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Performance Analysis of the MUSIC and Pencil-MUSIC Algorithms for Diversely Polarized Array

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***Abstract*—This paper presents an asymptotical analysis of the MUSIC and Pencil-MUSIC methods for estimating 2-D angles and polarizations using crossed dipoles. The explicit first order expressions for the variances of the MUSIC and Pencil-MUSIC estimates are derived. Both the theoretical and simulation results are used to analyze and compare the performances of the MUSIC and Pencil-MUSIC methods. A number of new insights into the two methods are revealed. In particular, the MUSIC and Pencil-MUSIC methods are shown to have comparable performances near the Cramer–Rao bound, although the latter is much more efficient in computation than the former.**

I. Introduction

SINCE the diversity in signal polarization can be exploited to improve the accuracy of angle estimates, using diversely polarized arrays for angle estimation has recently attracted considerable attention. As a result, a number of angle estimation techniques have been developed for diversely-polarized array, which include specialized versions of the MUSIC [1], [3] and ESPRIT [4]–[7], and ML [2], [8], as well as the Cramer–Rao bound analysis [9], [10]. Both the MUSIC and ESPRIT techniques require fewer computations than the ML. But, for coherent signals, the MUSIC [1], [3] and ESPRIT [4]–[7] fail. However, the MUSIC and Pencil-MUSIC methods proposed in [11] circumvent the coherent case.

We note that the "pencil" appears to be a key concept which connects a class of estimation methods such as the pencil-of-function method [19], the matrix pencil method [20], ESPRIT [21], the state-space method [22] and the SURE method [12]. The Pencil-MUSIC will remind one of the Root-MUSIC [23]. While the Root-MUSIC method can be modified and applied to the 2-D estimation problem shown in this paper, the Pencil-MUSIC method is much more efficient in computation because it solves a generalized eigenvalue problem of much smaller size than the polynomial rooting required by the Root-MUSIC. Another distinction between the Pencil-MUSIC and Root-MUSIC is that the former uses the signal subspace and the latter uses the noise subspace.

Although much work has been done in analyzing 1-D estimation techniques [12], [13], etc., there has been no comparable effort being put into the analysis of 2-D estimation methods. In this paper, we study two 2-D angle estimation methods: the MUSIC and Pencil-MUSIC methods [11]. For the MUSIC method, the analysis is made by means of the Taylor series expansion of a multivariate function, which is a generalization of the methodology used in [12], [13]. The analysis for the Pencil-MUSIC method is a further extension of the approach used for the SURE method in [12]. A better understanding of the MUSIC and Pencil-MUSIC methods is obtained as a result of the analysis.

The outline of the paper is as follows. Section II includes the problem considered in [11] and the introduction of the enhanced covariance matrix. Section III presents a brief review of the MUSIC and Pencil-MUSIC methods, a discussion on the consistency of the two methods and a couple of preliminary results. Section IV contains theoretical results for the MUSIC and Pencil-MUSIC estimates. Section V discusses numerical examples. Section VI is the conclusion.

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II. Problem Formulation

Consider a rectangular array consisting of $M \times N$ pairs of crossed dipoles, in the X–Y plane, with (1, 1) dipole pair situated in the origin of the coordinate system. Interelement distances along x-direction and y-direction are Δ_x and Δ_y, respectively. Suppose that I narrowband plane waves, centered at frequency ω_0, with elevations $\theta_1, \cdots, \theta_I$, and azimuths $\phi_1, \cdots, \phi_I$, impinge on the array. Denote by $x(m, n, t)$ and $y(m, n, t)$, respectively, the signal outputs at the (m, n)th x-dipole (parallel to the x-axis) and y-dipole (parallel to the y-axis), and by $w_x(m, n, t)$ and $w_y(m, n, t)$, respectively, the noise at the same x- and y-dipoles, for the tth snapshot. Noisy outputs at the (m, n)th x- and y-dipoles can be represented as [11]

$$\hat{x}(m, n, t) \triangleq x(m, n, t) + w_x(m, n, t) = \sum_{i=1}^{I} e_{xi} p_i^{m-1} q_i^{n-1} s_i(t) + w_x(m, n, t) \tag{1}$$

$$\hat{y}(m, n, t) \triangleq y(m, n, t) + w_y(m, n, t) = \sum_{i=1}^{I} e_{yi} p_i^{m-1} q_i^{n-1} s_i(t) + w_y(m, n, t)$$

$$m = 1, \cdots, M, \quad n = 1, \cdots, N, \quad t = 1, \cdots, T \tag{2}$$

where for $i = 1, \cdots, I$

$$e_{xi} = \cos\theta_i \cos\phi_i \sin\gamma_i \exp\{j\eta_i\} - \sin\phi_i \cos\gamma_i \tag{3}$$

$$e_{yi} = \cos\theta_i \sin\phi_i \sin\gamma_i \exp\{j\eta_i\} - \cos\phi_i \cos\gamma_i \tag{4}$$

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$$p_i = \exp\{-j\Delta_x\omega_0 \sin\theta_i \cos\phi_i\} \quad (5)$$
$$q_i = \exp\{-j\Delta_y\omega_0 \sin\theta_i \sin\phi_i\} \quad (6)$$

and $s_i(t)$ is the random signal. γ_i and η_i are polarization parameters for the ith signal.

The ranges of parameters for the above data model should be restricted. For the identifiability of $\eta_i, \theta_i \neq \pi/2$ and $\gamma_i \neq 0$, and $\gamma_i \neq \pi/2$ when using the Pencil-MUSIC method. Later we will see that, in the Pencil-MUSIC method, θ_i and ϕ_i are computed from p_i and q_i. For the identifiability of $\phi_i, \theta_i \neq 0$; to avoid ambiguity in θ_i and $\phi_i, 0 < \Delta_x, \Delta_y < \pi/\omega_0$ and θ_i is assumed in the upper half plane. Due to the above consideration, we confine the ranges of θ_i, γ_i to $0 < \theta_i < \pi/2$ and $0 < \gamma_i < \pi/2$. The ranges of ϕ_i, η_i are unchanged, $-\pi \leq \phi_i \leq \pi$ and $-\pi \leq \eta_i \leq \pi$.

Define $2MN$-element column vectors

$$\begin{aligned}\mathbf{z}(t) &= [\mathrm{vec}(x(m,n,t)), \mathrm{vec}(y(m,n,t))]^T \\ \mathbf{w}(t) &= [\mathrm{vec}(w_x(m,n,t)), \mathrm{vec}(w_y(m,n,t))]^T \end{aligned} \quad (7)$$

with ^{T} denoting transpose and $\mathrm{vec}(r(m,n,t))$ an MN-element row vector defined as

$$\begin{aligned}&\mathrm{vec}(r(m,n,t)) \\ &\quad = [r(1,1,t), r(2,1,t), \cdots, r(M,1,t), r(1,2,t), \\ &\quad r(2,2,t), \cdots, r(M,2,t), \cdots, r(1,N,t), \\ &\quad r(2,N,t), \cdots, r(M,N,t)]. \end{aligned} \quad (8)$$

Then the array output has the vector form

$$\hat{\mathbf{z}}(t) \triangleq \mathbf{z}(t) + \mathbf{w}(t) = \mathbf{A}'\mathbf{s}(t) + \mathbf{w}(t) \quad (9)$$

where

$$\mathbf{A}' = \begin{bmatrix} e_{x1}\mathbf{q}_1' \otimes \mathbf{p}_1' & \cdots & e_{xI}\mathbf{q}_I' \otimes \mathbf{p}_I' \\ e_{y1}\mathbf{q}_1' \otimes \mathbf{p}_1' & \cdots & e_{yI}\mathbf{q}_I' \otimes \mathbf{p}_I' \end{bmatrix}_{2MN \times I} \quad (10)$$
$$\mathbf{s}(t) = [s_1(t), \cdots, s_I(t)]^T \quad (11)$$

with

$$\mathbf{p}_i' = [1, p_i, \cdots, p_i^{M-1}]^T \quad (12)$$
$$\mathbf{q}_i' = [1, q_i, \cdots, q_i^{N-1}]^T \quad (13)$$

and $\otimes$ denoting the Kronecker product [17] defined as

$$\mathbf{X}_{m\times n} \otimes \mathbf{Y}_{k\times l} = \begin{bmatrix} x_{1,1}\mathbf{Y} & \cdots & x_{1,n}\mathbf{Y} \\ \vdots & \ddots & \vdots \\ x_{m,1}\mathbf{Y} & \cdots & x_{m,n}\mathbf{Y} \end{bmatrix}_{mk\times nl}. \quad (14)$$

Signals and noise are assumed circular Gaussian (see [18] for the definition). $\mathbf{s}(t)$ for different t are taken to be independent random vectors. $s_k(t)$ and $s_l(t) (1 \leq k, l \leq I)$ may be fully dependent on each other, which is the so-called coherent case. We assume $E[\mathbf{s}(t)\mathbf{s}^H(t)]$ are identical for different t (i.e., stationary) and unknown. The noise voltages $\mathbf{w}(t), t = 1, \cdots, T$ are independent with zero means and covariance matrix $\sigma^2\mathbf{I}_{2MN}$. Furthermore, the signals are assumed to be uncorrelated with noise.

Let

$$\mathbf{X}_e^l(t) = \begin{bmatrix} x(1,l,t) & x(2,l,t) & \cdots & x(M-J+1,l,t) \\ x(2,l,t) & x(3,l,t) & \cdots & x(M-J+2,l,t) \\ \vdots & \vdots & \ddots & \vdots \\ x(J,l,t) & x(J+1,l,t) & \cdots & x(M,l,t) \end{bmatrix}, \quad l = 1, \cdots, N \quad (15)$$

be a $J \times (M-J+1) (1 \leq J \leq M)$ Hankel matrix and (16), at the bottom of this page be a $K \times (N-K+1) (1 \leq K \leq N)$ Hankel block matrix. $\mathbf{Y}_e(t), \mathbf{W}_{ex}(t)$ and $\mathbf{W}_{ey}(t)$ are defined in the same way as $\mathbf{X}_e(t)$. Introduce

$$\mathbf{Z}_e(t) \triangleq \begin{bmatrix} \mathbf{X}_e(t) \\ \mathbf{Y}_e(t) \end{bmatrix}_{2JK \times (M-J+1)(N-K+1)} \quad (17)$$
$$\mathbf{W}_e(t) \triangleq \begin{bmatrix} \mathbf{W}_{ex}(t) \\ \mathbf{Y}_{ey}(t) \end{bmatrix}_{2JK \times (M-J+1)(N-K+1)} \quad (18)$$

the enhanced covariance matrix defined in [11] can be expressed as

$$\mathbf{R}_e = E[(\mathbf{Z}_e(t) + \mathbf{W}_e(t))(\mathbf{Z}_e^H(t) + \mathbf{W}_e^H(t))] \quad (19)$$

where ^{H} and E denote conjugate transpose and statistical expectation, respectively.

Let $\sigma_1, \sigma_2, \cdots, \sigma_{2JK}$ (arranged in decreasing order) be eigenvalues of $\mathbf{R}_e$, $\mathbf{U}_s$ and $\mathbf{U}_n$ be its orthonormal eigenvectors corresponding to $(\sigma_1, \sigma_2, \cdots, \sigma_I)$ and $(\sigma_{I+1}, \sigma_{I+2}, \cdots, \sigma_{2JK})$ respectively, and

$$L = (M-J+1)(N-K+1) \quad (20)$$

we can write

$$\mathbf{R}_e = E[\mathbf{Z}_e(t)\mathbf{Z}_e^H(t)] + L\sigma^2\mathbf{I}_{2JK} = \mathbf{U}_s\mathbf{\Sigma}_s\mathbf{U}_s^H + \mathbf{U}_n\mathbf{\Sigma}_n\mathbf{U}_n^H \quad (21)$$

where

$$\mathbf{\Sigma}_s = \mathrm{diag}[\sigma_1, \sigma_2, \cdots, \sigma_I]$$

and

$$\begin{aligned}\mathbf{\Sigma}_n &= \mathrm{diag}\ [\sigma_{I+1}, \sigma_{I+2}, \cdots, \sigma_{2JK}] \\ &= \mathrm{diag}[L\sigma^2, L\sigma^2, \cdots, L\sigma^2].\end{aligned}$$

We assume that (1) $I \leq \min(M-J+1, J, N-K+1, K)$[1] and (2) A (defined in Appendix A, having more rows than

[1] The choice of J and K also affects the noise robustness of the MUSIC and Pencil-MUSIC. However, it is not easy to determine the optimum J and K which lead to the minimum estimation variance, even for one-signal case. More details can be found in [20].

$$\mathbf{X}_e(t) = \begin{bmatrix} \mathbf{X}_e^1(t) & \mathbf{X}_e^2(t) & \cdots & \mathbf{X}_e^{N-K+1}(t) \\ \mathbf{X}_e^2(t) & \mathbf{X}_e^3(t) & \cdots & \mathbf{X}_e^{N-K+2}(t) \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{X}_e^K(t) & \mathbf{X}_e^{K+1}(t) & \cdots & \mathbf{X}_e^N(t) \end{bmatrix}_{JK\times(M-J+1)(N-K+1)} \quad (16)$$

columns) has full column rank (a sufficient condition[2] is presented in Appendix B). Then $E[(\mathbf{Z}_e(t)(\mathbf{Z}_e^H(t)]$ will be of rank I irrespective of coherency among signals (see [11] for a proof). We further assume that the first I principal eigenvalues of $\mathbf{R}_e$ are distinct.

In practical situations, $\mathbf{R}_e$ is not available and we can only obtain

$$\hat{\mathbf{R}}_e \triangleq \frac{1}{T}\sum_{t=1}^{T}(\mathbf{Z}_e(t)+\mathbf{W}_e(t))(\mathbf{Z}_e^H(t)+\mathbf{W}_e^H(t)) \triangleq \hat{\mathbf{U}}_s\hat{\mathbf{\Sigma}}_s\hat{\mathbf{U}}_s^H+\hat{\mathbf{U}}_n\hat{\mathbf{\Sigma}}_n\hat{\mathbf{U}}_n^H \tag{22}$$

where $\hat{\sigma}_1 \geq \hat{\sigma}_2 \geq \cdots \geq \hat{\sigma}_I \geq \hat{\sigma}_{I+1} \geq \hat{\sigma}_{I+2} \geq \cdots \geq \hat{\sigma}_{2JK}$ (arranged in decreasing order) are eigenvalues of $\hat{\mathbf{R}}_e, \hat{\mathbf{\Sigma}}_s = \mathrm{diag}[\hat{\sigma}_1, \hat{\sigma}_2, \cdots, \hat{\sigma}_I], \hat{\mathbf{\Sigma}}_n = \mathrm{diag}[\hat{\sigma}_{I+1}, \hat{\sigma}_{I+2}, \cdots, \hat{\sigma}_{2JK}]$, $\hat{\mathbf{U}}_s$ and $\hat{\mathbf{U}}_n$ are orthonormal eigenvectors corresponding to $\hat{\mathbf{\Sigma}}_s$ and $\hat{\mathbf{\Sigma}}_n$, respectively.

By multivariate central limit theorem, $\lim_{T\to\infty}\hat{\mathbf{R}}_e = \mathbf{R}_e$. From [16] (Theorem 1.1, p. 167) $\hat{\mathbf{\Sigma}}_s \overset{T\to\infty}{\rightarrow} \mathbf{\Sigma}_s, \hat{\mathbf{\Sigma}}_n \overset{T\to\infty}{\rightarrow} \mathbf{\Sigma}_n$. Since $\mathbf{U}_s$ contain eigenvectors (associated with simple eigenvalues) of $\mathbf{R}_e$, from [15, pp. 293–295], $\hat{\mathbf{U}}_s \overset{T\to\infty}{\rightarrow} \mathbf{U}_s$. $\hat{\mathbf{U}}_n\hat{\mathbf{U}}_n^H = \mathbf{I}_{2JK} - \hat{\mathbf{U}}_s\hat{\mathbf{U}}_s^H \overset{T\to\infty}{\rightarrow} \mathbf{I}_{2JK} - \mathbf{U}_s\mathbf{U}_s^H = \mathbf{U}_n\mathbf{U}_n^H$. (Note that one can NOT write $\hat{\mathbf{U}}_n \overset{T\to\infty}{\rightarrow} \mathbf{U}_n$ which has sometimes been mistakenly used in our literature.)

III. Preliminary Results

Since $\mathbf{U}_s$ and $\mathbf{A}$ (of full column rank, defined in Appendix A) belong to the same range, $\mathbf{U}_n^H\mathbf{U}_s = \mathbf{0}$ we obtain the following well-understood properties of $\mathbf{R}_e$.

Property 1: $\mathbf{U}_s\mathbf{B} = \mathbf{A}$, where $\mathbf{B}$ is an $I \times I$ nonsingular matrix.

Property 2: $\mathbf{U}_n^H\mathbf{A} = \mathbf{0}$.

A. MUSIC [11]

Property 2 is the basis of the MUSIC method. Let $\mathbf{a}(\theta_i, \phi_i, \gamma_i, \eta_i)$ denote the ith column of $\mathbf{A}$, we know

$$\mathbf{a}^H(\theta_i, \phi_i, \gamma_i, \eta_i)\mathbf{U}_n\mathbf{U}_n^H\mathbf{a}(\theta_i, \phi_i, \gamma_i, \eta_i) = \mathbf{0}. \tag{23}$$

The MUSIC estimates $\hat{\theta}_i, \hat{\phi}_i, \hat{\gamma}_i$ and $\hat{\eta}_i$ are defined as the locations of the I lowerest valleys of the following MUSIC null spectrum function

$$f(\theta, \phi, \gamma, \eta) = \mathbf{a}^H(\theta, \phi, \gamma, \eta)\hat{\mathbf{U}}_n\hat{\mathbf{U}}_n^H\mathbf{a}(\theta, \phi, \gamma, \eta). \tag{24}$$

(The apparent 4-D searching required here can be reduced to 2-D searching [1], [11].)

It is worth mentioning that (24) does not guarantee consistent polarization estimates in general. For example, given that $\theta_i = \theta_j$ and $\phi_i = \phi_j$ for $i \neq j$ (this case is allowed when using polarization-sensitive array), we have $\mathbf{a}_i^H\mathbf{U}_n = [e_{xi}^*\mathbf{q}_i^H \otimes \mathbf{p}_i^H, e_{yi}^*\mathbf{q}_i^H \otimes \mathbf{p}_i^H]\mathbf{U}_n = \mathbf{0}$ and $\mathbf{a}_j^H\mathbf{U}_n = [e_{xj}^*\mathbf{q}_i^H \otimes \mathbf{p}_i^H, e_{yj}^*\mathbf{q}_i^H \otimes \mathbf{p}_i^H]\mathbf{U}_n = \mathbf{0}$. Here we assume $e_{yi}/e_{xi} \neq e_{yj}/e_{xj}$, otherwise $\mathbf{a}_i$ and $\mathbf{a}_j$ have the same polarizations. Then $\mathbf{a}' = c_1\mathbf{a}_i + c_2\mathbf{a}_j = [(c_1e_{xi}+c_2e_{xj})\mathbf{q}_i^T \otimes \mathbf{p}_i^T, (c_1e_{yi}+c_2e_{yj})\mathbf{q}_i^T \otimes \mathbf{p}_i^T]^T$ is also orthogonal to $\mathbf{U}_n$ where $c_1 \neq 0$ and $c_2 \neq 0$. Using the fact that (a) $e_{yi}/e_{xi}, e_{yj}/e_{xj}$ and $(c_1e_{yi}+c_2e_{yj})/(c_1e_{xi}+c_2e_{xj})$ are three unequal quantities, and (b) $\mathbf{a}', \mathbf{a}_i$ and $\mathbf{a}_j$ have the identical 2-D angle pairs, it is readily shown that $\mathbf{a}'$ has polarizations different from (γ_i, η_i) and (γ_j, η_j). Thus minimizing function (24) may give polarization estimates with large variances even if the exact enhanced matrix $\mathbf{R}_e$ is available. This is due to the structure of $\mathbf{A}$ arising from using arrays of crossed dipoles.

[2] This condition for 1-D case was also pointed out in [7] but without proof.

For $i = 1, 2, \cdots, I$, define 4×4 matrix

$$\mathbf{R}_i = \Re e\{\mathbf{\Omega}_i^H\mathbf{U}_n\mathbf{U}_n^H\mathbf{\Omega}_i\} \tag{25}$$

where $\mathbf{\Omega}_i = [\frac{\partial\mathbf{a}(\theta_i,\phi_i,\gamma_i,\eta_i)}{\partial\theta}, \frac{\partial\mathbf{a}(\theta_i,\phi_i,\gamma_i,\eta_i)}{\partial\phi}, \frac{\partial\mathbf{a}(\theta_i,\phi_i,\gamma_i,\eta_i)}{\partial\gamma}, \frac{\partial\mathbf{a}(\theta_i,\phi_i,\gamma_i,\eta_i)}{\partial\eta}]_{2JK\times 4}$.

It is easy to see that positive definite $\mathbf{R}_i$'s ensure that the MUSIC (2-D angles and polarization) estimates are consistent. Our analysis of MUSIC is conducted under the condition that $\mathbf{R}_i$ *is positive definite*.

B. Pencil-MUSIC [11]

The Pencil-MUSIC estimates [11] are provided by the generalized eigenvalues of certain matrix pencils. Before describing the Pencil-MUSIC method, we define

$$\pi_i = \frac{e_{yi}}{e_{xi}}, \quad i = 1, \cdots, I \tag{26}$$

and let $z_{i,1}, z_{i,2}$ and $z_{i,3}$ denote p_i, q_i and π_i, respectively. Unlike MUSIC, the Pencil-MUSIC method first estimates p_i, q_i and π_i and then converts them to the estimates of $\theta_i, \phi_i, \gamma_i$ and η_i.

Define $\Phi_k = \mathrm{diag}[z_{1,k}, \cdots, z_{I,k}]$ and the following six submatrices of $\mathbf{U}_s$:

$$\mathbf{U}_{11} = [\mathbf{U}_s \text{ with its } kJ\text{th rows deleted}, k = 1, \cdots, 2K]_{2(J-1)K\times I}$$

$$\mathbf{U}_{12} = [\mathbf{U}_s \text{ with its } ((k-1)J+1)\text{th rows deleted}, k = 1, \cdots, 2K]_{2(J-1)K\times I}$$

$$\mathbf{U}_{21} = [\mathbf{U}_s \text{ with its } ((K-1)J+j)\text{th and } (K-1)J+j + JK)\text{th rows deleted}, j = 1, \cdots, J]_{2J(K-1)\times I}$$

$$\mathbf{U}_{22} = [\mathbf{U}_s \text{ with its } j\text{th and} (j+JK)\text{th rows deleted}, j = 1, \cdots, J]_{2J(K-1)\times I}$$

$$\mathbf{U}_{31} = [\mathbf{U}_s \text{ with its last } JK \text{ rows deleted}]_{JK\times I}$$

$$\mathbf{U}_{32} = [\mathbf{U}_s \text{ with its first } JK \text{ rows deleted}]_{JK\times I}$$

and let $\mathbf{A}_{kl}, k = 1, 2, 3, l = 1, 2$, be submatrices of $\mathbf{A}$ defined in the same way as $\mathbf{U}_{kl}$.

$\mathbf{U}_{kl}, k = 1, 2, 3, l = 1, 2$ (having more rows than columns) are required [11] to be of full column rank to guarantee the consistency of the Pencil-MUSIC estimates. Since $\mathbf{U}_{kl} = \mathbf{A}_{kl}\mathbf{B}$ (from Property 1) and $\mathbf{B}$ is nonsingular, $\mathbf{U}_{kl}$ and $\mathbf{A}_{kl}$ have the same rank. Under the condition stated in Theorem B1 in Appendix B, it can be proved that, $\mathbf{A}_{kl}(\mathbf{U}_{kl}), k = 1, 2, l = 1, 2$, are of full column rank. The requirement that $\mathbf{A}_{3l}(\mathbf{U}_{3l}), l = 1, 2$, are of full column rank is more demanding. Making use of Lemma B1 in Appendix B, we can prove that if there are no identical 2-D angle pairs, $\mathbf{A}_{3l}(\mathbf{U}_{3l}), l = 1, 2$, are of full column rank, and so are $\mathbf{A}_{kl}(\mathbf{U}_{kl}), k = 1, 2, l =$

1, 2. Our analysis of Pencil-MUSIC is conducted under the condition that *there are no identical 2-D angle pairs.* Note that this condition is sufficient for the Pencil-MUSIC to be consistent but not for the MUSIC.

Since

$$\mathbf{A}_{k2} = \mathbf{A}_{k1}\Phi_k \tag{27}$$

(see [11] for a proof) and $\mathbf{U}_{k1} = \mathbf{A}_{k1}\mathbf{B}^{-1}$ and $\mathbf{U}_{k2} = \mathbf{A}_{k2}\mathbf{B}^{-1}$ (from Property 1), then

$$\begin{aligned}\mathbf{F}_k &\triangleq (\mathbf{U}_{k1}^H\mathbf{U}_{k1})^{-1}\mathbf{U}_{k1}^H\mathbf{U}_{k2}\\ &= \mathbf{B}(\mathbf{A}_{k1}^H\mathbf{A}_{k1})^{-1}\mathbf{A}_{k1}^H\mathbf{A}_{k2}\mathbf{B}^{-1}\\ &= \mathbf{B}\Phi_k\mathbf{B}^{-1}.\end{aligned} \tag{28}$$

This means $z_{i,k}, i = 1, \cdots, I$, are eigenvalues of $\mathbf{F}_k$. Thus, the Pencil-MUSIC [11] estimates $\hat{z}_{i,k}, i = 1, \cdots, I, k = 1, 2, 3$, are defined as the eigenvalues of

$$\hat{\mathbf{F}}_k \triangleq (\hat{\mathbf{U}}_{k1}^H\hat{\mathbf{U}}_{k1})^{-1}\hat{\mathbf{U}}_{k1}^H\hat{\mathbf{U}}_{k2} \tag{29}$$

where $\hat{\mathbf{U}}_{kl}$ are submatrices of $\hat{\mathbf{U}}_s$ defined in the same way as $\mathbf{U}_{kl}$.

$-\Delta_x\omega_0 \sin\hat{\theta}_i \cos\hat{\phi}_i$ and $-\Delta_y\omega_0 \sin\hat{\theta}_i \sin\hat{\phi}_i$ are the phases of complex variables $\hat{p}_i$ and $\hat{q}_i$, respectively. Strictly speaking, $\hat{p}_i, \hat{q}_i$ and $\hat{\pi}_i$ need to be paired to complete the 2-D estimation which was discussed in [11]. The pairing issue is not considered in this paper.

C. Glossary of Lemmas

Due to the overlapping of noise in $\hat{\mathbf{Z}}_e(t)$, the statistical results on the perturbations of eigenvectors in [14] is not applicable here. Lemma 1 is thus developed concerning the perturbations of the first I principal eigenvectors of $\mathbf{R}_e$.

Lemma 1: $\mathbf{U}_n^H\Delta\mathbf{U}_s \approx \frac{1}{T}\sum_{t=1}^T \mathbf{U}_n^H(\mathbf{W}_e(t)\mathbf{Z}_e^H(t) + \mathbf{W}_e(t)\mathbf{W}_e^H(t))\mathbf{U}_s\Lambda_s^{-1}$, where $\Delta\mathbf{U}_s \triangleq \hat{\mathbf{U}}_s - \mathbf{U}_s, \Lambda_s = \text{diag}[\sigma_1 - L\sigma^2, \cdots, \sigma_I - L\sigma^2]$ (L is defined by (20)) and $\approx$ means keeping the first-order perturbed terms in equations.

Proof: Let $\mathbf{U}_s' = [\mathbf{u}_1, \cdots, \mathbf{u}_{i-1}, \mathbf{u}_{i+1}, \cdots, \mathbf{u}_I]$ (a submatrix of $\mathbf{U}_s$ with its ith column deleted) and $\Lambda' = \text{diag}[\sigma_i - \sigma_1, \cdots, \sigma_i - \sigma_{i-1}, \sigma_i - \sigma_{i+1}, \cdots \sigma_i - \sigma_I, \sigma_i - L\sigma^2, \cdots, \sigma_i - L\sigma^2]$. Since $\sigma_i, i = 1, \cdots, I$, are simple eigenvalue of $\mathbf{R}_e$, from [15, pp. 293–295], the perturbations on the eigenvectors associated with $\sigma_i, 1 \le i \le I$, are given by

$$\begin{aligned}\Delta\mathbf{u}_i &\approx [\mathbf{U}_s'\mathbf{U}_n](\Lambda')^{-1}\begin{bmatrix}\mathbf{U}_s'^H\\ \mathbf{U}_n^H\end{bmatrix}(\hat{\mathbf{R}}_e - \mathbf{R})\mathbf{u}_i\\ &= [\mathbf{U}_s'\mathbf{U}_n](\Lambda')^{-1}\begin{bmatrix}\mathbf{U}_s'^H\\ \mathbf{U}_n^H\end{bmatrix}\hat{\mathbf{R}}_e\mathbf{u}_i.\end{aligned} \tag{30}$$

Then

$$\mathbf{U}_n^H\Delta\mathbf{u}_i \approx \frac{1}{\sigma_i - L\sigma^2}\mathbf{U}_N^H\hat{\mathbf{R}}_e\mathbf{u}_i \tag{31}$$

in other words

$$\mathbf{U}_n^H\Delta\mathbf{U}_s \approx \mathbf{U}_N^H\hat{\mathbf{R}}_e\mathbf{U}_s\Lambda_s^{-1}. \tag{32}$$

From Appendix A, $\mathbf{U}_n^H\mathbf{Z}_e(t) = \mathbf{U}_n^H\mathbf{A}\Psi(t) = 0$, then substituting $\hat{\mathbf{R}}_e$ with (22) into the above equation, yields (33), at the bottom of this page. □

It should be noted that, to the first order approximation, $\hat{\mathbf{u}}_i(= \mathbf{u}_i + \Delta\mathbf{u}_i)$ is still of unit norm. A similar expression to (32) was developed in [12] via a different route. Lemma 1 is useful to express errors of the MUSIC and Pencil-MUSIC estimates in the next section.

The columns of $\mathbf{Z}_e(t)$ and $\mathbf{W}_e(t)$ can be expressed in $\mathbf{z}(t)$ and $\mathbf{w}(t)$ respectively as shown below.

Lemma 2: Let $\mathbf{w}_s(t)$ be the sth column of $\mathbf{W}_e(t)$ and $\mathbf{z}_s(t)$ be the sth column of $\mathbf{Z}_e(t)$, for $1 \le s \le L$. Then (1) $\mathbf{w}_s(t) = \mathbf{D}_s\mathbf{w}(t)$ and (2) $\mathbf{z}_s(t) = \mathbf{D}_s\mathbf{z}(t)$, where $\mathbf{D}_s$ is defined in Appendix C.

Proof: See Appendix C. □

Lemma 2 will be used in the proofs of Lemmas 3 and 4. Next Lemma involves the calculation of the fourth order moment of Gaussian noise vectors, which is also needed in the proof of Lemma 4.

Lemma 3: Let $\mathbf{c}_1, \mathbf{c}_2, \mathbf{c}_3$ and $\mathbf{c}_4$ are deterministic $2JK$-element column vectors, satisfying $\mathbf{c}_1^H\mathbf{c}_2 = 0$ or $\mathbf{c}_3^H\mathbf{c}_4 = 0$. Then for $1 \le s, r \le L$

$$\begin{aligned}&E[\mathbf{c}_1^H\mathbf{w}_s(t)\mathbf{w}_s^H(t)\mathbf{c}_2\mathbf{c}_3^H\mathbf{w}_r(t)\mathbf{w}_r^H(t)\mathbf{c}_4]\\ &\quad= \sigma^4\mathbf{c}_1^H\mathbf{D}_s\mathbf{D}_r^T\mathbf{c}_4\mathbf{c}_3^H\mathbf{D}_r\mathbf{D}_s^T\mathbf{c}_2\end{aligned} \tag{34}$$

where $\mathbf{w}_s(t)(\mathbf{w}_r(t))$ is as defined in Lemma 2.

Proof: See Appendix D. □

Since the signal covariance matrix $E[\mathbf{s}(t)\mathbf{s}^H(t)]$ is assumed to be stationary, we can write (for example, from eigendecomposition) $E[\mathbf{s}(t)\mathbf{s}^H(t)] = \mathbf{C}\mathbf{C}^H$, where $\mathbf{C}$ is an $I \times r_s(1 \le r_s \le I)$ matrix of full column rank with r_s denoting the rank of $E[\mathbf{s}(t)\mathbf{s}^H(t)]$. Then, from (9)

$$\mathbf{R}_z \triangleq E[\mathbf{z}(t)\mathbf{z}^H(t)] = \mathbf{A}'E[\mathbf{s}(t)\mathbf{s}^H(t)]\mathbf{A}'^H = \mathbf{A}'\mathbf{C}\mathbf{C}^H\mathbf{A}'^H. \tag{35}$$

Define $2JK \times 2JK$ matrices

$$\mathbf{G}_{i\ell} = \sum_{s=1}^{L}\mathbf{d}_{si}\mathbf{c}_\ell^H\mathbf{A}'^H\mathbf{D}_s^T \tag{36}$$

$$H_{ij} = \sum_{s=1}^{L}\mathbf{d}_{si}\mathbf{d}_{sj}^T \tag{37}$$

$$1 \le i, j \le 2MN, 1 \le \ell \le r_s$$

where $\mathbf{d}_{si}$ denotes the ith column of $\mathbf{D}_s$ and $\mathbf{c}_\ell$ the ℓth column of $\mathbf{C}$. With the introduction of $\mathbf{G}_{i\ell}$ and H_{ij}, we are able to present the following lemma that will be used in the proofs of Theorems 1 and 2, in the next section.

$$\begin{aligned}\mathbf{U}_n^H\Delta\mathbf{U}_s &\approx \frac{1}{T}\sum_{t=1}^{T}\mathbf{U}_n^H(\mathbf{Z}_e(t)\mathbf{Z}_e^H(t) + \mathbf{W}_e(t)\mathbf{Z}_e^H(t) + \mathbf{Z}_e(t)\mathbf{W}_e^H(t) + \mathbf{W}_e(t)\mathbf{W}_e^H(t))\mathbf{U}_s\Lambda_s^{-1}\\ &= \frac{1}{T}\sum_{t=1}^{T}\mathbf{U}_n^H(\mathbf{W}_e(t)\mathbf{Z}_e^H(t) + \mathbf{W}_e(t)\mathbf{W}_e^H(t))\mathbf{U}_s\Lambda_s^{-1}.\end{aligned} \tag{33}$$

Lemma 4: Let $\epsilon(t) = \mu^H(\mathbf{W}_e(t)\mathbf{Z}_e^H(t)+\mathbf{W}_e(t)\mathbf{W}_e^H(t))v$, where μ and v are orthogonal $2JK$-element column vectors. Then

$$(1)\quad E[\epsilon(t)\epsilon^*(t)] = \sigma^2 \sum_{ij=1,\ell=1}^{2MN,r_s} \{|\mu^H \mathbf{G}_{i\ell} v|^2 + \sigma^2 |\mu^H \mathbf{H}_{ij} v|^2\} \tag{38}$$

$$(2)\quad E[\epsilon(t)\epsilon(t)] = \sigma^4 \sum_{i,j=1}^{2MN} \{\mu^H \mathbf{H}_{ij} v \mu^H \mathbf{H}_{ji} v\} \tag{39}$$

$$(3)\quad E[\epsilon(t)] = 0 \tag{40}$$

where $*$ denotes conjugate.

Proof: See Appendix E. □

In the next section, we will see that each MUSIC or Pencil-MUSIC estimate error can be expressed as the real or imaginary part of a complex quantity which has the same form as $\epsilon(t)$ in Lemma 4. Combination of conclusions (1) and (2) gives the expression of estimate error variance and conclusion (3) indicates that the MUSIC and Pencil-MUSIC estimates are unbiased.

IV. Main Results

In this section, we will derive expressions for the variances of the MUSIC and Pencil-MUSIC estimates, by making use of Lemmas 1 and 4 developed in the previous section. For notational convenience, let $\omega_{i,1}, \omega_{i,2}, \omega_{i,3}$ and $\omega_{i,4}$ denote $\theta_i, \phi_i, \gamma_i$ and η_i respectively, estimate error $\Delta\omega_{i,k} = \hat{\omega}_{i,k} - \omega_{i,k}$, $\Re e\{\cdot\}$ and $\Im m\{\cdot\}$ be real and imaginary parts of complex numbers.

Theorem 1: The asymptotical ($T \gg 1$) MUSIC estimate errors $\Delta\omega_{i,k}$, $i = 1, \cdots, I, k = 1, 2, 3, 4$, have zero means and the variances

$$\mathrm{var}_{\mathrm{music}}(\Delta\omega_{i,k}) = \frac{\sigma^2}{2T} \sum_{i,j=1,\ell=1}^{2MN,r_s} (|\rho_{i,k}^H \mathbf{G}_{i\ell} v_i|^2 + \sigma^2 |\rho_{i,k}^H \mathbf{H}_{ij} v_i|^2 + \sigma^2 \Re e\{\rho_{i,k}^H \mathbf{H}_{ij} v_i \rho_{i,k}^H \mathbf{H}_{ji} v_i\}) \tag{41}$$

where $\rho_{i,k}^H$ is the kth row of $\mathbf{R}_i^{-1}\Omega_i^H \mathbf{U}_n \mathbf{U}_n^H$, $\mathbf{G}_{i\ell}$ and $\mathbf{H}_{ij}$ are previously defined by (36) and (37), $v_i = \mathbf{U}_s \Lambda_s^{-1} \mathbf{U}_s^H \mathbf{a}(\theta_i, \phi_i, \gamma_i, \eta_i)$, with $\mathbf{a}(\theta_i, \phi_i, \gamma_i, \eta_i)$ representing the ith row of $\mathbf{A}$. $\mathbf{R}_i$ is defined as in (25). (In Section 3, $\mathbf{R}_i^{-1}$ is assumed to exist.)

Proof: Let us first derive expressions of estimate errors. Since $\hat{\theta}_i, \hat{\phi}_i, \hat{\gamma}_i, \hat{\eta}_i$ are consistent estimates of $\theta_i, \phi_i, \gamma_i, \eta_i$ respectively, following the idea in [12], around $(\theta_i, \phi_i, \gamma_i, \eta_i)$, we obtain Taylor series expansion of $\frac{\partial f(\hat{\theta}_i, \hat{\phi}_i \hat{\gamma}_i, \hat{\eta}_i)}{\partial \theta}$ at the bottom of this page ($f(\theta, \phi, \gamma, \eta)$ is defined by (42)), where $\approx$ is as defined in Lemma 1.

From $\hat{\mathbf{U}}_n \hat{\mathbf{U}}_n^H \hat{\mathbf{U}}_s = 0$, we have $\hat{\mathbf{U}}_n \hat{\mathbf{U}}_n^H \hat{\mathbf{U}}_s = -\hat{\mathbf{U}}_n \hat{\mathbf{U}}_n^H \Delta\mathbf{U}_s \approx -\mathbf{U}_n \mathbf{U}_n^H \Delta\mathbf{U}_s$. Then

$$\begin{aligned}\frac{\partial f(\theta_i, \phi_i, \gamma_i, \eta_i)}{\partial \theta} &= 2\Re e\left\{\frac{\partial \mathbf{a}_i^H}{\partial \theta} \hat{\mathbf{U}}_n \hat{\mathbf{U}}_n^H \mathbf{a}_i\right\} \\ &= 2\Re e\left\{\frac{\partial \mathbf{a}_i^H}{\partial \theta} \hat{\mathbf{U}}_n \hat{\mathbf{U}}_n^H \mathbf{U}_s \mathbf{U}_s^H \mathbf{a}_i\right\} \\ &\approx -2\Re e\left\{\frac{\partial \mathbf{a}_i^H}{\partial \theta} \mathbf{U}_n \mathbf{U}_n^H \Delta\mathbf{U}_s \mathbf{U}_s^H \mathbf{a}_i\right\}\end{aligned} \tag{43}$$

and the constant (zero-order perturbed) terms of the second-order partial derivatives are given as follows

$$\begin{aligned}\left(\frac{\partial^2 f(\theta_i, \phi_i, \gamma_i, \eta_i)}{\partial \theta^2}\right)_0 &= \left(2\Re e\left\{\frac{\partial \mathbf{a}_i^H}{\partial \theta} \hat{\mathbf{U}}_n \hat{\mathbf{U}}_n^H \frac{\partial \mathbf{a}_i}{\partial \theta} + \frac{\partial^2 \mathbf{a}_i^H}{\partial \theta^2} \hat{\mathbf{U}}_n \hat{\mathbf{U}}_n^H \mathbf{a}_i\right\}\right)_0 \\ &= 2\Re e\left\{\frac{\partial \mathbf{a}_i^H}{\partial \theta} \mathbf{U}_n \mathbf{U}_n^H \frac{\partial \mathbf{a}_i}{\partial \theta} + \frac{\partial^2 \mathbf{a}_i^H}{\partial \theta^2} \mathbf{U}_n \mathbf{U}_n^H \mathbf{a}_i\right\} \\ &= 2\Re e\left\{\frac{\partial \mathbf{a}_i^H}{\partial \theta} \mathbf{U}_n \mathbf{U}_n^H \frac{\partial \mathbf{a}_i}{\partial \theta}\right\}\end{aligned} \tag{44}$$

$$\left(\frac{\partial^2 f(\theta_i, \phi_i, \gamma_i, \eta_i)}{\partial \phi \partial \theta}\right)_0 = 2\Re e\left\{\frac{\partial \mathbf{a}_i^H}{\partial \phi} \mathbf{U}_n \mathbf{U}_n^H \frac{\partial \mathbf{a}_i}{\partial \theta}\right\} \tag{45}$$

$$\left(\frac{\partial^2 f(\theta_i, \phi_i, \gamma_i, \eta_i)}{\partial \gamma \partial \theta}\right)_0 = 2\Re e\left\{\frac{\partial \mathbf{a}_i^H}{\partial \gamma} \mathbf{U}_n \mathbf{U}_n^H \frac{\partial \mathbf{a}_i}{\partial \theta}\right\} \tag{46}$$

$$\left(\frac{\partial^2 f(\theta_i, \phi_i, \gamma_i, \eta_i)}{\partial \eta \partial \theta}\right)_0 = 2\Re e\left\{\frac{\partial \mathbf{a}_i^H}{\partial \eta} \mathbf{U}_n \mathbf{U}_n^H \frac{\partial \mathbf{a}_i}{\partial \theta}\right\} \tag{47}$$

where $\mathbf{a}_i$ is the short notation of $\mathbf{a}(\theta_i, \phi_i, \gamma_i, \eta_i)$. Using (43)–(47) in (42), we obtain

$$0 \approx -2\Re e\left\{\frac{\partial \mathbf{a}_i^H}{\partial \theta} \mathbf{U}_n \mathbf{U}_n^H \Delta\mathbf{U}_s \mathbf{U}_s^H \mathbf{a}_i\right\} + 2r_{11}\Delta\theta_i + 2r_{12}\Delta\phi_i + 2r_{13}\Delta\gamma_i 2r_{14}\Delta\eta_i \tag{48}$$

with r_{kl} representing the (k, l)th element of $\mathbf{R}_i$.

Similar expressions can be derived from the Taylor series expansions of $\frac{\partial f(\hat{\theta}_i, \hat{\phi}_i, \hat{\gamma}_i, \hat{\eta}_i)}{\partial \phi}$, $\frac{\partial f(\hat{\theta}_i, \hat{\phi}_i, \hat{\gamma}_i, \hat{\eta}_i)}{\partial \gamma}$ and $\frac{\partial f(\hat{\theta}_i, \hat{\phi}_i, \hat{\gamma}_i, \hat{\eta}_i)}{\partial \eta}$. Omitting detailed derivations, we obtain

$$\mathbf{R}_i[\Delta\theta_i, \Delta\phi_i, \Delta\gamma_i, \Delta\eta_i]^T = \Re e\{\Omega_i^H \mathbf{U}_n \mathbf{U}_n^H \Delta\mathbf{U}_s \mathbf{U}_s^H \mathbf{a}_i\}. \tag{49}$$

Using Lemma 1, we get estimate errors

$$\begin{aligned}\Delta\omega_{i,k} &= \frac{1}{T} \sum_{t=1}^{T} \Re e\{\rho_{i,k}^H (\mathbf{W}_e(t)\mathbf{Z}_e^H(t) + \mathbf{W}_e(t)\mathbf{W}_e^H(t)) v_i\} \\ &\triangleq \frac{1}{T} \sum_{t=1}^{T} \Re e\{\epsilon_{i,k}(t)\}.\end{aligned} \tag{50}$$

$$\begin{aligned}\frac{\partial f(\hat{\theta}_i, \hat{\phi}_i, \hat{\gamma}_i, \hat{\eta}_i)}{\partial \theta} = 0 \approx \frac{\partial f(\theta_i, \phi_i \gamma_i, \eta_i)}{\partial \theta} + \Delta\theta_i \frac{\partial^2 f(\theta_i, \phi_i, \gamma_i, \eta_i}{\partial \theta^2} + \Delta\phi_i \frac{\partial^2 f(\theta_i, \phi_i, \gamma_i, \eta_i)}{\partial \phi \partial \theta} \\ + \Delta\gamma_i \frac{\partial^2 f(\theta_i, \phi_i, \gamma_i, \eta_i)}{\partial \gamma \partial \theta} + \Delta\eta_i \frac{\partial^2 f(\theta_i, \phi_i, \gamma_i, \eta_i)}{\partial \eta \partial \theta}\end{aligned} \tag{42}$$

Now we consider the error variances.

Observe that and $\rho_{i,k}$ and υ_i are orthogonal, Lemma 4 is applicable to $\epsilon_{i,k}(t)$. Therefore (a) $\Delta\theta_i, \Delta\phi_i, \Delta\gamma_i$ and $\Delta\eta_i$ have zero means, according to conclusion (3) of Lemma 4; (b) From $\Re e\{z_1\}\Re e\{z_2\} = \frac{1}{2}\Re e\{z_1 z_2^* + z_1 z_2\}$

$$\mathrm{var}_{\mathrm{music}}(\Delta\omega_{i,k}) = \frac{1}{2T^2}\sum_{s,t=1}^{T}\Re e\{E[\epsilon_{i,k}(s)\epsilon_{i,k}^*(t)] + E[\epsilon_{i,k}(s)\epsilon_{i,k}(t)]\}. \quad (51)$$

Since $\epsilon_{i,k}(s)$ and $\epsilon_{i,k}(t)$ are independent for $s \neq t$

$$\mathrm{var}_{\mathrm{music}}(\Delta\omega_{i,k}) = \frac{1}{2T}\Re e\{E[\epsilon_{i,k}(t)\epsilon_{i,k}^*(t)] + E[\epsilon_{i,k}(t)\epsilon_{i,k}(t)]\}. \quad (52)$$

Also from Lemma 4

$$\mathrm{var}_{\mathrm{music}}(\Delta\omega_{i,k}) = \frac{\sigma^2}{2T}\sum_{\imath,\jmath=1,\ell=1}^{2MN,r_s}\left(|\rho_{i,k}^H \mathbf{G}_{\imath\ell}\upsilon_i|^2 + \sigma^2|\rho_{i,k}^H \mathbf{H}_{\imath\jmath}\upsilon_i|^2 + \sigma^2\Re e\{\rho_{i,k}^H \mathbf{H}_{\imath\jmath}\upsilon_i \rho_{i,k}^H \mathbf{H}_{\jmath\imath}\upsilon_i\}\right). \quad (53)$$

□

Theorem 2: (1) For fixed i, if $p_i \neq p_j, q_i \neq q_j, j = 1, \cdots, I$ and $j \neq i$, the asymptotical ($T \gg 1$) Pencil-MUSIC estimate errors $\Delta\omega_{i,k}, k = 1, 2$, have zero means and the variances

$$\mathrm{var}_{\mathrm{pencil}}(\Delta\omega_{i,k}) = g_{i,k} \quad (54)$$

(2) For fixed i, if $p_i \neq p_j, q_i \neq q_j, \pi_i \neq \pi_j, j = 1, \cdots, I$ and $j \neq i$, the asymptotical ($T \gg 1$), Pencil-MUSIC estimate errors $\Delta\omega_{i,k}, k = 3, 4$, have zero means and the variances

$$\mathrm{var}_{\mathrm{pencil}}(\Delta\omega_{i,k}) = g_{i,k} \quad (55)$$

where for $i = 1, \cdots, I, k = 1, 2, 3, 4$

$$g_{i,k} = \frac{\sigma^2}{2T}\sum_{\imath,\jmath=1,\ell}^{2MN,r_s}\left(|\mu_{i,k}^H \mathbf{G}_{\imath\ell}\upsilon_i|^2 + \sigma^2\mu_{i,k}^H \mathbf{H}_{\imath\jmath}\upsilon_i|^2 - \sigma^2\Re e\{\mu_{i,k}^H \mathbf{H}_{\imath\jmath}\upsilon_i \mu_{i,k}^H \mathbf{H}_{\jmath\imath}\upsilon_i\}\right) \quad (56)$$

with $\mu_{i,k}^H = c_{k,1}(i)\tau_{i,1}^H + c_{k,2}(i)\tau_{i,2}^H + c_{k,3}(i)\tau_{i,3}^H$, $\mathbf{G}_{\imath\ell}$ and $H_{\imath\jmath}$ previously defined by (36) and (37), υ_i as defined in Theorem 1. $\tau_{i,l}^H$ denotes the ith row of $(\mathbf{A}_{l1}^H\mathbf{A}_{l1})^{-1}\mathbf{A}_{l1}^H(\mathbf{J}_{l2} - z_{i,l}\mathbf{J}_{l1})$ and $c_{k,l}(i) (l = 1, 2, 3)$ are given in Appendix F. $\mathbf{J}_{lj}, l = 1, 2, 3, j = 1, 2$, are selection matrices to represent the relations between $\mathbf{U}_{kj}$ and $\mathbf{U}_s$, and defined in Appendix G.

Proof: Let $\Delta\mathbf{F}_l = \hat{\mathbf{F}}_l - \mathbf{F}_l$($\mathbf{F}_l$ and $\hat{\mathbf{F}}_l$ are defined by (28) and (29)), κ_i^H be the ith row of $\mathbf{B}^{-1}$ and $\mathbf{b}_i$ the ith column of $\mathbf{B}$. It is known, from (28), that κ_i and $\mathbf{b}_i$ are left and right eigenvectors of $\mathbf{F}_l$ associated with $z_{i,l}$, having the property that $\kappa_i^H\mathbf{b}_i = 1$.

Due to the intricacy of the perturbation on multiple eigenvalues, we will only derive error expressions for simple eigenvalues of $\mathbf{F}_l$. Let us assume that $z_{i,l}$ is distinct from other eigenvalues of $\mathbf{F}_l$. For T large enough, the perturbation on simple eigenvalue of $\mathbf{F}_l$ is given by [15]

$$\Delta z_{i,l} \triangleq \hat{z}_{i,l} - z_{i,l} \approx \kappa_i^H\Delta\mathbf{F}_l\mathbf{b}_i \quad (57)$$

where $\approx$ is as defined in Lemma 1. Equation (5.13) of [12] gives

$$\Delta\mathbf{F}_l \approx (\mathbf{U}_{l1}^H\mathbf{U}_{l1})^{-1}\mathbf{U}_{l1}^H(\Delta\mathbf{U}_{l2} - \Delta\mathbf{U}_{l1}\mathbf{F}_l). \quad (58)$$

From the definition of $\mathbf{J}_{lj}$ in Appendix G, we know $\Delta\mathbf{U}_{l1} = \mathbf{J}_{l1}\Delta\mathbf{U}_s$ and $\Delta\mathbf{U}_{l2} = \mathbf{J}_{l2}\Delta\mathbf{U}_s$. Since $\mathbf{F}_l\mathbf{b}_i = z_{i,l}\mathbf{b}_i)(\Delta\mathbf{U}_{l2} - \Delta\mathbf{U}_{l1}\mathbf{F}_l)\mathbf{b}_i = (\mathbf{J}_{l2} - z_{i,l}\mathbf{J}_{l1})\Delta\mathbf{U}_s\mathbf{b}_i$ and $(\mathbf{U}_{l1} = \mathbf{A}_{l1}\mathbf{B}^{-1}, U_{l2} = \mathbf{A}_{l2}\mathbf{B}^{-1})\kappa_i^H(\mathbf{U}_{l1}^H\mathbf{U}_{l1})^{-1}\mathbf{U}_{l1}^H$ is the ith row of $(\mathbf{A}_{l1}^H\mathbf{A}_{l1})^{-1}\mathbf{A}_{l1}^H$, we obtain

$$\Delta z_{i,l} = \tau_{i,l}^H\Delta\mathbf{U}_s\mathbf{b}_i = \tau_{i,l}^H\Delta\mathbf{U}_s\mathbf{U}_s^H\mathbf{a}(\theta_i, \phi_i, \gamma_i, \eta_i). \quad (59)$$

In order to use Lemma 1, we have to verify that $\tau_{i,l}, l = 1, 2, 3$, belong to the column space of $\mathbf{U}_n$. From Property 1 and (27), $(\mathbf{A}_{l1}^H\mathbf{A}_{l1})^{-1}\mathbf{A}_{l1}^H(\mathbf{J}_{l2} - z_{i,l}\mathbf{J}_{l1})\mathbf{U}_s = (\mathbf{A}_{l1}^H\mathbf{A}_{l1})^{-1}\mathbf{A}_{l1}^H(\mathbf{U}_{l2} - z_{i,l}\mathbf{U}_{l1}) = (\mathbf{A}_{l1}^H\mathbf{A}_{l1})^{-1}\mathbf{A}_{l1}^H\mathbf{A}_{l1}(\Phi_l - z_{i,l}\mathbf{I}_I)\mathbf{B}^{-1} = (\Phi_l - z_{i,l}\mathbf{I}_I)\mathbf{B}^{-1}$. Thus $\tau_{i,l}^H\mathbf{U}_s = 0, l = 1, 2, 3$. Using Lemma 1, we get

$$\Delta z_{i,l} = \frac{1}{T}\sum_{t=1}^{T}\tau_{i,l}^H\{\mathbf{W}_e(t)\mathbf{Z}_e^H(t) + \mathbf{W}_e(t)\mathbf{W}_e^H(t)\}\upsilon_i. \quad (60)$$

From Lemma F1 in Appendix F, $\Delta\omega_{k,i} = \Im m\{c_{k,1}(i)\Delta p_i + c_{k,2}(i)\Delta q_i + c_{k,3}(i)\Delta\pi_i\}$, (1) For $k = 1, 2, c_{k,3}(i) = 0$. If p_i and q_i are simple eigenvalues of $\mathbf{F}_1$ and $\mathbf{F}_2$ respectively, the expressions of Δp_i and Δq_i can be obtained according to (60). We get, for $k = 1, 2$

$$\Delta\omega_{i,k} = \frac{1}{T}\sum_{t=1}^{T}\Im m\{(c_{k,1}(i)\tau_{i,1}^H + c_{k,2}(i)\tau_{i,2}^H + c_{k,3}(i)\tau_{i,3}^H) \times (\mathbf{W}_e(t)\mathbf{Z}_e^H(t) + \mathbf{W}_e(t)\mathbf{W}_e^H(t))\upsilon_i\}. \quad (61)$$

(2) If p_i, q_i and π_i are simple eigenvalues of $\mathbf{F}_1, \mathbf{F}_2$ and $\mathbf{F}_2$, respectively. Then for $k = 3, 4$

$$\Delta\omega_{i,k} = \frac{1}{T}\sum_{t=1}^{T}\Im m\{(c_{k,1}(i)\tau_{i,1}^H + c_{k,2}(i)\tau_{i,2}^H + c_{k,3}(i)\tau_{i,3}^H) \times (\mathbf{W}_e(t)\mathbf{Z}_e^H(t) + \mathbf{W}_e(t)\mathbf{W}_e^H(t))\upsilon_i\}. \quad (62)$$

Actually for cases (1) and (2), the estimate errors hold a uniform expression

$$\begin{aligned}\Delta\omega_{i,k} &= \frac{1}{T}\sum_{t=1}^{T}\Im m\{(c_{k,1}(i)\tau_{i,1}^H + c_{k,2}(i)\tau_{i,2}^H + c_{k,3}(i)\tau_{i,3}^H) \times (\mathbf{W}_e(t)\mathbf{Z}_e^H(t) + \mathbf{W}_e(t)\mathbf{W}_e^H(t))\upsilon_i\} \\ &= \frac{1}{T}\sum_{t=1}^{T}\Im m\{\mu_{i,k}^H(\mathbf{W}_e(t)\mathbf{Z}_e^H(t) + \mathbf{W}_e(t)\mathbf{W}_e^H(t))\upsilon_i\} \\ &\triangleq \frac{1}{T}\sum_{t=1}^{T}\Im m\{\varepsilon_{i,k}(t)\}. \end{aligned} \quad (63)$$

From $\mu_{i,k}^H\mathbf{U}_n = 0$, we have $\mu_{i,k}^H\upsilon_i = 0, k = 1, 2, 3, 4$. Lemma 4 is thus applicable to $\epsilon_{i,k}(t)$. Using conclusion (3) of Lemma 4, we have $E[\Delta\theta_i] = E[\Delta\phi_i] = E[\Delta\gamma_i] = E[\Delta\eta_i] = 0$. Using $\Im m\{z_1\} = \Im m\{z_2\} = \frac{1}{2}\Re e\{z_1 z_2^* - z_1 z_2\}$ and Lemma 4, the conclusion on the variances can be verified in a similar manner to that in Theorem 1. □

The theoretical analysis for the cases that are not covered by Theorem 2 is currently under investigation.

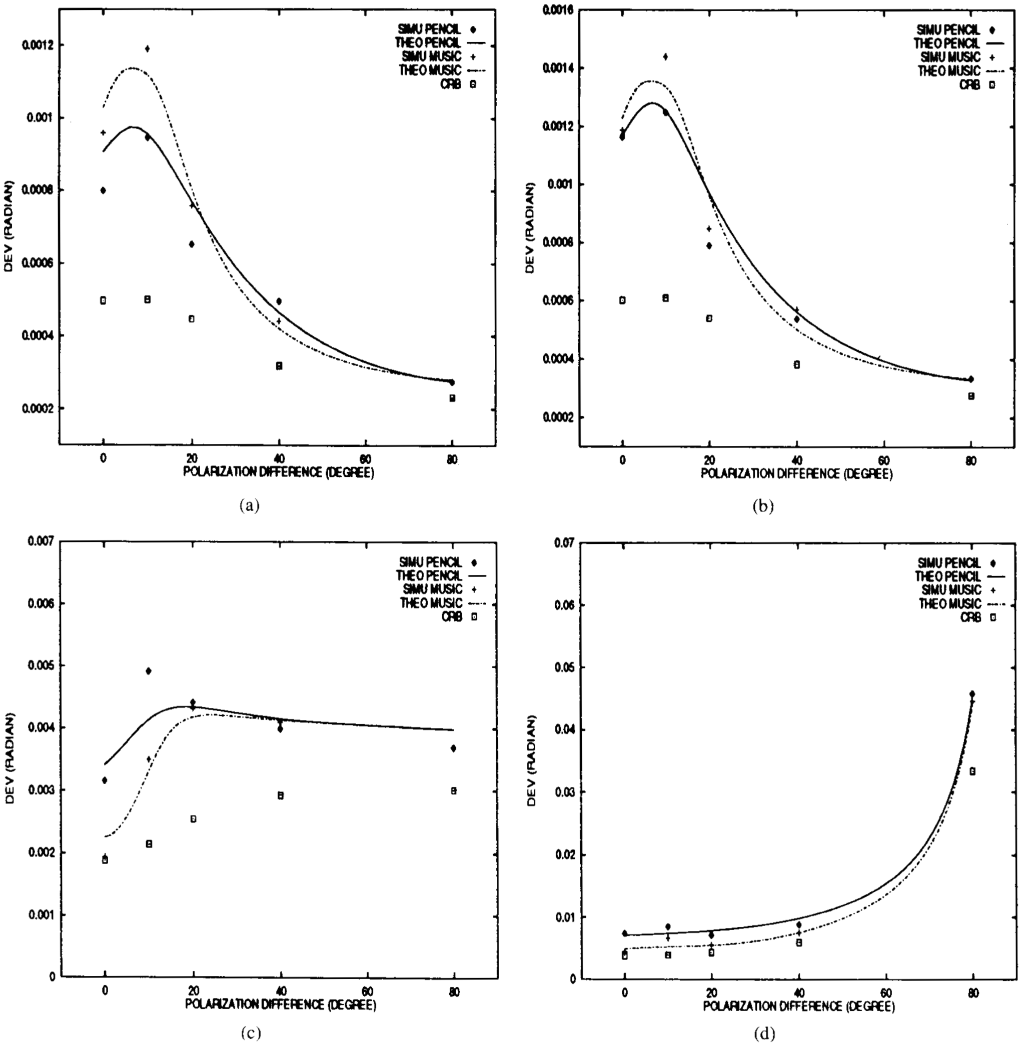


Fig. 1. CRB, theoretical and simulated deviations of (a) θ_1, (b) ϕ_1, (c) γ_1, (d) η_1 versus $\Delta\gamma$ (polarization difference). $(\theta_1, \phi_1, \eta_1) = (40°, 40°, -5°)$, $(\theta_2, \phi_2, \eta_2) = (45°, 45°, 5°)$, $\gamma_1 = 45° - \frac{1}{2}\Delta\gamma$, $\gamma_2 = 45° + \frac{1}{2}\Delta\gamma$, with $\Delta\gamma = 0°, \cdots, 80°$.

V. Simulation and Discussions

In this section, we present simulation results in order to further investigate the performances of the MUSIC and Pencil-MUSIC methods, as well as to verify theoretical work.

We consider two unit-power signals, $M = N = 20$ and $J = K = 7$. SNR $= -10 \log \sigma^2 = 10$ (dB). Fifty runs of Monte Carlo simulation were performed. Initial estimates of MUSIC were generated by the Pencil-MUSIC method. Simulation was conducted for the following three scenarios: (1) different polarization difference for uncorrelated signals, (2) different angle separation for uncorrelated signals, and (3) different correlations between two signals. The number T of snapshots is chosen as 50. Since the deviations of parameter estimates for the two signals follow the same pattern in scenarios (1) and (3), so only those for the first signal are plotted. Whereas in scenario (2), results for both signals are plotted. In all examples, the requirements by Theorems 1 and 2 are satisfied, i.e., $\mathbf{R}_1$ and $\mathbf{R}_2$ (defined by (25)) are positive definite, and $p_1 \neq p_2$, $q_1 \neq q_2$, $\pi_1 \neq \pi_2$.

Fig. 1 shows deviations of the angle and polarization estimates for the first signal when $(\theta_1, \phi_1, \eta_1) = (40°, 40°, -5°)$, $(\theta_2, \phi_2, \eta_2) = (45°, 45°, 5°)$, $\gamma_1 = 45° - \frac{1}{2}\Delta\gamma$, $\gamma_2 = 45° + \frac{1}{2}\Delta\gamma$, with *polarization difference* $\Delta\gamma$ varying from 0° to 80°. When $\Delta\gamma$ is small, there are fluctuations in the deviations of estimates of θ_1 and ϕ_1 however, as $\Delta\gamma$ is larger than 10°, the deviations are monotonically decreasing functions of $\Delta\gamma$. It is interesting to note that the deviation of the estimates of η_1 increases drastically when $\Delta\gamma$ approaches 90°. The reason is that if $\gamma = 0°$ and 90°, η becomes ambiguous, the deviation of its estimate takes large value as $\Delta\gamma$ approaches 90°.

Fig. 2 shows deviations of the angle and polarization estimates for the first signal when $(\theta_1, \phi_1, \gamma_1) = (40°, 40°, 45°)$, $(\theta_2, \phi_2, \gamma_2) = (45°, 45°, 45°)$, $\eta_1 = -\frac{1}{2}\Delta\eta$, $\eta_2 = -\frac{1}{2}\Delta\eta$, with *polarization difference* $\Delta\eta$ varying from 0° to 180°.

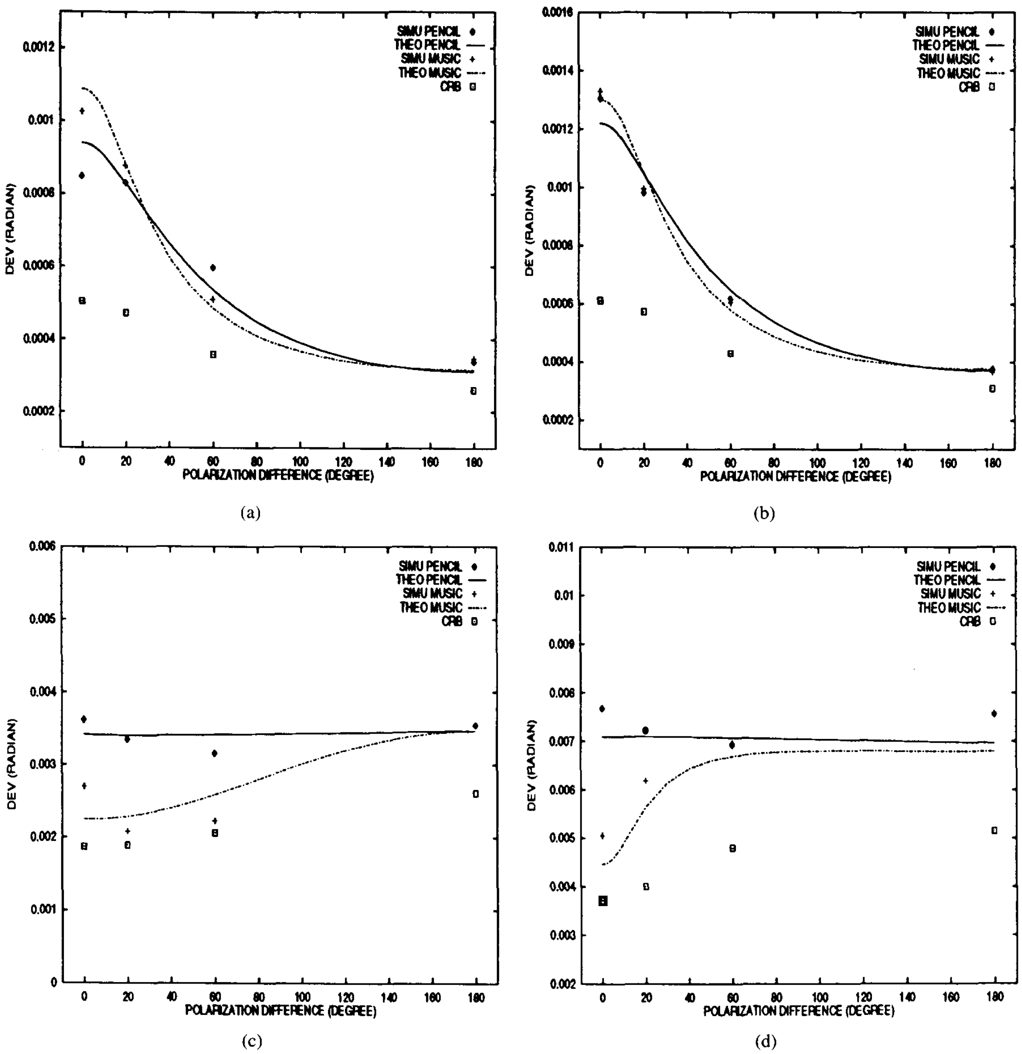
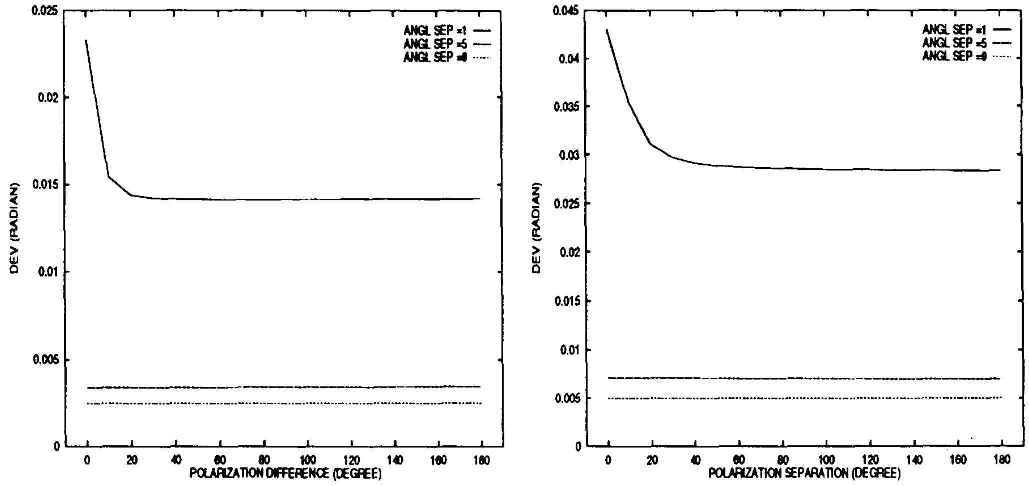


Fig. 2. CRB, theoretical and simulated deviations of (a) θ_1, (b) ϕ_1, (c) γ_1, (d) η_1 versus $\Delta\eta$ (polarization difference). $(\theta_1, \phi_1, \gamma_1) = (40°, 40°, 45°)$, $(\theta_2, \phi_2, \gamma_2) = (45°, 45°, 45°)$, $\eta_1 = -\frac{1}{2}\Delta\eta$, $\eta_2 = \frac{1}{2}\Delta\eta$, with $\Delta\eta = 0°, \cdots, 180°$.

We can see that increasing polarization difference improves the accuracies of angel estimates. Note that the deviations of the Pencil-MUSIC polarization estimates change very little in the whole range of $\Delta\eta$. A similar observation was found in [4]–[7] the authors there claimed that it is due to "small" angle separation between the two signals. However, as shown in the next figure, we found that it is due to the relatively "large" angle separation.

Fig. 3 shows theoretical deviations of the polarization estimates for the first signal, as a function of angle separations when $(\theta_1, \phi_1, \gamma_1) = (40°, 40°, 45°)$, $(\theta_2, \phi_2, \gamma_2) = (40° + \Delta, 40° + \Delta, 45°)$, $\eta_1 = -\frac{1}{2}\Delta\gamma$, $\eta_2 = \frac{1}{2}\Delta\gamma$, with *polarization difference* $\Delta\eta$ varying from 0° to 180° and *angle separation* $\Delta = 1°, 5°, 9°$. It can be seen that when Δ is 5° or 9°, the deviations of the Pencil-MUSIC polarization estimates are stable over the whole range of $\Delta\eta$; on the other hand, when Δ is decreased to 1°, the accuracies of the Pencil-MUSIC polarization estimates improve with the increase of $\Delta\eta$.

Figs. 4 and 5 present deviations for two signals when the angle separation changes. $(\theta_1, \phi_1, \gamma_1, \eta_1) = (45°, 45°, 45°, -10°)$, $(\theta_2, \phi_2, \gamma_2, \eta_2) = (45° + \Delta, 45° + \Delta, 45°, 10°)$, with *angle separation* Δ varying from 1° to 40°. In this case, the deviations for the two signals do not follow the same pattern. The deviations for the first signals decrease as Δ increases. However, the deviations of the estimates of θ_2, γ_2 and η_2 decrease with the increase of Δ for small Δ, but increase with the increase of Δ for large Δ. This observation can be explained as follows. When Δ is small, two signals are closely spaced, an increase in angle separation helps improve the accuracies of the estimates of θ_2, ϕ_2, γ_2 and η_2. When Δ is large, two signals are well separated, the deviations for the second signal are much like that for one signal case. Since $\theta_2 = \arcsin\{\cdot\}$, the sensitivity of θ_2 is nearly infinity when θ is close to 90°. From $\phi_2 = \arctan\{\cdot\}$, the sensitivity of ϕ_2 is nearly zero when ϕ_2 is close to 90°. When $(\theta_2, \phi_2) = (90°, 90°)$, γ_2 and η_2 are ambiguous, so the



(a) (b)

Fig. 3. Theoretical (Pencil-MUSIC) deviations of (a) γ_1, (b) η_1 versus $\Delta\eta$ for different angle separations. $(\theta_1, \phi_1, \gamma_1) = (40°, 40°, 45°)$, $(\theta_2, \phi_2, \gamma_2) = (45°, 45°, 45°)$, $\eta_1 = -\frac{1}{2}\Delta\eta$, $\eta_2 = \frac{1}{2}\Delta\eta$, with $\Delta\eta = 0°, \cdots, 180°$.

estimates of γ_2 and η_2 have large deviations as Δ approaches 45°.

Since the MUSIC and Pencil-MUSIC methods proposed in [11] can handle coherent case, in the following two figures, we consider correlated signals. Let the signal covariance matrix be

$$E[\mathbf{s}(t)\mathbf{s}^H(t)] = \begin{bmatrix} 1 & \rho\exp\{j\psi\} \\ \rho\exp\{-j\psi\} & 1 \end{bmatrix} \tag{64}$$

we will call ρ the correlation magnitude and ψ the correlation phase.

Figs. 6 and 7 show the deviations of the angle and polarization estimates for varying correlation magnitude (phase = 0°) and varying phase (magnitude = 1) respectively when $(\theta_1, \phi_1, \gamma_1, \eta_1) = (40°, 40°, 45°, -90°)$, $(\theta_2, \phi_2, \gamma_2, \eta_2) = (45°, 45°, 45°, 90°)$. The deviations of the MUSIC estimates are, in this case, very close to those of the Pencil-MUSIC estimates since polarization differences were set at large values.

Finally, we note that the polarization estimates of the MUSIC method tend to be more accurate than those of the Pencil-MUSIC method when the angle and/or polarization separations are small, but otherwise the MUSIC and the Pencil-MUSIC have comparable performance (while the latter is always more efficient in computation).

VI. Concluding Remarks

In this paper, we have studied the MUSIC and Pencil-MUSIC methods for finding 2-D angles and polarizations based on a rectangular array of crossed dipoles. In particular, we have obtained a closed form expression for the estimation accuracy of the two methods, compared the performance of the two methods via analysis and simulation, and revealed a number of new insights into the two methods. This study also represents another novel application of several statistical principles as previously applied in [9], [12]–[18]. This novelty includes a skill dealing with the 2-D aspect and the correlated noise (due to the enhanced covariance).

Appendix A

A Compact Expression of $\mathbf{Z}_e(t)$

From the definitions of $\mathbf{X}_e(t)$ and $\mathbf{Y}_e(t)$, it is easy to prove that

$$\mathbf{Z}_e(t) = \mathbf{A}\Psi(t) \tag{65}$$

where

$$\mathbf{A} = \begin{bmatrix} e_{x1}\mathbf{q}_1 \otimes \mathbf{p}_1 & \cdots & e_{xI}\mathbf{q}_I \otimes \mathbf{p}_I \\ e_{y1}\mathbf{q}_1 \otimes \mathbf{p}_1 & \cdots & e_{yI}\mathbf{q}_I \otimes \mathbf{p}_I \end{bmatrix}_{2JK \times I} \tag{66}$$

$$\Psi(t) = \begin{bmatrix} s_1(t) & & \\ & \ddots & \\ & & s_I(t) \end{bmatrix} \begin{bmatrix} \bar{\mathbf{q}}_1^T \otimes \bar{\mathbf{p}}_1^T \\ \vdots \\ \bar{\mathbf{q}}_I^T \otimes \bar{\mathbf{p}}_I^T \end{bmatrix}. \tag{67}$$

with

$$\mathbf{p}_i = [1, p_i, \cdots, p_i^{J-1}]^T \tag{68}$$
$$\mathbf{q}_i = [1, q_i, \cdots, q_i^{K-1}]^T \tag{69}$$
$$\bar{\mathbf{p}}_i = [1, p_i, \cdots, p_i^{M-J+1}]^T \tag{70}$$
$$\bar{\mathbf{q}}_i = [1, q_i, \cdots, q_i^{N-K+1}]^T \tag{71}$$

and $\otimes$ defined by (14).

Appendix B

A Sufficient Condition for $\mathbf{A}$ to Be of Full Column Rank

Theorem B1: If (a) there are no more than two signals with identical 2-D arriving angle pairs, and (b) $e_{xi}e_{yj} \neq e_{xj}e_{yi}$, for $(\theta_i, \phi_i) = (\theta_j, \phi_j)$, $i \neq j$, then $\mathbf{A}$ is of full column rank.

Before we prove Theorem B1, we need the following result.

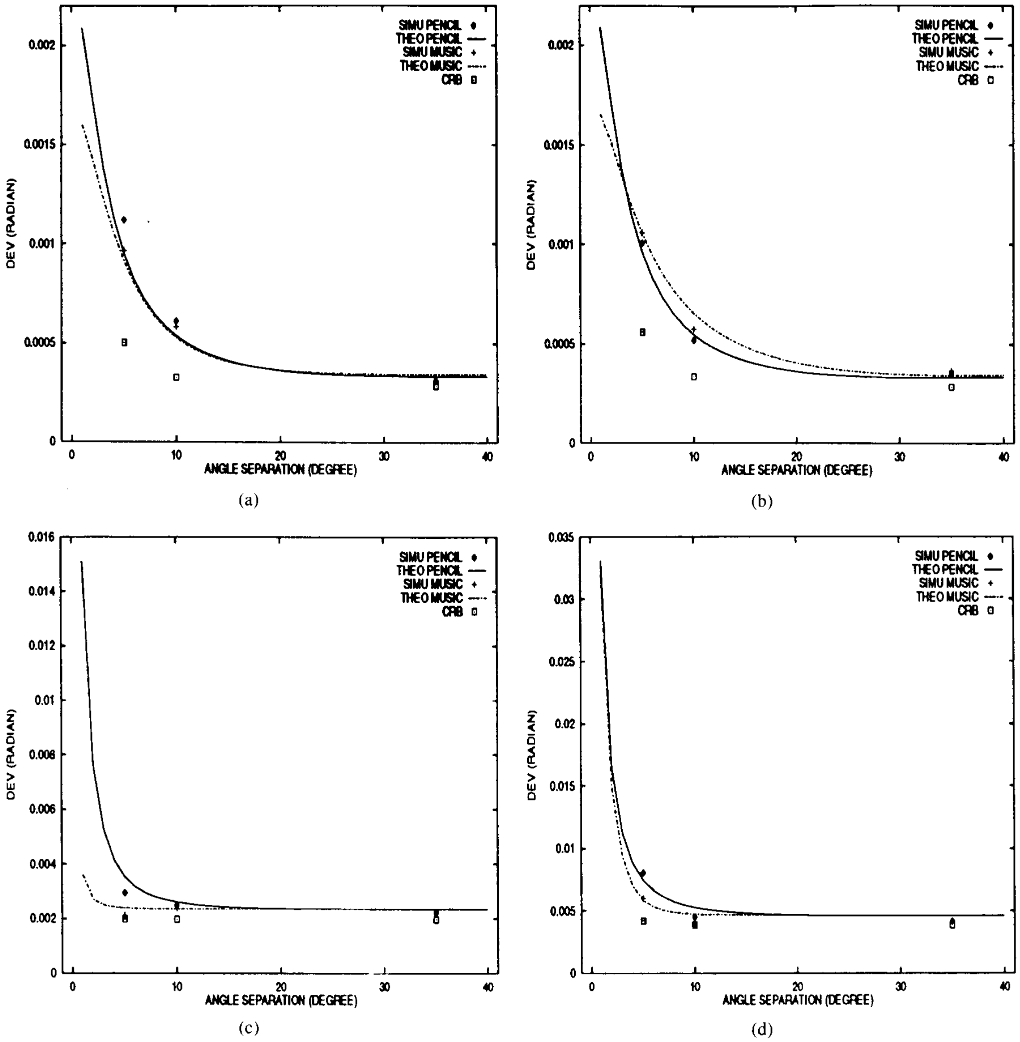


Fig. 4. CRB, theoretical and simulated deviations of (a) θ_1, (b) ϕ_1, (c) γ_1, (d) η_1 versus Δ (angle separation). $(\theta_1, \phi_1, \gamma_1, \eta_1) = (45°, 45°, 45°, -10°)$, $(\theta_2, \phi_2, \gamma_2, \eta_2) = (45° + \Delta, 45° + \Delta, 45°, 10°)$, with $\Delta = 1°, \cdots, 40°$.

Lemma B1: If there are no identical 2-D arriving angle pairs, i.e., $(\theta_i, \phi_i) \neq (\theta_j, \phi_j)$, for $i \neq j$ and $i, j = 1, 2, \cdots, l (\leq I)$, then $\mathbf{q}_1 \otimes \mathbf{p}_1, \mathbf{q}_2 \otimes \mathbf{p}_2, \cdots, \mathbf{q}_l \otimes \mathbf{p}_l$ are independent.

Proof: It can be proved that, $(\theta_i, \phi_i) \neq (\theta_j, \phi_j)$ is equivalent to $(p_i, q_i) \neq (p_j, q_j)$.

Without loss of generality, let $p_1 = \cdots = p_{d_1}$, $p_{d_1+1} = \cdots = p_{d_2}, \cdots, p_{d_{r-1}+1} = \cdots = p_{d_r}$, with $p_{d_1}, p_{d_2}, \cdots, p_{d_r}$ distinct and $d_r = l$.

From $[\mathbf{q}_1 \otimes \mathbf{p}_1, \mathbf{q}_2 \otimes \mathbf{p}_2, \cdots, \mathbf{q}_l \otimes \mathbf{p}_l][\mathbf{c}_1, c_2, \cdots, c_l]^T = \mathbf{0}$, we get $(c_1\mathbf{q}_1 + \cdots + c_{d_1}\mathbf{q}_{d_1}) \otimes \mathbf{p}_{d_1} + \cdots + (c_{d_{r-1}+1}\mathbf{q}_{d_{r-1}+1} + \cdots + \mathbf{c}_{d_r}\mathbf{q}_{d_r}) \otimes \mathbf{p}_{d_r} = \mathbf{0}$. Note $[\mathbf{p}_{d_1}, \cdots, \mathbf{p}_{d_r}]$ is a $J \times r (J \geq I \geq r)$ Vandermonde matrix of rank r, then $\mathbf{c}_{d_{i-1}+1}\mathbf{q}_{d_{i-1}+1} + \cdots + c_{d_i}\mathbf{q}_{d_i} = \mathbf{0}$, $i = 1, 2, \cdots, r$ with $d_0 \triangleq 0$. Note again $[\mathbf{q}_{d_{i-1}+1}, \cdots, \mathbf{q}_{d_i}]$ is a $K \times d_i (K \geq I \geq d_i)$ Vandermonde matrix of rank d_i, then $c_{d_{i-1}+1} = \cdots = c_{d_i} = 0$. Hence, $c_1 = \cdots = c_l = 0$ which leads to the sated result. □

Proof of Theorem B1: Let $(p_1, q_1) = (p_{s_1}, q_{s_1})$, $(p_{s_1+1}, q_{s_1+1}) = (p_{s_2}, q_{s_2})$, $\cdots$, $(p_{s_{t-1}+1}, q_{s_{t-1}+1}) = (p_{s_t}, q_{s_t})$, with $(p_{s_1}, q_{s_1}), (p_{s_2}, q_{s_2}), \cdots, (p_{s_t}, q_{s_t})$ distinct and $s_t = I$. According to (a), $s_i = 1$ or 2.

From

$$\begin{bmatrix} e_{x1}\mathbf{q}_1 \otimes \mathbf{p}_1 & e_{x2}\mathbf{q}_2 \otimes \mathbf{p}_2 & \cdots & e_{xI}\mathbf{q}_I \otimes \mathbf{p}_I \\ e_{y1}\mathbf{q}_1 \otimes \mathbf{p}_1 & e_{y2}\mathbf{q}_2 \otimes \mathbf{p}_2 & \cdots & e_{yI}\mathbf{q}_I \otimes \mathbf{p}_I \end{bmatrix} \times [c_1 \quad c_2 \quad \cdots \quad c_I]^T = \mathbf{0} \quad (72)$$

$$\begin{bmatrix} (c_1 e_{x1} + c_{s_1} e_{xs_1})\mathbf{q}_{s_1} \otimes \mathbf{p}_{s_1} + (c_{s_1+1} e_{x(s_1+1)} + c_{s_2} e_{xs_2})\mathbf{q}_{s_2} \otimes \mathbf{p}_{s_2} + \cdots + (c_{s_{t-1}+1} e_{x(s_{t-1}+1)} + c_{s_t} e_{xs_t})\mathbf{q}_{s_t} \otimes \mathbf{p}_{s_t} \\ (c_1 e_{y1} + c_{s_1} e_{ys_1})\mathbf{q}_{s_1} \otimes \mathbf{p}_{s_1} + (c_{s_1+1} e_{y(s_1+1)} + c_{s_2} e_{ys_2})\mathbf{q}_{s_2} \otimes \mathbf{p}_{s_2} + \cdots + (c_{s_{t-1}+1} e_{y(s_{t-1}+1)} + c_{s_t} e_{ys_t})\mathbf{q}_{s_t} \otimes \mathbf{p}_{s_t} \end{bmatrix} = \mathbf{0} \quad (73)$$

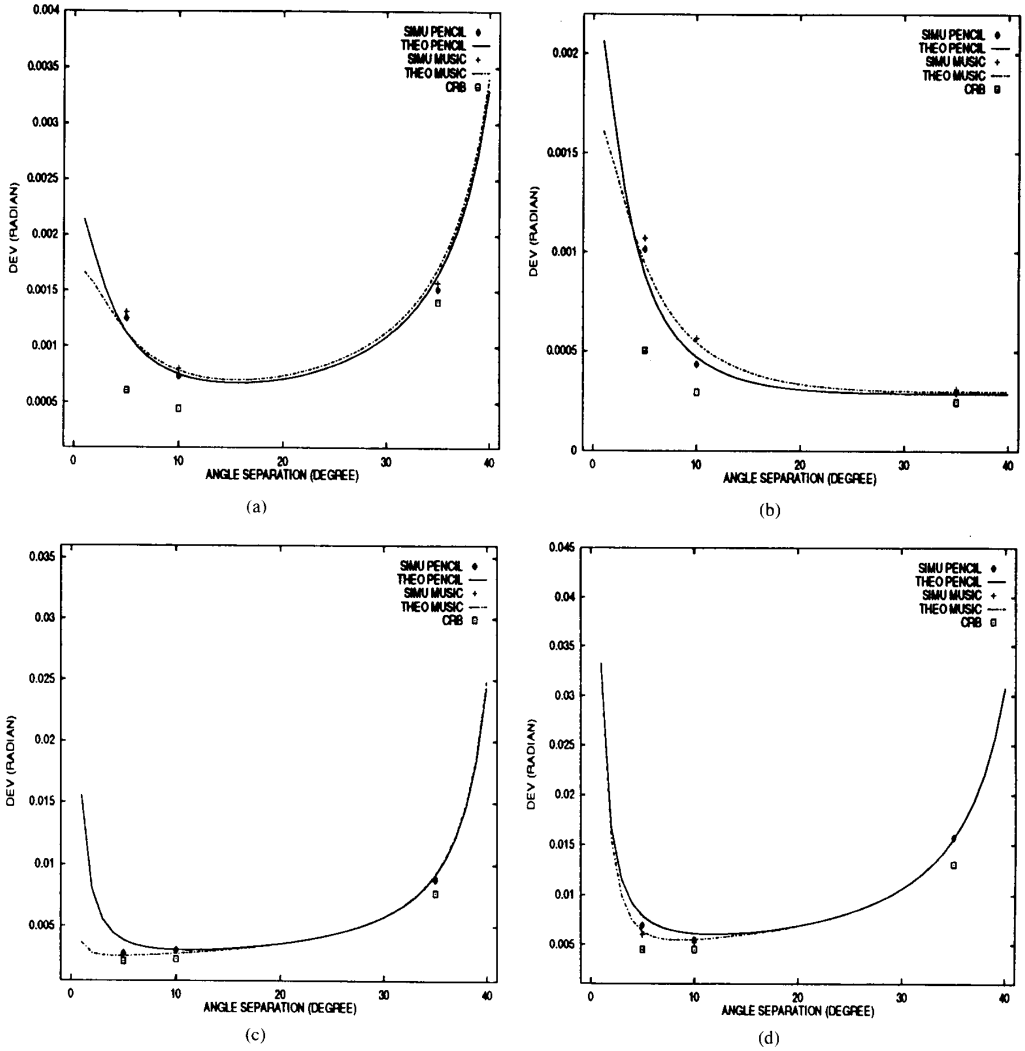


Fig. 5. CRB, theoretical and simulated deviations of (a) θ_2, (b) ϕ_2, (c) γ_2, (d) η_2 versus Δ (angle separation). $(\theta_1, \phi_1, \gamma_1, \eta_1) = (45°, 45°, 45°, -10°)$, $(\theta_2, \phi_2, \gamma_2, \eta_2) = (45° + \Delta, 45° + \Delta, 45°, 10°)$, with $\Delta = 1°, \cdots, 40°$.

we get (73), at the bottom of the previous page. From Lemma B1, we know $\mathbf{q}_{s_1} \otimes \mathbf{p}_{s_1}, \mathbf{q}_{s_2} \otimes \mathbf{p}_{s_2}, \cdots, \mathbf{q}_{s_t} \otimes \mathbf{p}_{s_t}$ are independent. Then if $s_i = 1$, we have

$$\begin{bmatrix} e_{xs_i} \\ e_{ys_i} \end{bmatrix} c_{s_i} = \mathbf{0}. \tag{74}$$

Note that $[e_{xs_i}, e_{ys_i}]^T$ is not a null vector (otherwise there will be $I-1$ signals), therefore $c_{s_i} = 0$; if $s_i = 2$, we have

$$\begin{bmatrix} e_{x(s_{i-1}+1)} & e_{xs_i} \\ e_{y(s_{i-1}+1)} & e_{ys_i} \end{bmatrix} \begin{bmatrix} c_{s_{i-1}+1} \\ c_{s_i} \end{bmatrix} = \mathbf{0}. \tag{75}$$

According to (b), $c_{s_{i-1}+1} = c_{s_i} = 0$. Hence, $c_1 = \cdots = c_I = 0$. This completes the proof.

Remark: If (a) is not satisfied, it is easy to prove that $\mathbf{A}$ is rank deficient. This fact tells us that using arrays of crossed dipoles, the eigendecomposition-based techniques can not be applied for cases of more than two signals with identical 2-D arriving angle pairs.

APPENDIX C

Let $s = i_s + (j_s - 1)(M - J + 1)(1 \leq s \leq L$, L is defined by (20)), with $1 \leq i_s \leq M - J + 1$ and $1 \leq j_s \leq N - K + 1$.

Definition of the Selection Matrix $\mathbf{D}_s$*:*

$$\mathbf{D}_s = [\mathbf{I}_2 \otimes (\mathbf{F}_1(j_s) \otimes \mathbf{F}_2(i_s))]_{2JK \times 2MN} \tag{76}$$

where

$$\mathbf{F}_1(j_s) = \begin{bmatrix} 0 & \cdots & 0 & 1 & & & & & \\ & & & \vdots & \ddots & & & & \\ & & & \vdots & & 1 & 0 & \cdots & 0 \\ & & & \uparrow & & \uparrow & & & \\ & & & j_s & & j_s + K - 1 & & & \end{bmatrix}_{K \times N} \tag{77}$$

$$((F_1(i_s))_{j, j_s + j - 1} = 1, j = 1, 2, \cdots, K)$$

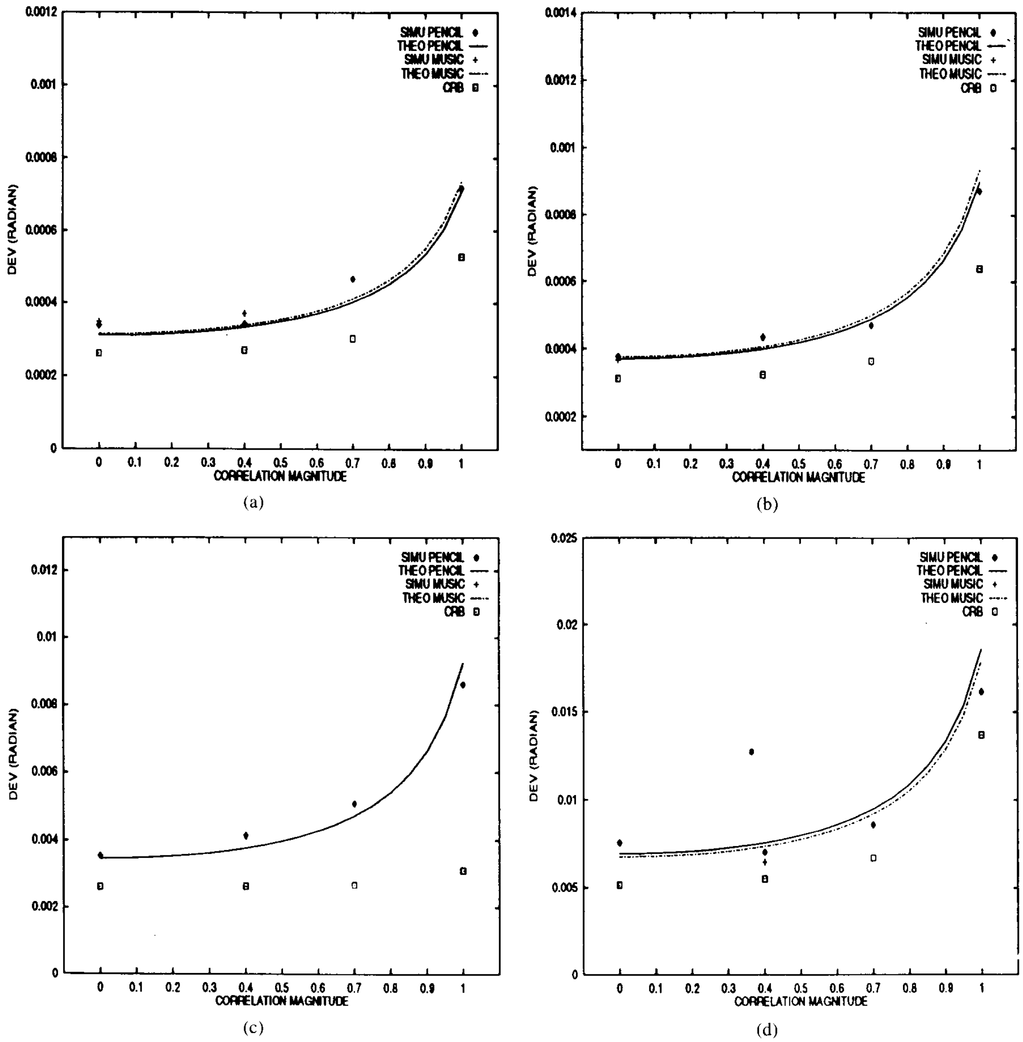


Fig. 6. CRB, theoretical and simulated deviations of (a) θ_1, (b) ϕ_1, (c) γ_1, (d) η_1 versus correlation magnitude (phase = $0°$). $(\theta_1, \phi_1, \gamma_1, \eta_1) = (40°, 40°, 45°, -90°)$, $(\theta_2, \phi_2, \gamma_2, \eta_2) = (45°, 45°, 45°, 90°)$.

and

$$\mathbf{F}_2(i_s) = \begin{bmatrix} 0 & \cdots & 0 & 1 & & & & & \\ & & & \vdots & \ddots & & & & \\ & & & \vdots & & 1 & 0 & \cdots & 0 \\ & & & \uparrow & & \uparrow & & & \\ & & & i_s & & i_s+J-1 & & & \end{bmatrix}_{J\times M} \tag{78}$$

$$((F_2(i_s))_{i, i_s+i-1} = 1, i = 1, 2, \cdots, J)$$

and $\otimes$ is defined by (14).

Notice that $\mathbf{D}_s$ is a (0, 1) matrix with its $((k-1)J+j, (j_s+k-2)M+i_s+j-1)$th and $((k-1)J+j+JK, (j_s+k-2)M+i_s+j-1+MN)$ elements equal to 1, for $j = 1, 2, \cdots, J$ and $k = 1, 2, \cdots, K$.

Proof of Lemma 2: Let

$$\mathbf{v}_{lx(y)}(t) = [w_{x(y)}(1, l, t), w_{x(y)}(2, l, t), \cdots, w_{x(y)}(M, l, t)]^T$$

and

$$\mathbf{w}_{x(y)}(t) = [\mathbf{v}_{1x(y)}(t), \mathbf{v}_{2x(y)}(t), \cdots, \mathbf{v}_{Nx(y)}(t)]^T.$$

One can write

$$\mathbf{W}^l_{ex(y)}(t) = [\mathbf{F}_2(1)\mathbf{v}_{1x(y)} \quad \mathbf{F}_2(2)\mathbf{v}_{2x(y)} \quad \cdots \quad \mathbf{F}_2(M-J+1)\mathbf{v}_{(M-J+1)x(y)}] \tag{79}$$

and

$$\begin{aligned}\mathbf{W}_{ex(y)}(t) &= [(\mathbf{F}_1(1)\otimes\mathbf{I}_J)\mathbf{w}_{x(y)}(t), (\mathbf{F}_1(2)\otimes\mathbf{I}_J) \\ &\quad \times \mathbf{w}_{x(y)}(t), \cdots, (\mathbf{F}_1(N-K+1)\otimes\mathbf{I}_J)\mathbf{w}_{x(y)}(t)] \\ &= [(\mathbf{F}_1(1)\otimes\mathbf{F}_2(1))\mathbf{w}_{x(y)}(t), (\mathbf{F}_1(1)\otimes\mathbf{F}_2(2)) \\ &\quad \times \mathbf{w}_{x(y)}(t), \cdots, (\mathbf{F}_1(1)\otimes\mathbf{F}_2(M-J+1))\mathbf{w}_{x(y)}(t), \\ &\quad (\mathbf{F}_1(2)\otimes\mathbf{F}_2(1))\mathbf{w}_{x(y)}(t), (\mathbf{F}_1(2)\otimes\mathbf{F}_2(2)) \\ &\quad \times \mathbf{w}_{x(y)}(t), \cdots, (\mathbf{F}_1(2)\otimes\mathbf{F}_2(M-J+1)) \\ &\quad \times \mathbf{w}_{x(y)}(t), \cdots, (\mathbf{F}_1(N-K+1)\otimes\mathbf{F}_2(1))\end{aligned}$$

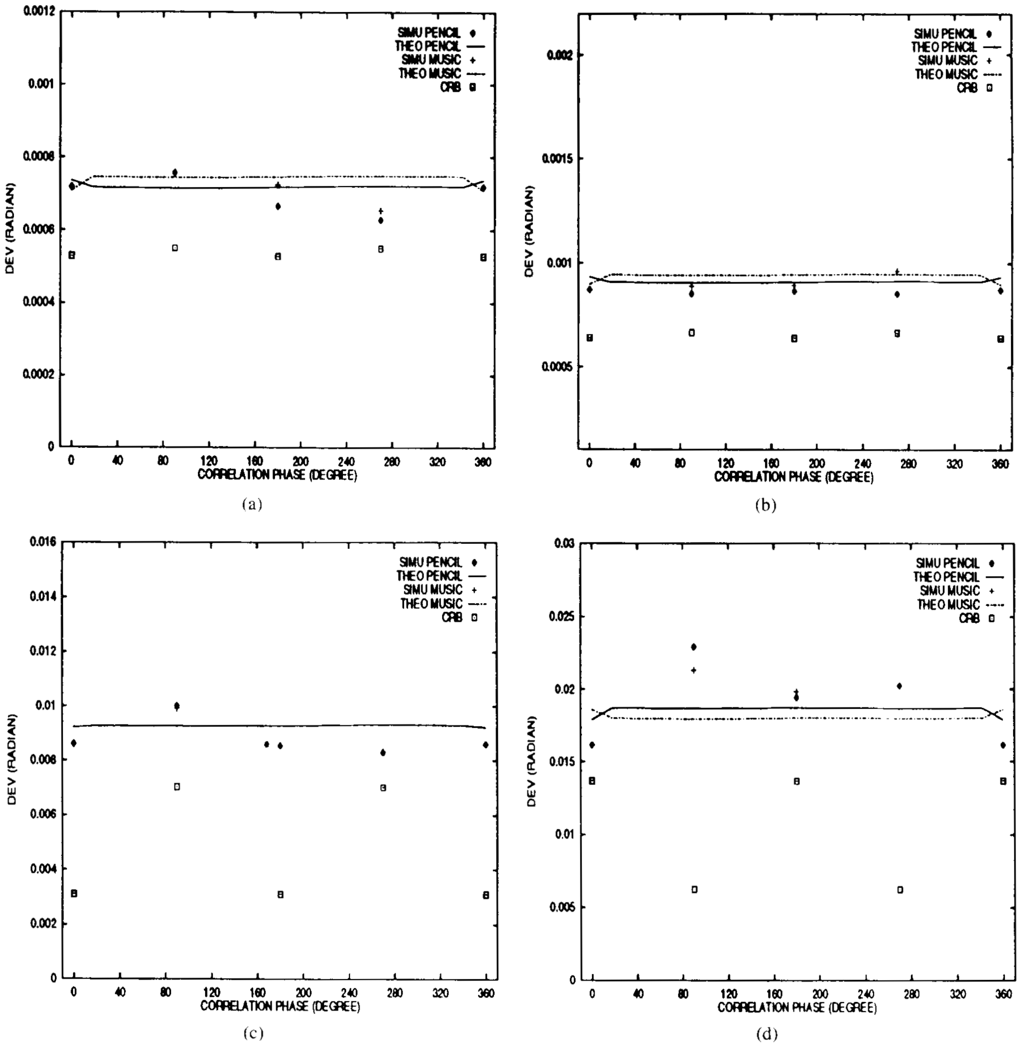


Fig. 7. CRB, theoretical and simulated deviations of (a) θ_1, (b) ϕ_1, (c) γ_1, (d) η_1 versus correlation phase (magnitude = 1). $(\theta_1, \phi_1, \gamma_1, \eta_1) = (40°, 40°, 45°, -90°)$, $(\theta_2, \phi_2, \gamma_2, \eta_2) = (45°, 45°, 45°, 90°)$.

$$\begin{aligned}&\times \mathbf{w}_{x(y)}(t), (\mathbf{F}_1(N-K+1)\otimes \mathbf{F}_2(2))\\ &\times \mathbf{w}_{x(y)}(t), \cdots, (\mathbf{F}_1(N-K+1)\otimes \mathbf{F}_2(M-J+1))\\ &\times \mathbf{w}_{x(y)}(t).\end{aligned} \tag{80}$$

Then the sth column of $\mathbf{W}_{cx(y)}(t))\mathbf{w}_{sx(y)}(t) = (\mathbf{F}_1(j_s)\otimes \mathbf{F}_2(i_s))\mathbf{w}_{x(y)}(t)$. Since

$$\mathbf{w}_s(t) = \begin{bmatrix}\mathbf{w}_{sx(t)}\\ \mathbf{w}_{sy(t)}\end{bmatrix}, \qquad \mathbf{w}(t) = \begin{bmatrix}\mathbf{w}_x(t)\\ \mathbf{w}_y(t)\end{bmatrix} \tag{81}$$

the relation between $\mathbf{w}_s(t)$ and $\mathbf{w}(t)$ can be expressed as

$$\begin{aligned}\mathbf{w}_s(t) &= \begin{bmatrix}\mathbf{F}_1(j_s)\otimes \mathbf{F}_2(i_s) & \\ & \mathbf{F}_1(j_s)\otimes \mathbf{F}_2(i_s)\end{bmatrix}\begin{bmatrix}\mathbf{w}_x(t)\\ \mathbf{w}_y(t)\end{bmatrix}\\ &= \mathbf{D}_s\mathbf{w}(t)\end{aligned} \tag{82}$$

which is conclusion (1) of Lemma 2.

Conclusion (2) can be similarly proved.

Appendix D

Proof of Lemma 3: Let $c_k(i)$ and $w_s(i)$ be the ith elements of $\mathbf{c}_k$ and $\mathbf{w}_s(t)$ respectively for $k = 1, 2, 3, 4$, and $s = 1, 2, \cdots, L$, then

$$\begin{aligned}&\mathbf{c}_1^H\mathbf{w}_s(t)\mathbf{w}_s^H(t)\mathbf{c}_2\mathbf{c}_3^H\mathbf{w}_r(t)\mathbf{w}_r^H(t)\mathbf{c}_4\\ &\quad = \sum_{i,j,k,l=1}^{2JK} c_1^*(i)c_2(j)c_3^*(k)c_4(l)w_s(i)w_s^*(j)w_r(k)w_r^*(l).\end{aligned} \tag{83}$$

It is shown that (see [18], for example), for any four jointly circular Gaussian random variables $\{x_1, x_2, x_3, x_4\}$ with zero means, the following equation

$$E[x_1x_2^*x_3x_4^*] = E[x_1x_2^*]E[x_3x_4^*] + E[x_1x_4^*]E[x_3x_2^*] \tag{84}$$

holds. We obtain

$$\begin{aligned}E[w_s(i)w_s^*(j)w_r(k)w_r^*(l)] &= E[w_s(i)w_s^*(j)]E[w_r(k)w_r^*(l)] \\ &+ E[w_s(i)w_r^*(l)]E[w_r(k)w_s^*(j)]. \quad (85)\end{aligned}$$

Hence

$$\begin{aligned}E[\mathbf{c}_1^H\mathbf{w}_s(t)\mathbf{w}_s^H(t)\mathbf{c}_2\mathbf{c}_3^H\mathbf{w}_r(t)\mathbf{w}_r^H(t)\mathbf{c}_4] &= \mathbf{c}_1^H E[\mathbf{w}_s(t)\mathbf{w}_s^H(t)]\mathbf{c}_2\mathbf{c}_3^H E[\mathbf{w}_r(t)\mathbf{w}_r^H(t)]\mathbf{c}_4 \\ &+ \mathbf{c}_1^H E[\mathbf{w}_s(t)\mathbf{w}_r^H(t)]\mathbf{c}_4\mathbf{c}_3^H E[\mathbf{w}_r(t)\mathbf{w}_s^H(t)]\mathbf{c}_2. \quad (86)\end{aligned}$$

From Lemma 2

$$E[\mathbf{w}_s(t)\mathbf{w}_s^H(t)] = \mathbf{D}_s E[\mathbf{w}(t)\mathbf{w}^H(t)]\mathbf{D}_s^T = \sigma^2\mathbf{D}_s\mathbf{D}_s^T \quad (87)$$

$$E[\mathbf{w}_s(t)\mathbf{w}_r^H(t)] = \sigma^2\mathbf{D}_s\mathbf{D}_r^T. \quad (88)$$

Using a few properties on the Kronecker product [10], we can prove that

$$\mathbf{D}_s\mathbf{D}_s^T = \mathbf{I}_{2JK} \quad (89)$$

for $s = 1, 2, \cdots, L$. Thus the first term on the right side of (86) is zero. The conclusion follows.

APPENDIX E

Proof of Lemma 4: It is known that

$$\begin{aligned}&E[\epsilon(t)\epsilon^*(t)] \\ &= \sum_{s,r=1}^{L}\{E[\mu^H\mathbf{w}_s(t)\mathbf{z}_s^H(t)\upsilon\upsilon^H\mathbf{z}_r(t)\mathbf{w}_r^H(t)\mu] \\ &\quad + E[\mu^H\mathbf{w}_s(t)\mathbf{w}_s^H(t)\upsilon\upsilon^H\mathbf{w}_r(t)\mathbf{w}_r^H(t)\mu]\} \\ &= \sum_{s,r=1}^{L}\{\mu^H E[\mathbf{w}_s(t)\mathbf{w}_r^H(t)]\mu\upsilon^H E[\mathbf{z}_r(t)\mathbf{z}_s^H(t)]\upsilon \\ &\quad + E[\mu^H\mathbf{w}_s(t)\mathbf{w}_s^H(t)\upsilon\upsilon^H\mathbf{w}_r(t)\mathbf{w}_r^H(t)\mu]\} \quad (90)\end{aligned}$$

(the first term on the right side of (90) follows from independence between signals and noise) and

$$E[\epsilon\epsilon] = \sum_{s,r=1}^{L} E[\mu^H\mathbf{w}_s(t)\mathbf{w}_s^H(t)\upsilon\mu^H\mathbf{w}_r(t)\mathbf{w}_r^H(t)\upsilon]. \quad (91)$$

Using Lemmas 2, 3 and (35) gives

$$\begin{aligned}&E[\epsilon(t)\epsilon^*(t)] \\ &= \sigma^2\sum_{s,r=1}^{L}\{\mu^H\mathbf{D}_s\mathbf{D}_r^T\mu\upsilon^H\mathbf{D}_r(\mathbf{R}_z + \sigma^2\mathbf{I}_{2MN})\mathbf{D}_s^T\upsilon\} \\ &= \sigma^2\sum_{s,r=1}^{L}\Big\{\mu^H\mathbf{D}_s\mathbf{D}_r^T\mu[\upsilon^H\mathbf{D}_r\mathbf{A}'\mathbf{C}\mathbf{C}^H\mathbf{A}'^H\mathbf{D}_s^H\upsilon \\ &\quad + \sigma^2\upsilon^H\mathbf{D}_r\mathbf{D}_s^T\upsilon]\Big\} \\ &= \sigma^2\sum_{i,j=1}^{2MN}\sum_{\ell=1}^{r_s}\sum_{s,r=1}^{L}\Big\{\mu^H\mathbf{d}_{si}\mathbf{d}_{ri}^T\mu\Big[\upsilon^H\mathbf{D}_r\mathbf{A}'\mathbf{c}_\ell\mathbf{c}_\ell^H\mathbf{A}'^H\mathbf{D}_s^T\upsilon \\ &\quad + \sigma^2\upsilon^H\mathbf{d}_{rj}\mathbf{d}_{sj}^T\upsilon\Big]\Big\} \\ &= \sigma^2\sum_{i,j=1}^{2MN}\sum_{\ell=1}^{r_s}\sum_{s,r=1}^{L}\Big\{\mu^H\mathbf{d}_{si}\mathbf{c}_\ell^H\mathbf{A}'^H\mathbf{D}_s^T\upsilon\upsilon^H\mathbf{D}_r\mathbf{A}'\mathbf{c}_\ell\mathbf{d}_{ri}^T\mu \\ &\quad + \sigma^2\mu^H\mathbf{d}_{si}\mathbf{d}_{sj}^T\upsilon\upsilon^H\mathbf{d}_{rj}\mathbf{d}_{rj}^T\mu\Big\} \\ &= \sigma^2\sum_{i,j=1}^{2MN}\sum_{\ell=1}^{r_s}\{|\mu^H\mathbf{G}_{i\ell}\upsilon|^2 + \sigma^2|\mu^H H_{ij}\upsilon|^2\} \quad (92)\end{aligned}$$

and

$$E[\epsilon(t)\epsilon(t)] = \sigma^4\sum_{i,j=1}^{2MN}\{\mu^H H_{ij}\upsilon\mu^H H_{ji}\upsilon\}. \quad (93)$$

(92) and (93) are conclusions (1) and (2), respectively.

Conclusion (3) can be shown by

$$\begin{aligned}E[\epsilon] &= \mu^H(E[\mathbf{W}_e(t)\mathbf{Z}_e(t)^H] + E[\mathbf{W}_e(t)\mathbf{W}_e^H(t)])\upsilon \\ &= L\sigma^2\mu^H\upsilon = 0. \quad (94)\end{aligned}$$

APPENDIX F

Since the Pencil-MUSIC method first gives estimates for the intermediate parameter p, q and π, we have to relate estimation errors of θ, ϕ, γ and η to that of p, q and π in this Appendix.

Lemma F1:

$$\Delta\omega_{k,i} = \Im m\{c_{k,1}(i)\Delta p_i + c_{k,2}(i)\Delta q_i + c_{k,3}(i)\Delta\pi_i\}$$

where $c_{k,1}(i)$ are given at the bottom of this page with

$$\begin{aligned}\lambda_i &= \tan\gamma_i\exp\{j\eta_i\} \\ \xi_1^i &= -\frac{\lambda_i\sin\phi_i\cos\phi_i}{\cos^2\theta_i} + \frac{(\lambda_i\cos\theta_i + \pi_i)\sin\phi_i}{\sin\theta_i\cos\theta_i\cos^2\phi_i(\pi_i - \tan\phi_i)} \\ \xi_2^i &= -\frac{\lambda_i\sin^2\phi_i}{\cos^2\theta_i} - \frac{\lambda_i\cos\theta_i + \pi_i}{\sin\theta_i\cos\theta_i\cos\phi_i(\pi_i - \tan\phi_i)} \\ \xi_3^i &= \frac{\tan\phi_i - \lambda_i\cos\theta_i}{\cos\theta_i(\pi_i - \tan\phi_i)}.\end{aligned}$$

$$\begin{aligned}c_{1,1}(i) &= -\frac{\cos\phi_i}{\Delta_x\omega_0 p_i\cos\theta_i} & c_{1,2}(i) &= -\frac{\sin\phi_i}{\Delta_y\omega_0 q_i\cos\theta_i} & c_{1,3}(i) &= 0 \\ c_{2,1}(i) &= \frac{\sin\phi_i}{\Delta_x\omega_0 p_i\sin\theta_i} & c_{2,2}(i) &= -\frac{\cos\phi_i}{\Delta_y\omega_0 q_i\sin\theta_i} & c_{2,3}(i) &= 0 \\ c_{3,1}(i) &= \frac{\tan\gamma_i}{\Delta_x\omega_0 p_i(1+\tan^2\gamma_i)}\Re e\left\{\frac{\xi_1^i}{\lambda_i}\right\} & c_{3,2}(i) &= \frac{\tan\gamma_i}{\Delta_y\omega_0 q_i(1+\tan^2\gamma_i)}\Re e\left\{\frac{\xi_2^i}{\lambda_i}\right\} & c_{3,3}(i) &= \frac{j\xi_3^i\tan\gamma_i}{\lambda_i(1+\tan^2\gamma_i)} \\ c_{4,1}(i) &= \frac{1}{\Delta_x\omega_0 p_i}\Im m\left\{\frac{\xi_1^i}{\lambda_i}\right\} & c_{4,2}(i) &= -\frac{1}{\Delta_y\omega_0 q_i}\Im m\left\{\frac{\xi_1^i}{\lambda_i}\right\} & c_{4,3}(i) &= \frac{\xi_1^i}{\lambda_i}\end{aligned}$$

Proof: To simplify the notation, the subscript $_i$ will be omitted. $\approx$ has the same meaning as defined in Lemma 1.

• *relations of $\Delta\theta$ and $\Delta\phi$ to Δp and Δq*

Since $-\Delta_x\omega_0 \sin\hat{\theta}\cos\hat{\phi}$ and $-\Delta_y\omega_0 \sin\hat{\theta}\cos\hat{\phi}$ are assumed as the angule positions of $\hat{p}$ and $\hat{q}$ respectively, from [12], we know

$$\begin{aligned}\Delta\left\{-\Delta_x\omega_0 \sin\hat{\theta}\cos\hat{\phi}\right\} &\triangleq -\Delta_x\omega_0\sin\hat{\theta}\cos\hat{\phi} \\ &\quad + \Delta_x\omega_0\sin\theta\cos\phi \approx \Im m\{p^{-1}\Delta p\} \quad (95)\\ \Delta\left\{-\Delta_y\omega_0 \sin\hat{\theta}\sin\hat{\phi}\right\} &\triangleq -\Delta_y\omega_0\sin\hat{\theta}\sin\hat{\phi} \\ &\quad + \Delta_y\omega_0\sin\theta\sin\phi \approx \Im m\{q^{-1}\Delta q\}. \quad (96)\end{aligned}$$

Solving the above systems of equations, we obtain

$$\Delta\theta \approx \Im m\left\{-\frac{\cos\phi}{\Delta_x\omega_0 p\cos\theta}\Delta p - \frac{\sin\phi}{\Delta_y\omega_0 q\cos\theta}\Delta q\right\} \quad (97)$$

$$\Delta\phi \approx \Im m\left\{\frac{\sin\phi}{\Delta_x\omega_0 p\sin\theta}\Delta p - \frac{\cos\phi}{\Delta_y\omega_0 q\sin\theta}\Delta q\right\}. \quad (98)$$

• *relations of $\Delta\gamma$ and $\Delta\eta$ to $\Delta p, \Delta q$ and $\Delta\pi$*

Let $\Delta\lambda \triangleq \hat{\lambda} - \lambda$. The relations of $\Delta\gamma$ and $\Delta\eta$ to $\Delta\lambda$ are derived as below: Since

$$\begin{aligned}\tan\hat{\gamma} = \frac{\sin\hat{\gamma}}{\cos\hat{\gamma}} &\approx \frac{\sin\gamma + \Delta\gamma\cos\gamma}{\cos\gamma - \Delta\gamma\sin\gamma} \\ &\approx \tan\gamma(1 + (\cot\gamma + \tan\gamma)\Delta\gamma)\end{aligned}$$

and $\exp\{j\Delta\eta\} \approx 1 + j\Delta\eta$, we have

$$\begin{aligned}\frac{\Delta\lambda}{\lambda} &= \frac{\tan\hat{\gamma}}{\tan\gamma}\exp\{j\Delta\eta\} - 1 \\ &\approx (\cot\gamma + \tan\gamma)\Delta\gamma + j\Delta\eta. \quad (99)\end{aligned}$$

Then

$$\Delta\gamma = \frac{\tan\gamma}{1+\tan^2\gamma}\Re e\left\{\frac{\Delta\lambda}{\lambda}\right\} \quad (100)$$

$$\Delta\eta = \Im m\left\{\frac{\Delta\lambda}{\lambda}\right\}. \quad (101)$$

The expression of $\Delta\lambda$ can be developed as follows: From (26)

$$\hat{\pi}(\hat{\lambda}\cos\hat{\theta} - \tan\hat{\phi}) = \hat{\lambda}\cos\hat{\theta}\tan\hat{\phi} + 1 \quad (102)$$

$$\begin{aligned}&\pi\left(\Delta\lambda\cos\theta - \Delta\theta\lambda\sin\theta - \frac{\Delta\phi}{\cos^2\phi}\right) + \Delta\pi(\lambda\cos\theta - \tan\phi) \\ &\approx \Delta\lambda\cos\theta\tan\phi - \Delta\theta\lambda\sin\theta\tan\phi + \Delta\phi\frac{\lambda\cos\theta}{\cos^2\phi}. \quad (103)\end{aligned}$$

Inserting (97) and (98) in (103) yields

$$\begin{aligned}\Delta\lambda = \xi_1^i\Im m\{(\Delta_x\omega_0 p)^{-1}\Delta p\} \\ +\xi_2^i\Im m\{(\Delta_y\omega_0 q)^{-1}\Delta q\} + \xi_3^i\Delta\pi. \quad (104)\end{aligned}$$

Thus

$$\begin{aligned}\Delta\gamma &= \frac{\tan\gamma}{1+\tan^2\gamma}\Re e\left\{\frac{\xi_1^i}{\lambda}\right\}\Im m\left\{\frac{\Delta p}{\Delta_x\omega_0 p}\right\} + \frac{\tan\gamma}{1+\tan^2\gamma} \\ &\quad \times \Re e\left\{\frac{\xi_2^i}{\lambda}\right\}\Im m\left\{\frac{\Delta q}{\Delta_y\omega_0 q}\right\} + \Re e\left\{\frac{\xi_3^i\tan\gamma}{\lambda(1+\tan^2\gamma)}\Delta\pi\right\} \\ &= \frac{\tan\gamma}{1+\tan^2\gamma}\Re e\left\{\frac{\xi_1^i}{\lambda}\right\}\Im m\left\{\frac{\Delta p}{\Delta_x\omega_0 p}\right\} + \frac{\tan\gamma}{1+\tan^2\gamma}\Re e\left\{\frac{\xi_2^i}{\lambda}\right\} \\ &\quad \times \Im m\left\{\frac{\Delta q}{\Delta_y\omega_0 q}\right\} + \Im m\left\{\frac{j\xi_3^i\tan\gamma}{\lambda(1+\tan^2\gamma)}\Delta\pi\right\} \quad (105)\\ \Delta\eta &= \Im m\left\{\frac{\xi_1^i}{\lambda}\right\}\Im m\left\{\frac{\Delta p}{\Delta_x\omega_0 p}\right\} + \Im m\left\{\frac{\xi_2^i}{\lambda}\right\} \\ &\quad \times \Im m\left\{\frac{\Delta q}{\Delta_y\omega_0 q}\right\} + \Im m\left\{\frac{\xi_3^i}{\lambda}\Delta\pi\right\}. \quad (106)\end{aligned}$$

Using $c_{k,l}$ to replace the coefficients in (97), (98), (105) and (106) completes the proof. □

Appendix G

Definition of selection matrices $\mathbf{J}_{lj}$

$$\mathbf{J}_{11} = \left[\mathbf{I_2} \otimes \left(\mathbf{I_K} \otimes \begin{bmatrix} 1 & & & \\ & \ddots & & \\ & & 1 & 0 \end{bmatrix}_{(J-1)\times J}\right)\right]_{2(J-1)K\times 2JK} \quad (107)$$

(with its $((k-1)J + j, (k-1)J + j)$th and $((k-1)J + j + JK, (k-1)J + j + JK)$th elements equal to 1, for $j = 1, 2, \cdots, J-1, k = 1, 2, \cdots, K$)

$$\mathbf{J}_{12} = \left[\mathbf{I_2} \otimes \left(\mathbf{I}_K \otimes \begin{bmatrix} 0 & 1 & & \\ & & \ddots & \\ & & & 1 \end{bmatrix}_{(J-1)\times J}\right)\right]_{2(J-1)K\times 2JK} \quad (108)$$

(with its $((k-1)J + j, (k-1)J + j + 1)$th and $((k-1)J + j + JK, (k-1)J + j + 1 + JK)$th elements equal to 1, for $j = 1, 2, \cdots, J-1, K = 1, 2, \cdots, K$)

$$\mathbf{J}_{21} = \left[\mathbf{I_2} \otimes \left(\begin{bmatrix} 1 & & & \\ & \ddots & & \\ & & 1 & 0 \end{bmatrix}_{(K-1)\times K} \otimes \mathbf{I}_J\right)\right]_{2J(K-1)K\times 2JK} \quad (109)$$

(with its (l, l)th and $(l + JK, l + Jk)$th elements equal to 1, for $l = 1, 2, \cdots, J(K-1)$)

$$\mathbf{J}_{22} = \left[\mathbf{I_2} \otimes \left(\begin{bmatrix} 0 & 1 & & \\ & & \ddots & \\ & & & 1 \end{bmatrix}_{(K-1)\times K} \otimes \mathbf{I}_J\right)\right]_{2J(K-1)K\times 2JK} \quad (110)$$

(with its $(l, l + J)$th and $(l + JK, l + J + Jk)$th elements equal to 1, for $l = 1, 2, \cdots, J(K-1)$)

$$\mathbf{J}_{31} = [\mathbf{I}_{JK} \quad \mathbf{0}]_{JK\times 2JK} \quad (111)$$

(with its (l, l)th elements equal to 1, for $l = 1, 2, \cdots, JK$) and

$$\mathbf{J}_{32} = [\mathbf{0} \quad \mathbf{I}_{JK}]_{JK \times 2JK} \tag{112}$$

(with its $(l, l + JK)$th elements equal to 1, for $l = 1, 2, \cdots, JK$).

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