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Author
Hanemann, W. Michael

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by

W. Michael Hanemann

California Agricultural Experiment Station
Giannini Foundation of Agricultural Economics
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By W. Michael Hanemann

1. Introduction

In recent years there has been considerable interest in econometric modeling of discrete choices, and significant progress has been made in the statistical formulation and estimation of such models.[1] However, the development of a methodology for conducting applied welfare analysis in discrete choice situations has proceeded more slowly. The issue was first raised in the transportation mode choice literature by Domencich and McFadden [5], Williams [15], Daly and Zachary [4], and Ben Akiva and Lerman [1]. A connection between this work and the conventional economic theory of consumer demand was recently established in an important paper by Small and Rosen [14]. They demonstrate that the conventional techniques for measuring the welfare effects of price or quality changes in a continuous choice setting can be generalized to handle cases where discrete choices are involved. The purpose of this paper is to extend their analysis, and to show how one can dispense with some of their more stringent assumptions about the structure of consumer preferences.

Small and Rosen (henceforth SR) discuss situations involving both purely qualitative choices and mixed qualitative-quantitative choices.[2] However, when they apply their results to empirical econometric models they focus primarily on logit and probit models, which pertain exclusively to qualitative choices. Since logit and probit are widely used in many areas of applied economics it is of great value to have practical procedures for ascertaining their welfare implications. Accordingly this paper will focus on these purely qualitative choices, termed "quantal choices" by McFadden [11].

In a binary quantal choice model the dependent variable may be thought of as dummy variable which takes the value 1 if one qualitative outcome occurs and the value 0 if another occurs. The selection probabilities are given by

(1.1) $\pi_1 \equiv \Pr\{\text{outcome 1 occurs}\} = F(w)$

$\pi_2 \equiv \Pr\{\text{outcome 2 occurs}\} = 1 - F(w)$

where $F(\cdot)$ is some function whose range is $[0, 1]$, and its argument takes the form $w = X'\beta$ where X is a vector of explanatory variables and β is a vector of coefficients to be estimated. In the binary logit model, for example, $F(w) = (1 + e^{-w})^{-1}$. Daly and Zachary [4] have developed necessary and sufficient conditions on the selection probability functions (1.1) in order for the underlying quantal choice model to be a <u>random utility maximization (RUM) model</u>.[3] In a RUM model w takes the form $w = \tilde{v}_1 - \tilde{v}_2$ where $\tilde{v}_1$ and $\tilde{v}_2$ are functions of regressors and coefficients to be estimated. The choice can be thought of as resulting from a utility maximization problem in which the agent's utility conditional on choice j is $\tilde{u}_j = \tilde{v}_j + \varepsilon_j$, $j = 1, 2$. Here, ε_1 and ε_2 are jointly distributed random variables with $E\{\varepsilon_j\} = 0$, and the selection probability function in (1.1) can be cast into the form: $\pi_1 = \Pr\{\tilde{u}_1 \geq \tilde{u}_2\}$. By contrast, the qualitative choice model considered by SR is an example of what might be called a <u>budget-constrained random utility maximization model</u>. That is, it is a RUM model with the additional property that the $\tilde{u}_j$'s satisfy the requirements to be <u>conditional indirect utility functions</u>. As explained below this imposes certain restrictions on which variables can appear as arguments of the $\tilde{u}_j$ functions and on their functional structure.

Imagine a situation in which a consumer must choose between two mutually exclusive items (courses of action) which differ with respect to their cost, p_j, and "quality," q_j. Let y denote the consumer's income and suppose that the consumer's indirect utility function conditional on the choice of act j may be written $\tilde{u}_j = \tilde{v}_j(p_j, q_j, y) + \varepsilon_j$, $j = 1, 2$. By fitting a quantal choice model based on (1.1) with $w \equiv \tilde{v}_1 - \tilde{v}_2$, one can estimate the $\tilde{v}_j$ functions. Now suppose that the quality of item 1 available to the consumer rises from q_1^0 to q_1^f while q_2, p_1, p_2, and y remain constant. Can one employ the fitted quantal choice model to derive a monetary measure of the welfare effects of this change? SR discuss how this can be done under three assumptions about the structure of consumer preferences: (A) $\lambda_1 \equiv \partial\tilde{v}_1/\partial y$ is independent of p_1 and q_1; (B) the income effect from the quality change is approximately zero; and (C) $\partial\tilde{v}_1/\partial q_1 \to 0$ as $p_1 \to \infty$. For the logit model, for example, SR use these assumptions to derive the following formula for the compensating variation measure of the change in the consumer's welfare:

$$\text{(1.2)} \qquad CV = -\frac{1}{\lambda_1} \ln \left[\left(e^{\tilde{v}_1^f} + e^{\tilde{v}_2} \right) / \left(e^{\tilde{v}_1^0} + e^{\tilde{v}_2} \right) \right]$$

where $\tilde{v}_2 = \tilde{v}_2\,(p_2, q_2, y)$ and $\tilde{v}_1^t = \tilde{v}_1(p_1, q_1^t, y)$, $t = 0, f$.

Although this is a valuable result, in the empirical literature one can find many applications of logit models which violate Assumptions A, B, or C. For example, the specification

$$\text{(1.3)} \qquad \tilde{v}_j = \alpha_j - \beta p_j + \gamma_j y + \delta q_j \qquad (j = 1, 2),$$

which leads to (1.1) with

$$\text{(1.4)} \qquad w = (\alpha_2 - \alpha_1) - \beta(p_2 - p_1) + (\gamma_2 - \gamma_1)y + \delta(q_2 - q_1),$$

violates C, since $\partial\tilde{v}_1/\partial q_1$ does not go to zero as p_1 goes to infinity. Similarly, the specification

$$\text{(1.5)} \qquad \tilde{v}_j = \alpha_j - \beta_j p_j + \gamma y p_j + \delta q_j \qquad (j = 1, 2),$$

which leads to (1.1) with

$$\text{(1.6)} \qquad w = (\alpha_2 - \alpha_1) - \beta(p_2 - p_1) + \gamma y(p_2 - p_1) + \delta(q_2 - q_1),$$

violates both A and C.[4] How might one calculate the compensating variation for a quality change in these cases? Does the welfare formula (1.2) still apply?

By exploiting the special structure of an indirect utility function in a purely qualitative choice context, I show that Assumption C is not needed in order to justify the formula (1.2), and is actually incompatible with Assumptions A and B. I show that Assumption B implies A, and *is* crucial to the validity of SR's welfare formula. However, it implies that the income variable does not appear in the quantal choice probability formula (1.1)—i.e., $w \equiv \tilde{v}_1 - \tilde{v}_2$ is independent of y. Thus, any quantal choice model which includes income as an explanatory variable must violate Assumption B. I develop general welfare formulas which are applicable whether or not Assumption B holds, and which reduce to SR's formula when it does hold. These formulas cover both the compensating and the equivalent variation welfare measures. Finally, I show how the welfare analysis may be extended from budget-constrained RUM models to general RUM models in which, for example, the individual is choosing among risky actions on the basis of von Neumann-Morgenstern expected utility maximization.

In deriving these results, I follow the same basic strategy as SR. They start with a purely deterministic choice situation similar to the "standard" utility maximization model, except that there is an element of discreteness in the consumer's choice. They establish the important result that the basic duality results of continuous demand analysis carry over to the discrete case. They then switch over to the set-up of the econometric quantal choice literature in which it is assumed that, although the consumer has a fixed utility function, some of its components are unobservable to the econometric investigator. This introduces an element of randomness into the consumer's utility and demand functions as they appear to the investigator. SR analyze discrete choices in this context by taking the expectation of certain relationships established in their analysis of the purely deterministic case. In deriving my results I show that there are some pitfalls in the transition from the deterministic to the random utility setting and that some results from the former do not carry over to the latter under the expectation operation.

This paper is organized as follows. The purely qualitative deterministic choice model is introduced and analyzed in Section 2. In Section 3 I turn to the random utility setting and explain the concepts of the RUM and budget-constrained RUM quantal choice models. The similarities with the deterministic utility model are explored. The concepts of the compensating and equivalent variation welfare measures are defined, and general formulas for calculating them are presented. In Section 4 these formulas are applied to the standard logit and probit models, discussed by SR, and also to the generalized logit and probit models recently introduced by McFadden and by Hausman and Wise.[5] The results in this Section provide a fully operational procedure for conducting welfare evaluations in the context of these quantal choice models. The conclusions are summarized in Section 5.

2. Deterministic Qualitative Choice

The deterministic choice model assumed by SR is as follows. A consumer has a twice differentiable, quasi-concave, increasing utility function u defined over the commodities x_1, x_2, and z, where z is taken as the numeraire. In addition, the consumer's utility depends on the quality of the non-numeraire goods, which is taken as exogenous; with no loss of generality this may be represented by the scalars q_1 and q_2. The consumer chooses x_1, x_2, and z so as to maximize

$$u = u(x_1, x_2, q_1, q_2, z) \tag{2.1}$$

subject to the budget and non-negativity constraints

(2.2) $p_1x_1 + p_2x_2 + z = y$

(2.3) $x_1, x_2, z \geq 0.$

SR make a useful distinction between three different ways in which an element of discreteness can be introduced into this "standard" utility model. One possibility is that nonconcavities in the utility function (2.1) lead to a corner solution in which one of the relations in (2.3) holds as an equality. A second case is where the two non-numeraire goods are for some reason mutually exclusive in consumption.[6] This leads to the imposition of an additional constraint on the utility maximization problem:

(2.4) $x_1x_2 = 0.$

A third case is where, perhaps because of their size, the non-numeraire goods are purchased only in discrete units. This can be represented by the constraints

(2.5) $x_j = \bar{x}_j$ or 0, $(j = 1, 2)$

where $\bar{x}_1$ and $\bar{x}_2$ are fixed numbers; the units of measurement for goods 1 and 2 could be chosen so that $\bar{x}_1 = \bar{x}_2 = 1$.[7] In their analysis of deterministic choice SR focus primarily on the model consisting of (2.1)-(2.4), although they emphasize that their results also apply to the other cases. By contrast, the purely qualitative choice implied by the logit and probit models corresponds to the utility model (2.1)-(2.5). Accordingly, I will focus here on the latter model.[8]

The starting point for the analysis of the deterministic qualitative choice model (2.1)-(2.5) is the concept of a conditional indirect utility function. Conditional on the choice of good j, the consumer's utility is $\tilde{u}_j$, $j = 1, 2$, where

$$(2.6) \qquad \tilde{u}_1 = u(\bar{x}_1, 0, q_1, q_2, y - p_1\bar{x}_1)$$
$$\tilde{u}_2 = u(0, \bar{x}_2, q_1, q_2, y - p_2\bar{x}_2).$$

SR make the further assumption, termed "weak complementarity" by Mäler [13], that the quality of good j does not matter unless good j is actually being consumed:

$$(2.7) \qquad \partial\tilde{u}_1/\partial q_2 \equiv \partial\tilde{u}_2/\partial q_1 \equiv 0.$$

Because of this assumption one can write

$$(2.8) \qquad \tilde{u}_j = \tilde{v}_j(p_j, q_j, y), \qquad (j = 1, 2).$$

The unconditional indirect utility function, which measures the utility actually achieved by the consumer when confronted with the given prices, qualities, and income, is

$$(2.9) \qquad u = v(p_1, p_2, q_1, q_2, y) \equiv \max\{\tilde{u}_1, \tilde{u}_2\}$$
$$= \delta_1\tilde{u}_1 + (1 - \delta_1)\tilde{u}_2$$

where $\delta_1(p_1, p_2, q_1, q_2, y)$ is a discrete choice index for the direct utility maximization problem; i.e., δ_1 is 1 if good 1 is preferred over good 2 given (p_1, p_2, q_1, q_2, y), and 0 otherwise.

Dual to the utility maximization problem is a cost minimization problem which yields a pair of conditional expenditure functions and an unconditional expenditure function. Under the assumption of (2.7) the conditional expenditure functions may be written

$$(2.10) \qquad \tilde{e}_j = \tilde{e}_j(p_j, q_j, u), \qquad (j = 1, 2),$$

and the unconditional expenditure function is defined by

$$(2.11) \qquad e = e(p_1, p_2, q_1, q_2, u) \equiv \min\{\tilde{e}_1, \tilde{e}_2\}$$
$$= \delta_1^c\,\tilde{e}_1 + (1 - \delta_1^c)\,\tilde{e}_2$$

where $\delta_1^c(p_1, p_2, q_1, q_2, u)$ is a compensated discrete choice index for the cost minimization problem. The unconditional expenditure and indirect utility functions are used to define monetary measures of the welfare effects of price and quality changes. For example, suppose that the quality of good 1 available to the consumer changes from q_1^0 to q_1^f, while the quality of good 2, prices, and income stay constant at (q_2, p_1, p_2, y). SR define the compensating variation for this change by

$$\text{(2.12)} \qquad cv = e(p_1, p_2, q_1^f, q_2, u^0) - e(p_1, p_2, q_1^0, q_2, u^0),$$

where $u^0 \equiv v(p_1, p_2, q_1^0, q_2, y)$. Because of the following identity

$$\text{(2.13)} \qquad e(p_1, p_2, q_1, q_2, v(p_1, p_2, q_1, q_2, y)) = y,$$

an alternative (implicit) definition of the compensating variation is

$$\text{(2.14)} \qquad v(p_1, p_2, q_1^f, q_2, y + cv) = v(p_1, p_2, q_1^0, q_2, y).$$

As in the continuous case, a goal of the analysis is to relate cv to areas under ordinary or compensated demand functions. The direct utility maximization problem yields a pair of conditional ordinary demand functions for goods 1 and 2, as well as a pair of unconditional ordinary demand functions. Let $\tilde{x}_j(p_j, q_j, y)$ denote the conditional ordinary demand function for good j; the unconditional demand function may be written

$$\text{(2.15)} \qquad x_j(p_1, p_2, q_1, q_2, y) = \delta_j(p_1, p_2, q_1, q_2, y)\tilde{x}_j(p_j, q_j, y), \qquad (j = 1, 2).$$

Similarly, the cost minimization problem yields a pair of conditional compensated demand functions, $\tilde{x}_j^c(p_j, q_j, u)$, j = 1, 2, and a pair of unconditional compensated demand functions which may be written

$$\text{(2.16)} \qquad x_j^c(p_1, p_2, q_1, q_2, u) = \delta_j^c(p_1, p_2, q_1, q_2, u)\tilde{x}_j^c(p_j, q_j, u), \qquad (j = 1, 2).$$

SR point out that, because the utility function (2.1) is well-behaved when viewed as a function of only x_1 and z or x_2 and z, the conditional indirect utility and ordinary demand functions are continuously differentiable and satisfy Roy's Identity. Similarly, the conditional expenditure and compensated demand functions are continuously differentiable and satisfy Shephard's Lemma. By contrast, the unconditional indirect utility and expenditure functions are nondifferentiable, and the unconditional demand functions are discontinuous. SR's important result is that, despite this, these unconditional functions also satisfy Roy's Identity and Shephard's Lemma:[9]

$$(2.17) \qquad x_j(p_1, p_2, q_1, q_2, y) = -\frac{\partial v(p_1, p_2, q_1, q_2, y)/\partial p_j}{\partial v(p_1, p_2, q_1, q_2, y)/\partial y}, \qquad (j = 1, 2),$$

$$(2.18) \qquad x_j^c(p_1, p_2, q_1, q_2, y) = \partial e(p_1, p_2, q_1, q_2, u)/\partial p_j \qquad (j = 1, 2).$$

These two relations are the key to establishing a link between the compensating variation, cv, and areas under demand curves.

Although SR's analysis primarily focuses on the qualitative-quantitative choice model (2.1)-(2.4), the results (2.17) and (2.18) also apply to the purely qualitative choice model (2.1) - (2.5) considered here. The main significance of the extra constraint (2.5) is that it imposes additional restrictions on the functional form of the conditional demand, expenditure, and indirect utility functions. Thus the conditional ordinary and compensated functions coincide and are constant: $\tilde{x}_j(p_j, q_j\ y) \equiv \tilde{x}_j^c(p_j, q_j, u) \equiv \bar{x}_j$.[10] Moreover, from (2.6) and (2.7), the conditional indirect utility and expenditure functions take the special form

$$(2.19) \qquad \tilde{v}_j(p_j, q_j, y) = h_j(q_j, y - p_j\bar{x}_j), \qquad (j = 1, 2),$$

$$(2.20) \qquad \tilde{e}_j(p_j, q_j, u) = g_j(q_j, u) + p_j\bar{x}_j, \qquad (j = 1, 2),$$

where $g_j(\cdot)$ is the inverse of $h_j(\cdot)$ with respect to its second argument. The special structure of (2.19) and (2.20) does <u>not</u> apply to the utility model (2.1)-(2.4).

This special structure has some implications for the three assumptions about the consumer's preferences which SR invoke in deriving their welfare formula. These assumptions are:

ASSUMPTION A: *The conditional marginal utility of income* $\partial\tilde{v}_j(p_j, q_j, y)/\partial y$ *is approximately independent of* p_j *and* q_j.

ASSUMPTION B: *The discrete goods are sufficiently unimportant that income effects from quality changes are negligible; i.e., the compensated demand function (2.16) is adequately approximated by the ordinary demand function (2.15).*

ASSUMPTION C: $\partial\tilde{v}_j(p_j, q_j, y)/\partial q_j \to 0$ *as* $p_j \to \infty$.

From (2.19) it can be seen that, for the purely qualitative choice model considered here, Assumption A implies that the conditional indirect utility functions may be approximated by

$$\tilde{v}_j(p_j, q_j, y) = h_j(q_j) - \gamma_j p_j \bar{x}_j + \gamma_j y, \qquad (j = 1, 2), \tag{2.21a}$$

where γ_j is a positive constant.[11] This in turn implies that the direct utility function (2.1) may be approximated by

$$u(x_1, x_2, q_1, q_2, z) = h(x_1, x_2, q_1, q_2) + \Theta\gamma_1 z + (1 - \Theta)\gamma_2 z \tag{2.21b}$$

for some function $h(\cdot)$, where $\Theta = 1$ if $x_1 > 0$ and $\Theta = 0$ otherwise.

If Assumption B is taken as an approximate restriction on the structure of the consumer's preferences, it is shown in the Appendix to imply that the conditional and unconditional utility functions have the same form as in (2.21a, b) with the added restriction that $\gamma_1 \equiv \gamma_2$; i.e.,

$$\tilde{v}_j(p_j, q_j, y) = h_j(q_j) - \gamma p_j \bar{x}_j + \gamma y, \qquad (j = 1, 2), \tag{2.22a}$$

$$u(x_1, x_2, q_1, q_2, z) = h(x_1, x_2, q_1, q_2) + \gamma z. \tag{2.22b}$$

In this case the income variable <u>cancels out</u> of the utility difference $\tilde{v}_1 - \tilde{v}_2$. Finally, it can be seen from (2.21a) and (2.22a) that Assumptions A and B each preclude Assumption C, since they imply that $\partial\tilde{v}_j/\partial q_j$ is independent of p_j. However, this is of no consequence because it will be shown in the following section that Assumption C is actually unnecessary for the derivation of SR's welfare formula.

3. Random Utility Qualitative Choice

A random utility model arises when one assumes that, although the consumer's utility function is deterministic for him, it contains some components which are unobservable to the econometric investigator and are treated by the investigator as random variables. As in the deterministic case discussed above, the starting point for the analysis is the concept of the consumer's utility conditional on the choice of good (action) j, $\tilde{u}_j$. In the random utility context this is a fixed number for the consumer, but because his preferences are incompletely observed it is a random variable for the econometric investigator. Its mean, $E\{\tilde{u}_j\}$, will be denoted by $\tilde{v}_j$. The situation may then be represented as

$$(3.1) \qquad \tilde{u}_j = \tilde{v}_j + \varepsilon_j, \qquad (j = 1, 2),$$

where ε_1 and ε_2 are fixed constants (functions) for the consumer representing the unobservable component of his preferences, but are treated by the investigator as jointly distributed random variables. Their joint density function will be denoted by $f_\varepsilon(\varepsilon_1, \varepsilon_2)$, and their joint cdf by $F_\varepsilon(\varepsilon_1, \varepsilon_2)$; by construction $E\{\varepsilon_j\} = 0$.[13] Although the consumer knows for sure which good (action) maximizes his utility, the econometric investigator does not know this because of the unobservable component of the consumer's preferences. Thus the utility maximization discrete choice index δ_1, which equals 1 if the consumer selects good 1 and 0 otherwise, is a random variable for the investigator with mean, $E\{\delta_1\} \equiv \pi_1$, given by

$$(3.2) \qquad \begin{aligned} \pi_1(\tilde{v}_1, \tilde{v}_2) &= \Pr\{\tilde{u}_1 \geq \tilde{u}_2\} \\ &= F_\eta(\tilde{v}_1 - \tilde{v}_2) \end{aligned}$$

where $F_\eta(\cdot)$, the cdf of the random variable $\eta \equiv \varepsilon_2 - \varepsilon_1$, is derived from $f_\varepsilon(\cdot)$ by change of variable.

The model consisting of (3.1) and (3.2), with no restrictions on the arguments or functional structure of $\tilde{v}_1$ and $\tilde{v}_2$, is known as a random utility maximization (RUM) quantal choice model. Its properties have been analyzed by Daly and Zachary [4], Williams [15], and Ben Akiva and Lerman [1]. These authors impose the additional assumption

ASSUMPTION D: *The distribution* $f_\varepsilon(\varepsilon_1, \varepsilon_2)$ *is independent of* $(\tilde{v}_1, \tilde{v}_2)$, which will be invoked at several points below. If one adds to (3.1) and (3.2) the requirement that $\tilde{u}_j$ be a conditional indirect utility function one obtains what might be called a budget-constrained RUM quantal choice model. Such a model is generated by converting the deterministic utility model (2.1)-(2.5) to a random utility setting. In place of (2.1) it would be natural to postulate the utility function

(3.3) $$u = u(x_1, x_2, q_1, q_2, z) + \Theta\varepsilon_1 + (1 - \Theta)\varepsilon_2$$

where Θ is 1 if $x_1 > 0$ and 0 otherwise.[14] As in (3.1) the terms ε_1 and ε_2 represent components of the consumer's utility function which are perceived as random variables by the econometric investigator. The consumer maximizes (3.3)—which is nonstochastic for him—subject to the same constraints as before:

(3.4a) $$p_1x_1 + p_2x_2 + z = y$$

(3.4b) $$z \geq 0$$

(3.4c) $$x_1x_2 = 0$$

(3.4d) $$x_j = \bar{x}_j \text{ or } 0, \qquad (j = 1, 2).$$

I also continue to assume that

(3.4e) $$\partial[u(\bar{x}_1, 0, q_1, q_2, y - p_1\bar{x}_1) + \Theta\varepsilon_1 + (1 - \Theta)\varepsilon_2]/\partial q_2 = 0$$
$$\partial[u(0, \bar{x}_2, q_1, q_2, y - p_2\bar{x}_2) + \Theta\varepsilon_1 + (1 - \Theta)\varepsilon_2]/\partial q_1 = 0.$$

The budget-constrained RUM quantal choice model consists of (3.3) and (3.4) together with a particular specification of the pdf $f_\varepsilon(\varepsilon_1, \varepsilon_2)$. By applying (3.4a-d) to (3.3) one obtains the quantal choice probability formula (3.2) with

$$(3.5) \qquad \tilde{v}_1 = u(\bar{x}_1,\ 0,\ q_1,\ q_2,\ y - p_1\bar{x}_1)$$
$$\tilde{v}_2 = u(0,\ \bar{x}_2,\ q_1,\ q_2,\ y - p_2\bar{x}_2).$$

By virtue of (3.4e) this can be simplified to

$$(3.6) \qquad \tilde{v}_j = h_j(q_j,\ y - p_j\bar{x}_j), \qquad (j = 1,\ 2).$$

where $h_j(\cdot)$ is increasing in both its arguments. Moreover, if Assumptions A or B are invoked, then (3.6) takes the forms given by the right-hand sides of (2.21a) and (2.22a), respectively. Thus, whereas the RUM quantal choice model consists of (3.1) and (3.2), the budget-constrained RUM model consists of (3.1), (3.2), and (3.6).[15]

Once the consumer has made an optimal choice his utility is $v = \max\{\tilde{u}_1,\ \tilde{u}_2\}$. In the context of a budget-constrained RUM model this is the unconditional indirect utility function. In that case there is additional structure on the functional form of v from (3.6); to signify this I will write $v = v(p_1,\ p_2,\ q_1,\ q_2,\ y)$. Although v is deterministic for the consumer, it is a random variable for the econometric investigator, with cdf $F_v(u) = F_\varepsilon(u - \tilde{v}_1,\ u - \tilde{v}_2)$. Thus, rather than knowing the consumer's true utility, the investigator knows its probability distribution function $F_v(u)$ whose parameters he estimates from the quantal choice probability formula (3.2). In these circumstances it would be natural for the investigator to focus on the mean of this distribution, $E\{v\} \equiv V$, where[16]

$$(3.7) \qquad V = \int_{-\infty}^{\infty} \int_{-\infty}^{\tilde{v}_1 - \tilde{v}_2 + \varepsilon_1} (\tilde{v}_1 + \varepsilon_1)\ f_\varepsilon(\varepsilon_1,\ \varepsilon_2)\, d\varepsilon_2 d\varepsilon_1$$
$$+ \int_{-\infty}^{\infty} \int_{\tilde{v}_1 - \tilde{v}_2 + \varepsilon_1}^{\infty} (\tilde{v}_2 + \varepsilon_2)\ f_\varepsilon(\varepsilon_1,\ \varepsilon_2)\, d\varepsilon_2 d\varepsilon_1$$
$$= E\{\tilde{u}_1 | \tilde{u}_1 \geq \tilde{u}_2\} + E\{\tilde{u}_2 | \tilde{u}_2 > \tilde{u}_1\}\ .$$

By differentiating (3.7) Daly and Zachary [4] have shown that, for any RUM which satisfies Assumption D,[17]

$$(3.8) \qquad \frac{\partial V}{\partial \tilde{v}_j} = \pi_j(\tilde{v}_1, \tilde{v}_2), \qquad (j = 1, 2).$$

In the context of the budget-constrained RUM, application of the chain rule to (3.8) yields:

$$(3.9a) \qquad \frac{\partial V}{\partial p_j}(p_1, p_2, q_1, q_2, y) = \pi_j \frac{\partial \tilde{v}_j}{\partial p_j} = -\pi_j \frac{\partial \tilde{v}_j}{\partial y} \bar{x}_j, \qquad (j = 1, 2)$$

$$(3.9b) \qquad \frac{\partial V}{\partial q_j}(p_1, p_2, q_1, q_2, y) = \pi_j \frac{\partial \tilde{v}_j}{\partial q_j}, \qquad (j = 1, 2)$$

$$(3.9c) \qquad \frac{\partial V}{\partial y}(p_1, p_2, q_1, q_2, y) = \pi_1 \frac{\partial \tilde{v}_1}{\partial y} + (1 - \pi_1) \frac{\partial \tilde{v}_2}{\partial y}.$$

Similarly, the consumer's unconditional ordinary demand function for good j, x_j, is a random variable for the investigator given by $x_j = \delta_j \bar{x}_j$ where δ_j is the random utility maximization discrete choice index. Its mean, $E\{x_j\} \equiv X_j$, is given by

$$(3.10) \qquad X_j = \pi_j \bar{x}_j, \qquad (j = 1, 2).$$

An interesting implication of (3.9) and (3.10) is that Roy's Identity, (2.9), does not generally hold in the budget-constrained RUM model when the random variables are replaced by their expectations:

$$(3.11) \qquad X_j \neq \frac{\partial V/\partial p_j}{\partial V/\partial y}.$$

It can be seen from (3.9c) that Roy's Identity holds only when $\partial \tilde{v}_1/\partial y \equiv \partial \tilde{v}_2/\partial y$, which corresponds to the case of Assumption B. This special case is further discussed below.

Welfare Evaluations

As in the deterministic qualitative choice model, the unconditional indirect utility function is an important tool for applied welfare analysis. Since v is now random, it would be natural to frame welfare evaluations in terms of its expectation, V. Suppose that q_1 changes from q_1^0 to q_1^f, while p_1, p_2, q_2 and y do not change. A compensating variation measure of the effect of this change on the consumer's welfare appropriate to the RUM context would be the quantity CV defined by [18]

(3.12) $V(p_1, p_2, q_1^f, q_2, y + CV) = V(p_1, p_2, q_1^0, q_2, y).$

CV is the amount of money that one would have to give the consumer after the quality change in order to render him as well off as he was before the change where, because the consumer's preferences are partially unobservable, the welfare comparison is based on the observer's expectation of his utility. By analogy with the standard welfare analysis of price changes, an alternative measure of the welfare effect of the quality change in RUM context would be the equivalent variation, EV, defined by

(3.13) $V(p_1, p_2, q_1^f, q_2, y) = V(p_1, p_2, q_1^0, q_2, y - EV).$

EV is the amount of money that one would have to give the consumer before the quality change in order to induce him to forego it. Since, by (3.9c), $\partial V/\partial y > 0$, it follows that sign (CV) = sign (EV). However, unless $V(\cdot)$ is quasilinear in y, CV $\neq$ EV.

An alternative way to define welfare measures is by working with expenditure functions. However, this is a little more complicated in the random utility case than in the deterministic utility case. The conditional expenditure function corresponding to (3.6), denoted $\tilde{e}_j$, is given by

(3.14) $\tilde{e}_j = g_j(q_j, u - \varepsilon_j) + p_j\bar{x}_j$, (j = 1, 2)

where $g_j(\cdot)$ is the inverse of $h_j(\cdot)$ in (3.6) with respect to its second argument.[19] The unconditional expenditure function is

(3.15) $e(p_1, p_2, q_1, q_2, u) = \min\{\tilde{e}_1, \tilde{e}_2\}.$

In the random utility context $\tilde{e}_1$, $\tilde{e}_2$, and e are all random variables; the mean of the latter, $E\{e\} \equiv E$, is given by

$$(3.16)\qquad E(p_1, p_2, q_1, q_2, u) = E\{\tilde{e}_1 | \tilde{e}_1 \leq \tilde{e}_2\} + E\{\tilde{e}_2 | \tilde{e}_2 < \tilde{e}_1\}$$

$$= \int_{-\infty}^{\infty} \int_{-\infty}^{\kappa} [g_1(q_1, u - \varepsilon_1) + p_1\bar{x}_1] f_\varepsilon(\varepsilon_1, \varepsilon_2) d\varepsilon_2\, d\varepsilon_1$$

$$+ \int_{\infty}^{\infty} \int_{\kappa}^{\infty} [g_2(q_2, u - \varepsilon_2) + p_2\bar{x}_2] f_\varepsilon(\varepsilon_1, \varepsilon_2) d\varepsilon_2\, d\varepsilon_1$$

where $\kappa \equiv u - h_2[q_2, g_1(q_1, u - \varepsilon_1) + p_1\bar{x}_1 - p_2\bar{x}_2]$.

By analogy with (2.12) it would be natural to define the compensating variation for the change in quality from q_1^0 to q_1^f in terms of the expected unconditional expenditure function, CV', as

$$(3.17a)\qquad CV' = E(p_1, p_2, q_1^f, q_2, V^0) - E(p_1, p_2, q_1^0, q_2, V^0)$$

where $V^0 \equiv V(p_1, p_2, q_1^0, q_2, y)$. Similarly one might define the equivalent variation, EV', by

$$(3.17b)\qquad EV' = E(p_1, p_2, q_1^f, q_2, V^f) - E(p_1, p_2, q_1^0, q_2, V^f)$$

where $V^f \equiv V(p_1, p_2, q_1^f, q_2, y)$. However, unlike the situation in the deterministic utility model, the welfare measures defined in terms of the expected indirect utility function do not necessarily coincide with those defined in terms of the expected expenditure function; i.e., $CV \neq CV'$ and $EV \neq EV'$. This arises because the deterministic utility identity (2.12) does <u>not</u> generally carry over to the budget-constrained RUM model when random variables are replaced by their expectations: on substituting (3.7) into (3.16) one finds that in general

$$(3.18)\qquad E(p_1, p_2, q_1, q_2, V(p_1, p_2, q_1, q_2, y)) \neq y.$$

It also follows from this that CV' and EV' cannot be expressed as differences between the consumer's actual income, y, and the expected income which would leave him equally well off before or after the quality change, using the investigator's expectation of his utility level.

Because of this, the welfare measures CV and EV seem to possess a more intuitive and appealing interpretation than the welfare measures

CV′ and EV′. Accordingly, I would recommend that one employ CV and EV for welfare evaluations in the budget-constrained RUM model. In order to calculate them one needs to obtain an explicit formula for $V(p_1, p_2, q_1, q_2, y)$, which can be derived by evaluating (3.7) directly or by integrating (3.8). Closed form expressions for V in some common quantal choice models are provided in the next section. Given this formula, one would solve (3.12) or (3.13) for CV or EV. Since these are in general nonlinear equations, one would have to employ numerical techniques such as Newton's method or its variants. But as a practical matter this method of welfare analysis is entirely feasible.

An important simplification can be obtained in the special case where the consumer's preferences satisfy Assumption B—i.e., the right-hand side of (3.6) takes the special form given in (2.22a):

(3.19) $$\tilde{v}_j = h_j(q_j) - \gamma p_j \bar{x}_j + \gamma y, \qquad (j = 1, 2).$$

This assumption is invoked by SR; I will show that no other assumption—including C—is required in order to derive their welfare formula. For the budget-constrained RUM model consisting of (3.1), (3.2), and (3.19), the quantal choice probabilities take the form

(3.20) $$\pi_1 = F_\eta[h_1(q_1) - h_2(q_2) - \gamma(p_1\bar{x}_1 - p_2\bar{x}_2)].$$

Denote the argument of $F_\eta(\cdot)$ by w. The expected value of the indirect utility function is given by

$$\begin{aligned} V(p_1, p_2, q_1, q_2, y) &= E\{\max\{h_1(q_1) - \gamma p_1\bar{x}_1 + \gamma y + \varepsilon_1, \\ &\qquad h_2(q_2) - \gamma p_2\bar{x}_2 + \gamma y + \varepsilon_2\}\} \\ &= \gamma y + E\{\max\{h_1(q_1) - \gamma p_1\bar{x}_1 + \varepsilon_1, \\ &\qquad h_2(q_2) - \gamma p_2\bar{x}_2 + \varepsilon_2\}\} \\ &= \gamma y + \int_{-\infty}^{\infty} \int_{-\infty}^{W} [h_1(q_1) - \gamma p_1\bar{x}_1 + \varepsilon_1] f_\varepsilon(\varepsilon_1, \varepsilon_2)\, d\varepsilon_2 d\varepsilon_1 \\ &\quad + \int_{-\infty}^{\infty} \int_{W}^{\infty} [h_2(q_2) - \gamma p_2\bar{x}_2 + \varepsilon_2] f_\varepsilon(\varepsilon_1, \varepsilon_2)\, d\varepsilon_2 d\varepsilon_1 \end{aligned}$$

(3.21) $\equiv \gamma y + T(p_1, p_2, q_1, q_2)$.

On substituting (3.21) into the definitions (3.12) and (3.13), one finds that

(3.22) $$CV = EV = \frac{1}{\gamma} [T(p_1, p_2, q_1^0, q_2) - T(p_1, p_2, q_1^f, q_2)].$$

In this case one can obtain the welfare measures directly without having to solve a nonlinear equation. Since the term in square brackets in (3.22) is equal to $[V(p_1, p_2, q_1^0, q_2, y) - V(p_1, p_2, q_1^f, q_2, y)]$, one can also write:[20]

$$CV = EV = -\frac{1}{\gamma} \int_{q_1^0}^{q_1^f} \frac{\partial V}{\partial \tilde{v}_1} \frac{\partial \tilde{v}_1}{\partial q_1} dq_1$$

$$= -\frac{1}{\gamma} \int_{\tilde{v}_1^0}^{\tilde{v}_1^f} \frac{\partial V}{\partial \tilde{v}_1} d\tilde{v}_1$$

(3.23) $$= -\frac{1}{\gamma} \int_{\tilde{v}_1^0}^{\tilde{v}_1^f} \pi_1(\tilde{v}_1, \tilde{v}_2) d\tilde{v}_1$$

where $\tilde{v}_1^t = \tilde{v}_1(p_1, q_1^t, y)$, $t = 0, f$. This is the form in which SR present their welfare formula [14, equation (5.5)].

It should also be noted that (3.19) combined with (3.16) implies

$$E(p_1, p_2, q_1, q_2, u) = E\{\min\{[\gamma p_1\bar{x}_1 - h_1(q_1) + u - \varepsilon_1]/\gamma\ ,\ [\gamma p_2\bar{x}_2 - h_2(q_2) + u - \varepsilon_2]/\gamma\}\}$$

$$= \frac{u}{\gamma} + \frac{1}{\gamma} \cdot E\{\min\{\gamma p_1\bar{x}_1 - h_1(q_1) - \varepsilon_1,\ \gamma p_2\bar{x}_2 - h_2(q_2) - \varepsilon_2\}\}$$

(3.24) $$= \frac{u}{\gamma} - \frac{1}{\gamma} T(p_1, p_2, q_1, q_2)\ .$$

On comparing (3.21) and (3.24) one observes that the inequality in (3.18) is removed and that $CV = CV' = EV = EV'$. It must be emphasized that the coincidence of all four welfare measures is a direct consequence of Assumption B. It does not occur, for example, under the weaker Assumption A.

It has been shown that under the "no income effects" Assumption B there is a single welfare measure which can be evaluated without resort to the solution of nonlinear equations. However, the practical significance of this result should not be exaggerated. Assumption B imposes strong restrictions on the structure of the consumer's preferences and, consequently, on the formula for the quantal choice probabilities. As can be seen from (3.20), it precludes models in which the consumer's income appears as a variable in the quantal choice probability formula.[21] This rules out many of the empirical quantal choice models which have appeared in the literature. The point to be emphasized is that one can still conduct welfare evaluations with budget-constrained RUM choice models which do not satisfy this assumption. As long as one has an explicit formula for $V(\cdot)$, the welfare measures CV and EV can always be obtained by solving (3.12) or (3.13).

It should also be emphasized that a similar approach to welfare analysis can be employed when one has a RUM quantal choice model which does not satisfy (3.6)—i.e., when the conditional indirect utility function (3.1) is some general function of income and other variables.[22] As an example, suppose that one is dealing with choices among actions with uncertain consequences and $\tilde{v}_j$ is a von Neumann-Morgenstern expected utility function. An individual has wealth y and a utility of wealth function whose nonstochastic component is denoted by $\psi(y)$. The individual must choose between two actions whose consequences depend on the state of the world, s.[23] Associated with act j is a vector of state probabilities, $\rho_j = (\rho_{j1}, \ldots, \rho_{jS})$, and a vector of monetary consequences, $z_j = (z_{j1}, \ldots, z_{jS})$. Conditional on the choice of act j the individual's utility is

$$\tilde{u}_j = \sum_s \rho_{js} \psi(y + z_{js}) + \varepsilon_j \equiv \tilde{v}_j + \varepsilon_j, \qquad (j = 1, 2). \tag{3.25}$$

Using this definition of $\tilde{v}_1$ and $\tilde{v}_2$, the probability that the individual selects act 1 is given by (3.2). The individual's unconditional utility (i.e., after making the optimal choice) is $v = \max\{\tilde{u}_1, \tilde{u}_2\}$. This is a random variable for the econometric investigator with a mean, $V(\rho_1, \rho_2, z_1, z_2, y)$, given by the right-hand side of (3.7). Now suppose, for example, that the state probabilities change from (ρ_1^0, ρ_2^0) to (ρ_1^f, ρ_2^f). By analogy with (3.12) and (3.13), the compensating and equivalent variation measures of the effect of this change on the individual's welfare are defined by

$$V(\rho_1^f, \rho_2^f, z_1, z_2, y + CV) = V(\rho_1^0, \rho_2^0, z_1, z_2, y) \tag{3.26}$$

and

$$V(\rho_1^f, \rho_2^f, z_1, z_2, y) = V(\rho_1^0, \rho_2^0, z_1, z_2, y - EV) \ . \tag{3.27}$$

An application of this methodology to the valuation of changes in mortality probabilities on the basis of private choices among risky actions is presented in [7].

4. Econometric Applications

The purpose of this section is to record the formulas for $V(\cdot)$ arising from several common econometric quantal choice models. With the aid of these formulas one can set up and solve the equations defining the compensating and equivalent variation measures of the welfare effects of any change in the variables influencing the individual's choice. For the sake of greater generality, in this section I will deal with multinomial quantal choices rather than the binary choices considered above. Conceptually these welfare measures presuppose that the quantal choice model is either a RUM model or a budget-constrained RUM model. In the general multinomial case there are N alternative outcomes. Let $\tilde{u}_j = \tilde{v}_j + \varepsilon_j$, $j = 1, \ldots, N$, where $E(\varepsilon_j) = 0$ and $\tilde{v}_j$ is some function of variables and coefficients to be estimated. In a multinomial RUM model the jth outcome occurs if $\tilde{u}_j = \max\{\tilde{u}_1, \ldots, \tilde{u}_N\}$, and the probability that this happens, π_j, takes the form

$$(4.1) \qquad \pi_j = F_j(\tilde{v}_1 - \tilde{v}_j, \ldots, \tilde{v}_{j-1} - \tilde{v}_j, \tilde{v}_{j+1} - \tilde{v}_j, \ldots, \tilde{v}_N - \tilde{v}_j) \qquad (j = 1, \ldots, N)$$

where $F_j(\cdot)$ is a joint distribution function of dimension $(N - 1)$. Let $F_\varepsilon(\varepsilon_1, \ldots, \varepsilon_N)$ be the joint distribution function of $(\varepsilon_1, \ldots, \varepsilon_N)$. The interpretation of (4.1) is that $F_j(\cdot)$ is the joint distribution function of the $(N - 1)$ differences $\eta_k \equiv \varepsilon_k - \varepsilon_j$, $k \neq j$, derived from $F_\varepsilon(\cdot)$ by change of variable.[24] Hence in a RUM model the quantal choice probabilities depend on a set of $(N - 1)$ differences, $\tilde{v}_k - \tilde{v}_j$, $k \neq j$. In a multinomial budget-constrained RUM model there is the additional constraint that the $\tilde{v}_j$'s have the form of (3.6). Below I will give formulas for $V \equiv E\{\max\{\tilde{u}_1, \ldots, \tilde{u}_N\}\}$ without specifying anything more about the $\tilde{v}_j$'s. Given the formula for V, the compensating and equivalent variations are obtained by solving equations such as (3.12) and (3.13), or (3.26) and (3.27).

Multinomial Logit Models

In the generalized multinomial logit model, introduced by McFadden [12], the random terms $(\varepsilon_1, \ldots, \varepsilon_N)$ are jointly distributed according to the generalized extreme value distribution; i.e.,

$$(4.2) \qquad F_\varepsilon(\varepsilon_1, \ldots, \varepsilon_N) = \exp\,[-G(e^{-\varepsilon_1}, \ldots, e^{-\varepsilon_N})]$$

where $G(t_1, \ldots, t_N)$ is an arbitrary non-negative linear homogenous function. Let its partial derivative with respect to the jth argument be denoted by $G_j(t_1, \ldots, t_N)$. McFadden shows that

$$(4.3) \qquad \pi_j = \frac{e^{\tilde{v}_j} G_j(e^{\tilde{v}_1}, \ldots, e^{\tilde{v}_N})}{G(e^{\tilde{v}_1}, \ldots, e^{\tilde{v}_N})} \qquad (j = 1, \ldots, N)$$

and

(4.4) $V = \ln G(e^{\tilde{v}_1}, \ldots, e^{\tilde{v}_N}) + 0.5722\ldots$ (Euler's constant).

The special case where $G(t_1, \ldots, t_N) = \Sigma_1^N t_j$ and $G_j = 1$ is the standard multinomial logit model, described in [10].

Multinomial Probit Models

In multinomial probit models $F_\varepsilon(\varepsilon_1, \ldots, \varepsilon_N)$ is a multivariate normal distribution with some covariance matrix $\Sigma = \{\sigma^2_{\varepsilon_j\varepsilon_k}\}$. Hence, the function $F_j(\cdot)$ in (4.1) is an $(N - 1)$ dimensional multivariate normal distribution function with zero mean and a covariance matrix Ω whose typical element is $w_{kl} = \sigma^2_{\varepsilon_j\varepsilon_j} + \sigma^2_{\varepsilon_k\varepsilon_l} - \sigma^2_{\varepsilon_k\varepsilon_j} - \sigma^2_{\varepsilon_l\varepsilon_j}$. In the standard multinomial probit model the elements of Σ are fixed constants independent of the $\tilde{v}_j$'s. Therefore, the model satisfies Assumption D.[25] In the generalized probit model, introduced by Hausman and Wise [9], this assumption is relaxed. As an example, consider the following random coefficient budget-constrained RUM model.

(4.5) $\tilde{u}_j = (\bar{\alpha} + \alpha)q_j + (\bar{\gamma} + \gamma)(y - p_j) + \upsilon_j \qquad (j = 1, \ldots, N),$

where $\bar{\alpha}$ and $\bar{\gamma}$ are fixed constants and $\alpha, \gamma, \upsilon_1, \ldots, \upsilon_n$ are independently distributed normal random variables with zero means and variances σ^2_α, σ^2_γ, $\sigma^2_{\upsilon_1}, \ldots, \sigma^2_{\upsilon_N}$. Collecting all the random terms together, (4.5) can be written

(4.6)
$$\tilde{u}_j = [\bar{\alpha}q_j + \bar{\gamma}(y - p_j)] + [\alpha q_j + \gamma(y - p_j) + \upsilon_j]$$
$$\equiv \tilde{v}_j + \varepsilon_j.$$

Hence,

(4.7)
$$\sigma^2_{\varepsilon_j\varepsilon_k} = \begin{cases} \sigma^2_\alpha q_j^2 + \sigma^2_\gamma(y - p_j)^2 + \sigma^2_{\upsilon_j}, & j = k \\ \sigma^2_\alpha q_j q_k + \sigma^2_\gamma(y - p_j)(y - p_k), & j \neq k \end{cases}.$$

In this case, Assumption D does not hold and the relations in (3.8) and (3.9) do not apply. Nevertheless, the indirect utility function is still a well-defined concept, and its mean, V, is still given by the multinomial generalization of (3.7).

It can be seen from (4.6) that the generalized probit model differs from the standard probit model through its specification of the elements of the covariance matrix Σ. In order to cover both models at the same time, I will present a single formula for V, defined in terms of a general matrix Σ. Now, V is the mean of the maximum of N dependent, nonidentically distributed random normal variables, $\tilde{u}_1, \ldots, \tilde{u}_N$. As explained in Clark [2], except for the case where N = 2, an exact closed-form expression for V does not exist, and one must resort to an approximation. Let $\phi(\cdot)$ be the standard univariate normal pdf, and $\Phi(\cdot)$ the corresponding cdf. Define $u_i^* \equiv \max\{\tilde{u}_1, \ldots, \tilde{u}_i\}$ and $V_i \equiv E\{u_i^*\}$. For N = 2, the bivariate standard or generalized probit model, the exact formula for V_2 is

$$(4.7) \qquad V_2 = \tilde{v}_1\Phi(\lambda_2) + \tilde{v}_2\Phi(-\lambda_2) + a_2\phi(\lambda_2)$$

where

$$a_2 = [\sigma^2_{\varepsilon_1\varepsilon_1} + \sigma^2_{\varepsilon_2\varepsilon_2} - 2\sigma^2_{\varepsilon_1\varepsilon_2}]^{\frac{1}{2}}$$

$$\lambda_2 = (\tilde{v}_1 - \tilde{v}_2)/a_2.$$

For N = 3, the trivariate probit model, Clark's approximation for V_3 is

$$(4.8) \qquad V_3 \approx \tilde{v}_3\Phi(\lambda_3) + V_2\Phi(-\lambda_3) + a_3\phi(\lambda_3)$$

where

$$a_3 = [\sigma^2_{\varepsilon_3\varepsilon_3} + S_2^2 - 2S^2_{2,3}]^{\frac{1}{2}}$$

$$\lambda_3 = (V_2 - \tilde{v}_3)/a_3$$

$$S^2_{2,3} = \sigma^2_{\varepsilon_2\varepsilon_3} + (\sigma^2_{\varepsilon_1\varepsilon_3} - \sigma^2_{\varepsilon_2\varepsilon_3})\,\Phi\,(\lambda_2)$$

$$S_2^2 = \tilde{v}_2^2 + \sigma^2_{\varepsilon_2\varepsilon_2} + (\tilde{v}_1^2 + \sigma^2_{\varepsilon_1\varepsilon_1} - \tilde{v}_2^2 - \sigma^2_{\varepsilon_2\varepsilon_2})\,\Phi\,(\lambda_2) + (\tilde{v}_1 + \tilde{v}_2)a_2\phi(\lambda_2) - V_2^2 .$$

A fuller account of Clark's approximation is given in the Appendix, together with the general recursion formulas.[26] Given these, one can set up the equations analogous to (3.12) and (3.13), or (3.26) and (3.27), and solve them for the compensating and equivalent variations, CV and EV.[27] Obviously an iterative solution procedure is required and, while still feasible, it would become tedious if N were large. However, most of the probit applications which have appeared in the literature are for cases where $N \leq 3$.

5. Conclusion

The results of the preceding two sections provide a fully operational procedure for conducting welfare evaluations on the basis of econometrically estimated quantal choice models. The procedure can be employed as long as the quantal choice model is a RUM model--i.e., the quantal choice probability formulas are recognized as being functions of utility differences of the form $\tilde{v}_1 - \tilde{v}_j$, $\tilde{v}_2 - \tilde{v}_j$, . . ., as in (3.2) or (4.1). From the fitted model one can obtain estimates of the coefficients of the $\tilde{v}_j$ functions and the parameters of the density function $f_\varepsilon(\varepsilon_1, \ldots, \varepsilon_N)$, usually up to some normalization convention. With this information one can evaluate the expected unconditional indirect utility function using either the general formula (3.7) or, in the case of logit or probit models, the specific formulas given in Section 4. Then one can set up and solve the equations such as (3.12) or (3.13) which define the compensating and equivalent variation measures of the welfare effects of some change in the variables appearing in the quantal choice model. Unless the model satisfies Assumptions B and D, the solution of these equations will generally require iterative numerical techniques. With current computer software, this should not be a serious obstacle.

The emphasis throughout this paper has been on the welfare theory of an individual consumer. An implication is that the welfare measures described above must be calculated separately for each consumer. The problems of estimating quantal choice models from aggregate choice data and developing welfare inferences on the basis of aggregate utility functions have not been addressed here. My approach presupposes that the quantal choice model is estimated from disaggregated micro data and that all the individuals in the sample have the same preferences—i.e., the nonstochastic component of their preferences is represented by the same utility function, and the random elements are governed by the same probability law. In fact, these conditions are met by virtually all of the empirical quantal choice models which have appeared in the literature. Hence, the technique of welfare analysis described here should be widely applicable.

University of California, Berkeley

APPENDIX

A. 1. *Implications of Assumption B*

Here I prove that the consumer's preferences must have the form given in (2.22a, b) if Assumption B holds. Without loss of generality I assume that $\bar{x}_1 = \bar{x}_2 = 1$. Consider the unconditional ordinary demand function for good 1. As a function of p_1, for given values of (p_2, q_1, q_2, y), it may be written in the form

(A.1)
$$x_1(p_1, p_2, q_1, q_2, y) = \begin{cases} 1 \text{ if } p_1 \leq p_1^* \\ 0 \text{ otherwise} \end{cases}$$

where $p_1^* = p_1^*(p_2, q_1, q_2, y)$ is defined by

(A.2) $\quad u(1, 0, q_1, q_2, y - p_1^*) = u(0, 1, q_1, p_2, y - p_2).$

Suppose that the actual price of good 1 is p_1^0. By virtue of (A.2), one can write

(A.3) $\quad p_1^* = p_1^0 - C^*$

where C^* is defined by

(A.4) $\quad u(1, 0, q_1, q_2, y - p_1^0 + C^*) = u(0, 1, q_1, q_2, y - p_2).$

The consumer's actual utility is $u^0 = v(p_1^0, p_2, q_1, q_2, y)$. Given (p_2, q_1, q_2, u^0), the consumer's unconditional compensated demand function for good 1, as a function of p_1, may be written

(A.5)
$$x_1^c(p_1, p_2, q_1, q_2, u^0) = \begin{cases} 1 \text{ if } p_1 \leq p_1^{**} \\ 0 \text{ otherwise} \end{cases}$$

where $p_1^{**} = p_1^{**}(p_2, q_1, q_2, u^0)$ is defined by

$$\tilde{e}_1(p_1^{**}, q_1, u^0) = \tilde{e}_2(p_2, q_2, u^0)$$

or, equivalently, by

(A.6) $\quad p_1^{**} + u^{-1}(u^0|1, 0, q_1, q_2) = p_2 + u^{-1}(u^0|0, 1, q_1, q_2)$

where $u^{-1}(u|x_1, x_2, q_1, q_2)$ is the inverse of $u(x_1, x_2, q_1, q_2, z)$ with respect to its last argument.

It follows from (A.1) and (A.5) that the ordinary and compensated demand functions coincide if and only if $p_1^* = p_1^{**}$. This occurs automatically when $p_1^0 > p_1^*$, since then $u^0 = u(0, 1, q_1, q_2, y - p_2)$, and

$$u^{-1}(u^0|0, 1, q_1, q_2) = y - p_2.$$

In this case, (A.6) becomes

$$u^{-1}(u^0|1, 0, q_1, q_2) = y - p_1^{**}$$

or

$$\text{(A.7)} \qquad u(0, 1, q_1, q_2, y - p_2) = u(1, 0, q_1, q_2, y - p_1^{**}).$$

Comparison with (A.2) shows that $p_1^* = p_1^{**}$. Accordingly, I will focus on the nontrivial case where $p_1^0 \leq p_1^*$. In this case $u^0 = u(1, 0, q_1, q_2, y - p_1^0)$ and, in general, $p_1^* \neq p_1^{**}$; hence Assumption B has substantive content. Since

$$u^{-1}(u^0|1, 0, q_1, q_2) = y_1 - p_1^0,$$

(A.6) may be written

$$p_1^{**} = p_2 + u^{-1}(u^0|0, 1, q_1, q_2) - y + p_1^0$$

$$\text{(A.8)} \qquad = p_1^0 - C^{**},$$

where

$$\text{(A.9)} \qquad C^{**} = y - p_2 - u^{-1}(u^0|0, 1, q_1, q_2).$$

It follows from (A.3) and (A.8) that $p_1^* = p_1^{**}$ if and only if $C^* = C^{**}$. Manipulation of (A.9) yields

$$u^{-1}(u^0|0, 1, q_1, q_2) = y - p_2 - C^{**}$$

or

$$\text{(A.10)} \qquad u(1, 0, q_1, q_2, y - p_1^0) = u(0, 1, q_1, q_2, y - p_2 - C^{**}).$$

From (A.4) and (A.10), $C^* = C^{**}$ independently of (p_2, q_1, q_2, y) only if the utility function has the quasilinear form given in (2.22a, b).

A. 2. *SR's Derivation of the Welfare Formula*

SR derive the welfare formula (3.23) by a different route from that followed in the text. Their starting point is (2.18). Integrating this from p_j to ∞ and then differentiating with respect to q_j, taking note of (2.7), yields

$$\text{(A.11)} \qquad \frac{\partial e}{\partial q_j} = -\frac{\partial}{\partial q_j} \int_{p_j}^{\infty} x_j^c(p_1, p_2, q_1, q_2, u)dp_j \qquad (j = 1, 2).$$

Applying Assumptions A and B to (A.11), substituting from (2.15), and invoking Roy's Identity yields, for j = 1,

$$\text{(A.12)} \qquad \frac{\partial e}{\partial q_1} = \frac{1}{\lambda_1} \frac{\partial}{\partial q_1} \int_{p_1}^{\infty} \delta_1 (p_1, p_2, q_1, q_2, y) \frac{\partial \tilde{v}_1}{\partial p_1} (p_1, q_1, y) \, dp_1 .$$

If one now switches to a random utility setting and substitutes expectations for the random terms in (A.12), which is a valid operation under Assumption B, one obtains

$$\frac{\partial E}{\partial q_1} = \frac{1}{\lambda_1} \frac{\partial}{\partial q_1} \int_{p_1}^{\infty} \pi_1(\tilde{v}_1, \tilde{v}_2) \frac{\partial \tilde{v}_1}{\partial p_1} dp_1$$

$$\text{(A.13)} \qquad = \frac{1}{\lambda_1} [\, \pi_1(\tilde{v}_1^{\infty}, \tilde{v}_2) \frac{\partial \tilde{v}_1^{\infty}}{\partial q_1} - \pi_1(\tilde{v}_1, \tilde{v}_2) \frac{\partial \tilde{v}_1}{\partial q_1} \,],$$

where $\tilde{v}_1^{\infty} = \tilde{v}_1 (\infty, p_1, y)$. SR then apply Assumption C to (A.13) to eliminate the first term inside the square brackets, yielding

$$\text{(A.14)} \qquad \frac{\partial E}{\partial q_1} = -\frac{1}{\lambda_1} \pi_1(\tilde{v}_1, \tilde{v}_2) \frac{\partial \tilde{v}_1}{\partial q_1} .$$

Integration of (A.14) from q_1^0 to q_1^f yields the welfare formula (3.23). However, Assumption C is <u>not</u> required in order to pass from (A.13) to (A.14). This is because in the budget-constrained RUM quantal choice model, since $\partial \tilde{v}_j / \partial p_j < 0$,

$$\lim_{p_1 \to \infty} \pi_1(\tilde{v}_1(p_1, q_1, y), \tilde{v}_2(p_2, q_2, y)) = 0.$$

Hence the first term inside the square brackets in (A.13) vanishes even without Assumption C.

A.3. *Clark's Approximation*

Let $\tilde{u}_1$, $\tilde{u}_2$, and $\tilde{u}_3$ be normally distributed random variables with means $\tilde{v}_1$, $\tilde{v}_2$, and $\tilde{v}_3$, variances $\sigma^2_{\varepsilon_1\varepsilon_1}$, $\sigma^2_{\varepsilon_2\varepsilon_2}$, and $\sigma^2_{\varepsilon_3\varepsilon_3}$, and covariances $\sigma^2_{\varepsilon_1\varepsilon_2}$, $\sigma^2_{\varepsilon_1\varepsilon_3}$, and $\sigma^2_{\varepsilon_2\varepsilon_3}$. Then $u^*_2 = \max\{\tilde{u}_1, \tilde{u}_2\}$ is not itself normally distributed, but the exact formula for its mean, V_2, is given in (4.7), and the exact formulas for its variance, S^2_2, and for its covariance with $\tilde{u}_3$, $S^2_{2,3}$, are given in (4.8) — these formulas are taken from [2]. Clark's procedure is to approximate the joint distribution of u^*_2 and $\tilde{u}_3$ by a bivariate normal distribution with means V_2 and $\tilde{v}_3$, variances S^2_2 and $\sigma^2_{\varepsilon_3\varepsilon_3}$, and covariance $S^2_{2,3}$, and then to apply the preceding results to obtain the moments of $u^*_3 = \max\{u^*_2, \tilde{u}_3\} = \max\{\tilde{u}_1, \tilde{u}_2, \tilde{u}_3\}$. The procedure is applied recusively to obtain the approximate moments of $u^*_N = \{\max u^*_{N-1}, \tilde{u}_N\} = \max\{\tilde{u}_1, \ldots, \tilde{u}_N\}$. The general recusion formulas are, for $i = 2, \ldots, N$,

$$V_i = V_{i-1}\,\Phi(\lambda_i) + \tilde{v}_i\,\Phi(-\lambda_i) + a_i\phi(\lambda_i)$$

where
$$a^2_i = S^2_{i-1} + \sigma^2_{\varepsilon_i\varepsilon_i} - 2S^2_{i-1,\,i}$$

$$\lambda_i = (V_{i-1} - \tilde{v}_i)/a_i\,.$$

Here $S^2_{i-1} = E\{(u^*_{i-1} - V_{i-1})^2\}$ and $S^2_{i-1,\,i} = E\{u^*_{i-1} - V_{i-1})(\tilde{u}_i - \tilde{v}_i)\}$ are computed from the formulas

$$S^2_i = (V^2_{i-1} + S^2_{i-1})\,\Phi(\lambda_i) + (\tilde{v}^2_i + \sigma^2_{\varepsilon_i\varepsilon_i})\Phi(-\lambda_i) + (V_{i-1} + \tilde{v}_i)a_i\phi(\lambda_i) - V^2_i$$

and

$$S^2_{ij} = \sigma^2_{\varepsilon_i\varepsilon_j} + (S^2_{i-1,\,j} - \sigma^2_{\varepsilon_i\varepsilon_j})\Phi(\lambda_i)\,.$$

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FOOTNOTES

[1]A valuable summary of this work is provided by McFadden [11].

[2]An example of a purely qualitative choice would be which of N mutually exclusive items to select; an example of a mixed qualitative-quantitative choice would be which of N mutually exclusive items to select and how much of the selected item to consume.

[3]Daly and Zachary actually deal with multiple qualitative choices but, following the example of SR, I will mainly focus on binary qualitative choices in order to simplify the exposition. The extension to multiple qualitative choices is straightforward. With minor changes I will use the same notation as SR.

[4]An example of a logit model consisting of (1.1) and (1.6) can be found in [6].

[5]See [12] and [9].

[6]As an example, SR suggest housing which can be purchased in either a rental or an owner mode but not both.

[7]SR suggest college degrees as an example: a person typically has either one BA degree or no degree, but not several BA degrees.

[8]Some random utility models corresponding to the first two cases, involving qualitative-quantitative choices, are presented in [8] together with a methodology for applied welfare analysis. A deterministic discrete choice model similar to that studied here is analyzed by Mäler [13, pp. 131-136]. In Mäler's model there is only one non-numeraire good and therefore, the constraint (2.4) is omitted; his model corresponds to (2.1)-(2.3) and (2.5). SR [14, fn. 16] note an error in part of Mäler's analysis.

[9]SR prove that v(•) and e(•) are always continuous and right- and left-differentiable.

[10] Recall that in some contexts it would be natural to impose the normalization that $\bar{x}_j \equiv 1$.

[11] By contrast, for the qualitative-quantitative choice model (2.1)-(2.4), Assumption A implies only that $\tilde{v}_j(p_j, q_j, y) = h_j(p_j, q_j) + \gamma_j y$.

[12] More correctly, there exists an increasing transformation $T(\cdot)$ such that $T(\tilde{v}_j)$ and $T(u)$ are approximated by the right-hand sides of (2.21a) and (2.21b) respectively. The same qualification also applies to (2.22a, b).

[13] In logit models $f_\varepsilon(\cdot)$ is taken to be a multivariate extreme value pdf; in probit models it is a multivariate normal pdf.

[14] This formulation of a random utility function is suggested by SR for a qualitative-quantitative choice setting as opposed to the purely qualitative choice considered here — i.e., for a model where constraints (3.4a)-(3.4c) are imposed, but not (3.4d). In that context, this formulation is unsatisfactory because it implies that the unobservable elements of the consumer's preferences affect only his qualitative choice and not his quantitative choice: by Roy's Identity, the conditional ordinary demand functions are nonstochastic from the point of view of the econometric investigator. I find this implausible. In [8] I consider random utility qualitative-quantitative choice models based on a different formulation of the random utility function which imply that both the quantitative and the qualitative choices are random for the investigator.

[15] In their analysis of random utility discrete choice econometric models SR formulate the nonstochastic component of the conditional indirect utility function as

$$\tilde{v}_j = \phi(y) + \psi_j(p_j, q_j, y) \qquad (j = 1, 2) \tag{3.6a}$$

where $\phi(\cdot)$ is an arbitrary increasing function. This is not generally valid for a purely qualitative choice model; $\phi(\cdot)$ and $\psi_j(\cdot)$ must be such that they can be cast in the form of (3.6), which imposes some restrictions on their functional forms.

[16]At one point in their discussion [14, text above eq. (5.6)] SR seem to imply that $V = \pi_1\tilde{v}_1 + (1 - \pi_1)\tilde{v}_2$. (3.7) shows that this is an incorrect formula.

[17]Proof: (3.7) may be written as

$$V = \int_{-\infty}^{\infty} \psi_1(\tilde{v}_1, \tilde{v}_2, \varepsilon_1, \varepsilon_2)d\varepsilon_1 + \int_{-\infty}^{\infty} \psi_2(\tilde{v}_1, \tilde{v}_2, \varepsilon_1, \varepsilon_2)d\varepsilon_1 .$$

Thus, $\partial V/\partial\tilde{v}_1 = \int_{-\infty}^{\infty}(\partial\psi_1/\partial\tilde{v}_1)d\varepsilon_1 + \int_{-\infty}^{\infty}(\partial\psi_2/\partial\tilde{v}_1)d\varepsilon_1$, where

$$\frac{\partial\psi_1}{\partial\tilde{v}_1} = \int_{-\infty}^{\tilde{v}_1 - \tilde{v}_2 + \varepsilon_1} f_\varepsilon(\varepsilon_1, \varepsilon_2)d\varepsilon_2 + (\tilde{v}_1 + \varepsilon_1)f_\varepsilon(\varepsilon_1, \tilde{v}_1 - \tilde{v}_2 + \varepsilon_1)$$

and

$$\frac{\partial\psi_2}{\partial\tilde{v}_1} = -(\tilde{v}_1 + \varepsilon_1)f_\varepsilon(\varepsilon_1, \tilde{v}_1 - \tilde{v}_2 + \varepsilon_1) .$$

Hence,

$$\frac{\partial V}{\partial\tilde{v}_1} = \int_{-\infty}^{\infty} \int_{-\infty}^{\tilde{v}_1 - \tilde{v}_2 + \varepsilon_1} f_\varepsilon(\varepsilon_1, \varepsilon_2)d\varepsilon_2 d\varepsilon_1 \equiv \pi_1 .$$

[18]For the general case where prices and quality all change from $(p_1^0, p_2^0, q_1^0, q_2^0)$ to $(p_1^f, p_2^f, q_1^f, q_2^f)$ the compensating and equivalent variations would be defined by

(3.12′) $\quad V(p_1^f, p_2^f, q_1^f, q_2^f, y + CV) = V(p_1^0, p_2^0, q_1^0, q_2^0, y)$

(3.13′) $\quad V(p_1^f, p_2^f, q_1^f, q_2^f, y) = V(p_1^0, p_2^0, q_1^0, q_2^0, y - EV)$.

In the text I focus mainly on the case where only q_1 changes because this is discussed by SR.

[19]Cf. (2.20).

[20]The second step follows by making a change of variable; the third step follows from (3.8). Note that Assumption C is not employed in deriving this result; the reason why it appears in SR's derivation is explained in

the Appendix. However, Assumption D is implicitly invoked in the derivation of both (3.21) and (3.23).

[21]Note that, although y does not appear as a variable in the quantal choice probability formula, the marginal utility of income, γ, can still be recovered because it is the coefficient of the price difference which does appear in quantal choice probability formula.

[22]The quantal choice models defined by (1.1) and (1.4) or (1.6) would fall into this category.

[23]State dependent preferences can be introduced by writing the nonstochastic utility of wealth function as $\psi_s(y)$.

[24]When N = 2 one obtains the formula given in (3.2).

[25]This is also true of the generalized logit model based on (4.2).

[26]The accuracy of Clark's approximation has been investigated by Daganzo and others; it is said to be reasonably satisfactory even when N > 3 [3, pp. 55-58].

[27]For the generalized probit model (4.5) the formula for V can be slightly simplified. Define $\hat{v}_j = \bar{\alpha} q_j - \bar{\gamma} p_j$. Then

$$V_N = \bar{\gamma} y + E\{\max \hat{v}_1 + \varepsilon_1, \ldots, \hat{v}_N + \varepsilon_N\}\} \equiv \bar{\gamma} y + \hat{V}_N$$

where the formula for $\hat{V}_N$ is obtained from (4.7) or (4.8) by substituting $\hat{v}_j$ for $\tilde{v}_j$. Note that, because Σ depends on y, $\hat{V}_N$ is not independent of y.