

Lawrence Berkeley National Laboratory

Recent Work

Title

THE N-LOOP STRING AMPLITUDE

Permalink

<https://escholarship.org/uc/item/7f391156>

Author

Mandelstam, S.

Publication Date

1986-06-01



Lawrence Berkeley Laboratory

UNIVERSITY OF CALIFORNIA

RECEIVED
LAWRENCE
BERKELEY LABORATORY

Physics Division

SEP 17 1986

LIBRARY AND
DOCUMENTS SECTION

Presented at the Second Nobel Symposium on
Elementary Particle Physics, Marstrand, Sweden,
June 2-7, 1986

THE N-LOOP STRING AMPLITUDE

S. Mandelstam

June 1986



LBL-21735
e.2

DISCLAIMER

This document was prepared as an account of work sponsored by the United States Government. While this document is believed to contain correct information, neither the United States Government nor any agency thereof, nor the Regents of the University of California, nor any of their employees, makes any warranty, express or implied, or assumes any legal responsibility for the accuracy, completeness, or usefulness of any information, apparatus, product, or process disclosed, or represents that its use would not infringe privately owned rights. Reference herein to any specific commercial product, process, or service by its trade name, trademark, manufacturer, or otherwise, does not necessarily constitute or imply its endorsement, recommendation, or favoring by the United States Government or any agency thereof, or the Regents of the University of California. The views and opinions of authors expressed herein do not necessarily state or reflect those of the United States Government or any agency thereof or the Regents of the University of California.

June 1986

UCB-PTH-86/16

LBL-21735

The n-Loop String Amplitude*

Stanley Mandelstam
Department of Physics
University of California
Berkeley, California 94720

Talk given at the Second Nobel
Symposium on Elementary Particle
Physics, Marstrand, June 1986.

* This work supported by the National Science Foundation under Research Grant No. PHY85-15857 and by the Director, Office of Energy Research, Office of High Energy and Nuclear Physics, Division of High Energy Physics of the U.S. Department of Energy under contract DE-AC03-76SF00098.

THE n-LOOP STRING AMPLITUDE

Stanley Mandelstam

Department of Physics
University of California
Berkeley, California 94720, U.S.A.

Introduction

In this talk I shall review some of the work of the last eighteen months on the derivation of an explicit perturbation series for string and superstring amplitudes. I shall concentrate on the light-cone approach which I have used personally and which, in my opinion, leads most directly to explicit formulas. I shall also mention some of the work on the Polyakov approach and shall indicate points of comparison between the two methods.

The perturbation series for the Bosonic string is now fairly well understood. In fact, the formula had almost been obtained during the previous incarnation of string theories, [1] but the models which were studied possessed ghosts. I gave the formula for the current Bosonic string models in my Santa Barbara lectures last year [2]; one point in the proof remains to be tightened up, but I believe the result is correct. Though the light-cone approach was used, the final formula is manifestly covariant. D'Hoker and Phong, [3] Belavin and Knizhnik [4] and Manin [5] have presented formulas based on the Polyakov approach. Their expressions are mathematically well defined, but an explicit formula for some of the factors in the integrands is not known at present.

No explicit formula for the n-loop superstring amplitude has been obtained to date. In the light-cone approach the difficulty was due to the

fact that the functional integral contained operators at the joining points of the strings. A new formulation by Berkovits [6] shows that this difficulty can be avoided if one integrates over supersheets instead of ordinary sheets. It should now be fairly straightforward to obtain an explicit formula for the n -loop amplitude and, if the external particles are vector bosons, the result should be manifestly covariant.

Some of the above-mentioned results on the Polyakov Bose string have been extended to the superstring by D'Hoker and Phong, [7] Friedan, Martinec and Shenker, [8] and Nelson, Moore and Polchinski. [9]

There are certain technical difficulties with amplitudes containing external fermions; as far as I am aware, there is no existing method of calculation, even for tree amplitudes, that does not break the manifest *transverse* rotational $SO(8)$ invariance down to $SO(4) \times SO(2)$. Also, all the superstring calculations I have mentioned use the N.S.R. formalism; the manifestly supersymmetric formalism has not yet been formulated as a functional integration over supersheets, and operators at the joining point appear explicitly. The difficulties, hopefully only technical, with external fermions and manifest spacetime supersymmetry should not stand in the way of a proof of the finiteness of the perturbation expansion. We have given handwaving arguments for such finiteness; we shall not repeat them here.

Approaches to String Perturbation Theory

Whether one uses the light-cone or Polyakov approach, one first performs the Gaussian functional integral for the general perturbation term; one thereby obtains the formula (for the Bose string)

$$A = -i(ig^2/4\pi) d^2Z_1 \dots d^2Z_N d^{3g-3}v M(Z_1, \dots, Z_N, v) \text{Exp}\{\sum P_i \cdot P_j N(Z_i, Z_j)\}. \quad (1)$$

The points Z_i ($1 \leq i \leq N$) represent the N external particles, and the integration is over the Riemann surface, parametrized by a complex variable Z , which represents the world-sheet traced out by the string. (We shall write all our formulae for the case of closed strings.) The factor $P_i \cdot P_j$ in the exponential is the *d-vector product* (even in the light-cone approach) between the i^{th} and j^{th} external momenta, and the function N is the Green's function of the Laplacian between the points Z_i and Z_j .

We still have to discuss the variables v and the factor M ; all recent work has been concerned with these factors. A Riemann surface of genus g , corresponding to a g -loop amplitude, is parametrized to within conformal equivalence by $3g-3$ complex parameters if $g \geq 2$ (and by one complex parameter if $g=1$; all surfaces with $g=0$ are conformally equivalent). This fact was originally discovered by Riemann himself, but the parameters are known as Teichmüller parameters, since Teichmüller initiated the recent mathematical work on the subject. There is no universally accepted "best way" of specifying the Teichmüller parameters, and the formula (1) leaves this question open. The measure function M depends on the choice of the v 's. The only difference between the light-cone and Polyakov approaches is in the calculation of M .

In the light-cone interacting-string picture one treats the strings as an ordinary quantum-mechanical system. In Fig. 1, one cuts the plane along the horizontal lines and identifies points above one another on adjacent horizontal lines (e.g., AA', BB', CC'). The diagram then represents an interacting closed-string system; σ parametrizes the string itself, while τ is

the light-cone time. The precise process depicted in the diagram is a two-to-three scattering process with two loops. Along each dotted line one breaks the diagram, displaces the string on one side by an arbitrary twisting angle θ , and reidentifies the points.

In terms of the string-diagram variables, the measure is simply

$$\mathcal{N}\mathcal{M} \int \prod d\tilde{\tau}_i \prod d\alpha_i \prod d\theta_i |\Delta|^{-(d-2)/2}, \quad (2)$$

where the $\tilde{\tau}_i$'s are the time co-ordinates of all joining points but one, the α_i 's are the lengths of one of the strings in each loop, and the θ_i 's are the twisting angles. The factor $|\Delta|^{-(d-2)/2}$, where $|\Delta|$ is the determinant of the Laplacian for the string diagram, results from the original Gaussian functional integral. \mathcal{N} is the Feynman normalization factor, while \mathcal{M} is an external-line factor which corrects for the fact that we have omitted the term with $i=j$ in the exponent of (1). We shall not give the form of \mathcal{M} (see ref. 2 for a full discussion); it diverges when the initial and final times became infinite, and it cancels an infinite factor in $|\Delta|^{-(d-2)/2}$. It is easily checked that the number of variables of integration is $2N+6g-6$, corresponding to the $N+3g-3$ complex variables of integration in Eq. (1).

As we shall see shortly, it is convenient to express the string-diagram variables in terms of a different set of variables. The measure will then contain the Jacobian J of the transformation as an additional factor. Since the "lengths" of the strings are proportional to the momentum in the + direction, the shape of the string diagram will depend on the Lorentz frame and none of the factors \mathcal{M} , $|\Delta|^{-(d-2)/2}$ and J will be Lorentz invariant. The product is Lorentz invariant if and only if $d=26$.

In the Polyakov approach, one embeds the world-sheet in d -dimensional

space, puts a metric on the space, and integrates the Polyakov action over all embeddings and metrics. The integration over embeddings is Gaussian and was performed by Polyakov. One is left with the integration over metrics, with the function-space metric [10]:

$$|\delta g_{ab}|^2 = \int_M d^2\xi \ g^{\frac{1}{2}} g^{ac} g^{bd} \delta g_{ab} \delta g_{cd} \quad (3)$$

One then has to factor out the gauge group, which consists of two classes of transformations: i) Diffeomorphisms of the co-ordinate system, and ii) Weyl conformal transformations $\delta g_{ab} = \delta\sigma g_{ab}$.

Any infinitesimal change in g can be decomposed into a Weyl conformal change and a traceless change. Most traceless changes can be obtained from diffeomorphisms $\xi^a \rightarrow \xi^a + \delta V^a$, with δV^a single-valued, followed by Weyl transformations

$$\delta g_{ab} = [P_1(\delta V)]_{ab} = \nabla_a \delta V_b + \nabla_b \delta V_a - (\nabla_c \delta V^c) g_{ab}. \quad (4)$$

However, not all traceless changes can be obtained in this way; one must supplement the changes (4) by a linear space of changes δg_{ab} , of dimension $3g-3$, in order to obtain all traceless changes. In mathematical terms, the operator P_1 defined by (4) has a cokernel, or its adjoint P_1^\dagger has a kernel, of dimension $3g-3$. Thus, after factoring out the orbits of the gauge group, one is left with a $(3g-3)$ -dimensional integral to perform. The $3g-3$ parameters are precisely the Teichmüller parameters. (For a fuller account see ref. 11.) The measure M will then be the product of the measure in Teichmüller space induced by the metric (3), the Faddeev-Popov determinant from the Weyl gauge group, and a factor $|\Delta|^{-d/2}$ which results from the original Gaussian integration over the embeddings.

It must be emphasized that the Polyakov S-matrix, unlike that obtained

from the light-cone picture, is not obviously unitary. It is clearly necessary to prove unitarity before we know that we have an acceptable theory.

Convenient Variables for the n-loop Problem

Now let us examine the interacting-string picture in a little more detail. We wish to replace the string-diagram variables by new variables Z_i , which have the property that the functions occurring in (1) can be calculated explicitly. A canonical set of such variables makes use of the theory of *automorphic functions* [12]. On the string diagram one draws g *A-cycles* A_r and g *B-cycles* B_r which have the property that A_r intersects B_r once, but no other pairs of cycles intersect. One then cuts the diagram along the *A-cycles* and transforms it conformally onto the complex plane with holes corresponding to the cut cycles (Fig. 2). A *B-cycle* thus takes us from one hole to another. One identifies corresponding points z and z' on corresponding *A-cycles* by *projective transformations*; one for each A_r ($1 \leq r \leq g$)

$$T_r: \quad z' = \frac{Az+B}{Cz+D} \quad (4)$$

The g transformations T_r thus correspond to the *B-cycles*. They and their reciprocals generate an infinite group of projective transformations; we denote a general member of the group by V_m ($1 \leq m < \infty$). One is only interested in those groups of transformations whose fundamental region is a region exterior to $2g$ holes as in Fig. 2. Such groups are called *Schottky groups*.

The transformation T_r depends only on the ratios of the constants A , B , C and D ; one usually fixes them (to within a sign) by setting $AD-BC=1$. Each transformation thus depends on three complex parameters. The generic projective transformation can be written in the form:

$$\frac{z'-z_1}{z'-z_2} = w \frac{z-z_1}{z-z_2} \quad (5)$$

The parameter w is known as the *multiplier*, z_1 and z_2 are known as the *invariant points*. By interchanging z_1 and z_2 if necessary we can ensure that $|w| \leq 1$, and we shall always do so.

Given one set of generators T_1, \dots, T_g , one may change them in two ways without changing the string diagram. One may subject them all to a projective transformation, i.e., one may define $T'_r = A^{-1}T_r A$, where A is a fixed projective transformation. One may also take a new set of A- and B-cycles; the corresponding transformations of the T's are the *modular* transformations.

Poincaré and Klein made the fundamental conjecture, which was subsequently proved by Koebe [12] that *a conformal transformation from a Riemann surface (and, in particular, from a string diagram) into the complex plane in the manner discussed above is always possible and is unique up to an overall projective transformation and a modular transformation*. Since each of the g projective transformations has $3g$ parameters and the arbitrary overall projective transformation has 3 parameters, the conformal class of the Riemann surface is characterized by $3g-3$ parameters, in agreement with Riemann's general result. We shall take as our Teichmüller parameters the $3g$ variables w_r, z_{1r}, z_{2r} , with three of the z 's fixed at arbitrary values.

Functions defined on the string diagram (or, in general, on a Riemann surface) must be unchanged when z is subjected to a projective transformation in the group. As in the theory of elliptic functions, it is of interest to consider multi-valued functions which have simple transformation properties when we traverse a cycle. In the z -plane, they must have simple transformation properties when z traverses an A-cycle or when it is subjected to a projective transformation in the group. One constructs such functions in the same way as one constructs the infinite series for the logarithm of the

Jacobi θ -function (or, alternatively, the infinite product for the θ -function itself). One starts with a given function, subjects it to a transformation in the group, and sums over all group elements. We shall write down the series for the functions we require; the verification that they have the desired properties is not difficult.

It is known that there exist g linearly independent functions which change by a constant when the variable traverses an A- or a B-cycle. The canonical basis for such functions is formed by the functions $v_r(z)$, where v_r changes by $2i\pi\delta_{rs}$ when the variable traverses the s^{th} A-cycle. The formula for v_r is:

$$v_r(z) = \sum_m^{(r)} \ln \frac{z - V_m z_{1r}}{z - V_m z_{2r}} \quad (6)$$

the superscript (r) indicating that we *omit* those values of m for which V_m , when expressed as a product of the generators T_s , has a factor T_r or T_r^{-1} at its right-hand end.

The differentials $\frac{1}{2i\pi} dv_r = \omega_r$ are the g single-valued holomorphic differentials on the Riemann surface. For our purposes the v_r 's are more convenient than the ω_r 's.

When the variable traverses the s^{th} B-cycle, v_r will change by a quantity which we denote by $2i\pi\tau_{rs}$, where

$$\tau_{rs} = \frac{1}{2i\pi} \left\{ \sum_m^{(r,s)} \ln \frac{(z_{1s} - V_m z_{1r})(z_{2s} - V_m z_{2r})}{(z_{1s} - V_m z_{2r})(z_{2s} - V_m z_{1r})} + \delta_{rs} \ln w_r \right\}, \quad (7)$$

the superscript (r,s) indicating that we omit those V_m 's which have $T_r^{\pm 1}$ as their right-most member or $T_s^{\pm 1}$ as their left-most member; we must also omit the identity transformation if $r=s$. The matrix of the τ 's is known as the *period matrix*.

Finally the Green's function is given by the formula:

$$N(z, z') = \ln |\phi(z, z')|, \quad (8a)$$

$$\ln \phi'(z, z') = \ln \phi'(z, z') - (2\pi)^{-1} \sum_{r,s} \operatorname{Re} \{v_r(z) - v_r(z')\} \{(\operatorname{Im} \tau)^{-1}\}_{rs},$$

$$\operatorname{Re} \{v_s(z) - v_s(z')\}, \quad (8b)$$

$$\ln \phi(z, z') = \ln(z - z') + \frac{1}{2} \sum_{m \neq 1} \ln \frac{(z - V_m z')(z' - V_m z)}{(z - V_m z)(z' - V_m z')}. \quad (8c)$$

It is not quite true that N remains unchanged when z is subjected to a projective transformation T_r ; the change will be

$$\delta_{B_r, z} = - \ln |C_r z + D_r|, \quad (9)$$

where C_r and D_r are the C- and D-parameters of T_r . The right-hand side of (9) does not depend on z' , and as a consequence, it is not difficult to show that one obtains the correct result if one uses N , defined by Eq. (8), in Eq. (1). The argument depends on momentum conservation.

We notice that the right-hand side of (8b) is not analytic. In fact, the presence of a zero mode requires us to define N by the equation

$$\Delta_z N = - 2\pi \delta^2(z - z') + f(z') \quad (10)$$

There exists no function satisfying (10) with $f=0$.

The series (6), (7) and (8c) are known to converge absolutely in a sub-region of the $(3g-3)$ -dimensional space of the Schottky region. The question whether they converge conditionally outside this sub-region has not yet been answered. If not, they must be defined outside the sub-region by analytic continuation. In fact, if one multiplies all the w_r 's by a factor λ , the series will converge as long as λ is sufficiently small. For larger values of λ the functions can be defined by a Padé approximation in the single variable λ .

Measure Factor in the Interacting-String Picture

Now let us outline the calculation of the factor M in (1) and, in particular, of $|\Delta|$. We use the formula

$$|\Delta| = \text{Exp} \{ \text{Tr} \ell n \Delta \}. \quad (11)$$

The operator $\ell n \Delta$ is singular when $\rho = \rho'$, ρ being any local co-ordinate on a Riemann surface:

$$\ell n \Delta = \pi^{-1} |\rho - \rho'|^{-2} - (12\pi)^{-1} R \ell n |\rho - \rho'| + \text{non-singular terms}, \quad (12)$$

where R is the scalar curvature. To regularize we evaluate $\ell n \Delta$ at small values of $\rho - \rho'$, subtract the first two terms of (12), and take the limit $\rho - \rho' = 0$. We also replace the joining point by a small region of large but finite curvature. The regularization adds to the energy of the string a term proportional to its "length" (i.e., to the + momentum), and also renormalizes the coupling constant. Neither of these changes is physically significant.

Another infinite contribution to $\ell n \Delta$ arises from the zero mode. The correct prescription is to replace the zero eigenvalue by $1/2\pi \mathcal{A}$, where \mathcal{A} is the area of the string diagram.

We evaluate $|\Delta|$ by examining its change under an infinitesimal change of the metric. The effect of *conformal* changes was considered several years ago by McKean and Singer [13]; formulas based on their work are given in Refs. [11] and [14]. To calculate the effect of a Teichmüller transformation, we shall consider a general diffeomorphism

$$\rho \rightarrow \rho + \delta V(\rho, \bar{\rho}). \quad (13)$$

Notice that this change differs from that in Eq. (4) in several respects.

We change the local co-ordinates and keep the g's constant rather than

vice versa, and we do not subtract out the change in the trace. More important, we do *not* require the V 's to be single-valued, but we allow them to change when the variables traverse a B-cycle. The change (13) is then sufficiently general to include the Teichmüller transformations.

Under a transformation (13), $\text{Tr } \ell n \Delta$ changes as follows:

$$\delta(\text{Tr } \ell n \Delta) = \text{Tr} \{ \partial_\rho (\mu \partial_\rho N) + \partial_{\bar{\rho}} (\bar{\mu} \partial_{\bar{\rho}} N) + 2(\delta\phi) \Delta N \} + M, \quad (14a)$$

where

$$\mu = \partial_{\bar{\rho}} \delta V, \quad (14b)$$

$$2\delta\phi = \partial_\rho \delta V + \partial_{\bar{\rho}} \delta \bar{V}, \quad (14c)$$

and M is an extra term, which we shall not specify, and which is necessary because of the limiting procedure used to define $\text{Tr } \ell n \Delta$. The quantity μ is known as the infinitesimal *Beltrami differential*. The function N is the Green's function, i.e., the reciprocal of Δ .

We now insert the expression (8) for N into Eq. (14). We emphasize that we are evaluating $|\Delta|$ on the Riemann surface itself and not in the z -plane; the variable z simply identifies points on the Riemann surface. Let us first consider the summation on the right of (8c), excluding the term $\ell n(z-z')$. As all the terms except $\ell n(z-z')$ are non-singular when $z=z'$, we can simply set z equal to z' . The trace can then be evaluated by a slight adaptation of a resummation procedure due to Selberg [15]. On integration we find that

$$|\Delta|_1 = \prod |1-w_m|^2, \quad (15)$$

where the product is over all *conjugacy classes* of elements V_m of the Schottky group of projective transformations, excluding the identity transformation. (All elements in the same conjugacy class have the same value of w .) Our remarks about the convergence of our previous summations apply

equally to the logarithm of (15).

The contribution from the second term on the right of (8b) is also evaluated without difficulty. The result is:

$$|\Delta|_2 = |\text{Im } \tau|^{-\frac{1}{2}}, \quad (16)$$

where $|\text{Im } \tau|$ is the determinant of the imaginary part of the period matrix.

The contribution from the first term on the right of (8c), though in principle the most straightforward, requires the most care because of limiting processes involved. One must separate ρ and ρ' (or z and z'), include the term M on the right of (14a), subtract the changes in the regularization terms on the right of (12), pass to the limit $\rho=\rho'$, and integrate over the Riemann surface. We find:

$$\ln|\Delta|_3 = - \frac{1}{12\pi} \int d^2z \partial_z \sigma \partial_{\bar{z}} \sigma, \quad (17)$$

where σ is the logarithm of the dilation in the conformal transformation from the z -plane, with Euclidean metric, to the Riemann surface. The integration is over one fundamental region of the z -plane. Eq. (17) is reminiscent of, and consistent with, the formula for the change of $|\Delta|$ under a conformal transformation given in Ref. 11.

On integration by parts, Eq. (17) takes the form

$$\ln|\Delta|_3 = - \frac{1}{12\pi} \oint dz (\sigma \partial_z \sigma) + \frac{1}{12\pi} \int d^2z (\sigma \partial_z \partial_{\bar{z}} \sigma), \quad (18)$$

where the first integral is over the curves bounding the fundamental region in the z -plane and over small curves surrounding the external particles.

The integrand of the second term on the right of (18) is zero if the Riemann surface is flat, so that the only contributions in a string diagram are from the joining points. On evaluating the contributions from the joining points, as

well as those from the circles surrounding the external particles in the first term on the right of (18), we obtain the equation:

$$\ln|\Delta|_3 - \frac{2}{d-2} \ln \mathcal{M} = -\frac{1}{12\pi} \oint dz (\sigma \partial_z \sigma) + \frac{1}{24} \sum \ln \left| \frac{d^2 \rho}{dz^2} \right| + \frac{1}{12} \sum \ln |\alpha_i|. \quad (19)$$

where the integral is now only over the curves bounding the fundamental region, the first summation is over all joining points and the second summation over all external particles. The last contribution includes that from the factor \mathcal{M} in (2). We recall that $\partial \rho / \partial z = 0$ at a joining point. In Eq. (19) we have dropped constant terms from the joining points, since they can be absorbed in the coupling constant.

The other factor in the measure M , eq. (1), is the Jacobian of the transformation from the string-diagram variables σ and τ to the new variables z . As far as we can see, the Jacobian is too complicated to calculate directly, but we can obtain it by making use of its analytic properties. The $2N+6g-6$ variables of integration in (2) cannot be replaced by $N+3g-3$ complex variables, so we proceed in two stages. We first transform to a new "string diagram", conformally equivalent to the old, where the variables α_i are regarded as fixed but the time intervals above and below the loops can differ. This means that the net twist θ on going around the loop is complex, and we can replace our variables by $N+2g-3$ complex joining points $\rho (= \tau + i\sigma)$ and g complex twists. The Jacobian from the old to the new string variables can be calculated explicitly.

The transformation from the new string variables to the variables z_i, w_r, z_{1r}, z_{2r} (with three z_r 's held fixed) is analytic except for

isolated singularities. The Jacobian is thus the square of the modulus of an analytic function f , again except for isolated singularities. It is not too difficult to show that the first term on the right of (19), and therefore the whole right-hand side of this equation, is the real part of a function which is analytic except for isolated singularities and which we denote by $2 \ln f$. By examining all possible isolated singularities one finds that, apart from some simple factors, the singularities of f precisely cancel those of \hat{f} , provided $d-2=24$. (The factors of 12 and 24 in the denominator of (19) are thus cancelled.) Furthermore, the product $f\hat{f}$ is invariant under modular transformations; the proof of this fact is tricky and, at the moment, not rigorous. Since an analytic, singularity-free, modular-invariant function on the Teichmüller space is a constant, these properties determine the result:

$$J' = \mathcal{K} J |\Delta|_3^{-(d-2)/2} = 2^{2-g} \pi^{-1} |\text{Im } \tau|^{-1} |(z_a - z_b)(z_b - z_c)(z_c - z_a)|^2 \prod_r |w_r(z_{1r} - z_{2r})|^{-4} \prod |\alpha_i|, \quad (20)$$

where z_a , z_b and z_c are the three z 's which are kept fixed. The factor $|\text{Im } \tau|^{-1}$, which is not the modulus of an analytic function, arises from the transformation from the old to the new string variables.

Our final formula for M , Eq. (1), is thus:

$$M = 2^{-g} \{|\Delta|_1 |\Delta|_2\}^{-12} J', \quad (21)$$

the factors being given by Eqs. (15), (16) and (20). The extra factor 2^{-g} corrects for some double-counting we have performed [8], about which we shall not elaborate.

One must integrate the Teichmüller parameters over one fundamental region of the modular group, since different such regions correspond to the

same string-diagram configurations with different choices of the A- and B-cycles. There is no simple formula for the change of our parameters w_r , z_{1r} and z_{2r} under a modular transformation. The period matrices change by a Siegel modular transformation:

$$\tau' = (A\tau + B)(C\tau + D)^{-1}, \quad (22)$$

where A, B, C and D are $g \times g$ matrices with integral entries such that $AD - BC = DA - CB = 1$. The τ 's must be calculated from our parameters using Eq. (7); one must then restrict the integration region to avoid two or more period matrices related by (22).

Results of the Polyakov Approach

Let us now very briefly compare our results with those of D'Hoker and Phong [3], Belavin and Knizhnik [4] and Manin [5]. These authors all use the Polyakov approach. They do not make a specific choice of parametrization for the Teichmüller space, but they express their results in terms of one of the canonical metrics used by mathematicians, the Weil-Petersson metric [16]. As far as we are aware, there is no specific parametrization of the Teichmüller space which leads to a specific formula for the Weil-Petersson metric *and* for the other factors in the integrand of the n-loop amplitude.

The expression for the measure function M given by D'Hoker and Phong may lead to a proof of the unitarity of the Polyakov *ansatz*. Their result is:

$$M = |P_1^\dagger P_1|^{1/2} |\Delta|^{-13} d(\text{Weil-Petersson}). \quad (23)$$

The first factor is the determinant of the tensor Laplacian $P_1^\dagger P_1$, where P_1 is defined by Eq. (4). It may be regarded as the contribution from the Faddeev-Popov ghosts. Each of the three factors in (23) depends on the metric on the Riemann surface; the Weil-Petersson metric is defined with respect to

the constant-curvature metric on the Riemann surface, which is the metric used by D'Hoker and Phong. However, it would be possible to define a "generalized Weil-Petersson metric" in terms of any metric on the Riemann surface, and the product in (23) is independent of the latter metric. In particular, we could use the string-diagram metric, where all the curvature is concentrated at the joining points. With this metric, we suspect that the determinant $|\mathbb{P}^+ \mathbb{P}_1|^{1/2}$ could easily be related to the determinant $|\Delta|$ of the ordinary Laplacian, and that the Weil-Petersson metric could easily be evaluated. The combination should yield the measure (2), thereby proving the unitarity of the Polyakov *ansatz*. The proof would be analogous to the often used proof of the unitarity of the Faddeev-Popov functional-integration formalism in gauge theories, where one relates the formalism to an operator formalism in a non-covariant gauge.

Extension to Superstrings

As we mentioned in the introduction, both approaches to the g -loop amplitude can be generalized to superstrings, though no explicit results have been obtained to date. All variables in (1) become replaced by supervariables, the Teichmüller space becomes replaced by a super-Teichmüller space, automorphic functions become replaced by super-automorphic functions, and so on. Obtaining a manifestly covariant amplitude with external fermions may possibly be more than a straightforward technical problem. The deepest problem is probably to obtain an amplitude with manifest Lorentz invariance *and* manifest space-time supersymmetry. Apart from that, we see no *fundamental* unsolved problems in the perturbation expansion of string amplitudes.

References

1. Alessandrini, V., *Nuovo Cimento* 2A, 321 (1971).
 Alessandrini, V., and Amati, D., *Nuovo Cimento* 4A, 793 (1971).
 Kaku, M. and Yu, L.P., *Phys. Lett.* 33B, 166 (1970); *Phys. Rev.* D3
 2992, 3007, 3020 (1971).
 Lovelace, C. *Phys. Lett.* 32B, 703 (1970).
2. Mandelstam, S. *Unified String Theories*, Proceedings of the Santa Barbara Workshop, August 1985, pp. 46, 577. (World Publishing, Singapore, 1986.)
3. D'Hoker, E. and Phong, D.H., Multiloop Amplitudes for the Bosonic Polyakov String. Columbia preprint.
4. Belavin, A.A. and Knizhnik, V.G., Algebraic Geometry and the Geometry of Quantum Strings. Landau Institute preprint.
5. Manin, Yu I., The Partition Function of the Polyakov String can be Expressed in Terms of Theta Functions. Preprint.
6. Berkovits, N., Calculation of Scattering Amplitudes for the Neveu-Schwarz Model using Supersheet Functional Integration. Berkeley preprint.
7. D'Hoker, E. and Phong, D.H., Loop Amplitudes for the Fermionic String. Columbia preprint.
8. Nelson, P., Moore, G. and Polchinski, J. Strings and Supermoduli. Harvard preprint.
9. Friedan, D., Shenker, S. and Martinec, E., Covariant Quantization of Superstrings. Chicago preprint.
10. Polchinski, J., Evaluation of the One-Loop String Path Integral. Texas preprint.
11. Alvarez, O., *Nucl. Phys.* B216, 125 (1983).
12. Ford, L.R., *Automorphic Functions*. (Chelsea, New York, 1951).
13. McKean, H. and Singer, I.M., *J. Diff. Geom.* 1, 43 (1973).
14. Durhuus, B. Nielsen, H.B., Olesen, P. and Petersen, J.L., *Nucl. Phys.* B196, 498 (1982).
 Durhuus, B., Olesen, P. and Petersen, J.L., *Nucl. Phys.* B198, 157; B201, 176 1982.
15. Selberg, A., *Journ. Ind. Math Soc.* 20, 47 (1956).
16. Bers, L. *Bull. Amer. Math Soc.* 5, 131 (1981).

Figure Captions

Fig. 1. A string diagram.

Fig. 2. The z -plane of the conformally transformed string diagram.

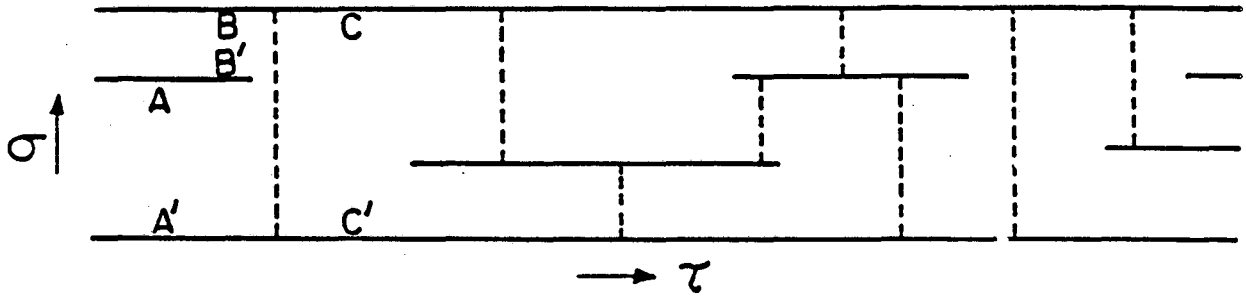


Figure 1

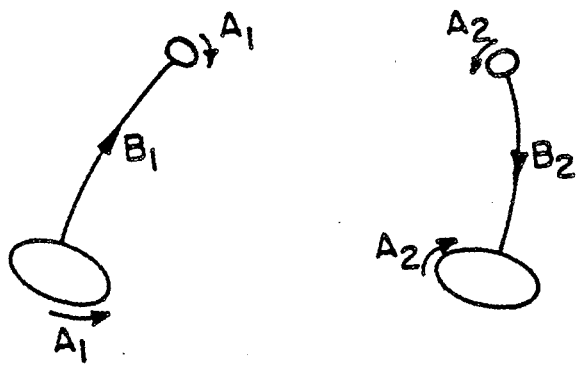


Figure 2

This report was done with support from the Department of Energy. Any conclusions or opinions expressed in this report represent solely those of the author(s) and not necessarily those of The Regents of the University of California, the Lawrence Berkeley Laboratory or the Department of Energy.

Reference to a company or product name does not imply approval or recommendation of the product by the University of California or the U.S. Department of Energy to the exclusion of others that may be suitable.

*LAWRENCE BERKELEY LABORATORY
TECHNICAL INFORMATION DEPARTMENT
UNIVERSITY OF CALIFORNIA
BERKELEY, CALIFORNIA 94720*