}_\eta\restriction\beta_\xi+1$.

2. $\beta_\xi \leq_{\mathcal{S}_\gamma} \delta_\gamma$ *and* $\hat{\imath}^{\mathcal{S}_\gamma}_{\beta_\xi,\delta_\gamma} = t^{\eta,\gamma}_{\beta_\xi}$*;*

3. in particular, if $(\eta, \xi+1]_{\mathbb{S}}$ *is the last drop along* $[0,\gamma]_{\mathbb{S}}$*, then letting* $Q = core(M^{\mathcal{S}_\gamma}_{\delta_\gamma})$*,*

(a) $t^{\eta,\gamma}_{\beta_\xi}$ *is the uncoring map from* Q^- *into* $M^{\mathcal{S}_\gamma}_{\delta_\gamma}$*, and*

(b) $Q \trianglelefteq M^{\mathcal{S}_\eta}_{\delta_\eta}$.

The main observation underlying this lemma is that, under the same hypotheses, we have

2*. the exit extenders used along $(\beta_\xi, \delta_\gamma]_{\mathcal{S}_\gamma}$ are exactly the meta-tree exit extenders used along $(\eta, \gamma]_{\mathbb{S}}$.

In our first application of this analysis of necessary drops, below, we'll use (2*) rather than the literal lemma statement.

Proof. To show (1) and (2) we proceed by induction on $\gamma \geq \xi + 1$, checking (2') as well.

For $\gamma = \xi + 1$ we're done, as $\mathcal{S}_{\xi+1} = W(\mathcal{S}_\eta, \mathcal{S}_\xi, F_\xi) = \mathcal{S}_\xi {}^\frown \langle F_\xi \rangle$ (since we have a necessary drop), so (1) holds and $\beta_\xi \leq_{\mathcal{S}_{\xi+1}} \delta_{\xi+1} = \alpha_\xi + 1$ giving us (2*) and part of (2). Moreover $\hat{\imath}^{\mathcal{S}_\gamma}_{\beta_\xi,\alpha_\xi+1}$ and $t^{\eta,\xi+1}_{\beta_\xi}$ are both just the F_ξ-ultrapower map on the initial segment of $M^{\mathcal{S}_\eta}_{\beta_\xi} = M^{\mathcal{S}_{\xi+1}}_{\beta_\xi}$ determined by normality, finishing (2).

At a successor stage $\gamma = \chi + 1 > \xi + 1$, letting $\zeta = \mathbb{S}\text{-pred}(\chi+1)$, we have that (2*) applies to ζ, i.e. the exit extenders of $[\beta_\xi, \delta_\zeta]_{\mathbb{S}}$ are all of the form F_θ for $\theta + 1 \in (\eta, \zeta]_{\mathbb{S}}$. Moreover, as F_χ doesn't overlap any of these F_θ, by the normality condition of meta-trees, F_χ must be applied to the last model of $\mathcal{S}_\zeta$, i.e. $\beta_\chi = \delta_\zeta$. It follows that $\mathcal{S}_{\chi+1} = \mathcal{S}_\chi {}^\frown \langle F_\chi \rangle$, so $\Phi_{\zeta,\chi+1}$ is total, giving (1). We also have (2*), as the main branch of $\mathcal{S}_{\chi+1}$ is just the main branch of $\mathcal{S}_\zeta$ with one more extender F_χ. As in the $\gamma = \xi + 1$ case, we have that the last t-map of $\Phi_{\zeta,\chi}$, $t^{\zeta,\chi+1}_{\beta_\xi}$, is $\hat{\imath}^{\mathcal{S}_{\chi+1}}_{\delta_\zeta,\delta_{\chi+1}}$ since they are both the appropriate F_χ-ultrapower. Using this we can check (2):

$$\begin{aligned}
t^{\eta,\chi+1}_{\beta_\xi} &= t^{\zeta,\chi+1}_{\beta_\xi} \circ t^{\eta,\zeta}_{\beta_\xi} \\
&= t^{\zeta,\chi+1}_{\beta_\xi} \circ \hat{\imath}^{\mathcal{S}_\zeta}_{\beta_\xi,\delta_\zeta} \\
&= \hat{\imath}^{\mathcal{S}_{\chi+1}}_{\delta_\zeta,\delta_{\chi+1}} \circ \hat{\imath}^{\mathcal{S}_\zeta}_{\beta_\xi,\delta_\zeta} \\
&= \hat{\imath}^{\mathcal{S}_{\chi+1}}_{\beta_\xi,\delta_{\chi+1}}.
\end{aligned}$$

The first line is just using that $\Phi_{\eta,\xi+1} = \Phi_{\zeta,\chi+1} \circ \Phi_{\eta,\zeta}$. The second uses that, by induction, (2) holds at ζ. The third just uses that $t^{\zeta,\chi+1}_{\beta_\xi} = \hat{\imath}^{\mathcal{S}_{\chi+1}}_{\delta_\zeta,\delta_{\chi+1}}$, as observed above. Finally, the last line uses that $\hat{\imath}^{\mathcal{S}_\zeta}_{\beta_\xi,\delta_\zeta} = \hat{\imath}^{\mathcal{S}_{\chi+1}}_{\beta_\xi,\delta_\zeta}$, also observed above (the main branch of of $\mathcal{S}_{\xi+1}$ is the main branch of $\mathcal{S}_\zeta$ followed by F_χ).

We leave it to the reader to check that (1),(2), (2*) pass through limits. This finishes the proof of (1), (2), and (2*).

For (3), we now assume that η-to-$\xi+1$ is the last drop along $[0,\gamma]_{\mathbb{S}}$. By definition, β_ξ-to-$\alpha_\xi + 1$ must be a drop in the tree $\mathcal{S}_\gamma$. By (2) (or maybe more clearly (2*)), any further drops along $[0, \delta_\gamma]_{\mathcal{S}_\gamma}$ would correspond to additional drops along $[0,\gamma]_{\mathbb{S}}$, so β_ξ-to-$\alpha_\xi+1$ is the last drop along $[0,\gamma]_{\mathbb{S}}$, and $t^{\eta,\gamma}_{\beta_\xi} = \hat{\imath}^{\mathcal{S}_\gamma}_{\beta_\xi,\delta_\gamma}$ is the uncoring map.

Finally, letting $Q = \operatorname{core}(M^{\mathcal{S}_\gamma}_{\delta_\gamma})$, we have that Q^- is the domain of $t^{\eta,\xi+1}_{\beta_\xi}$. First suppose $\beta_\xi = \delta_\eta$. Then $Q^- \lhd M^{\mathcal{S}_\eta}_{\delta_\eta}$, since we're in the dropping case of $W(\mathcal{S}_\eta, \mathcal{S}_\xi, F_\xi)$, so $Q \trianglelefteq M^{\mathcal{S}_\eta}_{\delta_\eta}$. Now suppose $\beta_\xi < \delta_\eta$. By the definition of the dropping case again, $Q^- \lhd M^{\mathcal{S}_\eta}_{\beta_\xi}|\mathrm{lh}(E)^{\mathcal{S}_\eta}_{\beta_\xi} = M^{\mathcal{S}_\eta}_{\delta_\eta}|\mathrm{lh}(E)^{\mathcal{S}_\eta}_{\beta_\xi}$. So again we have $Q \trianglelefteq M^{\mathcal{S}_\eta}_{\delta_\eta}$. This finishes (3)(b) and the lemma.

□

Proof of Theorem 3.1.46. We just do the case that $\theta = \omega_1$. The general case is basically the same. So fix $\{\mathcal{S}_i \mid i \in \omega\}$ a countable set of countable normal trees by Σ. We form meta-trees on $\mathcal{S}_i$ by transfinite recursion, extending meta-trees we have constructed so far by the least extender disagreement among their last models. A reflection argument shows that this process must terminate at some countable stage and our analysis of drops from Lemma 3.1.47 shows that at least one of these meta-trees doesn't drop along its main branch.

Again, this is a straightforward modification of the familiar process of comparison of premice by least extender disagreement, but the dropping analysis looks a little different; the reader may want to just skip to the last paragraph of the proof where this is done.

Here are the details. We define an increasing sequence of ordinals λ_α, extenders G_α, and, for $i < \omega$, meta-trees $\mathbb{S}^{i,\alpha}$ on $\mathcal{S}_i$ by Σ^* with last models $P^{i,\alpha}$, maintaining the following

1. for all $i < \omega$ and $\beta \leq \alpha$, $\mathbb{S}^{i,\beta} \trianglelefteq \mathbb{S}^{i,\alpha}$;
2. for all $i < \omega$ and $\beta < \alpha$, $P^{i,\beta}|\langle \lambda_\beta, -1\rangle = P^{i,\alpha}|\langle \lambda_\beta, -1\rangle$ and $P^{i,\alpha}|\langle \lambda_\beta, 0\rangle$ is extender-passive;
3. for all $i, j < \omega$, $\lambda_\alpha < o(P^{i,\alpha})$ and $P^{i,\alpha}|\langle \lambda_\alpha, -1\rangle = P^{j,\alpha}|\langle \lambda_\alpha, -1\rangle$;
4. for some $i, j < \omega$, $G_\alpha = E^{P^{i,\alpha}}_{\lambda_\alpha}$ and $P^{j,\alpha}|\langle \lambda_\alpha, 0\rangle$ is extender-passive.

Note that for j as in (4), we must have that G_α is used in $\mathcal{S}^{j,\alpha}_\infty$, since fixing an i as in (4) and letting $\xi = \alpha(\mathcal{S}^{i,\alpha}_\infty, G_\alpha)$, the minimality of λ_α implies that $M^{\mathcal{S}^{j,\alpha}_\infty}_\xi = M^{\mathcal{S}^{i,\alpha}_\infty}_\xi$, and $G_\alpha = E^{\mathcal{S}^{j,\alpha}_\infty}_\xi$ (since otherwise either G_α would be on the $P^{j,\alpha}$-sequence, a contradiction, or else there is some disagreement before $\lambda_\alpha = \mathrm{lh}(G_\alpha)$). It follows that for all $\beta < \alpha$ and all $i < \omega$, G_β is used in $\mathcal{S}^{i,\alpha}_\infty$.

Given $\{\mathbb{S}^{i,\alpha} \mid i < \omega\}$ and $\langle \lambda_\beta \mid \beta < \alpha\rangle$, $\langle G_\beta \mid \beta < \alpha\rangle$ we define λ_α, G_α, and $\mathbb{S}^{i,\alpha+1}$ for $i < \omega$ as follows. Let $\langle \lambda_\alpha, k_\alpha\rangle$ be lexicographically least such that for some i, j $P^{i,\alpha}|\langle \lambda_\alpha, k_\alpha + 1\rangle \neq P^{j,\alpha}|\langle \lambda_\alpha, k_\alpha + 1\rangle$. Since each of the $P^{i,\alpha}$ is a Σ-iterate, we must have that $k_\alpha = -1$, for all $i < \omega$, $\lambda_\alpha < o(P^{i,\alpha})$, and for some i, λ_α is the index of some extender G_α in $P^{i,\alpha}$. By the minimality of λ_α and since all of the $P^{i,\alpha}$ are Σ-iterates, G_α is the unique such extender, i.e. for all $i < \omega$, either $E^{P^{i,\alpha}}_{\lambda_\alpha} = G_\alpha$ or $P^{i,\alpha}|\langle \lambda_\alpha, 0\rangle$ is extender-passive; moreover, since $\langle \lambda_\alpha, 0\rangle$ is a disagreement, we must have for some j, $P^{j,\alpha}|\langle \lambda_\alpha, 0\rangle$ is extender-passive. By our induction hypothesis (1) and (3), we have $\mathrm{lh}(G_\alpha) > \mathrm{lh}(G_\beta)$ for all $\beta < \alpha$. So, we let

$$\mathbb{S}^{i,\alpha+1} = \begin{cases} \mathbb{S}^{i,\alpha} & \text{if } P^{i,\alpha}|\langle \lambda_\alpha, 0\rangle \text{ is extender-passive} \\ \mathbb{S}^{i,\alpha}{}^\frown\langle G_\alpha\rangle & \text{if } P^{i,\alpha}|\langle \lambda_\alpha, 0\rangle \text{ is extender-active.} \end{cases}$$

It's easy to verify that our inductive hypotheses are maintained.

Finally, at a limit stage λ, we simply let

$$\mathbb{S}^{i,\lambda} = \Sigma^*\big(\bigcup_{\alpha<\lambda} \mathbb{S}^{i,\alpha}\big),$$

which is easily seen to maintain (1).

We now check that this process must terminate at some countable stage. Suppose not, so that the process lasts $\omega_1 + 1$ stages producing meta-trees $\mathbb{S}^i = \mathbb{S}^{i,\omega_1}$, with component trees $\mathcal{S}^i_\xi = \mathcal{S}^{i,\omega_1}_\xi$ and last models $P^i = P^{i,\omega_1}$, together with ω_1-sequences $\langle \lambda_\alpha \mid \alpha < \omega_1\rangle$, $\langle G_\alpha \mid \alpha < \omega_1\rangle$.

Claim 1. *For all $i < \omega$, $lh(\mathbb{S}^i) = \omega_1 + 1$.*

Proof. It's clear that $\mathrm{lh}(\mathbb{S}^i)$ is a successor and at most $\omega_1 + 1$ (since the extenders of $\mathbb{S}^i$ are some subset of the G_α). So, towards a contradiction, suppose $\mathbb{S}^i$ is countable. Then the last tree of $\mathbb{S}^i$ must be countable. But all of the G_α are used in this last tree, a contradiction. □

A routine reflection argument will shows that some G_α must be used in all of the meta-trees $\mathbb{S}^i$, which is a contradiction.

Let μ large, H countable transitive with $\pi : H \to V_\mu$ elementary and $M, \Sigma^*, \langle \mathcal{S}_i \mid i < \omega\rangle, \langle \mathbb{S}^i \mid i < \omega\rangle \in \mathrm{ran}(\pi)$, each $\mathcal{S}_i, \mathbb{S}^i \in \mathrm{ran}(\pi)$, and $\mathrm{lh}(\mathcal{S}^i) < \alpha = \mathrm{crit}(\pi) = \omega_1^H$. As usual, we use $\bar{x}$ to denote the preimage of x under π, when $x \in \mathrm{ran}(\pi)$. As in the proof of Lemma 3.1.42, we get that for all $\xi < \alpha$, the copy trees $\pi\bar{\mathcal{S}}^i_\xi = \mathcal{S}^i_\xi$ and the copy maps are just restrictions of π.

Claim 2. *For all $i, j < \omega$,*

1. *$\alpha \leq_{\mathbb{S}^i} \omega_1$ and*
2. *for $\xi_i + 1$ is the successor of α in $[0, \omega_1)_{\mathbb{S}^i}$, $F^{\mathbb{S}^i}_{\xi_i} = F^{\mathbb{S}^j}_{\xi_j}$.*

Proof. Let $\bar{\mathbb{S}}^i$ be such that $\pi(\bar{\mathbb{S}}^i) = \mathbb{S}^i$. Then each $\bar{\mathbb{S}}^i$ has length $\alpha + 1$, by elementarity, and $\pi\bar{\mathbb{S}}^i = \mathbb{S}^i \restriction \alpha{+}1$ (as in the proof of Lemma 3.1.42). By elementarity, we get $[0, \alpha)_{\mathbb{S}^i} \subseteq [0, \omega_1)_{\mathbb{S}^i}$, so $\alpha \leq_{\mathbb{S}^i} \omega_1$, establishing (1).

Let $\xi_i + 1$ be the successor of α in $[0, \omega_1)_{\mathbb{S}^i}$. Let $\bar{\Phi}^i_{\eta,\xi}$ be the tree embeddings of $\bar{\mathbb{S}}^i$ and $\Phi^i_{\eta,\xi}$ the tree embeddings of $\mathbb{S}^i$. Since for any $\eta <_{\mathbb{S}^i} \alpha$, $\pi(u^{\bar{\Phi}^i_{\eta,\alpha}}) = u^{\Phi^i_{\eta,\omega_1}}$, it is easy to see that we must have $\alpha = \mathrm{crit}(u^{\Phi^i_{\alpha,\omega_1}})$ and $u^{\Phi^i_{\alpha,\omega_1}}(\alpha) = \omega_1$. It follows that $F^{\mathbb{S}^i}_{\xi_i}$ is the first extender used along $(\alpha, \omega_1]_{\mathcal{S}^i_{\omega_1}}$. Now the trees $\mathcal{S}^i_{\omega_1}$ all actually agree up to $\omega_1 + 1$, since they are all normal trees by the same strategy which use the same extenders below ω_1 (since otherwise we would have a disagreement in the last models below the sup of the λ_η). So we get that all of the $F^{\mathbb{S}^i}_{\xi_i}$ are equal (as they're the first extender used along the same branch in the same tree).

□

Since the $F^{\mathbb{S}^i}_\eta$ are just some subset of the G_γ, we get that some G_γ is used in every $\mathbb{S}^i$. Letting γ be such that $G_\gamma = F^{\mathbb{S}_i}_{\xi_i}$ for all i, we have that G_γ is used in every meta-tree $\mathbb{S}^i$. It follows that G_γ is on the sequence of every model $P^{i,\gamma}$, so that $\lambda_\gamma = \mathrm{lh}(G_\gamma)$ wasn't the index of a disagreement after all, a contradiction.

So this comparison process must terminate by some countable stage, i.e. at some countable α, $P^{i,\alpha} = P^{j,\alpha}$ for all $i, j < \omega$. Let $\mathbb{S}^i = \mathbb{S}^{i,\alpha}$. The last tree of $\mathbb{S}^i$ is the unique tree which is by Σ and has last model $P^{i,\alpha}$. So letting $\mathcal{T}$ be this tree, $\mathcal{T}$ is the last tree of all the $\mathbb{S}^i$.

Finally, we need to see that for some i, $\mathbb{S}^i$ doesn't drop along its main branch. Towards a contradiction, suppose that for all i, $\mathbb{S}^i$ has a drop along its main branch; let $\xi_i + 1$ be the index of some such drop, i.e. ξ_i+1 is on the main branch of $\mathbb{S}^i$ and letting $\eta_i = \mathbb{S}_i\text{-pred}(\xi_i+1)$, η_i-to-$\xi_i + 1$ is a drop.

Applying (2*) from the proof of Lemma 3.1.47, we have that the extenders F^i_χ for $\chi \geq \xi_i$ with $\chi + 1$ on the main branch of $\mathbb{S}^i$ are a tail of the extenders used along the main branch of $\mathcal{T}$. In particular, we have that each of the $F^i_{\xi_i}$ are used along the main branch of $\mathcal{T}$. As each of these extenders corresponds to a drop along this branch, there must be a largest index of one of them used $\mathcal{T}$, say χ is largest such that $E^{\mathcal{T}}_\chi = F^i_{\xi_i}$ for some i. But then for any i, there is a χ_i such that $E^{\mathcal{T}}_\chi = F^i_{\chi_i}$, since the extenders used along the main branch of $\mathcal{T}$ above η_i are all of this form. Since the extenders of the $\mathbb{S}^i$ are subsets of the G_α, there is some β such that $G_\beta = F^i_{\chi_i}$ for all i. But then G_β is used in all of the meta-trees $\mathbb{S}^i$, which is a contradiction, as before. $\square$

Finally, we can now prove Theorem 3.1.41.

Proof of Theorem 3.1.41. We again just consider the case $\theta = \omega_1$, so we define Σ^* a strategy for countable stacks of meta-trees by induction. Uniqueness will be clear.

Suppose we have a stack $\vec{\mathbb{S}} = \langle \mathbb{S}^\xi \mid \xi < \alpha \rangle$ which is so far according to our strategy Σ^*, where $\alpha < \omega_1$. Suppose first that α is a successor ordinal $\xi + 1$. Using Lemma 3.1.43, we define the tail meta-strategy $\Sigma^*_{\vec{\mathbb{S}}}$ for countable meta-trees on $\mathcal{S}^\xi_\infty$ by $\Sigma^*_{\vec{\mathbb{S}}} = \Sigma^*_{\mathcal{S}^\xi_\infty}$.[14] This works and is the unique extension with the property that every tree is by Σ.

Now suppose α is a limit ordinal. Let $\mathcal{T}_\xi$ be the first tree of $\mathbb{S}^\xi$ and $\Phi^{\eta,\xi} : \mathcal{T}_\eta \to \mathcal{T}_\xi$ the associated partial tree embeddings. It's enough to see that the direct limit $\lim\langle \mathcal{T}_\xi, \Phi^{\eta,\xi}\rangle$ is well-founded and by Σ, since then we may set $\mathcal{S}^\alpha_0 = \lim\langle \mathcal{T}_\xi, \Phi^{\eta,\xi}\rangle$ and proceed as in the successor case. Applying the tree comparison theorem (Theorem 3.1.46) to $\{\mathcal{T}_\xi \mid \xi < \alpha\}$, we get a countable tree $\mathcal{T}_\alpha$ which is by Σ and for each $\xi < \alpha$ a countable meta-tree $\mathbb{T}^\xi$ on $\mathcal{T}_\xi$ by Σ^* with last tree $\mathcal{T}_\alpha$. Moreover, for some $\xi < \alpha$, $\mathbb{T}^\xi$ doesn't drop along $\mathcal{T}_\xi$-to-$\mathcal{T}_\alpha$.

For every $\eta < \alpha$, we have $\mathcal{T}_\alpha$ is the last tree in the stack of meta-trees $\langle \mathbb{S}^\eta, \mathbb{T}^{\eta+1}\rangle$. Applying Theorem 3.1.44 to this stack, we get a meta-tree $\mathbb{U}$ by Σ^* on $\mathcal{T}_\eta$ with last tree $\mathcal{T}_\alpha$. By Lemma 3.1.45, $\mathbb{U} = \mathbb{T}^\eta$. So $\Phi^{\mathbb{T}^\eta} = \Phi^{\mathbb{U}} = \Phi^{\mathbb{T}^{\eta+1}} \circ \Phi^{\mathbb{S}^\eta}$ and if η is such that $\mathbb{T}^\eta$ doesn't drop along its main branch, then it follows that $\mathbb{T}^{\eta+1}$ and $\mathbb{S}^\eta$ don't either.

So, we get that $\Phi^{\mathbb{T}^\xi} = \Phi^{\eta,\xi} \circ \Phi^{\mathbb{T}^\eta}$ for all $\eta \leq \xi < \alpha$ and for all sufficiently large η, ξ, these are total extended tree embeddings.

Fixing ζ above which these are total, we have $\lim\langle \mathcal{S}_\xi, \Phi^{\eta,\xi} \mid \zeta \leq \eta \leq \xi < \alpha\rangle$ is well-founded, by Proposition 3.1.11. Since Σ has SHC$^-$, we also get that $\lim\langle \mathcal{S}_\xi, \Phi^{\eta,\xi} \mid \zeta \leq \eta \leq \xi < \alpha\rangle$ is by Σ. $\square$

[14] Technically, here we mean that $\Sigma^*_{\vec{\mathbb{S}}}$ is the restriction of $\Sigma^*_{\mathcal{S}^\xi_\infty}$ to single countable meta-trees.

As a corollary, we get Schlutzenberg's extension of Theorem 3.1.33 to countable stacks of iteration trees.

Theorem 3.1.48 (Schlutzenberg)**.** *Let M be a premouse, θ a regular cardinal, and Σ a $\theta+1$-strategy for M with SHC$^-$. Then there is a unique extension of Σ to an (ω_1, ω_1+1)-strategy for M which bottom-up normalizes well.*

Proof. By induction on length of $\vec{\mathcal{S}}$, we define an meta-tree $\mathbb{W}(\vec{\mathcal{S}})$ with last tree $W(\vec{\mathcal{S}})$ and a weakly elementary embedding $\sigma_{\vec{\mathcal{S}}}$ from the last model of $\vec{\mathcal{S}}$ to the last model of $W(\vec{\mathcal{S}})$.

Let $\vec{\mathcal{S}} = \langle \mathcal{S}_\xi \mid \xi < \alpha\rangle$. Suppose first that α is a successor ordinal $\xi+1$, so $\vec{\mathcal{S}}$ has last tree $\mathcal{S}_\xi$ and we've defined $W(\vec{\mathcal{S}})$ and $\sigma_{\vec{\mathcal{S}}}$ from the last model of $\mathcal{S}_\xi$, P_ξ, to the last model of $W(\vec{\mathcal{S}})$. We define the tail strategy $\Sigma_{\vec{\mathcal{S}},P_\xi}$ by

$$\mathcal{U} \text{ is by } \Sigma_{\vec{\mathcal{S}},P_\xi} \Leftrightarrow \mathbb{W}(W(\vec{\mathcal{S}}), \sigma_{\vec{\mathcal{S}}}\mathcal{U}) \text{ is by } \Sigma^*.$$

Now if α is a limit ordinal, by induction we have $W(\vec{\mathcal{S}} \upharpoonright \xi+1)$ is the last tree of an meta-tree $\mathbb{W}(\vec{\mathcal{S}}\upharpoonright\xi+1)$ by Σ^*, which is an meta-tree on $W(\vec{\mathcal{S}}\upharpoonright\xi)$ when ξ is a successor and we take direct limits at limits. So, we have that $\langle\mathbb{W}(\vec{\mathcal{S}}\upharpoonright\xi+1) \mid \xi<\alpha\rangle$ is a stack of meta-trees by Σ^*. We let $\mathbb{W}(\vec{\mathcal{S}}) = \lim\langle\mathbb{W}(\vec{\mathcal{S}}\upharpoonright\xi+1) \mid \xi<\alpha\rangle$. We have weakly elementary maps $\sigma_{\vec{\mathcal{S}}\upharpoonright\xi+1}$ from the last model of $\vec{\mathcal{S}}\upharpoonright\xi+1$ to the last model of $\mathcal{W}(\vec{\mathcal{S}}\upharpoonright\xi+1)$. Since the direct limit tree $W(\vec{\mathcal{S}})$ is well-founded, we must have that a tail of these maps are total (this uses that the $\sigma_{\vec{\mathcal{S}}}$ are factors of the t-maps). So, we can form the direct limit of the P_α along these maps and get a weakly elementary map $\sigma_{\vec{\mathcal{S}}}$ from P_α into the last model of $W(\vec{\mathcal{S}})$, the last tree of $\mathbb{W}(\vec{\mathcal{S}})$. We can then extend Σ as in the successor case:

$$\mathcal{U} \text{ is by } \Sigma_{\vec{\mathcal{S}},P_\alpha} \Leftrightarrow \mathbb{W}(W(\vec{\mathcal{S}}), \sigma_{\vec{\mathcal{S}}}\mathcal{U}) \text{ is by } \Sigma^*.$$

□

3.1.4 Copying meta-trees

In this section we'll continue to develop the basic theory of meta-trees. Pushing us forward is the following analogy with the usual inner model-theoretic objects.

$$\begin{aligned} \text{premice} &\leftrightsquigarrow \text{iteration trees} \\ \text{iteration trees} &\leftrightsquigarrow \text{meta-trees} \\ \text{strategies} &\leftrightsquigarrow \text{meta-strategies} \\ \text{elementary embeddings} &\leftrightsquigarrow \text{tree embeddings} \end{aligned}$$

Perhaps surprisingly, this analogy will produce useful results. Notice that one might have found the tree comparison theorem (Theorem 3.1.46) by pursuing this analogy: that theorem says any two normal iteration trees have a common meta-iterate, modulo the condition that we started with trees by the same strategy. Besides this (necessary) restriction, this looks like the analogue of the usual comparison theorem for premice.

Our main goal in this section is a kind of copying result. The usual copying construction is about lifting a normal iteration tree across an elementary embedding. Following the analogy, our result will be about lifting a meta-tree across an extended tree embedding.

Of course, a copying result like this must come from the appropriate version of a Shift Lemma, which is also motivated by the analogy. The Shift Lemma says that, under the right conditions, we can complete the following diagram.

$$\begin{tikzcd}
\bar{\mathcal{T}} \arrow[rr, "\Psi"] & & \mathcal{T} \\
\rotatebox[origin=c]{90}{$\in$} & & \rotatebox[origin=c]{90}{$\in$} \\
\bar{F} & \mapsto & F
\end{tikzcd}$$

$$\begin{tikzcd}
\bar{\mathcal{S}} \arrow[r, "\Pi"] \arrow[d, "\bar{F}"] & \mathcal{S} \arrow[d, "F"] \\
W(\bar{\mathcal{S}}, \bar{\mathcal{T}}, \bar{F}) \arrow[r, dashed, "\Gamma"] & W(\mathcal{S}, \mathcal{T}, F)
\end{tikzcd}$$

Unfortunately, the precise conditions on when this is possible are quite technical. Besides demanding that Ψ and Π have sufficient agreement, we must choose the image F of $\bar{F}$ carefully.

Definition 3.1.49. Let $\mathcal{T}$ be an iteration tree of successor length. An extender F on the $M_\infty^\mathcal{T}$-sequence is *λ-anomalous in $\mathcal{T}$* if for $\alpha = \alpha(\mathcal{T}, F)$, $\alpha + 1 < \mathrm{lh}(\mathcal{T})$ and for $\beta = \mathcal{T}\text{-pred}(\alpha + 1)$, letting P be the initial segment of $M_\beta^\mathcal{T}$ to which we apply $E_\alpha^\mathcal{T}$ in $\mathcal{T}$, there is G on the P-sequence such that $i^P_{E^\mathcal{T}_\alpha}(G) = F$ and $\mathrm{crit}(E_\alpha^\mathcal{T}) \leq \lambda(G) < \mathrm{lh}(G) < \mathrm{dom}(E_\alpha^\mathcal{T})$.

Note that an ostensible weakening of this definition is actually equivalent: if F is on the $M_\infty^\mathcal{T}$-sequence and there is some $\gamma + 1 < \mathrm{lh}(\mathcal{T})$ such that for $\beta = \mathcal{T}\text{-pred}(\gamma + 1)$ and P be the initial segment of $M_\beta^\mathcal{T}$ to which we apply $E_\gamma^\mathcal{T}$, there is G on the P-sequence such that $i^P_{E^\mathcal{T}_\gamma}(G) = F$ and $\mathrm{crit}(E_\gamma^\mathcal{T}) \leq \lambda(G) < \mathrm{lh}(G) < \mathrm{dom}(E_\gamma^\mathcal{T})$, then $\alpha = \gamma$ as $\alpha \leq \gamma$ (as F is on the $M_\gamma^\mathcal{T}$) but if $\alpha < \gamma$, then $\mathrm{lh}(F) < \mathrm{lh}(E_\alpha^\mathcal{T}) < \mathrm{lh}(E_\gamma^\mathcal{T})$ and $\lambda(E_\gamma^\mathcal{T}) \leq \lambda(F)$, since $\mathrm{crit}(E_\gamma^\mathcal{T}) \leq \lambda(G)$, so since $\lambda(E_\gamma^\mathcal{T})$ is the largest cardinal of $M_\gamma^\mathcal{T}|\mathrm{lh}(E_\gamma^\mathcal{T})$, $\mathrm{lh}(E_\alpha^\mathcal{T})$ cannot be a cardinal here, a contradiction.

Proposition 3.1.50. *Let $\bar{\mathcal{T}}$, $\mathcal{T}$ be normal trees on M of successor lengths and $\Psi : \bar{\mathcal{T}} \to \mathcal{T}$ an extended tree embedding. Let $\bar{F}$ be on the $M_\infty^{\bar{\mathcal{T}}}$-sequence with $\bar{F} \in dom(t_\infty^\Psi)$*[15] *and $\bar{\alpha} = \alpha(\bar{\mathcal{T}}, \bar{F})$. Let $\xi \in [v^\Psi(\bar{\alpha}), u^\Psi(\bar{\alpha})]_\mathcal{T}$, $F = s^\Psi_{\bar{\alpha},\xi}(\bar{F})$ and $\alpha = \alpha(\mathcal{T}, F)$. Then exactly one of the following holds.*

1. *$F = s^\Psi_{\bar{\alpha},\eta}(\bar{F})$ for some $\eta \in [v^\Psi(\bar{\alpha}), u^\Psi(\bar{\alpha})]_\mathcal{T}$ such that $\eta < \xi$,*
2. *$\xi = \alpha$, or*
3. *$v^\Psi(\bar{\alpha}) < \xi$, $\xi = \alpha + 1$, and F is λ-anomalous in $\mathcal{T}$.*

[15] The only case in which this is important is when $\bar{\alpha}+1 = \mathrm{lh}(\bar{\mathcal{T}})$, since $\bar{\alpha}+1 < \mathrm{lh}(\bar{\mathcal{T}})$, then $\mathrm{lh}(\bar{F}) < \mathrm{lh}(E^{\bar{\mathcal{T}}}_{\bar{\alpha}})$, so that it is trivial.

Moreover, there is a largest $\xi \in [v^\Psi(\bar{\alpha}), u^\Psi(\bar{\alpha})]_\mathcal{T}$ such that (2) holds.

Proof. It's easy to see that (1)-(3) are mutually exclusive, so we just want to see that one of them must hold. Suppose (1) fails, so that ξ is the least $\eta \in [v^\Psi(\bar{\alpha}), u^\Psi(\bar{\alpha})]_\mathcal{T}$ such that $F = s^\Psi_{\bar{\alpha},\eta}(\bar{F})$. Then for all $\eta < \xi$ with $\eta \in [v^\Psi(\bar{\alpha}), u^\Psi(\bar{\alpha})]_\mathcal{T}$, $\mathrm{crit}(\hat{\imath}^\mathcal{T}_{\eta,\xi}) \leq \lambda(s^\Psi_{\bar{\alpha},\eta}(\bar{F}))$, since we haven't finished blowing up $s^\Psi_{\bar{\alpha}}(\bar{F})$ to F along $[v^\Psi(\bar{\alpha}), \xi]_\mathcal{T}$.

First we'll consider the case that $\xi = v^\Psi(\bar{\alpha})$. Now, for all $\eta < \bar{\alpha}$, $\mathrm{lh}(E^{\bar{\mathcal{T}}}_\eta) < \mathrm{lh}(\bar{F})$, so since $s^\Psi_{\bar{\alpha}} \restriction \mathrm{lh}(E^{\bar{\mathcal{T}}}_\eta) + 1 = t^\Psi_\eta \restriction \mathrm{lh}(E^{\bar{\mathcal{T}}}_\eta) + 1$, by the agreement properties of tree embeddings, so $\mathrm{lh}(E^\mathcal{T}_{u(\eta)}) < \mathrm{lh}(F)$. Since $v^\Psi(\bar{\alpha}) = \sup\{u^\Psi(\eta) + 1 \mid \eta < \bar{\alpha}\}$, this implies that $\mathrm{lh}(F) > \mathrm{lh}(E^\mathcal{T}_\zeta)$ for all $\zeta < v^\Psi(\bar{\alpha})$. It follow $v^\Psi(\bar{\alpha}) = \alpha$.

Next, suppose that $\xi > v^\Psi(\bar{\alpha})$ is a limit ordinal. Then $\mathrm{lh}(F) \geq \sup\{s^\Psi_{\bar{\alpha},\eta}(\mathrm{lh}(\bar{F})) \mid \eta < \xi\} \geq \sup\{\mathrm{lh}(E^\mathcal{T}_\zeta) \mid \zeta < \xi\}$, since for any $\zeta < \xi$ with $v^\Psi(\bar{\alpha}) \geq \zeta$, letting χ the least element of $(v^\Psi(\bar{\alpha}), \xi]_\mathcal{T}$ with $\zeta < \chi$, χ is a successor ordinal, $\gamma + 1$, and for $\eta = \mathcal{T}\text{-pred}(\gamma + 1)$, $\mathrm{crit}(E^\mathcal{T}_\gamma) \leq \lambda(s^\Psi_{\bar{\alpha},\eta}(\bar{F}))$, so $s^\Psi_{\bar{\alpha},\gamma+1}(\mathrm{lh}(\bar{F})) > \lambda(E^\mathcal{T}_\gamma) > \mathrm{lh}(E^\mathcal{T}_\zeta)$. It follows that $\xi = \alpha$.

Suppose now that (2) fails as well, i.e. $\xi \neq \alpha$. We need to see that (3) holds. We have $\alpha < \xi$ since F is on the $M^\mathcal{T}_\xi$-sequence. By our observations above, we must have $\xi > v^\Psi(\bar{\alpha})$ and ξ is a successor ordinal, $\gamma + 1$. It follows that $\alpha \leq \gamma$, so $\mathrm{lh}(F) < \mathrm{lh}(E^\mathcal{T}_\gamma)$. Let $\beta = \mathcal{T}\text{-pred}(\gamma + 1)$. Since $\eta < \xi = \gamma + 1$, we have $\mathrm{crit}(E^\mathcal{T}_\gamma) \leq \lambda(s^\Psi_{\bar{\alpha},\beta}(\bar{F}))$. Letting P the level of $M^\mathcal{T}_\beta$ to which we apply $E^\mathcal{T}_\gamma$, we have that $i^P_{E^\mathcal{T}_\gamma}(s^\Psi_{\bar{\alpha},\beta}(\bar{F})) = F$. So since $\mathrm{lh}(F) < \mathrm{lh}(E^\mathcal{T}_\gamma)$, we must have $\mathrm{lh}(G) < \mathrm{dom}(E^\mathcal{T}_\gamma)$. It follows that $\gamma = \alpha$ and F is λ-anomalous in $\mathcal{T}$, by the remarks following that definition. So we've shown exactly one of (1)-(3) holds.

Note that (2) always holds for some $\xi \in [v^\Psi(\bar{\alpha}), u^\Psi(\bar{\alpha})]_\mathcal{T}$, namely $\xi = v^\Psi(\bar{\alpha})$. Now suppose that ξ is a limit ordinal for which there are cofinally many $\zeta < \xi$ such that $\zeta \in [v^\Psi(\bar{\alpha}), u^\Psi(\bar{\alpha})]_\mathcal{T}$ and (2) holds at ζ. Then $\xi \in [v^\Psi(\bar{\alpha}), u^\Psi(\bar{\alpha})]_\mathcal{T}$, since this is closed below $u^\Psi(\bar{\alpha})$, and (1) must fail at ξ since it fails at all $\zeta < \xi$ (the image of $\bar{F}$ cannot stablize below ξ). By our above observations, (2) must hold at ξ as well. So there is a largest $\xi \in [v^\Psi(\bar{\alpha}), u^\Psi(\bar{\alpha})]_\mathcal{T}$ for which (2) holds. $\square$

Definition 3.1.51. Let $\bar{\mathcal{T}}$, $\mathcal{T}$ be normal trees on M of successor lengths and $\Psi : \bar{\mathcal{T}} \to \mathcal{T}$ an extended tree embedding. Let $\bar{F}$ be on the $M^{\bar{\mathcal{T}}}_\infty$-sequence with $\bar{F} \in \mathrm{dom}(t^\Psi_\infty)$, $\bar{\alpha} = \alpha(\bar{\mathcal{T}}, \bar{F})$, and $\bar{\beta} = \beta(\bar{\mathcal{T}}, \bar{F})$.

An extender F on the $M^\mathcal{T}_\infty$-sequence is an *adequate Ψ-image of $\bar{F}$* if, letting $\alpha = \alpha(\mathcal{T}, F)$ and $\beta = \beta(\mathcal{T}, F)$,

1. $\alpha \in [v^\Psi(\bar{\alpha}), u^\Psi(\bar{\alpha})]_\mathcal{T}$,
2. $F = s^\Psi_{\bar{\alpha},\alpha}(\bar{F})$,
3. $\beta \in [v^\Psi(\bar{\beta}), u^\Psi(\bar{\beta})]_\mathcal{T}$,
4. $s^\Psi_{\bar{\alpha},\alpha} \restriction \mathrm{dom}(\bar{F}) = s^\Psi_{\bar{\beta},\beta} \restriction \mathrm{dom}(\bar{F})$,
5. if $\alpha < u^\Psi(\bar{\alpha})$ and, for $\gamma + 1$ the successor of α in $(\alpha, u^\Psi(\bar{\alpha})]_\mathcal{T}$, $\mathrm{crit}(E^\mathcal{T}_\gamma) < \mathrm{lh}(F)$, then $\mathrm{lh}(F) < \mathrm{dom}(E^\mathcal{T}_\gamma)$ and if $\bar{\alpha} + 1 < \mathrm{lh}(\bar{\mathcal{T}})$, $\mathrm{dom}(E^\mathcal{T}_\gamma) < s^\Psi_{\bar{\alpha},\alpha}(\mathrm{lh}(E^{\bar{\mathcal{T}}}_{\bar{\alpha}}))$.

We can use our previous proposition to show that there is always an adequate Ψ-image of $\bar{F}$.

Proposition 3.1.52. *Let $\bar{\mathcal{T}}$, $\mathcal{T}$ be normal trees on M of successor lengths and $\Psi : \bar{\mathcal{T}} \to \mathcal{T}$ an extended tree embedding. Let $\bar{F}$ be on the $M_\infty^{\bar{\mathcal{T}}}$-sequence with $\bar{F} \in dom(t_\infty^\Psi)$, $\bar{\alpha} = \alpha(\bar{\mathcal{T}}, \bar{F})$, and $\bar{\beta} = \beta(\bar{\mathcal{T}}, \bar{F})$. Let $F = s^\Psi_{\bar{\alpha},\xi}(\bar{F})$ for the largest $\xi \in [v^\Psi(\bar{\alpha}), u^\Psi(\bar{\alpha})]_\mathcal{T}$ such that $\xi = \alpha(\mathcal{T}, s^\Psi_{\bar{\alpha},\xi}(\bar{F}))$. Then F is an adequate Ψ-image of $\bar{F}$.*

Proof. Let $\alpha = \alpha(\mathcal{T}, F)$ and $\beta = \beta(\mathcal{T}, F)$. Before we start verifying (1)-(5), we need to see that F is on the $M_\infty^\mathcal{T}$-sequence. Suppose not. If $\alpha = u^\Psi(\bar{\alpha})$, then F is on the $M_\infty^\mathcal{T}$-sequence, a contradiction, since $F = t^\Psi_{\bar{\alpha}}(\bar{F}) = t^\Psi_\infty(\bar{F})$, as either $\alpha + 1 = \mathrm{lh}(\mathcal{T})$ or else this follows from the agreement of t-maps, since $\mathrm{lh}(\bar{F}) < \mathrm{lh}(E^{\bar{\mathcal{T}}}_{\bar{\alpha}})$. So $\alpha < u^\Psi(\bar{\alpha})$. Since F is not on the $M_\infty^\mathcal{T}$-sequence, $\mathrm{lh}(E^\mathcal{T}_\alpha) \leq \mathrm{lh}(F)$, so $\lambda(E^\mathcal{T}_\alpha) \leq \lambda(F)$. Now let $\gamma + 1$ be the successor of α in $[v^\Psi(\bar{\alpha}), u^\Psi(\bar{\alpha})]_\mathcal{T}$. Then $\mathrm{dom}(E^\mathcal{T}_\gamma) \leq \lambda(E^\mathcal{T}_\alpha) < \mathrm{lh}(F)$. Letting P be the initial segment of $M^\mathcal{T}_\alpha$ to which we apply $E^\mathcal{T}_\gamma$, we have $\mathrm{lh}(F) \leq o(P)$, since $\bar{F} \in \mathrm{dom}(t^\Psi_{\bar{\alpha}})$, so applying $i^P_{E^\mathcal{T}_\gamma}$ gives $\mathrm{lh}(E^\mathcal{T}_\gamma) < i^P_{E^\mathcal{T}_\gamma}(F) = s^\Psi_{\bar{\alpha},\gamma+1}(\bar{F})$. It follows that $\gamma + 1 = \alpha(\mathcal{T}, s^\Psi_{\bar{\alpha},\gamma+1}(\bar{F}))$, contradicting the maximality of α. So F is on the $M^\mathcal{T}_\infty$-sequence.

Conclusions (1) and (2) are trivial, so we just need to verify (3)-(5). We'll start by showing (3) and (4). Towards this, we'll show $v^\Psi(\bar{\beta}) \leq \beta \leq u^\Psi(\bar{\beta})$. If $\bar{\beta} + 1 = \mathrm{lh}(\bar{\mathcal{T}})$ then $u^\Psi(\bar{\beta}) + 1 = \mathrm{lh}(\mathcal{T})$ since Ψ is an extended tree embedding, so trivially $\beta \leq u^\Psi(\bar{\beta})$. So suppose $\bar{\beta} + 1 < \mathrm{lh}(\bar{\mathcal{T}})$. Then $\mathrm{crit}(\bar{F}) < \lambda(E^{\bar{\mathcal{T}}}_{\bar{\beta}})$, so as $t^\Psi_{\bar{\beta}} \restriction \mathrm{lh}(E^{\bar{\mathcal{T}}}_{\bar{\beta}}) + 1 = t^\Psi_{\bar{\alpha}} \restriction \mathrm{lh}(E^{\bar{\mathcal{T}}}_{\bar{\beta}}) + 1$, $\mathrm{crit}(F) \leq t^\Psi_{\bar{\beta}}(\mathrm{crit}(\bar{F})) < \lambda(E^\mathcal{T}_{u^\Psi(\bar{\beta})})$. So $\beta \leq u^\Psi(\bar{\beta})$. For any $\xi < \bar{\beta}$, we have $\lambda(E^{\bar{\mathcal{T}}}_\xi) \leq \mathrm{crit}(\bar{F})$, so using the agreement properties of the maps of Ψ, we get $\lambda(E^\mathcal{T}_{u^\Psi(\xi)}) \leq \mathrm{crit}(F)$, so $u^\Psi(\xi) < \beta$. Since $v^\Psi(\bar{\beta}) = \sup\{u^\Psi(\xi) + 1 \mid \xi < \bar{\beta}\}$, we have $v^\Psi(\bar{\beta}) \leq \beta$.

If $\beta = u^\Psi(\bar{\beta})$, (3) is trivial and (4) is easy to see: either $\bar{\beta} < \bar{\alpha}$ in which case the agreement of the maps of Ψ gives $t^\Psi_{\bar{\beta}} \restriction \mathrm{dom}(\bar{F}) = s^\Psi_{\bar{\alpha}} \restriction \mathrm{dom}(\bar{F}) = t^\Psi_{\bar{\alpha}} \restriction \mathrm{dom}(\bar{F})$, which easily implies (4), or else $\bar{\beta} = \bar{\alpha}$ and $u^\Psi(\bar{\alpha}) = u^\Psi(\bar{\beta}) = \beta = \alpha$, which makes (4) trivial.

So, suppose $\beta < u^\Psi(\bar{\beta})$. Let $\eta \in (v^\Psi(\bar{\beta}), u^\Psi(\bar{\beta})]_\mathcal{T}$ least such that $\beta < \eta$. η must be a successor ordinal, $\gamma + 1$. Let $\zeta = \mathcal{T}\text{-pred}(\gamma + 1)$. We'll show that $\zeta = \beta$ and that $s^\Psi_{\bar{\beta},\beta}(\mathrm{crit}(\bar{F})) = \mathrm{crit}(F)$, which suffices.

We consider cases. First suppose that $\bar{\beta} < \bar{\alpha}$. In this case we'll be able to show that $\beta = \zeta$ and $\mathrm{crit}(E^\mathcal{T}_\gamma) > \mathrm{crit}(F)$. Suppose this fails, i.e. that $\zeta < \beta$ or $\mathrm{crit}(E^\mathcal{T}_\gamma) \leq \mathrm{crit}(F)$. In either case, $\mathrm{crit}(F) \in [\mathrm{crit}(E^\mathcal{T}_\gamma), \lambda(E^\mathcal{T}_\gamma))$ (in the former case this is because $\mathrm{crit}(E^\mathcal{T}_\gamma) < \lambda(E^\mathcal{T}_\zeta) \leq \mathrm{crit}(F) < \lambda(E^\mathcal{T}_\beta) \leq \lambda(E^\mathcal{T}_\gamma)$; in the latter case we have assumed $\mathrm{crit}(E^\mathcal{T}_\gamma) \leq \mathrm{crit}(F)$ and still have $\mathrm{crit}(F) < \lambda(E^\mathcal{T}_\beta) \leq \lambda(E^\mathcal{T}_\gamma)$). Now since $\bar{\beta} < \bar{\alpha}$, the agreement between model maps in a tree embedding gives that $t^\Psi_{\bar{\alpha}} \restriction \mathrm{lh}(E^{\bar{\mathcal{T}}}_{\bar{\beta}}) = s^\Psi_{\bar{\alpha}} \restriction \mathrm{lh}(E^{\bar{\mathcal{T}}}_{\bar{\beta}}) = t^\Psi_{\bar{\beta}} \restriction \mathrm{lh}(E^{\bar{\mathcal{T}}}_{\bar{\beta}})$, so that $\mathrm{crit}(F) = t^\Psi_{\bar{\alpha}}(\mathrm{crit}(\bar{F})) = t^\Psi_{\bar{\beta}}(\mathrm{crit}(\bar{F})) = \hat{\imath}^\mathcal{T}_{v^\Psi(\bar{\beta}),u^\Psi(\bar{\beta})} \circ s^\Psi_{\bar{\beta}}(\mathrm{crit}(\bar{F}))$. But $E^\mathcal{T}_\gamma$ is used in $(v^\Psi(\bar{\beta}), u^\Psi(\bar{\beta}))_\mathcal{T}$, so that $[\mathrm{crit}(E^\mathcal{T}_\gamma), \lambda(E^\mathcal{T}_\gamma))$ is disjoint from $\mathrm{ran}(\hat{\imath}^\mathcal{T}_{v^\Psi(\bar{\beta}),u^\Psi(\bar{\beta})})$, a contradiction. So $\zeta = \beta$ and $\mathrm{crit}(E^\mathcal{T}_\gamma) > \mathrm{crit}(F)$, as claimed.

Now suppose $\bar{\beta} = \bar{\alpha}$. We easily have that $\beta \leq \alpha$. If $\beta = \alpha$, we're done. So suppose $\beta < \alpha$. Then $\zeta < \gamma + 1 \leq \alpha$. The argument for the previous case with $s^\Psi_{\bar{\alpha},\alpha}$ replacing $t^\Psi_{\bar{\beta}} = t^\Psi_{\bar{\alpha}}$ gives that $\beta = \zeta$ and $\mathrm{crit}(E^\mathcal{T}_\gamma) > \mathrm{crit}(F)$. This finishes (3) and (4).

For (5), suppose $\alpha < u^\Psi(\bar{\alpha})$ and $\mathrm{crit}(E^\mathcal{T}_\gamma) < \mathrm{lh}(F)$, for $\gamma + 1$ be the successor of α in $(\alpha, u^\Psi(\bar{\alpha})]_\mathcal{T}$. $\mathrm{dom}(E^\mathcal{T}_\gamma) \leq \mathrm{lh}(F)$, this contradicts the maximality of α, as in the beginning of the proof (applying $E^\mathcal{T}_\gamma$ together with coherence gives that $\mathrm{lh}(E^\mathcal{T}_\gamma) < s^\Psi_{\bar{\alpha},\gamma+1}(\mathrm{lh}(\bar{F}))$, so that $\gamma + 1 = \alpha(\mathcal{T}, s^\Psi_{\bar{\alpha},\gamma+1}(\bar{F}))$). So $\mathrm{lh}(F) < \mathrm{dom}(E^\mathcal{T}_\gamma)$, as desired.

For the rest of (5), we assume that $\bar{\alpha}+1 < \mathrm{lh}(\bar{\mathcal{T}})$. Towards a contradiction, suppose that $s^{\Psi}_{\bar{\alpha},\alpha}(\mathrm{lh}(E^{\bar{\mathcal{T}}}_{\bar{\alpha}})) < \mathrm{dom}(E^{\mathcal{T}}_{\gamma})$ (equality here is impossible, by coherence). Since $\mathrm{dom}(E^{\mathcal{T}}_{\gamma}) \lhd M^{\mathcal{T}}_{\alpha}|\lambda(E^{\mathcal{T}}_{\alpha})$, there is some $P \lhd M^{\mathcal{T}}_{\alpha}|\lambda(E^{\mathcal{T}}_{\alpha})$ projecting to $\mathrm{crit}(E^{\mathcal{T}}_{\gamma})$ with $s^{\Psi}_{\bar{\alpha},\alpha}(\mathrm{lh}(E^{\bar{\mathcal{T}}}_{\bar{\alpha}})) < \mathrm{dom}(E^{\mathcal{T}}_{\gamma}) < o(P)$. Applying $E^{\mathcal{T}}_{\gamma}$ gives that $s^{\Psi}_{\bar{\alpha},\gamma+1}(\mathrm{lh}(E^{\bar{\mathcal{T}}}_{\bar{\alpha}})) < \mathrm{lh}(E^{\mathcal{T}}_{\gamma})$, so we must have $\gamma+1 < u^{\Psi}(\bar{\alpha})$, as otherwise $E^{\mathcal{T}}_{\gamma+1} = s^{\Psi}_{\bar{\alpha},\gamma+1}(E^{\bar{\mathcal{T}}}_{\bar{\alpha}})$ so that $\mathrm{lh}(E^{\mathcal{T}}_{\gamma+1}) < \mathrm{lh}(E^{\mathcal{T}}_{\gamma})$, contradicting the normality of $\mathcal{T}$. Let $\gamma_0 = \gamma$. We just argued that $\gamma_0+1 < u^{\Psi}(\bar{\alpha})$, so γ_0+1 has some successor γ_1+1 in $[v^{\Psi}(\bar{\alpha}), u^{\Psi}(\bar{\alpha})]_{\mathcal{T}}$. Since we are still blowing up $E^{\bar{\mathcal{T}}}_{\bar{\alpha}}$, we must have that $\mathrm{crit}(E^{\mathcal{T}}_{\gamma_1}) \leq s^{\Psi}_{\bar{\alpha},\gamma_0+1}(\mathrm{lh}(E^{\bar{\mathcal{T}}}_{\bar{\alpha}}))$. So, by normality, we actually must have $\lambda(E^{\mathcal{T}}_{\gamma_0}) = \mathrm{crit}(E^{\mathcal{T}}_{\gamma_1})$ (normality immediately gives $\lambda(E^{\mathcal{T}}_{\gamma_0}) \leq \mathrm{crit}(E^{\mathcal{T}}_{\gamma_1})$ but $\lambda(E^{\mathcal{T}}_{\gamma_0}) < \mathrm{crit}(E^{\mathcal{T}}_{\gamma_1})$ is impossible since $\mathrm{crit}(E^{\mathcal{T}}_{\gamma_1}) < s^{\Psi}_{\bar{\alpha},\gamma_0+1}(\mathrm{lh}(E^{\bar{\mathcal{T}}}_{\bar{\alpha}})) < \mathrm{lh}(E^{\mathcal{T}}_{\gamma_0})$ so that $\mathrm{crit}(E^{\mathcal{T}}_{\gamma_1})$ is a cardinal of $M^{\mathcal{T}}_{\gamma_1}|\mathrm{lh}(E^{\mathcal{T}}_{\gamma_0}) = M^{\mathcal{T}}_{\gamma_1}|\mathrm{lh}(E^{\mathcal{T}}_{\gamma_0+1})$, where $\lambda(E^{\mathcal{T}}_{\gamma_0})$ is the largest cardinal). Since $\mathrm{lh}(E^{\mathcal{T}}_{\gamma_0})$ is a cardinal of $M^{\mathcal{T}}_{\gamma_0+1}$ and $M^{\mathcal{T}}_{\gamma_1}$, $E^{\mathcal{T}}_{\gamma_1}$ is total on $M^{\mathcal{T}}_{\gamma_0+1}$ and so $\hat{\imath}^{\mathcal{T}}_{\alpha,\gamma_0+1}(P) \lhd \mathrm{dom}(E^{\mathcal{T}}_{\gamma_1})$ and projects to $\mathrm{crit}(E^{\mathcal{T}}_{\gamma_1})$, by elementarity. So we have that

$$\mathrm{crit}(E^{\mathcal{T}}_{\gamma_1}) < s^{\Psi}_{\bar{\alpha},\gamma_0+1}(\mathrm{lh}(\bar{F})) < s^{\Psi}_{\bar{\alpha},\gamma_0+1}(\mathrm{lh}(E^{\bar{\mathcal{T}}}_{\bar{\alpha}})) < \mathrm{dom}(E^{\mathcal{T}}_{\gamma_1}),$$

This last inequality makes $\gamma_1+1 = u^{\Psi}(\bar{\alpha})$ impossible, as before, so that γ_1+1 also has a successor in $[v^{\Psi}(\bar{\alpha}), u^{\Psi}(\bar{\alpha})]_{\mathcal{T}}$, γ_2+1. Repeating this same reasoning allows us to conclude

$$\mathrm{crit}(E^{\mathcal{T}}_{\gamma_2}) < s^{\Psi}_{\bar{\alpha},\gamma_1+1}(\mathrm{lh}(\bar{F})) < s^{\Psi}_{\bar{\alpha},\gamma_1+1}(\mathrm{lh}(E^{\bar{\mathcal{T}}}_{\bar{\alpha}})) < \mathrm{dom}(E^{\mathcal{T}}_{\gamma_2}).$$

Continuing in this way, we get an ω-sequence $\langle \gamma_n \mid n \in \omega \rangle$ such that $\alpha = \mathcal{T}\text{-pred}(\gamma_0+1)$ and for all n,

- $\gamma_n + 1 \in [v^{\Psi}(\bar{\alpha}), u^{\Psi}(\bar{\alpha}))_{\mathcal{T}}$,
- $\gamma_n + 1 = \mathcal{T}\text{-pred}(\gamma_{n+1}+1)$, and
- $\mathrm{crit}(E^{\mathcal{T}}_{\gamma_{n+1}}) < s^{\Psi}_{\bar{\alpha},\gamma_n+1}(\mathrm{lh}(\bar{F}))$,

Letting $\xi = \sup\{\gamma_n+1 \mid n < \omega\}$, we have $\xi \in [v^{\Psi}(\bar{\alpha}), u^{\Psi}(\bar{\alpha})]_{\mathcal{T}}$, since this set is closed below its supremum, and so Proposition 3.1.50 implies that $\xi = \alpha(\mathcal{T}, s^{\Psi}_{\bar{\alpha},\xi}(\bar{F}))$. This contradicts the maximality of α. So we must have had $\mathrm{dom}(E^{\mathcal{T}}_{\gamma}) < s^{\Psi}_{\bar{\alpha},\alpha}(\mathrm{lh}(E^{\bar{\mathcal{T}}}_{\bar{\alpha}}))$ after all, finishing (5).

Although the full importance of condition (5) won't be apparent until we look at the copying construction, the next proposition is a preview which suffices for the Shift Lemma.

Proposition 3.1.53. *Let $\bar{\mathcal{T}}$, $\mathcal{T}$ be normal trees on M of successor lengths and $\Psi : \bar{\mathcal{T}} \to \mathcal{T}$ an extended tree embedding. Let $\bar{F}$ be on the $M^{\bar{\mathcal{T}}}_{\infty}$-sequence with $\bar{F} \in dom(t^{\Psi}_{\infty})$, $\bar{\alpha} = \alpha(\bar{\mathcal{T}}, \bar{F})$, and $\bar{\beta} = \beta(\bar{\mathcal{T}}, \bar{F})$. Let F be an adequate Ψ-image of $\bar{F}$, $\alpha = \alpha(\mathcal{T}, F)$, and $\beta = \beta(\mathcal{T}, F)$. Then*

1. *if $W(\mathcal{T}, F)$ is in the dropping case, then $W(\bar{\mathcal{T}}, \bar{F})$ is in the dropping case, and*
2. *if $W(\mathcal{T}, F)$ is not in the dropping case, then $t^{\Psi}_{\bar{\beta}} \restriction dom(\bar{F}) = s^{\Psi}_{\bar{\beta},\beta} \restriction dom(\bar{F})$*

Proof. For (1), suppose that $W(\mathcal{T}, F)$ is in the dropping case. We want to show $W(\bar{\mathcal{T}}, \bar{F})$ is, too. We consider cases.

First suppose that $\bar{\beta}+1 < \mathrm{lh}(\bar{\mathcal{T}})$. We must have $\beta+1 < \mathrm{lh}(\mathcal{T})$, too, and some level of $M^{\mathcal{T}}_\beta | \mathrm{lh}(E^{\mathcal{T}}_\beta)$ past $\mathrm{dom}(F)$ projects strictly across $\mathrm{dom}(F)$. Let P be the least such level. If $\beta = u^\Psi(\bar{\beta})$, then since $E^{\mathcal{T}}_{u^\Psi(\bar{\beta})} = t^\Psi_{\bar{\beta}}(E^{\bar{\mathcal{T}}}_{\bar{\beta}})$ and $\mathrm{dom}(F) = t^\Psi_{\bar{\beta}}(\mathrm{dom}(\bar{F}))$, the elementarity of $t^\Psi_{\bar{\beta}}$ gives that $P = t^\Psi_{\bar{\beta}}(\bar{P})$ for some $\bar{P}$ an initial segment of $M^{\bar{\mathcal{T}}}_{\bar{\beta}} | \mathrm{lh}(E^{\bar{\mathcal{T}}}_{\bar{\beta}})$ projecting strictly across $\mathrm{dom}(\bar{F})$. So $W(\bar{\mathcal{T}}, \bar{F})$ is in the dropping case, as desired. So suppose $\beta < u^\Psi(\bar{\beta})$. Let $\gamma+1$ be the successor of β in $[v^\Psi(\bar{\beta}), u^\Psi(\bar{\beta})]_{\mathcal{T}}$. Since $\rho(P) \leq \mathrm{crit}(F) < \mathrm{dom}(F) < o(P) < \mathrm{lh}(E^{\mathcal{T}}_\beta)$ and $\mathrm{dom}(F) \leq \lambda(E^{\mathcal{T}}_\beta)$, we must have $o(P) < \lambda(E^{\mathcal{T}}_\beta)$, as this is a cardinal in $M^{\mathcal{T}}_\beta | \mathrm{lh}(E^{\mathcal{T}}_\beta)$. Since $\mathrm{crit}(E^{\mathcal{T}}_\gamma)$ is also a cardinal here, we either have $o(P) < \mathrm{crit}(E^{\mathcal{T}}_\gamma)$ or else $\mathrm{crit}(E^{\mathcal{T}}_\gamma) \leq \rho(P)$. Since we are still blowing up $E^{\bar{\mathcal{T}}}_{\bar{\beta}}$ to $E^{\mathcal{T}}_{u^\Psi(\bar{\beta})}$, we have $\mathrm{crit}(E^{\mathcal{T}}_\gamma) < s^\Psi_{\bar{\beta},\beta}(\mathrm{lh}(E^{\bar{\mathcal{T}}}_{\bar{\beta}}))$. So if $o(P) < \mathrm{crit}(E^{\mathcal{T}}_\gamma)$, then, $o(P) < s^\Psi_{\bar{\beta},\beta}(\mathrm{lh}(E^{\bar{\mathcal{T}}}_{\bar{\beta}}))$ and, as before, the elementarity of $s^\Psi_{\bar{\beta},\beta}$ implies that some $\bar{P}$ below $\mathrm{lh}(E^{\bar{\mathcal{T}}}_{\bar{\beta}})$ projects across $\mathrm{dom}(\bar{F})$, and $W(\bar{\mathcal{T}}, \bar{F})$ is in the dropping case. So suppose $\mathrm{crit}(E^{\mathcal{T}}_\gamma) \leq \rho(P)$. Since $\rho(P) \leq \mathrm{crit}(F)$, we cannot have $\bar{\beta} < \bar{\alpha}$ or $\bar{\beta} = \bar{\alpha}$ but $\beta < \alpha$, as we showed that $\mathrm{crit}(E^{\mathcal{T}}_\gamma) > \mathrm{crit}(F)$ in these cases in the proof of (3) and (4) of the previous proposition. So we have $\bar{\beta} = \bar{\alpha}$ and $\beta = \alpha$ and it is enough to see that $o(P) < s^\Psi_{\bar{\alpha},\alpha}(\mathrm{lh}(E^{\bar{\mathcal{T}}}_{\bar{\alpha}}))$. But this follows from condition (5) since $o(P) < \mathrm{dom}(E^{\mathcal{T}}_\gamma) < s^\Psi_{\bar{\alpha},\alpha}(\mathrm{lh}(E^{\bar{\mathcal{T}}}_{\bar{\alpha}}))$.

Now suppose $\bar{\beta}+1 = \mathrm{lh}(\bar{\mathcal{T}})$ (we won't need to keep track of whether $\beta+1 < \mathrm{lh}(\mathcal{T})$). Then there is some $P \lhd M^{\mathcal{T}}_\beta$ beyond $\mathrm{dom}(F)$ projecting $\leq \mathrm{crit}(F)$. If $[v^\Psi(\bar{\beta}), \beta]_{\mathcal{T}}$ doesn't drop, then $s^\Psi_{\bar{\beta},\beta}$ is a total elementary map and so there is some $\bar{P} \trianglelefteq M^{\bar{\mathcal{T}}}_{\bar{\beta}} = M^{\bar{\mathcal{T}}}_\infty$ beyond $\mathrm{dom}(\bar{F})$ projecting $\leq \mathrm{crit}(\bar{F})$ by elementarity. This implies $W(\bar{\mathcal{T}}, \bar{F})$ is in the dropping case. Now suppose that there is a drop along $[v^\Psi(\bar{\beta}), \beta]_{\mathcal{T}}$. Let η-to-$\gamma+1$ be the first drop. Then since $\eta < \gamma+1 \leq \beta$, we have $\mathrm{crit}(E^{\mathcal{T}}_\gamma) \leq s^\Psi_{\bar{\beta},\eta}(\mathrm{crit}(\bar{F}))$, as we haven't finished blowing up $\mathrm{crit}(\bar{F})$ to $\mathrm{crit}(F)$ by η. Since η-to-$\gamma+1$ is a drop, there is a $P \lhd M^{\mathcal{T}}_\eta$ with $\rho(P) \leq \mathrm{crit}(E^{\mathcal{T}}_\gamma) \leq s^\Psi_{\bar{\beta},\eta}(\mathrm{crit}(\bar{F}))$. We must have $\mathrm{dom}(F) < o(P)$, as otherwise we would drop below the current image of $\mathrm{lh}(\bar{F})$, contradicting that $\bar{F} \in \mathrm{dom}(t^\Psi_\infty)$. Since $s^\Psi_{\bar{\beta},\eta}$ is a total elementary map, there is $\bar{P} \lhd M^{\bar{\mathcal{T}}}_{\bar{\beta}} = M^{\bar{\mathcal{T}}}_\infty$ beyond $\mathrm{dom}(\bar{F})$ projecting $\leq \mathrm{crit}(\bar{F})$, so $W(\bar{\mathcal{T}}, \bar{F})$, as before. This finishes (1).

For (2), Suppose that $t^\Psi_{\bar{\beta}} \restriction \mathrm{dom}(\bar{F}) \neq s^\Psi_{\bar{\beta},\beta} \restriction \mathrm{dom}(\bar{F})$. We must show $W(\mathcal{T}, F)$ is in the dropping case. We have that $\beta < u^\Psi(\bar{\beta})$, or else our hypothesis is impossible, so let $\gamma+1$ be the successor of β along $[v^\Psi(\bar{\beta}), u^\Psi(\bar{\beta})]_{\mathcal{T}}$. Since $t^\Psi_{\bar{\beta}} \restriction \mathrm{dom}(\bar{F}) \neq s^\Psi_{\bar{\beta},\beta} \restriction \mathrm{dom}(F)$, we must have $\mathrm{crit}(E^{\mathcal{T}}_\gamma) \leq \mathrm{crit}(F)$. But then we must have $\bar{\beta} = \bar{\alpha}$ and $\beta = \alpha$ (as mentioned before, via the proof for (3) and (4) of the previous proposition). By (5), we get $\mathrm{lh}(F) < \mathrm{dom}(E^{\mathcal{T}}_\gamma)$. So since $\mathrm{dom}(E^{\mathcal{T}}_\gamma) < \lambda(E^{\mathcal{T}}_\alpha) = \lambda(E^{\mathcal{T}}_\beta)$, there is some $P \lhd M^{\mathcal{T}}_\beta | \lambda(E^{\mathcal{T}}_\beta)$ with $\mathrm{dom}(F) < o(P)$ and $\rho(P) \leq \mathrm{crit}(F)$. So $W(\mathcal{T}, F)$ is in the dropping case, as desired. $\square$

Note that the proofs of (1) in the cases where $\bar{\beta}+1 = \mathrm{lh}(\bar{\mathcal{T}})$ or $\beta = u^\Psi(\bar{\beta})$ were very general—we only used clause (5) of adequate Ψ-image in the remaining case.

We'll briefly discuss two important cases where we can identify adequate Ψ-images of $\bar{F}$.

Proposition 3.1.54. *Let $\Psi : \bar{\mathcal{T}} \to \mathcal{T}$ be an extended tree embedding, $\bar{F}$ on the $M^{\bar{\mathcal{T}}}_\infty$-sequence, $\bar{\alpha} = \alpha(\bar{\mathcal{T}}, \bar{F})$, and $\alpha = \alpha(\mathcal{T}, t^\Psi_\infty(\bar{F}))$. Suppose that $\alpha \in ran(u^\Psi)$.*

Then $t^\Psi_\infty(\bar{F})$ is an adequate Ψ-image of $\bar{F}$ and $\alpha = u^\Psi(\bar{\alpha})$.

Proof. We'll just show that $\alpha = u^\Psi(\bar\alpha)$, which immediately implies that $t^\Psi_\infty(\bar F)$ is an adequate Ψ-image of $\bar F$, by our previous proposition.

Since $v^\Psi(\bar\alpha) = \sup\{u^\Psi(\zeta)+1 \mid \zeta < \bar\alpha\}$, we must have $v^\Psi(\bar\alpha) \leq \alpha$, as in the proof that (2) held for $\xi = v^\Psi(\bar\alpha)$ in the previous proposition. In particular, $\alpha \neq u^\Psi(\zeta)$ for any $\zeta < \bar\alpha$. But $t^\Psi_\infty(\bar F) = t^\Psi_{\bar\alpha}(\bar F)$ since either $\bar\alpha + 1 = \mathrm{lh}(\mathcal{T})$, and this is trivial, or else $\mathrm{lh}(\bar F) < \mathrm{lh}(E^{\bar{\mathcal{T}}}_{\bar\alpha})$ and so this follows by the agreement properties of the t-maps of a tree embedding. So $\alpha \leq u^\Psi(\bar\alpha)$. Since we assumed $\alpha \in \mathrm{ran}(u^\Psi)$, we must have $\alpha = u^\Psi(\bar\alpha)$. □

We now consider the case that Ψ is inflationary. In this case, we can identify an adequate Ψ-image of $\bar F$ which may differ from the image we proved was adequate in Proposition 3.1.52.

Definition 3.1.55. Let $\Psi : \bar{\mathcal{T}} \to \mathcal{T}$ be an extended tree embedding and $\bar F$ be on the $M^{\bar{\mathcal{T}}}_\infty$-sequence with $\bar F \in \mathrm{dom}(t^\Psi_\infty)$. Let $\bar\alpha = \alpha(\bar{\mathcal{T}}, \bar F)$. The *inflationary Ψ-image of $\bar F$* is $s^\Psi_{\bar\alpha,\xi}(\bar F)$ for ξ least such that $\xi \in [v^\Psi(\bar\alpha), u^\Psi(\bar\alpha)]_{\mathcal{T}}$ and either $\xi = u^\Psi(\bar\alpha)$ or else $\xi < u^\Psi(\bar\alpha)$ and for $\gamma+1$ the successor of ξ in $[v^\Psi(\bar\alpha), u^\Psi(\bar\alpha)]_{\mathcal{T}}$, $\mathrm{dom}(E^{\mathcal{T}}_\gamma) \ntrianglelefteq M^{\mathcal{T}}_\xi|\mathrm{lh}(s^\Psi_{\bar\alpha,\xi}(\bar F))$.

Proposition 3.1.56. *Let $\Psi : \bar{\mathcal{T}} \to \mathcal{T}$ be an inflationary extended tree embedding and $\bar F$ on the $M^{\bar{\mathcal{T}}}_\infty$-sequence. Then the inflationary Ψ-image of $\bar F$ is the minimal adequate Ψ-image of $\bar F$ (with respect to length or corresponding $\alpha(\mathcal{T}, F)$).*

Proof. Let $\bar\alpha = \alpha(\bar{\mathcal{T}}, \bar F)$ and $\bar\beta = \beta(\bar{\mathcal{T}}, \bar F)$. Let $\xi \in [v^\Psi(\bar\alpha), u^\Psi(\bar\alpha)]_{\mathcal{T}}$ least such that $\xi = u^\Psi(\bar\alpha)$ or $\xi < u^\Psi(\bar\alpha)$ and for $\gamma+1$ the successor of ξ in $[v^\Psi(\bar\alpha), u^\Psi(\bar\alpha)]_{\mathcal{T}}$, $s^\Psi_{\bar\alpha,\xi}(\mathrm{lh}(\bar F)) < \mathrm{dom}(E^{\mathcal{T}}_\gamma)$ and $F = s^\Psi_{\bar\alpha,\xi}(\bar F)$. So F is the inflationary Ψ-image of $\bar F$. If $\xi = u^\Psi(\bar\alpha)$, then $F = t^\Psi_{\bar\alpha}(\bar F)$ and the agreement of t-maps implies that F is on the $M^{\mathcal{T}}_\infty$-sequence. So suppose $\xi < u^\Psi(\bar\alpha)$. By normality and our choice of ξ we have that $\mathrm{lh}(F) < \mathrm{dom}(E^{\mathcal{T}}_\gamma) < \mathrm{lh}(E^{\mathcal{T}}_\xi)$, F is on the $M^{\mathcal{T}}_\infty$-sequence. Let $\alpha = \alpha(\mathcal{T}, F)$ and $\beta = \beta(\mathcal{T}, F)$.

For (1) and (2), we have that $\xi = \alpha$ by Proposition 3.1.50, since cases (1) and (3) of that proposition are not possible by how we chose ξ. In particular, if $\alpha = u^\Psi(\bar\alpha)$ or $\alpha < u^\Psi(\bar\alpha)$ and $\mathrm{lh}(F) < \mathrm{crit}(E^{\mathcal{T}}_\gamma)$, then we're done by Proposition 3.1.52, since $\xi = \alpha$ has the maximality property of the hypothesis of that proposition. So we can assume $\alpha < u^\Psi(\bar\alpha)$ and $\mathrm{crit}(E^{\mathcal{T}}_\gamma) \leq \lambda(F) < \mathrm{lh}(F) < \mathrm{dom}(E^{\mathcal{T}}_\gamma)$ for the rest of the proof.

The proof of (3) and (4) from Proposition 3.1.52 works here, even when $\alpha < u^\Psi(\bar\alpha)$, as it made no use of the maximality hypothesis of α. That argument also gives that if $\bar\beta < \bar\alpha$ or $\beta < \alpha$, then $t^\Psi_{\bar\beta} \restriction \mathrm{dom}(\bar F) = s^\Psi_{\bar\beta,\beta} \restriction \mathrm{dom}(\bar F)$.

(5) is immediate by our choice of α and the fact that Ψ is inflationary. □

Finally, we state and prove the Shift Lemma.

Lemma 3.1.57 (Shift Lemma)**.** *Let $\bar{\mathcal{T}}$, $\mathcal{T}$, $\bar{\mathcal{S}}$, $\mathcal{S}$ are normal iteration trees on M of successor lengths and $\Psi : \bar{\mathcal{T}} \to \mathcal{T}$, $\Pi : \bar{\mathcal{S}} \to \mathcal{S}$ extended tree embeddings. Let $\bar F$ be on the $M^{\bar{\mathcal{T}}}_\infty$-sequence with $\bar F \in dom(t^\Psi_\infty)$, $\bar\alpha = \alpha(\bar{\mathcal{T}}, \bar F)$, and $\bar\beta = \beta(\bar{\mathcal{T}}, \bar F)$. Let F be on the $M^{\mathcal{T}}_\infty$-sequence, $\alpha = \alpha(\mathcal{T}, F)$, and $\beta = \beta(\mathcal{T}, F)$.*

Suppose that

(i) $\alpha \in [v^\Psi(\bar\alpha), u^\Psi(\bar\alpha)]_{\mathcal{T}}$,

(ii) $F = s^\Psi_{\bar\alpha,\alpha}(\bar F)$,

(iii) $s^{\Psi}_{\bar{\alpha},\alpha} \restriction dom(\bar{F}) = s^{\Psi}_{\bar{\beta},\beta} \restriction dom(\bar{F})$,

(iv) $\Pi \restriction \bar{\beta}+1 \approx \Psi \restriction \bar{\beta}+1$,

(v) $\mathcal{S} \restriction \beta+1 = \mathcal{T} \restriction \beta+1$,

(vi) $\beta \in [v^{\Pi}(\bar{\beta}), u^{\Pi}(\bar{\beta})]_{\mathcal{S}}$,

(vii) *if* $\bar{\beta}+1 < lh(\bar{\mathcal{S}})$, *then* $dom(\bar{F}) \trianglelefteq M^{\bar{\mathcal{S}}}_{\bar{\beta}}|lh(E^{\bar{\mathcal{S}}}_{\bar{\beta}})$,

(viii) *if* $\beta+1 < lh(\mathcal{S})$, *then* $dom(F) \trianglelefteq M^{\mathcal{S}}_{\beta}|lh(E^{\mathcal{S}}_{\beta})$,

(ix) *if* $W(\mathcal{S},\mathcal{T},F)$ *is in the dropping case, then* $W(\bar{\mathcal{S}},\bar{\mathcal{T}},\bar{F})$ *is in the dropping case, and*

(x) *if* $W(\mathcal{S},\mathcal{T},F)$ *is not in the dropping case, then* $t^{\Pi}_{\bar{\beta}} \restriction dom(\bar{F}) = s^{\Pi}_{\bar{\beta},\beta} \restriction dom(\bar{F})$,

Then $W(\bar{\mathcal{S}},\bar{\mathcal{T}},\bar{F})$ *and* $W(\mathcal{S},\mathcal{T},F)$ *are defined and, letting* $\bar{\mu}$ *be the greatest ordinal* ξ *such that* $W(\bar{\mathcal{S}},\bar{\mathcal{T}},\bar{F}) \restriction \xi+1$ *is well-founded,* μ *the greatest ordinal* ξ *such that* $W(\mathcal{S},\mathcal{T},F) \restriction \xi+1$ *is well-founded,* $\bar{\mathcal{W}} = W(\bar{\mathcal{S}},\bar{\mathcal{T}},\bar{F}) \restriction \bar{\mu}+1$, $\mathcal{W} = W(\mathcal{S},\mathcal{T},F) \restriction \mu+1$, *and* $\bar{\Phi} = \Phi^{\bar{\mathcal{S}},\bar{\mathcal{T}},\bar{F}}$, $\Phi = \Phi^{\mathcal{S},\mathcal{T},F}$, *there is unique partial tree embedding* $\Gamma : \bar{\mathcal{W}} \to \mathcal{W}$ *with maximal domain such that*

1. $\Gamma \restriction \bar{\alpha}+1 \approx \Psi \restriction \bar{\alpha}+1$,
2. $u^{\Gamma}(\bar{\alpha}) = \alpha$,
3. $\Gamma \circ \bar{\Phi} = \Phi \circ \Pi$ *(on their common domain).*

Moreover, if $W(\mathcal{S},\mathcal{T},F)$ *is well-founded,* $W(\bar{\mathcal{S}},\bar{\mathcal{T}},\bar{F})$ *is well-founded and* Γ *is a total extended tree embedding from* $W(\bar{\mathcal{S}},\bar{\mathcal{T}},\bar{F})$ *into* $W(\mathcal{S},\mathcal{T},F)$*. Further,*

(a) *if* $W(\bar{\mathcal{S}},\bar{\mathcal{T}},\bar{F})$ *is not in the dropping case, then the ordinary Shift Lemma applies to* $(s^{\Psi}_{\bar{\alpha},\alpha}, t^{\Pi}_{\infty}, \bar{F})$ *and for* $\tau : Ult(M^{\bar{\mathcal{S}}}_{\infty}, \bar{F}) \to Ult(M^{\mathcal{S}}_{\infty}, F)$ *the copy map,* $t^{\Gamma}_{\infty} \circ \sigma^{\bar{\mathcal{S}},\bar{\mathcal{T}},\bar{F}} = \sigma^{\mathcal{S},\mathcal{T},F} \circ \tau$.

(b) *if* Ψ *and* Π *are non-dropping, then* Γ *is non-dropping, and*

(c) *if* Ψ *and* Π *are inflationary and* F *is the inflationary* Ψ*-image of* $\bar{F}$*, then* Γ *is inflationary.*

Note that by the definition of adequate Ψ-image and Proposition 3.1.53, hypotheses (i)-(x) hold when $\Pi = \Psi$ and F is an adequate Ψ-image of $\bar{F}$.

Also notice that if we assume that all of the trees $\bar{\mathcal{S}},\bar{\mathcal{T}},\mathcal{S},\mathcal{T}$ are all by some strategy Σ for M with SHC^{-}, then $W(\bar{\mathcal{S}},\bar{\mathcal{T}},\bar{F})$ and $W(\mathcal{S},\mathcal{T},F)$ are by Σ, so that we get a total extended tree embedding $\Gamma : W(\bar{\mathcal{S}},\bar{\mathcal{T}},\bar{F}) \to W(\mathcal{S},\mathcal{T},F)$.

Proof. We have $W(\bar{\mathcal{S}},\bar{\mathcal{T}},\bar{F})$ is defined by hypotheses (iv) and (vii) and $W(\mathcal{S},\mathcal{T},F)$ is defined by (v) and (viii). So all of the work is in identifying Γ and proving it is as desired, inductively. At bottom, we are able to do this because the s-maps of tree embeddings are given by the ordinary premouse Shift Lemma at successors.

Notice that $\bar{\mu} \geq \bar{\alpha}+1$ and $\mu \geq \alpha+1$ since $W(\bar{\mathcal{S}}, \bar{\mathcal{T}}, \bar{F}) \restriction \bar{\alpha}+1 = \bar{\mathcal{T}} \restriction \bar{\alpha}+1$ and $W(\mathcal{S}, \mathcal{T}, F) \restriction \alpha+1 = \mathcal{T} \restriction \alpha+1$, so the first models which are possibly ill-founded are the new models $M^{\bar{\mathcal{W}}}_{\bar{\alpha}+1}$ and $M^{\mathcal{W}}_{\alpha+1}$, which are obtained as ultrapowers by $\bar{F}$ and F, respectively.

Now, the u-map of Γ is totally determined by what we have demanded in (1)-(3). We must have

$$u^{\Gamma}(\zeta) = \begin{cases} u^{\Psi}(\zeta) & \text{if } \zeta < \bar{\alpha} \\ \alpha & \text{if } \zeta = \bar{\alpha} \\ u^{\Phi \circ \Pi}(\xi) & \text{if } \zeta > \bar{\alpha}, \text{ where } \xi \text{ is such that } u^{\bar{\Phi}}(\xi) = \zeta. \end{cases}$$

Recall that this third case makes sense since $u^{\bar{\Phi}}$ maps $[\bar{\beta}, \mathrm{lh}(\mathcal{S}))$ onto $[\bar{\alpha}+1, \mathrm{lh}(\mathcal{W}))$ and $u^{\Phi \circ \Pi}(\xi) > \alpha$ for all ξ such that $u^{\bar{\Phi}}(\xi) > \bar{\alpha}$ (since if $u^{\bar{\Phi}}(\xi) > \bar{\alpha}$, then $\xi \geq \bar{\beta}$, so $u^{\Phi \circ \Pi}(\xi) \geq u^{\Phi}(\beta) \geq \alpha+1$).

The definition of $u^{\Gamma}(\zeta)$ just given makes sense for any ordinal ζ, but the actual u-map of Γ has domain $\{\zeta \mid \zeta < \bar{\mu} \text{ and } u^{\Gamma}(\zeta) < \mu\}$ (since the domain can't include anymore than this and we want the domain of Γ to be maximal). In the course of the proof, we'll show that we can drop the condition "$\zeta < \bar{\mu}$" from the description of the domain of u^{Γ}.

Since we had to define u^{Γ} as above and tree embeddings are totally determined by their u-map, uniqueness of Γ is guaranteed. We just need to check that we actually find a tree embedding with this u-map. This amounts to identifying s-maps and t-maps that make the relevant diagrams commute.

We've stipulated $\Gamma \restriction \bar{\alpha}+1 \approx \Psi \restriction \bar{\alpha}+1$ and $u^{\Gamma}(\bar{\alpha}) = \alpha$, so since $u^{\bar{\Phi}}$ is maps $[\bar{\beta}, \mathrm{lh}(\bar{\mathcal{S}}))$ onto $[\bar{\alpha}+1, \mathrm{lh}(\bar{\mathcal{W}}))$, we just need to find appropriate $s^{\Gamma}_{u^{\bar{\Phi}}_{\xi}}$ and $t^{\Gamma}_{u^{\bar{\Phi}}_{\xi}}$ by induction on $\xi \in [\bar{\beta}, \mathrm{lh}(\bar{\mathcal{S}}))$. We also show that if $u^{\Phi \circ \Pi}(\xi) \leq \mu$, then $u^{\bar{\Phi}}(\xi) \leq \bar{\mu}$. We start with the base case.

Base case. $\xi = \bar{\beta}$.

As mentioned above, we have $u^{\bar{\Phi}}(\bar{\beta}) = \bar{\alpha}+1 \leq \bar{\mu}$ outright in this case. We first want to define $s^{\Gamma}_{\bar{\alpha}+1}$. For Γ to be a tree embedding, $s^{\Gamma}_{\bar{\alpha}+1}$ must be the copy map associated to the following situation.

$$M^{\bar{\mathcal{W}}}_{\bar{\alpha}} = M^{\bar{\mathcal{T}}}_{\bar{\alpha}} \xrightarrow{\quad s^{\Psi}_{\bar{\alpha},\alpha} \quad} M^{\mathcal{T}}_{\alpha} = M^{\mathcal{W}}_{\alpha}$$

$$\bar{F} \longmapsto F$$

$$\begin{tikzcd}
M^{\bar{\mathcal{W}}}_{\bar{\beta}} = M^{\bar{\mathcal{S}}}_{\bar{\beta}} \arrow[r, "s^{\Pi}_{\bar{\beta},\beta}"] \arrow[d, "\bar{F}"] & M^{\mathcal{S}}_{\beta} = M^{\mathcal{W}}_{\beta} \arrow[d, "F"] \\
M^{\bar{\mathcal{W}}}_{\bar{\alpha}+1} \arrow[r, dashed, "s^{\Gamma}_{\bar{\alpha}+1}"] & M^{\mathcal{W}}_{\alpha+1}
\end{tikzcd}$$

That is, $s^{\Gamma}_{\bar{\alpha}+1}$ is the copy map associated to $(s^{\Psi}_{\bar{\alpha},\alpha}, s^{\Pi}_{\bar{\beta},\beta}, \bar{F})$, which exists since the ordinary Shift Lemma applies, by hypotheses (iii)-(v).

If $W(\mathcal{S},\mathcal{T},F)$ is in the dropping case, then $W(\bar{\mathcal{S}},\bar{\mathcal{T}},\bar{F})$ is too, by (ix), and so we're done. For the remainder of the proof, suppose that $W(\mathcal{S},\mathcal{T},F)$ is not in the dropping case. If $\mu < u^{\Phi\circ\Pi}(\bar\beta)$, we stop, so suppose $\mu \geq u^{\Phi\circ\Pi}(\bar\beta)$. We must now put

$$t^{\Gamma}_{\bar\alpha+1} = \hat{\imath}^{\mathcal{W}}_{\alpha+1,u^{\Gamma}_{\bar\beta}} \circ s^{\Gamma}_{\bar\alpha+1}.$$

For this to make sense, we need that $\alpha+1 \leq_{\mathcal{W}} u^{\Gamma}(\bar\beta) = u^{\Phi\circ\Pi}(\bar\beta)$. Hypothesis (x) implies that any extender used along $(\beta, u^{\Pi}(\bar\beta)]_{\mathcal{S}}$ has critical point $> \mathrm{crit}(F)$, so u^{Φ} is tree-order preserving on $[\beta, u^{\Pi}(\bar\beta)]_{\mathcal{S}}$. So $\alpha+1 = u^{\Phi}(\beta) \leq_{\mathcal{W}} u^{\Phi\circ\Pi}(\bar\beta) = u^{\Gamma}(\bar\beta)$. We have the following picture.

$$\begin{tikzcd}
M^{\bar{\mathcal{W}}}_{\bar\beta} = M^{\bar{\mathcal{S}}}_{\bar\beta} \arrow[r, "s^{\Pi}_{\bar\beta,\beta}"] \arrow[d, "\bar{F}"] & M^{\mathcal{S}}_{\beta} = M^{\mathcal{W}}_{\beta} \arrow[r, "\mathcal{S}"] \arrow[d, "F"] & M^{\mathcal{S}}_{u^{\Pi}_{\bar\beta}} \arrow[d, "t^{\Phi}_{u^{\Pi}_{\bar\beta}}"] \\
M^{\bar{\mathcal{W}}}_{\bar\alpha+1} \arrow[r, dashed, "s^{\Gamma}_{\bar\alpha+1}"] & M^{\mathcal{W}}_{\alpha+1} \arrow[r, "\mathcal{W}"] & M^{\mathcal{W}}_{u^{\Phi\circ\Pi}_{\bar\beta}}
\end{tikzcd}$$

We already have that the left square commutes, by our choice of $s^{\Gamma}_{\bar\alpha+1}$, so we need to see that the right one does. We may assume $\beta < u^{\Pi}(\bar\beta)$, as otherwise this is trivial. For every $\zeta > \beta$, $u^{\Phi}(\zeta) = v^{\Phi}(\zeta)$ and $t^{\Phi}_{\zeta} = s^{\Phi}_{\zeta}$, letting $\zeta+1$ be the least element of $(\beta, u^{\Pi}(\bar\beta)]_{\mathcal{S}}$, so we can expand the right square as follows.

$$\begin{tikzcd}
M^{\mathcal{S}}_{\beta} \arrow[r, "E^{\mathcal{S}}_{\zeta}"] \arrow[d, "F"] & M^{\mathcal{S}}_{\zeta} \arrow[r, "\mathcal{S}"] \arrow[d, "s^{\Phi}_{\zeta}"] & M^{\mathcal{S}}_{u^{\Pi}_{\bar\beta}} \arrow[d, "s^{\Phi}_{u^{\Psi}_{\bar\beta}}"] \\
M^{\mathcal{W}}_{\alpha+1} \arrow[r, "E^{\mathcal{W}}_{u^{\Phi}_{\zeta}}"] & M^{\mathcal{W}}_{v^{\Phi}_{\zeta}} \arrow[r, "\mathcal{W}"] & M^{\mathcal{W}}_{u^{\Phi\circ\Psi}_{\bar\beta}}
\end{tikzcd}$$

But these squares commute since Φ is a tree embedding: the left square commutes since s^{Φ}_{ζ} is the appropriate copy map and the right square commutes since the s-maps of a tree embedding commute with branch embeddings, by definition. This finishes the base case.

Successor case. $\bar\beta < \xi+1$ and $u^{\Phi\circ\Pi}(\xi+1) \leq \mu$.

Since $v^{\Phi\circ\Pi}(\xi) < \mu$, we have $M^{\mathcal{W}}_{v^{\Phi\circ\Pi}(\xi)}$ is well-founded and so $M^{\bar{\mathcal{W}}}_{v^{\bar\Phi}(\xi)}$ is as well (since $s^{\Gamma}_{u^{\bar\Phi}_{\xi}} : M^{\bar{\mathcal{W}}}_{v^{\bar\Phi}(\xi)} \to M^{\mathcal{W}}_{v^{\Phi\circ\Pi}(\xi)}$ is a total elementary embedding). If $\xi > \bar\beta$, then $v^{\bar\Phi}(\xi) = u^{\bar\Phi}(\xi)$, so we have $u^{\bar\Phi}(\xi) < \bar\mu$. So $u^{\bar\Phi}(\xi+1) = v^{\bar\Phi}(\xi+1) = u^{\bar\Phi}(\xi)+1 \leq \bar\mu$. If $\xi = \bar\beta$, then we have $u^{\bar\Phi}(\xi) = \bar\alpha+1$, so we already had $u^{\bar\Phi}(\xi) \leq \bar\mu$.

Let $\bar\eta = \bar{\mathcal{S}}\text{-pred}(\xi+1)$ and $\eta = \mathcal{S}\text{-pred}(u^{\Pi}(\xi)+1)$. We have $\eta \in [v^{\Pi}(\bar\eta), u^{\Pi}(\bar\eta)]_{\mathcal{S}}$ since Π is a tree embedding.

There are two (similar) subcases depending on the critical point of $E^{\bar{\mathcal{S}}}_{\xi}$.

Subcase 1. $\mathrm{crit}(E^{\bar{\mathcal{S}}}_{\xi}) < \mathrm{crit}(\bar{F})$.

In this case $\bar\eta \leq \bar\beta$ and $\bar\eta = \bar{\mathcal{W}}\text{-pred}(u^{\bar\Phi}(\xi)+1)$. We also have that $\mathrm{crit}(E^{\mathcal{S}}_{u^{\Pi}(\xi)}) < \mathrm{crit}(F)$, so that $\eta \leq \beta$ and $\eta = \mathcal{W}\text{-pred}(u^{\Phi\circ\Pi}(\xi)+1)$, too. We have the following picture, in the case that we don't drop.

$$
\begin{tikzcd}
M^{\bar{\mathcal{S}}}_{\xi} \arrow[rrr, "t^{\Pi}_{\xi}"] \arrow[ddd, "t^{\bar\Phi}_{\xi}"'] \arrow[dr, phantom, "\ni"] & & & M^{\mathcal{S}}_{u^{\Pi}_{\xi}} \arrow[ddd, "t^{\Phi}_{u^{\Pi}_{\xi}}"] \arrow[dl, phantom, "\in"] \\
& E^{\bar{\mathcal{S}}}_{\xi} \arrow[r, mapsto] \arrow[d, mapsto] & E^{\mathcal{S}}_{u^{\Pi}_{\xi}} \arrow[d, mapsto] & \\
& E^{\bar{\mathcal{W}}}_{u^{\bar\Phi}_{\xi}} \arrow[r, mapsto] & E^{\mathcal{W}}_{u^{\Phi\circ\Pi}_{\xi}} & \\
M^{\bar{\mathcal{W}}}_{u^{\bar\Phi}_{\xi}} \arrow[rrr, "t^{\Gamma}_{u^{\bar\Phi}_{\xi}}"'] \arrow[ur, phantom, "\ni"] & & & M^{\mathcal{W}}_{u^{\Phi\circ\Pi}_{\xi}} \arrow[ul, phantom, "\in"]
\end{tikzcd}
$$

$$
\begin{tikzcd}
M^{\bar{\mathcal{S}}}_{\xi+1} \arrow[rrr, "s^{\Pi}_{\xi+1}"] \arrow[ddd, "s^{\bar\Phi}_{\xi+1}"'] & & & M^{\mathcal{S}}_{u^{\Pi}_{\xi}+1} \arrow[ddd, "s^{\Phi}_{u^{\Pi}_{\xi}+1}"] \\
& M^{\bar{\mathcal{S}}}_{\bar\eta} \arrow[ul, "E^{\bar{\mathcal{S}}}_{\xi}"] \arrow[r, "s^{\Pi}_{\bar\eta,\eta}"] \arrow[d, "id"'] & M^{\mathcal{S}}_{\eta} \arrow[ur, "E^{\mathcal{S}}_{u^{\Pi}_{\xi}}"'] \arrow[d, "id"] & \\
& M^{\bar{\mathcal{W}}}_{\bar\eta} \arrow[r, "s^{\Gamma}_{\bar\eta,\eta}"'] \arrow[dl, "E^{\bar{\mathcal{W}}}_{u^{\bar\Phi}_{\xi}}"'] & M^{\mathcal{W}}_{\eta} \arrow[dr, "E^{\mathcal{W}}_{u^{\Phi\circ\Pi}_{\xi}}"] & \\
M^{\bar{\mathcal{W}}}_{u^{\bar\Phi}_{\xi+1}} \arrow[rrr, dashed, "s^{\Gamma}_{u^{\bar\Phi}_{\xi+1}}"'] & & & M^{\mathcal{W}}_{u^{\Phi\circ\Pi}_{\xi}+1}
\end{tikzcd}
$$

Each of the maps along the outer square of the bottom diagram are the copy maps associated to the maps along corresponding side of the top square, inner square, and appropriate extender. In particular, we let $s^{\Gamma}_{u^{\bar\Phi}_{\xi+1}}$ be the copy map associated to $(t^{\Gamma}_{u^{\bar\Phi}_{\xi}},\ s^{\Gamma}_{\bar\eta,\eta},\ E^{\bar{\mathcal{W}}}_{u^{\bar\Phi}(\xi)+1})$, as we must. Note that the ordinary Shift Lemma applies in this case because we have assumed that, so far, Γ is a tree embedding. In particular, since $\mathrm{dom}(E^{\bar{\mathcal{W}}}_{u^{\bar\Phi}(\xi)}) \trianglelefteq M^{\bar{\mathcal{W}}}_{\bar\eta}|\lambda(E^{\bar{\mathcal{W}}}_{\bar\eta})$, the agreement properties of t-maps gives that

$$
\begin{aligned}
t^{\Gamma}_{u^{\bar{\Phi}}(\xi)} \restriction \mathrm{dom}(E^{\bar{\mathcal{W}}}_{u^{\bar{\Phi}}(\xi)}) &= t^{\Gamma}_{\bar{\eta}} \restriction \mathrm{dom}(E^{\bar{\mathcal{W}}}_{u^{\bar{\Phi}}(\xi)}) \\
&= s^{\Gamma}_{\bar{\eta},\eta} \restriction \mathrm{dom}(E^{\bar{\mathcal{W}}}_{u^{\bar{\Phi}}(\xi)}),
\end{aligned}
$$

using for this second equivalence that $\mathrm{crit}(\hat{\imath}^{\mathcal{W}}_{\eta,u^{\Gamma}(\bar{\eta})}) > \mathrm{crit}(E^{\mathcal{W}}_{u^{\Phi\circ\Pi}(\xi)})$, as, otherwise, we used an extender $E^{\mathcal{W}}_{\gamma}$ with $\mathrm{crit}(E^{\mathcal{W}}_{\gamma}) \leq \mathrm{crit}(E^{\mathcal{W}}_{u^{\Phi\circ\Pi}(\xi)}) < \lambda(E^{\mathcal{W}}_{\gamma})$, but then $\mathrm{crit}(E^{\mathcal{W}}_{u^{\Phi\circ\Pi}(\xi)})$ can't be in $\mathrm{ran}(\hat{\imath}^{\mathcal{W}}_{\eta,u^{\Gamma}(\bar{\eta})}) \supseteq \mathrm{ran}(t^{\Gamma}_{\bar{\eta}})$, a contradiction.

In the lower diagram, the inner square commutes by our induction hypothesis and all the trapezoids commute since the outer maps are copy maps associated to the relevant objects. We now want to see that the full outer square commutes. Let's look at the two ways of going around this outer square, $s^{\Phi}_{u^{\Phi}_{\xi}+1} \circ s^{\Pi}_{\xi+1}$ and $s^{\Gamma}_{u^{\bar{\Phi}}_{\xi+1}} \circ s^{\bar{\Phi}}_{\xi+1}$. Since $s^{\Pi}_{\xi+1}$ and $s^{\Phi}_{u^{\Phi}_{\xi}+1}$ are copy maps associated to the appropriate objects, Lemma 3.1.8 gives that $s^{\Phi}_{u^{\Phi}_{\xi}+1} \circ s^{\Pi}_{\xi+1}$ is the copy map associated to $(t^{\Phi}_{u^{\Phi}_{\xi}} \circ t^{\Pi}_{\xi},\ s^{\Psi}_{\bar{\eta},\eta},\ E^{\bar{\mathcal{S}}}_{\xi})$. Similarly, $s^{\Gamma}_{u^{\bar{\Phi}}_{\xi+1}} \circ s^{\bar{\Phi}}_{\xi+1}$ is the copy map associated to $(t^{\Gamma}_{u^{\bar{\Phi}}_{\xi}} \circ t^{\bar{\Phi}}_{\xi}$, $s^{\Gamma}_{\bar{\eta},\eta},\ E^{\bar{\mathcal{S}}}_{\xi})$. But $t^{\Phi}_{u^{\Phi}_{\xi}} \circ t^{\Pi}_{\xi} = t^{\Gamma}_{u^{\bar{\Phi}}_{\xi}} \circ t^{\bar{\Phi}}_{\xi}$ and $s^{\Pi}_{\bar{\eta},\eta} = s^{\Gamma}_{\bar{\eta},\eta}$, so the two ways of going around the outer square are both the copy map associated to the same objects, so $s^{\Phi}_{u^{\Phi}_{\xi}+1} \circ s^{\Pi}_{\xi+1} = s^{\Gamma}_{u^{\bar{\Phi}}_{\xi+1}} \circ s^{\bar{\Phi}}_{\xi+1}$.

Note that $u^{\Phi\circ\Pi}_{\xi} + 1 = v^{\Phi\circ\Pi}_{\xi+1} \leq_{\mathcal{W}} u^{\Phi\circ\Pi}_{\xi+1}$. We now define

$$
t^{\Gamma}_{u^{\bar{\Phi}}_{\xi+1}} = \hat{\imath}^{\mathcal{W}}_{u^{\Phi\circ\Pi}_{\xi}+1,\,u^{\Phi\circ\Pi}_{\xi+1}} \circ s^{\Gamma}_{u^{\bar{\Phi}}_{\xi+1}},
$$

as we must. Finally, we check that this assignment gives us a commuting square of the t-maps. We get the following diagram.

$$
\begin{tikzcd}
M^{\bar{\mathcal{S}}}_{\xi+1} \arrow[r, "s^{\Pi}_{\xi+1}"] \arrow[d, "t^{\bar{\Phi}}_{\xi+1}"'] & M^{\mathcal{S}}_{v^{\Pi}_{\xi+1}} \arrow[r, "\mathcal{S}"] \arrow[d, "t^{\Phi}_{v^{\Pi}_{\xi+1}}"'] & M^{\mathcal{S}}_{u^{\Pi}_{\xi+1}} \arrow[d, "t^{\Phi}_{u^{\Pi}_{\xi+1}}"] \\
M^{\bar{\mathcal{W}}}_{u^{\bar{\Phi}}_{\xi+1}} \arrow[r, "s^{\bar{\Phi}}_{u^{\bar{\Phi}}_{\xi+1}}"'] & M^{\mathcal{W}}_{v^{\Phi}(u^{\Pi}_{\xi}+1)} \arrow[r, "\mathcal{W}"'] & M^{\mathcal{W}}_{u^{\Phi\circ\Pi}_{\xi+1}}
\end{tikzcd}
$$

We just need to see that this diagram commutes, since $t^{\Psi}_{\xi+1}$ is just the map going across the top and $t^{\Gamma}_{u^{\bar{\Phi}}(\xi+1)}$ is the map going across the bottom (so this really is the relevant square of t-maps). The left square is just the outer square of the lower commuting diagram, above, though we used $u^{\Pi}(\xi) + 1 = v^{\Pi}(\xi + 1)$ and $u^{\Phi} \circ u^{\Pi}(\xi) + 1 = v^{\Phi}(u^{\Pi}(\xi) + 1)$ to change the labeled indices of the models in the middle column to emphasize how we knew they were tree-related to the appropriate models all the way on the right (we get these equivalences since $\xi + 1 > \bar{\beta}$ and $u^{\Pi}(\xi + 1), u^{\Pi}(\xi) + 1 > \beta$). We've also used that all the vertical t-maps are the same as the corresponding s-maps (by the equivalence of the indices just mentioned).

This last fact (that the vertical t-maps are the same as the corresponding s-maps) also gives us that the square on the right commutes, since Φ is a tree embedding.

If we drop when applying any of the $E^{\bar{\mathcal{S}}}_\xi, E^{\mathcal{S}}_{u^\Pi_\xi}$, etc, then we drop applying all of them and the initial segments to which we apply these extenders are all mapped to each other by the relevant maps. In this case, everything remains the same except that we must use the initial segments to which we apply the extenders instead of the models displayed in the above diagrams, e.g. some $P \trianglelefteq M^{\bar{\mathcal{S}}}_{\bar\eta}$ instead of $M^{\bar{\mathcal{S}}}_{\bar\eta}$.

Subcase 2. $\operatorname{crit}(E^{\bar{\mathcal{S}}}_\xi) \geq \operatorname{crit}(\bar F)$.

In this case $\bar\eta \geq \bar\beta$ and $u^{\bar\Phi}(\bar\eta) = \bar{\mathcal{W}}\text{-pred}(u^{\bar\Phi}(\xi)+1)$. We also get $\operatorname{crit}(E^{\mathcal{S}}_{u^\Pi(\xi)}) \geq \operatorname{crit}(F)$, so $\eta \geq \beta$ and $u^\Phi(\eta) = \mathcal{W}\text{-pred}(u^{\Phi\circ\Psi}(\xi)+1)$. We now have the model to which $E^{\bar{\mathcal{S}}}_\xi$ is applied is related to the model to which $E^{\bar{\mathcal{W}}}_{u^{\bar\Phi}(\xi)}$ is applied by a t-map of Φ, whereas they were just the same model in the previous case. Similarly on the $\mathcal{S}$-$\mathcal{W}$ side. The only thing this changes is that we replace the identity maps in the above previous diagram with these t-maps. This is the diagram for the non-dropping case (as before, dropping makes little difference).

$$\begin{tikzcd}
M^{\bar{\mathcal{S}}}_{\xi+1} \arrow[rrr, "s^\Pi_{\xi+1}"] \arrow[ddd, "s^{\bar\Phi}_{\xi+1}"'] & & & M^{\mathcal{S}}_{u^\Pi_\xi+1} \arrow[ddd, "s^\Phi_{u^\Pi_\xi+1}"] \\
& M^{\bar{\mathcal{S}}}_{\bar\eta} \arrow[ul, "E^{\bar{\mathcal{S}}}_\xi"] \arrow[r, "s^\Pi_{\bar\eta,\eta}"] \arrow[d, "t^{\bar\Phi}_{\bar\eta}"'] & M^{\mathcal{S}}_\eta \arrow[ur, "E^{\mathcal{T}}_{u^\Pi_\xi}"'] \arrow[d, "t^\Phi_\eta"] & \\
& M^{\bar{\mathcal{W}}}_{u^{\bar\Phi}_{\bar\eta}} \arrow[r, "s^\Gamma_{\bar\eta,\eta}"'] \arrow[dl, "E^{\bar{\mathcal{W}}}_{u^{\bar\Phi}_\xi}"'] & M^{\mathcal{W}}_{u^\Phi_\eta} \arrow[dr, "E^{\mathcal{W}}_{u^{\Phi\circ\Pi}_\xi}"] & \\
M^{\bar{\mathcal{W}}}_{u^{\bar\Phi}_{\xi+1}} \arrow[rrr, dashed, "s^\Gamma_{u^{\bar\Phi}_{\xi+1}}"'] & & & M^{\mathcal{W}}_{u^{\Phi\circ\Pi}_\xi+1}
\end{tikzcd}$$

The rest of the diagrams and arguments are as before. This finishes the successor case.

Limit case. $\lambda > \beta$ is a limit and $u^{\Phi\circ\Pi}(\lambda) \leq \mu$.

We have $u^{\Phi\circ\Pi}(\xi) < \mu$ for all $\xi < \lambda$, so that by our induction hypothesis, $u^{\bar\Phi}(\xi) < \bar\mu$. So, $u^{\bar\Phi}(\lambda) = v^{\bar\Phi}(\lambda) = \sup\{u^{\bar\Phi}(\xi) \mid \xi < \lambda\} \leq \bar\mu$.

Let $\bar c = [0, u^{\bar\Phi}(\lambda))_{\bar{\mathcal{W}}}$ and $c = [0, u^{\Phi\circ\Pi}(\lambda))_{\mathcal{W}}$. We need to see that c is the $\leq_{\mathcal{W}}$-downward closure of $v^\gamma[c]$. To do this, we just trace $\bar c$, c back to the branch $b = [0,\lambda)_{\bar{\mathcal{S}}}$. We have

$$\bar c = \{\xi \mid \exists \eta \in b\, (\xi \leq_{\bar{\mathcal{W}}} v^{\bar\Phi}(\eta))\},$$

and
$$c = \{\xi \mid \exists \eta \in b\,(\xi \leq_{\mathcal{W}} v^{\Phi\circ\Pi}(\eta))\}.$$
We also have that $v^\Gamma(v^{\bar{\Phi}}(\eta)) = v^{\Phi\circ\Pi}(\eta)$, so that $v^\Gamma[\bar{c}] = c$, as desired.

So, we get our map $s^\Gamma_{u^{\bar{\Phi}}_\lambda}$ commuting with the maps $s^\Gamma_{u^{\bar{\Phi}}(\xi)}$ since we are taking the direct limits along $\bar{c}$ and c on both sides. From here, we get $t^\Gamma_{u^{\bar{\Phi}}_\lambda}$ as in the successor case. This finishes our construction of Γ.

For the "moreover" clause, we've already shown that if $u^{\Phi\circ\Pi}(\xi) \leq \mu$, $u^{\bar{\Phi}}(\xi) \leq \bar{\mu}$. So, if the full $W(\mathcal{S}, \mathcal{T}, F)$ is well-founded, then for all $\xi < \mathrm{lh}(\bar{\mathcal{S}})$, $u^{\bar{\Phi}}(\xi) \leq \bar{\mu}$. But then $\bar{\mu} + 1 = \mathrm{lh}(W(\bar{\mathcal{S}}, \bar{\mathcal{T}}, \bar{F}))$, so Γ is a total extended tree embedding from $W(\bar{\mathcal{S}}, \bar{\mathcal{T}}, \bar{F})$ into $W(\mathcal{S}, \mathcal{T}, F)$. Finally, we handle the "further" clauses. For (a), note that conditions (iii), (x), and the agreement properties of t-maps of a tree embedding immediately give that the ordinary Shift Lemma applies to $(s^\Psi_{\bar{\alpha},\alpha}, t^\Pi_\infty, \bar{F})$. It is straightforward to use that $t^\Gamma_\infty \circ t^{\bar{\Phi}}_\infty = t^\Phi_\infty \circ t^\Pi_\infty$ and the definition of the embedding normalization maps to show $t^\Gamma_\infty \circ \sigma^{\bar{\mathcal{S}},\bar{\mathcal{T}},\bar{F}} = \sigma^{\mathcal{S},\mathcal{T},F} \circ \tau$, where τ is the copy map associated to $(s^\Psi_{\bar{\alpha},\alpha}, t^\Pi_\infty, \bar{F})$. For (b), suppose that Ψ and Π are non-dropping. If $W(\mathcal{S}, \mathcal{T}, F)$ is in the dropping case or $\beta + 1 = \mathrm{lh}(\mathcal{S})$, it is easy to see that Γ must be non-dropping (as our last t-map is just a copy map). Otherwise, Γ still must be non-dropping by our choice of t-maps, since for any $\xi < \mathrm{lh}(\bar{\mathcal{S}})$ we have $v^\Pi(\xi)$-to-$u^\Pi(\xi)$ drops in $\mathcal{S}$ iff $v^{\Phi\circ\Pi}(\xi)$-to-$u^{\Phi\circ\Pi}(\xi)$ drops in $\mathcal{W}$. For (c), suppose that Ψ and Π are inflationary and need to check that Γ is, too. By Proposition 3.1.16, $\Phi \circ \Pi$ is inflationary, so $\Gamma \circ \bar{\Phi}$ is, too, since they're equal (by (3)). Since Ψ is inflationary, we have $\Gamma \restriction \bar{\alpha}+1$ is too, by (1). Since we chose F to be the inflationary Ψ-image of $\bar{F}$, $\Gamma \restriction \bar{\alpha}+2$ is inflationary as well. So Proposition 3.1.14 (applied to $\Gamma \circ \bar{\Phi}$) gives that Γ is inflationary, since $[\bar{\alpha}+1, \mathrm{lh}(\bar{\mathcal{W}})) \subseteq \mathrm{ran}(u^{\bar{\Phi}})$. □

We will carry over our notation for applications of the ordinary Shift Lemma.

Definition 3.1.58. For extended tree embeddings $\Psi : \bar{\mathcal{T}} \to \mathcal{T}$ and $\Pi : \bar{\mathcal{S}} \to \mathcal{S}$ and an extender $\bar{F}$ on the $M^{\bar{\mathcal{T}}}_\infty$-sequence and F on the $M^{\mathcal{T}}_\infty$-sequence, we'll say *the Shift Lemma applies to* $(\Psi, \Pi, \bar{F}, F)$ iff the hypotheses of the Shift Lemma are met, i.e. (i)-(x) hold.

If $W(\mathcal{S}, \mathcal{T}, F)$, we'll say that an extended tree embedding $\Gamma : W(\bar{\mathcal{S}}, \bar{\mathcal{T}}, \bar{F}) \to W(\mathcal{S}, \mathcal{T}, F)$ is the *copy tree embedding associated to* $(\Psi, \Pi, \bar{F}, F)$ iff it is the unique extended tree embedding as in the conclusion of the Shift Lemma.

Next we will carry out the copying construction: given an extended tree embedding $\Psi : \mathcal{S} \to \mathcal{T}$, we will copy a meta-tree $\mathbb{S}$ on $\mathcal{S}$ to a meta-tree $\mathbb{T}$ on $\mathcal{T}$, using the Shift Lemma to determine the extenders of $\mathbb{T}$. Because there may be multiple adequate Ψ-images of some extender, there may be multiple ways to copy $\mathbb{S}$. We'll use the minimal one; this is occasionally important.

Proposition 3.1.59. *Let $\mathbb{S} = \langle \mathcal{S}_\xi, F_\xi, \Phi_{\eta,\xi} \rangle$ be a meta-tree. Let $\xi + 1 < lh(\mathbb{S})$ and $\eta = \mathbb{S}$-pred$(\xi + 1)$. Then η is the least $\zeta \leq \xi$ such that $\beta_\xi \leq \alpha_\zeta$.*

Proof. For any $\zeta < \xi$, $F_\zeta = E^{\mathcal{S}_\xi}_{\alpha_\zeta}$ and $\beta_\xi \leq \alpha_\xi$, trivially. So for any $\zeta \leq \xi$,
$$\beta_\xi \leq \alpha_\zeta \Leftrightarrow \mathrm{crit}(F_\xi) < \lambda(F_\zeta).$$
We have that η is the least $\zeta \leq \xi$ such that $\mathrm{crit}(F_\xi) < \lambda(F_\zeta)$, by definition, so it is also the least $\zeta \leq \xi$ such that $\beta_\xi < \alpha_\zeta$. □

Theorem 3.1.60 (Copying). *Let M be a premouse, $\mathcal{S}, \mathcal{T}$ be normal trees on M of successor lengths, and $\Gamma : \mathcal{S} \to \mathcal{T}$ a non-dropping extended tree embedding. Let $\mathbb{S} = \langle \mathcal{S}_\xi, F_\zeta, \Phi^{\eta,\xi} \mid \xi, \zeta+1 < lh(\mathbb{S})\rangle$ be a meta-tree on $\mathcal{S}$.*

Then there is a largest $\mu \leq lh(\mathbb{S})$ such that there is a meta-tree $\Gamma\mathbb{S} = \langle \mathcal{T}_\xi, G_\zeta, \Psi^{\eta,\xi} \mid \xi, \zeta+1 < \mu\rangle$ on $\mathcal{T}$ with tree-order $\leq_{\mathbb{S}}\restriction \mu$ and for $\xi < \mu$, non-dropping extended tree embeddings $\Gamma^\xi : \mathcal{S}_\xi \to \mathcal{T}_\xi$ such that

1. *$\Gamma^0 = \Gamma$,*
2. *for $\xi + 1 < \mu$, G_ξ is an adequate Γ^ξ-image of F_ξ, and*
3. *for all $\eta \leq_{\mathbb{S}} \xi < \mu$, $\Gamma^\xi \circ \Phi^{\eta,\xi} = \Psi^{\eta,\xi} \circ \Gamma^\eta$.*

Moreover, if $\mathcal{S}, \mathcal{T}$ are by some iteration strategy Σ for M with SHC⁻ and $\mathbb{S}$ is by Σ^*, then $\mu = lh(\mathbb{S})$ and $\Gamma\mathbb{S}$ is by Σ^*.

Proof. We define $\Gamma\mathbb{S}$ by induction, using the Shift Lemma at successors. μ will just be the least ordinal such that this process breaks down or the full $\mathrm{lh}(\mathbb{S})$ if it doesn't break down.

Let $\bar{\alpha}_\xi = \alpha(F_\xi, \mathcal{S}_\xi)$ and $\bar{\beta}_\xi = \beta(F_\xi, \mathcal{S}_\xi)$. Supposing we've define $\Gamma\mathbb{S} \restriction \xi + 1$, let $\alpha_\xi = \alpha(G_\xi, \mathcal{T}_\xi)$ and $\beta_\xi = \beta(G_\xi, \mathcal{T}_\xi)$. We'll maintain the following by induction for $\eta < \xi < \mu$.

1. $\Gamma^\eta \restriction \bar{\alpha}_\eta + 1 \approx \Gamma^\xi \restriction \bar{\alpha}_\eta + 1$,
2. $\alpha_\eta = u^{\Gamma^\xi}(\bar{\alpha}_\eta) \leq_{\mathcal{T}_\eta} u^{\Gamma^\eta}(\bar{\alpha}_\eta)$,
3. $\mathcal{T}_\xi \restriction \alpha_\eta + 1 = \mathcal{T}_\eta \restriction \alpha_\eta + 1$, and
4. $t^{\Gamma^\xi}_\infty$ is total.

Note that (3) is actually trivial, since $\Gamma\mathbb{S}$ will be be a meta-tree and (4) follows from the fact that the Γ^ξ will be non-dropping. This will allow us to verify that $\Gamma\mathbb{S}$ has the same tree order as $\mathbb{S}$ and show that we satisfy the hypotheses of the Shift Lemma at successor stages.

Suppose we've defined $\Gamma(\mathbb{S}\restriction \xi + 1)$, so we have $\Gamma^\xi : \mathcal{S}_\xi \to \mathcal{T}_\xi$ an extended tree embedding with total last t-map, by (4). In particular, $F_\xi \in \mathrm{dom}(t^{\Gamma^\xi}_\infty)$. We let G_ξ be the minimal adequate Γ^ξ-image of F_ξ. (1)-(3) easily imply that for $\eta < \xi$, $\mathrm{crit}(F_\xi) < \lambda(F_\eta)$ iff $\mathrm{crit}(G_\xi) < \lambda(G_\eta)$, so that, $\xi+1$ has the same tree-predecessor in both $\mathbb{S}$ and $\Gamma\mathbb{S}$. Let $\eta = \mathbb{S}\text{-pred}(\xi+1) = \mathbb{T}\text{-pred}(\xi + 1)$.

We want to let $\Gamma^{\xi+1}$ be the copy tree embedding associated to $(\Gamma^\xi, \Gamma^\eta, F_\xi, G_\xi)$. So we just need to show:

Claim 1. *The Shift Lemma applies to $(\Gamma^\xi, \Gamma^\eta, F_\xi, G_\xi)$.*

Proof. Recalling what this means, we need to verify the following ten (!) conditions are satisfied.

(i) $\alpha_\xi \in [v^{\Gamma^\xi}(\bar{\alpha}_\xi), u^{\Gamma^\xi}(\bar{\alpha}_\xi)]_{\mathcal{T}_\xi}$,

(ii) $G_\xi = s^{\Gamma^\xi}_{\bar{\alpha}_\xi, \alpha_\xi}(F_\xi)$,

(iii) $s^{\Gamma^\xi}_{\bar\alpha_\xi,\alpha_\xi}\restriction \mathrm{dom}(F_\xi) = s^{\Gamma^\xi}_{\bar\beta_\xi,\beta_\xi}\restriction \mathrm{dom}(F_\xi)$,

(iv) $\Gamma^\eta\restriction\bar\beta_\xi+1 \approx \Gamma^\xi\restriction\bar\beta_\xi+1$,

(v) $\mathcal{T}_\eta\restriction\beta_\xi+1 = \mathcal{T}_\xi\restriction\beta_\xi+1$,

(vi) $\beta_\xi\in[v^{\Gamma^\eta}(\bar\beta_\xi), u^{\Gamma^\eta}(\bar\beta_\xi)]_{\mathcal{T}_\eta}$,

(vii) if $\bar\beta_\xi+1<\mathrm{lh}(\mathcal{S}_\eta)$, then $\mathrm{dom}(F_\xi)\trianglelefteq M^{\mathcal{S}_\eta}_{\bar\beta_\xi}|\mathrm{lh}(E^{\mathcal{S}_\eta}_{\bar\beta_\xi})$,

(viii) if $\beta_\xi+1<\mathrm{lh}(\mathcal{T}_\eta)$, then $\mathrm{dom}(G_\xi)\trianglelefteq M^{\mathcal{T}_\eta}_{\beta_\xi}|\mathrm{lh}(E^{\mathcal{T}_\eta}_{\beta_\xi})$,

(ix) if $W(\mathcal{T}_\eta,\mathcal{T}_\xi,G_\xi)$ is in the dropping case, then $W(\mathcal{S}_\eta,\mathcal{S}_\xi,F_\xi)$ is in the dropping case,

(x) if $W(\mathcal{T}_\eta,\mathcal{T}_\xi,G_\xi)$ is not in the dropping case, then $t^{\Gamma^\eta}_{\bar\beta_\xi}\restriction\mathrm{dom}(F_\xi) = s^{\Gamma^\eta}_{\bar\beta_\xi,\beta_\xi}\restriction\mathrm{dom}(F_\xi)$,

This is all pretty easy. First, since we chose G_ξ to be an adequate Γ^ξ-image of F_ξ, all these conditions are satsified if $\eta=\xi$. So suppose $\eta<\xi$. We still get (i)-(iii) by how we chose G_ξ. (iv) is immediate by our induction hypothesis (1), since $\bar\beta_\xi\leq\bar\alpha_\eta$. Since $\beta_\xi\leq\alpha_\eta$, induction hypothesis (3) gives (v). (iv) also follows since G_ξ is an adequate Γ_ξ-image of F_ξ (using clause (4) of that definition). For (vi), we consider cases: if $\bar\beta_\xi<\bar\alpha_\eta$, (vi) is immediate from (1); if $\bar\beta_\xi=\bar\alpha_\eta$, this follows by (2), (3), and that G_ξ is adequate (clause (3) of that defintion). (vii) and (viii) are immediate since $\mathbb{S}$ and $\Gamma(\mathbb{S}\restriction\xi+1)^\frown\langle G_\xi\rangle$ are meta-trees.

For (ix), suppose $W(\mathcal{T}_\eta,\mathcal{T}_\xi,G_\xi)$ is in the dropping case. We consider cases. First suppose that $\bar\beta_\xi<\bar\alpha_\eta$ or $\bar\beta_\xi=\bar\alpha_\eta$ but $\beta_\xi<\alpha_\eta$. Then either $\beta_\xi+1=\mathrm{lh}(\mathcal{T}_\eta)$ or else $E^{\mathcal{T}_\eta}_{\beta_\xi}=E^{\mathcal{T}_\xi}_{\beta_\xi}$, and so in either case, $W(\mathcal{T}_\xi,G_\xi)$ is in the dropping case, too. By Proposition 3.1.53, since G_ξ is adequate, this implies $W(\mathcal{S}_\xi,F_\xi)$ is in the dropping case. Since either $\bar\beta_\xi+1=\mathrm{lh}(\mathcal{S}_\eta)$ or $\mathrm{lh}(E^{\mathcal{S}_\xi}_{\bar\beta_\xi})\leq\mathrm{lh}(E^{\mathcal{S}_\eta}_{\bar\beta_\xi})$ (as either $E^{\mathcal{S}_\xi}_{\bar\beta_\xi}=E^{\mathcal{S}_\eta}_{\bar\beta_\xi}$ or $\bar\beta_\xi=\bar\alpha_\eta$ and so $E^{\mathcal{S}_\xi}_{\bar\beta_\xi}=F_{\bar\alpha_\eta}$). So it follows that $W(\mathcal{S}_\eta,\mathcal{S}_\xi,F_\xi)$ is in the dropping case, as desired. If $\alpha_\eta=u^{\Gamma^\eta}(\bar\alpha_\eta)$ or $\bar\alpha_\eta+1=\mathrm{lh}(\mathcal{S}_\eta)$, then the arguments from the proof of (1) of Proposition 3.1.53, in the cases that $\bar\beta=u^\Psi(\bar\beta)$ or $\beta+1=\mathrm{lh}(\bar{\mathcal{T}})$, give $W(\mathcal{S}_\eta,\mathcal{S}_\xi,F_\xi)$ is in the dropping case, too. So suppose $\alpha_\eta<u^{\Gamma^\eta}(\bar\alpha_\eta)$ and $\bar\alpha_\eta<\mathrm{lh}(\mathcal{S}_\eta)$. We'll use that G_η is an adequate Γ^η-image of F_η. Since we assumed $W(\mathcal{T}_\eta,\mathcal{T}_\xi,G_\xi)$ is in the dropping case, there is a level $P\triangleleft M^{\mathcal{T}_\eta}_{\alpha_\eta}|\mathrm{lh}(E^{\mathcal{T}_\eta}_{\alpha_\eta})$ such that $\mathrm{dom}(G_\xi)<o(P)$ and $\rho(P)\leq\mathrm{crit}(G_\xi)$. Fix P the least such level. It is enough to see that $P\trianglelefteq M^{\mathcal{T}_\eta}_{\alpha_\eta}|s^{\Gamma^\eta}_{\bar\alpha_\eta,\alpha_\eta}(\mathrm{lh}(E^{\mathcal{S}_\eta}_{\bar\alpha_\eta}))$. Let $\gamma+1$ be the successor of α_η in $(\alpha_\eta,u^{\Gamma^\eta}(\bar\alpha_\eta)]_{\mathcal{T}_\eta}$. If $o(P)<\mathrm{crit}(E^{\mathcal{T}_\eta}_\gamma)$, then as $\mathrm{crit}(E^{\mathcal{T}_\eta}_\gamma)<s^{\Gamma^\eta}_{\bar\alpha_\eta,\alpha_\eta}(\mathrm{lh}(E^{\mathcal{S}_\eta}_{\bar\alpha_\eta}))$, we're done. So suppose $\mathrm{crit}(E^{\mathcal{T}_\eta}_\gamma)\leq o(P)$. Then we must have $\mathrm{crit}(E^{\mathcal{T}_\eta}_\gamma)\leq\rho(P)\leq\mathrm{crit}(G_\xi)$. Since G_η is an adequate Γ^η-image of F_η (specifically, clause (5)), we have $\mathrm{lh}(G_\eta)<\mathrm{dom}(E^{\mathcal{T}_\eta}_\gamma)<s^{\Gamma^\eta}_{\bar\alpha_\eta,\alpha_\eta}(\mathrm{lh}(E^{\mathcal{S}_\eta}_{\bar\alpha_\eta}))$. Since $\mathrm{dom}(G_\xi)\trianglelefteq M^{\mathcal{T}_\eta}_{\alpha_\eta}|\mathrm{lh}(G_\eta)$, we must have $P\trianglelefteq\mathrm{dom}(E^{\mathcal{T}_\eta}_\gamma)$, since we chose P to be minimal. So $P\trianglelefteq M^{\mathcal{T}_\eta}_{\alpha_\eta}|s^{\Gamma^\eta}_{\bar\alpha_\eta,\alpha_\eta}(\mathrm{lh}(E^{\mathcal{S}_\eta}_{\bar\alpha_\eta}))$, as desired. This finishes (ix).

For (x), suppose $t^{\Gamma^\eta}_{\bar\beta_\xi}\restriction\mathrm{dom}(F_\xi)\neq s^{\Gamma^\eta}_{\bar\beta_\xi,\beta_\xi}\restriction\mathrm{dom}(F_\xi)$. Then we must have $\bar\beta_\xi<u^\Psi(\bar\beta_\xi)$, but also that $\bar\beta_\xi=\bar\alpha_\eta$ and $\beta_\xi=\alpha_\eta$. So let $\gamma+1$ be the successor of $\beta_\xi=\alpha_\eta$ in $(\alpha_\eta,u^{\Gamma_\eta}(\bar\alpha_\eta)]_{\mathcal{T}_\eta}$. We must have $\mathrm{crit}(E^{\mathcal{T}_\eta}_\gamma)\leq\mathrm{crit}(G_\xi)$. So since G_η is adequate (clause (5), again), we have

$\mathrm{lh}(G_\eta) < \mathrm{dom}(E_\gamma^{\mathcal{T}_\eta})$. So some level $P \triangleleft M_{\alpha_\eta}^{\mathcal{T}_\eta}|\mathrm{lh}(E_{\alpha_\eta}^{\mathcal{T}_\eta}) = M_{\beta_\eta}^{\mathcal{T}_\eta}|\mathrm{lh}(E_{\beta_\eta}^{\mathcal{T}_\eta})$ projects across $\mathrm{dom}(G_\xi)$, i.e. $W(\mathcal{T}_\eta, \mathcal{T}_\xi, G_\xi)$ is in the dropping case.

□

So we let $\Gamma^{\xi+1} : \mathcal{S}_{\xi+1} \to \mathcal{T}_{\xi+1}$ be the copy tree embedding associated to $(\Gamma^\xi, \Gamma^\eta, F_\xi, G_\xi)$. So we have $\Gamma^{\xi+1} \restriction \bar{\alpha}_\xi + 1 = \Gamma^\xi \restriction \bar{\alpha}_\xi + 1$, $u^{\Gamma^{\xi+1}}(\bar{\alpha}_\xi) = \alpha_\xi \leq_{\mathcal{T}_\xi} u^{\Gamma^\xi}(\bar{\alpha}_\xi)$, $\Gamma^{\xi+1} \circ \Phi^{\eta,\xi+1} = \Psi^{\eta,\xi+1} \circ \Gamma^\eta$, and $\Gamma^{\xi+1}$ is non-dropping, by the Shift Lemma. This easily gives induction hypotheses (1)-(4) hold at $\xi + 1$. This finishes the successor step.

Let $\lambda < \mathrm{lh}(\mathbb{S})$ be a limit ordinal and let $b = [0, \lambda)_{\mathbb{S}}$. We put $\mathcal{T}_\lambda = \lim\langle \mathcal{T}_\xi, \Psi^{\eta,\xi} \mid \eta \leq_{\mathbb{S}} \xi \in b\rangle$, if this direct limit is well-founded. Otherwise, we put $\mu = \lambda$ and stop. Suppose it is well-founded. We let Γ_λ be the extended tree embedding guaranteed by Proposition 3.1.11. Now the normality clause in the definition of meta-tree gives that whenever $\zeta < \eta \leq_{\mathbb{S}} \xi \leq \lambda$, $\Phi^{\eta,\xi} \restriction \bar{\alpha}_\zeta + 1 \approx Id$, $\Psi^{\eta,\xi} \restriction \alpha_\zeta + 1 \approx Id$ (using that $\Gamma\mathbb{S} \restriction \lambda$ is a meta-tree with the same tree-order as $\mathbb{S}$, so far), and $u^{\Gamma^\eta}(\bar{\alpha}_\zeta) = \alpha_\zeta$. So since for any $\eta <_{\mathbb{S}} \lambda$, $\Gamma^\lambda \circ \Phi^{\eta,\lambda} = \Psi^{\eta,\lambda} \circ \Gamma^\eta$, we have for any $\zeta < \eta$, $\Gamma^\lambda \restriction \bar{\alpha}_\zeta + 1 \approx \Gamma^\eta \restriction \bar{\alpha}_\zeta + 1$. Combining this with (1) at $\eta < \xi < \lambda$ gives us (1) at λ. (2) is trivial since λ is a limit ordinal. (3) holds since $\Gamma\mathbb{S} \restriction \lambda$ is a meta-tree (this is an easy observation about the agreement of trees in a meta-tree which follows by induction, using at limit λ that $\Psi^{\eta,\lambda} \restriction \alpha_\zeta \approx Id$ for $\zeta < \eta <_{\mathbb{S}} \lambda$). (4) also follows since all of the Γ^ξ for $\xi < \lambda$ and a tail of the $\Psi^{\xi,\lambda}$ are (total) non-dropping tree embeddings and so Γ^λ must be as well. To see this, let $\xi <_{\mathbb{S}} \lambda$ be such that $\Psi_{\xi,\lambda}$ is a total, non-dropping extended tree embedding. Let $\bar{\chi} + 1 = \mathrm{lh}(\mathcal{S}_\xi)$, $\bar{\tau} + 1 = \mathrm{lh}(\mathcal{S}_\lambda)$, $\chi + 1 = \mathrm{lh}(\mathcal{T}_\xi)$, and $\tau + 1 = \mathrm{lh}(\mathcal{T}_\lambda)$. Since Γ^ξ is non-dropping, $v^{\Gamma^\xi}(\bar{\chi})$-to-$\chi$ doesn't drop in $\mathcal{T}_\xi$, so that $v^{\Psi^{\xi,\lambda} \circ \Gamma^\xi}(\bar{\chi})$-to-$v^{\Psi^{\xi,\lambda}}(\chi)$ doesn't drop in $\mathcal{T}_\lambda$. Since $\Psi_{\xi,\lambda}$ is non-dropping, $v^{\Psi^{\xi,\lambda}}(\chi)$-to-$\tau$ also does not drop. So $v^{\Psi^{\xi,\lambda} \circ \Gamma^\xi}(\bar{\chi})$-to-$\tau$ doesn't drop in $\mathcal{T}_\lambda$. But $[v^{\Gamma^\lambda}(\bar{\tau}), \tau]_{\mathcal{T}_\lambda} \subseteq [v^{\Psi^{\xi,\lambda} \circ \Gamma^\xi}(\bar{\chi}), \tau]_{\mathcal{T}_\lambda}$, so Γ^λ is non-dropping, too. This finishes the limit case.

We now turn to the "moreover" clause. Suppose $\mathcal{S}, \mathcal{T}$ are by Σ and $\mathbb{S}$ is by Σ^*, where Σ is some strategy for M with SHC$^-$. We show $\mu + 1 = \mathrm{lh}(\mathbb{S})$ and $\Gamma\mathbb{S}$ is by Σ^* simultaneously, by induction.

As long as $\Gamma\mathbb{S} \restriction \xi + 1$ is by Σ^*, we know $\xi \leq \mu$ since the process hasn't broken down. Successors cause no trouble, so we deal with limits. So we have $\Gamma\mathbb{S} \restriction \lambda$ is by Σ^* and we need to see that for $b = \Sigma^*(\Gamma\mathbb{S} \restriction \lambda)$, $b = [0, \lambda)_{\mathbb{S}}$. Since we take direct limits of both sides, by Proposition 3.1.11, we get a direct limit tree embedding from the last tree of $\mathbb{S}^\frown b$ to the last tree of $\Gamma(\mathbb{S} \restriction \lambda)^\frown b$, which is by Σ. So since Σ has SHC$^-$, the last tree of $\mathbb{S} \restriction \lambda^\frown b$ is by Σ, hence $b = \Sigma^*(\mathbb{S} \restriction \lambda) = [0, \lambda)_{\mathbb{S}}$ by the definition of Σ^*.

□

We will also need the analogue of Lemma 3.1.10, whose proof we omit.

Lemma 3.1.61. *Let $\mathcal{C} = \langle \{\mathcal{S}_a\}_{a \in A}, \{\Phi_{a,b}\}_{a \preceq b}\rangle$ and $\mathcal{D} = \langle \{\mathcal{T}_a\}_{a \in A}, \{\Psi_{a,b}\}_{a \preceq b}\rangle$ be directed systems of normal trees and $\{F_a\}_{a \in A}$ extenders such that*

(a) F_a is on the $M_\infty^{\mathcal{T}_a}$-sequence,

(b) for all $a, b \in A$ such that $a \preceq b$, $F_b = t_\infty^{\Psi_{a,b}}(F_a)$,

(c) for all $a, b \in A$ such that $a \preceq b$, the Shift Lemma applies to $(\Psi_{a,b}, \Phi_{a,b}, F_a, F_b)$.

For $a, b \in A$ such that $a \preceq b$, let $\Gamma_{a,b}$ be the copy tree embedding associated to $(\Psi_{a,b}, \Phi_{a,b}, F_a, F_b)$. Suppose $\lim \mathcal{C}$ and $\lim \mathcal{D}$ are well-founded. Let $\mathcal{S}_\infty = \lim \mathcal{C}$ and $\mathcal{T}_\infty = \lim \mathcal{D}$, let $\Phi_a : \mathcal{S}_a \to \mathcal{S}_\infty$ and $\Psi_a : \mathcal{T}_a \to \mathcal{T}_\infty$ be the direct limit tree embeddings, and let F_∞ the common value of $t^{\Psi_a}_\infty(F_a)$.

Let $\mathcal{P} = \lim\langle\{W(\mathcal{S}_a, \mathcal{T}_a, F_a)\}_{a\in A}, \{\Gamma_{a,b}\}_{a\preceq b}\rangle$. Suppose $\lim \mathcal{P}$ is well-founded. Let $\mathcal{W}_\infty = \lim \mathcal{P}$ and $\Gamma_a : W(\mathcal{S}_a, \mathcal{T}_a, F_a) \to \mathcal{W}_\infty$ be the direct limit extended tree embeddings, and $\Pi : \mathcal{S}_\infty \to \mathcal{W}_\infty$ be the unique extended tree embedding such that for every $a \in A$, $\Gamma_a \circ \Phi^{\mathcal{S}_a, \mathcal{T}_a, F_a} = \Pi \circ \Phi_a$ (such an extended tree embedding is guaranteed by Proposition 3.1.11).

Then $\mathcal{W}_\infty = W(\mathcal{S}_\infty, \mathcal{T}_\infty, F_\infty)$, $\Pi = \Phi^{\mathcal{S}_\infty, \mathcal{T}_\infty, F_\infty}$, and for all $a \in A$, Γ_a is the copy tree embedding associated to $(\Psi_a, \Phi_a, F_a, F_\infty)$.

3.2 Meta-tree embeddings

In this final section, we complete our proof of Schlutzenberg's Theorem 3.1.48 by proving Theorem 3.1.44 and also establish that SHC^- and bottom-upnormalizing well pass to tail strategies. These results are all proved by diving deeper into the kind of iteration tree combinatorics we've been exploring so far. Key to this is the notion of a meta-tree embedding, a natural analogue of tree embedding.

3.2.1 The definition

If the reader has been following along, they might be able to fill in the definition themselves.

Definition 3.2.1. Let $\mathbb{S}$, $\mathbb{T}$ be meta-trees on $\mathcal{S}$. A *meta-tree embedding* from $\mathbb{S}$ into $\mathbb{T}$ is a system $\vec{\Delta} = \langle U, V, \{\Gamma_\xi\}_{\xi<\mathrm{lh}(\mathbb{S})}, \{\Delta_\zeta\}_{\zeta+1<\mathrm{lh}(\mathbb{S})}\rangle$ such that

1. $V : \mathrm{lh}(\mathbb{S}) \to \mathrm{lh}(\mathbb{T})$ is tree-order preserving, $U : \{\eta \mid \eta + 1 < \mathrm{lh}(\mathbb{S})\} \to \mathrm{lh}(\mathbb{T})$, $V(\xi) = \sup\{U(\eta) + 1 \mid \eta < \xi\}$, and $V(\xi) \leq_{\mathbb{T}} U(\xi)$;
2. for all $\xi < \mathrm{lh}(\mathbb{S})$ and $\eta \leq_{\mathbb{S}} \xi$,
   (a) $\Gamma_\xi : \mathcal{S}_\xi \to \mathcal{T}_{V(\xi)}$ is an extended tree embedding and $\Gamma_0 = Id_{\mathcal{S}}$;
   (b) $\Phi^{\mathbb{T}}_{V(\eta),V(\xi)} \circ \Gamma_\eta = \Gamma_\xi \circ \Phi^{\mathbb{S}}_{\eta,\xi}$,
   (c) if $\xi + 1 < \mathrm{lh}(\mathbb{S})$, then $\Delta_\xi = \Phi^{\mathbb{T}}_{V(\xi),U(\xi)} \circ \Gamma_\xi$ and for all $\zeta \in [V(\xi), U(\xi)]_{\mathbb{T}}$, $u^{\Phi^{\mathbb{T}}_{V(\xi),\zeta}\circ\Gamma_\xi}(\alpha^{\mathbb{S}}_\xi) \in \mathrm{dom}(u^{\Phi^{\mathbb{T}}_{\zeta,U(\xi)}})$;[16]
3. for $\xi + 1 < \mathrm{lh}(\mathbb{S})$, $\eta = \mathbb{S}\text{-pred}(\xi + 1)$, and $\eta^* = \mathbb{T}\text{-pred}(U(\xi) + 1)$,
   (a) $F^{\mathbb{T}}_{U(\xi)} = t^{\Delta_\xi}_{\alpha^{\mathbb{S}}_\xi}(F^{\mathbb{S}}_\xi)$,
   (b) $\alpha^{\mathbb{T}}_{U(\xi)} = u^{\Delta_\xi}(\alpha^{\mathbb{T}}_\xi)$,
   (c) $\eta^* \in [V(\eta), U(\eta)]_{\mathbb{T}}$, and

[16] Here, $\Phi^{\mathbb{T}}_{V(\xi),U(\xi)}$ may only be a partial tree embedding, since we may drop along $[V(\xi), U(\xi)]_{\mathbb{T}}$, so Δ_ξ will only be partial in general, too. The additional condition explains that $\alpha^{\mathbb{S}}_\xi$ is in the domain of u^{Δ_ξ} in a strong sense.

(d) $\Gamma_{\xi+1}\restriction\alpha^{\mathbb{S}}_\xi+2\approx\Delta_\xi\restriction\alpha^{\mathbb{S}}_\xi+2$.

Note that (3) (a) and (b) imply that $F^{\mathbb{T}}_{U(\xi)}$ is an adequate Δ_ξ-image of $F^{\mathbb{S}}_\xi$ by Proposition 3.1.54. We'll see shortly that (3) implies $\Gamma_{\xi+1}$ is the copy map associated to $(\Delta_\xi, \Phi^{\mathbb{T}}_{V(\eta)\circ\Gamma_\eta,\eta^*}, F^{\mathbb{S}}_\xi, F^{\mathbb{T}}_{U(\xi)})$.

We'll carry over some our terminology and notation for tree embeddings in the natural way. If $\mathrm{lh}(\mathbb{S})=\gamma+1$, $\mathrm{lh}(\mathbb{T})=\delta+1$, and $V(\gamma)\leq_{\mathbb{T}}\delta$, then we define the associated *extended meta-tree embedding* by putting $U(\gamma)=\delta$ and $\Delta_\gamma=\Phi^{\mathbb{T}}_{V(\gamma),\delta}\circ\Gamma_\gamma$. We also define, for $\eta\in\mathrm{dom}(U)$ and $\xi\in[V(\eta),U(\eta)]_{\mathbb{T}}$, $\Gamma_{\eta,\xi}=\Phi^{\mathbb{T}}_{V(\eta),\xi}\circ\Gamma_\eta$.

We make the following easy observation about the agreement between the component tree embeddings of a meta-tree embedding.

Proposition 3.2.2. *Let $\vec{\Delta}:\mathbb{S}\to\mathbb{T}$ be a meta-tree embedding. Then for all $\zeta<\xi$,*

$$\Delta_\xi\restriction\alpha^{\mathbb{S}}_\zeta+2\approx\Gamma_\xi\restriction\alpha^{\mathbb{S}}_\zeta+2\approx\Delta_\zeta\restriction\alpha^{\mathbb{S}}_\zeta+2.$$

Proof. We prove this by induction on ξ. This is trivial at $\xi=0$, so suppose it holds at ξ, we need to show it holds at $\xi+1$. By (3)(d) and the fact that the $\alpha^{\mathbb{S}}_\zeta<\alpha^{\mathbb{S}}_\xi$ for $\zeta<\xi$, we have $\Gamma_{\xi+1}\restriction\alpha^{\mathbb{S}}_\zeta+2\approx\Delta_\zeta\restriction\alpha^{\mathbb{S}}_\zeta+2$. Since also either $V(\xi+1)=U(\xi+1)$ or else $\mathrm{crit}(u^{\Phi^{\mathbb{T}}_{V(\xi+1),U(\xi+1)}})\geq\alpha^{\mathbb{T}}_{U(\xi)}+1=v^{\Gamma_{\xi+1}}(\alpha^{\mathbb{S}}_\xi+1)$, we also have that $\Delta_{\xi+1}\restriction\alpha^{\mathbb{S}}_\zeta+2\approx\Gamma_\xi\restriction\alpha^{\mathbb{S}}_\zeta+2$ for $\zeta\leq\xi$, too.

At $\lambda<\mathrm{lh}(\mathbb{S})$ a limit ordinal, we have $V``[0,\lambda)_{\mathbb{S}}$ is a cofinal subset of $[0,V(\lambda))_{\mathbb{T}}$. Since $\mathcal{S}_\lambda=\lim_{[0,\lambda)_{\mathbb{S}}}(\mathbb{S}\restriction\lambda)$ and $\mathcal{T}_{V(\lambda)}=\lim_{[0,V(\lambda))_{\mathbb{T}}}(\mathbb{T}\restriction V(\lambda))$, the commutativity condition (2)(b) implies that $\Gamma_\lambda:\mathcal{S}_\lambda\to\mathcal{T}_\lambda$ is the direct limit tree embedding given by Proposition 3.1.11. Whenever $\zeta<\eta\leq_{\mathbb{S}}\xi\leq\lambda$, $\Phi^{\mathbb{S}}_{\eta,\xi}\restriction\alpha^{\mathbb{S}}_\zeta+1\approx Id$, $\Phi^{\mathbb{T}}_{\eta,\xi}\restriction\alpha^{\mathbb{T}}_{U(\zeta)}+1\approx Id$, and $u^{\Gamma^\eta}(\alpha^{\mathbb{S}}_\zeta)=\alpha^{\mathbb{T}}_{U(\zeta)}$. So since for any $\eta<_{\mathbb{S}}\lambda$, $\Gamma_\lambda\circ\Phi^{\mathbb{S}}_{\eta,\lambda}=\Phi^{\mathbb{T}}_{V(\eta),V(\lambda)}\circ\Gamma_\eta$, we have for any $\zeta<\eta$, $\Gamma_\lambda\restriction\alpha^{\mathbb{S}}_\zeta+2\approx\Gamma_\eta\restriction\alpha^{\mathbb{S}}_\zeta+2$. The argument which let us extend the desired agreement for $\Gamma_{\xi+1}$ to $\Delta_{\xi+1}$ also works to get the desired agreement for Δ_λ. $\square$

We need the following result about extending meta-tree embeddings.

Proposition 3.2.3. *Let $\vec{\Delta}:\mathbb{S}\to\mathbb{T}$ be an extended meta-tree embedding. Let $\xi+1=lh(\mathbb{S})$. Let F be on the $M^{\mathcal{S}_\xi}_\infty$-sequence and $\bar{\alpha}=\alpha(\mathcal{S}_\xi,F)$ such that for all $\zeta\in[V(\xi),U(\xi)]_{\mathbb{T}}$, $u^{\Gamma_{\xi,\zeta}}(\bar{\alpha})\in dom(u^{\Phi^{\mathbb{T}}_{\zeta,U(\xi)}})$. Let $G=t^{\Delta_\xi}_{\bar{\alpha}}(F)$ and $\alpha=\alpha(\mathcal{T}_{U(\xi)},G)$.*

Suppose that $lh(G)\geq\sup\{lh(F^{\mathbb{T}}_\eta)+1\mid\eta<U(\xi)\}$. Then

1. *G is the unique adequate Δ_ξ-image of F with length at least $\sup\{lh(F^{\mathbb{T}}_\eta)+1\mid\eta<U(\xi)\}$,*
2. *$\alpha=u^{\Delta_\xi}(\bar{\alpha})$, and*
3. *if $\mathbb{T}^\frown\langle G\rangle$ is well-founded, then $\mathbb{S}^\frown\langle F\rangle$ is well-founded and there is a unique meta-tree embedding from $\mathbb{S}^\frown\langle F\rangle$ into $\mathbb{T}^\frown\langle G\rangle$ extending $\vec{\Delta}$.*

Proof. First, since $\mathrm{lh}(G)\geq\sup\{\mathrm{lh}(F^{\mathbb{T}}_\eta)+1\mid\eta<U(\xi)\}$, we have $\alpha\geq\{\alpha^{\mathbb{T}}_\eta+1\mid\eta<U(\xi)\}$. It follows that $\alpha\in\mathrm{ran}(u^{\Phi^{\mathbb{T}}_{0,U(\xi)}})\subseteq\mathrm{ran}(u^{\Delta_\xi})$ so that Proposition 3.1.54 implies $\alpha=u^{\Delta_\xi}(\bar{\alpha})$ and that G is an adequate Δ_ξ-image of F. But this holds for any adequate Δ_ξ-image of F

with length at least $\sup\{\mathrm{lh}(F^{\mathbb{T}}_\eta)+1 \mid \eta < U(\xi)\}$, so that G is the unique such image. This gives (1) and (2).

Since $\mathrm{lh}(G) \geq \sup\{\mathrm{lh}(F^{\mathbb{T}}_\eta)+1 \mid \eta < U(\xi)\}$, we must have $\mathrm{lh}(F) \geq \sup\{\mathrm{lh}(F^{\mathbb{S}}_\zeta)+1 \mid \zeta < \xi\}$ and so both $\mathbb{S}^\frown\langle F\rangle$ and $\mathbb{T}^\frown\langle G\rangle$ are putative meta-trees. Let η least such that $\eta = \xi$ or $\mathrm{crit}(F) < \lambda(F^{\mathbb{S}}_\eta)$ and η^* least such that $\eta^* = U(\xi)$ or $\mathrm{crit}(G) < \lambda(F^{\mathbb{T}}_{\eta^*})$.

Claim 1. *η^* is the least $\zeta \in [V(\eta), U(\eta)]_{\mathbb{T}}$ such that either $\zeta = U(\eta)$ or else for $\gamma+1$ the successor of ζ in $[V(\eta), U(\eta)]_{\mathbb{T}}$, crit$(F^{\mathbb{T}}_\gamma) >$ dom(G).*

Proof. We have that η^* is least such that $\eta^* = U(\xi)$ or $\mathrm{crit}(G) < \lambda(F^{\mathbb{T}}_{\eta^*})$. Since $\eta = \xi$ or else $\mathrm{crit}(F) < \lambda(F^{\mathbb{S}}_\eta)$, we easily get $V(\eta) \leq \eta^* \leq U(\eta)$. Let $\zeta \in [V(\eta), U(\eta)]_{\mathbb{T}}$ least such that $\zeta = U(\eta)$ or $\eta^* < \zeta$. First suppose $\eta^* = \zeta = U(\eta)$. Then there can be no $\gamma+1 \in [V(\eta), U(\eta)]_{\mathbb{T}}$ with $\mathrm{crit}(F_\gamma) > \mathrm{dom}(G)$, as η^* is minimal with this property. So suppose $\eta^* < \zeta$. Then ζ is a successor ordinal, $\gamma+1$. Let $\rho = \mathbb{T}\text{-pred}(\gamma+1)$. Then $\rho \leq \eta^* \leq \gamma$. If $\rho < \eta^*$ or $\mathrm{crit}(F^{\mathbb{T}}_\gamma) \leq \mathrm{dom}(G)$, then actually $\mathrm{crit}(F^{\mathbb{T}}_\gamma) \leq \mathrm{dom}(G) < \lambda(F_\gamma)$ (since $\eta^* \leq \gamma$). Then as $F^{\mathbb{T}}_\gamma$ is (the trivial completion of) an initial segment of $t^{\Phi^{\mathbb{T}}_{\rho,U(\eta)}}_{u^{\Gamma_{\eta,\rho}}(\alpha^{\mathbb{S}}_\eta)}$, $\mathrm{dom}(G) \notin \mathrm{ran}(t^{\Phi^{\mathbb{T}}_{\rho,U(\eta)}}_{u^{\Gamma_{\eta,\rho}}(\alpha^{\mathbb{S}}_\eta)}) \subseteq \mathrm{ran}(t^{\Delta_\eta}_{\alpha^{\mathbb{S}}_\eta})$, contradicting that $t^{\Delta_\eta}_{\alpha^{\mathbb{S}}_\eta}(\mathrm{dom}(F)) = \mathrm{dom}(G)$. If $\eta = \xi$, then replacing $\alpha^{\mathbb{S}}_\eta$ with $\bar{\alpha}$ produces the same contradiction. So $\eta^* = \rho \in [V(\eta), U(\eta)]_{\mathbb{T}}$ and $\mathrm{crit}(F^{\mathbb{T}}_\gamma) > \mathrm{dom}(G)$. It follows that η^* is as claimed. $\square$

Now assume $\mathbb{T}^\frown\langle G\rangle$ is well-founded, so that it has well-founded final normal tree $W(\mathcal{T}_{\eta^*}, \mathcal{T}_{U(\xi)}, G)$. We want to show $\mathbb{S}^\frown\langle F\rangle$ is well-founded, too. For this, we just need to see that $W(\mathcal{S}_\eta, \mathcal{S}_\xi, F)$ is well-founded, which we'll get by producing a total extended tree embedding $\Gamma : W(\mathcal{S}_\eta, \mathcal{S}_\xi, F) \to W(\mathcal{T}_{\eta^*}, \mathcal{T}_{U(\xi)}, G)$. We'll also check that $\vec{\Delta}$ extended by Γ is a meta-tree embedding from $\mathbb{S}^\frown\langle F\rangle$ into $\mathbb{T}^\frown\langle G\rangle$.

Γ will be the copy tree embedding associated to $(\Delta_\xi, \Gamma_{\eta,\eta^*}, F, G)$. So we need to see the Shift Lemma applies to $(\Delta_\xi, \Gamma_{\eta,\eta^*}, F, G)$. Let $\bar{\beta} = \beta(\mathcal{S}_\xi, F)$ and $\beta = \beta(\mathcal{T}_{U(\xi)}, G)$. Recalling what this means, we need to verify the following.

(i) $\alpha \in [v^{\Delta_\xi}(\bar{\alpha}), u^{\Delta_\xi}(\bar{\alpha})]_{\mathcal{T}_{U(\xi)}}$,

(ii) $G = s^{\Delta_\xi}_{\bar{\alpha},\alpha}(F)$,

(iii) $s^{\Delta_\xi}_{\bar{\alpha},\alpha} \restriction \mathrm{dom}(F) = s^{\Delta_\xi}_{\bar{\beta},\beta} \restriction \mathrm{dom}(F)$,

(iv) $\Delta_\xi \restriction \bar{\beta}+1 \approx \Gamma_{\eta,\eta^*} \restriction \bar{\beta}+1$,

(v) $\mathcal{T}_{\eta^*} \restriction \beta+1 = \mathcal{T}_{U(\xi)} \restriction \beta+1$,

(vi) $\beta \in [v^{\Gamma_{\eta,\eta^*}}(\bar{\beta}), u^{\Gamma_{\eta,\eta^*}}(\bar{\beta})]_{\mathcal{T}_{\eta^*}}$,

(vii) if $\bar{\beta}+1 < \mathrm{lh}(\mathcal{S}_\eta)$, then $\mathrm{dom}(F) \trianglelefteq M^{\mathcal{S}_\eta}_{\bar{\beta}} | \mathrm{lh}(E^{\mathcal{S}_\eta}_{\bar{\beta}})$,

(viii) if $\beta+1 < \mathrm{lh}(\mathcal{T}_{\eta^*})$, then $\mathrm{dom}(F^{\mathbb{T}}_{U(\xi)}) \trianglelefteq M^{\mathcal{T}_{\eta^*}}_\beta | \mathrm{lh}(E^{\mathcal{T}_{\eta^*}}_\beta)$,

(ix) if $W(\mathcal{T}_{\eta^*}, \mathcal{T}_{U(\xi)}, G)$ is in the dropping case, then $W(\mathcal{S}_\eta, \mathcal{S}_\xi, F)$ is in the dropping case, and

(x) if $W(\mathcal{T}_{\eta^*}, \mathcal{T}_{U(\xi)}, G)$ is not in the dropping case, then $t^{\Gamma_{\eta,\eta^*}}_{\bar\beta} \restriction \mathrm{dom}(F) = s^{\Gamma_{\eta,\eta^*}}_{\bar\beta,\beta} \restriction \mathrm{dom}(F)$.

(i)-(iii) are immediate since G is an adequate Δ_ξ-image of F. (vii) is immediate since $\mathbb{S}$ is a meta-tree and (v) and (viii) are immediate since $\mathbb{T}$ is a meta-tree.

For (iv), if $\eta^* = U(\eta)$, then $\Gamma_{\eta,\eta^*} = \Delta_\eta$ and if $\eta^* < U(\eta)$, then Claim 1 gives $\beta^{\mathbb{T}}_\gamma \geq \beta$ which implies $\Delta_\eta \restriction \bar\beta + 1 \approx \Gamma_{\eta,\eta^*} \restriction \bar\beta + 1$ (since it implies that $v^{\Gamma_{\eta,\eta^*}}(\bar\beta) = v^{\Delta_\eta}(\bar\beta)$). Since either $\eta = \xi$ or else $\eta < \xi$ and $\bar\beta \leq \alpha^{\mathbb{S}}_\eta$ so that, by Proposition 3.2.2, $\Delta_\xi \restriction \bar\beta + 1 \approx \Delta_\eta \restriction \bar\beta + 1$, this gives (iv). Now, since G is an adequate Δ_ξ-image of F, we have $\beta \in [v^{\Delta_\xi}(\bar\beta), u^{\Delta_\xi}(\bar\beta)]_{\mathcal{T}_{U(\xi)}}$ so that Proposition 3.2.2 again implies $\beta \in [v^{\Delta_\eta}(\bar\beta), u^{\Delta_\eta}(\bar\beta)]_{\mathcal{T}_{U(\xi)}}$. Since either $\eta^* = U(\eta)$ or else $\beta^{\mathbb{T}}_\gamma \geq \beta$, we have $\beta \in [v^{\Gamma_{\eta,\eta^*}}(\beta^{\mathbb{S}}_\xi), u^{\Gamma_{\eta,\eta^*}}(\beta^{\mathbb{S}}_\xi)]_{\mathcal{T}_{\eta^*}}$ as well (since $\beta \leq \beta^{\mathbb{T}}_\gamma = \mathrm{crit}(u^{\Phi^{\mathbb{T}}_{\eta^*,U(\eta)}})$ and $\Delta_\eta = \Phi^{\mathbb{T}}_{\eta^*,U(\eta)} \circ \Gamma_\eta$), i.e. (vi) holds.

For (ix), suppose $W(\mathcal{T}_{\eta^*}, \mathcal{T}_{U(\xi)}, G)$ is in the dropping case. First suppose $\eta < \xi$. If $\bar\beta < \alpha^{\mathbb{S}}_\eta$ or $\bar\beta = \alpha^{\mathbb{S}}_\eta$ but $\beta < \alpha^{\mathbb{T}}_{\eta^*}$, the argument from the proof of (ix) in the copying construction (in the case $\bar\beta_\xi < \bar\alpha_\eta$ or $\bar\beta_\xi = \bar\alpha_\eta$ but $\beta_\xi < \alpha_\eta$) gives $W(\mathcal{S}_\eta, \mathcal{S}_\xi, F)$ is in the dropping case, too.

So suppose $\bar\beta = \alpha^{\mathbb{S}}_\eta$ and $\beta = \alpha^{\mathbb{T}}_{\eta^*}$. If $\eta^* = U(\eta)$, then since $F^{\mathbb{T}}_{U(\eta)} = t^{\Delta_\eta}_{\alpha^{\mathbb{S}}_\eta}(F^{\mathbb{S}}_\eta)$ is an adequate Δ_η-image of $F^{\mathbb{S}}_\eta$, the argument from the rest of the proof of (ix) from the copying construction gives that $W(\mathcal{S}_\eta, \mathcal{S}_\xi, F)$ is in the dropping case. So suppose $\eta^* < U(\eta)$ and let $\gamma + 1$ be the successor of η^* in $[V(\eta), U(\eta)]_{\mathbb{T}}$. By Claim 1, we have $\mathrm{dom}(G) < \mathrm{crit}(F^{\mathbb{T}}_\gamma)$ and $\beta^{\mathbb{T}}_\gamma \geq \beta$, so that $\beta^{\mathbb{T}}_\gamma = \alpha^{\mathbb{T}}_{\eta^*}$ as well. If there is a $P \lhd M^{\mathcal{T}_{\eta^*}}_{\alpha^{\mathbb{T}}_{\eta^*}} | \lambda(F^{\mathbb{T}}_{\eta^*})$ projecting across $\mathrm{dom}(G)$, then $W(\mathcal{T}_{U(\xi)}, G)$ is in the dropping case and we get $W(\mathcal{S}_\eta, \mathcal{S}_\xi, F)$ is too. So suppose there is no such P. Then the least level P projecting across $\mathrm{dom}(G)$ must be past $\mathrm{dom}(F^{\mathbb{T}}_\gamma)$. It follows that $W(\mathcal{T}_{\eta^*}, \mathcal{T}_\gamma, F^{\mathbb{T}}_\gamma)$ is in the dropping case as well. By hypothesis (2)(c), we have $u^{\Gamma_{\eta,\eta^*}}(\alpha^{\mathbb{S}}_\eta) \in \mathrm{dom}(u^{\Phi^{\mathbb{T}}_{\eta^*,U(\eta)}})$, so that $u^{\Gamma_{\eta,\eta^*}}(\alpha^{\mathbb{S}}_\eta) \leq \beta^{\mathbb{T}}_\gamma = \alpha^{\mathbb{T}}_{\eta^*}$. But $\alpha^{\mathbb{T}}_{\eta^*} = \beta^{\mathbb{T}}_\gamma \leq u^{\Gamma_{\eta,\eta^*}}(\alpha^{\mathbb{S}}_\eta)$ because we are still blowing up $F^{\mathbb{S}}_\eta$ to $F^{\mathbb{T}}_{U(\eta)}$ along $[V(\eta), U(\eta)]_{\mathbb{T}}$. So, since we're in the dropping case, $u^{\Gamma_{\eta,\eta^*}}(\alpha^{\mathbb{S}}_\eta) = \alpha^{\mathbb{T}}_{\eta^*} = \beta^{\mathbb{T}}_\gamma$. So either $\alpha^{\mathbb{S}}_\eta + 1 = \mathrm{lh}(\mathcal{S}_\eta)$ and there is a $P \trianglelefteq M^{\mathcal{T}_{\eta^*}}_{u^{\Gamma_{\eta,\eta^*}}(\alpha^{\mathbb{S}}_\eta)}$ projecting across $\mathrm{dom}(F_\gamma)$ or $\alpha^{\mathbb{S}}_\eta + 1 < \mathrm{lh}(\mathcal{S}_\eta)$ and there is a $P \lhd M^{\mathcal{T}_{\eta^*}}_{u^{\Gamma_{\eta,\eta^*}}(\alpha^{\mathbb{S}}_\eta)} | \lambda(E^{\mathcal{T}_{\eta^*}}_{u^{\Gamma_{\eta,\eta^*}}(\alpha^{\mathbb{S}}_\eta)})$ projecting across $\mathrm{dom}(F_\gamma)$. In either case, the elementarity of $t^{\Gamma_{\eta,\eta^*}}_{\alpha^{\mathbb{S}}_\eta}$ (or the fact that we drop along $v^{\Gamma_{\eta,\eta^*}}(\alpha^{\mathbb{S}}_\eta)$-to- $u^{\Gamma_{\eta,\eta^*}}(\alpha^{\mathbb{S}}_\eta)$, if we're in the former case but do drop along this branch, as in the proof of (1) from Proposition 3.1.53), gives that $W(\mathcal{S}_\eta, \mathcal{S}_\xi, F)$ is in the dropping case, too, as desired.

If $\eta = \xi$, the proof is basically the same, replacing $\alpha^{\mathbb{S}}_\eta$ with $\bar\alpha$ and $\alpha^{\mathbb{T}}_{\eta^*}$ with α if also $\eta^* = U(\eta) = U(\xi)$, $F^{\mathbb{S}}_\eta$ with F, and $F^{\mathbb{T}}_{U(\eta)}$ with G. This finishes (ix).

Finally, for (x), if $\eta^* = U(\eta)$ then the proof of (x) from the copying construction works, so we may assume $\eta^* < U(\eta)$. Suppose that $t^{\Gamma_{\eta,\eta^*}}_{\bar\beta} \restriction \mathrm{dom}(F) \neq s^{\Gamma_{\eta,\eta^*}}_{\bar\beta,\beta} \restriction \mathrm{dom}(F)$. So we must have $\beta < u^{\Gamma_{\eta,\eta^*}}(\bar\beta)$; let $\tau + 1$ be the successor of β in $[v^{\Gamma_{\eta,\eta^*}}(\bar\beta), u^{\Gamma_{\eta,\eta^*}}(\bar\beta)]_{\mathcal{T}_{\eta^*}}$. Our hypothesis gives $\mathrm{crit}(E^{\mathcal{T}_{\eta^*}}_\tau) \leq \mathrm{crit}(G)$. By Claim 1, we have $\mathrm{crit}(E^{\mathcal{T}_{\eta^*}}_\tau) < \mathrm{crit}(F^{\mathbb{T}}_\gamma)$. It follows that $\mathrm{crit}(E^{\mathcal{T}_{U(\eta)}}_{u^{\Phi^{\mathbb{T}}_{\eta^*,U(\eta)}}(\tau)}) = \mathrm{crit}(E^{\mathcal{T}_{\eta^*}}_\tau)$ and $\beta = \mathcal{T}_{U(\eta)}\text{-pred}(u^{\Phi^{\mathbb{T}}_{\eta^*,U(\eta)}}(\tau) + 1)$. This gives that $t^{\Delta_\eta}_{\bar\beta} \restriction \mathrm{dom}(F) \neq s^{\Delta_\eta}_{\bar\beta,\beta} \restriction \mathrm{dom}(F)$. So the proof of (x) from the copying construction gives that $W(\mathcal{T}_{U(\eta)}, \mathcal{T}_{U(\xi)}, G)$ is in the dropping case. Since either $\beta + 1 = \mathrm{lh}(\mathcal{T}_{\eta^*})$ or else $\mathrm{lh}(E^{\mathcal{T}_{U(\eta)}}_\beta) \leq \mathrm{lh}(E^{\mathcal{T}_{\eta^*}}_\beta)$, this implies $W(\mathcal{T}_{\eta^*}, \mathcal{T}_{U(\xi)}, G)$ is in the dropping case, too. This finishes (x).

By the Shift Lemma, we have $W(\mathcal{S}_\eta, \mathcal{S}_\xi, F)$ is well-founded and $\Gamma : W(\mathcal{S}_\eta, \mathcal{S}_\xi, F) \to W(\mathcal{T}_{\eta^*}, \mathcal{T}_{U(\xi)}, G)$ is the unique tree embedding such that $\Gamma \restriction \bar\alpha + 1 \approx \Delta_\xi \restriction \bar\alpha + 1$, $u^\Gamma(\bar\alpha) = \alpha$, and $\Gamma \circ \Phi^{\mathcal{S}_\eta, \mathcal{S}_\xi, F} = \Phi^{\mathcal{T}_{\eta^*}, \mathcal{T}_{U(\xi)}, G} \circ \Gamma_{\eta,\eta^*}$. It's easy to see that this implies $\vec\Delta^\frown\langle\Gamma\rangle$ is a meta-tree embedding from $\mathbb{S}^\frown\langle F\rangle$ into $\mathbb{T}^\frown\langle G\rangle$ and is the unique such meta-tree embedding extending $\vec\Delta$. □

This proposition immediately implies that the successor Γ-maps must be given by the Shift Lemma. Since the commutativity conditions guarantee that the limit Γ-maps are direct limit tree embeddings (as mentioned in the proof of Proposition 3.2.2), this implies that $\mathbb{S}$, $\mathbb{T}$, and the U-map totally determine $\vec\Delta$.

Proposition 3.2.4. *Let $\vec\Delta = \langle U, V, \{\Gamma_\xi\}_{\xi < lh(\mathbb{S})}, \{\Delta_\zeta\}_{\zeta+1 < lh(\mathbb{S})}\rangle$ be a meta-tree embedding from $\mathbb{S}$ into $\mathbb{T}$. Then for all $\xi + 1 < lh(\mathbb{S})$, letting $\eta = \mathbb{S}$-pred$(\xi+1)$ and $\eta^* = \mathbb{T}$-pred$(U(\xi)+1)$, the Shift Lemma applies to $(\Delta_\xi, \Gamma_{\eta,\eta^*}, F^{\mathbb{S}}_\xi, F^{\mathbb{T}}_{U(\xi)})$ and $\Gamma_{\xi+1}$ is the copy tree embedding associated to $(\Delta_\xi, \Gamma_{\eta,\eta^*}, F^{\mathbb{S}}_\xi, F^{\mathbb{T}}_{U(\xi)})$.*

Proof. The definition of meta-tree embedding immediately gives that the extended tree embedding $\vec\Delta \restriction \xi + 1 : \mathbb{S} \restriction \xi + 1 \to \mathbb{T} \restriction U(\xi) + 1$ together with $F = F^{\mathbb{S}}_\xi$ and $G = F^{\mathbb{T}}_{U(\xi)}$ are as in the hypothesis of the previous proposition, so that the proof of that proposition gives that the Shift Lemma applies to $(\Delta_\xi, \Gamma_{\eta,\eta^*}, F^{\mathbb{S}}_\xi, F^{\mathbb{T}}_{U(\xi)})$ and $\Gamma_{\xi+1}$ is the copy tree embedding associated to $(\Delta_\xi, \Gamma_{\eta,\eta^*}, F^{\mathbb{S}}_\xi, F^{\mathbb{T}}_{U(\xi)})$, as desired. □

3.2.2 Lifting tree embeddings

Meta-tree embeddings come up naturally in a couple contexts. First, we'll see that for $\Psi : \bar{\mathcal{T}} \to \mathcal{T}$ a tree embedding between normal trees on the last model of some normal tree $\mathcal{S}$, we can lift Ψ to a meta-tree embedding from $\mathbb{W}(\mathcal{S}, \bar{\mathcal{T}})$ into $\mathbb{W}(\mathcal{S}, \mathcal{T})$, assuming these are well-founded. This is the content of the next theorem.

Theorem 3.2.5. *Let $\mathcal{S}$, $\bar{\mathcal{T}}$, and $\mathcal{T}$ be normal tree of successor length with $\bar{\mathcal{T}}$ and $\mathcal{T}$ on $M^{\mathcal{S}}_\infty$ and $\Psi : \bar{\mathcal{T}} \to \mathcal{T}$ an extended tree embedding. Let $\bar\mu$ greatest such that $\mathbb{W}(\mathcal{S}, \bar{\mathcal{T}} \restriction \mu + 1)$ is well-founded and μ greatest such that $\mathbb{W}(\mathcal{S}, \mathcal{T} \restriction \mu + 1)$ is well-founded.*

Then $u^\Psi(\bar\mu) \geq \mu$ and there is a unique partial meta-tree embedding with maximal domain $\vec\Delta : \mathbb{W}(\mathcal{S}, \bar{\mathcal{T}} \restriction \bar\mu + 1) \to \mathbb{W}(\mathcal{S}, \mathcal{T} \restriction \mu + 1)$ with U-map u^Ψ.

Moreover, for $\xi \leq \bar\mu$ such that $U(\xi) \leq \mu$, letting $\bar R_\xi$ be the last model of $W(\mathcal{S}, \mathcal{T} \restriction \xi + 1)$ and $\bar\sigma_\xi : M^{\bar{\mathcal{T}}}_\xi \to R_\xi$ the embedding normalization map, $R_{U(\xi)}$ be the last model of $W(\mathcal{S}, \mathcal{T} \restriction U(\xi) + 1)$ and $\sigma_{U(\xi)} : M^{\mathcal{T}}_{U(\xi)} \to R_{U(\xi)}$ the embedding normalization map, $\sigma_{U(\xi)} \circ t^\Psi_\xi = t^{\Delta_\xi}_\infty \circ \bar\sigma_\xi$.

Proof. Let $\bar{\mathbb{W}} = \mathbb{W}(\mathcal{S}, \bar{\mathcal{T}} \restriction \bar\mu + 1) = \langle \bar{\mathcal{W}}_\xi, \bar F_\xi, \bar\Phi_{\xi,\eta}\rangle$, $\mathbb{W} = \mathbb{W}(\mathcal{S}, \mathcal{T} \restriction \mu + 1) = \langle \mathcal{W}_\xi, F_\xi, \Phi_{\xi,\eta}\rangle$, and $\bar\sigma_\xi : M^{\bar{\mathcal{T}}}_\xi \to M^{\bar{\mathcal{W}}_\xi}_\infty$, $\sigma_\xi : M^{\mathcal{T}}_\xi \to M^{\mathcal{W}_\xi}_\infty$ the associated embedding normalization maps. So we have that $\bar{F}_\xi = \bar{\sigma}_\xi(E_\xi^{\bar{\mathcal{T}}})$ and $F_\xi = \sigma_\xi(E_\xi^{\mathcal{T}})$. We also let $\bar{\alpha}_\xi = \alpha_\xi^{\bar{\mathbb{W}}}$, $\bar{\beta}_\xi = \beta_\xi^{\bar{\mathbb{W}}}$, $\alpha_\xi = \alpha_\xi^{\mathbb{W}}$, and $\beta_\xi = \beta_\xi^{\mathbb{W}}$.

Our meta-tree embedding $\vec{\Delta}$ will have $V = v^\Psi$ and $U = u^\Psi$. We just need to see that this works, by induction. Using the notation of the "moreover" clause, we have that $\bar{R}_\xi$ is the last model of $\bar{\mathcal{W}}_\xi$ and R_ξ is the last model of $\mathcal{W}_\xi$.

We maintain by induction on ξ that

1. $\vec{\Delta}\restriction\xi+1$ is an extended meta-tree embedding from $\bar{\mathbb{W}}\restriction\xi+1$ into $\mathbb{W}\restriction U(\xi)+1$,
2. for all $\zeta \in [V(\xi), U(\xi)]_{\mathbb{T}}$, $\sigma_\zeta \circ s^\Psi_{\xi,\zeta} = t_\infty^{\Gamma_{\xi,\zeta}} \circ \bar{\sigma}_\xi$

Note that the maps in (2) may be partial so we just mean that the maps commute on their common domain. If $\xi + 1 < \mathrm{lh}(\bar{\mathcal{T}})$, then (2) implies $t_\infty^{\Delta_\xi}(\bar{F}_\xi) = F_{U(\xi)}$.

We start with the successor case. Suppose (1) and (2) hold at ξ and $\xi + 1 < \bar{\mu}$. We first need to show the following.

Claim 1. *For all $\zeta \in [V(\xi), U(\xi)]_{\mathbb{W}}$, $u^{\Gamma_{\xi,\zeta}}(\alpha_\xi^{\mathbb{S}}) \in dom(u^{\Phi_{\zeta,U(\xi)}})$.*

This is not immediate from (2) alone, as it seems possible that $t_{\bar{\alpha}_\xi}^{\Gamma_{\xi,\zeta}}(\bar{F}_\xi) \in \mathrm{dom}(t_\infty^{\Phi_{\zeta,U(\xi)}})$ even though $u^{\Gamma_{\xi,\zeta}}(\bar{\alpha}_\xi) \notin \mathrm{dom}(u^{\Phi_{\zeta,U(\xi)}})$, as $t_{\bar{\alpha}_\xi}^{\Gamma_{\xi,\zeta}}(\bar{F}_\xi) \in \mathrm{dom}(t_\infty^{\Phi_{\zeta,U(\xi)}})$ may appear on an earlier model, to which we drop.

Proof. We'll show by induction that for all $\zeta \in [V(\xi), U(\xi)]_{\mathbb{W}}$,

i. if $\zeta < U(\xi)$, letting $\gamma + 1$ be the successor of ζ in $[V(\xi), U(\xi)]_{\mathbb{W}}$, either
   (a) $\beta_\gamma \leq \alpha(\mathcal{W}_\zeta, t_{\bar{\alpha}_\xi}^{\Gamma_{\xi,\zeta}}(\bar{F}_\xi)) = u^{\Gamma_{\xi,\zeta}}(\bar{\alpha}_\xi)$ or
   (b) $\alpha(\mathcal{W}_\zeta, t_{\bar{\alpha}_\xi}^{\Gamma_{\xi,\zeta}}(\bar{F}_\xi)) < \beta_\gamma = u^{\Gamma_{\xi,\zeta}}(\bar{\alpha}_\xi)$,

ii. $u^{\Gamma_{\xi,\zeta}}(\alpha_\xi^{\mathbb{S}}) \in \mathrm{dom}(u^{\Phi^{\mathbb{T}}_{\zeta,U(\xi)}})$.

First, assume that (ii) holds below ζ and (i) holds at ζ. We'll show that (ii) holds at ζ. This is trivially unless $\zeta < U(\xi)$, so suppose $\zeta < U(\xi)$ and let $\gamma + 1$ be the successor of ζ in $[V(\xi), U(\xi)]_{\mathbb{T}}$. Since $s^\Psi_{\xi,\zeta}(E_\xi^{\bar{\mathcal{T}}}) \in \mathrm{dom}(\hat{\imath}^{\mathcal{T}}_{\zeta,U(\xi)})$, there is no level of $M_\zeta^{\mathcal{T}}|\mathrm{lh}(s^\Psi_{\xi,\zeta}(E_\xi^{\bar{\mathcal{T}}}))$ projecting across $\mathrm{dom}(E_\gamma^{\mathcal{T}})$. By (ii) below ζ and (2), it follows that no level of $R_\zeta|\mathrm{lh}(t_{\bar{\alpha}_\xi}^{\Gamma_{\xi,\zeta}}(\bar{F}_\xi))$ projects across $\mathrm{dom}(F_\gamma)$. It follows that $t_{\bar{\alpha}_\xi}^{\Gamma_{\xi,\zeta}}(\bar{F}_\xi) \in \mathrm{dom}(t_\infty^{\Phi_{\zeta,U(\xi)}})$. So we must have $\alpha(\mathcal{W}_\zeta, t_{\bar{\alpha}_\xi}^{\Gamma_{\xi,\zeta}}(\bar{F}_\xi)) \in \mathrm{dom}(u^{\Phi^{\mathbb{T}}_{\zeta,U(\xi)}})$. So if (i)(a) holds, then $u^{\Gamma_{\xi,\zeta}}(\bar{\alpha}_\xi) \in \mathrm{dom}(u^{\Phi_{\zeta,U(\xi)}})$. If (i)(b) holds, we also have $u^{\Gamma_{\xi,\zeta}}(\bar{\alpha}_\xi) \in \mathrm{dom}(u^{\Phi_{\zeta,U(\xi)}})$, as $\beta_\gamma \in \mathrm{dom}(u^{\Phi_{\zeta,U(\xi)}})$. So (ii) holds at ζ.

Now suppose (i) and (ii) hold below ζ, $\zeta < U(\xi)$, and let $\gamma + 1$ be the successor of ζ in $[V(\xi), U(\xi)]_{\mathbb{W}}$. We'll show that $\beta_\gamma \leq u^{\Gamma_{\xi,\zeta}}(\bar{\alpha}_\xi)$. Suppose not. Then $\bar{\alpha}_\xi + 1 < \mathrm{lh}(\bar{\mathcal{W}}_\xi)$, as Γ_ξ is an extended tree embedding, $u^{\Gamma_{\xi,\zeta}}(\bar{\alpha}_\xi)$ is in the domain of $u^{\Phi_{\zeta,U(\xi)}}$, and $\mathrm{lh}(E^{\mathcal{W}_\zeta}_{u^{\Gamma_{\xi,\zeta}}(\bar{\alpha}_\xi)}) < \mathrm{lh}(F_\gamma)$. It follows that $t_\infty^{\Gamma_{\xi,\gamma+1}}\restriction\mathrm{lh}(E_{\bar{\alpha}_\xi}^{\bar{\mathcal{W}}_\xi})+1 = t_{\bar{\alpha}_\xi}^{\Gamma_{\xi,\gamma+1}}\restriction\mathrm{lh}(E_{\bar{\alpha}_\xi}^{\bar{\mathcal{W}}_\xi})+1$. But since $u^{\Gamma_{\xi,\zeta}}(\bar{\alpha}_\xi) < \beta_\gamma$, $t_{\bar{\alpha}_\xi}^{\Delta_\xi} = t_{\bar{\alpha}_\xi}^{\Gamma_{\xi,\zeta}}$, so that by (2), $F_{U(\xi)} = t_{\bar{\alpha}_\xi}^{\Gamma_{\xi,\zeta}}(\bar{F}_\xi)$. But since $t_{\bar{\alpha}_\xi}^{\Gamma_{\xi,\zeta}}(\bar{F}_\xi)$ is on the R_ζ-sequence, we must have $\mathrm{lh}(t_{\bar{\alpha}_\xi}^{\Gamma_{\xi,\zeta}}(\bar{F}_\xi)) < \mathrm{lh}(E^{\mathcal{W}_\zeta}_{u^{\Gamma_{\xi,\zeta}}(\bar{\alpha}_\xi)}) < \mathrm{lh}(F_\gamma)$. But $\mathrm{lh}(F_{U(\xi)}) > \mathrm{lh}(F_\gamma)$ since $U(\xi) > \gamma$ and $\mathbb{W}$ is a meta-tree. This is a contradiction. So $\beta_\gamma \le u^{\Gamma_{\xi,\zeta}}(\bar\alpha_\xi)$. We reach the same contradiction if $\mathrm{lh}(E^{\mathcal{W}_\zeta}_{u^{\Gamma_{\xi,\zeta}}(\bar\alpha_\xi)}) < \mathrm{crit}(F_\gamma)$, so $\mathrm{crit}(F_\gamma) < \mathrm{lh}(E^{\mathcal{W}_\zeta}_{u^{\Gamma_{\xi,\zeta}}(\bar\alpha_\xi)})$ as well. We'll use these facts to prove (i), by induction on ζ.

For the base case, we show (i)(a) holds at $V(\xi)$. Since $\mathrm{lh}(t^{\Gamma_\xi}_{\bar\alpha_\xi}(\bar F_\xi)) \ge \sup\{\mathrm{lh}(F_\eta)+1 \mid \eta < V(\xi)\}$,[17] we have by Proposition 3.1.54 that $\alpha(\mathcal{W}_{V(\xi)}, t^{\Gamma_\xi}_{\bar\alpha_\xi}(\bar F_\xi)) = u^{\Gamma_\xi}(\bar\alpha_\xi)$. Since $\beta_\gamma \le u^{\Gamma_\xi}(\bar\alpha_\xi)$, this gives (1)(a). By Proposition 3.1.50, we have that $\alpha(\mathcal{W}_\zeta, t^{\Gamma_{\xi,\zeta}}_{\bar\alpha_\xi}(\bar F_\xi)) = u^{\Gamma_{\xi,\zeta}}(\bar\alpha_\xi)$ whenever ζ is a limit ordinal, so that (i)(a) must also hold at limit $\zeta \in [V(\xi), U(\xi))_{\mathbb{W}}$. So suppose (i) holds at ζ; we want to show that (i) holds at $\gamma+1$, the successor of ζ in $[V(\xi), U(\xi)]_{\mathbb{W}}$. So suppose $\gamma+1 < U(\xi)$ and let $\tau+1$ be the successor of $\gamma+1$ in $[V(\xi), U(\xi)]_{\mathbb{W}}$. First suppose that (i)(a) holds at ζ. If $\mathrm{dom}(F_\gamma) \le \mathrm{lh}(t^{\Gamma_{\xi,\zeta}}_{\bar\alpha_\xi}(\bar F_\xi))$, then we must have $\alpha(\mathcal{W}_{\gamma+1}, t^{\Gamma_{\xi,\gamma+1}}_{\bar\alpha_\xi}(\bar F_\xi)) = u^{\Gamma_{\xi,\gamma+1}}(\bar\alpha_\xi)$, too, since every exit extender of $\mathcal{W}_{\gamma+1}$ used below $u^{\Gamma_{\xi,\gamma+1}}(\bar\alpha_\xi)$ has length $\le \mathrm{lh}(F_\gamma) < \mathrm{lh}(t^{\Gamma_{\xi,\gamma+1}}_{\bar\alpha_\xi}(\bar F_\xi))$ or else is the image under $t^{\Gamma_{\xi,\gamma+1}}_{\bar\alpha_\xi}$ of an extender of length less than $\mathrm{lh}(t^{\Gamma_{\xi,\zeta}}_{\bar\alpha_\xi}(\bar F_\xi))$, since (i)(a) holds at ζ. Since $\beta_\tau \le u^{\Gamma_{\xi,\gamma+1}}(\bar\alpha_\xi)$, this gives (i)(a) at $\gamma+1$. Now suppose $\mathrm{dom}(F_\tau) > \mathrm{lh}(t^{\Gamma_{\xi,\zeta}}_{\bar\alpha_\xi}(\bar F_\xi))$. Then we must have $\beta_\gamma = u^{\Gamma_{\xi,\zeta}}(\bar\alpha_\xi)$ since we've shown $\beta_\gamma \le u^{\Gamma_{\xi,\zeta}}(\bar\alpha_\xi)$ but cannot have $\mathrm{dom}(F_\tau) \le \mathrm{lh}(E^{\mathcal{W}_\zeta}_\rho)$ for any $\rho < u^{\Gamma_{\xi,\zeta}}(\bar\alpha_\xi)$, as $\mathrm{lh}(E^{\mathcal{W}_\zeta}_\rho) < \mathrm{lh}(t^{\Gamma_{\xi,\zeta}}_{\bar\alpha_\xi}(\bar F_\xi))$ for all such ρ, since we have $\alpha(\mathcal{W}_\zeta, t^{\Gamma_{\xi,\zeta}}_{\bar\alpha_\xi}(\bar F_\xi)) = u^{\Gamma_{\xi,\zeta}}(\bar\alpha_\xi)$. Since $\mathrm{crit}(F_\gamma) < \mathrm{lh}(t^{\Gamma_{\xi,\zeta}}_{\bar\alpha_\xi}(\bar F_\xi)) < \mathrm{dom}(F_\gamma)$, we have $\alpha(\mathcal{W}_{\gamma+1}, t^{\Gamma_{\xi,\gamma+1}}_{\bar\alpha_\xi}(\bar F_\xi)) = \alpha_\gamma$, by Proposition 3.1.50. Now $\beta_\tau \ge \alpha_\gamma+1$ by the normality of $\mathbb{W}$ and $u^{\Gamma_{\xi,\gamma+1}}(\bar\alpha_\xi) = \alpha_\gamma+1$. So since $\beta_\tau \le u^{\Gamma_{\xi,\gamma+1}}(\bar\alpha_\xi)$, we must actually have $\beta_\tau = u^{\Gamma_{\xi,\gamma+1}}(\bar\alpha_\xi)$. This gives (i)(b). This finishes the case that (i)(a) holds at ζ. So now suppose (i)(b) holds at ζ. Then since $\alpha(\mathcal{W}_\zeta, t^{\Gamma_{\xi,\zeta}}_{\bar\alpha_\xi}(\bar F_\xi)) < \beta_\gamma$, we have $\mathrm{dom}(F_\gamma) \ge \mathrm{lh}(E^{\mathcal{W}_\zeta}_{\alpha(\mathcal{W}_\zeta, t^{\Gamma_{\xi,\zeta}}_{\bar\alpha_\xi}(\bar F_\xi))}) > \mathrm{lh}(t^{\Gamma_{\xi,\zeta}}_{\bar\alpha_\xi}(\bar F_\xi)))$. So we have $\mathrm{crit}(F_\gamma) < \mathrm{lh}(t^{\Gamma_{\xi,\zeta}}_{\bar\alpha_\xi}(\bar F_\xi)) < \mathrm{dom}(F_\gamma)$, which gives $\alpha(\mathcal{W}_{\gamma+1}, t^{\Gamma_{\xi,\gamma+1}}_{\bar\alpha_\xi}(\bar F_\xi)) = \alpha_\gamma$ and $\beta_\tau = \alpha_\gamma + 1 = u^{\Gamma_{\xi,\gamma+1}}(\bar\alpha_\xi)$, as in the previous case. So (i)(b) holds at $\gamma+1$. This finishes the successor case ☐

Now let $\eta = \bar{\mathcal{W}}\text{-pred}(\xi+1)$ and $\eta^* = \mathcal{W}\text{-pred}(U(\xi)+1)$. Since $\bar{\mathcal{W}}$ and $\mathcal{W}$ have the same tree-order as $\mathcal{S}$ and $\mathcal{T}$, respectively, we have $\eta^* \in [V(\eta), U(\eta)]_{\mathbb{W}}$, since Ψ is a tree embedding. By condition (2) of our induction hypothesis, we have $t^{\Delta_\xi}_\infty(\bar F_\xi) = F_{U(\xi)}$. By our claim and since $\mathrm{lh}(F_{U(\xi)}) \ge \sup\{\mathrm{lh}(F_\eta)+1 \mid \eta < U(\xi)\}$, as $\mathbb{W}$ is a meta-tree, Proposition 3.2.3 implies that we can extend $\vec\Delta \restriction \xi+1$ to a meta-tree embedding from $\bar{\mathbb{W}} \restriction \xi+2$ into $\mathbb{W} \restriction U(\xi)+2$. Recalling the proof of that proposition, we just extend $\vec\Delta \restriction \xi+1$ by letting $\Gamma_{\xi+1}$ be the copy map associated to $(\Delta_\xi, \Gamma_{\eta,\eta^*}, \bar F_\xi, F_{U(\xi)})$. Using conclusion (a) of the Shift Lemma, hypothesis (2) at η, and the definition of the embedding normalization maps, it is straightforward to verify (2) holds at $\xi+1$ for $\zeta = V(\xi)$. We get our meta-tree embedding $\vec\Delta \restriction \xi+2 : \bar{\mathbb{W}} \restriction \xi+2 \to \mathbb{W} \restriction U(\xi+1)+1$ by setting $\Delta_{\xi+1} = \Phi_{V(\xi),U(\xi)} \circ \Gamma_{\xi+1}$. This gives (1). Using that (2) holds for $\zeta = V(\xi+1)$, the definitions of the embedding normalization maps of $\mathbb{W}$ give that (2) holds at all $\zeta \in [V(\xi+1), U(\xi+1)]_{\mathbb{W}}$. We leave it to the reader to verify this.

At a limit λ, we have that $V``[0,\lambda)_{\bar{\mathbb{W}}}$ is a cofinal subset of $[0, V(\lambda))_{\mathbb{W}}$, since Ψ is a tree

[17] We have $\mathrm{lh}(\bar F_\xi) \ge \sup\{\mathrm{lh}(\bar F_\eta)+1 \mid \eta < \xi\}$, which implies $\mathrm{lh}(\bar F_\xi) \ge \sup\{\mathrm{lh}(F_{U(\eta)})+1 \mid \eta < \xi\}$. This gives the desired inequality because $V(\xi) = \sup\{U(\eta) \mid \eta < \xi\}$.

embedding, $V = v^\Psi$, and $\bar{\mathbb{W}}$ and $\mathbb{W}$ have the same tree-order as $\mathcal{S}$ and $\mathcal{T}$, respectively, so that the commutativity conditions so far guarantee that there is a direct limit tree embedding $\Gamma_\lambda : \bar{\mathbb{W}} \restriction \lambda + 1 \to \mathbb{W} \restriction V(\xi) + 1$. We then let $\Delta_\lambda = \Phi_{V(\lambda),U(\lambda)} \circ \Gamma_\lambda$. It is easy to see that extending $\vec{\Delta} \restriction \lambda$ by Γ_λ and Δ_λ gives us our desired meta-tree embedding $\vec{\Delta} \restriction \lambda + 1 : \bar{\mathbb{W}} \restriction \lambda + 1 \to \mathbb{W} \restriction U(\lambda) + 1$, i.e. (1) holds at λ. (2) is straightforward to check using the definition of the embedding normalization maps; we leave this to the reader. □

We call the meta-tree embedding $\vec{\Delta}$ from the previous Theorem the *lift* of Ψ. As a corollary we get the following result.

Theorem 3.2.6. *Let M be a premouse, $\kappa \leq \theta$ regular cardinals, and Σ a (κ, θ)-strategy for M with SHC$^-$ which bottom-up normalizes well. Then all tails of Σ have SHC$^-$.*

Proof. Let $\vec{\mathcal{S}}$ be a stack on M by Σ with last model P. Let $\Psi : \bar{\mathcal{T}} \to \mathcal{T}$ be a tree embedding with $\bar{\mathcal{T}}, \mathcal{T}$ normal trees on P with $\mathcal{T}$ by Σ. We want to show $\bar{\mathcal{T}}$ is by Σ. By truncating $\mathcal{T}$ if necessary, we may assume Ψ is an extended tree embedding. Let $\mathcal{W} = W(\vec{\mathcal{S}})$, which is well-founded and by Σ, since Σ bottom-up normalizes well. Let $\sigma : P \to M_\infty^{\mathcal{W}}$ be the associated embedding normalization map. Then there is a tree embedding $\sigma\Psi : \sigma\bar{\mathcal{T}} \to \sigma\mathcal{T}$, and $\sigma\mathcal{T}$ is by $\Sigma_{\mathcal{W}, M_\infty^{\mathcal{W}}}$, since Σ bottom-up normalizes well. It suffices to show that $W(\mathcal{W}, \sigma\bar{\mathcal{T}})$ is by Σ, again by bottom-up normalizing well. We have that $\mathbb{W}(\mathcal{W}, \sigma\mathcal{T})$ is well-founded and all of its trees are by Σ (by bottom-up normalizing well). By Theorem 3.2.5, we have that $\mathbb{W}(\mathcal{W}, \sigma\bar{\mathcal{T}})$ is well-founded, too, and there is a meta-tree embedding $\vec{\Delta}$ from $\mathbb{W}(\mathcal{W}, \sigma\bar{\mathcal{T}})$ into $\mathbb{W}(\mathcal{W}, \sigma\mathcal{T})$. In particular, we have the last Γ-map of $\vec{\Delta}$ is a total extended tree embedding from $W(\mathcal{W}, \sigma\bar{\mathcal{T}})$ into a tree by Σ, so that SHC$^-$ gives $W(\mathcal{W}, \sigma\bar{\mathcal{T}})$ is by Σ, too, as desired. □

3.2.3 Normalizing stacks of meta-trees

In this subsection, we study another source of meta-tree embeddings: the analogue of embedding normalization for meta-trees. As one might expect, this will be used in the proof of Theorem 3.1.44. We start with the one-step case.

Given meta-trees $\mathbb{S}, \mathbb{T}$ of lengths and an extender F on $M_\infty^{\mathbb{T}}$, we want to define a meta-tree $\mathbb{W} = \mathbb{W}(\mathbb{S}, \mathbb{T}, F)$ and an extended meta-tree embedding $\vec{\Delta} = \vec{\Delta}^{\mathbb{S},\mathbb{T},F}$ from $\mathbb{S}$ into $\mathbb{W}$. Moreover, we will have that the last tree of $\mathbb{W}$ is just $W(\mathcal{S}_\infty, \mathcal{T}_\infty, F)$ and Δ_∞, the last Δ-map of $\vec{\Delta}$, is $\Phi^{\mathcal{S}_\infty, \mathcal{T}_\infty, F}$, so that this analogue of embedding normalization is really producing an analogue of full normalization. It's hard to point to an intuitive explanation of this fact, but one reasonable intuition is that meta-trees are basically *coarse-structural* objects—all the fine structure occurs one-level down, in some sense. Ordinary embedding normalization also coincides with full normalization in the coarse setting, i.e. for nice trees on V.

For a meta-tree $\mathbb{T} = \langle \mathcal{T}_\xi, F_\xi, \Phi_{\eta,\xi} \rangle$ of successor length and F on the sequence of the last model of $\mathbb{T}$, we define

$$\begin{aligned} a(\mathbb{T}, F) &= \text{the least } \xi \text{ such that } F \text{ is on the } M_\infty^{\mathcal{T}_\xi}\text{-sequence,} \\ b(\mathbb{T}, F) &= \text{the least } \eta \leq a(\mathbb{T}, F) \text{ such that } \eta = a(\mathbb{T}, F) \text{ or} \\ &\qquad \eta < \xi \leq a(\mathbb{T}, F) \text{ and } \mathrm{crit}(F) < \lambda(F_\xi). \end{aligned}$$

Note that $a(\mathbb{T}, F)$ is also the least ξ such that $\xi+1 = \mathrm{lh}(\mathbb{T})$ or $\xi+1 < \mathrm{lh}(\mathbb{T})$ and $\alpha(\mathcal{T}_\xi, F_\xi) \leq \alpha(\mathcal{T}_\infty, F)$.

Let $\mathbb{S} = \langle \mathcal{S}_\xi, F_\xi, \Phi_{\eta,\xi}\rangle$, $\mathbb{T} = \langle \mathcal{T}_\xi, G_\xi, \Psi_{\eta,\xi}\rangle$, and F on the sequence of the last model of $\mathbb{T}$. Let $a = a(\mathbb{T}, F)$ and $b = b(\mathbb{T}, F)$. Suppose that $\mathbb{S}\restriction b+1 = \mathbb{T}\restriction b+1$ and if $b+1 < \mathrm{lh}(\mathbb{S})$, suppose that $\mathrm{dom}(F) \leq \mathrm{lh}(F_b)$.

We define $\mathbb{W} = \langle \mathcal{W}_\xi, H_\xi, \Pi_{\eta,\xi}\rangle$ and $\vec{\Delta} = \langle U, V, \Gamma_\xi, \Delta_\xi\rangle$ inductively as follows. First, we let $\mathbb{W}\restriction a+1 = \mathbb{T}\restriction a+1$. We put

$$U(\xi) = \begin{cases} \xi & \text{if } \xi < b \\ a+1+(\xi - b) & \text{if } \xi \geq b \end{cases},$$

and $V(\xi) = \sup\{U(\eta)+1 \mid \eta < \xi\}$.

We let $H_a = F$, $W_{a+1} = W(\mathcal{S}_b, \mathcal{T}_a, F)$, and let $\vec{\Delta}\restriction b+1$ be the identity tree embedding on $\mathbb{S}\restriction b+1 = \mathbb{T}\restriction b+1$ followed by $\Delta_b = \Phi^{\mathcal{S}_b,\mathcal{T}_b,F}$.

The dropping case. Suppose that $b+1 < \mathrm{lh}(\mathbb{S})$ and some $P \trianglelefteq M_\infty^{\mathcal{W}_p}|\mathrm{lh}(F_b)$ projects across $\mathrm{dom}(F)$ or $b+1 = \mathrm{lh}(\mathbb{S})$ and some $P \trianglelefteq M_\infty^{\mathcal{W}_p}$ projects across $\mathrm{dom}(F)$. Then we stop, so that $\mathbb{W} = \mathbb{T}\restriction a+1^\frown\langle F\rangle$. In this case, we must have $W(\mathcal{S}_b, \mathcal{T}_a, F)$ is in the dropping case and $W(\mathcal{S}_b, \mathcal{T}_a, F) = W(\mathcal{S}_\infty, \mathcal{T}_\infty, F)$ is the last tree of $\mathbb{W}$ (using here that $\mathcal{S}_b\restriction\alpha_b^{\mathbb{S}}+1 = \mathcal{S}_\infty\restriction\alpha_b^{\mathbb{S}}+1$). We also have that and $\vec{\Delta}$ is an extended tree embedding from $\mathcal{S}\restriction b+1$ into $\mathbb{W}$ with last tree embedding $\Delta_\xi = \Phi^{\mathcal{S}_b,\mathcal{T}_a,F} = \Phi^{\mathcal{S}_b,\mathcal{T}_a,F}$.

The non-dropping case. Suppose we're not in the dropping case. Then if $b+1 < \mathrm{lh}(\mathbb{S})$, we continue building $\mathbb{W}$ by using images of the extenders of $\mathbb{S}$. We'll maintain the following by induction on $\xi \geq b$.

1. $\vec{\Delta} = \langle U\restriction\xi+1, V\restriction\xi+1, \{\Gamma_\eta\}_{\eta\leq\xi}, \{\Delta_\eta\}_{\eta\leq\xi}\rangle$ is an extended meta-tree embedding from $\mathbb{S}\restriction\xi+1$ into $\mathbb{W}\restriction U(\xi)+1$,
2. $\mathcal{W}_{U(\xi)} = W(\mathcal{S}_\xi, \mathcal{T}_a, F)$ and $\Delta_\xi = \Phi^{\mathcal{S}_\xi,\mathcal{T}_a,F}$.

Note that there is a lot built into (1); for example, we must have $\Gamma_\xi = \Delta_\xi$ for $\xi > b$ by our choice of U.

We've already established the base case $\xi = b$. So suppose $\xi \geq b$ and (1) and (2) hold at all $\zeta \leq \xi$. Let $\eta = \mathbb{S}\text{-pred}(\xi+1)$. There are two subcases depending on the critical point of F_ξ.

First suppose $\mathrm{crit}(F_\xi) < \mathrm{crit}(F)$. In this case $\eta \leq b$ and $\mathrm{crit}(H_{U(\xi)}) = \mathrm{crit}(t_\infty^{\Delta_\xi}(F_\xi)) = \mathrm{crit}(F_\xi)$, since $t_\infty^{\Delta_\xi}$ has critical point $\mathrm{crit}(F)$, by our induction hypothesis (2).

Since $\mathbb{W}\restriction b+1 = \mathbb{S}\restriction b+1$, we must put $\eta = \mathbb{W}\text{-pred}(U(\xi)+1)$, as dictated by normality. We let $\mathcal{W}_{U(\xi)+1} = W(\mathcal{W}_\eta, \mathcal{W}_{U(\xi)}, H_{U(\xi)})$ and, we let $\Gamma_{\xi+1}$ be the copy tree embedding given by associated to $(\Delta_\xi, \Gamma_\eta = Id, F_\xi, H_{U(\xi)})$. It is easy to see that Proposition 3.2.3 applies here since either $\Delta_\xi = \Gamma_\xi$, so is total, or else $\xi = b$ and $\alpha(\mathcal{S}_b, F_b) = \alpha_b^{\mathbb{S}} \in \mathrm{dom}(u^{\Delta_b})$ because we're in the non-dropping case (even though $W(\mathcal{S}_b, \mathcal{T}_a, F)$ may be in the dropping case, we have either $\alpha_b^{\mathbb{S}}+1 = \mathrm{lh}(\mathcal{S}_b)$ or $\alpha_b^{\mathbb{S}}+1 < \mathrm{lh}(\mathcal{S}_b)$ and no level of $M_{\alpha_b^{\mathbb{S}}}^{\mathcal{S}_b}|\mathrm{lh}(F_b)$ projects across $\mathrm{dom}(F)$ so that if $W(\mathcal{S}_b, \mathcal{T}_a, F)$ is in the dropping case, we must have $\alpha_b^{\mathbb{S}} = \beta(\mathcal{T}_a, F) \in \mathrm{dom}(u^{\Delta_b})$).

So the Shift Lemma does apply to these objects, by the proof of that proposition. Since $\xi+1>b$, we also let $\Delta_{\xi+1}=\Gamma_{\xi+1}$. This maintains (1) by Proposition 3.2.3. To see (2), we show that $\mathcal{W}_{U(\xi)+1}=W(\mathcal{S}_{\xi+1},\mathcal{T}_a,F)$ and $\Gamma_{\xi+1}=\Phi^{\mathcal{S}_{\xi+1},\mathcal{T}_a,F}$ simultaneously.

Let $\bar{\mathcal{W}}=W(\mathcal{S}_{\xi+1},\mathcal{T}_a,F)$ and $\bar{\Gamma}=\Phi^{\mathcal{S}_{\xi+1},\mathcal{T}_a,F}$. Let $\bar{\alpha}=\alpha(\mathcal{S}_\xi,F_\xi)$ and $\alpha=\alpha(\mathcal{W}_{U(\xi)},H_{U(\xi)})$. Let $\bar{\Phi}=\Phi^{\mathcal{S}_\eta,\mathcal{S}_\xi,F_\xi}$ and $\Phi=\Phi^{\mathcal{W}_\eta,\mathcal{W}_{U(\xi)},H_{U(\xi)}}$. We show $\bar{\Gamma}=\Gamma_{\xi+1}$ by showing it satisfies the conditions which uniquely determine $\Gamma_{\xi+1}$ in the conclusion of the Shift Lemma. First we show that $\bar{\Gamma}\restriction\bar{\alpha}+1=\Gamma_\xi\restriction\bar{\alpha}+1$ and $u^{\bar{\Gamma}}(\bar{\alpha})=\alpha$, which guarantees that $\bar{\Gamma}\restriction\bar{\alpha}+2=\Gamma_{\xi+1}\restriction\bar{\alpha}+2$. From here, we'll show by induction on $\zeta<\mathrm{lh}(\mathcal{S}_\eta)$ that $\bar{\Gamma}\circ\bar{\Phi}\restriction\zeta+1=\Phi\restriction\zeta+1$ and $\bar{\mathcal{W}}\restriction u^{\bar{\Phi}}(\zeta)+1=\mathcal{W}_{U(\xi)+1}\restriction u^{\Phi}(\zeta)+1$, which establishes $\bar{\Gamma}=\Gamma_{\xi+1}$ by the remaining commutativity condition $\Gamma_{\xi+1}\circ\bar{\Phi}=\Phi$, which uniquely determines the rest of $\Gamma_{\xi+1}$. Note that we're using in several places that $\Gamma_\eta=Id$.

We have that $\mathcal{S}_{\xi+1}\restriction\bar{\alpha}+1=\mathcal{S}_\xi\restriction\bar{\alpha}+1$, so that $\bar{\mathcal{W}}\restriction\alpha+1=\mathcal{W}_{U(\xi)}\restriction\alpha+1$ and $\bar{\Gamma}\restriction\bar{\alpha}+1=\Gamma_{\xi+1}\restriction\bar{\alpha}+1$, since both are just given by one-step embedding normalization $W(\mathcal{S}_\xi\restriction\bar{\alpha}+1,\mathcal{S}_a,F)$, by (2) at ξ. Moreover, $u^{\bar{\Gamma}}(\bar{\alpha})=u^{\Delta_\xi}(\bar{\alpha})$, since both of these u-maps are just the u-map of the embedding normalization by F (for Δ_ξ, this is our induction hypothesis (2)). We have that $F_\xi=E^{\mathcal{S}_{\xi+1}}_{\bar{\alpha}}$ and so $t^{\Gamma_{\xi+1}}_{\bar{\alpha}}=t^{\bar{\Gamma}}_{\bar{\alpha}}$ agrees with $t^{\Delta_\xi}_\infty$ on F_ξ (as either $\mathrm{lh}(F_\xi)<\mathrm{lh}(E^{\mathcal{S}_\xi}_{\bar{\alpha}})$ or $\bar{\alpha}+1=\mathrm{lh}(\mathcal{S}_\xi)$). It follows that $H_{U(\xi)}=E^{\bar{\mathcal{W}}}_{u^{\bar{\Gamma}}(\bar{\alpha})}$, so that $u^{\bar{\Gamma}}(\bar{\alpha})=\alpha$. This establishes $\bar{\Gamma}\restriction\bar{\alpha}+2=\Gamma_{\xi+1}\restriction\bar{\alpha}+2$.

For the rest, we show by induction on $\zeta<\mathrm{lh}(\mathcal{S}_\eta)$ that $\bar{\Gamma}\circ\bar{\Phi}\restriction\zeta+1=\Phi\restriction\zeta+1$, $u^{\bar{\Gamma}\circ\bar{\Phi}}(\zeta)=u^{\Phi}(\zeta)$, and $\bar{\mathcal{W}}\restriction u^{\Phi}(\zeta)+1=\mathcal{W}_{U(\xi)+1}\restriction u^{\Phi}(\zeta)+1$. Let $\bar{\beta}=\beta(\mathcal{S}_\xi,F_\xi)$ and $\beta=\beta(\mathcal{W}_{U(\xi)},H_{U(\xi)})$. We have that $\bar{\beta}=\beta$ by our case hypothesis and so $v^{\bar{\Phi}}\restriction\beta+1=v^{\Phi}\restriction\beta+1=id$. Also by our case hypothesis, we have $v^{\bar{\Gamma}}\restriction\beta+1=id$, so $\bar{\Gamma}\circ\bar{\Phi}\restriction\beta+1=\Phi\restriction\beta+1=Id_{\mathcal{S}_\eta\restriction\beta+1}$. Since $u^{\bar{\Gamma}}$ agrees with $v^{\bar{\Gamma}}$ on $\bar{\alpha}+1$ and $u^{\bar{\Gamma}}(\bar{\alpha})=\alpha$, as already established, we get $u^{\bar{\Phi}}(\beta)=\bar{\alpha}+1$ so $u^{\bar{\Gamma}\circ\bar{\Phi}}(\beta)=u^{\bar{\Gamma}}(\bar{\alpha}+1)=\alpha+1$. We also clearly have $u^{\Phi}(\beta)=\alpha+1$ (since Φ is just one-step normalization by $H_{U(\xi)}$), so $u^{\bar{\Gamma}\circ\bar{\Phi}}(\beta)=u^{\Phi}(\beta)$. Moreover, we already established $\bar{\mathcal{W}}\restriction\alpha+1=\mathcal{W}_{U(\xi)}\restriction\alpha+1=\mathcal{W}_{U(\xi)+1}\restriction\alpha+1$, as desired.

Suppose now $\zeta\geq\beta$ and our induction hypothesis holds up to ζ. We have that the exit extenders $E^{\bar{\mathcal{W}}}_{u^{\Phi}(\zeta)}$ and $E^{\mathcal{W}_{\xi+1}}_{u^{\Phi}(\zeta)}$ are equal since they are both images of $E^{\mathcal{S}_\eta}_\zeta$ under the same t-map (our induction hypothesis implies the ζth t-maps of $\bar{\Gamma}\circ\bar{\Phi}$ and Φ are the same). It follows that $\bar{\mathcal{W}}$ and $\mathcal{W}_{U(\xi)+1}$ agree up to $v^{\Phi}(\zeta+1)=u^{\Phi}(\zeta)+1$ and $\bar{\Gamma}\circ\bar{\Phi}$ and Φ agree up to $\zeta+2$. Since $\zeta+1\geq\beta$, we get that $u^{\bar{\Phi}}$ and u^{Φ} agree with their corresponding v-maps on $\zeta+1$. Moreover, since $u^{\Phi}(\zeta)>\alpha>\alpha(\mathcal{T}_a,F)$ (using here that $\xi\geq b$ so $U(\xi)>a$), we have that $u^{\bar{\Gamma}}$ agrees with $v^{\bar{\Gamma}}$ above $u^{\bar{\Phi}}(\zeta)$. So $u^{\bar{\Gamma}\circ\bar{\Phi}}(\zeta+1)=v^{\bar{\Gamma}\circ\bar{\Phi}}(\zeta+1)=v^{\Phi}(\zeta+1)$ and the trees agree this far, too.

Both trees must pick the same branches at limits, as well, since at limit $\lambda=v^{\Phi}(\bar{\lambda})=v^{\bar{\Gamma}\circ\bar{\Phi}}(\bar{\lambda})$, both trees must pick the image of $[0,\bar{\lambda})_{\mathcal{S}_\eta}$ under the same map. So as long as we don't reach ill-founded models, this agreement continues through limits. This finishes the induction, establishing (2) holds at $\xi+1$, in the case $\mathrm{crit}(F_\xi)<\mathrm{crit}(F)$.

Now suppose $\mathrm{crit}(F_\xi)\geq\mathrm{crit}(F)$. In this case $\eta\geq b$ and $\mathrm{crit}(H_{U(\xi)})\geq\lambda(F)$. It follows that $a+1\leq U(\eta)=\mathbb{W}\text{-pred}(U(\xi)+1)$. Where we used Γ_η above, we must now use Δ_η, which is, in particular, *not* the identity. We let $\mathcal{W}_{U(\xi)+1}=W(\mathcal{W}_{U(\eta)},\mathcal{W}_{U(\xi)},H_{U(\xi)})$ and $\Gamma_{\xi+1}$ the copy map associated to $(\Delta_\xi,\Delta_\eta,F_\xi,H_{U(\xi)})$, which is possible via Proposition 3.2.3, as before. We let $\Delta_{\xi+1}=\Gamma_{\xi+1}$, which maintatins (1). Let $\bar{\mathcal{W}}=W(\mathcal{S}_{\xi+1},\mathcal{T}_a,F)$ and $\bar{\Gamma}=\Phi^{\mathcal{S}_{\xi+1},\mathcal{T}_a,F}$. Again, we must show that $\bar{\Gamma}$ satisfies the properties in the conclusion of the Shift Lemma which uniquely specify $\Gamma_{\xi+1}$.

Getting $\bar{\Gamma} \restriction \bar{\alpha} + 2 = \Gamma_{\xi+1} \restriction \bar{\alpha} + 2$ is the same as before. For the rest, we now want to see that $\bar{\Gamma} \circ \bar{\Phi} = \Phi \circ \Delta_\eta$, where $\bar{\Phi}$ is as before and $\Phi = \Phi^{\mathcal{W}_{U(\eta)}, \mathcal{W}_{U(\xi)}, H_{U(\xi)}}$. The argument here is the same as in the previous case: we get agreement up to $\bar{\beta} + 2$ for free and then use that the remainder of our trees and tree embeddings are given by images of $\mathcal{S}_\eta$ under the same maps. We leave the details to the reader. This gives (2) at $\xi + 1$ and finishes the successor case.

Now suppose our induction hypothesis holds below some limit $\lambda > b$. We must let $\Delta_\lambda = \Gamma_\lambda$ be the unique tree embedding from $\mathcal{S}_\lambda = \lim_{[0,\lambda)_{\mathbb{S}}} (\mathbb{S} \restriction \lambda)$ to $\mathcal{W}_\lambda = \lim_{[0,V(\lambda))_{\mathbb{W}}} (\mathbb{W} \restriction V(\lambda))$ which commutes with the rest of our embeddings. By Lemma 3.1.61, we must have that $\Gamma_\lambda = \Phi^{\mathcal{S}_\lambda, \mathcal{T}_a, F}$. This maintains (1) and (2) and finishes the one-step normalization.

For normalizing an arbitrary stack of meta-trees $\langle \mathbb{S}, \mathbb{T} \rangle$, we'll need to talk about direct limits of systems of meta-trees under meta-tree embeddings. Our analysis of direct limits of trees under extended tree embeddings from §3.1 carries over to meta-trees under meta-tree embeddings in the natural way.

Definition 3.2.7. A *directed system of meta-trees* is a system $\mathcal{D} = \langle \{\mathbb{T}_a\}_{a \in A}, \{\vec{\Delta}^{a,b}\}_{a \preceq b} \rangle$, where $\preceq$ is a directed partial order on some set A and

(a) for any $a \in A$, $\mathbb{T}_a$ is a meta-tree of successor length,

(b) for any $a, b \in A$ with $a \prec b$, $\vec{\Delta}^{a,b} : \mathbb{T}_a \to \mathbb{T}_b$ is an extended meta-tree embedding,

(c) for any $a, b, c \in A$ such that $a \preceq b \preceq c$, $\vec{\Delta}^{a,c} = \vec{\Delta}^{b,c} \circ \vec{\Delta}^{a,b}$.

We define $\lim \mathcal{D}$ similarly to before, except we replace the parts of the tree embeddings with the corresponding parts of our meta-tree embeddings, e.g. we form U-threads x using the $U_{a,b}$ and form trees $\mathcal{T}_x$ by taking direct limits along $\Delta^{a,b}_{x(a)}$ instead of the t-maps, provided that enough of these are total. We also define systems $\vec{\Lambda}^a$ which, when the direct limit is well-founded, are extended meta-tree embeddings from $\mathbb{T}_a$ into $\lim \mathcal{D}$.

We say $\lim \mathcal{D}$ is *well-founded* if all the $\mathcal{T}_x$ are defined and are actually normal trees, the order on U-threads is well-founded, and the direct limit object is an meta-tree. Like in the case of direct limits of trees under extended tree embeddings, the last two conditions follow from the first.

We get that this construction really identifies the direct limit in the category of meta-trees (of successor lengths) and extended meta-tree embeddings between them, i.e. we have

Proposition 3.2.8. *Let $\mathcal{D} = \langle \{\mathbb{T}_a\}_{a \in A}, \{\vec{\Delta}^{a,b}\}_{a \preceq b} \rangle$ be a directed system of meta-trees.*

Suppose there is a meta-tree $\mathbb{S}$ and for all $a \in A$ extended meta-tree embeddings $\vec{\Pi}^a : \mathbb{T}_a \to \mathbb{S}$ such that whenever $a \preceq b$, $\vec{\Pi}^b = \vec{\Delta}^{a,b} \circ \vec{\Pi}^a$.

Then the direct limit $\lim \mathcal{D}$ is well-founded and there is a unique extended meta-tree embedding $\vec{\Pi} : \lim \mathcal{D} \to \mathbb{S}$ such that $\vec{\Pi}^a = \vec{\Pi} \circ \vec{\Lambda}^a$ for all $a \in A$.

Now, given a stack of meta-trees $\langle \mathbb{S}, \mathbb{T} \rangle$, with $\mathbb{S} = \langle \mathcal{S}_\xi, F_\xi, \Phi_{\eta,\xi} \rangle$, $\mathbb{T} = \langle \mathcal{T}_\xi, G_\xi, \Psi_{\eta,\xi} \rangle$, we define $\mathbb{W}(\mathbb{S}, \mathbb{T})$ as the last meta-tree in a sequence of meta-trees $\mathbb{W}^\xi = \langle \mathcal{W}^\xi_\zeta, F^\xi_\zeta \rangle$ of successor lengths, for $\xi < \mathrm{lh}(\mathbb{T})$. We also define (partial) extended meta-tree embeddings

$\vec{\Delta}^{\eta,\xi} : \mathbb{W}^\eta \to \mathbb{W}^\xi$ for $\eta \leq_{\mathbb{T}} \xi$. Of course, our construction only makes sense as long as we never reach ill-founded models, in which case we'll say that $\mathbb{W}(\mathbb{S}, \mathbb{T})$ is *well-founded.*

We maintain the following by induction.

1. $\mathcal{W}^\xi_\infty = \mathcal{T}_\xi$,
2. for $\eta \leq \xi$, $\mathbb{W}^\eta \restriction a(\mathbb{W}^\eta, G_\eta) + 1 = \mathbb{W}^\xi \restriction a(\mathbb{W}^\eta, G_\eta) + 1$,
3. for $\eta < \xi$, $G_\eta = F^\xi_{a(\mathbb{W}^\eta, G_\eta)}$,
4. for $\zeta \leq_{\mathbb{T}} \eta \leq_{\mathbb{T}} \xi$, $\vec{\Delta}^{\zeta,\xi} = \vec{\Delta}^{\eta,\xi} \circ \vec{\Delta}^{\zeta,\eta}$.
5. for $\eta \leq_{\mathbb{T}} \xi$, $\Delta^{\eta,\xi}_\infty = \Psi_{\eta,\xi}$.

To start, $\mathbb{W}^0 = \mathbb{S}$. Given everything up to $\mathbb{W}^\xi$, let $\eta = \mathbb{T}\text{-pred}(\xi + 1)$. We want to set $\mathbb{W}^{\xi+1} = \mathbb{W}(\mathbb{W}^\eta, \mathbb{W}^\xi, G_\xi)$, so we need to see that the agreement hypotheses of the one-step case are met. If $\eta = \xi$, this is trivial, so assume $\eta < \xi$. By our induction hypothesis (3), we have that $G_\eta = F^\xi_{a(\mathbb{W}^\eta, G_\eta)}$. By the normality of $\mathbb{T}$, we have that $\text{crit}(G_\xi) < \lambda(G_\eta)$, so $b(\mathbb{W}^\xi, G_\xi) \leq a(\mathbb{W}^\eta, G_\eta)$. If $b(\mathbb{W}^\xi, G_\xi) < a(\mathbb{W}^\eta, G_\eta)$ we're done by our induction hypothesis (2) and if $b(\mathbb{W}^\xi, G_\xi) = a(\mathbb{W}^\eta, G_\eta) = \text{lh}(\mathbb{W}^\eta) - 1$, we're also done (since $F^\eta_b(\mathbb{W}^\xi, G_\xi)$ is undefined). So assume $b(\mathbb{W}^\xi, G_\xi) = a(\mathbb{W}^\eta, G_\eta) < \text{lh}(\mathbb{W}^\eta) - 1$. Then $F^\eta_{a(\mathbb{W}^\eta, G_\eta)}$ is defined, but as G_η is on the sequence of the last models of both $\mathcal{W}^\eta_{a(\mathbb{W}^\eta, G_\eta)}$ and $\mathcal{T}_\eta$, the last tree of $\mathcal{W}^\eta$, we must have that $\text{lh}(F^\eta_{a(\mathbb{W}^\eta, G_\eta)}) \geq \text{lh}(G_\eta)$. So the hypotheses of the one-step case still apply. We also put $\vec{\Delta}^{\eta,\xi+1} = \vec{\Delta}^{\mathbb{W}^\eta, \mathbb{W}^\xi, G_\xi}$ and $\vec{\Delta}^{\zeta,\xi+1} = \vec{\Delta}^{\eta,\xi+1} \circ \vec{\Delta}^{\zeta,\eta}$ whenever $\zeta \leq_{\mathbb{T}} \eta$. By our work in the one-step case and our induction hypothesis at η and ξ, it's easy to see all our induction hypotheses still hold at $\xi + 1$.

At limit λ we take the extended meta-tree embedding direct limit along the branch chosen by $\mathcal{T}$. That is, letting $\mathcal{D}_\lambda = \langle \{\mathbb{W}^\eta\}_{\eta <_{\mathbb{T}} \lambda}, \{\vec{\Delta}^{\eta,\xi}\}_{\eta \leq_{\mathbb{T}} \xi <_{\mathbb{T}} \lambda} \rangle$, we let $\mathbb{W}^\lambda = \lim \mathcal{D}_\lambda$, if this is well-founded. The last tree of $\mathbb{W}^\lambda$ is the direct limit of the $\mathcal{T}_\eta$ under $\Psi_{\eta,\xi}$ for $\eta \leq_{\mathbb{T}} \xi <_{\mathbb{T}} \lambda$ by our induction hypotheses (1) and (5), which is just $\mathcal{T}_\lambda$ (since $\mathbb{T}$ is an meta-tree). We also let $\vec{\Delta}^{\eta,\lambda}$ be the direct limit meta-tree embeddings. It's easy to see that this maintains the rest of our induction hypotheses. This finishes the limit case and the definition of $\mathbb{W}(\mathbb{S}, \mathbb{T})$.

For a finite stack of meta-trees $\vec{\mathbb{S}}$, we also define, by induction, $\mathbb{W}(\vec{\mathbb{S}}) = \mathbb{W}(\mathbb{W}(\mathbb{S} \restriction n), \mathbb{S}_n)$. This definition makes sense since, by induction, $W(\vec{\mathbb{S}})$ and $\vec{\mathbb{S}}$ have the same same last tree.

Definition 3.2.9. Let $\mathcal{S}$ be a normal tree of successor length, θ a regular cardinal, and Σ an (ω, θ)-meta-iteration strategy for $\mathcal{S}$. Σ *normalizes well* iff for any finite stack of meta-trees $\vec{\mathbb{S}}$ by Σ, $\mathbb{W}(\vec{\mathbb{S}})$ is by Σ (in particular, it is well-founded).

Proposition 3.2.10. *Let M be a premouse, θ a regular cardinal, and Σ a θ-iteration strategy for M with SHC$^-$. Let $\mathcal{S}$ be a normal tree on M by Σ of successor length ¡θ. Then $\Sigma^*_{\mathcal{S}}$ normalizes well.*

Proof. By induction, we just need to verify this for stacks of length 2. By our characterization of $\Sigma^*_{\mathcal{S}}$, we just need to see that all the trees in all the $\mathbb{W}^\xi = \mathbb{W}(\mathbb{S}, \mathbb{T} \restriction \xi + 1)$ are by Σ. We do this by induction. At successors, all our new trees are all of the form $W(\mathcal{U}, \mathcal{V}, G)$ for trees

$\mathcal{U}, \mathcal{V}$ which are by Σ, so $W(\mathcal{U}, \mathcal{V}, G)$ is by Σ by SHC$^-$. At limit λ, we have all the $\mathbb{W}^\xi$ for $\xi < \lambda$ are by Σ^* and we want to see the direct limit along $[0, \lambda)_\mathbb{T}$ is by Σ^*. The trees of the direct limit are either trees of $\mathbb{W}^\xi$ or else are (non-trivial) direct limits along one-step embedding normalization tree embeddings by the extenders of $[0, \lambda)_\mathbb{T}$. All these non-trivial direct limit trees agree with $\mathcal{T}_\lambda$ up to $\delta_\lambda + 1$, where $\delta_\lambda = \sup_{\xi<\lambda}\{\alpha(\mathcal{T}_\xi, G_\xi) \mid \xi < \lambda\}$, which is by Σ. At limit ordinals $\gamma > \delta_\lambda$ in these direct limit trees, the branches are images under the v-maps of earlier trees which are by Σ and so must be by Σ by SHC$^-$. □

We can now easily prove Theorem 3.1.44.

Proof of Theorem 3.1.44 Let $\mathcal{S}$ be a normal tree by Σ of successor length and $\langle \mathbb{S}, \mathbb{T}\rangle$ be a stack by $\Sigma^*_\mathcal{S}$ with last tree $\mathcal{U}$. Since $\Sigma^*_\mathcal{S}$ normalizes well, $\mathbb{U} = \mathbb{W}(\mathbb{S}, \mathbb{T})$ is by $\Sigma^*_\mathcal{S}$, and is a meta-tree with last tree $\mathcal{U}$. Since $\vec{\Delta}^{0,\infty} : \mathbb{S} \to \mathbb{U}$ is a meta-tree embedding, we have $\Delta^{0,\infty}_\infty \circ \Phi_{0,\infty} = \Phi^\mathbb{U}_{0,\infty}$. We have $\Phi^\mathbb{S} = \Phi_{0,\infty}$, $\Phi^\mathbb{U}_{0,\infty} = \Phi^\mathbb{U}$, and, by (5), $\Delta^{0,\infty}_\infty = \Psi_{0,\infty} = \Phi^\mathbb{T}$. So $\Phi^\mathbb{U} = \Phi^\mathbb{T} \circ \Phi^\mathbb{S}$. Any drop along the main branch of $\mathbb{U}$ comes from being in the dropping case at some stage in forming $\mathbb{W}(\mathbb{S}, \mathbb{T})$ along the main branch of $\mathbb{T}$ (and so comes from a drop of $\mathbb{T}$), or else is the image of a drop coming from $\mathbb{S}$ under the resulting meta-tree embedding. It follows that $\mathbb{U}$ drops along its main branch iff $\langle \mathbb{S}, \mathbb{T}\rangle$ drops from $\mathcal{S}$-to-$\mathcal{U}$.

□

3.2.4 Uniqueness of embedding normalization

When we defined the embedding normalization of stack of normal trees $\vec{\mathcal{S}}$, $W(\vec{\mathcal{S}})$, we chose to iteratively embedding normalize pairs of normal trees, going from left-to-right. This is basically an arbitrary decision: for example, we could normalize the stack $\langle \mathcal{S}, \mathcal{T}, \mathcal{U}\rangle$ from right-to-left as well, producing the normal tree $W(\mathcal{S}, W(\mathcal{T}, \mathcal{U}))$ (assuming this is well-founded). In this section, we'll prove that any way of normalizing a stack actually produces the same final normal tree, answering a question of Steel. We'll use this result to show that bottom-up normalizing well passes to tail strategies.

We need some terminology for discussing different potential embedding normalizations of a stack of normal trees.

Definition 3.2.11. Let $\vec{\mathcal{S}}$ be a stack of normal iteration trees of length $n \in \omega$. A *putative embedding normalization sequence* for $\vec{\mathcal{S}}$ is any sequence of stacks of putative normal trees obtained by iteratively normalizing pairs of adjacent trees until we reach an ill-founded model or end up with a single normal tree, that is a sequence $\langle \vec{\mathcal{S}}^0, \ldots, \vec{\mathcal{S}}^m\rangle$ such that

(i) $\vec{\mathcal{S}}^0 = \vec{\mathcal{S}}$,

(ii) either $m = n - 1$ or $m < n - 1$ and the last tree of $\vec{\mathcal{S}}^{m-1}$ has an ill-founded last model,

(iii) for $i + 1 < m$, $\vec{\mathcal{S}}^i$ is a stack of normal trees and there is a $k < n - i$ such that $\vec{\mathcal{S}}^{i+1}$ is

$$\vec{\mathcal{S}}^i \upharpoonright k^\frown \langle W(\mathcal{S}^i_k, \mathcal{S}^i_{k+1})\rangle^\frown \sigma_i\left(\vec{\mathcal{S}}^i \upharpoonright (k + 1, n - i)\right),$$

where we build $W(\mathcal{S}^i_k, \mathcal{S}^i_{k+1})$ until we reach an ill-founded model, and if $W(\mathcal{S}^i_k, \mathcal{S}^i_{k+1})$ is well-founded, then $\sigma_i : M^{\mathcal{S}^i_{k+1}}_\infty \to M^{W(\mathcal{S}^i_k, \mathcal{S}^i_{k+1})}_\infty$ is the associated embedding normalization map and $\sigma_i\left(\vec{\mathcal{S}}^i \upharpoonright (k + 1, n - i)\right)$ is the stack of putative normal trees obtained by copying $\vec{\mathcal{S}}^i \restriction (k+1, n-i)]$ under σ_i as long as possible (i.e. we stop when we reach an ill-founded model).

An *embedding normalization sequence* for $\vec{\mathcal{S}}$ is a putative embedding normalization sequence for $\vec{\mathcal{S}}$ consisting of stacks normal tree, i.e. in which we never reach ill-founded models. A normal tree $\mathcal{T}$ is an *embedding normalization* of $\vec{\mathcal{S}}$ if there is an embedding normalization sequence for $\vec{\mathcal{S}}$ with final stack $\langle \mathcal{T} \rangle$. It is easy to see that the last stack of any embedding normalization sequence must consist of a single normal tree.

We'll show that all embedding normalizations of a stack are the same, as are the resulting embedding normalization maps. This is an easy consequence of a kind of associativity of embedding normalization: for any stack $\langle \mathcal{S}, \mathcal{T}, \mathcal{U} \rangle$, $W(\mathcal{S}, W(\mathcal{T}, \mathcal{U})) = W(W(\mathcal{S}, \mathcal{T}), \sigma\mathcal{U})$, where σ is the embedding normalization map from the last model of $\mathcal{T}$ to the last model of $W(\mathcal{S}, \mathcal{T})$.

First, we'll show that embedding normalization is *continuous* in the sense that it commutes with taking direct limits. This will be important for dealing with limit stages in our inductive proof of associativity.

Theorem 3.2.5 implies that we can lift directed systems of trees to directed systems of meta-trees via the embedding normalization process.

Proposition 3.2.12. *Let $\mathcal{S}$ be a normal tree with last model P and $\mathcal{D} = \langle \{\mathcal{T}\}_{a \in A}, \{\Psi_{a,b}\}_{a \preceq b} \rangle$ is a directed system of trees on P and suppose for all $a \in A$, $\mathbb{W}(\mathcal{S}, \mathcal{T}_a)$ is well-founded.*

Then there are (unique) meta-tree embeddings $\vec{\Delta}^{a,b}$ such that $\mathcal{D}^* = \langle \{\mathbb{W}(\mathcal{S}, \mathcal{T}_a)\}_{a \in A}, \{\vec{\Delta}^{a,b}\}_{a \preceq b} \rangle$ is a directed system of meta-trees, $U^{\vec{\Delta}^{a,b}} = u^{\Psi_{a,b}}$, and for σ^a_ξ the embedding normalization map from $M^{\mathcal{T}_a}_\xi$ into the last model of $W(\mathcal{S}, \mathcal{T}_a \restriction \xi + 1)$, $\sigma^b_{u^{\Psi_{a,b}}(\xi)} \circ t^{\Psi_{a,b}}_\xi = t^{\Delta^{a,b}_\xi}_\infty \circ \sigma^a_\xi$.

We call this $\mathcal{D}^*$ the *lift* of $\mathcal{D}$.

For $\mathcal{D} = \langle \{\mathbb{T}_a\}_{a \in A}, \{\vec{\Delta}^{a,b}\}_{a \preceq b} \rangle$ a directed system of meta-trees, and $B \subseteq A$, we define $\mathcal{D} \restriction B = \langle \{\mathbb{T}_a\}_{a \in B} \{\vec{\Delta}^{a,b}\}_{a \preceq b \wedge a,b \in B} \rangle$.

The following is immediate from Proposition 3.2.8.

Proposition 3.2.13. *Let $\mathcal{D} = \langle \{\mathbb{T}_a\}_{a \in A}, \{\vec{\Delta}^{a,b}\}_{a \preceq b} \rangle$ be a well-founded directed system of meta-trees such that $\lim \mathcal{D}$ is well-founded. Suppose that $B \subseteq A$ is such that for every $a \in A$ there are extended meta-tree embeddings $\vec{\Xi}^a : \mathbb{T}_a \to \lim(\mathcal{D} \restriction B)$ such that for every $a, b \in A$ such that $a \preceq b$, $\vec{\Xi}^a \circ \vec{\Delta}^{a,b} = \vec{\Xi}^a$.*

Then $\lim \mathcal{D} = \lim(\mathcal{D} \restriction B)$.

Note that any $\preceq$-cofinal $B \subseteq A$ satisfies the hypothesis of the proposition, for example. Now we'll show that embedding normalization is continuous.

Theorem 3.2.14. *Let $\mathcal{S}$ be a normal tree with last model P, $\mathcal{D} = \langle \{\mathcal{T}_a\}_{a \in A}, \{\Psi_{a,b}\}_{a \preceq b} \rangle$ be a directed system of trees on P such that $W(\mathcal{S}, \mathcal{T}_a)$ is well-founded for every $a \in A$, and let $\mathcal{D}^*$ be the lift of of $\mathcal{D}$.*

Suppose that $\lim \mathcal{D}$ is well-founded. Then for any μ, $\lim \mathcal{D}^* \restriction \mu + 1$ is well-founded iff $\mathbb{W}(\mathcal{S}, \lim \mathcal{D} \restriction \mu + 1)$ is well-founded and $\lim \mathcal{D}^* \restriction \mu + 1 = \mathbb{W}(\mathcal{S}, \lim \mathcal{D} \restriction \mu + 1)$.

Note that this implies, in particular, that if $\lim \mathcal{D}^*$ is well-founded, then the full $\mathbb{W}(\mathcal{S}, \lim \mathcal{D})$ is well-founded, and vice-versa. In this case, looking at the last tree on either sides gives us $\lim\langle\{W(\mathcal{S}, \mathcal{T}_a)\}_{a\in A}, \{\Delta^{a,b}_\infty\}_{a\preceq b}\rangle = W(\mathcal{S}, \lim \mathcal{D})$.

Proof. First notice that since $\lim \mathcal{D}$ is well-founded and the underlying partial orders of $\mathcal{D}^*$ and $\mathcal{D}$ are the same, we have that the order on U-threads of $\mathbb{D}$ (which are just u-threads of $\mathcal{D}$) is well-founded. So we can let μ^* be largest such that $\lim \mathcal{D}^* \restriction \mu^* + 1$ is well-founded. We also let μ be largest such that $\mathbb{W}(\mathcal{S}, \lim \mathcal{D}) \restriction \mu + 1$ is well-founded. For every $a, b \in A$ such that $a \preceq b$ and every $\xi < \mathrm{lh}(\mathcal{T}_a)$, we have $\Delta^{a,b}_\xi : W(\mathcal{S}, \mathcal{T}_a \restriction \xi + 1) \to W(\mathcal{S}, \mathcal{T}_b \restriction u^{a,b}(\xi) + 1)$, where $u^{a,b} = u^{\Psi_{a,b}}$. For x a u-thread, we let $\mathcal{D}_x = \langle W(\mathcal{S}, \mathcal{T}_a \restriction x(a)+1), \Delta^{a,b}_{x(a)} \mid a, b \in \mathrm{dom}(x) \wedge a \preceq b\rangle$, and $\mathcal{D}^*_x$ the lift of $\mathcal{D}_x$, so $\mathcal{D}^*_x$ is just the system $\langle \mathbb{W}(\mathcal{S}, \mathcal{T}_a \restriction x(a) + 1), \vec{\Delta}^{a,b} \restriction x(a) + 1 \mid a, b \in \mathrm{dom}(x) \wedge a \preceq b\rangle$. We also let $\tau^{a,b}_\xi = t_\infty^{\Delta^{a,b}_\xi}$.

We let M_x and E_x be the models and extenders of $\lim \mathcal{D}$. If $x + 1$ has rank $< \mathrm{lh}(\lim \mathcal{D})$ and x has rank $\leq \mu$, we let σ_x be the embedding normalization map from M_x into the last model of $\mathbb{W}(\mathcal{S}, \lim \mathcal{D} \restriction x + 1)$. Also, if $x + 1$ has rank $< \mathrm{lh}(\lim \mathcal{D})$ and $x \leq \mu^*$, we let F_x be the meta-tree exit extender of $\lim \mathcal{D}^* \restriction \mu + 1$. For $a \in A$ and $\xi < \mathrm{lh}(\mathcal{T}_a)$, let σ^a_ξ be the embedding normalization map from $M^{\mathcal{T}_a}_\xi$ into the last model of $W(\mathcal{S}, \mathcal{T}_a \restriction \xi + 1)$ and for $a \preceq b$, let $t^{a,b}_\xi = t^{\Psi_{a,b}}_\xi$. We also let $t^a_\xi : M^{\mathcal{T}_a}_\xi \to M_{[a,\xi]_\mathcal{D}}$ be the t-maps of the direct limit tree embedding from $\mathcal{T}_a$ into $\lim \mathcal{D}$.

We also $\vec{\Lambda}^a = \langle U^a, V^a, \Pi^a_\eta, \Lambda^a_\eta\rangle$ be the partial tree embedding from $\mathbb{T}_a$ into $\lim \mathcal{D}^* \restriction \mu^* + 1$. If $[a, \xi]_\mathcal{D} \leq \mu^*$, then $\vec{\Lambda}^a \restriction \xi + 1$ is a total extended tree embedding from $\mathbb{T}_a \restriction \xi + 1$ into $\lim \mathcal{D}^* \restriction [a, \xi]_\mathcal{D} + 1$. Let $\tau^a_\xi = t_\infty^{\Lambda^a_\xi}$. For x a u-thread and $a, b \in \mathrm{dom}(x)$ with $a \preceq b$, we have $\tau^{a,b}_{x(a)} \circ \sigma^a_{x(a)} = \sigma^b_{x(b)} \circ t^{a,b}_{x(a)}$, and since the last model of $\lim \mathcal{D}^*_x$ is a direct limit of the last models of the $W(\mathcal{S}, \mathcal{T}_a \restriction x(a) + 1)$ under the $\tau^{a,b}_{x(a)}$ and $\tau^a_x(a)$ is the resulting direct limit map, we get an elementary embedding σ^*_x from M_x into the last model of $\lim \mathcal{D}^*_x$ with the obvious commutativity properties. If $x + 1 < \mathrm{lh}(\lim \mathcal{D})$, then since $F^{\mathbb{W}(\mathcal{S}, \mathcal{T}_a \restriction x(a)+1)}_{x(a)} = \sigma^a_{x(a)}(E^{\mathcal{T}_a}_{x(a)})$ for all $a \in \mathrm{dom}(x)$, we get that $F_x = \sigma^*_x(E_x)$.

Finally, if x has rank $\leq \mu$, we let σ_x be the embedding normalization map from M_x into the last model of $W(\mathcal{S}, \lim \mathcal{D}_x)$. We'll show by induction on u-threads x that the rank of x is $\leq \mu$ iff it is $\leq \mu^*$ and if rank of $x \leq \mu, \mu^*$,

1. $\lim \mathcal{D}^*_x = \mathbb{W}(\mathcal{S}, \lim \mathcal{D}_x)$
2. $\sigma^*_x = \sigma_x$.

Base case. Let x be the $\mathcal{D}$ u-thread of rank 0.

So $x(a) = 0$ for all $a \in \mathrm{dom}(x)$, so $\mathcal{T}_a \restriction x(a) + 1$ is the trivial tree on P, the last model of $\mathcal{S}$. $\lim \mathcal{D}_x$ is also the trivial tree on P. So

$$\mathbb{W}(\mathcal{S}, \mathcal{T}_a \restriction x(a) + 1) = \mathbb{W}(\mathcal{S}, \lim \mathcal{D}_x) = \mathcal{S}.$$

This gives (1). For (2), we have that $\sigma^*_x = \sigma_x = id$.

Successor case. Let x have successor rank in $\leq$, say x is the successor of y, where y has rank $\leq \mu, \mu^*$.

By our induction hypothesis (1) at y, we have that $\lim \mathcal{D}^*_y = \mathbb{W}(\mathcal{S}, \lim \mathcal{D}_y)$. By (2) at y, we have $F_y = \sigma_y(E_y) = \sigma^*_y(E_y)$. So we get

$$\begin{aligned}\lim \mathcal{D}^*_x &= \lim \mathcal{D}^{*\frown}_y \langle F_y \rangle \\ &= \mathbb{W}(\mathcal{S}, \lim \mathcal{D}_y)^\frown \langle F_y \rangle \\ &= \mathbb{W}(\mathcal{S}, \lim \mathcal{D}_x).\end{aligned}$$

This shows (1) and, since we didn't assume $x \leq \mu, \mu^*$, we get that

$$\begin{aligned}x \leq \mu^* &\Leftrightarrow \lim \mathcal{D}^*_x \text{ is well-founded} \\ &\Leftrightarrow \mathbb{W}(\mathcal{S}, \lim \mathcal{D}_x) \text{ is well-founded} \\ &\Leftrightarrow x \leq \mu.\end{aligned}$$

(2) follows since σ_x and σ^*_x are determined in the same way from $\sigma_y = \sigma^*_y$ and $\sigma_z = \sigma^*_z$ and the last t-map of the common relevant tree embedding, where $z = \lim \mathcal{D}\text{-pred}(y+1)$.

Limit case. Suppose x has limit rank and for all $y < x$, $y \leq \mu, \mu^*$. We get that $x \leq \mu, \mu^*$ for free in this case. For all $a \in \operatorname{dom}(x)$ and all ξ we have

$$\xi \in [0, x(a))_{\mathcal{T}_a} \Leftrightarrow [a, \xi]_{\mathcal{D}} <^* x.$$

Moreover, any $y <^* x$ has the form $[a, \xi]_{\mathcal{D}}$ for some $a \in \operatorname{dom}(x)$ and $\xi \in [0, x(a))_{\mathcal{T}_a}$. We define a system of meta-trees $\mathcal{C}$ with underlying partial order $\preceq_{\mathcal{C}}$ on some set C. $\preceq_{\mathcal{C}}$ won't be directed on the full set C, but we'll look at $C_0, C_1 \subseteq C$ such that $\preceq_{\mathcal{C}}\restriction C_0$ and $\preceq_{\mathcal{C}}\restriction C_1$ are directed. We'll have that $\lim(\mathcal{C} \restriction C_0) = \lim \mathcal{D}^*_x$ and $\lim(\mathcal{C} \restriction C_1) = \mathbb{W}(\mathcal{S}, \lim \mathcal{D}_x)$. We then show that Proposition 3.2.13 applies to $B = C_0 \cap C_1$, so that

$$\lim \mathcal{D}^*_x = \lim(\mathcal{C} \restriction B) = \mathbb{W}(\mathcal{S}, \lim \mathcal{D}_x).$$

Let $C = \{(a, \xi) \mid \xi \in [0, x(a)]_{\mathcal{T}_a}\} \cup \{y \mid y <^* x\}$, $C_0 = \{(a, \xi) \mid \xi \in [0, x(a)]_{\mathcal{T}_a}\}$, and $C_1 = \{(a, \xi) \mid \xi \in [0, x(a))_{\mathcal{T}_a}\} \cup \{y \mid y <^* x\}$. So $B = C_0 \cap C_1 = \{(a, \xi) \mid \xi \in [0, x(a))_{\mathcal{T}_a}\}$.

For all $y <^* x$, $a, b \in \operatorname{dom}(x)$, $\xi \in [0, x(a)]_{\mathcal{T}_a}$ and $\eta \in [0, x(b)]_{\mathcal{T}_b}$, we put

$$\begin{aligned}(a, \xi) \preceq_{\mathcal{C}} (b, \eta) &\Leftrightarrow a \preceq b \wedge u^{a,b}(\xi) \leq_{\mathcal{T}_b} \eta \\ (a, \xi) \preceq_{\mathcal{C}} y &\Leftrightarrow [a, \xi]_{\mathcal{D}} \leq^* y \\ y \preceq_{\mathcal{C}} z &\Leftrightarrow y \leq^* z.\end{aligned}$$

We also put $y \not\preceq_{\mathcal{C}} (a, \xi)$ for all $y <^* c$, $a \in \operatorname{dom}(x)$, and $\xi \in [0, x(a)]_{\mathcal{T}_a}$.

We define meta-trees $\mathbb{U}_c$ for $c \in C$ as follows. For $a \in \operatorname{dom}(x)$, and $\xi \in [0, x(a)]_{\mathcal{T}_a}$, we put $\mathbb{U}_{(a,\xi)} = \mathbb{W}(\mathcal{S}, \mathcal{T}_a \restriction \xi + 1)$. For $y <^* x$, we put $\mathbb{U}_y = \lim \mathcal{D}^*_y = \mathbb{W}(\mathcal{S}, \lim \mathcal{D}_y)$, using our induction hypothesis (1) at y.

We define the extended meta-tree embeddings $\vec{\Omega}^{c,d} : \mathbb{U}_c \to \mathbb{U}_d$ for $c \preceq_{\mathbb{C}} d$ as follows. If $(a, \xi) \preceq_{\mathcal{C}} (b, \eta)$, we put

$$\Omega^{(a,\xi),(b,\eta)}_\zeta = \Phi^{\mathbb{W}(\mathcal{S}, \mathcal{T}_b)}_{u^{a,b}(\xi), \eta} \circ \Delta^{a,b}_\zeta.$$

If $y \preceq_{\mathcal{C}} z$, and for $\zeta <$ rank of y, we let

$$\Omega^{y,z}_\zeta = Id$$

and for $\zeta =$ the rank of y ($= \mathrm{lh}(\mathbb{U}_y) - 1$), we let

$$\Omega^{y,z}_\zeta = \Phi^{\mathbb{W}(\mathcal{S}, \lim \mathcal{D})}_{y,z}$$

Finally, if $(a, \xi) \preceq_{\mathcal{C}} y$, we put

$$\vec{\Omega}^{(a,\xi),y} = \vec{\Omega}^{[a,\xi]_{\mathcal{D}},y} \circ \vec{\Lambda}^a \restriction \xi + 1.$$

It is easy to check that whenever $c \preceq_{\mathcal{C}} d \preceq_{\mathcal{C}} e$, $\vec{\Omega}^{c,e} = \vec{\Omega}^{d,e} \circ \vec{\Omega}^{c,d}$.

We have that $A = \{(a, x(a)) \mid a \in \mathrm{dom}(x)\}$ is a $\preceq_{\mathcal{C}}$-cofinal subset of C_0, and $\mathcal{D}^*_x = \mathcal{C} \restriction A$, so by Proposition 3.2.13, $\lim \mathcal{D}^*_x = \lim(\mathcal{C} \restriction C_0)$. Similarly, $\{y \mid y <^* x\}$ is a $\preceq_{\mathcal{C}}$-cofinal subset of C_1, so $\mathbb{W}(\mathcal{S}, \lim \mathcal{D}_x) = \lim(\mathcal{C} \restriction C_1)$.

We now show that Proposition 3.2.13 applies to $\mathcal{C} \restriction C_0$ and B. There are two cases depending on how often $x(a)$ is a limit ordinal.

Subcase 1. $x(a)$ is a limit ordinal on a $\preceq$-cofinal set.

This is the easy case. For any a such that $x(a)$ is a limit ordinal,

$$\mathbb{U}_{(a,x(a))} = \lim \langle \{\mathbb{U}_{(a,\xi)}\}_{\xi <_{\mathcal{T}_a} x(a)}, \{\vec{\Omega}^{(a,\xi),(a,\eta)}\}_{\xi \leq_{\mathcal{T}_a} \eta <_{\mathcal{T}_a} x(a)} \rangle.$$

Since $\langle \{\mathbb{U}_{(a,\xi)}\}_{\xi <_{\mathcal{T}_a} x(a)}, \{\vec{\Omega}^{(a,\xi),(a,\eta)}\}_{\xi \leq_{\mathcal{T}_a} \eta <_{\mathcal{T}_a} x(a)} \rangle \subseteq (\mathcal{C} \restriction B)$, we get a meta-tree embedding $\vec{\Xi}^{(a,x(a))}$ with the commutativity properties of 3.2.13. By our case hypothesis, we can do this for $\preceq$-cofinally many a's, so we can do it for all of them by composing with the $\vec{\Delta}^{b,a}_{x(b)}$.

Subcase 2. There is $a \in \mathrm{dom}(x)$ such that for all $b \succeq a$, $x(b)$ is a successor ordinal.

Fix such an a. Since x doesn't have successor rank, for all $b \succeq a$ there is a u-thread $y_b <^* x$ such that for all $c \succeq b$ with $c \in \mathrm{dom}(y)$,

$$v^{b,c}(x(b)) \leq_{\mathcal{T}_c} y_b(c) <_{\mathcal{T}_c} x(c).$$

It follows that there is an extended tree embedding from $\mathcal{T}_b \restriction x(b) + 1$ into $\mathcal{T}_c \restriction y_b(c) + 1$ and so this lifts to an extended meta-tree embedding $\vec{\Gamma}^{b,c}$ from $\mathbb{W}(\mathcal{S}, \mathcal{T}_b \restriction x(b) + 1)$ into $\mathbb{W}(\mathcal{S}, \mathcal{T}_c \restriction y_b(c) + 1)$. We let

$$\vec{\Xi}^{(b,x(b))} = \vec{\Xi}^{(c,y_b(c))} \circ \vec{\Gamma}^{b,c},$$

where $\vec{\Xi}^{(c,y_b(c))}$ is the direct limit meta-tree embedding from $\mathbb{U}_{(c,y_b(c))} = \mathbb{W}(\mathcal{S}, \mathcal{T}_c \restriction y_b(c)+1)$ into $\lim \mathcal{C} \restriction B$. It's easy to see that the $\vec{\Xi}^{(b,x(b))}$ meet the commutativity condition of Proposition 3.2.13.

So in either case, we have $\lim(\mathcal{C} \restriction C_0) = \lim(\mathcal{C} \restriction B)$.

Now for $y <^* x$, $\mathcal{D}^*_y \subseteq \mathcal{C} \restriction B$ and $\mathbb{U}_y = \lim \mathcal{D}_y$, so we get extended meta-tree embeddings $\vec{\Xi}^y$ from $\mathbb{U}_y$ into $\lim(\mathcal{C} \restriction B)$ with the commutativity properties of Proposition 3.2.13, as in Subcase 1 for C_0. This gives $\lim(\mathcal{C} \restriction C_1) = \lim(\mathcal{C} \restriction B)$.

So $\lim \mathcal{D}_x^* = \lim(\mathcal{C} \restriction B) = \mathbb{W}(\mathcal{S}, \lim \mathcal{D}_x)$, giving (1). For (2), σ_x and σ_x^* are both just the direct limit map from M_x into the last model of this common meta-tree (such a map exists because we're obtaining the models on either side via direct limits of maps which commute with the $\sigma_y = \sigma_y^*$ for $y <^* x$).

This finishes the limit case and the induction. $\square$

The main step in the proof of the uniqueness of embedding normalizations is to show that the two embedding normalizations of stacks of length 3 are the same, i.e. $W(W(\mathcal{S}, \mathcal{T}), \sigma\mathcal{U}) = W(\mathcal{S}, W(\mathcal{T}, \mathcal{U}))$. We think of this as a kind of *associativity* of the embedding normalization operation. We prove this by induction on the length of $\mathcal{U}$. The next lemma will get us through the successor step of that induction.

Lemma 3.2.15. *Let $\mathcal{S}$ be a normal tree with last model P and $\bar{\mathcal{T}}, \mathcal{T}$ normal trees on P of successor length. Let F be an extender on the P-sequence and suppose that $\mathbb{W}(\bar{\mathcal{T}}, \mathcal{T}, F)$ is defined. Let $\sigma : M_\infty^{\mathcal{T}} \to M_\infty^{W(\mathcal{S},\mathcal{T})}$ be the associated embedding normalization map.*

Let μ largest such that $\mathbb{W}(\mathbb{W}(\mathcal{S}, \bar{\mathcal{T}}), \mathbb{W}(\mathcal{S}, \mathcal{T}), \sigma(F)) \restriction \mu + 1$ is well-founded and μ^* largest such that $\mathbb{W}(\mathcal{S}, W(\bar{\mathcal{T}}, \mathcal{T}, F)) \restriction \mu^* + 1$ is well-founded.

Then $\mu = \mu^*$ and

$$\mathbb{W}(\mathbb{W}(\mathcal{S}, \bar{\mathcal{T}}), \mathbb{W}(\mathcal{S}, \mathcal{T}), \sigma(F)) \restriction \mu + 1 = \mathbb{W}(\mathcal{S}, W(\bar{\mathcal{T}}, \mathcal{T}, F)) \restriction \mu + 1.$$

Moreover, $\vec{\Delta}^{\mathbb{W}(\mathcal{S},\bar{\mathcal{T}}),\mathbb{W}(\mathcal{S},\mathcal{T}),\sigma(F)}$ is the lift of $\Phi^{\bar{\mathcal{T}},\mathcal{T},F}$.[18]

Of course this implies that if $\mathbb{W}(\mathbb{W}(\mathcal{S}, \bar{\mathcal{T}}), \mathbb{W}(\mathcal{S}, \mathcal{T}), \sigma(F))$ is well-founded, so is $\mathbb{W}(\mathcal{S}, W(\bar{\mathcal{T}}, \mathcal{T}, F))$, and vice-versa.

Proof. We will just assume all of the meta-trees are well-founded; it is straightforward to check that we are below μ on one side iff we are below μ^* on the other. We'll prove $\mathbb{W}(\mathbb{W}(\mathcal{S}, \bar{\mathcal{T}}), \mathbb{W}(\mathcal{S}, \mathcal{T}), \sigma(F)) = \mathbb{W}(\mathcal{S}, W(\bar{\mathcal{T}}, \mathcal{T}, F))$ by induction. To start, we need to name all of the objects involved.

Let $\beta = \beta(\mathcal{T}, F)$, and $\alpha = \alpha(\mathcal{T}, F)$. Let $\mathbb{W}(\mathcal{S}, \bar{\mathcal{T}}) = \langle \bar{\mathcal{T}}_\xi, \bar{F}_\xi, \bar{\Phi}_{\eta,\xi} \rangle$, $\bar{\sigma}_\xi : M_\xi^{\bar{\mathcal{T}}} \to M_\infty^{\bar{\mathcal{T}}_\xi}$ the embedding normalization map, $\mathbb{W}(\mathcal{S}, \mathcal{T}) = \langle \mathcal{T}_\xi, F_\xi, \Phi_{\eta,\xi} \rangle$, and $\sigma_\xi : M_\xi^{\mathcal{T}} \to M_\infty^{\mathcal{T}_\xi}$ the embedding normalization map. Note that $\sigma = \sigma_\infty$, the last of these maps, and $\sigma \restriction \mathrm{lh}(F) + 1 = \sigma_\alpha \restriction \mathrm{lh}(F) + 1$.

Let $\mathcal{U} = W(\bar{\mathcal{T}}, \mathcal{T}, F)$ and $\Phi = \Phi^{\bar{\mathcal{T}},\mathcal{T},F}$. So we have $\mathcal{U} \restriction \alpha+1 = \mathcal{T} \restriction \alpha+1$, $\beta = \mathcal{U}\text{-pred}(\alpha+1)$, $u^\Phi(\beta) = \alpha + 1$, and $u^\Phi \restriction [\beta, \mathrm{lh}(\bar{\mathcal{T}}))$ is an order isomorphism and preserves tree-order strictly above β.

Let $\mathbb{W}(\mathcal{S}, \mathcal{U}) = \langle \mathcal{W}_\xi, G_\xi, \Psi_{\eta,\xi} \rangle$ and $\pi_\xi : M_\xi^{\mathcal{U}} \to M_\infty^{\mathcal{W}_\xi}$ the associated embedding normalization map. Notice that for $\xi \geq \beta$, $\mathcal{W}_{u^\Phi(\xi)} = W(\mathcal{S}, W(\bar{\mathcal{T}} \restriction \xi + 1, \mathcal{T}, F))$ and for $\xi < \beta$, $\mathcal{W}_{u^\Phi(\xi)} = \mathcal{W}_\xi = \bar{\mathcal{T}}_\xi$.

Let $\mathbb{W}^* = \mathbb{W}(\mathbb{W}(\mathcal{S}, \bar{\mathcal{T}}), \mathbb{W}(\mathcal{S}, \mathcal{T}), \sigma(F)) = \langle \mathcal{W}_\xi^*, G_\xi^*, \Psi_{\eta,\xi}^* \rangle$ and $\vec{\Delta} : \mathbb{W}(\mathcal{S}, \bar{\mathcal{T}}) \to \mathbb{W}^*$ be the meta-tree embedding coming from the meta-tree embedding normalization. We have that $\mathbb{W}(\mathcal{S}, \mathcal{T})$ and $\mathcal{T}$ have the same tree-order, $\alpha = a(\mathbb{W}(\mathcal{S}, \mathcal{T}), \sigma(F))$, and $\beta = b(\mathbb{W}(\mathcal{S}, \mathcal{T}), \sigma(F))$.

[18] Recall that this just means that $\vec{\Delta}^{\mathbb{W}(\mathcal{S},\bar{\mathcal{T}}),\mathbb{W}(\mathcal{S},\mathcal{T}),\sigma(F)}$ is the meta-tree embedding as in Theorem 3.2.5 for $\Psi = \Phi^{\bar{\mathcal{T}},\mathcal{T},F}$.

It follows that $U^{\vec{\Delta}} = u^\Phi$. We let $u = U^{\vec{\Delta}} = u^\Phi$. We have that for $\xi \geq \beta$, $\Delta_\xi = \Phi^{\bar{\mathcal{T}}_\xi, \mathcal{T}_\alpha, \sigma(F)}$ and $\mathcal{W}^*_{u(\xi)} = W(\bar{\mathcal{T}}_\xi, \mathcal{T}_\alpha, \sigma(F))$ and for $\xi < \beta$, $u(\xi) = \xi$, and $\mathcal{W}^*_{u(\xi)} = \mathcal{W}_\xi = \bar{\mathcal{T}}_\xi$. We also have, by Proposition 3.2.4, that, at successors, $\Delta_{\xi+1}$ is a copy tree embedding associated to the appropriate objects, which we'll use later on.

Note that, since $\mathcal{U} \restriction \alpha + 1 = \mathcal{T} \restriction \alpha + 1$, $\mathbb{W}(\mathcal{S}, \mathcal{T}) \restriction \alpha + 1 = \mathbb{W}(\mathcal{S}, \mathcal{U}) \restriction \alpha + 1$. It follows that $\mathbb{W}^* \restriction u(\alpha) + 1 = \mathbb{W}(\mathcal{S}, \mathcal{U}) \restriction u(\alpha) + 1$, by the definition of the one-step meta-tree embedding normalization. So we just need to show by induction on $\xi < \mathrm{lh}(\bar{\mathcal{T}})$ with $\xi \geq \beta$ that $\mathbb{W}^* \restriction u(\xi) + 1 = \mathbb{W}(\mathcal{S}, \mathcal{U}) \restriction u(\xi) + 1$.

We need an easy preliminary observation.

Claim 1. *$\mathbb{W}^*$ and $\mathbb{W}(\mathcal{S}, \mathcal{U})$ have the same tree-order, $\leq_{\mathcal{U}}$.*

Proof. We have that $\mathbb{W}(\mathcal{S}, \mathcal{U})$ has the same tree-order as $\mathcal{U}$, so we just need to see that $\mathbb{W}^*$ does, too.

$\mathcal{U} = W(\bar{\mathcal{T}}, \mathcal{T}, F)$, so that $\mathcal{U} \restriction \alpha + 1 = \mathcal{T} \restriction \alpha + 1$. We also have that $\mathbb{W}^* = \mathbb{W}(\mathbb{W}(\mathcal{S}, \bar{\mathcal{T}}), \mathbb{W}(\mathcal{S}, \mathcal{T}), \sigma(F))$, so that $\mathbb{W}^* \restriction \alpha + 1 = \mathbb{W}(\mathcal{S}, \mathcal{T}) \restriction \alpha + 1$. Since $\mathbb{W}(\mathcal{S}, \mathcal{T})$ has the same tree-order as $\mathcal{T}$, we have that $\mathcal{U}$ and $\mathbb{W}^*$ have the same tree-order up to α. We also have $\beta = \mathcal{U}\text{-pred}(\alpha + 1) = \mathbb{W}^*\text{-pred}(\alpha + 1)$. If $W(\bar{\mathcal{T}}, \mathcal{T}, F)$ is in the droppin case, then so is $\mathbb{W}^*$, and we're done. So suppose we're not in the dropping case. Then it is enough to see that $\mathbb{W}^*$ and $\mathcal{U}$ assign the same tree-predecessors and make the same branch choices in $u``[\beta, \mathrm{lh}(\bar{\mathcal{T}}))$.

Let $\xi \geq \beta$ with $\xi + 1 < \mathrm{lh}(\bar{\mathcal{T}})$. Let $\eta = \bar{\mathcal{T}}\text{-pred}(\xi + 1)$. Since $W(\bar{\mathcal{T}}, \mathcal{T}, F)$ is defined and $\xi \geq \beta$, we get that

$$\begin{aligned} \sigma \restriction \mathrm{dom}(F) &= \sigma_\beta \restriction \mathrm{dom}(F) \\ &= \bar{\sigma}_\beta \restriction \mathrm{dom}(F) \\ &= \bar{\sigma}_\xi \restriction \mathrm{dom}(F). \end{aligned}$$

Since $\bar{F}_\xi = \bar{\sigma}_\xi(E^{\bar{\mathcal{T}}}_\xi)$, it follows that $\mathrm{crit}(E^{\bar{\mathcal{T}}}_\xi) \geq \mathrm{crit}(F)$ iff $\mathrm{crit}(\bar{F}_\xi) \geq \mathrm{crit}(\sigma(F))$. So, recalling how the one-step embedding normalization works, if $\mathrm{crit}(E^{\bar{\mathcal{T}}}_\xi) \geq \mathrm{crit}(F)$, then $u(\eta) = \mathcal{U}\text{-pred}(u(\xi) + 1) = \mathbb{W}^*\text{-pred}(u(\xi) + 1)$ and if $\mathrm{crit}(E^{\bar{\mathcal{T}}}_\xi) < \mathrm{crit}(F)$, then $\eta = \mathcal{U}\text{-pred}(u(\xi) + 1) = \mathbb{W}^*\text{-pred}(u(\xi) + 1)$.

Now let $\lambda > \beta$ be a limit ordinal and suppose the tree-orders agree below $u(\lambda) = v(\lambda) = \sup\{u(\eta) \mid \eta < \lambda\}$. Then $[0, v(\lambda))_{\mathcal{U}}$ and $[0, v(\lambda))_{\mathbb{W}^*}$ are both the common downwards closure of $v``[0, \lambda)_{\bar{\mathcal{T}}}$, using here that $[0, \lambda)_{\bar{\mathcal{T}}} = [0, \lambda)_{\mathbb{W}(\mathcal{S}, \bar{\mathcal{T}})}$. $\square$

Now we'll show the following by induction on ξ.

(i) $\mathcal{W}^*_{u(\xi)+1} = \mathcal{W}_{u(\xi)+1}$,

(ii) for all η such that $\eta \leq_{\mathcal{U}} u(\xi)$, $\Psi_{\eta, u(\xi)} = \Psi^*_{\eta, u(\xi)}$, and

(iii) $t^{\Delta_\xi}_\infty \circ \bar{\sigma}_\xi = \pi_{u(\xi)} \circ t^\Phi_\xi$

Note that (ii) makes sense by our previous claim and (i) and (ii) suffice to get $\mathbb{W}^* \restriction u(\xi) + 1 = \mathbb{W}(\mathcal{S}, \mathcal{U}) \restriction u(\xi) + 1$. (iii) will be important for seeing that, if they're defined, the next meta-tree exit extenders are the same, i.e. $G^*_{u(\xi)} = G_{u(\xi)}$.

Base case. $\xi = \beta$.

Recall that $u(\beta) = \alpha+1$ and $\mathcal{U}\restriction\alpha+1 = \mathcal{T}\restriction\alpha+1$, so $\mathbb{W}(\mathcal{S},\mathcal{U}\restriction\alpha+1) = \mathbb{W}(\mathcal{S},\mathcal{T}\restriction\alpha+1)$. In particular, for $\eta \leq_{\mathcal{T}} \xi \leq \alpha$, $\mathcal{W}_\xi = \mathcal{T}_\xi$, $\Psi_{\eta,\xi} = \Phi_{\eta,\xi}$, and $\pi_\xi = \sigma_\xi$. In particular, $\pi_\alpha(F) = \sigma_\alpha(F) = \sigma(F)$. Recall also that $\bar{\mathcal{T}}\restriction\beta+1 = \mathcal{T}\restriction\beta+1$, so also $\mathcal{W}_\xi = \bar{\mathcal{T}}_\xi$ and $\Psi_{\eta,\xi} = \bar{\Phi}_{\eta,\xi}$ for $\eta \leq_{\bar{\mathcal{T}}} \xi \leq \beta$. We also have $\beta = \mathcal{U}$-pred$(\alpha+1)$ and $F = E^{\mathcal{U}}_\alpha$, so we get

$$\begin{aligned}\mathcal{W}_{\alpha+1} &= W(\mathcal{W}_\beta, \mathcal{W}_\alpha, \pi_\alpha(E^{\mathcal{U}}_\alpha))\\ &= W(\bar{\mathcal{T}}_\beta, \mathcal{T}_\alpha, \sigma(F))\\ &= \mathcal{W}^*_{\alpha+1}.\end{aligned}$$

This observation also gives us that $\Delta_\beta = \Psi^*_{\beta,\alpha+1} = \Psi_{\beta,\alpha+1}$, since all of these tree embeddings are just $\Phi^{\bar{\mathcal{T}}_\beta,\mathcal{T}_\alpha,\sigma(F)}$. Since we have already established $\mathbb{W}^*\restriction\alpha+1 = \mathbb{W}(\mathcal{S},\mathcal{U})\restriction\alpha+1$, this is the only new instance of (ii) we need to verify.

For (iii), we have that $t^\Phi_\beta = \hat{\imath}^{\mathcal{U}}_{\beta,\alpha+1}$, which is just the F ultrapower embedding on (some initial segment of) $M^{\bar{\mathcal{T}}}_\beta = M^{\mathcal{U}}_\beta$. We need to recall how $\pi_{\alpha+1}$ is defined. First, since $\Delta_\beta = \Psi_{\beta,\alpha+1}$, $t^{\Delta_\beta}_\infty$, factors as $\psi \circ i^{M^{\mathcal{W}_\beta}_\infty}_{\pi_\alpha(F)}$[19] for $\psi = \sigma^{\mathcal{W}_\beta,\mathcal{W}_\alpha,\pi_\alpha(F)} = \sigma^{\bar{\mathcal{T}}_\beta,\bar{\mathcal{T}}_\alpha,\sigma(F)}$ and, letting φ be the copy map associated to $(\sigma_\alpha, \bar{\sigma}_\beta, F)$, we have that $\pi_{\alpha+1} = \psi\circ\varphi$. Since φ was the relevant copy map,

$$i^{M^{\mathcal{W}_\beta}_\infty}_{\pi_\alpha(F)} \circ \bar{\sigma}_\beta = \varphi \circ t^\Phi_\beta.$$

Applying ψ to both sides, we get

$$\psi \circ i^{M^{\mathcal{W}_\beta}_\infty}_{\pi_\alpha(F)} \circ \bar{\sigma}_\beta = \psi\circ\varphi\circ t^\Phi_\beta.$$

Since $t^{\Delta_\beta}_\infty = \psi \circ i^{M^{\mathcal{W}_\beta}_\infty}_{\pi_\alpha(F)}$ and $\pi_{\alpha+1} = \varphi\circ\psi$, we get

$$t^{\Delta_\beta}_\infty \circ \bar{\sigma}_\beta = \pi_{\alpha+1} \circ t^\Phi_\beta,$$

which is the relevant instance of (iii).

Here's the commutative diagram illustrating the situation just discussed.

$$\begin{tikzcd}
M^{\bar{\mathcal{T}}}_\beta \arrow[r, "\bar{\sigma}_\beta"] \arrow[d, "F"'] & M^{\bar{\mathcal{T}}_\beta}_\infty \arrow[d, "\pi_\alpha(F)"] & \\
M^{\mathcal{U}}_{\alpha+1} \arrow[r, "\varphi"'] \arrow[rr, bend right, "\pi_{\alpha+1}"'] & Ult(M^{\bar{\mathcal{T}}_\beta}_\infty, \pi_\alpha(F)) \arrow[r, "\psi"'] & M^{\mathcal{W}_{\alpha+1}}_\infty
\end{tikzcd}$$

We now turn to the successor case.

[19]Here and elsewhere we are ignoring dropping, but this just requires working with initial segments of the displayed models. Since this only occurs in the dropping case of the corresponding embedding normalization, some of the displayed factors also end up being trivial. We leave the details to the reader.

Successor case. $\xi + 1 > \beta$.

Suppose $\xi + 1 < \mathrm{lh}(\bar{\mathcal{T}})$, otherwise we're done. By our induction hypothesis (iii) at ξ, $t_\infty^{\Delta_\xi} \circ \bar{\sigma}_\xi = \pi_{u(\xi)} \circ t_\xi^\Phi$. We have $G_{u(\xi)} = \pi_{u(\xi)} \circ t_\xi^\Phi(E_\xi^{\bar{\mathcal{T}}})$ and, since $\vec{\Delta}$ is a meta-tree embedding with U-map u, $G^*_{u(\xi)} = t_\infty^{\Delta_\xi} \circ \bar{\sigma}_\xi(E_\xi^{\bar{\mathcal{T}}})$. So $G_{u(\xi)} = G^*_{u(\xi)}$. Also note that $u(\xi+1) = u(\xi)+1$, as $\xi \geq \beta$. Let $\eta^* = \mathbb{U}\text{-pred}(u(\xi)+1) = \mathbb{W}^*\text{-pred}(u(\xi)+1)$, using our previous claim to get that these predecessors are the same. We have that $\mathcal{W}_{\eta^*} = \mathcal{W}^*_{\eta^*}$ and $\mathcal{W}_{u(\xi)} = \mathcal{W}^*_{u(\xi)}$ by induction (or outright when these ordinals are below α), so that

$$\begin{aligned} \mathcal{W}_{u(\xi)+1} &= W(\mathcal{W}_{\eta^*}, \mathcal{W}_{u(\xi)}, G_{u(\xi)}) \\ &= W(\mathcal{W}^*_{\eta^*}, \mathcal{W}^*_{u(\xi)}, G^*_{u(\xi)}) \\ &= \mathcal{W}^*_{u(\xi)+1} \end{aligned}$$

and

$$\begin{aligned} \Psi_{\eta^*,u(\xi)+1} &= \Phi^{\mathcal{W}_{\eta^*}, \mathcal{W}_{u(\xi)}, G_{u(\xi)}} \\ &= \Phi^{\mathcal{W}^*_{\eta^*}, \mathcal{W}^*_{u(\xi)}, G^*_{u(\xi)}} \\ &= \Psi^*_{\eta^*,u(\xi)+1}. \end{aligned}$$

This gives (i) and the new instance of (ii). For (iii), let $\eta = \bar{\mathcal{T}}\text{-pred}(\xi+1)$. We'll consider subcases based on whether $\mathrm{crit}(E_\xi^{\mathcal{T}}) \geq \mathrm{crit}(F)$.

Subcase (a). $\mathrm{crit}(E_\xi^{\mathcal{T}}) \geq \mathrm{crit}(F)$.

In this case we have that $\eta^* = u(\eta)$. We need to recall how we get the maps $\bar{\sigma}_{\xi+1}$ and $\pi_{u(\xi)+1}$. Let $\psi_{\xi+1} = \sigma^{\bar{\mathcal{T}}_\eta, \bar{\mathcal{T}}_\xi, \bar{F}_\xi}$ so that the last t-map of $\bar{\Phi}_{\eta,\xi+1}$ factors as $\psi_{\xi+1}$ composed with an ultrapower of $\bar{F}_\xi = \bar{\sigma}_\xi(E_\xi^{\bar{\mathcal{T}}})$. Let $\varphi_{\xi+1}$ be the copy map associated to $(\bar{\sigma}_\xi, \bar{\sigma}_\eta,\ E_\gamma^{\bar{\mathcal{T}}})$. We have that

$$\bar{\sigma}_{\xi+1} = \psi_{\xi+1} \circ \varphi_{\xi+1}.$$

Similarly, let $\psi^*_{\gamma+1} = \sigma^{\mathcal{W}_{u(\eta)}, \mathcal{W}_{u(\xi)}, G^*_{u(\xi)}}$ so that the last t-map of $\Psi_{u(\eta),u(\xi)+1}$ factors as $\psi^*_{u(\xi)+1}$ composed with an ultrapower of $G^*_{u(\xi)}$. Let $\varphi^*_{u(\xi)+1}$ be the copy map associated to $(\pi_{u(\xi)}, \pi_{u(\eta)}, E^{\mathcal{U}}_{u(\xi)})$. We have that $\pi_{u(\xi)+1} = \psi^*_{u(\xi)+1} \circ \varphi^*_{u(\xi)+1}$. Also let $\tau_{\xi+1}$ be the copy map associated to $(t_\infty^{\Delta_\xi}, t_\infty^{\Delta_\eta}, \bar{F}_\xi)$. We have the following diagram.

$$
\begin{tikzcd}
M^{\bar{\mathcal{T}}}_{\xi+1} \arrow[rrr, "\varphi_{\xi+1}"] \arrow[rrrr, bend left, "\bar{\sigma}_{\xi+1}"] \arrow[ddd, "t^{\Phi}_{\xi+1}"'] & & & \mathrm{Ult}(M^{\bar{\mathcal{T}}_\eta}_\infty, \bar{F}_\xi) \arrow[r, "\psi_{\xi+1}"] \arrow[ddd, "\tau_{\xi+1}"] & M^{\bar{\mathcal{T}}_{\xi+1}}_\infty \arrow[ddd, "t^{\Delta_{\xi+1}}_\infty"] \\
& M^{\bar{\mathcal{T}}}_\eta \arrow[ul, "E^{\bar{\mathcal{T}}}_\xi"'] \arrow[r, "\bar{\sigma}_\eta"] \arrow[d, "t^{\Phi}_\eta"'] & M^{\bar{\mathcal{T}}_\eta}_\infty \arrow[ur, "\bar{F}_\xi"'] \arrow[d, "t^{\Delta_\eta}_\infty"] & & \\
& M^{\mathcal{U}}_{u(\eta)} \arrow[r, "\pi_{u(\eta)}"'] \arrow[dl, "E^{\mathcal{U}}_\xi"] & M^{\mathcal{W}_{u(\eta)}}_\infty \arrow[dr, "G^*_{u(\xi)}"] & & \\
M^{\mathcal{U}}_{u(\xi)+1} \arrow[rrr, "\varphi^*_{u(\xi)+1}"'] \arrow[rrrr, bend right, "\pi_{\xi+1}"'] & & & \mathrm{Ult}(M^{\mathcal{W}_{u(\eta)}}_\infty, G^*_{u(\xi)}) \arrow[r, "\psi^*_{u(\xi)+1}"'] & M^{\mathcal{W}_{u(\xi)+1}}_\infty
\end{tikzcd}
$$

We just need to see that the outer most square commutes. Using induction hypothesis (iii) at η and ξ, we get that $\tau_{\xi+1} \circ \varphi_{\xi+1} = \varphi^*_{\xi+1} \circ t^{\Phi}_{\xi+1}$, since the left-hand side and the right-hand side are both the copy map to $(t^{\Delta_\xi}_\infty \circ \bar{\sigma}_\xi, t^{\Delta_\eta}_\infty \circ \bar{\sigma}_\eta, E^{\bar{\mathcal{T}}}_\xi) = (\pi_{u(\xi)} \circ t^{\Phi}_\xi, \pi_{u(\eta)} \circ t^{\Phi}_\eta, E^{\bar{\mathcal{T}}}_\xi)$, by Lemma 3.1.8. Since $\Delta_{\xi+1}$ is the copy tree embedding associated to $(\Delta_\xi, \Delta_\eta, \bar{F}_\xi, G^*_{u(\xi)})$, we have that the right square commutes, i.e. $t^{\Delta_{\xi+1}}_\infty \circ \psi_{\xi+1} = \psi^*_{\xi+1} \circ \tau_{\xi+1}$ (this is just conclusion (a) of the Shift Lemma). Combining these facts gives that the whole outer square commutes, i.e. $t^{\Delta_{\xi+1}}_\infty \circ \bar{\sigma}_{\xi+1} = \pi_{\xi+1} \circ t^{\Phi}_{\xi+1}$, which is (iii) at $\xi+1$. This finishes Subcase (a).

Subcase (b). $\mathrm{crit}(E^{\mathcal{U}}_\xi) < \mathrm{crit}(F)$.

In this case $\eta^* = \eta \leq \beta$. Checking (iii) is similar to Subcase (a), except we must replace t^{Φ}_η and $t^{\Delta_\eta}_\infty$ with the identity on $M^{\bar{\mathcal{T}}}_\eta$ and $M^{\mathcal{W}_\eta}_\infty$ (using that the last t-map of Γ_η is the identity, since $\Gamma_\eta = Id_{\bar{\mathcal{T}}_\eta}$), respectively, and $\pi_{u(\eta)}$ with $\pi_\eta = \bar{\sigma}_\eta$. This new inner square commutes trivially and the rest of the argument is the same. We leave the details to the reader.

This finishes Subcase (b) and the successor case. All that remains is the limit case.

Limit case. $\xi > \beta$ is a limit ordinal.

Since $\xi > \beta$, we have that $u(\xi) = v(\xi) = \sup\{u(\xi) \mid \xi < \gamma\}$. Since $\mathcal{U}$ and $\mathbb{W}^*$ have the same tree-order, (i) and (ii) are immediate, as we new trees and tree embeddings come from the same direct limit. For (iii), we have each of the maps $\bar{\sigma}_\gamma$, $t_\infty^{\Delta_\xi}$, t_ξ^Φ, and π_ξ is the unique map such that for all $\eta \in (\beta, \xi)_{\mathcal{T}}$,[20] the relevant trapezoid in the following diagram commute.

$$\begin{tikzcd}
M_\xi^{\bar{\mathcal{T}}} \arrow[rrr, "\bar{\sigma}_\xi"] \arrow[ddd, "t_\xi^\Phi"'] & & & M_\infty^{\bar{\mathcal{T}}_\xi} \arrow[ddd, "t_\infty^{\Delta_\xi}"] \\
& M_\eta^{\bar{\mathcal{T}}} \arrow[ul, "i^{\bar{\mathcal{T}}}_{\eta,\xi}"'] \arrow[r, "\bar{\sigma}_\eta"] \arrow[d, "t_\eta^\Phi"'] & M_\infty^{\bar{\mathcal{T}}_\eta} \arrow[ur, "t_\infty^{\bar{\Phi}_{\eta,\xi}}"] \arrow[d, "t_\infty^{\Delta_\eta}"] & \\
& M^{\mathcal{U}}_{u(\eta)} \arrow[r, "\pi_{u(\eta)}"'] \arrow[dl, "i^{\mathcal{U}}_{u(\eta),u(\xi)}"'] & M_\infty^{\mathcal{W}_{u(\eta)}} \arrow[dr, "t_\infty^{\Psi_{u(\eta),u(\xi)}}"] & \\
M^{\mathcal{U}}_{u(\xi)} \arrow[rrr, "\pi_{u(\xi)}"'] & & & M_\infty^{\mathcal{W}_{u(\xi)}}
\end{tikzcd}$$

For example, since $M_\xi^{\bar{\mathcal{T}}}$ is the direct limit of the $M_\eta^{\bar{\mathcal{T}}}$ under the $i^{\bar{\mathcal{T}}}_{\eta,\zeta}$ and $M_\infty^{\bar{\mathcal{T}}_\xi}$ is the direct limit of the $M_\infty^{\bar{\mathcal{T}}_\eta}$ under the $t_\infty^{\bar{\Phi}_{\eta,\zeta}}$, and $\bar{\sigma}_\zeta \circ i^{\bar{\mathcal{T}}}_{\eta,\zeta} = t_\infty^{\bar{\Phi}_{\eta,\zeta}} \circ \bar{\sigma}_\eta$, for $\eta \leq_{\bar{\mathcal{T}}} \zeta <_{\bar{\mathcal{T}}} \xi$, $\bar{\sigma}_\xi : M_\xi^{\bar{\mathcal{T}}} \to M_\infty^{\bar{\mathcal{T}}_\xi}$ is the unique map such that $\bar{\sigma}_\xi \circ i^{\bar{\mathcal{T}}}_{\eta,\xi} = t_\infty^{\bar{\Phi}_{\eta,\xi}} \circ \bar{\sigma}_\eta$ for all $\eta <_{\bar{\mathcal{T}}} \xi$.

For any $\eta <_{\bar{\mathcal{T}}} \xi$, we have each of the trapezoids commute and the inner squares commute, by our induction hypothesis (iii), so that the outer square commutes on points in the range of $\hat{i}^{\bar{\mathcal{T}}}_{\eta,\xi}$. Since every point in $M_\xi^{\bar{\mathcal{T}}}$ is in the range of $i^{\bar{\mathcal{T}}}_{\eta,\xi}$ for some $\eta <_{\bar{\mathcal{T}}} \xi$, the whole outer square commutes, i.e. $t_\infty^{\Delta_\xi} \circ \bar{\sigma}_\xi = \pi_{u(\xi)} \circ t_\xi^\Phi$. So (iii) holds at ξ. This finishes the limit case and the induction.

The "moreover" clause is immediate because $u^{\vec{\Delta}} = u^\Phi$. □

We can now prove our associativity result.

Theorem 3.2.16. *Let $\langle \mathcal{S}, \mathcal{T}, \mathcal{U} \rangle$ be a stack of normal trees on M. Suppose $\mathbb{W}(\mathcal{S}, \mathcal{T})$ is well-founded and and let $\sigma = \sigma^{\mathcal{S},\mathcal{T}}$. Let μ greatest such that $\mathbb{W}(\mathcal{W}(\mathcal{S},\mathcal{T}), \sigma\mathcal{U} \restriction \mu + 1)$ and $\mathbb{W}(\mathbb{W}(\mathcal{S},\mathcal{T}), \mathbb{W}(\mathcal{W}(\mathcal{S},\mathcal{T}), \sigma\mathcal{U} \restriction \mu + 1))$ are well-founded. Let μ^* greatest such that $\mathbb{W}(\mathcal{T}, \mathcal{U} \restriction \mu^* + 1)$ and $\mathbb{W}(\mathcal{S}, W(\mathcal{T}, \mathcal{U} \restriction \mu^* + 1))$ are well-founded. Then $\mu = \mu^*$ and*

$$\mathbb{W}(\mathbb{W}(\mathcal{S},\mathcal{T}), \mathbb{W}(\mathcal{W}(\mathcal{S},\mathcal{T}), \sigma\mathcal{U} \restriction \mu + 1)) = \mathbb{W}(\mathcal{S}, W(\mathcal{T}, \mathcal{U} \restriction \mu + 1)).$$

[20] We're assuming for convenience that there is no dropping above β. In general, we just need to look at the tail of this branch above the last drop.

Proof. The proof is by induction on the length of $\mathcal{U}$. We will assume $\mu+1 = \mu^*+1 = \mathrm{lh}(\mathcal{U})$. Inspecting the proof, it is easy to see that we are below μ on one side iff we are below μ^* on the other, as usual.

Let $\mathbb{W}(\mathcal{T},\mathcal{U}) = \langle \mathcal{U}_\xi, F_\xi, \Phi_{\eta,\xi}\rangle$ and $\tau_\xi : M^{\mathcal{U}}_\xi \to M^{\mathcal{U}_\xi}_\infty$ the associated embedding normalization maps. Let $\sigma_\xi : M^{\mathcal{U}}_\xi \to M^{\sigma\mathcal{U}}_\xi$ be the copy maps, so $\sigma_0 = \sigma$. Let $\pi^\xi_\eta : M^{\mathcal{U}_\xi}_\eta \to M^{W(\mathcal{S},\mathcal{U}_\xi \restriction \eta+1)}_\infty$ be the embedding normalization maps of $\mathbb{W}(\mathcal{S},\mathcal{U}_\xi)$. Let $\mathbb{W}(W(\mathcal{S},\mathcal{T}),\sigma\mathcal{U}) = \langle \mathcal{W}_\xi, G_\xi, \Psi_{\eta,\xi}\rangle$ and $\varphi_\xi : M^{\sigma\mathcal{U}}_\xi \to M^{\mathcal{W}_\xi}_\infty$ the embedding normalization map. Finally, let $\mathbb{W}^\xi = \mathbb{W}(\mathbb{W}(\mathcal{S},\mathcal{T}),\mathbb{W}(W(\mathcal{S},\mathcal{T}),\sigma\mathcal{U}) \restriction \xi+1)$ and, for $\eta \leq_\mathcal{U} \xi$, $\vec{\Delta}^{\eta,\xi} : \mathbb{W}^\eta \to \mathbb{W}^\xi$ the meta-tree embedding normalization meta-tree embeddings (using here that $\sigma\mathcal{U}$ has the same tree-order as $\mathcal{U}$). Since this is the meta-tree embedding normalization of the stack $\langle \mathbb{W}(\mathcal{S},\mathcal{T}),\mathbb{W}(W(\mathcal{S},\mathcal{T}),\sigma\mathcal{U})\restriction \xi+1\rangle$, we have that $\mathcal{W}_\xi$ is the last tree of $\mathbb{W}^\xi$.

We show by induction on $\xi < \mathrm{lh}(\mathcal{U})$ that $\xi \leq \mu$ iff $\xi \leq \mu^*$ and

(i) $\mathbb{W}^\xi = \mathbb{W}(\mathcal{S},\mathcal{U}_\xi)$

(ii) $\varphi_\xi \circ \sigma_\xi = \pi^\xi_\infty \circ \tau_\xi$,

(iii) for $\eta \leq_\mathcal{U} \xi$, $\vec{\Delta}^{\eta,\xi}$ is the lift of $\Phi_{\eta,\xi}$.

The base case is trivial since the two meta-trees are just $\mathbb{W}(\mathcal{S},\mathcal{T})$ and $\varphi_0 = \tau = id$ and $\sigma_0 = \pi^0_\infty = \sigma$, so (i) and (ii) hold. (iii) is trivial.

Lemma 3.2.15 will take us through the successor step. Suppose we have (i) and (ii) hold for all $\zeta \leq \xi$. Let $\eta = \mathcal{U}\text{-pred}(\xi+1)$. Then we also have that $\eta = \sigma\mathcal{U}\text{-pred}(\xi+1)$, since these trees have the same tree-order. So, since $\mathbb{W}(W(\mathcal{S},\mathcal{T}),\sigma\mathcal{U})$ has the same tree-order as $\sigma\mathcal{U}$, we have that

$$\mathbb{W}^{\xi+1} = \mathbb{W}(\mathbb{W}^\eta, \mathbb{W}^\xi, G_\xi).$$

By our induction hypothesis (i) at η and ξ, we get

$$\mathbb{W}^{\xi+1} = \mathbb{W}(\mathbb{W}(\mathcal{S},\mathcal{U}_\eta), \mathbb{W}(\mathcal{S},\mathcal{U}_\xi), G_\xi).$$

We have that $G_\xi = \varphi_\xi \circ \sigma_\xi(E^{\mathbb{U}}_\xi)$ so that induction hypothesis (ii) at ξ gives

$$G_\xi = \pi^\xi_\infty \circ \tau_\xi(E^{\mathcal{U}}_\xi) = \pi^\xi_\infty(F_\xi).$$

Applying Lemma 3.2.15 (with $\bar{\mathcal{T}} = \mathcal{U}_\eta$, $\mathcal{T} = \mathcal{U}_\xi$, $F = F_\xi$, and $\sigma = \pi^\xi_\infty$) gives

$$\mathbb{W}^{\xi+1} = \mathbb{W}(\mathcal{S},\mathcal{U}_{\xi+1}),$$

using here that $\mathcal{U}_{\xi+1} = W(\mathcal{U}_\eta,\mathcal{U}_\xi,F_\xi)$. This gives (i) at $\xi+1$. For (ii), let τ^* be the copy map associated to $(\tau_\xi,\tau_\eta,E^{\mathcal{U}}_\xi)$, φ^* the copy map associated to $(\varphi_\xi,\varphi_\eta,E^{\sigma\mathcal{U}}_\xi)$, and π^* the copy map associated to $(\pi^\xi_\infty,\pi^\eta_\infty,F_\xi)$. We have the following diagram.

$$
\begin{tikzcd}
M^{\mathcal{U}}_{\xi+1} \arrow[rrr, "\tau^*"] \arrow[rrrr, bend left, "\tau_{\xi+1}"] \arrow[ddd, "\sigma_{\xi+1}"'] & & & \mathrm{Ult}(M^{\mathcal{U}_\eta}_\infty, F_\xi) \arrow[r, "\sigma^{\mathcal{U}_\eta,\mathcal{U}_\xi,F_\xi}"] \arrow[ddd, "\pi^*"] & M^{\mathcal{U}_{\xi+1}}_\infty \arrow[ddd, "\pi^{\xi+1}_\infty"] \\
& M^{\mathcal{U}}_\eta \arrow[ul, "E^{\mathcal{U}}_\xi"] \arrow[r, "\tau_\eta"] \arrow[d, "\sigma_\eta"'] & M^{\mathcal{U}_\eta}_\infty \arrow[ur, "F_\xi"'] \arrow[d, "\pi^\eta_\infty"] & & \\
& M^{\sigma\mathcal{U}}_\eta \arrow[r, "\varphi_\eta"'] \arrow[dl, "E^{\sigma\mathcal{U}}_\xi"'] & M^{\mathcal{W}_\eta}_\infty \arrow[dr, "G_\xi"] & & \\
M^{\sigma\mathcal{U}}_{\xi+1} \arrow[rrr, "\varphi^*"'] \arrow[rrrr, bend right, "\tau_{\xi+1}"'] & & & \mathrm{Ult}(M^{\mathcal{W}_\eta}_\infty, G_\xi) \arrow[r, "\sigma^{\mathcal{W}_\eta,\mathcal{W}_\xi,G_\xi}"'] & M^{\mathcal{W}_{\xi+1}}_\infty
\end{tikzcd}
$$

We just need to see that the outermost square commutes. Our induction hypothesis (ii) at η and ξ gives that $\varphi^* \circ \sigma_{\xi+1} = \pi^* \circ \tau^*$, since the left-hand side and right-hand side maps are both the copy map associated to the same objects (using Lemma 3.1.8). So it is enough to see that $\sigma^{\mathcal{W}_\eta,\mathcal{W}_\xi,G_\xi} \circ \pi^* = \pi^{\xi+1}_\infty \circ \sigma^{\mathcal{U}_\eta,\mathcal{U}_\xi,F_\xi}$. Now, $t^{\Psi_{\eta,\xi+1}}_\infty = \sigma^{\mathcal{W}_\eta,\mathcal{W}_\xi,G_\xi} \circ i^{M^{\mathcal{W}_\eta}_\infty}_{G_\xi}$, $t^{\Phi_{\eta,\xi+1}}_\infty = \sigma^{\mathcal{U}_\eta,\mathcal{U}_\xi,F_\xi} \circ i^{M^{\mathcal{U}_\eta}_\infty}_{F_\xi}$, and the proof of Lemma 3.2.15 gives $t^{\Psi_{\eta,\xi+1}}_\infty \circ \pi^\eta_\infty = \pi^{\xi+1}_\infty \circ t^{\Phi_{\eta,\xi+1}}_\infty$ (the reader can check that this is an instance of clause (iii) of the inductive hypothesis in that proof). It follows that $\sigma^{\mathcal{W}_\eta,\mathcal{W}_\xi,G_\xi} \circ \pi^*$ and $\pi^{\xi+1}_\infty \circ \sigma^{\mathcal{U}_\eta,\mathcal{U}_\xi,F_\xi}$ agree on $\mathrm{ran}(i^{M^{\mathcal{U}_\eta}_\infty}_{F_\xi})$. So it suffices to check that these maps agree on the sup of the generators of F_ξ.

We have $\pi^* \restriction \mathrm{lh}(F_\xi)+1 = \pi^\xi_\infty \restriction \mathrm{lh}(F_\xi)+1$, because it is a copy map, and $\mathrm{crit}(\sigma^{\mathcal{W}_\eta,\mathcal{W}_\xi,G_\xi}) \geq \lambda(G_\xi) = \pi^\xi_\infty(\lambda(F_\xi))$, so that $\sigma^{\mathcal{W}_\eta,\mathcal{W}_\xi,G_\xi} \circ \pi^* \restriction \lambda(F_\xi) = \pi^\xi_\infty \restriction \lambda(F_\xi)$. We also have $\mathrm{crit}(\sigma^{\mathcal{U}_\eta,\mathcal{U}_\xi,F_\xi}) \geq \lambda(F_\xi)$, so we just need to see that $\pi^{\xi+1}_\infty \restriction \lambda(F_\xi) = \pi^\xi_\infty \restriction \lambda(F_\xi)$. Let $\alpha_\xi = \alpha_\xi^{\mathbb{W}(\mathcal{T},\mathcal{U})}$. We have that $\mathcal{U}_\xi \restriction \alpha_\xi + 1 = \mathcal{U}_{\xi+1} \restriction \alpha_\xi + 1$. It follows that $\pi^\xi_{\alpha_\xi} = \pi^{\xi+1}_{\alpha_\xi}$. Since either $\alpha_\xi + 1 = \mathrm{lh}(\mathcal{U}_\xi)$ or $\alpha_\xi + 1 < \mathrm{lh}(\mathcal{U}_\xi)$ and $\mathrm{lh}(F_\xi) < \mathrm{lh}(E^{\mathcal{U}_\xi}_{\alpha_\xi})$, the agreement properties of embedding normalization maps gives that $\pi^\xi_{\alpha_\xi} \restriction \mathrm{lh}(F_\xi)+1 = \pi^\xi_\infty \restriction \mathrm{lh}(F_\xi)+1$. Similarly, $\pi^{\xi+1}_{\alpha_\xi} \restriction \mathrm{lh}(F_\xi)+1 = \pi^{\xi+1}_\infty \restriction \mathrm{lh}(F_\xi)+1$, using here that $F_\xi = E^{\mathcal{U}_{\xi+1}}_{\alpha_\xi}$. So $\pi^{\xi+1}_\infty \restriction \mathrm{lh}(F_\xi)+1 = \pi^\xi_\infty \restriction \mathrm{lh}(F_\xi)+1$, which is more agreement than we needed. So the outer square commutes, i.e. $\varphi_{\xi+1}\circ\sigma_{\xi+1} = \pi^{\xi+1}_\infty \circ \tau_{\xi+1}$. This is (ii) at $\xi+1$.

For (iii), it suffices to show $\vec{\Delta}^{\eta,\xi+1}$ is the lift of $\Phi_{\eta,\xi+1}$, since the other instances of (iii) follow by induction. But this is immediate from Lemma 3.2.15 since $\vec{\Delta}^{\eta,\xi+1} = \vec{\Delta}^{\mathbb{W}(\mathcal{S},\mathcal{U}_\eta),\mathbb{W}(\mathcal{S},\mathcal{U}_\xi),G_\xi}$. This finishes the successor step.

Finally, suppose we're at limit $\gamma < \mathrm{lh}(\mathcal{U})$ and (i)-(iii) hold at all $\xi<\gamma$. Let $b = [0,\gamma)_\mathcal{U} = [0,\gamma)_{\sigma\mathcal{U}}$. Let $\mathcal{D} = \langle \{\mathcal{U}_\xi\}_{\xi\in b}, \{\Phi_{\eta,\xi}\}_{\eta\leq_\mathcal{U}\xi\in b}\rangle$[21] and $\mathcal{D}^*$ be the lift of $\mathcal{D}$ to a directed system of meta-trees. By our induction hypothesis (iii), we have that $\mathcal{D}^* = \langle\{\mathbb{W}^\xi\}_{\xi\in b}, \{\vec{\Delta}^{\eta,\xi}\}_{\eta\leq_\mathcal{U}\xi\in b}\rangle$. We have that $\mathcal{U}_\gamma = \lim \mathcal{D}$ and $\mathbb{W}^\gamma = \lim\mathcal{D}^*$, so Theorem 3.2.14 gives $\mathbb{W}^\gamma = \mathbb{W}(\mathcal{S},\mathcal{U}_\gamma)$, $\mathcal{U}_\gamma = \lim_b \langle \mathcal{U}_\xi, \Phi_{\eta,\xi}\rangle$ and $\mathbb{W}^\gamma = \lim_b\langle \mathbb{W}_\xi, \vec{\Delta}_{\eta,\xi}\rangle$. So (i) holds. Since the u-threads of $\mathcal{D}$ are the same as the U-threads of $\mathcal{D}^*$, we get that, for $\eta <_\mathcal{U} \gamma$, the u-map of the new direct limit tree embedding $\Phi_{\eta,\gamma}$ is the same as the U-map of the new meta-tree embedding $\vec{\Delta}^{\eta,\gamma}$. It follows that $\vec{\Delta}^{\eta,\gamma}$ is the lift of $\Phi_{\eta,\gamma}$, giving the new instances of (iii). It is straightforward to see that (ii) holds at γ because all of the relevant maps are direct limit maps, just like in the proof of Lemma 3.2.15 that condition (iii) (of that proof) held in the limit case. This finishes the induction.

□

Combining this result with Steel's result that embedding normalization commutes with copying, Theorem 3.1.34, will give our uniqueness result about embedding normalizations of a stack of normal trees.

Theorem 3.2.17. *Let $\vec{\mathcal{S}}$ be a finite stack of normal trees. Suppose that $\vec{\mathcal{S}}$ has an embedding normalization, $\mathcal{T}$. Then $\mathcal{T}$ is the unique embedding normalization of $\vec{\mathcal{S}}$ and every putative embedding normalization sequence for $\vec{\mathcal{S}}$ is an embedding normalization sequence for $\vec{\mathcal{S}}$.*

Proof. Before we start, let us just mention that the core of this proof is nothing more than the fact that we can omit parentheses when expressing terms generated out of an associative binary operation. We encourage the reader to convince themselves that this is all that is needed, modulo the previous theorem and Theorem 3.1.34.

Let $\vec{\mathcal{S}}$ be a stack of length $n>2$ (the theorem is trivial for $n\leq 2$). Let $\langle \vec{\mathcal{S}}^0, \vec{\mathcal{S}}^1,\ldots,\vec{\mathcal{S}}^{n-1}\rangle$ be an embedding normalization sequence for $\vec{\mathcal{S}}$ with last stack $\langle\mathcal{T}\rangle$. Let $\langle\vec{\mathcal{T}}^0,\ldots,\vec{\mathcal{T}}^m\rangle$ be the left-to-right putative embedding normalization sequence for $\vec{\mathcal{S}}$, that is, the unique sequence such that for all $i+1<m$, $\vec{\mathcal{T}}^{i+1} = \langle \mathcal{W}(\mathcal{T}^i_0,\mathcal{T}^i_1)\rangle^\frown \sigma_i(\vec{\mathcal{T}}\restriction(1,n-i))$, for σ_i the appropriate embedding normalization map. Recalling notation from earlier, we have that $\vec{\mathcal{T}}^i = \langle W(\vec{\mathcal{S}}\restriction i+1)\rangle^\frown\sigma_{\vec{\mathcal{S}}\restriction i+1}\big(\vec{\mathcal{S}}\restriction[i+1,n-1)\big)$. Our first goal is to show the following.

Claim 1. *$\langle\vec{\mathcal{T}}^0,\ldots,\vec{\mathcal{T}}^m\rangle$ is an embedding normalization sequence with final stack $\langle\mathcal{T}\rangle$.*

Proof. The idea is to iteratively use our associativity result, Theorem 3.2.16, to convert $\langle\vec{\mathcal{S}}^0,\vec{\mathcal{S}}^1,\ldots,\vec{\mathcal{S}}^{n-1}\rangle$ into $\langle\vec{\mathcal{T}}^0,\ldots,\vec{\mathcal{T}}^m\rangle$.

[21] We actually need to restrict to ξ above the last drop along b, but this causes no trouble, so we'll ignore it.

We define embedding normalization sequences $\langle\vec{\mathcal{S}}^{i,0},\vec{\mathcal{S}}^{i,1},\dots,\vec{\mathcal{S}}^{i,n-1}\rangle$ and indices (k_i,j_i) by recursion, maintaining that $\langle\mathcal{T}\rangle$ is the last stack of $\langle\vec{\mathcal{S}}^{i,0},\vec{\mathcal{S}}^{i,1},\dots,\vec{\mathcal{S}}^{i,n-1}\rangle$.

First, we let

$$\langle\vec{\mathcal{S}}^{0,0},\vec{\mathcal{S}}^{0,1},\dots,\vec{\mathcal{S}}^{0,n-1}\rangle=\langle\vec{\mathcal{S}}^{0},\vec{\mathcal{S}}^{1},\dots,\vec{\mathcal{S}}^{n-1}\rangle.$$

Given $\langle\vec{\mathcal{S}}^{i,0},\vec{\mathcal{S}}^{i,1},\dots,\vec{\mathcal{S}}^{i,n-1}\rangle$, we let j_i be the largest j such that

$$\vec{\mathcal{S}}^{i,j+1}=\vec{\mathcal{S}}^{i,j}\restriction k^\frown\langle W(\mathcal{S}^{i,j}_k,\mathcal{S}^{i,j}_{k+1})\rangle^\frown\sigma_j\left(\vec{\mathcal{S}}^{i,j}\restriction(k+1,n-j]\right),$$

for some $k\geq 1$, where σ_j the appropriate embedding normalization map, if such a j exists. In this case, we also let k_i be the witnessing k.

If no such j exists, then we stop. In this case we must have $\langle\vec{\mathcal{T}}^0,\dots,\vec{\mathcal{T}}^m\rangle$ is an initial segment of $\langle\vec{\mathcal{S}}^{i,0},\vec{\mathcal{S}}^{i,1},\dots,\vec{\mathcal{S}}^{i,n-1}\rangle$. It follows that have that they must be equal, since we cannot have $m<n-1$, as we do not reach an ill-founded model. Since $\langle\mathcal{T}\rangle$ is the last stack of $\langle\vec{\mathcal{S}}^{i,0},\vec{\mathcal{S}}^{i,1},\dots,\vec{\mathcal{S}}^{i,n-1}\rangle$, we're done.

Now suppose that j_i and k_i are defined. By our choice of j_i, for any $l<n-j_i$,

$$\vec{\mathcal{S}}^{i,j_i+l+1}=\langle W(\mathcal{S}^{i,j_i+l}_0,\mathcal{S}^{i,j_i+l}_1)\rangle^\frown\sigma_{j_i+l}\left(\vec{\mathcal{S}}^{i,j_i+l}\restriction(1,n-j_i-l)\right).$$

It follows that

$$\vec{\mathcal{S}}^{i,j_i+k_i}=\langle W(\vec{\mathcal{S}}^{i,j_i}\restriction k_i)\rangle^\frown\sigma_{\vec{\mathcal{S}}^{i,j_i}\restriction k_i}\left(\langle W(\mathcal{S}^{i,j_i}_{k_i},\mathcal{S}^{i,j_i}_{k_i+1})\rangle^\frown\sigma_{j_i}\left(\vec{\mathcal{S}}^{i,j_i}\restriction(k_i+1,n-j_i)\right)\right).$$ [22]

Let $\pi_0=\sigma_{\vec{\mathcal{S}}^{i,j_i}\restriction k_i}$ and π_1,π_2 the resulting copy maps such that

$$\pi_0\left(\vec{\mathcal{S}}^{i,j_i}\restriction[k_i,n-j_i)\right)=\langle\pi_0\mathcal{S}^{i,j}_{k_i},\pi_1\mathcal{S}^{i,j}_{k_i+1}\rangle^\frown\pi_2\left(\vec{\mathcal{S}}^{i,j_i}\restriction(k_i+1,n-j_i)\right).$$

Since normalizing commutes with copying, by Theorem 3.1.34, we get

$$\pi_0 W(\mathcal{S}^{i,j_i}_{k_i},\mathcal{S}^{i,j_i}_{k_i+1})=W(\pi_0\mathcal{S}^{i,j_i}_{k_i},\pi_1\mathcal{S}^{i,j_i}_{k_i+1}),$$

and actually, letting $\sigma=\sigma^{\pi_0\mathcal{S}^{i,j_i}_{k_i},\pi_1\mathcal{S}^{i,j_i}_{k_i+1}}$,

$$\pi_0\left(\langle W(\mathcal{S}^{i,j_i}_{k_i},\mathcal{S}^{i,j_i}_{k_i+1})\rangle^\frown\sigma_{j_i}\left(\vec{\mathcal{S}}^{i,j_i}\restriction(k_i{+}1,n{-}j_i)\right)\right)=\langle W(\pi_0\mathcal{S}^{i,j_i}_{k_i},\pi_1\mathcal{S}^{i,j_i}_{k_i+1})\rangle^\frown\sigma\circ\pi_2\left(\vec{\mathcal{S}}^{i,j_i}\restriction(k_i{+}1,n{-}j_i)\right).$$

So we get

$$\vec{\mathcal{S}}^{i,j_i+k_i+1}=\langle W(W(\vec{\mathcal{S}}^{i,j_i}\restriction k_i),W(\pi_0\mathcal{S}^{i,j_i}_{k_i},\pi_1\mathcal{S}^{i,j_i}_{k_i+1}))\rangle^\frown\sigma_{k_i}\circ\sigma\circ\pi_2\left(\vec{\mathcal{S}}^{i,j_i}\restriction(k_i+1,n-j_i)\right).$$

By Theorem 3.2.16,

$$\begin{aligned}W(W(\vec{\mathcal{S}}^{i,j_i}\restriction k_i),W(\pi_0\mathcal{S}^{i,j_i}_{k_i},\pi_1\mathcal{S}^{i,j_i}_{k_i+1}))&=W(W(W(\vec{\mathcal{S}}^{i,j_i}\restriction k_i),\pi_0\mathcal{S}^{i,j_i}_{k_i}),\tau\circ\pi_1\mathcal{S}^{i,j_i}_{k_i+1})\\&=W(\vec{\mathcal{S}}^{i,j_i}\restriction k_i+2),\end{aligned}$$

[22] Recall that for a stack $\vec{\mathcal{S}}$, $W(\vec{\mathcal{S}})$ is the (putative) normal tree obtained by iteratively embedding normalizing from left-to-right and $\sigma_{\vec{\mathcal{S}}}$ is the resulting embedding normalization map.

where $\tau = \sigma^{W(\vec{\mathcal{S}}^{i,j_i}|k_i),\pi_0\mathcal{S}^{i,j_i}_{k_i}}$. Theorem 3.2.16 also gives that $\sigma_{\vec{\mathcal{S}}^{i,j_i}|k_i+2} = \sigma_{k_i}\circ\sigma\circ\pi_2$. It follows that for any $l \geq k_i+1$,

$$\vec{\mathcal{S}}^{i,j_i+l} = \langle W(\mathcal{S}^{i,j_i}\restriction l+1)\rangle^\frown \sigma_{\mathcal{S}^{i,j_i}|l+1}\big(\vec{\mathcal{S}}^{i,j_i}\restriction(l+1,n-j_i)\big).$$

In particular, our induction hypothesis gives that $\mathcal{T} = W(\vec{\mathcal{S}}^{i,j_i})$ and so $W(\vec{\mathcal{S}}^{i,j_i}\restriction l)$ is well-founded for all $l \leq n-j_i$.

We now define $\langle \vec{\mathcal{S}}^{i+1,0},\dots,\vec{\mathcal{S}}^{i+1,n-1}\rangle$ as follows.

- for $p \leq j_i$, $\vec{\mathcal{S}}^{i+1,p} = \vec{\mathcal{S}}^{i,p}$,
- for $p > j_i$, $\vec{\mathcal{S}}^{i+1,p} = \langle W(\vec{\mathcal{S}^{i,j_i}}\restriction(p-j_i)+1)\rangle^\frown \sigma_{\vec{\mathcal{S}}^{i,j_i}|(p-j_i)+1}\big(\vec{\mathcal{S}}^{i,j_i}\restriction[(p-j_i)+1,n-j_i)\big)$.

By our above observations, this is an embedding normalization sequence with last stack $\langle\mathcal{T}\rangle$, as desired.

Now, it is easy to see that the j_i must be strictly decreasing, so that at some stage i, j_i is undefined and $\langle\vec{\mathcal{T}}^0,\dots,\vec{\mathcal{T}}^m\rangle = \langle\vec{\mathcal{S}}^{i,0},\vec{\mathcal{S}}^{i,1},\dots,\vec{\mathcal{S}}^{i,n-1}\rangle$. □

In particular, it follows that $n = m+1$.

To finish, we just need to show that any putative embedding normalization sequence for $\vec{\mathcal{S}}$ is actually an embedding normalization sequence and has last stack $\langle\mathcal{T}\rangle$. So suppose that $\langle\vec{\mathcal{U}}^0,\dots,\vec{\mathcal{U}}^p\rangle$ is a putative embedding normalization sequence.

Claim 2. *$\langle\vec{\mathcal{U}}^0,\dots,\vec{\mathcal{U}}^p\rangle$ is an embedding normalization sequence with final stack $\langle\mathcal{T}\rangle$.*

Proof. The proof is basically the same as that of the last claim. We define putative embedding normalization sequences $\langle\vec{\mathcal{S}}^{i,0},\dots,\vec{\mathcal{S}}^{i,p}\rangle$ by recursion, maintaining that $\vec{\mathcal{S}}^{i,p} = \vec{\mathcal{U}}^p$. We'll stop when we reach an i such that $\langle\vec{\mathcal{T}}^0,\dots,\vec{\mathcal{T}}^{n-1}\rangle = \langle\vec{\mathcal{S}}^{i,0},\vec{\mathcal{S}}^{i,1},\dots,\vec{\mathcal{S}}^{i,n-1}\rangle$. Since $\langle\mathcal{T}\rangle$ is the last stack of $\vec{\mathcal{T}}^0,\dots,\vec{\mathcal{T}}^{n-1}\rangle$, this gives the claim.

To start, we let $\langle\vec{\mathcal{S}}^{0,0},\dots,\vec{\mathcal{S}}^{0,p}\rangle = \langle\vec{\mathcal{U}}^0,\dots,\vec{\mathcal{U}}^p\rangle$. Given $\langle\vec{\mathcal{S}}^{i,0},\dots,\vec{\mathcal{S}}^{i,p}\rangle$, we define $\langle\vec{\mathcal{S}}^{i+1,0},\dots,\vec{\mathcal{S}}^{i+1,p}\rangle$ just as we did in the proof of Claim 1. Similarly to that proof, this must terminate at some i such $\langle\vec{\mathcal{T}}^0,\dots,\vec{\mathcal{T}}^{n-1}\rangle = \langle\vec{\mathcal{S}}^{i,0},\vec{\mathcal{S}}^{i,1},\dots,\vec{\mathcal{S}}^{i,n-1}\rangle$, as desired. □

□

We get the following theorem is an easy corollary.

Theorem 3.2.18. *Let M be a premouse, θ a regular cardinal, and Σ an (ω,θ)-strategy on M which has SHC⁻ and bottom-up normalizes well. Then all tails of Σ bottom-up normalize well.*

Proof. Fix $\vec{\mathcal{S}}$ a stack on M by Σ of length n and $\vec{\mathcal{T}}$ a stack on $M^{\vec{\mathcal{S}}}_\infty$ by $\Sigma_{\vec{\mathcal{S}},M^{\vec{\mathcal{S}}}_\infty}$ of length m. Since Σ bottom-up normalizes well, $W(\vec{\mathcal{S}}^\frown\vec{\mathcal{T}})$ is by Σ. In particular, $W(\vec{\mathcal{S}}^\frown\vec{\mathcal{T}})$ is an embedding normalization of $\vec{\mathcal{S}}^\frown\vec{\mathcal{T}}$ (as witnessed by the left-to-right embedding normalization sequence). Now consider the putative embedding normalization sequence $\langle\vec{\mathcal{S}}^0,\dots,\vec{\mathcal{S}}^p\rangle$ such that for $i \leq p$,

- if $i < m$, $\vec{\mathcal{S}}^i = \vec{\mathcal{S}}^\frown\langle W(\vec{\mathcal{T}}\upharpoonright i+1)\rangle^\frown\sigma_{\vec{\mathcal{T}}\upharpoonright i+1}\big(\vec{\mathcal{T}}\upharpoonright[i+1,m)\big)$,
- if $i \geq m$, $\vec{\mathcal{S}}^i = \langle W(\vec{\mathcal{S}}\upharpoonright(i+1-m))\rangle^\frown\sigma_{\vec{\mathcal{S}}\upharpoonright(i+1-m)}\big(\vec{\mathcal{S}}\upharpoonright[i+1-m,n)^\frown\langle W(\vec{\mathcal{T}})\rangle\big)$,

where we stop building the trees in the displayed stacks if we reach an ill-founded model. By Theorem 3.2.17, $\langle\vec{\mathcal{S}}^0,\ldots,\vec{\mathcal{S}}^p\rangle$ is actually an embedding normalization sequence, so we never actually reach an ill-founded model, $p+1=n+m$, and $\vec{\mathcal{S}}^p = \langle W(\vec{\mathcal{S}}^\frown\vec{\mathcal{T}})\rangle$. Since we never reach ill-founded models, this gives that $W(W(\vec{\mathcal{S}}), \sigma_{\vec{\mathcal{S}}}W(\vec{\mathcal{T}})) = W(\vec{\mathcal{S}}^\frown\vec{\mathcal{T}})$. It follows that $W(W(\vec{\mathcal{S}})), \sigma_{\vec{\mathcal{S}}}W(\vec{\mathcal{T}})$ is by Σ, so that $W(\vec{\mathcal{T}})$ is by $\Sigma_{\vec{\mathcal{S}},M^{\vec{\mathcal{S}}}_\infty}$, as desired. $\square$

In earlier versions of [24], one version of Steel's notion of *mouse pair* was a pair (P,Σ) such that Σ is an (ω,ω_1)-strategy for P and all tails of Σ bottom-up normalize well and have SHC$^-$, in our terminology. Theorems 3.2.6 and 3.2.18 reveal that this is equivalent to the ostensible weakening that Σ itself has SHC$^-$ and bottom-up normalizes well.

Steel revised the notion of mouse pair from earlier versions of [24] to the final version which appears in [25]. These modifications, including leaving ordinary Jensen-indexed premice for the modified pfs-premice, are used to secure the central comparison theorem for mouse pairs in [25]. The work in this chapter shows that one can get pretty far working with pairs of the form (P,Σ), for Σ is an (ω,ω_1)-strategy for P and all tails of Σ bottom-up normalize well and have SHC$^-$. We believe that one should be able to further develop the theory of pairs (P,Σ) of this form, including proving a comparison theorem for these pairs (i.e. a strategy comparison theorem). One route to such a theorem would be to rely on Steel's mouse pair comparison theorem from [24] and attempt to translate between mouse pairs in the above sense and mouse pairs in the sense of [24]—this involves translating between the ordinary Jensen-indexed premice and pfs-premice, as well as between the corresponding strategies. Another route would be to try to modify Steel's proof to obtain a direct comparison theorem for these mouse pairs. Steel has made progress on this front. Modulo a direct comparison theorem of this form, we believe we can recover many other nice properties of mouse pairs. For example, we think we see how to prove that Σ moves itself correctly, which suffices for Dodd-Jensen and, ultimately, very strong hull condensation. Very strong hull condensation suffices for full normalization and, consequently, positionality, but also can be used to show that the s-maps of (weak) tree embeddings are mouse pair elementary via direct combinatorial arguments, similar to those in this chapter.

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