Title

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Journal

Mathematical Geosciences, 40(2)

ISSN

1874-8961

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Publication Date

2008-02-01

DOI

10.1007/s11004-007-9136-2

Peer reviewed

Math Geosci (2008) 40: 199–211
DOI 10.1007/s11004-007-9136-2

Phreatic Surface in Island Aquifers with Regular Geometry and Time-Independent Recharge and Pumping

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Received: 15 March 2007 / Accepted: 4 November 2007 / Published online: 12 January 2008

Abstract The equation of groundwater flow in marine island aquifers in which there is time-independent, spatially-variable recharge and pumping is solved in closed form for rectangular, circular, and elliptical island geometries. The solution of the groundwater flow equation is expressed in terms of the elevation of the phreatic surface within the flow domain. The depth of the seawater-freshwater interface below mean sea level follows from the Dupuit–Ghyben–Herzberg relation. The method of solution presented in this work relies on expanding the hydraulic head and forcing function (recharge and groundwater extraction) as Fourier series that transforms the two-dimensional Poisson-type flow equations into second-order ordinary differential equations solvable using classical theory. The important case of constant recharge (without groundwater extraction) leads to solutions in which the hydraulic head is expressible as the product of a flow factor equal to the squared root of the ratio of recharge over hydraulic conductivity times a geometric factor involving island shape parameters and flow boundary conditions. Estimability conditions for the hydraulic conductivity are derived for the cases of constant recharge and spatially variable recharge with pumping.

Keywords Groundwater flow · Phreatic surface · Island aquifers · Fourier series · Elliptical coordinates

1 Introduction

The study of groundwater flow and phreatic-surface geometry in marine island aquifers and coastal areas has a long tradition. Island and coastal aquifers are in many cases vulnerable to seawater intrusion (Fetter 1972; Barlow 1972; Sato et al. 1999;

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Loáiciga and Leipnik 2000; Won et al. 2006). This concern has been heightened by sea-level rise caused by (i) ocean warming and its associated thermal expansion, and (ii) the melting of terrestrial ice caused by warming planetary trends (California Department of Water Resources 2006; Loáiciga 2006a, 2006b).

This paper presents closed-form solutions to the groundwater flow equation in marine island aquifers subject to time-independent and spatially variable recharge and groundwater pumping. Rectangular, circular, and elliptical geometries are considered in this work. The solutions to the groundwater flow equation are given in terms of the elevation of the phreatic surface above mean sea level, from which the depth to the seawater-freshwater interface below mean sea level follows at once by virtue of the Dupuit–Ghyben–Herzberg relation. The solution method presented in this paper relies on Fourier series expansions of hydraulic head and forcing function that transform the two-dimensional Poisson-type flow equations into second-order ordinary differential equations (ODEs) solvable using classical theory. The closed-form solutions for the groundwater flow equation in island aquifers provides a rapid assessment approach to the problem of determining phreatic-surface geometry and seawater-freshwater interface configuration caused by recharge and pumping once steady-state is reached. The closed-form solutions to the island (unconfined) groundwater flow problem driven by spatially variable recharge and pumping are novel. The case of elliptical island geometry is tackled using orthogonal elliptical coordinates. The important case of constant recharge (without groundwater extraction) leads to solutions in which the hydraulic head is expressible as the product of a flow factor equal to the squared root of the ratio of recharge over hydraulic conductivity times a geometric factor involving island shape parameters and flow boundary conditions. Estimability conditions for the hydraulic conductivity are presented for the cases of constant recharge and spatially variable recharge with pumping.

2 The Equation of Groundwater Flow in Island Aquifers

2.1 Equation for Rectangular Island Aquifers

Let γ denote the following ratio of water densities

$$\gamma = \frac{\rho_w}{\rho_s - \rho_w}, \tag{1}$$

in which ρ_s (1025 $\text{kg}\,\text{m}^{-3}$) and ρ_w (1000 $\text{kg}\,\text{m}^{-3}$) denote the average density of seawater and freshwater, respectively. For a static seawater-freshwater interface it is known that the relation between the elevation of the phreatic surface (h) above mean sea level and the depth of the seawater-freshwater interface below mean sea level (z) (Fig. 1) is

$$z = \gamma h. \tag{2}$$

The specific discharge moving through the saturated aquifer thickness $h + z$ is denoted by q_w, where w denotes a coordinal direction $w = x$ or y, is obtained

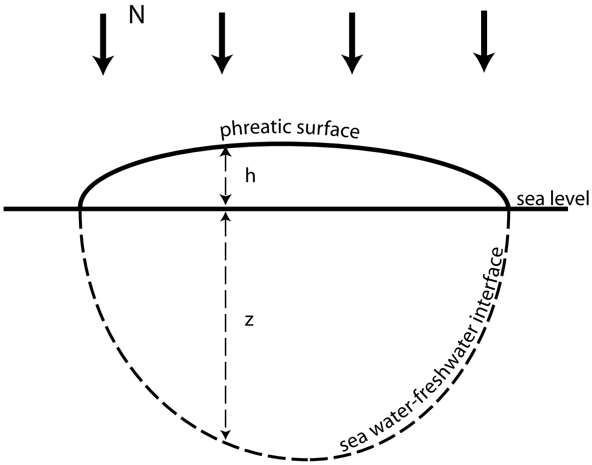

Fig. 1 A cross-sectional (elevation) view of an island aquifer showing the height of the phreatic surface (h) and the depth to the sea-water freshwater interface (z)

with Darcy's law and using the Dupuit–Ghyben–Herzberg simplified description of groundwater flow over a seawater-freshwater interface

$$q_w = -K \cdot (\gamma + 1) h \frac{\partial h}{\partial w}. \tag{3}$$

The steady-state groundwater flow equation with recharge $N(x, y)$ and groundwater pumping rates Q_k, at locations (x_k, y_k), $k = 0, 1, 2, \ldots, M$, is given by the following expression

$$-\frac{\partial q_x}{\partial x} - \frac{\partial q_y}{\partial y} = -\left(N(x, y) - \sum_{k=0}^{M} Q_k \cdot \delta(x - x_k, y - y_k) \right), \tag{4}$$

in which $\delta(x - x_k, y - y_k)$ denotes Dirac's delta function evaluated at the point (x_k, y_k). Using (3) in (4) produces the steady-state equation of groundwater flow

$$\frac{\partial^2 h^2}{\partial x^2} + \frac{\partial^2 h^2}{\partial y^2} = -H(x, y), \tag{5}$$

in which the forcing function $H(x, y)$ is as follows

$$H(x, y) = \frac{2}{K \cdot (\gamma + 1)} \left(N(x, y) - \sum_{k=0}^{M} Q_k \cdot \delta(x - x_k, y - y_k) \right). \tag{6}$$

The flow equation (5) is 2-dimensional with respect to the coordinates (x, y). It is assumed (following Dupuit's (1863) approximation) that pumping and recharge do not deviate the flow field so as to warrant making the dependent variable h vary along the vertical for fixed x, y. This paper deals with analytical solutions to 2-D groundwater flow fields.

The boundary condition of (5) is $h = 0$ on the aquifer's rectangular perimeter

$$h^2(x, y) = 0 \quad \text{on } x = 0,\ 0 \le y \le Y;\ x = X, 0 \le y \le Y, \tag{7}$$

$$h^2(x, y) = 0 \quad \text{on } y = 0,\ 0 \le x \le X;\ y = Y, 0 \le x \le X. \tag{8}$$

It shall prove advantageous to scale the x coordinate so that its range be $[0, \pi]$. To this end, define the scaled coordinates $\bar{x} = x(\pi/X)$ and $\bar{y} = y(\pi/X)$, so that $0 \le \bar{x} \le \pi$ and $0 \le \bar{y} \le \pi(Y/X) \equiv A$. The hydraulic head ($\bar{h}$) and recharge ($\bar{N}$) are also defined in terms of the scaled coordinates $\bar{x}$ and $\bar{y}$: $\bar{h}(\bar{x}, \bar{y}) = h(x, y)$ and $\bar{N}(\bar{x}, \bar{y}) = N(x, y)$. The problem (5)–(8) is rewritten in terms of scaled coordinates as follows

$$\begin{aligned} &\frac{\partial^2 \bar{h}^2}{\partial \bar{x}^2} + \frac{\partial^2 \bar{h}^2}{\partial \bar{y}^2} \\ &\quad = -\frac{2}{K \cdot (\gamma + 1)} \left(\frac{X}{\pi}\right)^2 \left(\bar{N}(\bar{x}, \bar{y}) - \sum_{k=0}^{M} Q_k \cdot \delta(\bar{x} - \bar{x}_k, \bar{y} - \bar{y}_k)\right) \\ &\quad \equiv -\bar{F}(\bar{x}, \bar{y}), \end{aligned} \tag{9}$$

with boundary conditions

$$\bar{h}^2(\bar{x}, \bar{y}) = 0 \quad \text{on } \bar{x} = 0,\ 0 \le \bar{y} \le A;\ \bar{x} = \pi, 0 \le \bar{y} \le A, \tag{10}$$

$$\bar{h}^2(\bar{x}, \bar{y}) = 0 \quad \text{on } \bar{y} = 0,\ 0 \le \bar{x} \le \pi;\ \bar{y} = A, 0 \le \bar{x} \le \pi. \tag{11}$$

Equations (9)–(11) are linear on $\bar{h}^2$. They typify the classical Poisson equation for diffusion processes (Weinberger 1965). They are solved for $\bar{h}^2(\bar{x}, \bar{y})$, whose square root produces $\bar{h}(\bar{x}, \bar{y}) = h(x, y)$. The solution is presented in a subsequent section of this paper.

2.2 Equation for Circular Island Aquifer

The groundwater flow equation in a circular island aquifer of radius R is obtained from (5) by using radial coordinates (r, θ), where $x = r\cos\theta$, $y = r\sin\theta$, and $\theta = \tan^{-1}(y/x)$, with flow domain $0 \le r \le R$ and $0 \le \theta \le 2\pi$,

$$\begin{aligned} &\frac{\partial^2 h^2}{\partial r^2} + \frac{1}{r}\frac{\partial h^2}{\partial r} + \frac{1}{r^2}\frac{\partial^2 h^2}{\partial \theta^2} \\ &\quad = -\frac{2}{K \cdot (\gamma + 1)} \left(N(r, \theta) - \sum_{k=0}^{M} Q_k \cdot \frac{\delta(r - r_k)\delta(\theta - \theta_k)}{r}\right) \equiv -F(r, \theta), \end{aligned} \tag{12}$$

with boundary condition

$$h^2(R, \theta) = 0, \quad 0 \le \theta \le 2\pi. \tag{13}$$

The groundwater flow problem embodied by (12)–(13) is solved in a subsequent section of this paper.

2.3 Equation for Elliptical Island Aquifer

The elliptical geometry is incorporated in the derivation of the groundwater flow equation by introducing elliptical coordinates u and v (Lamb 1945; Obdam and Veling 1987; Zimmerman 1996; Zhao et al. 2006). The elliptical coordinates are related to the Cartesian coordinates x and $y: x = c\cosh u \cos v$, $y = c\sinh u \sin v$, where, in general, $0 \le u \le \infty$ and $0 \le v \le 2\pi$, and in which $\cosh u = (1/2)(\exp(u)+\exp(-u))$ denotes the hyperbolic cosine, and $\sinh u = (1/2)(\exp(u) - \exp(-u))$ is the hyperbolic sine. For constant $u = u_0$, the elliptical coordinates give rise to an ellipse with semi-axes $a = c\cosh u_0$ and $b = c\sinh u_0$, and foci $(\pm c, 0)$, where $c = \pm(a^2 - b^2)^{1/2}$. For constant $v = v_0$, the elliptical coordinates define a hyperbola with foci $(\pm c, 0)$, where $c = \pm(a_h^2 + b_h^2)^{1/2}$, $a_h = c\cos v_0$, and $b_h = c\sin v_0$. These conics (ellipses and hyperbolae) have common foci $(\pm c, 0)$. Figure 2 shows a system of elliptical coordinates.

Define the coordinal scale factors $g_u = g_v = g = c((\cosh u \sin v)^2 + (\sinh u \cos v)^2)^{1/2}$. With these definitions, the groundwater flow equation (5) is rewritten in elliptical coordinates as follows

$$\frac{\partial^2 h^2(u,v)}{\partial u^2} + \frac{\partial^2 h(u,v)}{\partial v^2} = -g^2 H(u,v) = -G(u,v). \tag{14}$$

Assume that the flow domain associated with (14) is a ellipse with given semi-axes a, b and area πab. That is, the flow domain includes all x and y such that

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} \le 1. \tag{15}$$

Let $m = b/a$. In elliptical coordinates, the perimeter of the ellipse implied by (15) is described by $u_0 = \tanh^{-1} m = (1/2)\ln[(1+m)/(1-m)]$, in which $\tanh^{-1}$ denotes

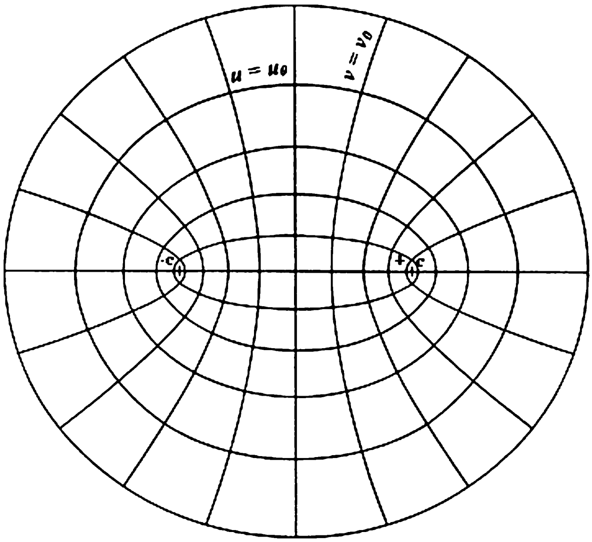

Fig. 2 Geometry of elliptical coordinates (u, v). The curves $u = u_0$, where u_0 is a constant, are ellipses, and the curves $v = v_0$, where v_0 is a constant, are hyperbolas. These conics have the common foci $(\pm c, 0)$. See text for details

the inverse hyperbolic tangent. The flow domain associated with (14) is expressible in elliptical coordinates by all u, v such that $0 \le u \le u_0$ and $0 \le v \le 2\pi$. The constant-head boundary condition associated with (14) is

$$h^2(u, v) = 0 \quad \text{on} \quad u = u_0,\ 0 \le v \le 2\pi. \tag{16}$$

The solution of (14) requires a second boundary condition which in this case is provided by using conservation of water volume. The total groundwater flow leaving though the aquifer's perimeter must equal the net of total recharge (N_T) and total pumping (Q_T)

$$\int_{v=0}^{2\pi} q_u(u_0, v) g\, dv = \int_0^{2\pi}\int_0^{u_0} N(u, v) g^2\, du\, dv - \sum_{k=1}^{M} Q_k \equiv N_T - Q_T. \tag{17}$$

The solution to (14) with boundary conditions (16) and (17) is presented in a subsequent section of this paper.

3 Solutions to the Groundwater Flow Equation

3.1 Solution for Rectangular Island Aquifer

The problem at hand is embodied by (9)–(11). The square of the hydraulic head $\bar{h}^2(\bar{x}, \bar{y})$ and forcing function $\bar{F}(\bar{x}, \bar{y})$ are expanded as sine Fourier series in the interval $0 \le \bar{x} \le \pi$

$$\bar{h}^2(\bar{x}, \bar{y}) = \sum_{n=1}^{\infty} b_n(\bar{y}) \sin n\bar{x}, \quad 0 \le \bar{x} \le \pi, \tag{18}$$

where the Fourier coefficients b_n are given by the following equation

$$b_n(\bar{y}) = \frac{2}{\pi}\int_0^{\pi} \bar{h}^2(\bar{x}, \bar{y}) \sin n\bar{x}\, d\bar{x}. \tag{19}$$

Likewise,

$$\bar{F}(\bar{x}, \bar{y}) = \sum_{n=1}^{\infty} B_n(\bar{y}) \sin n\bar{x}, \quad 0 \le \bar{x} \le \pi, \tag{20}$$

with Fourier coefficients

$$B_n(\bar{y}) = \frac{2}{\pi}\int_0^{\pi} \bar{F}(\bar{x}, \bar{y}) \sin n\bar{x}\, d\bar{x}. \tag{21}$$

Substitution of the sine Fourier series (18) and (20) into the groundwater flow equation (9) leads to the following ordinary differential equation (ODE) for the coefficients $b_n(\bar{y})$

$$-n^2 b_n(\bar{y}) + \frac{d^2 b_n(\bar{y})}{d\bar{y}^2} = -B_n(\bar{y}), \quad n = 1, 2, 3, \ldots;\ 0 \le \bar{y} \le A, \tag{22}$$

that has boundary conditions

$$b_n(0) = b_n(A) = 0. \tag{23}$$

Solving (22) and (23) yields the Fourier coefficients $b_n(\bar{y})$. These are used to construct the hydraulic head $\bar{h}(\bar{x}, \bar{y}) = (\bar{h}^2(\bar{x}, \bar{y}))^{1/2}$ obtained by taking the square root of (18).

From the theory of second-order ODEs, the solution of (22) and (23) is

$$b_n(\bar{y}) = -\frac{1}{n}\int_0^{\bar{y}} \sinh[n(\bar{y}-u)]B_n(u)\,du + C_1 e^{-n\bar{y}} + C_2 e^{n\bar{y}}, \quad n = 1, 2, 3, \ldots, \tag{24}$$

in which the coefficients C_1 and C_2 are

$$C_1 = -\frac{\frac{1}{n}\int_0^A \sinh[n(A-u)]B_n(u)\,du}{2\sinh[nA]}, \tag{25}$$

$$C_2 = -C_1. \tag{26}$$

The hydraulic head in a rectangular aquifer takes a relatively simpler form when the forcing function $\bar{F}(\bar{x}, \bar{y})$ contains only constant recharge as a stress factor, that is, when $\bar{F}(\bar{x}, \bar{y}) = (X/\pi)^2(2N/K \cdot (\gamma+1))$. In this case the hydraulic head can be expressed as the product of a flow factor and a geometric factor

$$\bar{h}(\bar{x}, \bar{y}) = \alpha\phi_r(\bar{x}, \bar{y}), \quad 0 \le \bar{x} \le \pi;\ 0 \le \bar{y} \le A, \tag{27}$$

in which the flow factor α is

$$\alpha = \sqrt{\frac{N}{K}} \tag{28}$$

and the geometric factor ϕ_r is written as follows

$$\phi_r(\bar{x}, \bar{y}) = \frac{2X}{\pi}\left[\frac{2}{(\gamma+1)\pi}\sum_{n=1,3,5,\ldots}^{\infty}\frac{1}{n^3}\left\{\frac{(\cosh[nA]-1)}{\sinh[nA]}\sinh[n\bar{y}] + \big(1-\cosh[n\bar{y}]\big)\right\}\sin n\bar{x}\right]^{1/2} \tag{29}$$

whose convergence can be established using the remarkable equality $\pi/4 = \sum_{n=1,3,5,\ldots}(\sin n\bar{x})/n$. Equation (27) shows that the hydraulic head in a rectangular island aquifer with constant recharge is proportional to the square root of N/K. The flow factor so defined was shown by Loáiciga (2005) to also play a key role on the geometry of phreatic surface of aquifers in sloping terrain.

3.2 Solution for Circular Island Aquifer

The solution of the groundwater flow equations (12) and (13) is obtained by expanding $h^2(r, \theta)$ as a complete Fourier series in the interval $0 \le \theta \le 2\pi$

$$h^2(r, \theta) = \frac{1}{2}a_0(r) + \sum_{n=1}^{\infty}\big[a_n(r)\cos n\theta + b_n(r)\sin n\theta\big], \tag{30}$$

with Fourier coefficients

$$a_0(r) = \frac{1}{\pi}\int_0^{2\pi} h^2(r,\theta)\,d\theta, \tag{31}$$

$$a_n(r) = \frac{1}{\pi}\int_0^{2\pi} h^2(r,\theta)\cos n\theta d\theta, \quad n = 1, 2, 3, \ldots, \tag{32}$$

$$b_n(r) = \frac{1}{\pi}\int_0^{2\pi} h^2(r,\theta)\sin n\theta d\theta, \quad n = 1, 2, 3, \ldots. \tag{33}$$

The forcing function $F(r,\theta)$ on the right-hand side of (12) is also expanded as a complete Fourier series

$$F(r,\theta) = \frac{1}{2}A_0(r) + \sum_{n=1}^{\infty}\big[A_n(r)\cos n\theta + B_n(r)\sin n\theta\big], \tag{34}$$

with Fourier coefficients

$$A_0(r) = \frac{1}{\pi}\int_0^{2\pi} F(r,\theta)\,d\theta, \tag{35}$$

$$A_n(r) = \frac{1}{\pi}\int_0^{2\pi} F(r,\theta)\cos n\theta d\theta, \quad n = 1, 2, 3, \ldots, \tag{36}$$

$$B_n(r) = \frac{1}{\pi}\int_0^{2\pi} F(r,\theta)\sin n\theta d\theta, \quad n = 1, 2, 3, \ldots. \tag{37}$$

Substitution of the Fourier series (30) and (34) into the flow equation (12) leads to the following set of ODEs for the Fourier coefficients a_0, a_n, b_n

$$\frac{d^2a_0(r)}{dr^2} + \frac{1}{r}\frac{da_0(r)}{dr} = -A_0(r), \tag{38}$$

$$\frac{d^2a_n(r)}{dr^2} + \frac{1}{r}\frac{da_n(r)}{dr} - n^2a_n(r) = -A_n(r), \quad n = 1, 2, 3, \ldots, \tag{39}$$

$$\frac{d^2b_n(r)}{dr^2} + \frac{1}{r}\frac{db_n(r)}{dr} - n^2b_n(r) = -B_n(r), \quad n = 1, 2, 3, \ldots. \tag{40}$$

The boundary conditions of the ODEs (38)–(40) are obtained from the constant-head boundary on the perimeter of the circular aquifer, $h^2(r,\theta) = 0$,

$$a_0(R) = a_n(R) = b_n(R) = 0. \tag{41}$$

The ODEs (38)–(40) are solved using classic theory to yield

$$a_0(r) = -\int_0^r \ln\left(\frac{r}{v}\right)vA_0(v)\,dv + \int_0^R \ln\left(\frac{R}{v}\right)vA_0(v)\,dv, \tag{42}$$

$$a_n(r) = -\frac{1}{2n}\int_0^r \left[\left(\frac{r}{v}\right)^n - \left(\frac{v}{r}\right)^n\right]vA_n(v)\,dv$$

$$+\frac{r^n}{2nR^n}\int_0^R\left[\left(\frac{R}{v}\right)^n-\left(\frac{v}{R}\right)^n\right]vA_n(v)\,dv, \tag{43}$$

$$b_n(r)=-\frac{1}{2n}\int_0^r\left[\left(\frac{r}{v}\right)^n-\left(\frac{v}{r}\right)^n\right]vB_n(v)\,dv$$
$$+\frac{r^n}{2nR^n}\int_0^R\left[\left(\frac{R}{v}\right)^n-\left(\frac{v}{R}\right)^n\right]vB_n(v)\,dv. \tag{44}$$

The coefficients a_0, $a_n(r)$, $b_n(r)$ are used in (30) to form $h^2(r,\theta)$ from which the hydraulic head follows by taking the square root. The important case of forcing function containing only constant recharge as a stress leads to a solution in which the hydraulic head is expressible as the product of a flow factor $\alpha=\sqrt{N/K}$ and a geometric factor $\phi_c(r)$

$$h(r,\theta)=\alpha\phi_C(r),\quad 0\le r\le R,\ 0\le\theta\le 2\pi, \tag{45}$$

in which the geometric factor is as follows

$$\phi_c(r)=\left(\frac{R^2-r^2}{2(\gamma+1)}\right)^{1/2}. \tag{46}$$

Equations (45)–(46) establish that the lines of equal hydraulic head are concentric circles. The (radial) flow crossing the aquifer's circular perimeters is given by the following equation

$$q_R=-\frac{1}{2}K\cdot(\gamma+1)\frac{\partial h^2}{\partial r}\bigg|_{r=R}=\frac{1}{2}NR. \tag{47}$$

Integration of the perimetric flow equals πR^2N, which is equal to the total recharge to the aquifer as required by conservation of mass in steady-state flow

$$\int_0^{2\pi}q_R R\,d\theta=\pi R^2N. \tag{48}$$

3.3 Solution for Elliptical Island Aquifer

The groundwater flow equations to be solved in this case are (14), (16), and (17). The hydraulic head is expanded in a Fourier series in the interval $0\le v\le 2\pi$

$$h^2(u,v)=\frac{1}{2}\alpha_0(u)+\sum_{n=1}^{\infty}\big[\alpha_n(u)\cos nv+\beta_n(u)\sin nv\big], \tag{49}$$

with Fourier coefficients

$$\alpha_0(u)=\frac{1}{\pi}\int_0^{2\pi}h^2(u,v)\,dv, \tag{50}$$

$$\alpha_n(u)=\frac{1}{\pi}\int_0^{2\pi}h^2(u,v)\cos nv\,dv,\quad n=1,2,3,\ldots, \tag{51}$$

$$\beta_n(u) = \frac{1}{\pi}\int_0^{2\pi} h^2(u,v)\sin nv\,dv, \quad n = 1,2,3,\dots. \tag{52}$$

Likewise, the forcing function $G(u,v)$ on the right-hand side of (14) is also expanded as complete Fourier series

$$G(u,v) = \frac{1}{2}\phi_0(u) + \sum_{n=1}^{\infty}\left[\phi_n(u)\cos nv + \psi_n(u)\sin nv\right], \tag{53}$$

with Fourier coefficients

$$\phi_0(u) = \frac{1}{\pi}\int_0^{2\pi} G(u,v)\,dv, \tag{54}$$

$$\phi_n(u) = \frac{1}{\pi}\int_0^{2\pi} G(u,v)\cos nv\,dv, \quad n = 1,2,3,\dots, \tag{55}$$

$$\psi_n(u) = \frac{1}{\pi}\int_0^{2\pi} G(u,v)\sin nv\,dv, \quad n = 1,2,3,\dots. \tag{56}$$

Substitution of the Fourier series (49) and (53) into the flow equation (14) leads to the following set of ODEs for the Fourier coefficients α_0, α_n, and β_n

$$\frac{d^2\alpha_0(u)}{du^2} = -\phi_0(u), \tag{57}$$

$$\frac{d^2\alpha_n(u)}{du^2} - n^2\alpha_n(u) = -\phi_n(u), \quad n = 1,2,3,\dots, \tag{58}$$

$$\frac{d^2\beta_n(u)}{du^2} - n^2\beta_n(u) = -\psi_n(u), \quad n = 1,2,3,\dots. \tag{59}$$

The constant-head boundary condition $h^2(u,v) = 0$ on the perimeter of the ellipse implies that

$$\alpha_0(u = u_0) = \alpha_n(u = u_0) = \beta_n(u = u_0) = 0. \tag{60}$$

In addition, conservation of mass in the island aquifer (cf. (17)) provides a second boundary condition

$$\left.\frac{dh^2}{du}\right|_{u=u_0} = -\frac{(N_T - Q_T)}{\pi K\cdot(\gamma+1)} \equiv \Psi. \tag{61}$$

The solutions for the Fourier coefficients α_0, α_n, and β_n are as follows

$$\alpha_0(u) = -\int_0^u (u-\xi)\phi_0(\xi)\,d\xi + C_1 u + C_2, \tag{62}$$

$$C_1 = 2\Psi + \int_0^{u_0}\phi_0(\xi)\,d\xi, \tag{63}$$

$$C_2 = -C_1 u_0 + \int_0^{u_0}(u_0-\xi)\phi_0(\xi)\,d\xi, \tag{64}$$

$$\alpha_n(u) = -\frac{1}{n}\int_0^u \sinh\big[n(u-\xi)\big]\phi_n(\xi)\,d\xi + D_1 e^{-nu} + D_2 e^{nu},$$
$$n = 1, 2, 3, \ldots, \tag{65}$$

with integration constants

$$D_1 = -D_2 e^{2nu_0} + \frac{e^{nu_0}}{n}\int_0^{u_0} \sinh\big[n(u_0-\xi)\big]\phi_n(\xi)\,d\xi, \tag{66}$$

$$D_2 = \frac{e^{-nu_0}}{2n}\int_0^{u_0} e^{n(u_0-\xi)}\phi_n(\xi)\,d\xi. \tag{67}$$

Lastly,

$$\beta_n(u) = -\frac{1}{n}\int_0^u \sinh\big[n(u-\xi)\big]\psi_n(\xi)\,d\xi + K_1 e^{-nu} + K_2 e^{nu}, \quad n = 1, 2, 3, \ldots, \tag{68}$$

with integration constants

$$K_1 = -K_2 e^{2nu_0} + \frac{e^{nu_0}}{n}\int_0^{u_0} \sinh\big[n(u_0-\xi)\big]\psi_n(\xi)\,d\xi, \tag{69}$$

$$K_2 = \frac{e^{-nu_0}}{2n}\int_0^{u_0} e^{n(u_0-\xi)}\psi_n(\xi)\,d\xi. \tag{70}$$

For constant recharge N and zero groundwater extraction, (62)–(70) used in conjunction with the Fourier expansion of h^2 (cf. (49)) produce the following solution for $h(u, v)$

$$h(u, v) = \alpha\phi_E(u), \quad 0 \le u \le u_0;\; 0 \le v \le 2\pi, \tag{71}$$

in which $\alpha = \sqrt{N/K}$ is the flow factor and the geometric factor $\phi_E(u)$ is as follows

$$\phi_E(u) = \left\{\frac{1}{\gamma+1}\Big[c^2\big(e^{u_0} - e^u\big) + c^2 e^{u_0}(u - u_0) + ab(u_0 - u)\Big]\right\}^{1/2}, \tag{72}$$

in which a, b are the semi-axes of the elliptical aquifer, that is, $u_0 = \tanh^{-1}(b/a)$. The hydraulic head calculated at arbitrary coordinates u and v with foci of magnitude c is converted to the hydraulic head at corresponding Cartesian coordinates by the relation

$$h(u, v) = h(x = c\cosh u \cos v, y = c \sinh u \sin v). \tag{73}$$

Lines of equal hydraulic head h calculated using (71) give rise to a set of concentric ellipses in which the boundary ellipse corresponds to $h = 0$. The set of such concentric ellipses is similar to that shown in Fig. 2 for elliptical coordinates.

4 Caveats on the Estimation of Hydraulic Conductivity

The hydraulic head in a marine island aquifer with recharge $N(x, y)$ and pumping rates $Q_k, k = 1, 2, 3, \ldots,$ can be expressed as follows

$$h(x_j, y_j) = \frac{\phi_r(x_j, y_j; N; Q_k)}{\sqrt{K}}. \tag{74}$$

Equation (74) was derived explicitly for the case of rectangular geometry and forcing function with constant recharge and no pumping rates ((27) evaluated at Cartesian coordinates (x_j, y_j)). Analogous explicit equations (45) and (71) were derived for circular and elliptical geometries, respectively. Assume that hydraulic head measurements $\hat{h}_j$ are available at locations (x_j, y_j), $j = 1, 2, 3, \ldots, M$. The sum of squared deviations among measured hydraulic heads and theoretical hydraulic heads is minimized with respect to the hydraulic conductivity K (and denoting $\phi_r(x_j, y_j; N; Q_k)$ by ϕ^*_{rj}) to produce the following least-squared estimate of K when the recharge N is known

$$\hat{K} = \left(\frac{\sum_{j=1}^{M} \phi^{*2}_{rj}}{\sum_{j=1}^{M} \hat{h}_j \phi^*_{rj}} \right)^2. \tag{75}$$

In the case of constant recharge with no pumping, (74) simplifies to

$$h(x_j, y_j) = \frac{\sqrt{N}}{\sqrt{K}} \phi_r(x_j, y_j) \tag{76}$$

and the estimate of hydraulic conductivity becomes (letting $\phi_r(x_j, y_j) = \phi_{rj}$)

$$\hat{K} = N \cdot \left(\frac{\sum_{j=1}^{M} \phi^{2}_{rj}}{\sum_{j=1}^{M} \hat{h}_j \phi_{rj}} \right)^2. \tag{77}$$

Equation (77) shows that if the hydraulic-head measurements were error free, that is, if $\hat{h}_j = \sqrt{N}\phi_{rj}/\sqrt{K}$, $j = 1, 2, 3, \ldots,$ then the estimate of the hydraulic conductivity would be exact, that is, $\hat{K} = K$. The same conclusion applies to the case of spatially variable recharge. In either case, the hydraulic conductivity is estimable only if the recharge N is known.

5 Conclusions

Novel, complete, and closed-form solutions for the steady-state elevation of the phreatic surface in marine island aquifers were developed in this paper. The solutions correspond to spatially variable recharge and groundwater extraction via wells. Three island shapes were considered, namely, rectangular, circular, and elliptical shapes. The solution for the phreatic surface was achieved using Fourier series expansions of the hydraulic head and the forcing function in the groundwater flow equation. In the case of constant recharge and no groundwater pumping, the elevation of the hydraulic head was found to be equal to the product of a flow factor (equal to the square root of the recharge over the hydraulic conductivity) times a geometric factor that depends on the aquifer's geometry and flow boundary conditions. It was also shown that the hydraulic conductivity is estimable in closed form by least squares only if the recharge function N is known.

Acknowledgements The work leading to this paper was partly supported by the University of California Water Resources Center through grant WR-2007-09 to research the impacts of sea-level rise on coastal aquifers.

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