Derivation of Equivalent Continuous Dilution for Cyclic, Unsteady Driving Forces

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Abstract

This article uses an analytical approach to determine the dilution of an unsteadily-generated solute in an unsteady solvent stream, under cyclic temporal boundary conditions. The goal is to find a simplified way of showing equivalence of such a process to a reference case where equivalent dilution is defined as a weighted average concentration. This derivation has direct applications to the ventilation of indoor spaces where indoor air quality and energy consumption cannot in general be simultaneously optimized. By solving the equation we can specify how much air we need to use in one ventilation pattern compared to another to obtain same indoor air quality. Because energy consumption is related to the amount of air exchanged by a ventilation system, the equation can be used as a first step to evaluate different ventilation patterns effect on the energy consumption. The use of the derived equation is demonstrated by representative cases of interest in both residential and non-residential buildings.

Keywords

Dilution; Concentration; Unsteady Ventilation

Nomenclature

A	dilution rate
C	concentration
f	fractional time in the period T
N	dilution
r	fractional dilution rate in the step
S	solute source strength
t	time
T	period
W	weighting

Greek

ε	ventilation effectiveness

Subscript

*	reference case

Abbreviations

ACH	Air Change Rate

VOC Volatile Organic Compounds

1 Introduction

The issue being addressed in this article is how to treat a cyclic process in which a variably generated solute is being diluted by a variable flow-rate solvent in a stream. The goal is to find a simplified way of showing equivalence of such a process to a steady-state one.

1.1 Target Application

The motivation for this work is to find ways of showing equivalence in determining indoor air quality in buildings when both sources and ventilation rates vary over time. A key step in designing a building is determining the correct amount of ventilation and the optimal system with which to provide it. There is no shortage of guidance on how much ventilation to use. The standard of care for ventilation system design in the US is probably the 62 series of ASHRAE standards (62.1-2004 for non-residential buildings [1] and 62.2-2007 for residential buildings [2]). In Europe the standard EN15251-2007 [3] gives recommended ventilation rates for non-residential buildings but countries specify design ventilation rates in their national codes.

When ventilation rates are stated in terms of airflow rate per person (e.g. l/s pr. person) or airflow rate per floor area (e.g. l/s pr. m^2), we generally assume a constant airflow during the entire period of interest. There are, however, a variety of reasons why one might want to design and operate the ventilation system with variable amounts of ventilation airflow. For example:

- There may be periods of the day when the outdoor air quality is poor and one wishes to reduce the amount of outdoor air entering the building;
- Equipment operating for other reasons (e.g. economizer operation) can provide exogenous ventilation from the point of view of indoor air quality and energy savings can be achieved by lowering the designed mechanical ventilation to account for it
- Energy or power costs may make it advantageous to reduce ventilation for certain periods of the day;

- Some HVAC equipment may make cyclic operation more attractive than steady-state operation such as residential or small commercial systems that tie ventilation to heating and cooling system operation; and
- The generation of pollutant indoors may vary over time, e.g., depending on occupancy such that adjusting the ventilation rate according to the demand can improve indoor air quality and potentially save energy.
- Different times of the day may be more important than others (e.g. because different number of occupants are present) and can be weighted differently.

Regardless of the reason, the designer or decision-maker needs a method to determine how two ventilation systems compare for the purposes of providing acceptable indoor air quality. ASHRAE Standard 62.1-2004 does not directly address these issues; 62.2-2004 does address intermittent ventilation compared to continuously operated ventilation system in a limited way. The standard EN15665 [4] sets out criteria to assess the performance of residential ventilation systems concerning hygiene and indoor air quality.

1.2 Ventilation Background

Ventilation is principally used to maintain acceptable indoor air quality by controlling indoor contaminant concentrations and minimizing occupant exposures to the contaminants. Whole-building ventilation dilutes contaminants in the indoor air with air that does not contain those contaminants, and is normally used for controlling unavoidable, generic or non-specific contaminants. When specific contaminant sources can be identified, they are best dealt with directly through source control methods including local exhaust. For example, bathroom and cooking contaminants (including water vapor) are best addressed by exhaust fans in those spaces. Volatile Organic Compounds (VOC) are often best addressed by changes in composition or use of specific materials.

If ventilation rate and contaminant concentration were linearly related, the average concentration would be proportional to the average ventilation and straightforward methods could be used to determine the effectiveness of a ventilation system with variable flow rates. Unfortunately, ventilation and concentration are dynamically and inversely related through the mass continuity equation, which leads to a typically non-linear relationship between ventilation and concentration.

Solutions to the continuity equation always involve an air change rate (ACH)[1] appropriate to the problem at hand. Although we are often more accustomed to dealing with ventilation in terms of specific airflow rates, the efficacy of a ventilation system with variable air flow rates will depend on the air change rate, so it is important to keep typical rates in mind for some specific, but common occupancies. A related parameter of interest is the *turn-over time*, which is the inverse of the air change rate. It is the characteristic time in which the concentration of a contaminant responds to a change in ventilation rate.

One can derive typical air change rates and turn-over times from literature and from standards using specific ventilation rates, typical occupant densities, and typical geometry of the space in question.

TABLE 1

The turn-over times in table 1 vary from 6 minutes to 6 hours, indicating that different occupancies will behave quite differently at a variety of configurations. The use of such quantities to explore the spatial dependency of ventilation is also important for large spaces, but will not be discussed here. Sandberg and Sjoberg [9] developed much of the nomenclature used in this field to deal principally with spatial variation.

Sherman and Wilson [10] followed by Yuill [11, 12] have already solved the continuity equation for the general case and defined (temporal) ventilation effectiveness,[2] ε, as a measure of how good a given, time-varying, ventilation pattern is at providing acceptable IAQ. As in those cases, we limit our analysis to contaminants with a linear dose-response and no other loss mechanism (e.g. sorption or deposition). ASHRAE Standard 136 [13] uses this kind of approach to convert time-varying envelope air leakage into an effective seasonal ventilation rate. Sherman [14] studied the case of equivalent dilution of a steadily generated source for intermittent ventilation compared to constant ventilation. These results have been included in ASHRAE standard 62.2 by allowing intermittent ventilation provided that the ventilation rate is raised outside the off period. This paper expands on the work by Sherman to also consider variable source generation, variable flow rates and variable weighting of the concentration. The purpose of this paper is to develop

[1] The term "air change rate" does not discriminate between, infiltration, natural ventilation, or mechanical ventilation. It simply denotes that air is being exchanged between indoors and outdoors—somehow.
[2] We will also use the term *efficacy* as a synonym for (temporal) ventilation effectiveness.

approaches for determining the indoor air quality equivalency of different ventilation systems based on fundamental principles of mass balance. The approaches are demonstrated using a few representative cases of interest in both residential and non-residential buildings.

2 Problem Definition

Consider the situation in which we have a small amount of solute being generated at a known time-varying rate $\dot{m}_{solute}$ inside a solvent filled space of volume *V*. This space is being flushed by the solvent at a known, time-varying rate $\dot{m}_{solvent}$ to yield a time varying solute concentration, *C*. The relationship between these quantities is constrained by the conservation of mass as follows:

$$V\dot{C} + \frac{\dot{m}_{solvent}}{\rho_{solvent}} C = \frac{\dot{m}_{solute}}{\rho_{solute}} \tag{1}$$

Assuming constant densities for solvent and solute eq. 1 can be expressed by:

$$\dot{C} + AC = S \tag{2}$$

Where $A \equiv \dot{m}_{solvent} / V\rho_{solvent}$ is the dilution rate of the solvent stream and $S \equiv \dot{m}_{solute} / V\rho_{solute}$ is the source strength of the solute.

We are investigating the problem of a system that is unsteady, but in cyclic equilibrium over some known period, T. This means that the concentration, the solute source strength and the dilution rate take on same value when the time changes one period: $C(t-T) = C(t)$, $S(t-T) = S(t)$ and $A(t-T) = A(t)$.

We seek to evaluate some test system that performs dilution equivalent to a reference case. We define the dose, *d*, as the quantity that we wish to hold constant in determining equivalent dilution and the dose is calculated as the weight-integrated concentration over the cyclic period, T:

$$d = \int C(t)W(t)dt \tag{3}$$

Where W is the weighting function: $\int W(t)dt = T$. The weighting function allows us to emphasize parts of the cyclic period heavier than others or to omit parts of the cyclic period by using a zero value weighting factor when solving for equivalent dose.

2.1 Reference case

Before proceeding further, we consider the reference case to compare our test case with. We select as our reference the case conventionally called *perfect dilution*, which we define as that time varying reference dilution rate $A_*(t)$ that holds the concentration constant at some steady state value, C_*. By inspection of equation 2 the time-varying reference dilution rate is:

$$A_*(t) \equiv \frac{S(t)}{C_*} \tag{4}$$

In our reference case the dose is then:

$$d = \int C_* W(t)dt = C_* T \tag{5}$$

Let us now define the *efficacy* as the ratio of the amount of solvent required in the reference case compared the amount in the test case under consideration. The amount of solvent used in the reference case, N_*, is the integrated reference dilution rate over the cyclic period:

$$\mathrm{N}_* \equiv \int A_*(t)dt \tag{6}$$

In the same ways is the amount of solvent used in the test dilution system, N, given:

$$\mathrm{N} \equiv \int A(t)dt \tag{7}$$

From equation 9 and 10 we calculate the efficacy:

$$\varepsilon \equiv \frac{\int A_*(t)dt}{\int A(t)dt} = \frac{\mathrm{N}_*}{\mathrm{N}} \tag{8}$$

The efficacy is a measure of how much solvent we need to use in our test system compared to the reference case to obtain the same dose. The efficacy can be used as a target or optimization parameter in the design process. We can design our test system to match some target efficacy that often will be unity, as we then provide dilution equivalent to our reference case. The efficacy can also be used as an optimization parameter in designing systems that provide same dose but use less solvent.

3 Derivation of dose for cyclic, unsteady driving forces

We want to derive an equation for equivalent dilution in our test and reference case and we use dose as the quantity that we wish to hold constant in determining equivalent dilution. To evaluate the dose (eq. 6) in our test case, we need to solve the continuity equation (eq. 2) for the concentration, *C(t)*. The standard integral form of an inhomogeneous, first-order, linear differential with arbitrary coefficients can be used to do this:

$$C(t) = C(t_o)\xi(t,t_o) + \int_{t_0}^{t} S(t')\xi(t,t')dt' \tag{9}$$

Where $C(t_o)$ is the known constant of integration representing the concentration at some reference time, t_o. For simplicity we have defined the following function:

$$\xi(t,t') \equiv e^{-\int_{t'}^{t} A(u)du} \tag{10}$$

Our process is cyclic over the period T and the concentration therefore takes on the same value when the time changes one period. This also means that t_0 in our constant of integration is arbitrary and the constant of integration itself is a solution to the differential equation. The time-varying concentration in our cyclic process is then given by:

$$C(t) = \frac{\int_{t-T}^{t} S(t')\xi(t,t')dt'}{(1-\xi(T,0))} \tag{11}$$

The derivation of the time-varying concentration can be found online in the paper's supplementary data in section 1.

By substituting the time varying concentration (eq. 11) into the expression of the dose (eq. 3) we can calculate the dose for any test system with variable dilution rates, solute source strength and weightings by the following double integral:

$$d = \frac{\int W(t) \int_{t-T}^{t} S(t')\xi(t,t')dt'dt}{(1-\xi(T,0))} \tag{12}$$

With the derived expression of the dose (eq. 12) for an unsteady but cyclic test system we can compare this to the dose in our reference case of perfect dilution (eq. 5). Depending on which parameters are known we can use the equivalency equation in different ways.

If we know the solute source strength, *S(t)*, and the time-varying dilution rate in our test system, *A(t)*, is completely specified we can determine what the steady-state concentration, C_*, in our reference case of perfect dilution would be. Sometimes we are trying to design a system that produces a dose equal to that in our reference case and we can use equation 12 as the constraint on the test system that makes that true. Our problem then reduces to finding that test dilution pattern that gives us the target dose.

Because we defined our time varying reference dilution rate $A_*(t)$ as that which holds the concentration constant at some steady state value, C_*, see equation 4, we do not need to individually know the solute source strength, *S(t)*, and the steady state concentration, C_*, but only the presumed dilution for perfect dilution, $A_*(t)$.

4 Step Function

The applications we consider further on will only involve situations in which the weightings, solute source strength and dilution rates are all step-wise constant with one step at time *t.*, see figure 1. Any of the three parameters can change at the step t_1 or they must remain the same through the cyclic period, T. In other words, they have to change at the same time or not change at all.

FIGURE 1 (black and white)

Because of the step-wise constant profiles we can expand the dose equation (eq. 12) into a sum of integrals where the parameters are constant. We can thereby set up an analytical expression for equivalent dose in our reference and test case (eq. 5 equals eq. 12). This analytical expression can be simplified using the following definitions: weighting of the two periods: $W_1 t_1 + W_2(T - t_1) = T$, reference dilution: $\mathrm{N}_* = A_{*1} t_1 + A_{*2}(T - t_1)$, test dilution: $\mathrm{N} = A_1 t_1 + A_2(T - t_1)$, fraction of time in the step: $f \equiv t_1 / T$, non-dimensional test dilution rate in the step: $r \equiv A_1 T / \mathrm{N}$, and a non-dimensional reference dilution rate in the step: $r_* \equiv A_{*1} T / \mathrm{N}_*$. The non-dimensionalized dilution rates, *r* and *r**, will be 1 when the dilution rate does not change through the cyclic period and they will be 0 when the dilution rate during one of the two periods is zero. The parameters *r*, *r** and *W1* can take on values in the interval $[0, T/t_1]$.

By introducing the variables: $Z \equiv f r$ and $\phi \equiv f^2[(r - r_*)(r - W_1)]$ the equation for equivalent dilution can be reduced to:

$$\mathrm{N}_* = \frac{\mathrm{N}}{1 + \frac{\phi}{Z(1-Z)} - \frac{2}{\mathrm{N}} \frac{\phi}{(Z(1-Z))^2} \Big/ \left(\mathrm{Coth}\left[\frac{\mathrm{N}Z}{2}\right] + \mathrm{Coth}\left[\frac{\mathrm{N}(1-Z)}{2}\right] \right)} \tag{13}$$

For detailed derivation see the online supplemental data section 2.

Φ and Z are only introduced to simplify the equation and they also allow us to more easily investigate the behavior of our step wise constant problem in its space of solutions. In equation 13 it is worth noting that Z is symmetrical around ½ as replacing Z by (1-Z) yields the same result.

We can derive a recursive expression for the efficacy, but at the expense of breaking the symmetries of Z:

$$\varepsilon = \frac{1 - \left(\frac{\phi}{(1-Z)} - Z(1/\varepsilon - 1) \right) \left(\frac{\mathrm{N}_*}{2} \right) \left(\mathrm{Coth}\left[\frac{\mathrm{N}_* Z}{2\varepsilon}\right] + \mathrm{Coth}\left[\frac{\mathrm{N}_*(1-Z)}{2\varepsilon}\right] - \frac{2\varepsilon}{\mathrm{N}_* Z} \right)}{1 - \frac{\phi}{(1-Z)^2}} \tag{14}$$

Because we consider a step-wise constant system we only need to know the ratio of the reference dilution rates (or solute source strength) in the two periods to estimate the efficacy of the system. However, if we only know the ratio we cannot calculate the actual dose.

5 Discussion

5.1 Phase space of Φ and Z

Before discussing the actual phase space of the efficacy it is important to realize that the allowable phase space of the parameters Φ and Z is limited. The maximum value Φ occurs when the product: $(r - r_*)(r - W1)$ is as large as possible. This occurs at two points; the first is when r_* and $W1$ equals 0, hence $\Phi_{max} = Z^2$. Because Z is symmetrical around ½, $\Phi_{max} = (1\text{-}Z)^2$ for $Z > ½$. The minimum value of Φ occurs when one of the differences: $(r - r_*)$ or $(r - W1)$ is positive and the other is negative. Because r_* and $W1$ take on values in the interval [0;1/f], $\Phi_{min} = -Z(1\text{-}Z)$. The limits of Φ are thereby given by: $-Z(1-Z) \le \phi \le Maximum(Z^2,(1-Z)^2)$. Figure 2 shows the allowed phase space of Φ and Z. It is seen that Φ goes from zero to unity when Z=0 (or Z=1), but when Z=1/2, Φ goes between ±0.25.

FIGURE 2 (black and white)

5.2 Intermittent Dilution

In the limiting situation where there is no dilution during one of the two steps Z equals either 0 or 1. We call this limit *Intermittent Dilution*. If we take the limit of equation 14 when Z approaches zero (or unity) we get the following expression for the efficacy:

$$\varepsilon_0 = \frac{1}{1-\phi+\phi\ (\mathrm{N}/2)\mathrm{Coth}\left[\frac{\mathrm{N}}{2}\right]} \qquad (15)$$

Where we have used the subscript on the efficacy to show that it is for a solution where one of the two steps has no dilution. Sherman [14] solved the case of intermittent dilution when the solute source strength and weighting were constant during the cyclic period. In that specific case of intermittent dilution Φ approaches the following limit:

$$\phi \to f^2 \tag{16}$$

For that application the solution was more conveniently expressed as a recursive relationship between the efficacy and the reference dilution, N_*. We can also express the more general intermittent dilution solution (eq. 15) in that form as follows:

$$\varepsilon_0 = \frac{1\text{-}\phi\,(N_*/2)\mathrm{Coth}\left[\dfrac{N_*}{2\varepsilon_0}\right]}{1-\phi} \tag{17}$$

5.3 Phase space of efficacy

Let us know examine the phase space of the efficacy by the equation for equivalent dose (eq. 13). Figure 3 is a plot of the efficacy vs. reference dilution rate at a representative value of Φ= 0.25. A mesh of curves for different values of *Z* is graphed spanning the full range of Z. The lower bounding curve is for *Z*=0 (or Z=1 corresponding to the intermittent dilution limit) and the upper curve is for *Z*=1/2.

FIGURE 3 (black and white)

For reference dilution values, below approximately 2, the efficacy is independent of *Z* and the solution for intermittent dilution provides sufficiently accurate results. As the reference dilution gets higher the efficacy approaches an asymptote that is very much dependent on *Z*. Taking the limit of the general equivalency equation (eq. 14) for high dilution rates we find that efficacy asymptotically approaches a limit given by:

$$\varepsilon_\infty = \frac{1}{1+\dfrac{\phi}{(Z(1-Z))}} \tag{18}$$

This suggests that for practical problems one may choose to use the efficacy equation for intermittent dilution until the efficacy approaches the limited given by the asymptote. Inspection of eq. 18 also shows us that because Z always is positive, the efficacy is below unity for positive values of Φ and above unity for negative values of Φ. In figure 3 we saw

how the efficacy depended on Z and not Φ which was maintained at a fixed value of 0.25. Figure 4 shows the efficacy at three different values of Φ each spanning their individual range of Z.

FIGURE 4 (black and white)

Again we see that for a test dilution of less than about 2, the intermittent dilution equation (eq. 15) provides sufficiently accurate results as Z in this range has little effect on the efficacy. Furthermore we also see how the efficacy can take on values above unity when Φ is negative. At low dilution the efficacy for Φ=-0.1 starts near unity but it can slowly grow without bound for increasing dilution. Efficacies above unity means that a test case can perform better (i.e. use less solvent) than our reference case of perfect dilution. For Φ to be negative, *r* must be between the values of r_* and *W1*. This could occur, for example, in a case where the source was high during a period when the weighting was low. The most negative Φ occurs when the sum of the two difference (r-r_*) and (r-*W1*) is a big as possible. This happens when *r* take on a value exactly between r_* and *W1*, hence: $r = (r_* + W_1)/2$ which means that: $\phi_{low} = -(f^2/4)(r_* - W_1)^2$.

Figure 5 shows the dependence on Φ a bit more clearly for representative values of test dilution (N=2, 5 and 10) and spanning the full range of Z. The lower bounding curve is for *Z*=0 when Φ is positive whereas Z for the upper bounding curve changes depending on Φ. When Φ is negative the lower bounding curve is for Z=½ and Z changes for the upper bounding curve depending on Φ.

FIGURE 5 (black and white)

In figure 5 we again see that for low values of test dilution there is not much dependence on Φ for the efficacy but at higher values there is (as is there on Z). Again our results show that for a test dilution of two or below the efficacy is independent of Z and the equation for intermittent ventilation (eq. 15) will provide sufficiently accurate results. From figure 5 we also see that to design systems with high efficacies the strategy would be to minimize Φ as much as possible. If more degrees of freedom are available, Z can be optimized after that; an optimal Z should be as close to ½ as possible for efficacies below unity and as low (or high) as possible for efficacies above unity.

5.4 Approximate solution

For some applications it may be desirable to have an approximate solution. We note two results from above that suggest an approximate solution. The first result is that high test dilutions solutions are generally low efficacy and relatively independent of efficacy (see figure 3). The second result is that low test dilution solutions have efficacies near unity and are relatively independent of Z (see figure 3, figure 4 and figure 5), and thus equal to the intermittent dilution solution. Accordingly, an approximate solution for the efficacy that works over a broad spectrum combines the limiting solutions is as follows:

$$\varepsilon \approx \varepsilon_0 + \varepsilon_\infty (1-\varepsilon_0)^2 \tag{19}$$

This approximate solution does not work well when any of the efficacies are much greater than unity—which can only happen when ϕ is negative. Other reasonable approximations are possible, but were not investigated as this appears sufficiently accurate for most purposes.

Figure 6 shows the approximate solution of efficacy for Φ=0.5 and Z=0.2. The largest difference between the approximate and exact solution occurs in the region where the intermittent and asymptotical solution intersect.

FIGURE 6 (black and white)

6 Practical examples

The discussion above was quite general. We will now work two practical examples from the field of ventilation. In the terminology of ventilation the solute source strength, *S(t)*, corresponds to the emission of pollutants in a room. The dilution rate, *A(t)*, is the air change rates in the room. The weighting function can be used to represent the presence of occupants in the room, so that it is possible only to evaluate the dose when the room is occupied. We assume that the dose is linearly proportional to the pollutant concentration because the vast majority of ventilation standards, are limited to chronic, long term exposure and do not address short term exposures to highly toxic substances with non-linear dose response for human health. Therefore dose is used as the metric for equivalent air quality in the following examples.

6.1 Example 1 - Intermittent ventilation with variable source generation

Consider a home ventilated at constant rate of $0.5h^{-1}$ and assume the emission of pollutants is constant during the day. The home is occupied at all times, hence *W1*=1. The owner of the house now wants to start up a business and establishes an office in the home. It is estimated that the emission of pollutants is increased by a factor of 4 during the 8 hours each day that the office is used. To maintain the dose at the same pre-office level, the ventilation rate can be changed proportional to the pollutant emission rate during the office hours. Hence the rate during office hours would be $2.0h^{-1}$ and outside office hours it would be $0.5h^{-1}$. The amount of dilution air is therefore 24 calculated by eq. 7. The efficacy is 1 because this is perfect dilution.

Alternatively, the ventilation could be constant during the day and adjusted to give the same dose. Because the total quantity of air over the day would be the same as the above variable ventilation rate, this new, higher, constant ventilation rate can be determined by simple averaging. The average ventilation rate that would give same dose is therefore $(16h \cdot 0.5h^{-1} + 8h \cdot 2.0\ h^{-1})/24h = 1.0h^{-1}$. The dilution is still 24, and the efficacy is 1. The difference between them is that in the *perfect dilution* case the concentration is the same for the whole period, while in the constant dilution case, the concentration varies over the period. Since we have assumed acute exposures are not an issue, these two cases are equivalent.

As an alternative to these perfect dilution cases, we could increase the ventilation rate outside office hours by 50% (from 0.5 h^{-1} to 0.75 h^{-1}). Using Equation 13 to solve for equivalent dose, we find that the rate during office hours would have be 1.25 h^{-1} to obtain same dose in the two systems on a daily basis. The amount of dilution air is then 22 (eq. 7) and the efficacy of this system compared to that of perfect dilution is 1.09. By increasing the ventilation rate outside office hours by 50% we use approximately 8% less air in total each day.

6.2 Example 2 - Demand controlled ventilation

As a first step towards evaluating a systems' energy performance, a designer wants to know the total volume of air exchanged on a daily basis in a demand controlled ventilation system (DCV) compared to a continuously operated system. The systems are to be operated in a home that is occupied for 16 hours a day. Because the occupants are not present all times the systems only need to provide equivalent air quality during occupied hours. The occupied period is given the index 1 and the weighting parameters will therefore be *W1*=1.5 and *W2*=0. Pollutants are emitted by the

building itself together with pollutants from the occupants and their activities. The emission of pollutants is assumed to be 4 times higher at occupied hours compared to unoccupied hours.

The continuous ventilation system is operated at an air change rate of $0.5h^{-1}$ corresponding to the ventilation required in residential buildings in Denmark (see [7]). The test dilution in this system is therefore 12. Because all parameters needed to calculate that reference dilution, N_*, that gives us equivalent dose in our test and reference case of perfect dilution are given, we do not need to know the actual dilution rates in the reference system. Using Equation 13 shows that the reference dilution is 9.9 for the continuous system.

The DCV system is operated at half rate of the continuous rate during unoccupied hours (A_2 is $0.25h^{-1}$). We solve equation 13 to find the ventilation rate that during occupied hours gives us a reference dilution of 9.9. We find that this ventilation rate must be 5% higher than the rate in our continuously operated system and the test dilution, N, in the DCV case is 10.4 (eq. 7). Because we evaluated our two test systems relative to the same reference system, we can evaluate the two systems to each other and we find that the efficacy of the continuously operated system compared to the DCV system is 1.16. Hence, the DCV use approximately 14% less air in total every day compared to the continuously operated system to provide same indoor air quality. Further evaluations of efficacy of demand controlled ventilation with variable emission ratios are given in [15].

7 Summary and Conclusions

In this paper we have derived an expression for dilution of an unsteadily-generated solute in an unsteady solvent stream, under cyclic boundary conditions. We determined an analytical relationship showing equivalence of such a process across a step-wise constant function with one step to a steady-state one. This expression was used to evaluate the efficacy i.e. how much air we needed to use in one case compared to another. Investigating the phase space of efficacy we found that a simple equation for intermittent dilution provides sufficiently accurate results at low dilution. Furthermore we found that at high dilution the efficacy approaches an asymptote. We have demonstrated how the expression can be applied to the problem of determining equivalency for different approaches to ventilation in a building where contaminants, air flows, weightings are variable.

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Tables

Table 1: Example Air Change Rates and Turn-Over Times

ACH (1/h)	*Turn-over time (h)*	*DESCRIPTION*
0.15	6.67	Assumed infiltration rate of homes [2]
0.25	4.00	Infiltration rate of commercial buildings [5]
0.3	3.33	Ventilation requirement of almost empty commercial buildings [1]; estimated infiltration rate of new homes [6]
0.45	2.22	Ventilation requirement for small a homes of 90m^2 including default 62.2 infiltration credit [2]
0.5	2.00	Office space requirement[1]; also large home [2]; requirements in residential buildings in Denmark [7]
1	1	Infiltration rate of older home [8]
2	0.50	Conference room requirement [1]
4	0.25	High density space (e.g. theater lobby) [1]
6	0.17	School – Low polluted building, Indoor environment category B [3]
10	0.10	School – Non-low polluted building, Indoor environment category A [3]

Figure captions

Figure 1: Step wise constant weighting, solute source strength and dilution rate during the cyclic period, T.

Figure 2: Allowable phase space of Φ vs. Z.

Figure 3: Efficacy vs. reference dilution for Φ=1/2.

Figure 4: Efficacy vs. test Dilution for Φ=-0.1, 0.1, 0.9.

Figure 5: Efficacy vs. Φ for three different test dilutions (N=2, 5, 10 from flattest to steepest). Mesh is for full range of allowable values of Z

Figure 6: Approximate, exact, intermittent and asymptotical solution of ε for Φ=0.5 and Z=0.2.

Figures

Figure 1

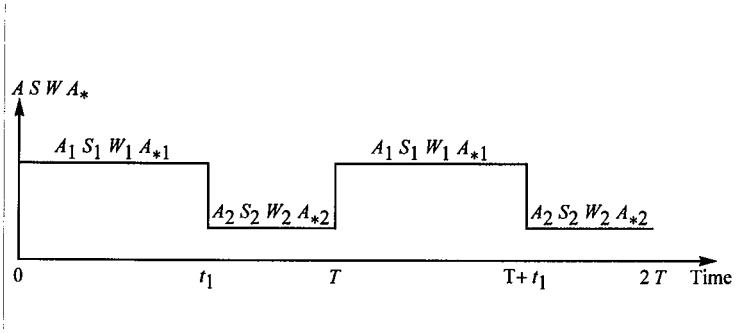


Figure 2

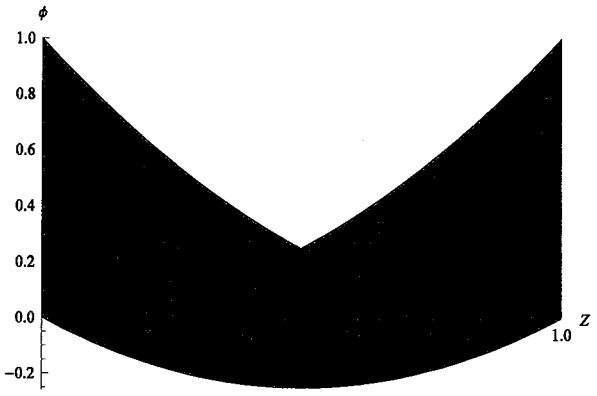


Figure 3

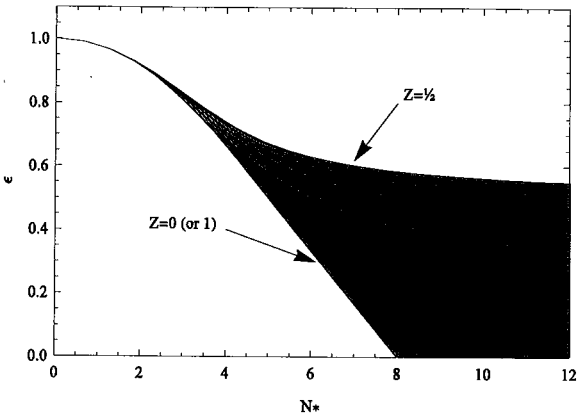

1.0
0.8
0.6
0.4
0.2
0.0
ϵ
0
2
4
6
8
10
12
N*
Z=½
Z=0 (or 1)

Figure 4

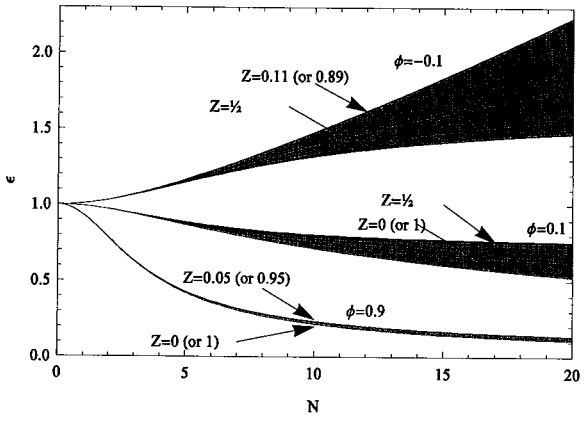


Figure 5

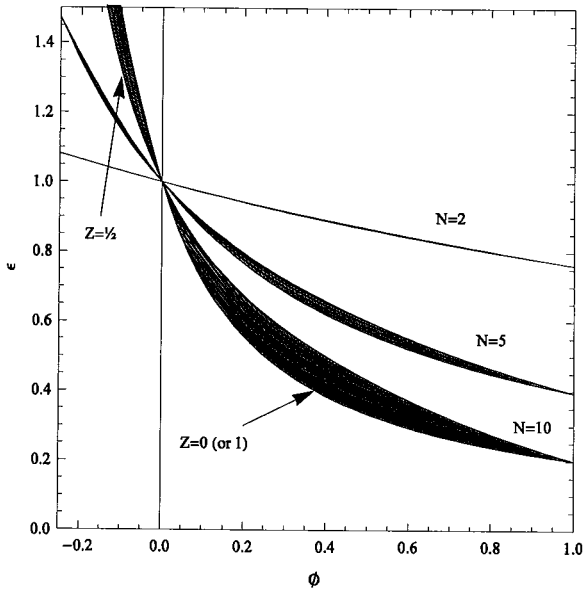
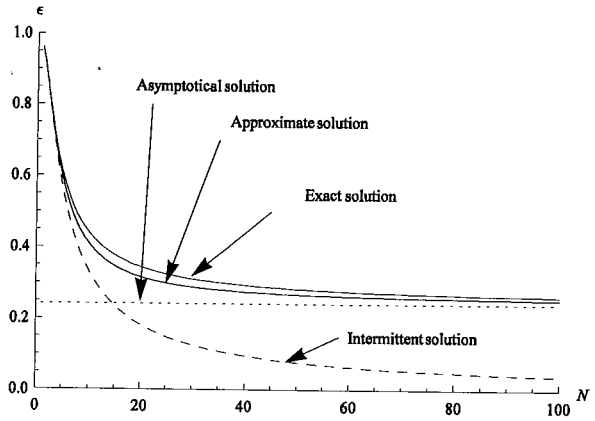


Figure 6



Supplementary data

10 Derivation of the time-varying concentration C(t)

To derive an expression for the time-varying concentration, *C(t)*, we must solve the continuity equation for this. The standard integral form of an inhomogeneous, first-order, linear differential with arbitrary coefficients can be used to do this:

$$C(t) = C(t_o)\xi(t,t_o) + \int_{t_0}^{t} S(t')\xi(t,t')dt' \qquad (1)$$

Where $C(t_o)$ is the known constant of integration representing the concentration at some reference time, t_o. For simplicity we have defined the following function:

$$\xi(t,t') \equiv e^{-\int_{t'}^{t} A(u)du} \qquad (2)$$

Because our system is in cyclic equilibrium over the period T, some useful identities of the function are:

$\xi(t,u)\xi(u,t') = \xi(t,t')$ and $\xi(t+T,t'+T) = \xi(t,t')$.

To calculate the time-varying concentration, *C(t)*, we need find the constant of integration $C(t_0)$. To do this we make us of the fact that the concentration at time zero, *C(0)*, must be the same as that at the end of the period, *C(T)*, due to our cyclic process.

$$C(0) = C(T) \tag{3}$$

$$C(t_0)\xi(0,t_0) + \int_{t_0}^{0} S(t')\xi(0,t')dt' = C(t_0)\xi(T,t_0) + \int_{t_0}^{T} S(t')\xi(T,t')dt' \tag{4}$$

Rearranging eq. 4 we find that the constant of integration is given by:

$$C(t_0) = \frac{\int_{t_0}^{T} S(t')\xi(T,t')dt' + \int_{0}^{t_0} S(t')\xi(0,t')dt'}{\xi(0,t_0) - \xi(T,t_0)} \tag{5}$$

In a first step towards simplifying the expression further we substitute, *t'*, by *u +T* and change our limits accordingly. In the denominator we make use of the fact that: $\xi(T,t_0) = \xi(T,0)\xi(0,t_0)$.

$$C(t_0) = \frac{\int_{t_0-T}^{0} S(u+T)\xi(T,u+T)du + \int_{0}^{t_0} S(t')\xi(0,t')dt'}{\xi(0,t_0)(1-\xi(T,0))} \tag{6}$$

We then make use of the fact that: $S(u+T) = S(u)$ and $\xi(T,u+T) = \xi(0,u)$ and in the end we substitute *u* by *t'*:

$$C(t_0) = \frac{\int_{t_0-T}^{0} S(t')\xi(0,t')dt' + \int_{0}^{t_0} S(t')\xi(0,t')dt'}{\xi(0,t_0)(1-\xi(T,0))} \tag{7}$$

We then expand the function $\xi(0,t')$ to $\xi(0,t_0)\xi(t_0,t')$ and make us of the fact that $\xi(0,t_0)$ is a constant we can take outside the integral.

$$C(t_0)=\frac{\xi(0,t_0)\left(\int_{t_0-T}^{0} S(t')\xi(t_0,t')dt'+\int_{0}^{t_0} S(t')\xi(t_0,t')dt'\right)}{\xi(0,t_0)(1-\xi(T,0))} \tag{8}$$

The constant of integration is then expressed by:

$$C(t_0)=\frac{\int_{t_0-T}^{0} S(t')\xi(t_0,t')dt'+\int_{0}^{t_0} S(t')\xi(t_0,t')dt'}{(1-\xi(T,0))} \tag{9}$$

The constant of integration (eq. 9) can now be put back into the solution for the concentration (eq. 1). However, because t_o is arbitrary and we have a cyclic process the constant of integration itself is a solution to the differential equation and we can remove the "0" subscript. Applying this to equation 9 and merging the integrals find that the time-varying concentration for a cyclic process can be expressed by:

$$C(t)=\frac{\int_{t-T}^{t} S(t')\xi(t,t')dt'}{(1-\xi(T,0))} \tag{10}$$

11 Derivation of an analytical expression for equivalent dose in a step wise constant process with one step

The derived expression of the dose (eq. 12 in the paper) for an unsteady but cyclic test system is compare to the dose in our reference case of perfect dilution (eq. 5 in the paper) to set up an expression for equivalent dilution:

$$C_* T = \frac{\int_{t-T}^{t} W(t)\xi(t,t')S(t')dt'dt}{(1-\xi(T,0))} \quad (11)$$

Because we defined our time varying reference dilution rate $A_*(t)$ as that which holds the concentration constant at some steady state value, C_*, see equation 4, we do not need to individually know the solute source strength, $S(t)$, and the steady state concentration, C_{*seper}, but only the presumed dilution for perfect dilution, $A_*(t)$. Rearranging equation 11 gives us the following expression for equivalent dilution:

$$T(1-\xi(T,0)) = \int_{t-T}^{t} W(t)\xi(t,t')A_*(t')dt'dt \quad (12)$$

This equation can be expanded into a sum of integrals because of the step-wise constant profiles for the solute source strength, dilution rates and weightings changing at time t_1. An analytical solution to equation 12 is found by expanding the double integral on the right side into 6 integrals each with constant weighting, solute source strength and dilution rate:

$$T(1-\xi(T,t_1)\xi(t_1,0)) =$$
$$\int_0^{t_1}\int_0^{t} W(t)\xi(t,t')A_*(t')dt'dt + \int_0^{t_1}\int_{t_1-T}^{0} W(t)\xi(t,t')A_*(t')dt'dt + \int_0^{t_1}\int_{t-T}^{t_1-T} W(t)\xi(t,t')A_*(t')dt'dt + \quad (13)$$
$$\int_{t_1}^{T}\int_{t_1}^{t} W(t)\xi(t,t')A_*(t')dt'dt + \int_{t_1}^{T}\int_{0}^{t_1} W(t)\xi(t,t')A_*(t')dt'dt + \int_{t_1}^{T}\int_{t-T}^{0} W(t)\xi(t,t')A_*(t')dt'dt$$

The analytical solution to the left side of is given by equation 14 and the 6 integrals on the right side are given in equation 15 to 20:

$$T(1-\xi(T,t_1)\xi(t_1,0)) = T\left(1-e^{-A_2(T-t_1)}e^{-A_1 t_1}\right) \tag{14}$$

$$\int_0^{t_1}\int_0^{t} W(t)\xi(t,t')A_*(t')dt'dt = W_1 A_{*1}\int_0^{t_1}\int_0^{t} e^{-A_1 t}e^{A_1 t'}dt'dt = \frac{W_1 A_{*1}}{A_1^2}\left(A_1 t_1 - 1 + e^{-A_1 t_1}\right) \tag{15}$$

$$\int_0^{t_1}\int_{t_1-T}^{0} W(t)\xi(t,t')A_*(t')dt'dt = W_1 A_{*2}\int_0^{t_1}\int_{t_1-T}^{0} e^{-A_1 t}e^{A_2 t'}dt'dt = \frac{W_1 A_{*2}}{A_1 A_2}\left(1-e^{-A_1 t_1}\right)\left(1-e^{-A_2(T-t_1)}\right) \tag{16}$$

$$\begin{aligned}\int_0^{t_1}\int_{t-T}^{t_1-T} W(t)\xi(t,t')A_*(t')dt'dt &= W_1 A_{*1}\int_0^{t_1}\int_{t-T}^{t_1-T} e^{-A_1 t}e^{-A_2(T-t_1)}e^{-A_1(t_1-t'-T)}dt'dt \\ &= \frac{W_1 A_{*1}}{A_1^2}e^{-A_2(T-t_1)}\left(1-e^{-A_1 t_1}-A_1 t_1 e^{-A_1 t_1}\right)\end{aligned} \tag{17}$$

$$\int_{t_1}^{T}\int_{t_1}^{t} W(t)\xi(t,t')A_*(t')dt'dt = W_2 A_{*2}\int_{t_1}^{T}\int_{t_1}^{t} e^{-A_2 t}e^{A_2 t'}dt'dt = \frac{W_2 A_{*2}}{A_2 A_2}\left(A_2\left(T-t_1\right)+e^{-A_2(T-t_1)}-1\right) \tag{18}$$

$$\int_{t_1}^{T}\int_{0}^{t_1} W(t)\xi(t,t')A_*(t')dt'dt = W_2 A_{*1}\int_{t_1}^{T}\int_{0}^{t_1} e^{-A_2(t-t_1)}e^{-A_1(t_1-t')}dt'dt = \frac{W_2 A_{*1}}{A_1 A_2}\left(1-e^{-A_1 t_1}\right)\left(1-e^{-A_2(T-t_1)}\right) \tag{19}$$

$$\int_{t_1}^{T}\int_{t-T}^{0} W(t)\xi(t,t')A_*(t')dt'dt = W_2A_{*2}\int_{t_1}^{T}\int_{t-T}^{0} e^{-A_2(t-t_1)}e^{-A_1t_1}e^{+A_2t'}dt'dt$$
$$= \frac{W_2A_{*2}}{A_2A_2}e^{-A_1t_1}\left(1-e^{-A_2(T-t_1)} - A_2\left(T-t_1\right)e^{-A_2(T-t_1)}\right) \quad (20)$$

The analytical equation for equivalent dose is the given by:

$$T\left(1-e^{-A_2(T-t_1)}e^{-A_1t_1}\right) =$$
$$\frac{W_1A_{*1}}{A_1^2}\left(A_1t_1-1+e^{-A_1t_1}\right)+\frac{W_1A_{*2}}{A_1A_2}\left(1-e^{-A_1t_1}\right)\left(1-e^{-A_2(T-t_1)}\right)+\frac{W_1A_{*1}}{A_1^2}e^{-A_2(T-t_1)}\left(1-e^{-A_1t_1}-A_1t_1e^{-A_1t_1}\right)$$
$$+\frac{W_2A_{*2}}{A_2A_2}\left(A_2\left(T-t_1\right)-1+e^{-A_2(T-t_1)}\right)+\frac{W_2A_{*1}}{A_1A_2}\left(1-e^{-A_1t_1}\right)\left(1-e^{-A_2(T-t_1)}\right) \quad (21)$$
$$+\frac{W_2A_{*2}}{A_2A_2}e^{-A_1t_1}\left(1-e^{-A_2(T-t_1)}-A_2\left(T-t_1\right)e^{-A_2(T-t_1)}\right)$$

This equation can be rewritten by substitution of the following variables:

Weighting: $W_1t_1 + W_2(T-t_1) = T$

Reference dilution: $\mathrm{N}_* = A_{*1}t_1 + A_{*2}(T-t_1)$

Actual dilution: $\mathrm{N} = A_1t_1 + A_2(T-t_1)$

Fraction of time in the step: $f \equiv t_1/T$

Fractional dilution rate in the step: $r \equiv A_1T / \mathrm{N}$

Fractional source strength in the step: $r_* \equiv A_{*1}T / \mathrm{N}_*$

Substitution of the variables and grouping of term gives the following:

$$0 = \frac{2e^{-\frac{N}{2}}\begin{pmatrix} \frac{1}{2}\left(-e^{-\frac{N}{2}}+e^{\frac{N}{2}}\right)Nr(-1+fr)\left(r\left(N-Nfr+N_*(-1+fr_*)\right)+N_*f\left(r-r_*\right)W1\right) \\ -\frac{1}{2}\left(-e^{-\frac{Nfr}{2}}+e^{\frac{Nfr}{2}}\right)\left(e^{\frac{N(1-fr)}{2}}-e^{-\frac{N(1-fr)}{2}}\right)N_*\left(r-r_*\right)\left(r-W1\right) \end{pmatrix}T}{N^2r^2\left(-1+fr\right)^2} \tag{22}$$

Equation 22 can equivalently be expressed by the hyperbolic sine function: $\sinh(x)=\frac{1}{2}\left(e^x-e^{-x}\right)$ as:

$$0 = \frac{2e^{-\frac{N}{2}}\begin{pmatrix} Nr(-1+fr)\left(r\left(N-Nfr+N_*(-1+fr_*)\right)+N_*f\left(r-r_*\right)W1\right)Sinh\left[\frac{N}{2}\right] \\ -2N_*\left(r-r_*\right)\left(r-W1\right)Sinh\left[\frac{Nfr}{2}\right]Sinh\left[\frac{N(1-fr)}{2}\right] \end{pmatrix}T}{N^2r^2\left(-1+fr\right)^2} \tag{23}$$

Solving for N_* gives us the following:

$$N_* = \frac{N^2r^2\left(-1+fr\right)^2 Sinh\left[\frac{N}{2}\right]}{\begin{pmatrix} Nr(-1+fr)\left(r\left(-1+fr\right)-f\left(r-r_*\right)\left(r-W1\right)\right)Sinh\left[\frac{N}{2}\right] \\ -2\left(r-r_*\right)\left(r-W1\right)Sinh\left[\frac{Nfr}{2}\right]Sinh\left[\frac{N(1-fr)}{2}\right] \end{pmatrix}} \tag{24}$$

Because $\frac{\sinh(A+B)}{\sinh(A)\sinh(B)}=\coth(A)+\coth(B)$ equation 24 can be expressed as:

$$N_* = \frac{N^2r^2\left(-1+fr\right)^2}{Nr(-1+fr)\left(\left(r\left(-1+fr\right)-f\left(r-r_*\right)\left(r-W1\right)\right)\right)-\frac{2\left(r-r_*\right)\left(r-W1\right)}{\left[Coth\left[\frac{Nfr}{2}\right]+Coth\left[\frac{N(1-fr)}{2}\right]\right]}} \tag{25}$$

Introducing the two variables: $\theta = f^2\left(r - r_*\right)\left(r - W1\right)$ and $Z = fr$ and rearranging equation 25 we can simplify the equation to:

$$N_* = \frac{N}{1 + \frac{\theta}{(1-Z)Z} - \frac{2}{N}\frac{\theta}{(1-Z)^2 Z^2} \Big/ \left[Coth\left[\frac{NZ}{2}\right] + Coth\left[\frac{N(1-Z)}{2}\right]\right]} \qquad (26)$$