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Fujikawa's Jacobian and Berry's phase for fermion hopping models defined on lattices

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Abstract

After briefly reviewing Fujikawa's derivation of chiral anomaly, we study fermions hopping on a lattice. *Under the specific conditions considered in this paper*, we show that chiral gauge transformation induces a Berry's phase which is equal to Fujikawa's Jacobian[1].

1. Introduction

Anomaly in fermion field theories has a long history in particle physics. The anomaly occurs when a symmetry of a low energy theory can not be maintained with a cutoff (or lattice regularization). A theory is said to have the 't Hooft anomaly with respect to a *continuous* global symmetry if there is obstruction against gauging this symmetry [2].

The best-known example of 't Hooft anomaly is the chiral anomaly of the massless Dirac theory in odd space dimensions. The Lagrangian density of such a theory is given by

$$\mathcal{L} = \psi^\dagger(x^\mu) \left[\partial_0 - i \sum_{j=1}^d \Gamma_j \partial_j \right] \psi(x^\mu), \quad (1)$$

where d stands for the space dimension. $\psi(x^\mu)$ is a q -component fermion field (for flavorless and colorless Dirac fermions $q = 2$ for $d = 1$ and $q = 4$ for $d = 3$) and Γ_j are $q \times q$ hermitian matrices satisfying $\{\Gamma_j, \Gamma_k\} = 2\delta_{jk}$. Eq. [1] is invariant under the following global gauge transformation

$$\psi \rightarrow \left(P_+ \otimes e^{i\theta_+} + P_- \otimes e^{i\theta_-} \right) \psi \quad (2)$$

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where

$$P_{\pm} := \frac{I \pm \Gamma_5}{2} \quad \text{and} \quad \Gamma_5 = \prod_{j=1}^d \Gamma_j.$$

Gauging the above symmetry amounts to introducing the $U(1)$ gauge fields $A_{\pm}$ associated with the $e^{i\theta_{\pm}}$ gauge transformations.

One manifestation of the chiral anomaly is that in the presence of a diagonal (i.e., $A = A_+ = A_-$) $U(1)$ gauge field, the axial current is not conserved, namely,

$$\partial_{\mu} j_5^{\mu} = \begin{cases} \frac{1}{\pi} F_{01} & d = 1 \\ \frac{1}{16\pi^2} \epsilon^{\mu\nu\lambda\rho} F_{\mu\nu} F_{\lambda\rho} & d = 3, \end{cases} \quad (3)$$

where $F_{\mu\nu}$ is the field strength associated with A . This famous result was obtained by Adler[3], Bell, and Jackiw[4].¹

Recently, in condensed matter physics, there is a surge of interest in topological insulators and superconductors [5]. The novelty of these materials is that they have symmetry-protected gapless fermion modes on the boundary. Moreover, the symmetry that protects the gapless modes possesses anomalies at the boundary. Ref.[6] relates the non-conserved part of boundary current with the inflow of a bulk current. This is the so-called ‘‘anomaly inflow’’ picture. The classification of topological insulators and superconductors [7, 8] amounts to the classification of the boundary anomaly.

In the rest of the paper, we shall focus on the chiral anomaly of topological insulators in one and three space dimensions. (In two dimensions, the the anomaly is associated with discrete symmetries (e.g. time reversal or parity). It is very different from the continuous symmetry anomaly discussed above. We shall not address this subject in the current paper.)

2. Fujikawa’s Jacobian

An elegant derivation of Eq. [3] using field theory is due to Fujikawa[1]. In the following, we briefly review such derivation. The starting Lorentz invariant

¹A clarification is in order here. While the $\partial_{\mu} j_5^{\mu}$ in the lefthand side of Eq. [3] is a fermion operator, but for abelian gauge field (which is our concern in this paper) the righthand side is a pure number. How can they be equal? The answer is that the lefthand side should be interpreted as the vacuum expectation value. ‘‘Vacuum’’ means the ground state in many-body physics. Thus the chiral anomaly is also a property of the ground state.

action and the associated partition function read

$$\begin{aligned} S &= \int d^{d+1}x \bar{\psi} [-i\gamma^\mu (\partial_\mu - iA_\mu)] \psi \\ Z &= \int D[\psi, \bar{\psi}] e^{-S} \end{aligned} \quad (4)$$

The path integral in Eq. [4] is computed under the cutoff $e^{-\not{D}^2/M^2}$ where $\not{D} = -i\gamma^\mu (\partial_\mu - iA_\mu)$. Moreover, in Eq. [4] $\bar{\psi} = \psi^\dagger(-i\gamma^0)$ and $\gamma^j \equiv -i\gamma^0\Gamma_j$, where γ^0 is the mass matrix which anti-commutes with all Γ_j . In terms of γ^μ the generator of the chiral transformation is $\gamma_5 = \Gamma_5 = i\gamma^0\gamma^1\gamma^2\gamma^3$.² Upon chiral gauge transformation

$$\begin{aligned} \psi &\rightarrow e^{i\frac{\alpha}{2}\gamma_5}\psi \\ \bar{\psi} &\rightarrow \bar{\psi}e^{i\frac{\alpha}{2}\gamma_5}, \end{aligned} \quad (5)$$

the partition function transforms as

$$\begin{aligned} Z &\rightarrow J[A_\mu] \int D[\psi, \bar{\psi}] e^{-S'} \\ S' &= S - \frac{i}{2} \int d^d x dt (\partial_\mu \alpha) j_5^\mu. \end{aligned} \quad (6)$$

In Eq. [6] $J[A_\mu]$ is the Jacobian of the path integral measure. For *infinitesimal* α it is given by

$$J[A_\mu] = \begin{cases} \exp \left[-\frac{i}{2\pi} \int d^d x dt \alpha F_{01} \right] & d = 1 \\ \exp \left[-\frac{i}{32\pi^2} \int d^3 x dt \alpha (\epsilon^{\mu\nu\lambda\rho} F_{\mu\nu} F_{\lambda\rho}) \right] & d = 3. \end{cases} \quad (7)$$

The partition function should be independent of the change of integration variables, hence the value of α . This means Eq. [7] implies Eq. [3].

3. Fujikawa's Jacobian and Berry's phase

The chiral anomaly finds important application in condensed matter physics, especially in the field of topological insulators and superconductors. Elegant as

²Under a suitable choice of basis we have, say,

$$d = 1, \quad \Gamma_1 = Z, \gamma_0 = X, \gamma_1 = -Y, \gamma_5 = \Gamma_5 = Z,$$

and

$$d = 3, \quad \Gamma_1 = ZX, \Gamma_2 = ZY, \Gamma_3 = ZZ, \gamma_0 = XI, \gamma_1 = -YX, \gamma_2 = -YY, \gamma_3 = -YZ, \gamma_5 = \Gamma_5 = ZI.$$

Here I, X, Y, Z stand for Pauli matrices $\sigma_0, \sigma_x, \sigma_y, \sigma_z$. Two Pauli matrices next to each other denote tensor product.

Fujikawa's derivation is, there are questions that a condensed matter physicist will ask. For example, in Fujikawa's derivation, the fermion cutoff is sent to infinity. However, in a condensed matter system the bandwidth is always finite. Wouldn't the finite bandwidth lead to the correction of the chiral anomaly? The $J[A_\mu]$ in Eq. [6] and Eq. [7] originates from the Jacobian associated with a change in the integration variables when computing the partition function. It is a phase factor. The physical meaning of such phase factor is a bit mysterious. Is it related to the Berry phase[9]? In the rest of the paper we answer these questions. We emphasize that the purpose of this paper is to offer a different point of view of a well-known theory, not to report any unknown result.

To simplify things we shall focus on the case where α and $\vec{B}$ in Eq. [7] are independent of space and time. We shall assume the electric field to be uniform in space, but varies slowly in time. The reason we can derive Fujikawa's result under these restrictions is because a topological result should not depend on details of physical parameters. Under these conditions, we can rewrite Eq. [7] as

$$J[A_\mu] = \begin{cases} \exp \left[-i\alpha L \int dt \frac{E(t)}{2\pi} \right] & d = 1 \\ \exp \left[-i\alpha V \int dt \frac{\vec{E}(t) \cdot \vec{B}}{4\pi^2} \right] & d = 3 \end{cases} \quad (8)$$

Here we assume the 1D and 3D systems to be subjected to the periodic boundary condition, with L, V being the length (1D) and volume (3D), respectively.

3.1. One spatial dimension

Let's begin with one spatial dimension. In momentum space, the gapless Dirac Hamiltonian density read ³

$$\int \frac{dk}{2\pi} \psi^\dagger(k) [k \cdot Z] \psi(k), \quad (9)$$

(See footnote 2 for the definition of symbols.) It is apparent that Eq. [9] is invariant under the chiral gauge transformation

$$\psi \rightarrow e^{i\frac{\alpha}{2}Z} \psi. \quad (10)$$

Eq. [9] is not a legitimate lattice Hamiltonian density because it fails to respect the periodicity of the Brillouin zone $k \equiv k + 2\pi$. Naively, one might replace k by $\sin k$ to restore the periodicity

$$H \rightarrow \sum_{k \in \text{BZ}} \psi^\dagger(k) [\sin k \cdot Z] \psi(k). \quad (11)$$

³This is the low energy Hamiltonian for a nearest-neighbor tight-binding model where the hopping matrix element $t_{i,i+1} = -J(i+1)/2$. In Eq. [9], Eq. [11], Eq. [12], Eq. [13], Eq. [14] and Eq. [16] we have set $J = 1$, i.e., expressing energy eigenvalues in unit of J .

However, by doing so we encounter the fermion doubling problem[10], namely, there is an extra Dirac point at $k = \pi$. To remove it we need to gap out the Dirac point at $k = \pi$ while maintaining Eq. [9] as the low-energy Hamiltonian. One way to achieve this is to use Wilson's regularization[11], namely,

$$H \rightarrow \sum_{k \in \text{BZ}} \psi^\dagger(k) [\sin k Z + (1 - \cos k)X] \psi(k). \quad (12)$$

The second (regularization) term in Eq. [12] is proportional to a “mass matrix” X . Under the chiral gauge transformation Eq. [12] becomes

$$H \rightarrow \sum_{k \in \text{BZ}} \psi^\dagger(k) \left[\sin(k) Z + (1 - \cos k)(\cos \alpha X + \sin \alpha Y) \right] \psi(k). \quad (13)$$

In Fig. 1 we show the spectrum of Eq. [11] (panel(a)) and Eq. [12] (panel(b)). It is apparent that the number of Dirac point in panel (a) is doubled.

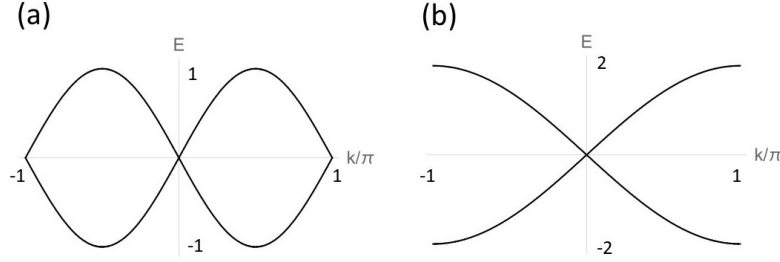


Figure 1: The spectrum of Eq. [11] (panel (a)) and Eq. [12] (panel (b)). See footnote 3 for the explanation of the energy unit.

Unlike Eq. [9], because of the regularization, Eq. [13] is no longer invariant under the chiral gauge transformation! In fact so long as the chiral symmetry is to be respected, no mass matrices (X or Y) are allowed to appear in the Hamiltonian. Under such condition lattice regularization can not be achieved [12].

For the convenience of future discussion, let's rewrite Eq. [13] as

$$H = \sum_{k \in \text{BZ}} \psi^\dagger(k) \left[\vec{b}(k, \alpha) \cdot (X, Y, Z) \right] \psi(k) \quad (14)$$

where

$$\vec{b}(k, \alpha) = \left((1 - \cos k) \cos \alpha, (1 - \cos k) \sin \alpha, \sin k \right).$$

To introduce a spatial uniform electric field we turn on a time-dependent (but spatial uniform) vector potential in the $A_0 = 0$ gauge so that

$$E = \dot{A}_1. \quad (15)$$

In the presence of A_1 the Hamiltonian becomes time-dependent, namely,

$$H \rightarrow \sum_{k \in \text{BZ}} \sum_{k \in \text{BZ}} \psi^\dagger(k) \left\{ \vec{b}(k + A_1(t), \alpha) \cdot (X, Y, Z) \right\} \psi(k). \quad (16)$$

hence

$$\vec{b}(k, A_1, \alpha) = \left((1 - \cos(k + A_1)) \cos \alpha, (1 - \cos(k + A_1)) \sin \alpha, \sin(k + A_1) \right). \quad (17)$$

In the following we shall consider a closed loop in the parameter space spanned by (Φ, α) (where $\Phi = \oint dx A_1 = L A_1$), namely,

$$(0, 0) \rightarrow (2\pi, 0) \rightarrow (2\pi, \alpha) \rightarrow (0, \alpha) \rightarrow (0, 0) \quad (18)$$

and compute the Berry phase (in the current unit 2π corresponds to a Dirac flux quantum). The trajectory of $\vec{b}$ corresponds to such a loop is shown in Fig. 2, where the portion of the loop corresponds to $(0, 0) \rightarrow (2\pi, 0)$ and $(2\pi, \alpha) \rightarrow (0, \alpha)$ are shown in blue and orange, respectively. The red point marks $\vec{b} = (0, 0, 0)$. It is where the Hamiltonian is gapless.

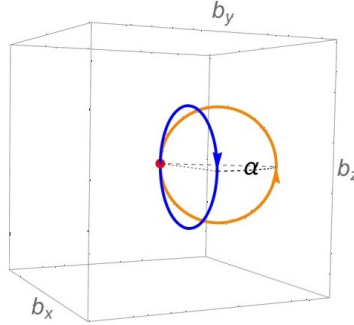


Figure 2: The loops traced out by $\vec{b}$ in Eq. [17], as (A_1, α) varies through Eq. [18]. The portion of the loop corresponds to $(0, 0) \rightarrow (2\pi, 0)$ and $(2\pi, \alpha) \rightarrow (0, \alpha)$ are shown in blue and orange, respectively. The planes where the blue and red loop lies make an angle α . The red point marks where the Hamiltonian is gapless. In this figure $b_{x,y,z}$ axes ranges from -2 to 2.

The subsequent computation of Berry's phase⁴ requires us to compute $\hat{b} :=$

⁴The Berry phase is defined as the integral of the Berry connection $\frac{1}{i} \langle \Psi_0(\{q_i\}) | \frac{\partial}{\partial q_i} | \Psi_0(\{q_i\}) \rangle$ (where $|\Psi_0(\{q_i\})\rangle$ is the ground state) when the parameters $\{q_i\}$ in the Hamiltonian adiabatically vary through a loop in the parameter space.

$\vec{b}/|\vec{b}|$. There, we need to apply the L'Hospital's rule as both $\vec{b}$ and $|\vec{b}|$ vanish as $\vec{b} \rightarrow (0, 0, 0)$. After doing so we obtain

$$\hat{b}(k, A_1, \alpha) = \left(\cos(\alpha) \sin\left(\frac{k + A_1}{2}\right), \sin(\alpha) \sin\left(\frac{k + A_1}{2}\right), \cos\left(\frac{k + A_1}{2}\right) \right). \quad (19)$$

To compute the Berry connection we write down the instantaneous ground state wavefunction of Eq. [16]. For a given α it is the tensor product of the following two-component wavefunctions for each k

$$|\Psi_G(t, \alpha)\rangle = \otimes_{k \in BZ} |\phi_{k(t), \alpha}^-\rangle, \quad (20)$$

where $k(t) = k + A_1(t)$. Here $|\phi_k^- \rangle$ is the negative-energy eigenvector of the $\vec{b}(k, A_1, \alpha) \cdot (X, Y, Z)$. If we adiabatic vary the flux $\Phi(t) = A_1(t) \times L$ from 0 to 2π , the accumulated Berry's phase associated with Eq. [18] is

$$\Upsilon(\alpha) = \frac{1}{i} \int dt \langle \Psi_G(t, \alpha) | \partial_t \Psi_G(t, \alpha) \rangle - \frac{1}{i} \int dt \langle \Psi_G(t, 0) | \partial_t \Psi_G(t, 0) \rangle, \quad (21)$$

where

$$\begin{aligned} \langle \Psi_G(t, \theta) | \partial_t \Psi_G(t, \theta) \rangle &= \sum_k \dot{A}_1 \langle \phi_{k(t), \theta}^- | \partial_k \phi_{k(t), \theta}^- \rangle \\ &= E(t) \frac{L}{2\pi} \left(\int dk \langle \phi_{k, \theta}^- | \partial_k \phi_{k, \theta}^- \rangle \right) \end{aligned} \quad (22)$$

The factor $L/2\pi$ in Eq. [22] originates from converting $\sum_k$ to $\int dk$.

The difference

$$\frac{1}{i} \int dt \langle \Psi_G(t, \alpha) | \partial_t \Psi_G(t, \alpha) \rangle - \frac{1}{i} \int dt \langle \Psi_G(t, 0) | \partial_t \Psi_G(t, 0) \rangle$$

is half of the solid angle Ω sustained by the blue and orange arcs with respect to the origin (red point). This is shown in Fig. 3. Note that the blue and orange circles in Fig. 2 become half circles after we normalize $\vec{b}(k)$. The angle between the planes of where the blue and orange half circle lies is still α . Thus we obtain $\Omega = 2\alpha$. As the result Eq. [21] becomes

$$\exp(-i\Upsilon(\alpha)) = \exp \left[-\frac{iL}{4\pi} 2\alpha \int dt E(t) \right], \quad (23)$$

which agrees with Eq. [8].

It is important to note that had we turned off the regularization term (term proportional to $(1 - \cos k)$ in Eq. [12] and Eq. [13]) $\hat{b}(k)$ becomes $\text{sign}(k)$, which

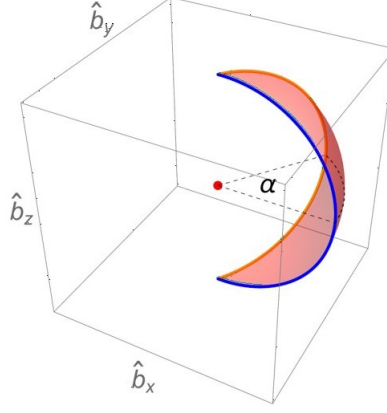


Figure 3: The surface (pink) sustained by the $\hat{b}$ loop correspond to Fig. 2. In this figure the $\hat{b}_{x,y,z}$ axes ranges from -1 to 1.

is independent of α . Consequently the Berry phase vanishes and there is no chiral anomaly. This is because the anomaly cancels among the two Dirac points.

Fujikawa's result is consistent with the well-known physical picture shown in Fig. 4 – the EMF of a time-dependent magnetic flux accelerates the electrons which causes the k values for the filled states to linearly increase with time. Within a cutoff around the Fermi energy, this is manifested as an increase in the number of right-moving electrons minus the number of left-moving electrons, hence the chiral anomaly. We shall refer to this picture as the “ k acceleration picture”.

3.2. Three space dimensions

From the preceding section, we see that the chiral anomaly corresponds to the change in the Berry phase induced by the chiral rotation. In this section, we generalize the result to three space dimensions.

The 3D analog of the Hamiltonian in Eq. [12] is⁵

$$H = \sum_{\mathbf{k} \in \text{BZ}} \psi^\dagger(\mathbf{k}) \left[\sum_{j=1}^3 \sin k_j \Gamma_j + \left(3 - \sum_{j=1}^3 \cos k_j \right) M \right] \psi(\mathbf{k}). \quad (24)$$

⁵This is the tight-binding Hamiltonian where the nearest-neighbor hopping is $t_{i,i+x} = t_{i,i+y} = t_{i,i+z} = -J(i+1)/2$. In Eq. [24], Eq. [25], Eq. [26], Eq. [28] we have set $J = 1$, i.e. express the energy in unit of J .

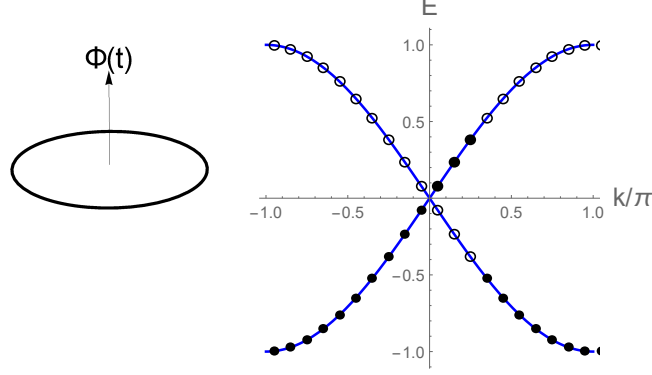


Figure 4: The solid circles represent occupied momentum states while the open circles represent the unoccupied momentum state. The electrons exit from the right side and reenters through the left side due to the periodicity of the Brillouin zone. See footnote 5 for the explanation of the energy unit.

Here $\Gamma_{1,2,3}$ are given in footnote 2. We shall consider the situation where the magnetic field in Eq. [8] is in the $\hat{z}$ direction, namely, $\vec{B} = B\hat{z}$. In the Landau gauge the vector potential is given by $\vec{A} = (0, Bx, 0)$, and the real space Hamiltonian is given by

$$\begin{aligned}
H &= \sum_{x,y,z} \left\{ \psi^\dagger(x+1, y, z) \left(-\frac{M}{2} - \frac{i\Gamma_1}{2} \right) \psi(x, y, z) \right. \\
&\quad \left. + \psi^\dagger(x, y, z+1) \left(-\frac{M}{2} - \frac{i\Gamma_2}{2} \right) \psi(x, y, z) + h.c. \right\} \\
&\quad + \sum_{x,y,z} \left\{ \psi^\dagger(x, y+1, z) \left(-\frac{M}{2} - \frac{i\Gamma_2}{2} \right) e^{iBx} \psi(x, y, z) + h.c. \right\} \\
&\quad + 3 \sum_{x,y,z} \psi^\dagger(x, y, z) M \psi(x, y, z). \tag{25}
\end{aligned}$$

Because translation symmetry in $\hat{y}$ and $\hat{z}$ are preserved, we can Fourier transform in y and z so that Eq. [25] becomes

$$\begin{aligned}
H &= \sum_{x,k_y,k_z} \left\{ \psi^\dagger(x+1, k_y, k_z) \left(-\frac{M}{2} - \frac{i\Gamma_1}{2} \right) \psi(x, k_y, k_z) + h.c. \right. \\
&\quad \left. + \psi^\dagger(x, k_y, k_z) \left[\sin(k_y - Bx)\Gamma_2 + (2 - \cos(k_y - Bx))M \right. \right. \\
&\quad \left. \left. + \sin k_z \Gamma_3 + (1 - \cos k_z)M \right] \psi(x, k_y, k_z) \right\} \tag{26}
\end{aligned}$$

In Eq. [26] the terms proportional to M originate from the lattice regularization[11]. In the continuum limit, the rest of the terms are precisely the Hamiltonian of

3D Dirac fermion in the presence of a magnetic field. To treat the lattice regularization exactly, we numerically diagonalize Eq. [26] on a finite x-direction lattice with k_y and k_z as parameters. The result is shown in Fig. 5.

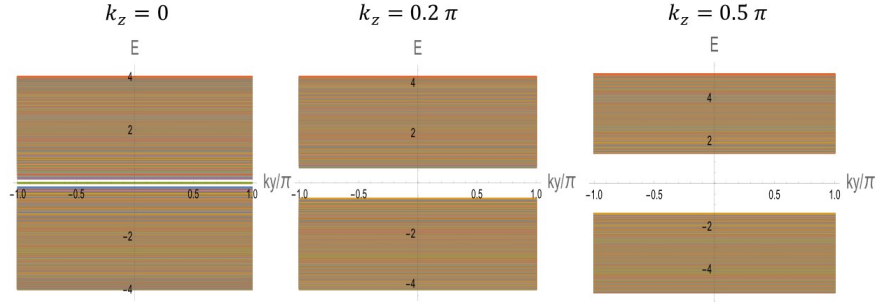


Figure 5: The energy spectrum of Eq. [26] as a function of k_y for three different values of k_z . For each of the spectrum, and a fixed k_y , each energy eigenvalue is $BL_x/2\pi$ fold degenerate, where L_x is the linear dimension of the system in the x direction (the lattice constant is set to 1). This degeneracy plus the fact that the spectrum is independent of k_y gives rise to a total $N_L = \frac{BL_x L_y}{2\pi}$ fold degeneracy for each energy eigenvalue of Eq. [25]. In addition, for $k_z = 0$ there is a $2 \times BL_x/2\pi$ degenerate zero-mode for each k_y . See footnote 5 for the explanation of the energy unit.

We note that the spectrum is *independent* of k_y . In Fig. 5 we show the spectrum as a function of k_y for three different values of k_z . In addition to the k_y -independence, we checked that for a fixed k_y each eigen energy in Fig. 5 is $BL_x/2\pi$ fold degenerate⁶. This gives rise to a total of N_L -fold degeneracy for each energy eigenvalue of Eq. [25], where

$$N_L = \frac{BL_x L_y}{2\pi}, \quad (27)$$

and L_x and L_y are the linear dimensions of the system in the $\hat{x}$ and $\hat{y}$ directions. Eq. [27] is the degeneracy of the Landau bands, hence Fig. 5 is the Landau band structure in the presence of lattice regularization.

In Fig. 6 we plot the spectrum at an arbitrary value of k_y (here chosen as $k_y = 0$) as a function of k_z . A pair of chiral gapless modes are apparent. For any value

⁶To use the periodic boundary condition (which is necessary to avoid edge modes) $BL_x/2\pi$ needs to be an integer.

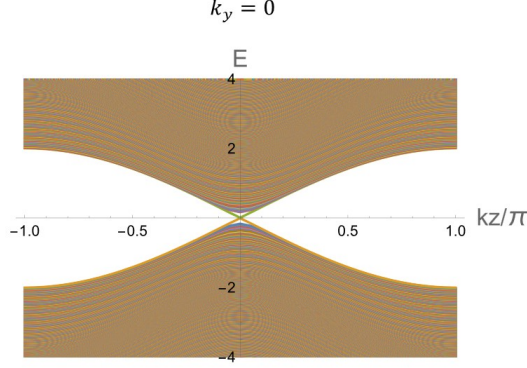


Figure 6: The energy spectrum of Eq. [26] as a function of k_z for arbitrary values of k_y . Note the difference in the vertical scale between Fig. 5 and Fig. 6. The left and right dispersing low energy modes stem from the zero modes at $k_z = 0$. See footnote 5 for the explanation of the energy unit.

of k_y , projecting Eq. [26] onto the Hilbert space spanned by the in-gap mode we obtain the following 2×2 Hamiltonian

$$h_{2 \times 2}(k_z) = \sin k_z \ Z + (1 - \cos k_z) \ X. \quad (28)$$

In Fig. 7 (a) we plot 60 of the energy bands in Fig. 6 that are the closest to $E = 0$. In Fig. 7(b) we overlaid the bands of Eq. [28] (dashed red curves) on top of the pair of energy bands in panel (a) that are closest to $E = 0$. The agreement attests to the fact that the gapless modes are described by Eq. [28] (which is the same as Eq. [12]).

Now we turn on an electric field in the $\hat{z}$ by introducing time-dependent A_z . (Note since the magnetic field is in $\hat{z}$, according to Eq. [8] only the z-component electric field is capable of producing the chiral anomaly.) We repeat the Berry phase calculation for the gapless modes. The reason we only need to focus on the gapless modes is that according to the “ k acceleration picture” in Fig. 4 only the electrons occupying the gapless modes can be accelerate above (or below) E_F under the adiabatic evolving A_z . These are the only electrons that contribute to the chiral anomaly.

Thus the chiral anomaly of the 3D system is the same as that of N_L independent

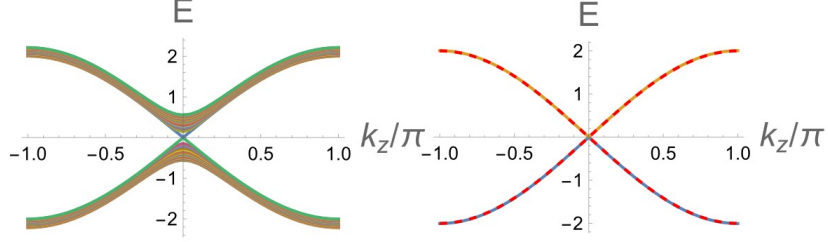


Figure 7: (a) For arbitrary k_y 60 of the energy bands in Fig. 6 that straddle $E = 0$ are plotted. (b) The bands of Eq. [28] (dashed red curves) are overlaid on top of the pair of energy bands in panel (a) that are closest to $E = 0$. See footnote 5 for the explanation of the energy unit.

1D systems studied in the subsection 3.1, i.e.,

$$\begin{aligned}
 \text{Berry's phase factor} &= \prod_{n=1}^{N_L} \left\{ \exp \left[-i \alpha L_z \int dt \frac{E_z(t)}{2\pi} \right] \right\} \\
 &= \exp \left[-i N_L \alpha L_z \int dt \frac{E_z(t)}{2\pi} \right] \\
 &= \exp \left[-i \left(\frac{B L_x L_y}{2\pi} \right) \alpha L_z \int dt \frac{E_z(t)}{2\pi} \right] \\
 &= \exp \left[-i \alpha V \int dt \frac{\vec{E}(t) \cdot \vec{B}}{4\pi^2} \right].
 \end{aligned}$$

This last result is consistent with Eq. [8].

4. Conclusion

In this paper, we shed light on Fujikawa's Jacobian by showing that it is a Berry's phase. The derivation is performed for fermions hopping on a lattice with finite bandwidth. Our result implies Fujikawa's Jacobian is the change in Berry's phase induced by a chiral gauge transformation. We hope these results shed some light on the origin of the chiral anomaly for non-particle physicists.

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