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ABSTRACT

Simplified formulas approximating the shell-model theoretical results of Mang and of Harada are derived and tested.

Numerical calculations of alpha reduced widths from ground and excited states of Po^{212} are made using wave functions of Band, Sliv, and Kharitonov. The calculated reduced widths for decay from Po^{212} excited states are lower than the width from ground by factors ranging from 4-7 except that the second excited 2+ state shows a very small width. The usual analyses of long-range alpha intensities to derive excited state lifetimes must be modified, making the lifetime estimates longer by the appropriate hindrance factor.

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1. Introduction

In recent years there have been rather successful theoretical calculations of alpha decay rates of nuclei near Pb^{208} based on shell-model wave functions of the nucleons outside the closed shells. This linking of alpha decay rate theory with the powerful concepts of the shell-model is an advance of great value. Both Mang[1] and Harada[2] have carried through their calculations with a finite Gaussian internal wave function for the alpha particle. Brussard and Tolhoek[3] outlined a simpler delta-function approximation for such calculations, and Mang[4] has made reference to this approximation, but it has not yet been seriously tested. It is the main purpose of this paper to develop and test the delta-function approximation further. With simpler formulas we may then hope to gain further understanding and physical intuition about the results, and we may more readily carry out calculations involving wave functions with much configuration mixture.

Zeh[5] has shown that the formally different expressions of Mang (eqs. 1.19, 1.20 of ref. 4) are equivalent to those of Thomas[6]. Here we shall begin as Harada has done, with the expression involving the usual reduced width γ_L^2 of nuclear reaction theory rather than the reduced derivative width δ_L^2 of Thomas.

$$\lambda_L = \frac{2\gamma_L^2}{\hbar}\left(\frac{\rho}{G_L^2 + F_L^2}\right)_{r=R_0} \qquad (1)$$

where λ_L is the alpha decay constant, ρ is $\underline{kr}$, with $\underline{k}$ the wave number of the alpha particle at large distance, and $\underline{G_L}$ and $\underline{F_L}$ the irregular and regular Coulomb functions, or their continuation into the region of attractive nuclear potential. The spherical surface at $\underline{R_o}$ where inner and outer solutions should join is ideally to be chosen at a distance sufficiently large that there is no coupling between different alpha exit channels in the exterior region. For shell model alpha decay calculations it usually is a poor approximation to choose the distance $\underline{R_o}$ sufficiently large, since the shell model wave functions, even including considerable configuration mixture, fail to represent the amount of clustering in the outer fringes of the nucleus. As Wilkinson has pointed out[7], and as Zeh's calculations clearly show[5], the shell-model theory using $\underline{R_o}$ values near 9f or greater vastly underestimates the absolute alpha decay rate, where Igo's optical model potential[8] is used to define the barrier. At R_o values of 8f or less the calculated absolute transition probabilities appear closer to experiment. Of course, as pointed out[9] by Mang and Rasmussen, the results would be independent of R_o if the wave functions for initial and final states had a region of overlapping validity. Because of a failure of the regions of validity to meet properly, the shell-model alpha rate calculations to date show a dependence on R_o and generally much too small absolute values. We shall make no attempt to evaluate absolute decay rates in this paper, but will look only at relative transition probabilities. We shall use an R_o value of 8f that will set the alpha cluster dependence on the shell-model functions not far from their last radial maximum. The factor in parentheses in eq. (1) will for our relative decay rate calculations be taken as the WKB penetrability factors P from the diffuse, optical-model nuclear potential, although in absolute value the factor in eq. (1) exceeds by more than an order of magnitude the tabulated P values which are just the WKB exponential[8]. This choice for R_o

and P raises questions of inconsistency, since the optical-model barrier begins around 9.3f for Igo's exponential potential although the beginning of the barrier is at somewhat shorter distance for some of the Woods-Saxon potentials fitting scattering data. In the appendix we show that it is possible to formulate the theory for the choice of R_o in a region where the kinetic energy of the alpha particle is positive. Also the refinement of coupling between exit channels is introduced formally.

2. Alpha Decay Reduced Widths in the Delta Function Approximation

Harada's eq. (2) for γ_L and Mang's eq. (I.19) involve integrations over the coordinates of the A - 4 nucleons in the daughter and the relative coordinates of the four nucleons of the alpha particle. With the delta-function approximation the integration over relative coordinates is replaced by the product of the alpha particle volume to the 3/2 power times the function with the relative separation coordinates set to zero. Harada's equation simplifies to

$$\gamma_{TL} = \left(\frac{\hbar^2 R_o}{2M}\right)^{\frac{1}{2}} \left(\frac{4\pi S_\alpha^3}{3}\right)^{\frac{3}{2}} \sum_{j_1 j_2 j_3 j_4} b_{T\, j_n\, j_p\, L}^{j_1 j_2 j_3 j_4} R_1 R_2 R_3 R_4$$

$$\int d\omega\, dx_d \left\{ \Psi_{I_f T}^{*}(x_d),\ Y_L^{\dagger}(\omega) \right\}_{IM} \pi_s(1,2)\, \pi_s(3,4) \left\{ \Psi_{I_f T},\ \left\{ \left\{ \chi_{j_1}(\omega,\sigma_1),\ \chi_{j_2}(\omega,\sigma_2) \right\}_{j_N},\ \left\{ \chi_{j_3}(\omega,\sigma_3),\ \chi_{j_4}(\omega,\sigma_4) \right\}_{j_p} \right\}_{L} \right\}_{IM} \qquad (2)$$

The bracketed notation is an abbreviation for the coupling of two angular momenta to a good total angular momentum and projection.

$$\chi_{jm} = \left\{ Y_\ell,\ \chi_{\frac{1}{2}} \right\}_{jm} = \sum_{m_\ell\, m_s} (\ell\, \tfrac{1}{2}\, m_\ell\, m_s | jm)\, Y_{\ell m_\ell}\, \chi_{\frac{1}{2} m_s}$$

The R_i functions are the values of the nucleon radial wave functions evaluated at R_o and they will later in this paper usually be equated to $R_o^{-1}\, y(R_o)$, with the y functions the normalized nucleonic radial functions. One might use three-dimensional harmonic oscillator wave functions, as we shall for comparisons with Mang and Harada calculations. Usually we shall take the $y_j(R)$ from Blomqvist and Wahlborn's tables[10] of shell model radial functions in a realistic rounded square-well potential. The χ_j functions are spin and angular parts of the nucleon wave function, and the single index implies the sufficient set of quantum numbers n, l, j. The b coefficient is a product of fractional parentage coefficient and recoupling coefficient appropriate to splitting off the four-nucleon configuration in the brackets on the rightmost factorof eq. (2). For the examples of this paper the fractional parentage coefficient is unity. The symbol $\pi_s(1,2)$ is the singlet-spin projection operator for the neutrons (and 3,4 for the protons). The volume of the sphere $\frac{4}{3}\pi S_\alpha^3$ is of such size that with one nucleon at the center and the three others spread uniformly over the volume the correct r.m.s. radius of the alpha is obtained; S_α is taken as 2.34 f, corresponding to the electron-scattering r.m.s. radius[11] of 1.57 f. The daughter wave function is symbolized by $\Psi_{I_f\ \tau}(x_d)$.

The integration over daughter coordinates is readily made, simplifying the integral to

$$\text{Int.} = \int d\omega\ Y^*_{L\nu}\ \pi_s(1,2)\pi_s(3,4)\Big\{\big\{\chi_{j_1}(\omega,\sigma_1),\chi_{j_2}(\omega,\sigma_2)\big\}_{J_n},\ \big\{\chi_{j_3}(\omega,\sigma_3),\chi_{j_4}(\omega,\sigma_4)\big\}_{J_p}\Big\}_{L\nu}$$

The singlet spin projection can readily be carried out by recoupling from j-j to L-S coupling. In general the recoupling involves a 9j coefficient, but with S = 0 the 9j coefficient reduces to a Racah coefficient.

$$\pi_s(1,2)\left\{\chi_{j_1},\chi_{j_2}\right\}_{J_n} = (-)^{J_n-\ell_2-j_1+\frac{1}{2}}\left[\frac{(2j_1+1)(2j_2+1)}{2}\right]^{\frac{1}{2}} W(j_1\ell_1 j_2\ell_2;\tfrac{1}{2}J_n)\left\{Y_{\ell_1}(\omega),Y_{\ell_2}(\omega)\right\}_{J_n} \quad (3)$$

In the bracketed expression with the spherical harmonics the angular arguments are the same, and it is possible to derive a simple and useful identity based on a Clebsch-Gordan expansion and orthogonality properties of the Clebsch-Gordan coefficients.

$$\left\{Y_{\ell_1}(\omega),Y_{\ell_2}(\omega)\right\}_{J_n\mu} = \left[\frac{(2\ell_1+1)(2\ell_2+1)}{4\pi(2J_n+1)}\right]^{\frac{1}{2}}(\ell_1\ell_2 00|J_n 0)\,Y_{J_n\mu}(\omega) \quad (4)$$

Combining eqs. (3) and (4) we get

$$\pi_s(1,2)\left\{\chi_{j_1},\chi_{j_2}\right\}_{J_n\mu} = (-)^{J_n-\ell_2-j_1+\frac{1}{2}}\left[\frac{(2j_1+1)(2j_2+1)(2\ell_1+1)(2\ell_2+1)}{2\pi(2J_n+1)}\right]^{\frac{1}{2}} W(j_1\ell_1 j_2\ell_2;\tfrac{1}{2}J_n)\,Y_{J_n\mu}(\omega) \quad (5)$$

This expression can be further simplified by making use of eq. (A.17) of Devons and Goldfarb,[12] where a Racah coefficient with at least one index equal to 1/2 is related to a simple ratio of Clebsch-Gordan coefficients. This simplifying relation has been used earlier by Ford and Levinson[13].

The Devons and Goldfarb relation is not an identity, but is conditional on satisfaction of certain relations among the indices. We may more properly write it as follows:

$$(\ell_1 \ell_2 00 | J_n 0)(2\ell_1+1)^{\frac{1}{2}}(2\ell_2+1)^{\frac{1}{2}} W(j_1 \ell_1 j_2 \ell_2 ; \tfrac{1}{2} J_n)$$

$$= \begin{cases} (j_1 j_2 \tfrac{1}{2} -\tfrac{1}{2} | J_n 0), & \text{if } |j_1-\ell_1| = |j_2-\ell_2| = \tfrac{1}{2} \\ & \text{and } \ell_1+\ell_2+J_n \text{ is even} \\ 0, \text{ otherwise.} \end{cases}$$

The use of this relation to simply eq. (5) gives us the following:

$$\Pi_s(1,2)\left\{\chi_{j_1}(1)\,\chi_{j_2}(2)\right\}_{J_n M} = (-)^{\ell_1} \frac{(2j_1+1)^{\frac{1}{2}}}{(8\pi)^{\frac{1}{2}}} (J_n j_1 0 -\tfrac{1}{2} | j_2 -\tfrac{1}{2}) Y_{J_n M}^{(\omega)} \quad (6)$$

It is interesting to note that the same product of Racah and Clebsch-Gordan coefficients occurs in the general expression[14] for the "statistical factor" in the single particle gamma transition rate formulas for electric multipoles. In an analogous fashion to the simplification of eq. (5) we can simplify the general expression for the statistical factor to

$$S(j_i L j_f) = (2j_f+1)(j_i j_f \tfrac{1}{2} -\tfrac{1}{2} | L 0)^2 \quad (7)$$

with the only dependence on the orbital angular momenta being in the selection rule that $\ell_i + \ell_f + L$ must be even, which is just the parity selection rule for electric radiation.

Before using our approximation on new calculations we must test it carefully against calculations of Mang[1], of Harada[2], and of Zeh[5]. Mang has noted in his eq. (II.7) of ref. 1 the form of the delta-function approximation and has correctly cautioned that it may systematically overestimate the contribution of high-j orbitals. Let us calculate with our approximation the reduced widths for different even polonium isotopes with assumed pure shell model orbital assignments for the neutron pair (where we restrict ourselves

to having parent or daughter with closed configurations so that the fractional parentage coefficient will be unity). We consider for simplicity the ratios to Po^{210} with assumed pure $(p_{1/2})^2$ neutrons participating

$$\frac{\gamma_j^2}{\gamma_{p\frac{1}{2}}^2} = \frac{2j+1}{2}\left(\frac{y_j}{y_{p\frac{1}{2}}}\right)^4_{r=R_0} \qquad (8)$$

where the y_j are the nucleon-radial wave functions discussed earlier.

In Table 1 we show the calculation of ratios from eq. (8) compared to the finite alpha calculations of Zeh. For comparing with Zeh, who renormalized his harmonic oscillator radial functions at 9.5f to Blomqvist and Wahlborn's functions[10] for a rounded square well potential, we used in eq. (8) these numerical functions at 9.5f and also at 9.0f for a second set of calculations. The second set was made because it seems likely that the finite alpha calculation feels the effective nuclear wave function at something less than the separation distance of the alpha center.

Table 1. Theoretical alpha reduced width ratios.

Neutron pair orbital	$(y_j/y_{p\ 1/2})^4$		$\left(\frac{\gamma_j}{\gamma_{p\ 1/2}}\right)^2$ delta		$\left(\frac{\gamma_j}{\gamma_{p\ 1/2}}\right)^2$ finite	Error Ratio for Delta Function Model (col. 6 divided by cols. 4 and 5, respectively).	
	R_0=9f	9.5f	9f	9.5f		9f	9.5f
$i_{13/2}$	0.116	0.062	0.81	0.44	0.102	0.13	0.23
$i_{11/2}$	0.074	0.053	0.44	0.32	0.075	0.17	0.23
$g_{9/2}$	1.21	1.50	6.0	7.5	3.73	0.62	0.50
$f_{5/2}$	0.30	0.24	0.91	0.73	0.549	0.60	0.75
$p_{3/2}$	1.00	0.95	2.0	1.89	1.89	0.94	1.0
$p_{1/2}$*	1.	1.	1.	1.	1.	1.	1.

* Ratios normalized to unity for $(p_{1/2})^2$ neutron case.

In addition to the comparison of Table 1 we have compared with Mang's and with Harada's Po^{212} to Po^{210} reduced-width ratios with pure $(g_{9/2})^2$ and pure $(p_{1/2})^2$ neutron configurations, respectively. Here we used harmonic oscillator wave functions at the same value of radius and scale parameters as they used. Mang's calculated ratio is about 6, compared to a delta-function value of 10.0. Harada's calculated ratio is 4.2, compared to our delta-function ratio of 7.1. The difference between the Mang and Harada values is probably due to their different choice of radial parameters, but in both cases their finite-size treatments give reduced widths 60% of the respective delta-function calculations, in good agreement with the last two columns of the third line of Table 1.

It is evident that the delta-function model indeed overestimates the contributions of high-j orbitals, but fortunately the deviation is systematic, so that we may introduce a correction factor depending on the relative angular momentum of the nucleons forming the correlated wave packet. It can be shown that to lowest order in the relative angular momentum ℓ the correction factor to γ should have the form

$$B = 1 - a\,\ell\,(\ell+1) + \ldots \qquad (9)$$

An exponential factor $\exp[-a\,\ell\,(\ell+1)]$ has a more reasonable asymptotic behavior, but for simplicity we choose the first expression. In Fig. 1 are plotted the square roots of the correction ratios from the final two columns of Table 1 vs. $\ell(\ell+1)$. The straight line represents thechoice of 0.013 for a, and we use this correction factor throughout the remainder of this paper.

Parenthetically, we call attention to the sensitivity to R_o of the relative contributions of different orbitals. Zeh's calculations show very weak relative contributions from the i 11/2 orbital, for example, compared

to the p 1/2. Yet if the radius R_o is chosen at 8f instead of 9f, the relative contribution of the i 11/2 orbital increases a factor of about 3. Using a smaller radius would have lessened Zeh's difficulty of explaining the large relative intensity of decay of the Po^{211} isomer to the i 13/2 state in Pb^{207}, but we cannot regard this as decisive evidence in favor of smaller R_o values for shell-model alpha decay calculations. In the subsequent calculations of this paper we will use R_o of 8f.

3. Alpha Rate Enhancement from Residual Interaction between like Nucleons

For Po^{212} and Po^{210} we now use the delta-function formulas to estimate the alpha rate enhancement due to configuration mixing induced by n-n and p-p attractive residual force. First the calculation is made with the same configuration admixture with which Harada made his calculations.

In the next section a calculation of the alpha widths from Po^{212} states is given, based on the wave functions[15] of Band, Sliv, and Kharitonov (hereafter referred to as BSK). Their configuration admixture was calculated with n-p residual forces as well as those between like nucleons. In order to estimate the specific effect on alpha widths of bringing in n-p correlations we make at the end of this section a calculation of the Po^{212} alpha width for a wave function such as BSK calculations should have yielded if the n-p force were turned off.

Harada[2] found a theoretical alpha rate enhancement for Po^{212} of a factor of 5.5 by using the following wave function, where the proton configuration is that of single-closed-shell nucleus Po^{210} and the neutron configuration is that of Pb^{210}:

$$\text{neutron pair} \quad 0.935\,(g_{9/2})_o^2 + 0.325\,(i_{11/2})_o^2 + 0.141\,(d_{5/2})_o^2$$

$$\text{proton pair} \quad 0.914\,(h_{9/2})^2 + 0.256\,(f_{7/2})_o^2 - 0.317\,(i_{13/2})_o^2$$

Such a wave function possesses no correlation between neutrons and protons, such as would be the case if there were no neutron-proton residual force.

As a reference value we first calculate the reduced width for ground state decay assuming a pure shell model configuration $(g_{9/2})_o^2 (h_{9/2})_o^2$. By substitution of eq. (6) twice into eq. (2) we readily obtain

$$\gamma^2_{pure} = C^2 (2j_n+1) y_n^4 B_n^2 (2j_p+1) y_p^4 B_p^2 \qquad (10)$$

where C is a constant, equal to 7.0 $ev^{1/2}$ f^2 for an R_o of 8f and s_α of 2.34f, with the y functions being the normalized radial wave functions $r\Psi(r)$ of dimensionality $(fermi)^{-1/2}$. B_n and B_p are the correction factors for finite size from eq. (9) with a of 0.013.

With eq. (10) we calculate the reduced width γ^2 of 0.00131 C^2 for the pure state.

For the mixture Harada used we have a somewhat more complicated equation that still factors between neutron and proton parts.

$$\gamma^2_{pair} = C^2 \left| \sum_{j_n} (-)^{l_n} c(j_n) (2j_n+1)^{\frac{1}{2}} B_n y_{j_n}^2 \sum_{j_p} (-)^{l_p} c(j_p) (2j_p+1)^{\frac{1}{2}} B_p y_{j_p}^2 \right|^2 \qquad (11)$$

With the wave functions above which Harada used we get $\gamma^2_{pair} = 0.0131\ C^2$, exactly a factor of 10 enhancement, whereas Harada obtained 5.5. The main reason for our higher enhancement is that the Blomqvist-Wahlborn $f_{7/2}$ radial function is considerably larger relative to the $h_{9/2}$ than is the case for harmonic oscillator functions that Harada used.

Doing a similar mixing enhancement comparison with Harada for Po^{210} we get a factor of 26 compared to his factor of 11. Zeh obtained about a factor of 10.

Let us turn now to an examination of the BSK wave function for the ground state of Po^{212} (their Table 2). In addition to configuration admixture where protons and neutrons are separately coupled to zero there are components, such as, $[(g_{9/2})^2_J (h_{9/2})^2_J]_o$, with J = 2, 4, 6. Indeed the component with J = 2 is the second largest component in the wave function. Before carrying out an alpha width calculation with the full BSK wave function we wish to make a reference calculation with a wave function approximately equal to what BSK might have obtained by omitting the n-p interaction. We estimate such a wave function by deleting the components arising from n-p interaction and renormalizing. The BSK components with surface phonons are also deleted, since they should mainly contribute to decay to Pb^{208} excited states and not to ground. The abbreviated and renormalized wave function is as follows:

$$\psi_{Po^{212}} = 0.97 (g_{9/2})^2_0 (h_{9/2})^2_0 + 0.12 (i_{11/2})^2_0 (h_{9/2})^2_0 + 0.11 (g_{9/2})^2_0 (f_{7/2})^2_0$$
$$+ 0.15 (g_{7/2})^2_0 (h_{9/2})^2_0 - 0.01 (g_{9/2})^2_0 (i_{13/2})^2_0$$

Using this wave function and eq. (11) with slight and obvious modification for a single sum over configurations we obtain a reduced width $\gamma^2 = (0.0588)^2 c^2$. If we take into account that 17% of the BSK ground state wave function involves configurations with surface phonons, we must lower this value slightly to

$$\gamma^2_{\text{BSK no n-p}} = 0.00286\ c^2$$

This value represents an enhancement of a factor of 2.2 over the unmixed shell model case. There is much less enhancement due to configuration mixing here than in the previous case with the product function used by Harada. The difference arises from the considerably lower mixing in the BSK wave function.

The reduced widths are especially sensitive to proton admixture of $(f_{7/2})^2$ and $(i_{13/2})^2_o$, since the radial wave functions of these orbitals at

8f are much larger than that of $h_{9/2}$, and the reduced width contributions have a fourth-power dependence on the radial-wave function. This great sensitivity argues against using harmonic oscillator radial functions without correction for finite well effects. It also calls for more careful studies of the radial wave functions in realistic potential wells. It seems likely that the radial wave functions in the outer regions $R_o \gtrsim 8f$ will depend rather sensitively on the separation energy for the nucleon orbital in question especially in the case of neutrons, which see no Coulombic potential.

Let us now go on to use the full BSK wave functions. In addition to terms of the simple form of eq. (11), there will be additional terms for the configurations in which neutrons, protons, or both are not coupled to angular momentum zero.

Let us define a quantity $F_n(j_1\ j_2\ J_n)$ for the neutron factor in the reduced width, consisting of the coefficient of the spherical harmonic in eq. (6) times the radial functions and the finite size correction times $(8\pi)^{1/2}$.

$$F_n = (-)^{l_1}(2j+1)^{\frac{1}{2}}\left(J_n\, j_1\, 0\, -\tfrac{1}{2} \,|\, j_2\, -\tfrac{1}{2}\right) B_n\, Y_{j_1}\, Y_{j_2} \qquad (12)$$

A similar factor for the protons F_p is also defined. Secondly, we define a factor of G, dependent on the coupling of neutron and proton motion.

$$G(J_n J_p L) = (4\pi)^{\frac{1}{2}} \int d\omega\, Y^*_{LM} \left\{ Y_{J_n}, Y_{J_p} \right\}_{LM} = (2J_n+1)^{\frac{1}{2}} \left(L J_p 00 \,|\, J_n 0\right) \qquad (13)$$

If we consider as a good approximation that for the double-closed shell case of the Pb^{208} daughter we can neglect any inter-channel coupling, then the BSK wave function components with quadrupole phonons will not contribute to decay to the ground state of Pb^{208}. This assumption amounts to assuming the k-matrix in eq. (A2) has no off-diagonal elements connecting the ground state. In terms of the BSK wave function coefficients $c^I(\alpha)$ and the quantities defined in our eqs. (12) and (13) the expression for the reduced alpha width from a state in Po^{212} becomes

$$\gamma_{LI}^2 = C^2 \left| \sum_{\alpha} \delta_{NO}\ c^I(\alpha) F_n(j_1 j_2 J_n) F_p(j_3 j_4 J_p) G(J_n J_p L) \right|^2 \qquad (14)$$

where N is the number of phonons of core excitation and α represents the set of all quantum numbers specifying the basis vectors of the wave function.

In Table 2 we retabulate for convenience the BSK coefficients and also the quantities F_n, F_p, and G and their product, applying to the decay of the ground and first excited 0+ states.

For decay of the ground state all eight terms are of the same sign and we get a reduced width γ^2 of 0.0034 C^2, a factor of 2.6 enhancement over the unmixed state calculation.

This factor of 2.6 enhancement is about 20 percent greater than the enhancement we estimated without the n-p residual interaction. Thus, the probability of an alpha cluster in the shell model wave function in the nuclear surface region is increased by the effects of an attractive residual force between unlike nucleons. The correlation between nucleons is built up when the n-p force is turned on, but at some expense in loss of correlation between like nucleons. The overall enhancement of the alpha emission width in this case is rather small.

Table 2. Alpha width calculation from 0+ states.

Configuration	F_n	F_p	G	$F_n F_p G$ ($\times 10^2$)	BSK Coefficients 0_1^+	BSK Coefficients 0_2^+
$(g_{9/2})^2\ 0(h_{9/2})^2_0$	0.496	-0.0734	1	-3.65	0.80	-0.39
$(g_{9/2})^2\ 2(h_{9/2})^2_2$	-0.243	0.0400	$\sqrt{5}$	-2.18	0.36	-0.77
$(g_{9/2})^2\ 4(h_{9/2})^2_4$	0.176	-0.0290	$\sqrt{9}$	-1.53	0.12	0.26
$(g_{9/2})^2\ 6(h_{9/2})^2_6$	-0.145	0.0184	$\sqrt{13}$	-0.96	0.04	0.05
$(i_{11/2})^2\ 0(h_{9/2})^2_0$	0.131	-0.0734	1	-0.96	0.10	-0.17
$(g_{9/2})^2\ 0(f_{7/2})^2_0$	0.496	-0.308	1	-15.3	0.09	-0.12
$(g_{7/2})^2\ 0(h_{9/2})^2_0$	0.415	-0.0734	1	- 3.05	0.12	-0.09
$(g_{9/2})^2\ 0(i_{13/2})^2_0$	0.496	0.170	1	8.43	-0.01	0.09

We note that in the calculation of the alpha width from the excited 0+ state the components with zero seniority all contribute with the same sign but are partially cancelled by higher seniority terms, so that the theoretical prediction is for a reduced width a factor of 6.3 smaller than ground. This theoretical hindrance factor is probably to be regarded as of qualitative significance, since the terms involving $(f\ 7/2)^2$ not coupled to zero are probably significant contributors, but their amplitudes are not given by BSK. It is qualitatively clear that the hindrance of the excited 0+ state affects the oft-made analysis of long-range alpha abundances in terms of lifetimes of the excited states. The lifetimes analyzed on the basis of reduced widths being the same as from ground must be lengthened by the hindrance factor. Since the cancellation giving rise to the hindrance comes from wave function components mixed by the n-p residual force, the theoretical hindrance may be rather sensitive to the ratio of residual force between unlike nucleons compared to that between like nucleons.

The reduced widths for decay from the first three 2+ excited states were calculated in like manner with detailed numerical coefficients tabulated in Table 3. The resultant reduced widths are 0.00080 C^2, 5.6×10^{-6} C^2, and 0.00052 C^2. respectively, corresponding to reduced hindrance factors of 4.3, 610, and 6.6, respectively.

With the hindrance factor of 4.3 we correct Griffioen and Rasmussen's lifetime estimate[16] for the first excited state to 1.8×10^{-11} sec; their analysis was based on long-range alpha abundance and beta decay scheme data determining the fraction of decay populating the first excited state, and their alpha barrier penetrability factors were from the diffuse optical model potential. This new lifetime estimate is readily compared with the theoretical gamma transition probability calculated by BSK and given in their Table 3; for a

Table 3. Alpha width calculations from 2+ states.

Configuration	F_n	F_p	G	F_nF_pG (×100)	BSK Coefficients 2_1^+	2_2^+	2_3^+
$(g_{9/2})^2\ 0(h_{9/2})^2_2$	0.496	0.040	1	1.98	0.63	-0.60	0.18
$(g_{9/2})^2\ 2(h_{9/2})^2_0$	-0.243	-0.0734	1	1.78	0.55	0.66	0.20
$(g_{9/2})^2\ 2(h_{9/2})^2_2$	-0.243	0.040	-1.20	1.17	-0.26	-0.04	0.88
$(g_{9/2})^2\ 2(h_{9/2})^2_4$	-0.243	-0.0290	1.60	1.13	0.18	-0.18	-0.01
$(g_{9/2})^2\ 4(h_{9/2})^2_2$	0.176	0.0400	1.60	1.13	0.17	-0.21	0.02
$(g_{9/2})^2\ 4(h_{9/2})^2_4$	0.176	-0.0290	-1.53	0.782	-0.04	-0.01	0.29
$(g_{7/2})^2\ 0(h_{9/2})^2_2$	0.415	0.0400	1	1.66	0.10	-0.11	0.04
$(g_{9/2})^2\ 2(i_{13/2})^2_0$	-0.243	0.170	1	-4.13	-0.08	-0.12	-0.05
$(i_{11/2})^2\ 0(h_{9/2})^2_2$	0.131	0.0400	1	0.524	0.07	-0.09	0.05

surface tension constant of 1000 MeV their half life estimate is 7.9×10^{-12} sec., and for an alternative constant of 2000 MeV they estimate 1.5×10^{-11} sec. The revised long-range alpha estimate is clearly in satisfactory agreement with the latter estimate. If instead of the diffuse-barrier estimate we use Hanna's estimate[17] for the square potential barrier with neglect of centrifugal effects, we get a corrected lifetime of 8.6×10^{-12} sec in agreement with the former BSK estimate.

The high alpha hindrance for the second-excited 2+ state, together with the BSK theoretical estimate of a strong M1 transition probability to the first excited state clearly lead us to expect exceedingly low long-range alpha branching from the second 2+ state. It is thus consistent to assign the 1.51 MeV state this character, as is done in several papers; it is not necessary to invoke an assignment of 3+, as has sometimes been done, in order to explain the absence of long-range alpha emission. From examination of the first two lines of our Table 3 we see evidence again of the effects of the n-p residual interaction. As pointed out by BSK and illustrated in their Fig. 2, in the absence of off-diagonal elements of the residual n-p force there would be two closely-lying 2+ first and second excited states, corresponding respectively to the breaking of a proton and of a neutron pair. It is clear that these pure-unmixed states would exhibit rather comparable alpha widths. The effect of the off-diagonal part of the n-p interaction in the BSK calculation is to mix thoroughly these states, and the lower of the two states gets an enhanced alpha width, while the alpha width of the upper member nearly vanishes.

Decay schemes in the literature vary in assignment of the higher alpha emitting states at 1.68 MeV and at 1.80 MeV. It seems likely that they represent the first excited 0+ and third excited 2+ states, respectively. The recent paper of Bertolini et al.[18] reports definitely on gamma radiation of

1.8 MeV, apparently ruling out the assignment of 0+ to this state, as made in some earlier analyses.

The above assignments are also consistent with results of other recent studies,[19] including γ-γ angular correlations. There is sufficient experimental uncertainty about multipolarity admixtures in radiations from some of the higher states, that is seems premature to try to make a complete comparison of relative alpha and gamma transition rates between theory and experiment. Furthermore, the configuration mixing in the shell model wave functions for Po^{212} may be uncertain and rather sensitive to details of assumptions about the character and strength of the residual forces. Glendenning[20] has recently made a theoretical shell model calculation on Po^{212}, primarily to explain the high-spin, alpha-emitting isomer,[21] and without attention to fitting the lower energy levels. Glendenning uses a strong residual Gaussian force of Serber type with singlet-even depth of 32.9 MeV and triplet-even depth of 51.9 MeV and no interaction in odd states. BSK use a force which has singlet strength in excess of triplet, and with no distinction between even and odd states of relative motion. Glendenning's calculations would have the lowest 0+ and 2+ states of Po^{212} primarily made up of $(i_{11/2})^2$ neutrons rather than $(g_{9/2})^2$. The general fit of low lying states is not very good, but a preliminary examination of his published and unpublished results indicate the possibility of a fit, retaining his force mixture but somewhat weakening it and increasing his single particle spacing of $i_{11/2}$ above $g_{9/2}$ to the 0.75 MeV used by BSK and indicated from levels of Pb^{209}. In such a case his lowest 0+ and 2+ states would become predominantly $(g_{9/2})^2$ like BSK, but his force mixture seems to produce more of the higher-seniority components in the wave function. Glendenning's force mixture applied to Po^{212}, considering only $\left[(h_{9/2})^2_J\ (g_{9/2})^2_J\right]_0$ components, gives for the lowest eigenstate the

coefficients 0.73, 0.57, 0.31, 0.18, and 0.12 for J = 0, 2, 4, 6, 8, respectively. These coefficients can be compared roughly with those in the second-from-the-right hand column in our Table 2. It is obvious that Glendenning's wave function would show a greater enhancement of the alpha width from ground upon turning on the n-p residual force than does the BSK wave function.

Another difference between BSK and an analysis based on Glendenning's work is that the latter leads almost surely to an interpretation of the second 0+ state as predominantly $\left[(h_{9/2})^2_0\ (i_{11/2})^2_0\right]_0$, whereas BSK have it mainly $\left[(h_{9/2})^2_2\ (g_{9/2})^2_2\right]_0$. The same remarks apply to the first two 2+ states, where the analysis from Glendenning's work would have them mainly involve different neutron orbitals.

It appears that the qualitative features of our calculations of alpha widths would remain the same, even with the greatly changed wave function. There is still a constructive addition of terms for the lowest 0+ and 2+ states and a destructive subtraction of major terms for the second 0+ and 2+ states.

4. Summary

We would reemphasize these results of the studies:

The shell-model theoretical relative reduced alpha widths are very sensitive to nucleon radial wave function values in the surface region.

Configuration mixing of lowest-seniority configurations as a consequence of the attractive residual interaction between like nucleons causes important enhancement of the alpha widths for ground to ground decay. Mixing of higher seniority configurations, due to the n-p attractive residual interaction, causes some additional enhancement of the ground-to-ground decay.

The abruptly increased alpha widths for polonium isotopes going from below to above 126 neutrons is to be attributed mainly to the increased correlation in relative angle between neutrons for the $(g_{9/2})^2_0$ configuration as compared to the $(p_{1/2})^2_0$ configuration—to first approximation a simple (2j+1) dependence. This simple dependence may be overridden in other cases by radial wave function variations and by configuration mixing.

In general we must expect the reduced widths for alpha decay involving excited states to be less than for ground-to-ground transitions, and the hindrance factors may not show a monotonic dependence on energy of excitation.

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Appendix

Formulation of Theory for Small Connection Radii

Our intention here is to try to point out a way to make a consistent application of the theory at smaller connection radii ($\leq 8f$) than usually used and still retain the diffuse optical model barrier which has its inner turning point at some larger distance.

First of all, we are faced with the unusual necessity of connecting solutions in a region inside the barrier, where the kinetic energy of the alpha is positive. Formally, the connection is easy if we come inside the turning point a sufficient distance that the WKB approximation again becomes valid. For substitution in eq. (1) we now have for the irregular solution

$$G_L(R_0) = 2\left[\frac{k(\infty)}{k(R_0)}\right]^{\frac{1}{2}} \cos\left(\int_{R_0}^{R_i} k(r)\,dr - \frac{\pi}{4}\right) \exp\left(\int_{R_i}^{R_t} q_L(r)\,dr\right) \tag{A1}$$

where $k(r)$ is the wave number at the distance r, R_i and R_t are inner and outer turning points of the barrier, respectively, and $q_L(r)$ is the imaginary wave number within the barrier. The regular solution is completely negligible for eq. (1). From the references which tabulated penetrability factors P for the Igo potential we may conveniently obtain the exponential factor of eq. (A1) as the reciprocal square root of the tabulated P values. The pre-exponential factor depends on assumption of a potential extending in further than the stated limits of validity of the Igo potential, but the factor may be taken as unity in the absence of more detailed information.

Secondly, we need to generalize eq. (1) to provide for inter-channel coupling beyond R_o. We define $\gamma_{\tau L}$ as the reduced width at a channel entrance and a real matrix with elements $k_{\tau L \tau' L'}$ that gives the effective integrated scattering between channels. The symbol τ stands for all quantum numbers

specifying the states of the daughter nucleus. Then the generalized eq. (1) becomes

$$\lambda_{\tau L} = \frac{2}{\hbar} P_{\tau L} \left| \sum_{\tau' L'} k_{\tau L \tau' L'} \gamma_{\tau' L'} \right|^2 \qquad \text{(A2)}$$

where $P_{\tau L}$ stands for the penetration factor $\frac{k R_o}{G^2(R_o)}$

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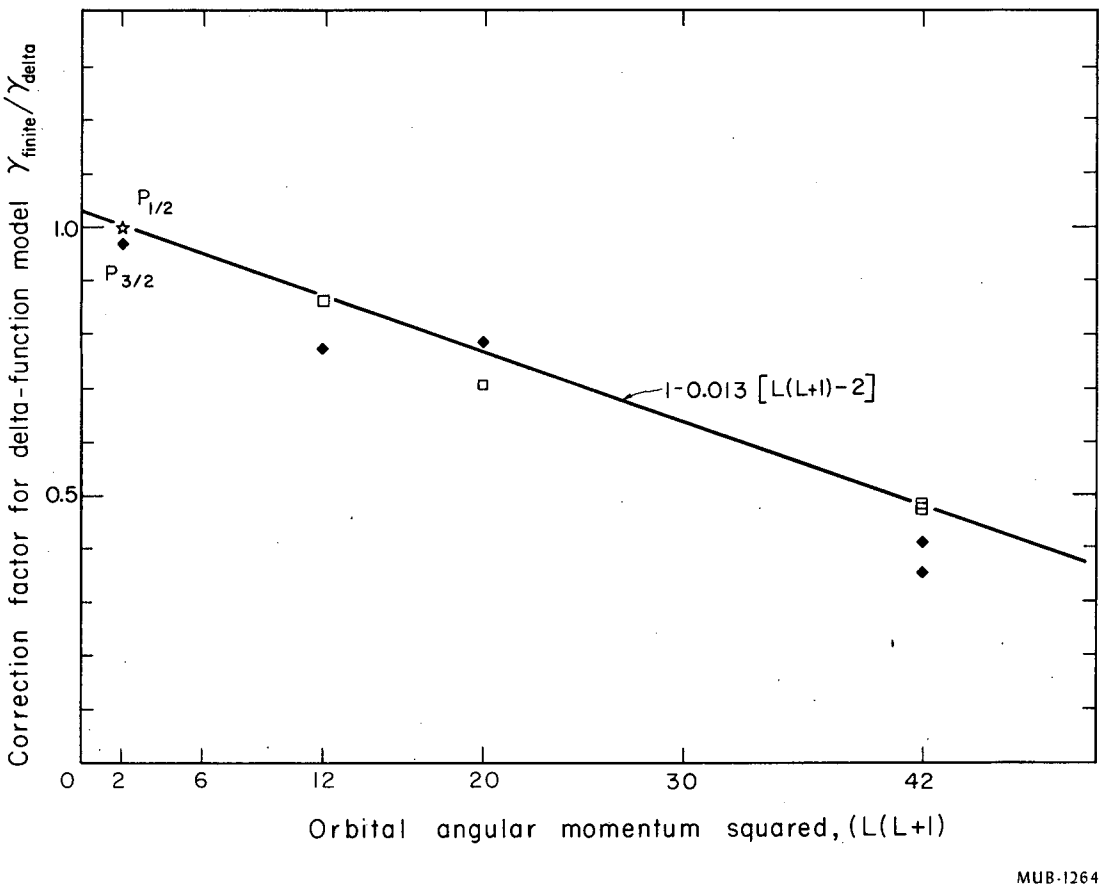

Fig. 1. Plot of correction factor to the reduced alpha widths of the delta-function model, as derived by comparison with finite-size calculations of Zeh. The square roots in the last two columns of Table 1 are plotted against $\ell(\ell+1)$ for the neutron orbital. Open squares are for delta function calculation at R_o = 9.5f and solid diamonds for 9.0f. The star is the intersection of the two points of normalization for $(p_{1/2})^2$ and also the $(p_{3/2})^2$ point at 9.5f.