Sets of Points with Many Halving Lines

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Abstract

We used a genetic search algorithm to find sets of points with many halving lines. There are sets of 10 points with 13 halving lines, 12 points with 18 halving lines, 14 points with 22 halving lines, 16 points with 27 halving lines, and 18 points with 32 halving lines. We find a construction generalizing the 12 point configuration and show that, for any $n = 3 \cdot 2^i$, there are configurations of n points with $n \log_4(2n/3) = 3(i+1)2^{i-1}$ halving lines.

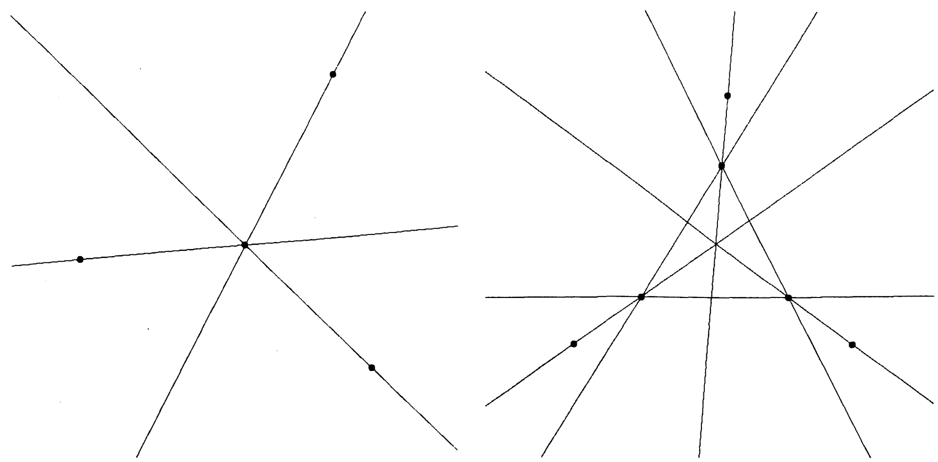

Figure 1. (a) 4 points, 3 halving lines; (b) 6 points, 6 halving lines.

1 Introduction

Counting *halving lines* is one of the important open problems of combinatorial geometry. A halving line for a set of n points (n even) is a line passing through two of the points, and cutting the remaining set of $n-2$ points in half. Given any set S, we define $H(S)$ to be the set of such lines. We are interested in bounding the worst-case size of this set,

$$h(n) = \max_{|S|=n} |H(S)|.$$

The best known lower bound gives a construction for sets of n points with $\Omega(n \log n)$ lines [1], while the best known upper bound is slightly smaller than $O(n^{3/2})$ [3], so there remains a wide gap. Thus finding data from which we can estimate the true asymptotic behavior of $h(n)$ becomes important.

Clearly, $h(2) = 1$ and $h(4) = 3$. The known $\Omega(n \log n)$ construction shows that $h(6) \geq 6$ and $h(18) \geq 27$; the first is tight but the second is not. It was also known that $h(8) = 9$. In this paper we show that $h(10) = 13$, $h(12) \geq 18$, $h(14) \geq 22$, $h(16) \geq 27$, and $h(18) \geq 32$. We generalize certain patterns in these examples to find arbitrarily large configurations with $n \log_4(2n/3)$ halving lines, improving the previous construction (which produced $n \log_9(3n/2)$ lines) by an asymptotic factor of $\log_2 3 \approx 1.585$.

There are a number of equivalent ways in which this problem has been defined: First, we might instead consider lines which pass through no point,

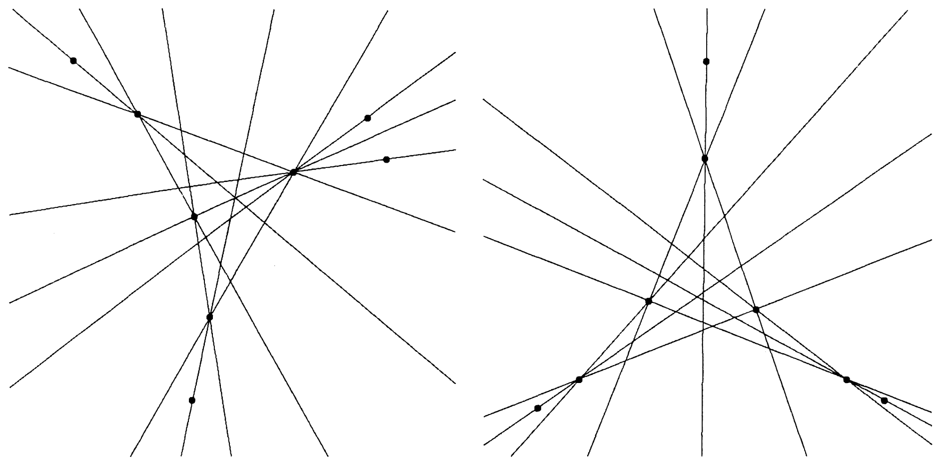

Figure 2. Two sets of 8 points with 9 halving lines each.

and split the entire point set in half. To count such lines, we treat as equivalent any two lines that can be continuously transformed into each other by a motion that does not pass the line across any point, and count equivalence classes under this relation. Second, we may count the different sets of points which can be separated by lines in this way. Finally, we may use projective duality to consider an arrangement of n lines, and points that are above half the lines and below the other half. The halving lines of our original definition dualize to vertices of intersection in the line arrangement; the equivalence classes of our first alternate definition dualize to cells in the arrangement (counting the leftmost and rightmost regions as being a single cell connected through infinity). In this dual formulation it is easy to see that the two definitions are equivalent: each halving vertex touches two halving cells and each halving cell touches two halving vertices, so for any configuration the numbers of either type of object are the same. Figures 6(a) and 6(b) show the dual line arrangements for Figures 1(a) and 1(b) respectively.

Stoeckl [4] performed an exhaustive computer search for the similar problem of enumerating *circular sequences* of permutations; these are sequences in which each permutation is related to the next by a single flip of adjacent objects, and each pair of objects is flipped exactly once. Every point or line arrangement gives rise to a circular sequence, but not vice versa. The quantity corresponding to the number of halving lines is the number of flips occurring between the middle two elements of the permutations. Stoeckl uses his search results to show that $h(6) = 6$ and $h(8) = 9$. He also bounds

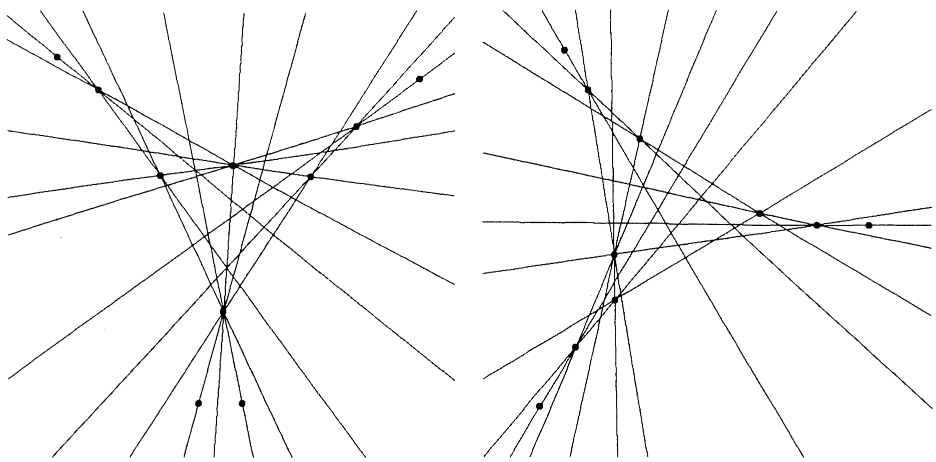

Figure 3. Two sets of 10 points with 13 halving lines each.

$h(10) \leq 13$; together with the examples we give this bound is again seen to be tight. Stoeckl was unable to complete the enumeration for $n = 12$, but he found circular sequences corresponding to a value of 18 for $h(12)$; we show that in fact this value can be realized by halving lines in a point set.

Klawe et al. [2] show that the number of central flips in a circular sequence can be at least $n \exp(c \log^{1/2} n)$ for some c. In terms of points and halving lines, their construction would replace each point of a configuration by a regular $2k$-gon; this would produce $2k$ bundles of k and $k-1$ parallel lines. If these bundles of lines can be aligned with the halving lines of the original configuration, the total number of lines in the new configuration could be multiplied by a factor as large as $4k-1$, while the number of points would only be multiplied by a factor of $2k$. It is not known if this lower bound can be realized by geometric configurations of points. Part of the difficulty is in the step of aligning the bundles of lines from each regular polygon with the lines of the original configuration: it seems there are too few degrees of freedom to get all of the lines going in the right directions. However we were able to salvage the idea of replacing single points by small numbers of points, in a new general lower bound for halving lines.

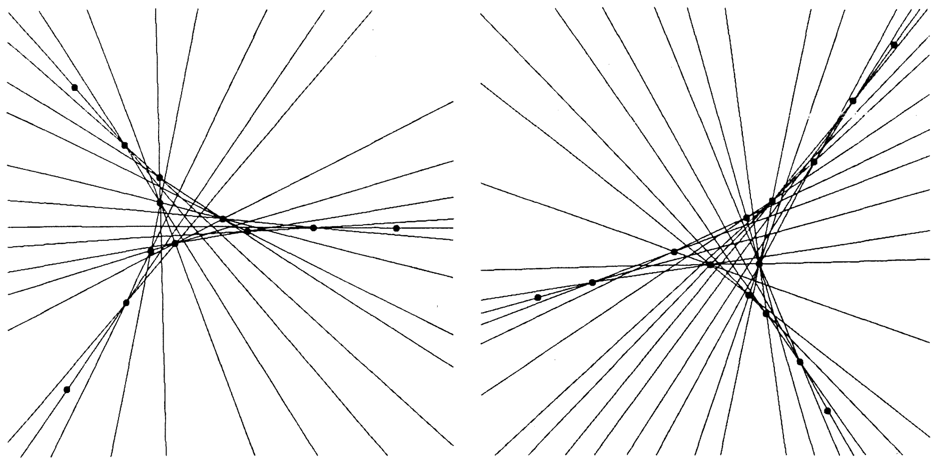

Figure 4. (a) 12 points, 18 halving lines; (b) 14 points, 22 halving lines.

2 Methodology

We performed a non-exhaustive computer search for configurations of points with many halving lines, using the technique of *genetic search*. Our program started by generating a collection of M random point sets, each with n points, for given parameters M and n. Then the algorithm iteratively removed the point set having the least number of halving lines, and replaced it by combining two other sets. Combination was performed by mixing up some points from one set with some other points of the other set, to reach a total of n points; some random variation in coordinate values was also introduced in this stage.

To break ties when selecting the point set to remove, we added a term summing the squares of the numbers of points on either side of non-halving lines. Such a term would be large when many lines are close to halving the point set, and small when many lines are far from halving the set. Curiously, the best performance was achieved when this term was negated, so that the program attempted to make lines other than halving lines cut the point set as unevenly as possible. In experiments with $n = 12$ and $M = 200$, the program typically found an 18-line configuration after approximately 25000 iterations using the negated tie-breaking term. When the term was missing, the program took more iterations and was successful less often, and when the term was expressed positively the program never found a configuration with more than 15 halving lines.

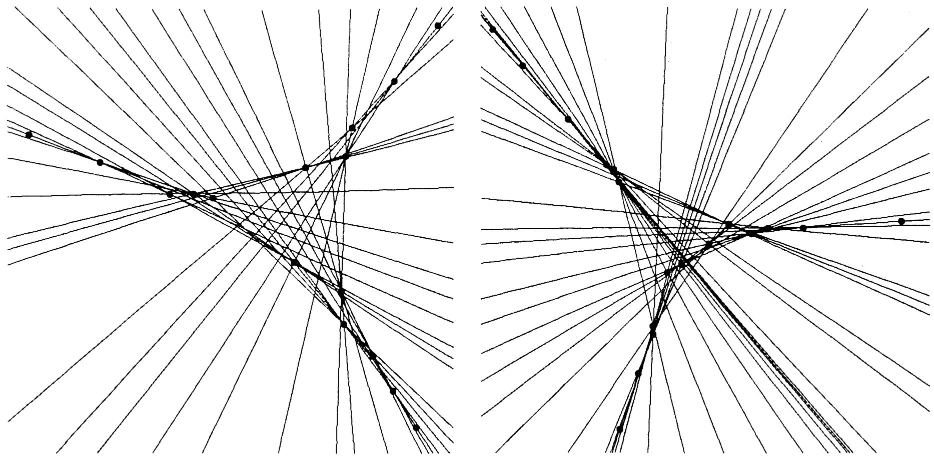

Figure 5. (a) 16 points, 27 halving lines; (b) 18 points, 32 halving lines.

For $n \leq 12$, we were able to duplicate the program's performance by hand, and find configurations with matching numbers of halving lines. For $n > 12$, we were unable to find such configurations before examining the program's output.

The sets of points displayed in the figures were found by starting with the output of our genetic search, and then shifting the points by hand to move them further from the lines and make the configurations more legible.

3 Results

Our results are shown in the figures. We list the coordinates of the point sets in an appendix.

We found sets of 10 points with 13 halving lines (Figures 3(a) and 3(b)), 12 points with 18 halving lines (Figure 4(a)), 14 points with 22 halving lines (Figure 4(b)), 16 points with 27 halving lines, (Figure 5(a)), and 18 points with 32 halving lines (Figure 5(b)).

For completeness we also include figures for sets of 4 points with 3 halving lines (Figure 1(a)), 6 points with 6 halving lines (Figure 1(b)) and 8 points with 9 halving lines (Figures 2(a) and 2(b)). In several instances there were multiple combinatorially different configurations achieving the same numbers of halving lines; we illustrate this in the figures for 8 and 10 points.

Figures 6(a) and 6(b) show the dual line arrangements of the 4- and

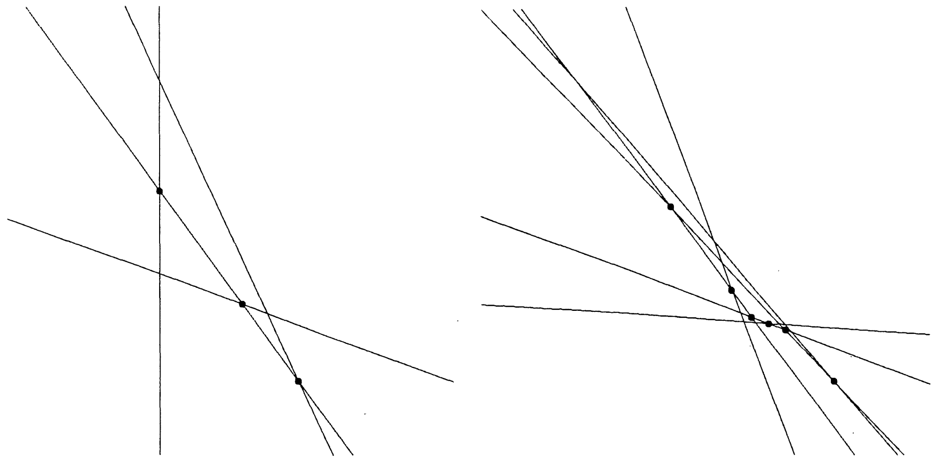

Figure 6. Dual line arrangements for 4 and 6 point sets, showing points dual to halving lines.

6-point configurations under the transformation mapping points (x, y) to lines $\{(x', y') : xx' + yy' = 1\}$. The duals of the higher order configurations involved many almost parallel lines, and it proved difficult to draw the entire line arrangements and still allow the halving points to be seen.

4 General Construction

The 12-point, 18-line configuration depicted in Figure 4(a) can be thought of as having 6 pairs of points, in approximately similar positions to the 6 points in Figure 1(b). We now show how to generalize this, in a construction that takes as input configurations with n points and k lines, satisfying some weak conditions, and produces as output new configurations with $2n$ points and $2k + n$ lines (and again satisfying the same conditions). Iterating this operation produces arbitrarily large configurations with $n \log_4(n) - O(n)$ points, improving the best previous construction by a factor of $\log_2 3$.

The method described here differs from the best previous construction in an important detail: that construction replaced an n-point configuration by 3 well-separated copies of the configuration, or in other words replaced every point of a 3-point configuration by an n-point configuration. Our construction replaces every point of an n-point configuration by a 2-point configuration, or in other words replaces a 2-point configuration by n well-

separated copies of the configuration.

Define the *underlying graph* of a configuration of points and halving lines as follows. Each point corresponds to a vertex of the graph; each halving line through two points corresponds to an edge connecting the corresponding vertices. For instance, the underlying graph of Figure 1(a) is the 3-star $K_{1,3}$.

Lemma 1. *Let S be a configuration of n points and k halving lines for which the underlying graph is connected and contains a cycle. Then there is a configuration of $2n$ points with $2k+n$ halving lines for which the underlying graph is again connected and contains a cycle.*

Proof: For each vertex in the underlying graph, pick an adjacent edge, such that no two vertices pick the same edge. This can be done as follows. Start by finding a cycle in the graph. For each vertex in the cycle, pick the next edge in some consistent orientation of the cycle. For each vertex not on the cycle, pick the edge on a shortest path from the vertex to the cycle.

Now replace each point p of the original configuration by two points p_1 and p_2, spaced at some small distance of ϵ from p, on either side of p along the line corresponding to the edge pq picked by p.

If ϵ is small, line $p_1p_2 = pq$ will be a halving line, since it cuts between q_1 and q_2, and since the number of other pairs of points on either side of the line in the new configuration is the same as the number of single points on either side of the line in the original configuration. Further, if p_2 is closer to q than p_1, lines p_2q_1 and p_2q_2 will halve the four-point configuration $p_1p_2q_1q_2$. If ϵ is small, these two lines will be halving lines, configuration, since they will behave the same as pq with respect to the points outside this four-point configuration. Thus line pq has been replaced by three halving lines.

If some other line rs is not chosen by either r or s, then $r_1r_2s_1s_2$ will form a convex quadrilateral, which has two halving lines. As above, if ϵ is small enough these lines will be halving lines of the entire configuration.

Thus each of the n chosen halving lines is replaced by three such lines, and each unchosen line is replaced by two lines. The new edges p_2q_2 connect all vertices p_2, since the chosen edges of the original graph form a connected subgraph. Each vertex p_1 is then connected by the edge p_1p_2. The old configuration has at least n lines, so the new one must have more than $2n-1$ and thus contains a cycle. □

Figures 1(a), 2(a) and 2(b) can also be viewed as results of the construction described in Lemma 1; in each of these cases the original configuration (of two or four points) does not contain a cycle, so one point is unable to

choose a line, and $2k+n-1$ instead of $2k+n$ lines are produced. In this view, both Figures 2(a) and 2(b) are based on the same original configuration; they differ in the selection of the point without a chosen line. In configurations satisfying the lemma there may similarly be several ways to assign edges to points.

Theorem 1. *For any $i > 1$, there is a configuration of $3 \cdot 2^i$ points having $3(i+1)2^{i-1}$ halving lines.*

Proof: By induction, there is a configuration of $3 \cdot 2^{i-1}$ points with $3i \cdot 2^{i-2}$ lines, satisfying the conditions of Lemma 1. As a base case, the 6-point, 6-line configuration of Figure 1(b) meets these conditions. Applying Lemma 1 produces a new configuration with $2 \cdot 3 \cdot 2^{i-1} = 3 \cdot 2^i$ points, and $2 \cdot 3i \cdot 2^{i-2} + 3 \cdot 2^{i-1} = 3(i+1)2^{i-1}$ lines. □

It seems likely that we can further improve this construction by replacing points with almost-symmetric quadrilaterals, creating two pairs of almost-parallel lines and a fifth line which can be simultanously aligned with three halving lines of the original configuration. We might then expect $4k+5n-O(1)$ halving lines in the quadrupled configuration, instead of the $4k+4n$ found by iterating Lemma 1. Alternately, it may be possible to replace each point with six, forming three families of three almost parallel lines. However the details involved in choosing three lines per point and in the precise alignment of the new lines seem to be much more complicated.

Acknowledgements

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References

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[4] G. Stoeckl. Gesammelte und neue Ergebnisse ueber extreme k-Mengen ebener Punktmengen. Master's thesis, T.U. Graz, 1984.

Appendix: Coordinates of Points

10 points with 13 halving lines (Figure 3(a)):

```
(0671475459,0602662965)    (0866393159,0790162152)
(1755985859,1252437042)    (1396006925,1148161791)
(0406797296,1823624192)    (1755985859,1071955826)
(0534402779,1659842416)    (0822654030,1106636502)
(0493356573,0345861517)    (0865171914,1408088207)
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10 points with 13 halving lines (Figure 3(b)):

```
(0501596025,2922267901)    (0800158555,2762315833)
(1157894161,2409920931)    (1704618741,1591416200)
(1798651637,0854302324)    (1795385990,1201657914)
(2016353732,2584642799)    (2354749141,2580266712)
(2705238301,2848589647)    (3149142988,3091964229)
```

12 points with 18 halving lines (Figure 4(a)):

```
(0102451081,2106657104)    (0477587180,1807524123)
(0680748732,1598072998)    (0841055706,1594993850)
(0944160504,1212602994)    (0998123159,0659637087)
(1000853802,0159637087)    (1024160504,1062602994)
(1101502272,1503296173)    (1148414542,1652267310)
(1477698542,1803086239)    (2032803036,2159412470)
```

14 points with 22 halving lines (Figure 4(b)):

```
(0915614039,0937003394)    (1251183799,1160778355)
(1623529318,1377729192)    (1859335300,1611196868)
(1965012256,1754730321)    (2172482408,2150741026)
(2250552630,1952576094)    (2244046118,1683238476)
(2359932857,2604784235)    (2434250509,1741821384)
(2544177914,1646844078)    (2450503123,2904469335)
(2841476718,1459731235)    (3137303259,1309436786)
```

16 points with 27 halving lines (Figure 5(a)):

(0346677906,0515312099)	(0600845440,0742231463)
(0816067128,0960581132)	(0855545120,2635284987)
(0946783094,0993704913)	(0981703110,2269545580)
(0998507170,1199092935)	(1127438295,1909153458)
(1125062073,1784496810)	(1142235379,1683563705)
(1441457110,1260815591)	(1575230969,1016596579)
(1727043299,1003097457)	(1875315345,0852517541)
(2037972576,0750760531)	(2210235774,0629203588)

18 points with 32 halving lines (Figure 5(b)):

(0046418756,2118677361)	(0235501865,1993735372)
(0520934855,1806046244)	(0750315458,1652192759)
(0777409514,1621565112)	(0850282468,1595205536)
(1055809821,0421855284)	(1085702707,0973178912)
(1083964441,0826950740)	(1065200723,1135189109)
(1114812091,1040972412)	(1170373582,1219938774)
(1270949952,1334701350)	(1314273780,1393042395)
(1614755517,1452534926)	(1644522577,1454093161)
(1860083884,1514073986)	(2154666371,1592058869)