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Authors
Urzay, Javier
Glover, Beverley J.
Smith, Stefan G. Llewellyn

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The elastohydrodynamic force on a sphere near a soft wall

Javier Urzay[a)] and Stefan G. Llewellyn Smith[b)]
Department of Mechanical and Aerospace Engineering, Jacobs School of Engineering, University of California San Diego, 9500 Gilman Drive, La Jolla, California 92093-0411, USA

Beverley J. Glover[c)]
Department of Plant Sciences, University of Cambridge, Downing Street, Cambridge CB2 3EA, United Kingdom

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The influence of soft boundaries on the forces experienced by a small sphere undergoing slow translation and rotation near a wall is investigated using asymptotic and numerical methods. The clearance between the sphere and the wall is assumed to be small, so that the lubrication approximation holds in the gap. The forces induced by boundary deformation break the symmetry of the Stokes equations, leading to irreversibility of the motion of the sphere and yielding a nonzero lift force. A general formulation, applicable to any constitutive equation of the wall, is presented, and an asymptotic analysis for slightly soft boundaries is developed and applied to a thin compressible elastic layer coating a rigid surface. Expressions are derived for the elastohydrodynamic lift exerted on a sphere moving parallel to the wall, which include the influence of the sphere rotation in the direction of its motion. The results extend and correct previous work, and are different from those for the elastohydrodynamic motion of a cylinder near a wall, in that the geometry eliminates the occurrence of maximum lift and optimum choice of the material properties of the boundary. © *2007 American Institute of Physics*. [DOI: 10.1063/1.2799148]

I. INTRODUCTION

The study of the influence of soft boundaries in fluid mechanics is motivated by a number of problems, including their importance in biological systems.[1] Examples of current interest involving elastohydrodynamic flows are the lubrication of red blood cells and endothelial cells in small capillaries, lubrication of the eye in the ocular cavity, and lubrication of synovial joints. Other recent investigations are concerned with the motion and propulsion of cell and bacteria.[2,3] Microorganisms that undergo low-Reynolds-number swimming motions usually rely on propulsion mechanisms that break the time reversibility of the Stokes equations, such as rotating flagellas or flexible oars.[4] The effect of a nearby nonslip rigid wall on the motion of these micro-swimmers has been found to be important.[5] The purpose of the present investigation is to study the dynamics of a solid spherical body in the presence of a soft wall, and to give a general derivation of the elastohydrodynamic lubrication equations that describe the flow.

The reversibility properties of the Stokes equations yield a zero lift force L on a sphere translating and rotating near a wall. This can be demonstrated by observing that the imaginary problem of a sphere near a wall translating at the reversed translational velocity $-U$ and at the reversed angular velocity $-\Omega$ turns out to be the same as the original one if the change of variables in velocity $\boldsymbol{v}^*=-\boldsymbol{v}$ and pressure $p^*=-p$ is performed in the equations of fluid motion. This implies that $L^*=-L$, from which it follows that the lift force must be zero by symmetry. Some nonlinear mechanisms that break the symmetry of the Stokes equations have been proposed and tested experimentally, such as fluid inertia,[6] electrokinetics,[7] and non-Newtonian effects.[8] As pointed out previously for cylindrical journal bearings,[9] elastic deformations of the boundaries also represent a source of irreversibility and nonlinearity of the flow.

The nonlinear effect of a soft boundary on linear problems treated previously in Refs. 10–12 for the case of rigid boundaries is studied in the present investigation. An additional elastohydrodynamic lift force, which does not appear with rigid boundaries, is found. Critical differences are found compared to the cylindrical case.

II. GENERAL FORMULATION

Consider a sphere of radius a translating at constant velocity U and rotating at constant angular velocity Ω perpendicular to the streamwise coordinate x, in a Newtonian fluid of constant density ρ and constant viscosity μ (see Fig. 1). The Reynolds numbers of translation $\mathrm{Re}_t=\rho U a/\mu \ll 1$ and rotation $\mathrm{Re}_r=\rho\Omega a^2/\mu \ll 1$ are assumed to be small enough that the inertial terms are negligible, so that the flow is described by the Stokes equations to leading order. The minimum gap width between the sphere and the wall is $\delta=\epsilon a$, where $\epsilon \ll 1$. The motion can be considered steady in the reference frame shown in Fig. 1 as long as the time scale of the sphere motion $t_s \sim \delta/U(1+\omega)$ is much longer than the viscous time scale $t_\nu \sim \delta^2/\nu$ and the wall response time scale $t_w \sim \delta/c$, where c is the speed of sound in the solid material. Here $\omega=\Omega a/U$ is a kinematic parameter of the sphere motion. That is, for $\epsilon \mathrm{Re}_t \ll 1$, $\epsilon \mathrm{Re}_r \ll 1$, and $U(1+\omega)/c \ll 1$, the

[a)]Electronic mail: jurzay@ucsd.edu
[b)]Electronic mail: sgls@ucsd.edu
[c)]Electronic mail: bjg26@cam.ac.uk

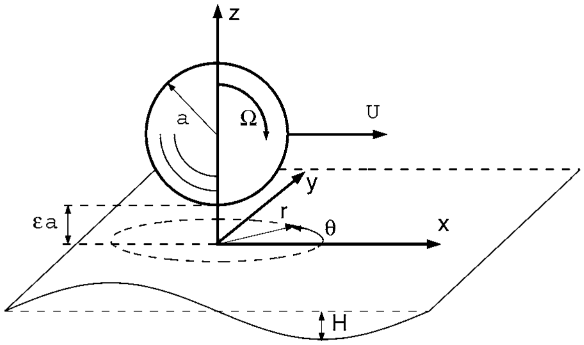

FIG. 1. Schematic diagram of the problem. A solid sphere undergoes slow translational and rotational motion through a liquid of viscosity μ and density ρ near a soft wall. The pressure in the gap produces a wall deformation, which breaks the symmetry and induces a lift force on the sphere. Small clearances $\epsilon \ll 1$ are assumed, but large enough such that slip effects need not be considered.

flow can be considered steady. A softness parameter is defined as $\eta = H_c/\delta$, where H_c is the characteristic wall deflection caused by the flow. The wall deflection is in general a function of the flow pressure at the wall, and the exact dependence is determined by the constitutive equations of the wall.

The value of the kinematic parameter ω is found to have a great relevance on the lift force trend, as will be shown later. When $\omega=-1$, rotation and translation drag fluid into the gap in opposite directions, such that the induced mass flows are equal in magnitude but of opposite signs. No overpressure is produced in this limit, and since the wall remains flat, a zero lift force results. When $\omega=1$, the relative velocity between the bottom point of the sphere and the wall surface is zero, its minimum value. Such a configuration is kinematically favorable for the lift force, since both translation and rotation motions locally produce a leading-order overpressure of the same sign, and in the thin elastic-layer limit they produce a leading-order surface deflection of the same sign too, breaking the local symmetry of the flow. These arguments are demonstrated in what follows.

The conservation equations are written in nondimensional variables, with a the unit of length, $U(1+\omega)$ the unit of velocity, and $\mu U(1+\omega)/a$ the unit of pressure. In these variables, a cylindrical coordinate system $\boldsymbol{x}=(r,\theta,z)$ is introduced, and the mass and momentum conservation equations for the velocity vector $\boldsymbol{v}=(v_r,v_\theta,v_z)$ and pressure p become

$$\nabla \cdot \boldsymbol{v} = 0 \tag{1}$$

and

$$\nabla p = \nabla^2 \boldsymbol{v}. \tag{2}$$

These equations are to be solved with no-slip conditions on the sphere and the wall: $\boldsymbol{v}=[1+\omega(z-1-\epsilon)]\boldsymbol{e}_x/(1+\omega) - \omega x \boldsymbol{e}_z/(1+\omega)$ on the sphere and $\boldsymbol{v}=0$ on the wall, where the $\boldsymbol{e}_i$ are unit vectors. The fluid is assumed to be at rest far from the sphere.

Following earlier work,[10–12] $\epsilon \ll 1$ is treated as a singular perturbation parameter of the problem. As $\epsilon \to 0$, the pressure becomes singular because of the vanishing thickness of the lubrication layer. The flow field is divided into an inner region inside the gap and an outer region, although it is found that the elastic deformations are negligibly small outside the gap where $r \gg \epsilon^{1/2}$. A suitable coordinate scaling in the inner region is then given by $R=r/\epsilon^{1/2}$ and $Z=z/\epsilon$. In addition, both radial and azimuthal velocities are expected to be $O(1)$, whence a dominant balance in Eq. (1) shows that the normal velocity v_z remains bounded as $\epsilon \to 0$, so that $v_z = O(\epsilon^{1/2})$. Equation (2) yields values for the pressure p of order $\epsilon^{-3/2}$. In the inner region, appropriate variables therefore are the inner velocities $V_r=v_r$, $V_\theta=v_\theta$, $V_z=v_z\epsilon^{-1/2}$, and the inner pressure $P=p\epsilon^{3/2}$. In terms of the inner coordinates, the location of the sphere surface $Z=h_0(R)$ is given by

$$h_0(R) = 1 + \frac{R^2}{2} + O(\epsilon), \tag{3}$$

which also represents the gap profile in the case of rigid boundaries.

When the conservation equations are written in inner variables and expanded in powers of ϵ, the problem is reduced to a hydrodynamic lubrication flow to leading order in ϵ,

$$\begin{gathered} \frac{1}{R}\frac{\partial}{\partial R}(RV_r) + \frac{1}{R}\frac{\partial V_\theta}{\partial \theta} + \frac{\partial V_z}{\partial Z} = 0, \\ \frac{\partial P}{\partial R} = \frac{\partial^2 V_r}{\partial Z^2}, \quad \frac{1}{R}\frac{\partial P}{\partial \theta} = \frac{\partial^2 V_\theta}{\partial Z^2}, \quad \frac{\partial P}{\partial Z} = 0. \end{gathered} \tag{4}$$

In an inertial reference frame translating with the sphere, the boundary conditions become, again to leading order in ϵ,

$$V_r = \frac{-\omega}{1+\omega}\cos\theta, \quad V_\theta = \frac{\omega}{1+\omega}\sin\theta, \quad V_z = \frac{-\omega R}{1+\omega}\cos\theta \tag{5}$$

on the sphere surface $Z=h_0$, and

$$V_r = \frac{-1}{1+\omega}\cos\theta, \quad V_\theta = \frac{1}{1+\omega}\sin\theta, \quad V_z = \frac{\eta}{1+\omega}\frac{\partial H}{\partial x} \tag{6}$$

on the wall surface $Z=-\eta H$, where H denotes the wall deflection nondimensionalized with H_c. A constitutive equation relating H to P completes the specification of the problem.

Although Eqs. (4) are linear, the coordinate of the wall surface $Z=-\eta H$ where the boundary conditions (6) are imposed, along with the constitutive equation $H=H(P)$, renders the problem nonlinear. If the wall were rigid and the sphere soft, the mathematical analysis would be similar; in this case, the forces on the sphere would be found to be the same as obtained below if the constitutive equation for the soft sphere were the same as that assumed below for the wall. In general, for a soft sphere and a soft wall of the same material properties, the force is the sum of the forces associated with each departure from rigidity.

Eliminating V_r, V_θ, and V_z from (4), we obtain the nonlinear Reynolds equation

$$\frac{\partial}{\partial R}\left(R\frac{\partial P}{\partial R}h^3+6Rh\cos\theta\right)+\frac{1}{R}\frac{\partial}{\partial\theta}\left(\frac{\partial P}{\partial\theta}h^3-6Rh\sin\theta\right)=0, \tag{7}$$

in which ω no longer appears and $h=h_0+\eta H$ is the gap profile. The function $P(R,\theta)$ is bounded at the origin, periodic, and decays for large R, such that

$$|P(0,\theta)|<\infty,\quad P(\infty,\theta)=0,\quad P(R,0)=P(R,2\pi)$$

and

$$\frac{\partial P}{\partial\theta}(R,0)=\frac{\partial P}{\partial\theta}P(R,2\pi). \tag{8}$$

The formulation given here does not impose any constraint on the constitutive equation for the wall. Equation (7) may thus be used for elastic walls or for rough walls, for example if the characteristic height H_c of the roughness elements is small compared with δ. If the wall is rough with no elastic response, the topography H is a function only of the spatial coordinates and the problem is linear. Equation (7) may also be used for poroelastic walls with sufficiently small permeability, such that $H_1kU(1+\omega)/\mu a^2\epsilon^2\ll 1$, where H_1 is the wall thickness and k is the permeability of the porous medium. The formulation thus may be applied to a wide range of problems.

III. PERTURBATION ANALYSIS FOR $\eta\ll 1$

Consider a slightly soft material with $\eta\ll 1$ and expand the pressure, velocities, and the elastic deflection in powers of η, such that $P=P_0+\eta P_1+O(\eta^2)$, $\boldsymbol{V}=\boldsymbol{V}_0+\eta\boldsymbol{V}_1+O(\eta^2)$, and $H=H_0+\eta H_1+O(\eta^2)$. To leading order in η the conservation equations are (4) with $\boldsymbol{V}$ replaced by $\boldsymbol{V}_0$. The boundary conditions on the sphere surface (5) also hold, but (6) is transferred to $Z=0$ such that $V_{z0}=0$ at $Z=0$. The resulting Reynolds equation is

$$\frac{\partial}{\partial R}\left(R\frac{\partial P_0}{\partial R}h_0^3+6Rh_0\cos\theta\right)+\frac{1}{R}\frac{\partial}{\partial\theta}\left(\frac{\partial P_0}{\partial\theta}h_0^3-6Rh_0\sin\theta\right)=0, \tag{9}$$

where $h_0=h_0(R)$ is given by (3) and subject to (8). Equation (9) has a solution

$$P_0=\frac{6R}{5h_0^2}\cos\theta, \tag{10}$$

and the leading-order velocity field in the present notation is

$$V_{r0}=\left[\frac{3(2-3R^2)}{10h_0^3}Z^2-\frac{3(2-3R^2)}{10h_0^2}Z+\left(\frac{1-\omega}{1+\omega}\right)\frac{Z}{h_0}-\frac{1}{1+\omega}\right]\cos\theta,$$

$$V_{\theta0}=\left[-\frac{3Z^2}{5h_0^2}+\frac{3Z}{5h_0}-\left(\frac{1-\omega}{1+\omega}\right)\frac{Z}{h_0}+\frac{1}{1+\omega}\right]\sin\theta, \tag{11}$$

$$V_{z0}=\left[\frac{8R-2R^3}{5h_0^4}Z^3+\frac{1}{1+\omega}\left(\frac{4R^3-16R-\omega(26R+R^3)}{10h_0^3}\right)Z^2\right]\cos\theta.$$

The leading-order hydrodynamic drag force F_{x0} and torque T_{y0} exerted on the sphere, which exhibit a $\ln\epsilon$ singularity, are given in Refs. 10–12. Note that $F_{z0}=F_{y0}=T_{x0}=T_{z0}=0$, and F_{x0} and T_{y0} are reversible in the rigid wall limit since P_0 scales linearly with the sphere velocity.

To $O(\eta)$, the conservation equations are given by (4) with $\boldsymbol{V}$ replaced by $\boldsymbol{V}_1$. Since $\boldsymbol{V}_0$ already satisfies the boundary conditions (5), we have $\boldsymbol{V}_1=0$ at $Z=h_0$. To $O(\eta)$, $\boldsymbol{V}_0+\eta\boldsymbol{V}_1$ is equal to the prescribed velocity of the wall at $Z=-\eta H_0$. Therefore, we transfer (6) to $Z=0$ yielding

$$V_{r1}|_{Z=0}=H_0\left.\frac{\partial V_{r0}}{\partial Z}\right|_{Z=0}=A(\omega,R)\frac{H_0}{h_0}\cos\theta,$$

$$V_{\theta1}|_{Z=0}=H_0\left.\frac{\partial V_{\theta0}}{\partial Z}\right|_{Z=0}=B(\omega)\frac{H_0}{h_0}\sin\theta, \tag{12}$$

$$V_{z1}|_{Z=0}=\frac{1}{1+\omega}\left(\frac{\partial H_0}{\partial R}\cos\theta-\frac{1}{R}\frac{\partial H_0}{\partial\theta}\sin\theta\right),$$

with the functions $A(\omega,R)$ and $B(\omega)$ given by

$$A(\omega,R)=\frac{3}{10}\frac{(3R^2-2)}{h_0}+\left(\frac{1-\omega}{1+\omega}\right),$$

$$B(\omega)=\frac{3}{5}-\left(\frac{1-\omega}{1+\omega}\right).$$

Eliminating the velocities from (4) by using (12) together with $\boldsymbol{V}_1=0$ at $Z=h_0$ gives the Reynolds equation

$$\begin{aligned}R^2\frac{\partial^2P_1}{\partial R^2}+\left(R+\frac{3R^3}{h_0}\right)\frac{\partial P_1}{\partial R}+\frac{\partial^2P_1}{\partial\theta^2}
&=-\frac{3H_0}{h_0}\left[R^2\frac{\partial^2P_0}{\partial R^2}+\left(R+\frac{2R^3}{h_0}\right)\frac{\partial P_0}{\partial R}+\frac{\partial^2P_0}{\partial\theta^2}\right]\\
&\quad-\frac{3}{h_0}\left(R^2\frac{\partial H_0}{\partial R}\frac{\partial P_0}{\partial R}+\frac{\partial H_0}{\partial\theta}\frac{\partial P_0}{\partial\theta}\right)\\
&\quad-\frac{6R^2}{h_0^3}\left(\frac{\partial H_0}{\partial R}\cos\theta-\frac{1}{R}\frac{\partial H_0}{\partial\theta}\sin\theta\right)\\
&=\frac{18}{5}\frac{R^3(6+R^2)}{h_0^5}H_0\cos\theta+\frac{48}{5}\frac{R}{h_0^3}\frac{\partial H_0}{\partial\theta}\sin\theta\\
&\quad+\frac{12}{5}\frac{R^2(R^2-4)}{h_0^4}\frac{\partial H_0}{\partial R}\cos\theta\end{aligned} \tag{13}$$

for the pressure perturbation P_1, which is subject to (8).

The second-order velocity field $\boldsymbol{V}_1$ is given by

$$V_{r1} = \frac{1}{2}\frac{\partial P_1}{\partial R}(Z^2 - h_0 Z) + A(\omega,R)\left(1 - \frac{Z}{h_0}\right)\frac{H_0}{h_0}\cos\theta,$$

$$V_{\theta 1} = \frac{1}{2R}\frac{\partial P_1}{\partial \theta}(Z^2 - h_0 Z) + B(\omega)\left(1 - \frac{Z}{h_0}\right)\frac{H_0}{h_0}\sin\theta, \quad (14)$$

$$V_{z1} = \frac{1}{2R^2}\left(\frac{Z^2 h_0}{2} - \frac{Z^3}{3}\right)\left(R^2\frac{\partial^2 P_1}{\partial R^2} + R\frac{\partial P_1}{\partial R} + \frac{\partial^2 P_1}{\partial \theta^2}\right) + \frac{Z^2 R}{4}\frac{\partial P_1}{\partial R} + \left\{A(\omega,R)\frac{\partial H_0}{\partial R}\cos\theta + \frac{B(\omega)}{R}\frac{\partial H_0}{\partial \theta}\sin\theta - \frac{RH_0\cos\theta}{5h_0^2(1+\omega)}[11R^2 - 14 + \omega(R^2-34)]\right\}\frac{Z^2}{2h_0^2} - \left\{A(\omega,R)\frac{\partial H_0}{\partial R}\cos\theta + \frac{B(\omega)}{R}\frac{\partial H_0}{\partial \theta}\sin\theta + \frac{RH_0\cos\theta}{5h_0^2(1+\omega)}[16 - 4R^2 + \omega(26+R^2)]\right\}\frac{Z}{h_0} + \frac{1}{1+\omega}\left(\frac{\partial H_0}{\partial R}\cos\theta - \frac{1}{R}\frac{\partial H_0}{\partial \theta}\sin\theta\right).$$

Since (13) indicates that the pressure perturbation P_1, unlike P_0, in general is not proportional to $\cos\theta$ dependence, we no longer necessarily have vanishing lift.

IV. THIN COMPRESSIBLE ELASTIC LAYER APPROXIMATION

Assume a thin compressible elastic layer of an isotropic material, for instance a gel layer, coating a rigid planar wall. The nondimensional thickness of the gel is assumed to be small, $\zeta = \epsilon^{-1/2}H_1/a \ll 1$, where H_1 is the unperturbed thickness and $\epsilon^{1/2}a$ is the contact length. The constitutive properties are given by the two Lamé constants $\lambda = \nu E/[(1+\nu)(1-2\nu)]$ and $G = E/[2(1+\nu)]$, where E is the Young modulus and ν is the Poisson coefficient. The elasticity equations are nondimensionalized, with H_1 for Z, $\epsilon^{1/2}a$ for R, $\mu U(1+\omega)\epsilon^{-3/2}/a$ for the solid stress tensor $\boldsymbol{\sigma_s}$, and ϵa for the elastic displacement $\boldsymbol{u} = (u_r, u_\theta, u_z)$. The solid stress tensor is related to the displacement field by the linear constitutive equation[13]

$$\boldsymbol{\sigma}_s = G(\nabla \boldsymbol{u} + \nabla \boldsymbol{u}^T) + \lambda \nabla \cdot \boldsymbol{u}\boldsymbol{I}, \quad (15)$$

and must satisfy the internal elastic equilibrium equation

$$\nabla \cdot \boldsymbol{\sigma_s} = 0. \quad (16)$$

Equations (15) and (16) are to be solved with compatibility boundary conditions on the rigid-elastic solid interface and on the linearized solid-fluid interface: $\boldsymbol{u}=0$ at $Z=-1$ and $\boldsymbol{\sigma_s}\cdot\boldsymbol{n} = \boldsymbol{\sigma_f}\cdot\boldsymbol{n}$ at $Z=0$. Here $\boldsymbol{\sigma_f}$ is the nondimensionalized fluid stress tensor based on inner variables described in Sec. II, and $\boldsymbol{n} = (-\epsilon^{1/2}\partial u_z/\partial R|_{Z=0}, -\epsilon^{1/2}R^{-1}\partial u_z/\partial\theta|_{Z=0}, 1) + O(\epsilon)$ is the normal unit vector to the layer. When the solid stress tensor and displacement field are expanded in powers of ζ with $\boldsymbol{\sigma_s} = \boldsymbol{\sigma_{s0}} + \zeta\boldsymbol{\sigma_{s1}} + O(\zeta^2)$ and $\boldsymbol{u} = \boldsymbol{u_0} + \zeta\boldsymbol{u_1} + O(\zeta^2)$, to leading-order in ζ, Eqs. (15) and (16) are reduced to

$$\frac{\partial^2 u_{r0}}{\partial Z^2} = \frac{\partial^2 u_{\theta 0}}{\partial Z^2} = \frac{\partial^2 u_{z0}}{\partial Z^2} = 0. \quad (17)$$

Equations (17) are subject to the compatibility conditions $u_{r0} = u_{\theta 0} = u_{z0} = 0$ at $Z=-1$, and $\partial u_r/\partial Z = \partial u_\theta/\partial Z = 0$ and $\partial u_z/\partial Z = -\eta P$ at $Z=0$. These boundary conditions at the solid-fluid interface are found to hold for $\eta \ll \epsilon^{-1/2}$. For values $\eta = O(\epsilon^{-1/2})$, which represents large deformations of the soft surface, the displacements caused by the viscous stresses are no longer negligible. According to the proposed scaling, the softness parameter

$$\eta = \frac{\mu U(1+\omega)H_1 a^{1/2}}{(2G+\lambda)\delta^{5/2}} \quad (18)$$

is obtained. To leading order in ζ, the displacements

$$u_{r0} = u_{\theta 0} = 0, \quad u_{z0} = -\eta P(Z+1) \quad (19)$$

are obtained, so that the nondimensional elastic deflection H is then found to be equal to the nondimensional local pressure to every order in η,

$$H(P) = P, \quad H_0(P_0) = P_0, \ldots, H_n(P_n) = P_n. \quad (20)$$

It is important to notice that, to leading order in ζ, (20) represents a well-posed deformation law for thin compressible elastic layers, for which $\lambda \sim G$. On the other hand, for incompressible elastic layers such as elastomeric layers, $\nu \sim 1/2$, $G \sim E/3$, and $\lambda \to \infty$, which yields a zero leading-order displacement field. A third-order correction is needed to achieve a normal displacement u_{z2} which remains nonzero in the incompressible limit $\lambda \to \infty$.

Assuming $\eta \ll 1$ and substituting (20) into (13), the Reynolds equation for the pressure perturbation

$$R^2\frac{\partial^2 P_1}{\partial R^2} + \left(R + \frac{3R^3}{h_0}\right)\frac{\partial P_1}{\partial R} + \frac{\partial^2 P_1}{\partial \theta^2} = \frac{72}{25}\frac{R^4(20+R^2)}{h_0^7}\cos^2\theta - \frac{288}{25}\frac{R^2}{h_0^5} \quad (21)$$

is obtained. A particular integral of the form $P_1 = f(R)\cos^2\theta + g(R)$ exists, such that (21) reduces to the coupled system

$$R^2 f'' + \left(R + \frac{3R^3}{h_0}\right)f' - 4f = \frac{72}{25}\frac{R^4(20+R^2)}{h_0^7},$$

$$R^2 g'' + \left(R + \frac{3R^3}{h_0}\right)g' = -\frac{288}{25}\frac{R^2}{h_0^5} - 2f,$$

with $f(0)=0$, $f(\infty)=0$, $g'(0)=0$, and $g(\infty)=0$. The homogeneous solution of $f(R)$ can be shown to be related to a hypergeometric series ${}_2F_1$ if the change of variables $\beta = -R/2$ and $f = R^{-3-\sqrt{13}}F(\beta)$ is performed. Therefore, obtaining an exact solution for the pressure P_1 is a formidable task, and is in fact not needed for computing the forces exerted on the sphere. An auxiliary function $\gamma(R)$ can be defined by $\gamma = 2g + f$ such that $\int_0^{2\pi} P_1 d\theta = \pi\gamma$, which is found to satisfy the differential equation

$$R^2\gamma'' + \left(R + \frac{3R^3}{h_0}\right)\gamma' = \frac{72}{25}\frac{R^4(20+R^2)}{h_0^7} - \frac{576}{25}\frac{R^2}{h_0^5} \tag{22}$$

subject to $\gamma'(0)=0$ and $\gamma(\infty)=0$. The solution to (22) is found to be

$$\gamma(R) = \frac{18}{125}\frac{14-5R^2}{h_0^5}. \tag{23}$$

From (23), it follows that $P_1=O(R^{-8})$ for $R\gg 1$. The leading order in ϵ of the outer pressure calculated in tangent-sphere coordinates for the rigid case is of $O(r^{-3})$ in the overlapping region $r\sim\epsilon^{1/2}$,[10–12] which also follows from (10). Here the rapid decay of the pressure perturbation outside the gap, $r\gg\epsilon^{1/2}$, indicates that the elastic deformations can be considered negligible outside the inner region. Hence we do not have to solve the outer problem.

Substituting P_1 into (14), the second-order velocity field $\boldsymbol{V}_1$ may be obtained as a function of the unknown $f(R)$. The following angular dependences are found: $V_{r1}=a_1(R,Z)\cos^2\theta+a_4(R,Z)$, $V_{z1}=a_3(R,Z)\cos^2\theta+a_4(R,Z)$, and $V_{\theta 1}=a_5(R,Z)\sin\theta\cos\theta+a_6(R,Z)$. The fluid stress tensor is integrated over a spherical inner element surface in cylindrical coordinates, $d\boldsymbol{S}=(a^2\epsilon^{3/2}R^2dRd\theta,0,-a^2\epsilon RdRd\theta)$. Thus, after lengthy algebra, the force and torque perturbations, $\boldsymbol{F}_1=(F_{x1},F_{y1},F_{z1})$ and $\boldsymbol{T}_1=(T_{x1},T_{y1},T_{z1})$, are obtained, such that

$$\begin{aligned} &F_{x1}=F_{y1}=T_{x1}=T_{y1}=T_{z1}=0,\\ &F_{z1}=\frac{\mu U(1+\omega)a}{\epsilon^{1/2}}\left\{\int_0^\infty\int_0^{2\pi}\left[\eta P_1R+\epsilon\eta\left(R^2\frac{\partial V_{r1}}{\partial Z}\right.\right.\right.\\ &\qquad\left.\left.\left.-2R\frac{\partial V_{z1}}{\partial Z}\right)\right]_{Z=h_0}dRd\theta+O(\epsilon^2\eta)\right\}\\ &\quad=\frac{\mu U(1+\omega)a}{\epsilon^{1/2}}\left(\frac{48\pi}{125}\eta+\frac{4\pi(19+14\omega)}{25(1+\omega)}\epsilon\eta+O(\epsilon^2\eta)\right)\\ &\quad=\frac{\mu^2U^2(1+\omega)^2H_1a^2}{(2G+\lambda)\delta^3}\left(\frac{48\pi}{125}+\frac{4\pi(19+14\omega)}{25(1+\omega)}\epsilon\right.\\ &\qquad\left.+O(\epsilon^2)\right), \end{aligned} \tag{24}$$

where the first and second terms in the expansion of F_{z1} represent the pressure and viscous lift, respectively.

From the last equality in (24), it is observed that if the signs of U and Ω are reversed, so that the motion of the sphere is reversed, the sign and magnitude of the lift F_{z1} is unchanged. Hence the motion of the sphere is no longer reversible.

Notice also the presence in (24) of mixed-order terms of $O(\epsilon\eta,\epsilon^2\eta)$ from the viscous stresses. Equations (4) are valid to leading order in ϵ, so that mixed-order terms are relevant for $\eta\ll 1$ and $\eta\ll\epsilon^{-1}$, respectively. Previous work[14] quotes the leading-order term of (24) but the coupled rotation-translation effects appear to be new.

The elastohydrodynamic lift (24) and softness parameter (18) scalings obtained here differ from those obtained previously[9] for the corresponding cylinder problem. There it

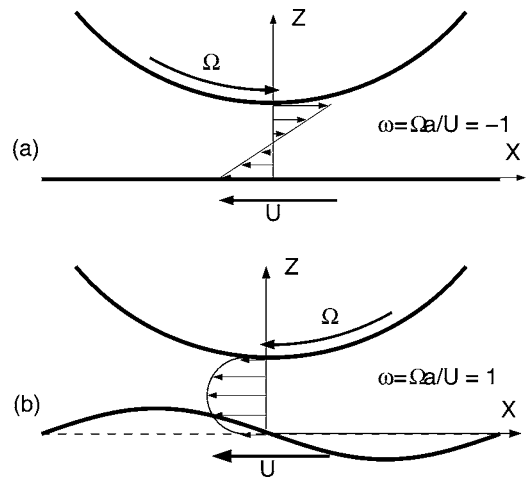

FIG. 2. Wall deformation and velocity schematics. (a) $\omega=-1$, the local velocity profile is a Couette flow, with no pressure gradient since the net fluid entrainment is zero, and no lift force produced. (b) $\omega=1$, the local flow is driven by a pressure gradient generated by the fluid entrainment in the gap and by the shear stresses exerted by the walls, producing a combined Poiseuille-Couette flow and a net upwards lift force.

was stated that the relative velocity of the surfaces $U(1-\omega)$, rather than the absolute velocity $U(1+\omega)$, is what contributes to the lift. That this statement (which may result from a slip in their derivation of the Reynolds equation) is erroneous may be seen by viewing the flow in a coordinate system translating with the velocity U of the cylinder, as illustrated in Fig. 2. In this coordinate system, for $\omega=-1$ the flow in the gap locally is a Couette flow, for which $P_0=0$,[10–12] so that, in view of (13), P_1 and higher-order terms must also be zero, leading to zero lift, consistent with (24) but not with previous results.[9] For $\omega=1$, on the other hand, where the previous result would give zero lift,[9] the flow approaching from infinity at velocity U must pass through a channel of smaller width, and therefore by continuity must have a higher average velocity in the gap, leading to a Couette-Poiseuille velocity profile as illustrated. Now there is a pressure gradient producing a nonzero P_0 and a positive lift force. It is therefore not true that the lift vanishes when the relative velocity of the two adjacent surface vanishes. These considerations may not be applicable to a rough wall, since the leading order of the topography H_0 may represent a source term in (13). In general this gives a nonzero pressure perturbation to the zero leading-order pressure.

Equation (7) was integrated numerically, subject to boundary conditions (8) and to the thin compressible elastic layer approximation (20), and using a second-order finite-difference method on a stretched grid along the R coordinate. The numerical results are shown in Figs. 3 and 4. The elastohydrodynamic pressure lift increases monotonically with η, and agrees well with the pressure lift calculated from the leading-order term in (24). As η increases, the effect of the soft wall is to diffuse the pressure profile radially. This radial diffusion cannot be accomplished in the two-dimensional case, where the lift per unit length of the cylinder decreases after reaching an optimum point,[9] unlike the result for the sphere shown in (24). From the conditions for the validity of (20), the value of η cannot increase indefinitely. Therefore,

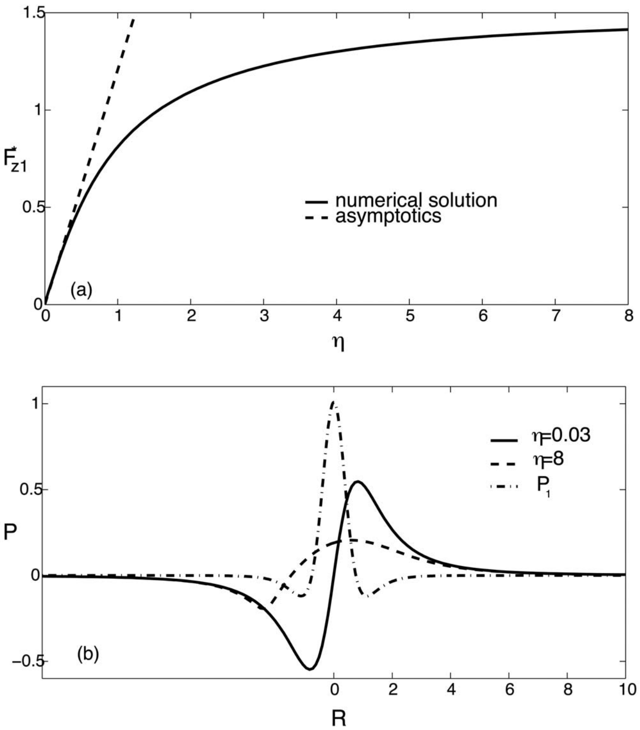

FIG. 3. (a) Nondimensional elastohydrodynamic pressure lift $F_z^*=F_z\epsilon^{-1/2}/\mu U(1+\omega)a$ as a function of the softness parameter η. (b) Numerical pressure profiles and asymptotic pressure perturbation $P_1=f(R)\cos^2\theta+g(R)$ along the axis $\theta=0$.

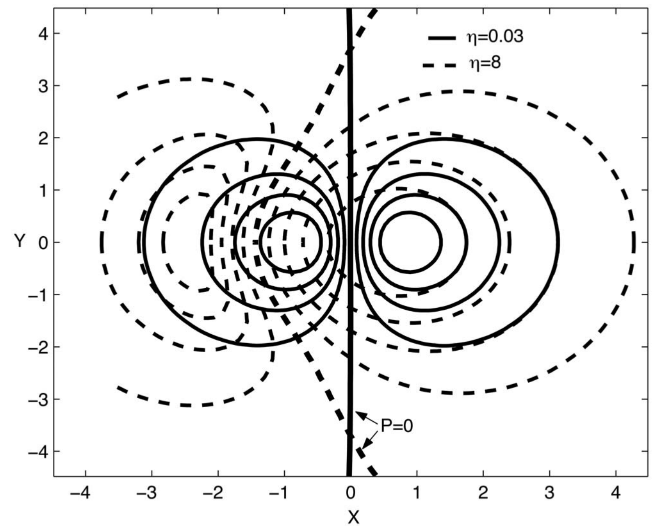

FIG. 4. Pressure contours. The sphere motion is directed toward positive x.

conclusions obtained from Fig. 3 cannot be applied when $\eta=O(\epsilon^{-1/2})$.

Pressure contours obtained from the numerical integration are shown in Fig. 4. As the softness of the wall increases, the motion of the sphere generates a diffusion front of positive pressure that travels upstream and downstream to produce the lift force. The pressure distortion also produces an additional elastohydrodynamic drag F_x and torque T_y, which, based on (19) and on the symmetry of the parallel motion, must be of $O(\eta^2)$ and yield a zero value for every odd power of η, with the F_x and T_y scale containing odd powers of the sphere velocity.

Similar trends to these shown in this section are found in earlier investigations about the motion of a sphere near a fluid interface,[15] in which a lift force that pulls the sphere away from the interface is calculated for $\epsilon \geqslant O(1)$.

As a practical application, the elastohydrodynamic lift (24) may be used to calculate the equilibrium gap distance for a sphere rolling down an inclined plane of slope $\tan\alpha$ in a viscous fluid. Torque and drag force balances may be calculated using the formulation shown in earlier works[10–12] for F_{x0} and T_{y0}. Since no net torque is applied to the sphere, the value $\omega=1/4$ is found in the limit $\epsilon \ll 1$, which shows that the sphere must slip. The component of the gravitational force tangential and normal to the plane surface must be balanced by the drag force and elastohydrodynamic lift, respectively, so that a single equilibrium equation $\epsilon^{1/2}\ln\epsilon \sim -4\eta\tan\alpha/25$ for the gap distance is found. Note that for $\eta=0$, then $\epsilon=0$, resulting in an infinite drag, so that no motion of the sphere is possible in this limit, an inconsistency previously found in Ref. 12 when there was no opposing force to the normal gravitational component. For softness parameters $\eta=O(1)$ and $\tan\alpha\sim 1$, equilibrium values of $\epsilon=O(10^{-3})$ are found, so that the sphere does not touch the elastic wall while rolling down.

V. CONCLUSIONS

The nonlinear effect of a soft boundary on the slow motion of a sphere near a wall has been investigated. A general formulation, which is applicable to any constitutive equation of the wall, is presented, and an asymptotic analysis is developed in the case of slightly rigid walls $\eta \ll 1$. This formulation is applied to a thin compressible elastic layer coating a rigid wall, and an asymptotic expression for the elastohydrodynamic lift F_{z1} is derived, which agrees well with the numerical results, resulting in a positive force normal to the soft surface. The results are different from the cylinder problem: there is no optimum choice of the soft materials to obtain the maximum lift. The influence of the sphere rotation on the direction of its motion is investigated: rotation can suppress the elastohydrodynamic lift in the case $\omega=-1$. The lift F_{z1} scales with the total velocity squared $U^2(1+\omega)^2$, so the sign and modulus of F_{z1} remain invariant if the sphere motion is reversed: the motion is no longer reversible. Equation (24) may thus be used for experimental surface characterization of a soft substrate.

The formulation developed in Secs. II and III can also be applied to other kinds of walls, such as elastomeric layers, semi-infinite compressible elastic walls, or poroelastic layers. In these cases, obtaining a closed solution of the leading-order deflection H_0 for the sphere problem is a more difficult task, and solutions might have to be obtained numerically. It would be of some interest to study the influence of these other kinds of soft walls and the influence of wall roughness on the sphere motion.

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