Title
Holomorphic extension of solutions to homogeneous analytic partial dierential equations

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Publication Date
2009

Peer reviewed|Thesis/dissertation

UNIVERSITY OF CALIFORNIA, SAN DIEGO

Holomorphic Extension of Solutions to Homogeneous Analytic Partial Differential Equations

A dissertation submitted in partial satisfaction of the
requirements for the degree
Doctor of Philosophy

in

Mathematics

by

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2009

The dissertation of Jonathan J. Armel is approved, and it is acceptable in quality and form for publication on microfilm:

Chair

University of California, San Diego

2009

DEDICATION

For Ananda.

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ACKNOWLEDGEMENTS

I would like to thank my advisor, Peter Ebenfelt, not only for teaching how to do mathematical research, but for teaching me how to be a good mentor and advisor. And of course, for his infinite patience with me. I could not have done this without him.

I would also like to thank all of the other wonderful teachers I have had over the years. Salah Baouendi for giving me my real introduction to Complex Analysis, and my first taste of Several Complex Variables. Jim Walsh, Robert Young, Michael Henle, Susan Colley, and Oliver Schirokauer for showing me the beauty of mathematics, and helping me find my path.

I thank all of my friends here in the math department for helping me maintain a semblance of a real life, and providing me with indispensable recreation and relaxation. The UCSD Math Department Swim Team and UCSD Math Department Swim Team Alumni, especially co-captain Oded Yacobi, and my Budo teacher Kevin McGown. Manda Riehl and Jamie Browning for their companionship, conversation, and hot tub.

Thanks most of all to Ananda. I cannot imagine the last five years without you.

VITA

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ABSTRACT OF THE DISSERTATION

Holomorphic Extension of Solutions to Homogeneous Analytic Partial Differential Equations

by

Jonathan J. Armel
Doctor of Philosophy in Mathematics

University of California San Diego, 2009

Professor Peter Ebenfelt, Chair

In this dissertation we derive sufficient conditions on a pseudoconvex domain Ω and a linear, analytic differential operator P for the existence of solutions to $Pu = 0$ which are holomorphic in Ω near $p \in \partial\Omega$, but cannot be prolonged across p.

We first define the notion of strong P-convexity of Ω, given by the existence of a supporting everywhere characteristic analytic hypersurface, and show how under the assumption of strong P-convexity a theorem of Tsuno [Tsu74] can be used to construct our desired solutions.

The rest of the work consists of finding necessary conditions for strong P-convexity. In Chapter 2 we consider the case in which Ω is strictly pseudoconvex at p and show that strong P-convexity follows from a positivity condition given by Kawai and Takei [KT90].

In Chapter 3 we discuss bicharacteristic convexity, and show that strong P-convexity follows from the combination of bicharacteristic convexity and the convexifiability of a local projection along bicharacteristics.

In the final two chapters we extend the results of Chapter 2 to the case in which Ω is weakly pseudoconvex. Under some simplifying assumptions, we find new invariants of the pair (P, M), and use them to give sufficient conditions for strong P-convexity in terms of a defining equation for Ω and the coefficients of P.

Chapter 1

Preliminaries

1.1 Introduction

Broadly construed, this dissertation is an extension of the classical Levi problem: determining conditions on the boundary of a domain Ω in $\mathbb{C}^n$ which imply that it is a domain of holomorphy. Domains of holomorphy are interesting for many reasons, and in particular they are natural domains of definition for holomorphic functions. At a boundary point p of a domain which is not a domain of holomorphy, any function holomorphic on Ω near p can be extended to a holomorphic function in a full neighborhood of p.

The situation for us is similar. In addition to a domain Ω, we are also given a linear partial differential operator P with coefficients holomorphic near p, and we wish to know not when *some* holomorphic function cannot be extended across p, but when there is such a function that also solves a homogeneous differential equation $Pu = 0$.

The prolongation of holomorphic solutions to PDE has been studied extensively in many contexts. The noncharacteristic case is well-understood. On an analytic hypersurface, if a point p is noncharacteristic with respect to P, then the Cauchy-Kovalevsky theorem tells us that for any f holomorphic near p, there is a unique holomorphic solution to $Pu = f$ near p when initial data is prescribed on the hypersurface.

On a real hypersurface M, such as $\partial\Omega$ for a domain Ω,we have a similar result. If the supporting complex hyperplane at p is noncharacteristic (such points are called Zerner characteristic), then Zerner [Zer71] has shown that for any f holomorphic near p, any solution to $Pu = f$ which is holomorphic in Ω near p may be prolonged holomorphically across p. Bony and Schapira [BS71] have proven similar results for systems of equations.

Prolongation across characteristic points is more delicate. In 1973, Sato, Kawai, and Kashiwara published [SKK73], developed the theory of microlocal analysis and hyperfunctions, and used them to study the semi-global solvability of PDE near characteristic points. We say that an operator P is semi-globally solvable at p if $P : \mathcal{O}_p(\Omega)/\mathcal{O}_p \to \mathcal{O}_p(\Omega)/\mathcal{O}_p$ is surjective. Combined with a result of Kashiwara and Kawai [KK72] they give a positivity condition (condition (Pos), as we define in Chapter 2) which guarantees the semi-global solvability of P at p if Ω is strictly pseudoconvex.

Building on these results, Trépreau [Tré84] showed that semi-global solvability is equivalent to condition (Ψ), as defined by Nirenburg and Treves [NT70a]. Condition (Ψ) is a positivity condition on the bicharacteristic strips emanating from characteristic points on $\partial\Omega$.

Bicharacteristic strips are the most important geometric object associated to a differential operator, and thus it is natural that they should be important when examining solutions. In [Ler57], Leray considers the Cauchy problem with initial data on an analytic hypersurface Γ, and shows that the singularities of solutions to $Pu = f$ lie on an everywhere characteristic variety tangent to Γ and foliated by bicharacteristic curves.

Later, in 1990 Kawai and Takei [KT90] used bicharacteristics to give further geometric insight into the results of [SKK73] and [Tré84]. Specifically, they showed that the positivity condition (Pos) is equivalent to assuming that $\partial\Omega$ is bicharacteristically convex at p, and that a local projection along the bicharacteristics emanating from characteristic points of $\partial\Omega$ is strictly pseudoconvex.

The above results focus on proving existence of solutions to $Pu = f$ for any f, and are thus concerned with surjectivity of P. In this work, we will be looking for solutions to $Pu = 0$ which are in a certain sense nontrivial, and thus are really proving results about a lack of injectivity. In this area, less work has been done. There is a theorem by Tsuno [Tsu74] which gives sufficient conditions in the case where Ω is strictly pseudoconvex, but they are far from necessary, and in chapter 2 we relax the assumptions considerably.

All of the work that has been done thus far has also focused on case where Ω is strictly pseudoconvex at p. They essentially begin by working with a simple operator such as $\frac{\partial}{\partial z_1}$, then use a microlocal transformation provided by [SKK73] to transform it to an arbitrary operator. If Ω is only weakly pseudoconvex, however, the symplectic form used to generate the transformation is degenerate, and the theory breaks down.

As we describe in detail below, our work relies on a theorem of Tsuno [Tsu74] which allows us to bypass the microlocal transformation theory. Tsuno gives a way to construct a solution to $Pu = 0$ which has singularities along a given everywhere characteristic hypersurface, and so if we can create such a hypersurface which does not intersect $\overline{\Omega}$ (except at p), we get the desired solution. This allows us somewhat direct proofs of many of our results in the strictly pseudoconvex case, and gives a way to extend to the weakly pseudoconvex case as well.

1.2 Hypersurfaces in $\mathbb{C}^n$

The two primary objects of study in this dissertation are a domain Ω in $\mathbb{C}^n$ (or sometimes $\mathbb{C}^{n+1}$), with a point $p \in \partial\Omega$, and a linear differential operator

$$P(z, \partial) = \sum_{|\alpha| \leq m} a_\alpha(z) \left(\frac{\partial}{\partial z}\right)^\alpha \tag{1.1}$$

with coefficients $a_\alpha(z)$ holomorphic in some neighborhood of p.

Our goal will be to determine conditions on Ω and P that guarantee the existence of a solution u to $Pu = 0$ which is holomorphic in Ω near p, but cannot be prolonged across p. That is, for some neighborhood U of p, u is holomorphic in $U \cap \Omega$, but there is no function $H(z)$ such that H is holomorphic in some neighborhood W of p and $h(z) = H(z)$ for $z \in \Omega \cap W$.

As this question is local in nature, we will often be somewhat loose with the neighborhoods of p, and will often just say "near p" to mean in some small enough neighborhood of p, or in $\Omega \cap U$ for some small enough neighborhood U.

Of course, with an arbitrary domain and operator, finding such a u will not be possible, and thus we must restrict the class of object that we will consider. We first look at necessary conditions on the domain Ω, and will require a few definitions.

Definition 1.2.1. A subset M of $\mathbb{C}^n$ is a *real-analytic (or $\mathcal{C}^k$) hypersurface* if near each point p in M, there is a real-valued, real-analytic (resp. $\mathcal{C}^k$) function $\rho(z, \bar{z})$ such that for some neighborhood U of p, $M = \{z \in U \mid \rho(z, \bar{z}) = 0\}$, and $\mathrm{d}\rho(z, \bar{z}) \neq 0$ in U. An *analytic hypersurface* is a subset S of $\mathbb{C}^n$ such that for each $p \in S$ there is a holomorphic function $f(z)$ and a neighborhood U of p with $S \cap U = \{z \in U \mid f(z) = 0\}$ and $\mathrm{d}f(z) \neq 0$ for all $z \in U$. The functions ρ and f are called *defining functions* for M and S, respectively.

We denote by M the boundary of Ω, assume that M is a real-analytic hypersurface, and take $\rho(z, \bar{z})$ to be a real-analytic defining function for M near p. We will further assume that $\Omega = \{z \mid \rho(z, \bar{z}) < 0\}$ near p. The most fundamental assumption we must have for Ω is pseudoconvexity.

Definition 1.2.2. A domain $\Omega \subset \mathbb{C}^n$ given by $\{z \mid \rho(z, \bar{z}) < 0\}$ with ρ real-analytic near a point $p \in \partial\Omega$ is *pseudoconvex at p* if for every $q \in \partial\Omega$ near p,

$$\sum_{j,k=1}^{n} \frac{\partial^2 \rho}{\partial z_j \bar{z}_k}(q, \bar{q}) w_j \bar{w}_k \geq 0 \tag{1.2}$$

for all $w \in \mathbb{C}^n$ such that $\sum_{j=1}^{n} \frac{\partial \rho}{\partial z_j}(q, \bar{q}) w_j = 0$.
Ω is *strictly pseudoconvex at p* if the inequality (1.2) can be taken to be strict, and Ω is *weakly pseudoconvex at p* if it is pseudoconvex but not strictly pseudoconvex at p.

Definition 1.2.3. The quadratic form in equation (1.2) is the *Levi form* of $\partial\Omega$ at q, and we shall refer to a point $w \in \mathbb{C}^n$ such that $\sum_{j=1}^{n} \frac{\partial \rho}{\partial z_j}(q, \bar{q}) w_j = 0$ as a *holomorphic tangent vector* to $\partial\Omega$ at q. We denote the collection of such vectors as $T_q^{1,0}\partial\Omega$.

Thus one may also define pseudoconvexity by saying that the Levi form is positive semidefinite on $T_q^{1,0}$ for all q near p.

If $\partial\Omega$ is weakly pseudoconvex at p, then the Levi form at p must have a non-trivial null space. This null space, and its relation to P, will be important when we consider weakly pseudoconvex domains, and it will be denoted by $N(\mathcal{L}(p))$, or if the point p is clear, simply $N(\mathcal{L})$.

The importance of pseudoconvexity is furnished by the following well-known theorem.

Theorem 1.2.4. *If a domain $\Omega \subset \mathbb{C}^n$ is pseudoconvex at $p \in \partial\Omega$, and f is a function holomorphic in Ω near p, then there is a neighborhood U of p and a function F which is holomorphic in U such that $F(z) = f(z)$ for all $z \in \Omega \cap U$.*

In fact pseudoconvexity at each point in $\partial\Omega$ is equivalent to Ω being a domain of holomorphy, but we shall not really need this, and will not discuss it further here. For our purposes, we see that if we are to have any chance of finding a solution to $Pu = 0$ which does not extend across p, we must assume Ω is pseudoconvex at p. If we do not, then every holomorphic function extends across p, whether it is a solution or not.

We shall often like to work in coordinates that make our calculations simple, and allow us to focus on the important quantities in question. Our favorite coordinates for the purposes of this dissertation are so-called *normal coordinates*. A proof of the existence of such coordinates can be found in Baouendi, Ebenfelt, and Rothschild [BER99].

Definition 1.2.5. Let $M \subset \mathbb{C}^{n+1}$ be a real-analytic hypersurface and let $p \in M$. Coordinates $(z, w) \in \mathbb{C}^n \times \mathbb{C}$ are called *normal coordinates* near p if they form a holomorphic system of coordinates near p with $p = (0, 0)$, and there is a real-valued real-analytic function $\phi(z, \bar{z}, s)$ defined in a neighborhood of 0 in $\mathbb{R}^{2n+1}$ such that M is given by

$$M = \{(z, w) \,|\, 2\mathrm{Re}\ w = \phi(z, \bar{z}, \mathrm{Im}\ w)\}$$

and $\phi(z, 0, s) \equiv \phi(0, z, s) \equiv 0$.

1.3 Differential Equations

Even when Ω is pseudoconvex, we do not want to consider any arbitrary P. In order to discuss which operators it will be interesting to consider, we need some notation and a few definitions.

For $P(z,\partial)$ as given in (1.1), the principle symbol of P will be denoted by

$$P_m(z,\zeta) = \sum_{|\alpha|=m} a_\alpha(z)\zeta^\alpha$$

where $\zeta \in \mathbb{C}^n$. $P_m^{(j)}(z,\zeta)$ and $P_{m,j}(z,\zeta)$ will denote $\frac{\partial P_m}{\partial \zeta_j}(z,\zeta)$ and $\frac{\partial P_m}{\partial z_j}(z,\zeta)$, respectively.

For a function $f(x,y)$, with $(x,y) \in \mathbb{C}^N \times \mathbb{C}^d$, we will denote the holomorphic gradient of f with respect to the x variables by

$$\partial_x f(x,y) = \left(\frac{\partial f}{\partial x_1}(x,y), \ldots, \frac{\partial f}{\partial x_N}(x,y)\right)$$

and $\partial f(x,y)$ will denote the holomorphic gradient of f with respect to all of its variables. $\partial_x f(x_0,y_0)$ will denote $\partial_x f(x,y)$ evaluated at (x_0,y_0). Thus for a real-analytic function $\rho(z,\bar{z})$, $\partial\rho(z,\bar{z}) = \left(\frac{\partial\rho}{\partial z_1}, \ldots, \frac{\partial\rho}{\partial z_n}\right)$.

The most important relationship between a hypersurface and a differential operator that of being characteristic.

Definition 1.3.1. An analytic hypersurface S is *characteristic at a point* $p \in S$ *with respect to* P if given any defining function $f(z)$ for S, we have $P_m(p,\partial f(p)) = 0$.

Definition 1.3.2. Let $M \subset \mathbb{C}^n$ be a real, $\mathcal{C}^1$ hypersurface with defining function ρ. Then $p \in M$ is a characteristic point of M with respect to P if $P_m(p,\partial\rho(p,\bar{p})) = 0$.

As $P_m(z,\zeta)$ is a homogeneous polynomial in ζ, it is clear that this definition is independent of the choice of defining function ρ. We note that in some literature such points are called Zerner characteristic, but if there is no fear of confusion, we shall simply call them characteristic. We also often drop the "of M" and "with respect to P", as there will usually be only one hypersurface and one operator in question. If every point on a hypersurface, either analytic or real-analytic, is characteristic we say that the hypersurface is *everywhere characteristic.* When the meaning is clear, we may also say that p is a characteristic point of S or M.

The importance of characteristic points on an analytic hypersurface comes from the Cauchy-Kovalevsky theorem, which is perhaps the only general theorem in all of PDE. It essentially states that away from characteristic points, solving analytic differential equations is not too difficult. For a proof of this particular version, see e.g. [Fol95].

Theorem 1.3.3 (The Cauchy-Kovalevsky Theorem)**.** *Let G and $\phi_0, \ldots, \phi_{m-1}$ be holomorphic functions near the origin. Then there is a unique holomorphic solution u to the Cauchy problem*

$$\left\{ \begin{array}{rcll} \partial_{z_n}^m u &=& G\left(z, (\partial^\alpha u)_{|\alpha| \leq m, \alpha_n < m}\right) & \\ \partial_{z_n}^j u(z_1, \ldots, z_{n-1}, 0) &=& \phi_j(z_1, \ldots, z_{n-1}), & \textit{for } j < m \end{array} \right. \tag{1.3}$$

in some neighborhood of the origin.

If we wish to solve a differential equation with initial data on an analytic hypersurface S near a point p, to apply the Cauchy-Kovalevsky theorem we simply change coordinates to turn S into the hyperplane $\{z_n = 0\}$. However, we must then also be able to isolate $\partial_{z_n}^m u$, and this is possible exactly when S is not characteristic at p. Thus if S is not characteristic at p, the Cauchy-Kovalevsky theorem guarantees a holomorphic solution in a neighborhood of p.

For real-analytic hypersurfaces, we have a theorem due to M. Zerner, which gives information about solutions to differential equations near non-characteristic points.

Theorem 1.3.4 (Zerner's Theorem)**.** *Let $\Omega \subset \mathbb{C}^n$ be a domain with $\mathcal{C}^1$ boundary M. Let $p \in M$ and let U be a neighborhood of p. If p is not characteristic with respect to P, then every function $u(z)$ which is holomorphic in $\Omega \cap U$ and satisfies $Pu = 0$ in U can be prolonged across p.*

Thus we see that if $p \in M$ is not characteristic, there is no solution u to $Pu = 0$ which is holomorphic in Ω but not in a neighborhood of p. Thus for the remainder of the paper, we restrict ourselves to solutions near characteristic points. Further, we will assume that p is simply characteristic with respect to P.

Definition 1.3.5. M is *simply characteristic with respect to* P at p if M is characteristic at p, and

$$\partial_\zeta P_m(p, \partial\rho(p, \bar{p})) \neq (0, \dots, 0).$$

1.4 Strong P-Convexity

The primary tool we will use to construct our desired solution u is given by the following theorem, proven by Tsuno in [Tsu74].

Theorem 1.4.1 (Tsuno). *Let $P(z, \partial) = \sum_{|\alpha| \le m} a_\alpha(z)(\frac{\partial}{\partial z})^\alpha$ where a_α are holomorphic in a neighborhood U of 0 in $\mathbb{C}^n$, and let $P_m(z, \zeta) = \sum_{|\alpha|=m} a_\alpha(z)\zeta^\alpha$ denote it's symbol. Let S be an analytic hypersurface through 0 with defining function $f(z)$ in U. Assume that 0 is a simply characteristic point of S with respect to P. Further, assume that*

$$P_m(z, \partial f(z)) = 0 \text{ for all } z \in U. \tag{1.4}$$

Then there exists solution $u(z)$ to the equation $Pu = 0$ in a neighborhood of zero such that

$$u(z) = \frac{F(z)}{f(z)} + G(z) \log f(z) + H(z)$$

where $F(z)$, $G(z)$, and $H(z)$ are holomorphic and $u(z)$ is not holomorphic at 0.

The assumption in equation (1.4) can be weakened considerably, however. In particular, we only need $P_m(z, \partial f(z))$ to vanish on S. That is, we require S to be everywhere characteristic. In particular, we have the following result.

Theorem 1.4.2. *Let $P(z, \partial)$, $P_m(z, \zeta)$, and U be as in Theorem 1.4.1, and let S be a complex analytic hypersurface through 0 which is everywhere simply characteristic with respect to P. Then the conclusion of Theorem 1.4.1 still holds.*

Proof. Assume that S is everywhere characteristic with respect to P. We would like to find a new defining function ψ for S such that $P_m(z, \partial\psi(z)) = 0$ for all $z \in U$. To do this, we let Γ be any analytic hypersurface through zero which is not characteristic with respect to P at 0, and solve the following Cauchy problem for a holomorphic function $c(z)$

$$\begin{cases} P_m(z, \partial(a(z)f(z)) = 0 \\ \qquad\qquad\qquad c(z) = 1 \quad \text{for all } z \text{ in } \Gamma \end{cases} \tag{1.5}$$

Then the function $\psi(z) = c(z)f(z)$ is a defining function for S, and we still have $(P_m^{(1)}(z, \partial\psi(z)), \ldots, P_m^{(n)}(z, \partial\psi(z))) = (P_m^{(1)}(z, \partial f(z)), \ldots, P_m^{(n)}(z, \partial f(z))) \neq (0, \ldots, 0)$. Then by Theorem 1.4.1, there is a solution $u(z)$ to $Pu = 0$ such that

$$u(z) = \frac{F(z)/c(z)}{f(z)} + G(z)\log(c(z)f(z)) + (H(z) + \log c(z))$$

and since $c(0) \neq 0$, u satisfies the required conditions. $\square$

Given this theorem, the following definition is natural.

Definition 1.4.3. Ω is *strongly P-convex at a point $p \in \partial\Omega$* if there is an everywhere characteristic, analytic hypersurface S such that $S \cap \overline{\Omega} = \{p\}$.

We then have a corollary that tells us the importance of strong P-convexity.

Corollary 1.4.4. *Let $\Omega \subset \mathbb{C}^n$ be a domain, $p \in \partial\Omega$, and let $P(z, \partial)$ be a linear differential operator with coefficients holomorphic at p such that $\partial\Omega$ is simply characteristic at p. If Ω is strongly P-convex at p, then there is a solution u to the equation $P(z, \partial)u(z) = 0$ which is holomorphic in Ω near p but does not extend holomorphically across p.*

Proof. Since Ω is strongly P-convex, we have an everywhere characteristic hypersurface $S = \{z \mid f(z) = 0\}$ with $\bar{\Omega} \cap S = \{0\}$. By Theorem 1.4.2, there is a neighborhood W of p and a germ of a solution u at some point $q \in W$ of the form $u(z) = F(z)/f(z) + G(z)\log f(z) + H(z)$. We may assume W is small enough that $W \cap \Omega$ is simply connected, and note that u can be analytically continued along any path in $W \cap \Omega$, since $W \cap \Omega \cap S = \emptyset$. By the monodromy theorem, then, there is function $U(z)$, holomorphic in $W \cap \Omega$, which coincides with u at q. $\square$

Thus in order to find solutions to $Pu = 0$ which do not extend holomorphically across p, it suffices to produce an everywhere characteristic analytic hypersurface through p which does not enter into Ω.

1.5 P as an Injective Map

The problem of finding a solution to $Pu = 0$ which cannot be holomorphically extended across p can be considered as one regarding uniqueness. To discuss this further, we again need some further notation.

We use $\mathcal{O}_p$ and $\mathcal{O}_p(\Omega)$ to denote the germs at p of functions holomorphic either in a full neighborhood of p or in Ω near p, respectively. Specifically, $\mathcal{O}_p$ is the set of functions u which are holomorphic in a neighborhood of p, with two such functions identified if they agree on some such neighborhood. $\mathcal{O}_p(\Omega)$ is the set of functions u for which there is a neighborhood U such that u is holomorphic in $\Omega \cap U$, again identifying two functions that agree in $\Omega \cap W$ for some neighborhood W.

If f is holomorphic in a neighborhood of p, then there is a solution $u \in \mathcal{O}_p(\Omega)$ to $Pu = f$ exactly when there is a solution to $Pu = 0$. We may find, by the Cauchy-Kovalevsky theorem, a function v holomorphic near p solving $Pv = f$. Then for $u \in \mathcal{O}_p(\Omega)$, $Pu = 0$ if and only if $P(u + v) = f$. Thus in this language, our problem becomes to determine conditions under which the map

$$P : \mathcal{O}_p(\Omega)/\mathcal{O}_p \to \mathcal{O}_p(\Omega)/\mathcal{O}_p$$

is injective.

1.6 Further Notation

Finally, we set a bit more notation that will clarify the discussions in the remaining chapters.

We will always use $\|\cdot\|$ to denote the Euclidean norm, so for $z \in \mathbb{C}^n$ and $z' = (z_1, \ldots, z_{n-1})$, $\|z\|^2 = \sum_1^n |z_j|^2$, and $\|z'\|^2 = \sum_1^{n-1} |z_j|^2$.

For a power series in N variables, and $k \in \mathbb{N}$, we will use the notation $O(k)$ to denote the terms in the series which have at least degree k. If f is a function on $\mathbb{C}^N$, we will say that another function $g(z)$ is $o(f(z))$ if $\lim_{z \to 0} g(z)/f(z) = 0$. We often abuse the notation slightly and use $o(f(z))$ to denote the function g itself. Usually this will appear as $o(\|z'\|^2)$, with $z' = (z_1, \ldots, z_{n-1})$. In a power series in

z and $\bar{z}$, then, it would denote the terms in the series that are $O(3)$, and contain either $z_j z_k$, $z_j \bar{z}_k$, or $\bar{z}_j \bar{z}_k$ for $j \neq n$ and $k \neq n$.

For a multiindex $\alpha = (\alpha_1, \ldots, \alpha_N)$ and a function $f(x_1, \ldots, x_N)$, we use subscripts to denote partial derivatives:

$$f_{x^\alpha} = \frac{\partial^{|\alpha|} f}{\partial x_1^{\alpha_1} \partial x_2^{\alpha_2} \cdots \partial x_N^{\alpha_N}}.$$

With this, we are ready to address the problem at hand. We first consider the case in which Ω is strictly pseudoconvex at p, where much work has already been done. We then move on to the case when Ω is weakly pseudoconvex, in which considerably less is known.

Chapter 2

The Strictly Pseudoconvex Case

In this chapter we will assume that Ω is strictly pseudoconvex at $p \in \partial\Omega$, and that $\partial\Omega$ is characteristic at p with respect to an operator P. The existence of solutions to $Pu = f$ has been studied extensively in this case, primarily using the tools of microlocal analysis and hyperfunctions given by Sato, Kawai, and Kashiwara in [SKK73]. Using these, Kashiwara and Kawai derive a Hermitian form Q, given below, such that if Q is positive definite on $T_p^{1,0}M \times \mathbb{C}$ then $P : \mathcal{O}_p(\Omega)/\mathcal{O}_p \to \mathcal{O}_p(\Omega)/\mathcal{O}_p$ is surjective.

Building on these results, Trépreau [Tré84] has shown that the surjectivity of P is equivalent to condition (Ψ), as given by Nirenburg and Treves [NT70a]. (Ψ) is essentially a positivity condition concerning the bicharacteristic strips of P emanating from characteristic points of M. Kawai and Takei [KT90] later give another equivalent condition, concerning local projections of bicharacteristics. These will be discussed in detail and expanded upon in the next chapter.

Our results concern the injectivity of $P : \mathcal{O}_p(\Omega)/\mathcal{O}_p \to \mathcal{O}_p(\Omega)/\mathcal{O}_p$. Rather than using microlocal analysis, we will find sufficient conditions under which we can construct an everywhere characteristic hypersurface that does not intersect $\overline{\Omega}$, use Theorem (1.4.2) to prove injectivity. This approach has the advantage that while microlocal analysis breaks down if $\partial\Omega$ is only weakly pseudoconvex, our methods extend rather well, although the calculations are somewhat more involved. We remark that in the same paper in which he proves Theorem 1.4.1, Tsuno finds conditions which imply the existence of a solution to $Pu = 0$ which does not extend across p. As such, the results of this chapter may be considered as a strengthening of his theorem.

2.1 Condition (Pos)

Following Kawai and Takei [KT90], for each $z \in \mathbb{C}^{n+1}$ near p we define the Hermitian form

$$Q_z(\zeta) = \sum_{0 \le j,k \le n+1} q_{j,k}(z,\bar{z})\zeta_j\bar{\zeta}_k \tag{2.1}$$

for $\zeta \in \mathbb{C}^{n+2}$, where the coefficients $q_{j,k}$ are given by

$$\begin{aligned}
q_{j,k}(z,\bar{z}) &= \rho_{z_j\bar{z}_k}(z,\bar{z}) && 0 \le j,k \le n \\
q_{j,n+1}(z,\bar{z}) &= P_{m,j}(z,\partial\rho(z,\bar{z})) + \sum_{k=0}^{n} P_m^{(k)}(z,\partial\rho(z,\bar{z}))\rho_{z_j z_k}(z,\bar{z}) && 0 \le j \le n \\
q_{n+1,j}(z,\bar{z}) &= \overline{q_{j,n+1}(z,\bar{z})} && 0 \le j \le n \\
q_{n+1,n+1}(z,\bar{z}) &= \sum_{0 \le j,k \le n} P_m^{(j)}(z,\partial\rho(z,\bar{z}))\overline{P_m^{(k)}(z,\partial\rho(z,\bar{z}))}\rho_{z_j\bar{z}_k}(z,\bar{z})
\end{aligned}$$

Definition 2.1.1. We will say that *condition (Pos) holds at p* if $Q_p(\zeta)$ is positive definite on $T_p^{1,0}M \times \mathbb{C}$. That is, if

$$\sum_{j,k=0}^{n} q_{j,k}(p,\bar{p})\zeta_j\bar{\zeta}_k > 0 \qquad \text{when} \qquad \zeta \ne 0,\ \sum_{j=1}^{n} \rho_{z_j}(p,\bar{p})\zeta_j = 0.$$

The existence of solutions to $Pu = f$ in this situation is well understood. In particular, we have the following Theorem found in [Hör94], combining results from [SKK73] and [KK72].

Theorem 2.1.2. *Assume that Ω is strictly pseudoconvex and is simply characteristic at p with respect to P. Then the map*

$$P : \mathcal{O}_p(\Omega)/\mathcal{O}_p \to \mathcal{O}_p(\Omega)/\mathcal{O}_p$$

is surjective if (Pos) holds at p. if P is surjective, then Q_p is positive semidefinite on $T_p^{1,0}M \times \mathbb{C}$.

2.2 (Pos) Implies Strong P-Convexity

We now prove the main theorem of this chapter, showing that if (Pos) holds, Ω is strongly P-convex. Thus we have that at a simply characteristic point, unless we are on the borderline case where Q_p is semidefinite but not positive definite, then $Pu = f$ has a solution for every $f \in \mathcal{O}_p(\Omega)/\mathcal{O}_p$, but u can never be unique, even in the quotient space $\mathcal{O}_p(\Omega)/\mathcal{O}_p$.

Theorem 2.2.1. *Let $\Omega \subset \mathbb{C}^{n+1}$ be a strictly pseudoconvex domain and let $p \in \partial\Omega$. Let $P(z, \partial)$ be a linear, differential operator with coefficients holomorphic near p, and assume that p is simply characteristic with respect to P. If condition (Pos) holds at p, then Ω is strongly P-convex at p.*

Proof. As shown in [Hör94], Corollary 7.4.9, condition (Pos) is independent of the choice of coordinates, we may assume $p = 0$ and choose normal coordinates $(z_0, z') \in \mathbb{C} \times \mathbb{C}^n$ for $M = \partial\Omega$. That is, $\Omega = \{(z_0, \ldots, z_n) \in \mathbb{C}^{n+1} |\ \rho(z, \bar{z}) < 0\}$, where

$$\rho(z, \bar{z}) = -2\mathrm{Re}\ z_0 + \sum_1^n |z_j|^2 + O(3). \tag{2.2}$$

We also write the symbol of P (for $(\tau, \chi) \in \mathbb{C} \times \mathbb{C}^n$) as

$$P_m(z, \tau, \chi) = \tilde{a}(z)\tau^m + (\sum_1^n \tilde{b}_j(z)\chi_j)\tau^{m-1} + O(\|\chi\|^2).$$

Then it is easy to see that 0 is a characteristic point of M with respect to P exactly when $\tilde{a}(0) = 0$, and M is simply characteristic when $\tilde{b}_j(0) \neq 0$ for some j. Since the vector $(\tilde{b}_1(0), \ldots, \tilde{b}_n(0))$ is nonzero, by a rotation in the z' variables we may further assume that $\tilde{b}_n(0) \neq 0$ and $\tilde{b}_j(0) = 0$ for $j = 1, \ldots, n-1$.

Now, if $c(z)$ is holomorphic and nonzero near 0, and the equation $cPu = 0$ has a solution u which is holomorphic in Ω but not at 0, then u also solves $Pu = 0$. Thus we may divide P by $\tilde{b}_n(z)$, and without loss of generality we may assume that

$$P_m = a(z)\tau^m + \chi_n + (\sum_1^{n-1} b_j(z)\chi_j)\tau^{m-1} + O(\|\chi\|^2), \tag{2.3}$$

where $a(0) = b_j(0) = 0$ for $j = 1, \ldots, n-1$. We now compute condition (Pos) in these coordinates. We have (Pos) satisfied if $Q(\zeta, \bar{\zeta}) = \sum_{j,k=0}^{n+1} q_{j,k}(0)\zeta_j\bar{\zeta}_k > 0$ for all nonzero $\zeta = (\zeta_0, \ldots, \zeta_{n+1})$ such that $\zeta_0 = 0$, where:

$$
\begin{aligned}
P_{m,j}(0, \partial\rho(0)) &= (-1)^m a_{z_j}(0) && j = 1, \ldots, n \\
P_m^{(k)}(0, \partial\rho(0)) &= \delta_k^n && k = 0, \ldots, n \\
q_{j,k}(0) &= \delta_j^k && j = 1, \ldots, n \\
q_{j,0}(0) &= q_{0,j}(0) = 0 && j = 0, \ldots, n \\
q_{j,n+1}(0) &= \overline{q_{n+1,j}(0)} = (-1)^m a_{z_j}(0) && j = 0, \ldots, n \\
q_{n+1,n+1}(0) &= 1.
\end{aligned}
$$

Since the tangent space to M at 0 is $\{z \in \mathbb{C}^{n+1} \mid z_0 = 0\}$, Q_0 will be positive definite on $T_p^{1,0}M \times \mathbb{C}$ exactly when the form $Q_0'(\zeta_1, \ldots, \zeta_{n+1}) = \sum_1^{n+1} q_{j,k}(0)\zeta_j\bar{\zeta}_k$ is positive definite on $\mathbb{C}^{n+1}$. The smallest eigenvalue of Q_0' is $1 - \sqrt{\sum_{j=1}^n |a_{z_j}(0)|^2}$, and thus in these coordinates, (Pos) is equivalent to

$$\sum_1^n |a_{z_j}(0)|^2 < 1. \tag{2.4}$$

To show that under (Pos), Ω is strongly P-convex, we construct an analytic hypersurface S in $\mathbb{C}^{n+1}$ which, under inequality (2.4), intersects $\bar{\Omega}$ only at 0. As $\partial\rho(0) = (-1, 0, \ldots, 0)$, with $P_m(z, \tau, \chi)$ as in equation (2.3) we have $P_m(0, \partial\rho(0)) = a(0) = 0$ and $\frac{\partial P_m}{\partial \chi_n}(0, \partial\rho(0)) = 1$. By the implicit function theorem, there is a holomorphic function $F(z, \chi_1, \ldots, \chi_{n-1})$ such that $P_m(z, -1, \chi) = 0$ exactly when $\chi_n = F(z, \chi_1, \ldots, \chi_{n-1})$. Thus we obtain the Cauchy problem

$$\begin{cases} f_{z_n}(z') = F(z', f(z'), f_{z_1}(z'), \ldots, f_{z_{n-1}}(z')) \\ f(z_1, \ldots, z_{n-1}, 0) = g(z_1, \ldots, z_{n-1}) \end{cases} \tag{2.5}$$

where g is a holomorphic function to be determined. This has a unique holomorphic solution $f(z')$, and we see that $f(0) = f_{z_j}(0) = 0$ for $j = 1, \ldots, n$, and that $f_{z_j z_k}(0) = 0$ if $j \neq n$ and $k \neq n$. Further, by the definition of F we have that

$$P_m(f(z'), z', -1, f_{z_1}(z'), \ldots, f_{z_n}(z')) \equiv 0. \tag{2.6}$$

And thus the analytic hypersurface $S = \{z \in \mathbb{C}^{n+1} \mid z_0 = f(z')\}$ is everywhere characteristic with respect to P. It now remains to show that S stays outside of Ω. That is, $\rho(f(z'), z', \overline{f(z')}, \overline{z'}) > 0$ for $z' \neq 0$.

Differentiating equation (2.6) with respect to z_j and evaluating at 0 gives

$$f_{z_j z_n}(0) = a_{z_j}(0), \qquad j = 1, \ldots, n. \tag{2.7}$$

Thus the Taylor expansion of f at 0 is given by

$$f(z') = \sum_{j=1}^{n-1} a_{z_j}(0) z_j z_n + \frac{1}{2} a_{z_n}(0) z_n^2 + g(z_1, \ldots, z_{n-1}) + O(3). \tag{2.8}$$

The proof of the theorem then follows from the following lemma, which will be useful throughout this paper.

Lemma 2.2.2. *Let $a_1, \ldots, a_n \in \mathbb{C}$ such that*

$$\sum_{j=1}^{n} |a_j|^2 < 1. \tag{2.9}$$

Then there is a quadratic function $g(z_1, \ldots, z_{n-1})$ such that

$$\sum_{j=1}^{n} |z_j|^2 - 2\mathrm{Re} \left(\sum_{j=1}^{n-1} a_j z_j z_n + \frac{1}{2} z_n^2 + g(z_1, \ldots, z_{n-1}) \right) > 0 \tag{2.10}$$

for all $z \neq 0$.

Proof. By multiplying z_1 by a unimodular constant, and then doing the same to each z_j, we may assume without loss of generality that $a_j \geq 0$ for $j = 1, \ldots, n$. In this case, we let $g(z_1, \ldots, z_{n-1}) = -\frac{1}{2} a_n \sum_{j=1}^{n-1} z_j^2$. Then the real quadratic form

$$\sum_{j=1}^{n} |z_j|^2 - 2\mathrm{Re} \left(\sum_{j=1}^{n-1} a_j z_j z_n + g(z_1, \ldots, z_{n-1}) \right)$$

has eigenvalues $1 \pm |a_n|$ and $1 \pm \sqrt{\sum_1^n |a_j|^2}$. Thus under assumption (2.9), it is positive definite. $\square$

To complete the proof, we choose $g(z_1, \ldots, z_{n-1})$ given by lemma (2.2.2) as initial data in the Cauchy problem (2.5), and let

$$H(z, \bar{z}) = \sum_{j=1}^{n} |z_j|^2 - 2\mathrm{Re} \left(\sum_{j=1}^{n-1} a_j z_j z_n + \frac{1}{2} z_n^2 + g(z_1, \ldots, z_{n-1}) \right). \tag{2.11}$$

Then $H \geq 0$ for $z \neq 0$, and thus

$$\rho(z, f(z), \bar{z}, \overline{f(z)}) = H(z, \bar{z}) + O(3) > 0 \tag{2.12}$$

for $|z|$ small. □

Combining this result with Theorem 1.4.4, we have the following corollary.

Corollary 2.2.3. *If (Pos) holds at p, there is a solution u to $Pu = 0$ which is holomorphic in Ω and cannot be prolonged across p.*

2.3 A Similar Result

We remark that Tsuno has shown ([Tsu74], Theorem 3) the existence of a solution to $Pu = 0$ which cannot be prolonged across p, but under considerably stronger assumptions. Specifically, we have

Theorem 2.3.1 (Tsuno)**.** *Let Ω be a strictly pseudoconvex domain with $0 \in \partial\Omega$ be given by $\Omega = \{\rho(z, \bar{z}) < 0\}$, and let $P(z, \partial)$ be a linear, analytic differential operator such that $\partial\Omega$ is simply characteristic with respect to P at 0. Then under the two assumptions below, there exists a solution u to $Pu = 0$ which is holomorphic in Ω near 0 but not in any neighborhood of 0.*

Assumptions: Assume that for every complex number $t_0 \neq 0$, and for every $\lambda = (\lambda_0, \ldots, \lambda_n) \in \mathbb{C}^{n+1}$ such that $\sum_j \lambda_j P_m^{(j)}(0, \partial\rho(0)) = 0$, we have

$$\begin{aligned}\sum_{j,k=0}^{n} \Big(&\rho_{z_j z_k}(0) P_m^{(j)}(0, \partial\rho(0)) P_m^{(k)}(0, \partial\rho(0)) t_0^2 \\ &+ \rho_{\bar{z}_j \bar{z}_k}(0) \overline{P_m^{(j)}(0, \partial\rho(0))}\ \overline{P_m^{(k)}(0, \partial\rho(0))} \bar{t}_0^2 \\ &+ 2\rho_{z_j \bar{z}_k}(0) P_m^{(j)}(0, \partial\rho(0)) \overline{P_m^{(k)}(0, \partial\rho(0))} |t_0|^2 \Big) \\ &+ \sum_{j=0}^{n} \Big(P_m^{(j)}(0, \partial\rho(0)) P_{m,j}(0, \partial\rho(0)) t_0^2 \\ &+ \overline{P_m^{(j)}(0, \partial\rho(0))}\ \overline{P_{m,j}(0, \partial\rho(0))} \bar{t}_0^2 \Big) > 0\end{aligned} \tag{2.13}$$

and

$$\sum_{j,k=0}^{n}\Big(\lambda_k\rho_{z_jz_k}(0)+\bar{\lambda}_k\rho_{z_j\bar{z}_k}(0)\Big)P_m^{(j)}(0,\partial\rho(0))+\sum_{k=0}^{n}\lambda_kP_{m,k}(0,\partial\rho(0))=0. \qquad (2.14)$$

Tsuno shows that the quantities in these assumptions are invariantly defined, and thus we compute them in our normalization, with ρ and P_m given by (2.2) and (2.3), respectively.

$$\begin{aligned}
\rho_{z_jz_k}(0)=\rho_{\bar{z}_j\bar{z}_k}(0)&=0 && j,k=0,\ldots,n\\
\rho_{z_j\bar{z}_k}(0)&=\delta_j^k && j,k=1,\ldots,n\\
\rho_{z_j\bar{z}_k}(0)&=0 && \text{if } j=0 \text{ or } k=0\\
P_m^{(j)}(0,\partial\rho(0))&=\delta_j^n && j=0,\ldots,n\\
P_{m,j}(0,\partial\rho(0))&=(-1)^m a_{z_j}(0) && j=0,\ldots,n,
\end{aligned}$$

The restriction that $\sum_j \lambda_j P_m^{(j)}(0,\partial\rho(0))=0$ becomes $\lambda_n=0$, and (2.13) and (2.14) become, respectively

$$2|t_0|^2\pm 2\mathrm{Re}\ (a_{z_n}(0)t_0^2)>0 \qquad (2.15)$$

and

$$\sum_{k=0}^{n-1}\lambda_k a_{z_k}(0)=0. \qquad (2.16)$$

It is easy to see that assuming these two condition is equivalent to assuming that $a_{z_j}(0)=0$, for $j=0,\ldots,n-1$, and $|a_{z_n}|<1$. This is certainly much stronger than $\sum_{j=1}^n |a_{z_j}(0)|^2<1$, as we assume in Theorem 2.2.1.

Chapter 3

Projection Along Bicharacteristics

In this chapter we will find another condition that implies the existence of solutions to $Pu = 0$ which do not extend across $p \in \partial\Omega$. This result is phrased in more geometric terms, saying that we have such a solution if $\partial\Omega$ is bicharacteristically convex, and the local projection of the set of characteristic points of $\partial\Omega$ onto a hyperplane is convexifiable. We note that for this result, unlike in the previous chapter, we assume that Ω is pseudoconvex, but not strictly so. We then conclude with an example in C^3 to illustrate the relationship between our two theorems in the strictly pseudoconvex case.

Many of the definitions and constructions in this chapter come from Kawai and Takei [KT90], where they show that assuming a strict form of bicharacteristic convexity, and Ω strictly pseudoconvex, the strict pseudoconvexity of the projection is equivalent to (Pos). As we have shown that (Pos) implies strong P-convexity, this is not surprising.

Our goal here will be, of course, to construct an everywhere characteristic analytic hypersurface S which does not enter into Ω. We will do this by flowing the characteristic points of $\partial\Omega$ along bicharacteristic curves, then cutting this real hypersurface with a transverse hyperplane H. If the result is convexifiable, we find a codimension 2 analytic submanifold Σ which lies in H. Flowing Σ along suitably chosen bicharacteristics will then give us the desired hypersurface S.

Of course, to make sense of all this, we require some definitions and constructions.

3.1 Bicharacteristic Curves

Again we will have a $\Omega \subset \mathbb{C}^n$, pseudoconvex at $p \in \partial\Omega$ with a defining function $\rho(z, \bar{z})$ such that $\Omega = \{z \,|\, \rho(z, \bar{z}) < 0\}$, and a linear, analytic partial differential operator $P(z, \partial)$ such that $\partial\Omega$ is simply characteristic at p with respect to P.

The most important geometric objects one can associate with a differential operator are the bicharacteristic strips. For a neighborhood U of p we denote by $T^\star U$ the complex cotangent bundle of U, with coordinates (z, ζ). Then it is well known that the principle symbol of P, $P_m(z, \zeta)$ is a well-defined function on $T^\star U$. Since $\partial\Omega$ is simply characteristic at p, we have $P_m(p, \partial\rho(p, \bar{p})) = 0$ and $P_\zeta(p) = \partial_\zeta P_m(p, \partial\rho(z, \bar{z})) \neq 0$, and thus we may assume that U is small enough so that $P_\zeta(z) \neq 0$ in U. We may now define the bicharacteristic strips.

Definition 3.1.1. Let $(z_0, \zeta_0) \in T^\star U$ such that $P_m(z_0, \zeta_0) = 0$. The *bicharacteristic strip emanating from* (z_0, ζ_0) is the integral curve $\{z(t; z_0, \zeta_0), \zeta(t; z_0, \zeta_0)\}$ of the differential equations

$$\begin{aligned}
\frac{dz_j(t; z_0, \zeta_0)}{dt} &= P_m^{(j)}(z(t; z_0, \zeta_0)), \quad j = 1, \ldots, n \\
\frac{d\zeta_j(t; z_0, \zeta_0)}{dt} &= -P_{m,j}(z(t; z_0, \zeta_0)), \quad j = 1, \ldots, n \\
z(0; z_0, \zeta_0) &= z_0, \\
\zeta(0; z_0, \zeta_0) &= \zeta_0.
\end{aligned} \tag{3.1}$$

The projection of $(z(t; z_0, \zeta_0), \zeta(t; z_0, \zeta_0))$ to U is called the *bicharacteristic curve* of P passing through (z_0, ζ_0), and is denoted by $z(t; z_0, \zeta_0)$.

We will primarily be concerned with bicharacteristic curves emanating from characteristic points on $\partial\Omega$. We let

$$\Gamma_\Omega = \{q \in \partial\Omega \,|\, \partial\Omega \text{ is characteristic at } q \text{ with respect to } P\}$$

If $q \in \Gamma_\Omega$, the bicharacteristic curve of P passing through $(q, \partial\rho(q))$ is called the *principle bicharacteristic*, and will be denoted by b_q.

3.2 Bicharacteristic Convexity

As discussed in the introduction, the bicharacteristic curves of P are in a sense the carriers of singularities of solutions to $Pu = f$. As such, a first requirement that we should make is for the principle bicharacteristics not to enter into Ω, and we make the following definition.

Definition 3.2.1. Ω is *bicharacteristically convex* at a point $q \in \Gamma_\Omega$ if $b_q \cap W \cap \overline{\Omega} = \{q\}$ for some neighborhood W of q.

In terms of the defining function ρ, this is equivalent to $\rho(z(t; q, \partial\rho(q, \bar{q}))) > 0$ when $0 < |t| < \epsilon$ for some ϵ.

We note here that our definition of bicharacteristic convexity is different than that given by Kawai and Takei ([KT90], Definition 2.2.1). They define Ω to be bicharacteristically convex at q if, writing t in polar form as $t = re^{i\phi}$,

$$\left.\frac{\partial^2 \rho(z(t; q, \partial\rho(q, \bar{q})), \overline{z(t; q, \partial\rho(q, \bar{q}))})}{\partial r^2}\right|_{r=0} > 0 \qquad \text{for any real } \phi. \tag{3.2}$$

This condition says that not only does the principle bicharacteristic not enter into Ω, but that its order of contact with $\partial\Omega$ is 1. While it is clear that this is sufficient condition to have $b_q \cap \overline{\Omega} = \{q\}$, it is certainly not necessary. As such, if Ω and P satisfy (3.2), we shall say that Ω is *strictly bicharacteristically convex* at q.

In fact, we have encountered (3.2) before; Tsuno ([Tsu74], Corollary 1) shows that this is equivalent to (2.13) above.

There are several nice properties of Γ_Ω and the bicharacteristics emanating from it which always hold when Ω is strictly bicharacteristically convex, but may not be so when it is only bicharacteristically convex as we have defined it. The first is that if Ω is strictly bicharacteristically convex at q, then it is so for all points in Γ_Ω near q. They also prove ([KT90], Lemma 2.4.1) that Γ_Ω is a non-singular submanifold of real codimension 3, and that at each $q \in \Gamma_\Omega$, b_q is transverse to Γ_Ω.

As the following example shows, it is possible for Ω to have all of these properties, but not be strictly bicharacteristically convex.

Example 3.2.2. Let

$$\Omega = \{z = (z_1, z_2, z_3) \in \mathbb{C}^3 \,|\, 2\mathrm{Re}\ z_3 > |z_1|^2 + |z_2|^4\}$$

and

$$P(z, \partial) = \frac{\partial}{\partial z_2}.$$

Then we see that $\Gamma_\Omega = \{z \in \partial\Omega \,|\, 2z_2\bar{z}_2^2 = 0\} = \{(z_1, 0, z_3) \in \mathbb{C}^3 \,|\, 2\mathrm{Re}\ z_3 = |z_1|^2\}$, and that the principle bicharacteristic curves emanating from Γ_Ω are complex lines parallel to the z_2-axis. Taking $\rho(z, \bar{z}) = -2\mathrm{Re}\ z_3 + |z_1|^2 + |z_2|^4$ as a defining function for Ω, we have that

$$\rho(z(t; 0, \partial\rho(0)), \overline{z(t; 0, \partial\rho(0))}) = \rho(0, t, 0, 0, \bar{t}, 0) = |t|^4$$

and in polar coordinates $t = re^{i\phi}$ it is clear that

$$\left.\frac{d^2\rho(0, t, 0, 0, \bar{t}, 0)}{r^2}\right|_{r=0} = 0.$$

Thus Ω is not strictly bicharacteristically convex at 0, but it is clear that Ω is bicharacteristically convex at every point on Γ_Ω, Γ_Ω is a non-singular submanifold of real codimension 3, and that the principle bicharacteristics are transverse to Γ_Ω at every point.

As we would like to include situations such as that given in Example 3.2.2, we will not assume that Ω is strictly bicharacteristically convex at p. Rather, we will assume the following conditions, which taken together are somewhat weaker.

(i) Ω is bicharacteristically convex for all $q \in \Gamma_\Omega$in a neighborhood U of p.

(ii) Γ_Ω is a real-analytic submanifold of codimension 3.

(iii) For every $q \in \Gamma_\Omega \cap U$, b_q is transverse to Γ_Ω.

(3.3)

3.3 A Bicharacteristic Hull

We would now like to flow Γ_Ω along the principle bicharacteristics to obtain a subvariety of codimension 1 to define a sort of bicharacteristic hull of Ω. Again following Kawai and Takei, we make the following definition.

Definition 3.3.1.

$$\tilde{\Gamma}_\Omega(W) = \{w \in W \mid \text{ there exists a point } q \in \Gamma_\Omega \text{ for which } w \text{ lies on } b_q\}$$

Given assumptions (3.3) above, we have the following lemma. It is proved in [KT90], so we omit it the proof here.

Lemma 3.3.2. *There is a neighborhood W of p such that $\tilde{\Gamma}_\Omega(W)$ is a real-analytic hypersurface.*

Now, since Ω is bicharacteristically convex at all $q \in \Gamma_\Omega$ near p, it is clear that we may take W small enough that $\overline{\Omega} \cap \tilde{\Gamma}_\Omega(W) = \Gamma_\Omega$, and that $W \setminus \tilde{\Gamma}_\Omega(W)$ has two connected components, one of which contains Ω. This is the domain that we are really interested in, and we give it a name.

Definition 3.3.3.

$$B(\Omega) = \text{ the connected component of W } \setminus \tilde{\Gamma}_\Omega(W) \text{ which contains } \Omega.$$

The main result in [KT90] is essentially that if Ω is strictly pseudoconvex and strictly bicharacteristically convex, and H is an analytic hypersurface transverse to b_p, then (Pos) (see Definition 2.1.1) holds at p if and only if $H \cap B(\Omega)$ is a strictly pseudoconvex subset of H. Thus given the results in Chapter 2, we have that if $H \cap B(\Omega)$ is strictly pseudoconvex, there is a solution to $Pu = 0$ which is holomorphic in Ω near p but cannot be prolonged across p. However, more is in fact true. Before we can state the main theorem of this chapter, we need one more definition.

Definition 3.3.4. A domain U in $\mathbb{C}^n$ is *convexifiable* at $p \in \partial U$ if there is an analytic hypersurface Σ such that $\Sigma \cap \bar{U} = \{p\}$.

It is well known that convexifiability is weaker than strict pseudoconvexity, as we may write the defining equation for a strictly pseudoconvex domain in normal coordinates as in (2.2), and take $z_0 = 0$ as Σ. Convexifiability is stronger than weak pseudoconvexity, however, as shown by the example of Kohn and Nirenburg in [KN73]. As strong P-convexity is given by the existence of a supporting everywhere characteristic hypersurface, it is not surprising that this condition should come into play.

3.4 The Main Theorem

Theorem 3.4.1. *Let Ω be a domain in $\mathbb{C}^n$ with real-analytic boundary which is pseudoconvex at $p \in \partial\Omega$, and let $P(z, \partial)$ be a linear, analytic differential operator such that $\partial\Omega$ is simply characteristic at p with respect to P, and assume that the conditions (3.3) are satisfied. Let H be an analytic hypersurface through p such that the principle bicharacteristic curve, b_p is transversal to H.*

If $H \cap B(\Omega)$ is pseudoconvex at p, then Ω is strongly P-convex at p, and thus there is a solution to $Pu = 0$ which is holomorphic in Ω, near p but cannot be prolonged across p.

When we say $H \cap B(\Omega)$ is convexifiable at p, we mean that there is an analytic submanifold Σ of complex codimension 2 through p such that $\Sigma \subset H$, and $S \cap \overline{B(\Omega)} = \{p\}$.

Proof. Given Σ as above, we would like to flow it along bicharacteristic curves of P to create an everywhere characteristic hypersurface S in $\mathbb{C}^n$ that stays outside $B(\Omega)$, which we know stays outside Ω. Since $\partial B(\Omega) = \tilde{\Gamma}_\Omega(W)$ is a union of bicharacteristic curves, this would be simple if bicharacteristics could not intersect. Unfortunately, this is only true when P is first order, and the bicharacteristic curves are the integral curves of P. Thus we must show that for some choice of initial data ζ_0 as z varies in H, the bicharacteristics with those initial data cannot intersect.

The idea is to think of the bicharacteristic curves emanating from points on H as the image of a function $b(t, z, \zeta)$, and show that this function is injective. We will use $C(P)$ to denote the characteristic variety of P in $T^\star\mathbb{C}^n$, so

$$C(P) = \{(z, \zeta) \in T^\star\mathbb{C}^n \,|\, \zeta \neq 0 \text{ and } P_m(z, \zeta) = 0\}.$$

Under the assumption that $\partial\Omega$ is simply characteristic at 0 with respect to P, if ρ is a defining function for Ω, we have that $C(P)$ is an analytic hypersurface in $T^\star\mathbb{C}^n$ in a neighborhood of $(0, \partial\rho)$. As we are concerned with bicharacteristics emanating from H, we set

$$C(P)_H = \{(z, \zeta) \in T^\star\mathbb{C}^n \,|\, \zeta \neq 0, \; z \in H, \; \text{and } P_m(z, \zeta) = 0\}.$$

As b_p is transverse to H, $C(P)_H$ is a submanifold of dimension $2n-2$, and thus $\mathbb{C} \times C(P)|^H$ has dimension $2n-1$. We may now define the map

$$b : \mathbb{C} \times C(P)_H \to \mathbb{C}^n; \qquad b(t, z, \zeta) = z(t; z, \zeta), \tag{3.4}$$

where $z(t; z, \zeta)$ is the solution to (3.1), with initial data z and ζ. With this notation, then, $b_p = b(t, p, \partial\rho(p))$, and it is easy to see that $b(t, z, \zeta)$ has rank n at $(0, p, \partial\rho(p))$, so it is a submersion.

Consider the fiber

$$\begin{aligned} b^{-1}(0) &= \{(t, z, \zeta) \in \mathbb{C} \times C(P)_H \,|\, t = 0,\ z = p,\ \text{and } P_m(0, \zeta) = 0\} \\ &= \{0\} \times C(P)|_{z=p}. \end{aligned} \tag{3.5}$$

This is an $n-1$-dimensional submanifold of $\mathbb{C} \times C(P)_H$, and $(0, p, \partial\rho(p)) \in b^{-1}(0)$. Thus we may find a submanifold V of $\mathbb{C} \times C(P)_H$ of codimension $n-1$ which is transverse to $b^{-1}(0)$ at $(0, p, \partial\rho(p)) \in b^{-1}(0)$. Then $b|_V : V \to \mathbb{C}^n$ is a biholomorphism, and in particular, is injective in a neighborhood of $(0, p, \partial\rho(p))$.

By (3.5) it is clear that we may take (t, z) as coordinates on V. Now, if $\phi(z, \bar{z})$ is a defining function for $\tilde{\Gamma}_\Omega(W)$, then for any holomorphic function $\zeta(t, z)$ on $\mathbb{C} \times H$ such that $\zeta(0, z) = \partial\phi(z))$ when $z \in \tilde{\Gamma}_\Omega(W) \cap H$, we have

$$b|_V(t, z) = b(t, z, \zeta(t, z)).$$

We may take $\zeta(t, z)$ to be independent of t, thus for every $z \in H$ near 0, we have a $\zeta(z)$ and we may choose $(z, \zeta(z))$ as our initial conditions in (3.1) to get bicharacteristic curves that do not intersect, at least for $|t| < \epsilon$ for some ϵ.

We may now define S as

$$S = \{b(t, z, \zeta(z)) \,|\, |t| < \epsilon \text{ and } z \in \Sigma\}.$$

and we see that S is an analytic hypersurface in $\mathbb{C}^n$ which only intersects $\tilde{\Gamma}_\Omega(W)$ at p, and thus only intersects $\overline{\Omega}$ at p. Since S is a union of bicharacteristic curves, it is everywhere characteristic, and thus Ω is strongly P-convex with respect to P at p. $\square$

3.5 An Illustrative Example

To illustrate the method described above, we now compute in local coordinates the projection of bicharacteristics on to a transverse hyperplane, in the somewhat simple case where $\Omega \subset \mathbb{C}^3$ is strictly pseudoconvex at p, and P is first order.

We will work in normal coordinates, $(z, w) \in \mathbb{C}^2 \times \mathbb{C}$, so that $p = (0, 0)$, and $\Omega = \{\rho(z, w, \bar{z}, \bar{w}) < 0\}$, where

$$\rho(z, w, \bar{z}, \bar{w}) = -2\mathrm{Re}\ w + |z_1|^2 + |z_2|^2 + O(3). \tag{3.6}$$

We assume that $\partial\Omega$ is simply characteristic at 0, and thus we may write

$$P(z, w, \partial) = a(z, w)\frac{\partial}{\partial w} + b(z, w)\frac{\partial}{\partial z_1} + \frac{\partial}{\partial z_2} \tag{3.7}$$

where $a(0) = b(0) = 0$.

To simplify the notation, we will denote the first derivatives of a and b at 0 by

$$\begin{aligned} \frac{\partial a}{\partial z_j}(0) &= a_j \qquad & \frac{\partial a}{\partial w}(0) &= a_w \\ \frac{\partial b}{\partial z_j}(0) &= b_j \qquad & \frac{\partial b}{\partial w}(0) &= b_w \end{aligned}$$

We will also assume that Ω is bicharacteristically convex at 0 with respect to P. As we stated above, this is equivalent to assumption (2.13), which in this example simply means that

$$|a_2| < 1. \tag{3.8}$$

In order to project along the bicharacteristics, it will be useful to "straighten P". That is, to make a change of coordinates $(z, w) \to (y, x)$ such that P transforms to $\frac{\partial}{\partial y_2}$. To do this, we solve the (noncharacteristic) Cauchy problems

$$\begin{cases} Pf_1(z, w) = 0 \\ f_1(z_1, 0, w) = z_1 \end{cases} \quad \text{and} \quad \begin{cases} Pg(z, w) = 0 \\ g(z_1, 0, w) = w \end{cases} \tag{3.9}$$

and set $f_2(z, w) = z_2$. We then make the change of coordinates

$$\begin{aligned} y_j &= f_j(z, w) \qquad j = 1, 2 \\ x &= g(z, w) \end{aligned} \tag{3.10}$$

and we clearly have that P transforms to

$$\tilde{P} = \frac{\partial}{\partial y_2}. \tag{3.11}$$

To determine what happens to ρ, we must use the inverse function theorem to solve (3.10) for z and w. It is clear that $f_2(z, w) = z_2 = y_2$. By (3.9) we see that the linear terms in f_1 and g are simply z_1 and w, respectively, and we may also compute the quadratic terms in f_1 and g to be

$$\begin{aligned} f_1(z, w) &= z_1 - b_1 z_1 z_2 - \frac{1}{2} b_2 z_2^2 - b_w z_2 w + O(3) \\ g(z, w) &= w - a_1 z_1 z_2 - \frac{1}{2} a_2 z_2^2 - a_w z_2 w + O(3). \end{aligned} \tag{3.12}$$

It is clear that the Jacobian of (3.10) is the identity, so we may solve for z and w to get functions $F(y, x)$ and $G(y, x)$ such that

$$\begin{aligned} f_1(F(y, x), y_2, G(y, x)) &= y_1 \\ g(F(y, x), y_2, G(y, x)) &= x. \end{aligned} \tag{3.13}$$

Plugging (3.12) into these equations and equating power series coefficients, we find that

$$\begin{aligned} z_1 &= F(y, x) = y_1 + b_1 y_1 y_2 + \frac{1}{2} b_2 y_2^2 + b_w y_2 x + O(3) \\ z_2 &= y_2 \\ w &= G(y, x) = x + a_1 y_1 y_2 + \frac{1}{2} a_2 y_2^2 + a_w y_2 x + O(3). \end{aligned} \tag{3.14}$$

In these coordinates, then, we have Ω given by the defining function

$$\begin{aligned} \tilde{\rho}(y, x) &= \rho(F(y, x), G(y, x)) \\ &= -2\mathrm{Re}\left(x + a_1 y_1 y_2 + a_w y_2 x - \frac{1}{2} a_2 y_2^2\right) + |y_1|^2 + |y_2|^2 + O(3). \end{aligned} \tag{3.15}$$

Now, since P is first order, it is easy to see that $\partial\Omega$ is characteristic at a point (y_0, x_0) if it is on $\partial\Omega$ and $\tilde{P}\tilde{\rho}(x_0, y_0) = 0$. So we let Γ denote the set of characteristic points of $\partial\Omega$, and have that $(y, x) \in \Gamma$ if (y, x) solves the system

$$\begin{aligned} 0 &= \tilde{\rho}(y, x) \\ 0 &= \tilde{\rho}_{y_2}(y, x) = -a_1 y_1 - a_2 y_2 - a_w x + \bar{y}_2 + O(2) \\ 0 &= \overline{\tilde{\rho}_{y_2}(y, x)} = -\bar{a}_1 \bar{y}_1 - \bar{a}_2 \bar{y}_2 - \bar{a}_2 \bar{y}_2 - \bar{a}_w \bar{x} + y_2 + O(2) \end{aligned} \tag{3.16}$$

It is clear that the principle bicharacteristic curves of $\tilde{P}$ are lines parallel to the y_2 axis, and thus we want to project onto the $y_2 = 0$ hyperplane. This is equivalent to solving the system (3.16) to eliminate y_2 and $\bar{y}_2$. In fact, we would like to simultaneously eliminate y_2 from the equations and solve for Re x to get a nice defining function for $\Gamma \cap \{z_2 = 0\}$.

By the implicit function theorem, from equations (3.16) we see that this is possible if $|a_2| \neq 1$. In fact, we have from (3.8) that $|a_2| < 1$, and so, denoting Im x by t, we can find a real-valued function $\phi(y_1, \bar{y}_1, t)$ and a complex-valued $\psi(y_1, \bar{y}_1, t)$ such that $\phi(0) = 0$ and $\psi(0) = 0$, and

$$2\phi = a_1 y_1 \psi + a_w \psi(\phi + it) - \frac{1}{2} a_2 \psi^2 + |y_1|^2 + |\psi|^2 + O(3) \tag{3.17}$$

$$0 = -a_1 y_1 - a_2 \psi - a_w(\phi + it) + \bar{\psi} + O(2) \tag{3.18}$$

$$0 = -\bar{a}_1 \bar{y}_1 - \bar{a}_2 \bar{\psi} - \bar{a}_w(\phi - it) + \psi + O(2). \tag{3.19}$$

Now, we will not be too concerned with t, and everything on the right side of (3.17) is $O(2)$, so ϕ is $O(2)$, and thus we may write equations (3.18) and (3.19) as

$$\begin{aligned} \bar{\psi} &= a_1 y_1 + a_2 \psi + O(t) + O(2) \\ \psi &= \bar{a}_1 \bar{y}_1 + \bar{a}_2 \bar{\psi} + O(t) + O(2). \end{aligned} \tag{3.20}$$

We can use this to find the linear terms of ψ. To clarify notation, we will suppress evaluation of the derivatives at 0, and define

$$\psi_{y_1} = \psi_{y_1}(0), \qquad \text{and} \qquad \psi_{\bar{y}_1} = \psi_{\bar{y}_1}(0).$$

Then (3.20) becomes

$$\begin{aligned} \overline{\psi_{y_1}} \bar{y}_1 + \overline{\psi_{\bar{y}_1}} y_1 &= a_1 y_1 + a_2(\psi_{y_1} y_1 + \psi_{\bar{y}_1} \bar{y}_1 + O(t) + O(2) \\ \psi_{y_1} y_1 + \psi_{\bar{y}_1} \bar{y}_1 &= \bar{a}_1 \bar{y}_1 + \bar{a}_2(\overline{\psi_{y_1}} \bar{y}_1 + \overline{\psi_{\bar{y}_1}} y_1 + O(t) + O(2). \end{aligned} \tag{3.21}$$

Equating coefficients, we find that

$$\psi_{y_1} = \bar{a}_2 \overline{\psi_{\bar{y}_1}} \qquad \text{and} \qquad \psi_{\bar{y}_1} = \bar{a}_1 + \bar{a}_1 \overline{\psi_{y_1}} \tag{3.22}$$

from which we derive

$$\psi_{y_1}(0) = \frac{a_1 \bar{a}_2}{1 - |a_2|^2} \qquad \text{and} \qquad \psi_{\bar{y}_1}(0) = \frac{\bar{a}_1}{1 - |a_2|^2} \tag{3.23}$$

We can now put these values into (3.17). After some simplification, we find

$$2\phi = \Big(\frac{1-|a_1|^2-|a_2|^2}{1-|a_2|^2}\Big)|y_1|^2 + 2\mathrm{Re}\ Cy_1^2 + O(t) + O(3) \tag{3.24}$$

where C is a constant.

Since $|a_2| < 1$ by assumption, the domain Λ in the C^2 given by $\{\sigma > 0\}$, where

$$\sigma(y_1, x, \bar{y}_1, \bar{x}) = -2\mathrm{Re}\ x + \Big(\frac{1-|a_1|^2-|a_2|^2}{1-|a_2|^2}\Big)|y_1|^2 + 2\mathrm{Re}\ Cy_1^2 + O(t) + O(3),$$

is strictly pseudoconvex exactly when $|a_1|^2+|a_2|^2 < 1$, as we know from [KT90]. If $|a_1|^2 + |a_2|^2 = 1$, Λ may still be weakly pseudoconvex or convexifiable, depending on higher order terms in (3.24).

Chapter 4

Calculations in Local Coordinates for Ω Weakly Pseudoconvex

In the last chapter we found that there is a solution to $Pu = 0$ which does not extend holomorphically across $p \in \partial\Omega$ if and only if Ω is bicharacteristically convex at p, and if the local projection of the bicharacteristics nearby is convexifiable. This has that advantage that it is true even if Ω is only weakly pseudoconvex, and is thus stronger than the results in Chapter 2.

If one has a differential operator given in local coordinates, however, computing this projection can be quite difficult, and determining if it is convexifiable even more so. As such, in this chapter and the next we give sufficient conditions for the existence of solutions that do not extend across p, based on the defining equation for Ω and the coefficients of P. Thus they are a direct extension of the results of Chapter 2 to the case in which Ω is weakly pseudoconvex.

Writing down a defining function for an arbitrary pseudoconvex domain is difficult, and a complete classification of such domains has not yet been found. We therefore restrict our discussion to the case in which the null space of the Levi form at p is one dimensional, and there is positivity in the fourth order terms of the defining function.

4.1 Admissible Changes of Coordinates

We take a defining function ρ for $M = \partial\Omega$ such that $\Omega = \{\rho < 0\}$, and we work in normal coordinates $(z, w) \in \mathbb{C}^n \times \mathbb{C}$. That is, coordinates such that $p = (0, 0)$, and $M = \{(z, w) \mid \rho(z, w, \bar{z}, \bar{w}) = 0\}$, where ρ is of the form

$$\rho(z, w, \bar{z}, \bar{w}) = -2\text{Re } w + \phi(z, \bar{z}, \text{Im } w) \tag{4.1}$$

with $\phi(0, 0, 0) = 0$, $\mathrm{d}\phi(0, 0, 0) = 0$, and $\phi(z, 0, s) \equiv \phi(0, \bar{z}, s) \equiv 0$. The existence of such coordinates can be found in [BER99], in chapter 4. To simplify notation, in what follows we will usually just write 0 instead of $(0, 0)$ for the origin in $\mathbb{C}^n \times \mathbb{C}$.

As we stated above, working in local coordinates with an arbitrary weakly pseudoconvex hypersurface is very challenging, and a full classification of such has still not been given. We will therefore restrict our discussion to the somewhat simple case in which ϕ is of the form

$$\phi(z, \bar{z}, s) = \sum_{j=1}^{n-1} \lambda_j |z_j|^2 + O(3) \tag{4.2}$$

where the cubic terms in ϕ do not depend on s.

We first note that assuming P to be simply characteristic at 0 is equivalent to being able to write P_m in the following form, with $\zeta = (\chi, \tau) \in \mathbb{C}^n \times \mathbb{C}$:

$$P_m(z, w, \chi, \tau) = a(z, w)\tau^m + \tau^{m-1} \sum_{j=1}^{n} b_j(z, w)\chi_j + o(\|\chi\|^2) \tag{4.3}$$

where $a(0) = 0$ and the vector $P_\zeta(0) = \partial_\zeta P_m(0, \partial\rho(0)) = (b_1(0), \ldots, b_n(0), 0) \neq (0, \ldots, 0)$.

As in chapter 2, we would like to assume that $P_\zeta(0)$ lies along a z_j axis. However, unlike the strictly pseudoconvex case, there are now two distinct possibilities. If $P_\zeta(0)$ lies in the null space of the Levi form of $\partial\Omega$ at 0, which we denote by $N(\mathcal{L})$, then we may assume that $b_j(0) = 0$ for $j = 1, \ldots, n-1$, so that $P_\zeta(0) = (0, \ldots, 0, b_n(0), 0)$. If $P_\zeta \notin N(\mathcal{L})$, then after a rotation in the z variables we may assume that $P_\zeta(0) = (b_1(0), 0, \ldots, 0)$, so that $b_1(0) \neq 0$ and $b_j(0) = 0$ for $j \geq 2$. Further, we can scale each z_j so that we may assume $\lambda_j = 1$ for $j = 1, \ldots, n-1$.

We now wish to make a change of coordinates to simplify the third and fourth order terms in ρ, without changing the normalization we have already made. As such we make the following definition.

Definition 4.1.1. A change of coordinates $(z, w) = F(y)$ is *admissible* if it preserves the normality of the coordinates, the Levi form of M, and the direction of $P_\zeta(0)$.

Our primary concern is to eventually show that a particular analytic hypersurface $S = \{(z, w) \,|\, w = f(z)\}$ intersects $\overline{\Omega}$ only at 0. As in the proof of Theorem **??** we will find a holomorphic function $f(z)$, determined by the coefficients of P, such that $f(0) = 0$, $\partial f(0) = 0$, and

$$\rho(z, f(z), \bar{z}, \overline{f(z)}) > 0$$

for z near zero. Given the form of ϕ in (4.2), we will not be concerned with $O(3)$ terms in ϕ which are $o(\|z'\|^2)$, where $z' = (z_1, \ldots, z_{n-1})$, and throughout this discussion we will simply collect them all together and not give them much attention. We also note that if there are any $O(4)$ terms in ϕ which contain s, they will become $O(5)$ when $w = f(z)$, and thus we will discard those terms as well. Finally, we note that any change of coordinates of the form $z_j = z_j + cw$ will change the Levi form, and $z_j + cw^2$ will only give us terms of the form $|cw|^4$ and $2\text{Re}\,(cz_jw^2)$, both of which will be $O(5)$ when $w = f(z)$. Thus we need not consider any change of coordinates involving w at all.

Therefore we need only to consider the holomorphic change of coordinates $z = \tilde{G}(y), w = w$, where

$$\begin{cases} z_j &= y_j + \tilde{F}_j(y) \qquad \text{for } j = 1, \ldots, n-1 \\ z_n &= \sum_{j=1}^n c_j y_j + \tilde{F}_n(y) \end{cases} \tag{4.4}$$

with $c_n \neq 0$, and $\tilde{F}_j(0) = \mathrm{d}\tilde{F}_j(0) = 0$. Not every such transformation will preserve the direction of $P_\zeta(0)$, however. To determine what is required for this, we solve equations (4.4) for y, and by the implicit function theorem have

$$\begin{cases} y_j &= z_j + F_j(z) \qquad \text{for } j = 1, \ldots, n-1 \\ y_n &= \sum_{j=1}^n d_j z_j + F_n(z) \end{cases} \tag{4.5}$$

with $d_j = -\dfrac{c_j}{c_n}$ for $j = 1, \dots n-1$ and $d_n = \dfrac{1}{c_n}$, and $F_j(0) = \mathrm{d}F_j(0) = 0$. We also have for each $j = 1, \dots, n$

$$\frac{\partial}{\partial z_j} = \sum_{k=1}^{n} \frac{\partial y_k}{\partial z_j} \frac{\partial}{\partial y_k}$$

and thus evaluated at zero,

$$\left.\frac{\partial}{\partial z_j}\right|_0 = \left.\frac{\partial}{\partial y_j}\right|_0 + d_j \left.\frac{\partial}{\partial y_n}\right|_0 \tag{4.6}$$

To find the resulting $P_\zeta(0)$, we then compute

$$\sum_{j=1}^{n} b_j(0) \left.\frac{\partial}{\partial z_j}\right|_0 = \sum_{j=1}^{n-1} b_j(0) \left.\frac{\partial}{\partial y_j}\right|_0 + \left(b_n(0) + \sum_{j=1}^{n-1} d_j b_j(0)\right) \left.\frac{\partial}{\partial y_n}\right|_0 . \tag{4.7}$$

And so we see that if $P_\zeta(0) \in N(\mathcal{L})$, the coefficients on $\frac{\partial}{\partial y_j}$ are the same as those of $\frac{\partial}{\partial z_j}$, when evaluated at 0, and so any change of coordinates of the form in equations (4.4) is admissible. If $P_\zeta(0) \notin N(\mathcal{L})$, however, then in order for the coefficient on $\frac{\partial}{\partial y_n}$ to vanish at zero, we must have $c_1 = 0$. We collect these results in to a proposition.

Proposition 4.1.2. *Let M and P be as above. If $P_\zeta(0) \in N(\mathcal{L})$, then any change of coordinates of the form*

$$\begin{cases} z_j &= y_j + F_j(y) \qquad \text{for } j = 1, \dots, n-1 \\ z_n &= \sum_{j=1}^{n} c_j y_j + F_n(y) \end{cases} \tag{4.8}$$

is admissible. If $P_\zeta(0) \notin N(\mathcal{L})$, such a change of coordinates is admissible if and only if $c_1 = 0$ (Recall that we are assuming $P_\zeta(0) = (b_1(0), 0, \dots, 0)$).

4.2 Cubic Terms

We are now ready to determine how much we may simplify the cubic terms in ϕ with an admissible change of coordinates. As the Levi form is zero in the z_n direction, we begin with ϕ in the form

$$\begin{aligned} \phi(z, \bar{z}, s) = \|z'\|^2 + 2\mathrm{Re}\Bigg[&\sum_{|\alpha|=2} A_\alpha (z')^\alpha \bar{z}_n + C_n z_n \bar{z}_n^2 \\ &+ \sum_{j=1}^{n-1} \left[\sum_{k=1}^{n-1} J_{jk} z_j \bar{z}_k \bar{z}_n + B_j z_j z_n \bar{z}_n + C_j z_j \bar{z}_n^2 \right] \Bigg] + o(\|z'\|^2) + O(4) \end{aligned} \tag{4.9}$$

where α is a multiindex in $\mathbb{N}^{n-1}$. Under the change of coordinates in (4.4), for $j, k = 1, \ldots, n-1$, we have:

$$|z_j|^2 = |y_j|^2 + 2\text{Re } (y_j \bar{F}_j(y)) + |F_j(y)|^2 \tag{4.10}$$

$$z_j z_k \bar{z}_n = \bar{c}_n y_j F_k(y) \bar{y}_n + \bar{c}_n y_k F_j(y) \bar{z}_n + o(\|y'\|^2) + O(5) \tag{4.11}$$

$$\begin{aligned} z_j \bar{z}_n^2 &= \bar{c}_n^2 y_j \bar{y}_n^2 + 2\bar{c}_n y_j \bar{y}_n \overline{F_n(y)} + \bar{c}_n^2 F_j(y) \bar{y}_n^2 \\ &\quad + 2\sum_{l=1}^{n-1} \bar{c}_n \bar{c}_l F_j \bar{y}_l \bar{y}_n + o(\|y'\|^2) + O(5). \end{aligned} \tag{4.12}$$

From these equations, we see that there is no way in which we can obtain terms of the form $y_n \bar{y}_n^2$, except from $z_n \bar{z}_n^2$. If we are to get positivity from ρ when $w = f(z)$, then, we must assume that the coefficient on this term is zero, as there is no $|z_n|^2$ term to dominate it. Assuming this, we cannot obtain any terms of the form $y_j y_n \bar{y}_n$. Thus we must also assume these to be zero. From this point on, then, we shall assume that

$$\rho_{z_j z_n \bar{z}_n}(0) = 0 \qquad \text{for } j = 1, \ldots, n. \tag{4.13}$$

We remark that assuming these derivatives are zero is equivalent to assuming that ρ is not 2-nondegenerate, as defined by Baouendi, Ebenfelt and Rothschild in [BER99]. The fourth order invariants we discuss below, however, are not (in any way know to the author) related to finite nondegeneracy, and so we will continue to state our assumptions in terms of derivatives of ρ.

4.3 Fourth Order Terms

We also see that if any of the terms in $F_j(y)$ is of the form $y_j y_k$, for $j \neq n$ and $k \neq n$, it will only contribute to the $o(|y'\|^2)$ terms, and they are thus unnecessary. Further, $F_n(y)$ only appears in the term $2\bar{c}_n y_j \bar{y}_n \overline{F_n(y)}$ in equation (4.12). We lose nothing, then, in assuming that $F_j(y) = o(y_n)$ for $j = 1, \ldots, n-1$ and $F_n(y) = g y_n^2$.

Specifically, we only consider changes of coordinates of the form

$$z_j = y_j + \sum_{j=1}^{n-1} d_{jl} y_l y_n + e_j y_n^2 + f_j y_n^3 \qquad \text{for } j = 1, \ldots, n-1 \tag{4.14}$$

$$z_n = h y_n + \sum_{l=1}^{n-1} c_l y_l + g y_n^2 \tag{4.15}$$

where $h \neq 0$, and the other variables are arbitrary. Putting these into ϕ (still assuming, as we must, that $\rho_{z_j z_n \bar{z}_n}(0) = 0$), we obtain

$$\begin{aligned}
\phi(y, \bar{y}, w, \bar{w}) &= \|y'\|^2 + \left(\sum_{k=1}^{n-1} |e_k|^2\right)|y_n|^4 + H|h|^4|y_n|^4 \\
&+ \sum_{j=1}^{n-1} 2\mathrm{Re}\, \Bigg[\bar{e}_j y_j \bar{y}_n^2 + f_j y_j \bar{y}_n^3 + \left(\sum_{k=1}^{n-1} d_{kj}\bar{e}_k\right) y_j y_n \bar{y}_n^2 + \left(\sum_{k=1}^{n-1} A_{jk} e_k \bar{h}\right) y_j y_n^2 \bar{y}_n \\
&+ \left(\sum_{k=1}^{n-1} \bar{J}_{kj}\bar{e}_k h\right) y_j y_n \bar{y}_n^2 + \left(\sum_{k=1}^{n-1} J_{jk}\bar{e}_k \bar{h}\right) y_j \bar{y}_n^3 + C_j \bar{h}^2 y_j \bar{y}_n^2 \\
&+ 2C_j \bar{h}\bar{g} y_j \bar{y}_n^3 + \left(\sum_{k=1}^{n-1} C_k d_{kj}\bar{h}^2\right) y_j y_n \bar{y}_n^2 + \left(\sum_{k=1}^{n-1} 2\bar{C}_k \bar{e}_k h c_j\right) y_j y_n \bar{y}_n^2 \\
&+ C_j e_j \bar{h}^2 y_n^2 \bar{y}_n^2 + D_j y_j \bar{y}_n^3 + E_j y_j y_n \bar{y}_n^2 + F_j y_j y_n^2 \bar{y}_n + G h \bar{h}^3 y_n \bar{y}_n^3 \\
&+ 3\bar{G}\bar{h}h^2 c_j y_j y_n^2 \bar{y}_n + G\bar{h}^3 c_j y_j \bar{y}_n^3 + 2\bar{H}\bar{h}^2 h c_j y_j y_n \bar{y}_n^2\Bigg] + o(\|y'\|^2) + O(5).
\end{aligned} \tag{4.16}$$

Gathering like terms together, we have

$$\begin{aligned}
\phi(y, \bar{y}, w, \bar{w}) &= \|y'\|^2 + 2\mathrm{Re}\, \Bigg[\sum_{j=1}^{n-1} \Bigg[\left(\bar{e}_j + C_j \bar{h}^2\right) y_j \bar{y}_n^2 \\
&+ \left(f_j + \sum_{k=1}^{n-1} J_{jk}\bar{e}_k \bar{h} + 2C_j \bar{h}\bar{g} + D_j + G\bar{h}^3 c_j\right) y_j \bar{y}_n^3 \\
&+ \left(\sum_{k=1}^{n-1} (d_{kj}\bar{e}_k + \bar{J}_{kj}\bar{e}_k h + C_k d_{kj}\bar{h}^2 + 2\bar{C}_k \bar{e}_k h c_j) + E_j + 2H\bar{h}^2 h c_j\right) y_j y_n \bar{y}_n^2 \\
&+ \left(\sum_{k=1}^{n-1} A_{jk} e_k \bar{h} + F_j + 3\bar{G}\bar{h}h_2 c_j + \right) y_j y_n^2 \bar{y}_n\Bigg] + G h \bar{h}^3 y_n \bar{y}_n^3\Bigg] \\
&+ \left(H|h|^4 + \sum_{k=1}^{n-1} (|e_k|^2 + 2\mathrm{Re}\,(C_k e_k \bar{h}^2))\right)|y_n|^4 + o(\|y'\|^2) + O(5).
\end{aligned} \tag{4.17}$$

4.4 Invariants for the Pair (P, Ω)

Now, we will need any coefficient with a y_j to vanish, and the coefficient on $|y_n|^4$ to be nonzero, thus we obtain the system of equations, for $j = 1, \ldots, n-1$,

$$0 = \bar{e}_j + C_j \bar{h}^2 \tag{4.18}$$

$$0 = f_j + 2C_j \bar{h}\bar{g} + D_j + G\bar{h}^3 c_j + \sum_{k=1}^{n-1} \bar{J}_{jk} \bar{e}_k \bar{h} \tag{4.19}$$

$$0 = 2H\bar{h}^2 h c_j + h\bar{h}^2 E_j + \sum_{k=1}^{n-1} (d_{kj}\bar{e}_k + \bar{J}_{kj}\bar{e}_k h + C_k d_{kj} \bar{h}^2 + 2\bar{C}_k \bar{e}_k h c_j) \tag{4.20}$$

$$0 = 3\bar{G}\bar{h}h^2 c_j + \Big(\sum_{k=1}^{n-1} A_{jk} e_k \bar{h}\Big) h_2 \bar{h} F_j. \tag{4.21}$$

$$0 \neq H|h|^4 + \sum_{k=1}^{n-1} (|e_k|^2 + 2\mathrm{Re}\ (C_k e_k \bar{h}^2)) \tag{4.22}$$

We first note that the variables f_j only appear in equations (4.19), and so these equations will always be solvable. We may also divide equations (4.20), (4.21), and (4.22) by $h\bar{h}^2$, $h^2\bar{h}$, and $|h|^4$ respectively, and see that the value of h has no effect on the solvability of the system (recall that $h \neq 0$). Equation (4.18) forces us to set $e_j = -\bar{C}_j h^2$, for $j = 1 \ldots, n-1$, and from this, we see that (4.22) is satisfied exactly when $H \neq \sum_{k=1}^{n-1} |C_k|^2$, and we may choose h such that the coefficient on $|y_n|^4$ is unity. Then we have reduced our problem to solving the system

$$\begin{aligned} 0 &= 2E_j + 4Hc_j - \sum_{k=1}^{n-1} \big(2\bar{J}_{kj} C_k + 4|C_k|^2 c_j\big) \\ 0 &= 2F_j + 6\bar{G}c_j - \sum_{k=1}^{n-1} 2A_{jk}\bar{C}_k \\ H &\neq \sum_{k=1}^{n-1} |C_k|^2 \end{aligned} \tag{4.23}$$

This is a system of $2(n-1)$ equations in $n-1$ variables, and has a solution exactly when all of the determinants

$$\Delta_j = \Big(4H - \sum_{k=1}^{n-1} 4|C_k|^2\Big)\Big(\sum_{k=1}^{n-1} 2A_{jk}\bar{C}_k - 2F_j\Big) - 6\bar{G}\Big(\sum_{k=1}^{n-1} 2\bar{J}_{kj} C_k - 2E_j\Big) \tag{4.24}$$

vanish. Recall, now, that the variables above are given by the power series expansion of ρ, and so we have

$$\begin{aligned} 4H &= \rho_{z_n^2\bar{z}_n^2}(0), & 6\bar{G} &= \rho_{z_n^3\bar{z}_n}(0), \quad 2E_j = \rho_{z_jz_n\bar{z}_n^2}(0) \\ 2\bar{C}_kA_{jk} &= \rho_{\bar{z}_kz_n^2}(0)\rho_{z_jz_k\bar{z}_n}(0), & 2F_j &= \rho_{z_jz_n^2\bar{z}_n}(0), \\ 2C_k\bar{J}_{kj} &= \rho_{z_k\bar{z}_n^2}(0)\rho_{z_jz_n\bar{z}_k}(0), & 4|C_k|^2 &= |\rho_{z_k\bar{z}_n^2}(0)|^2 \end{aligned} \tag{4.25}$$

and (4.24) becomes, for $j = 1, \dots, n-1$,

$$\begin{aligned} \Delta_j = &\Big(\rho_{z_n^2\bar{z}_n^2}(0) - \sum_{k=1}^{n-1} |\rho_{z_k\bar{z}_n^2}(0)|^2\Big)\Big(\sum_{k=1}^{n-1} \rho_{\bar{z}_kz_n^2}(0)\rho_{z_jz_k\bar{z}_n}(0) - \rho_{z_jz_n^2\bar{z}_n}(0)\Big) \\ &- \rho_{z_n^3\bar{z}_n}(0)\Big(\sum_{k=1}^{n-1} 2\bar{\rho}_{z_k\bar{z}_n^2}(0)\rho_{z_jz_n\bar{z}_k}(0) - \rho_{z_jz_n\bar{z}_n^2}(0)\Big). \end{aligned} \tag{4.26}$$

We must now distinguish between the cases where $P_\zeta(0) \in N(\mathcal{L})$ and $P_\zeta(0) \notin N(\mathcal{L})$. By Proposition 4.1.2, if $P_\zeta(0) \in N(\mathcal{L})$, then the variables c_j in (4.23) can be anything we like, and to solve the system of equations we only need that the determinants Δ_j in (4.26) vanish for $j = 1, \dots, n-1$.

If $P_\zeta(0) \notin N(\mathcal{L})$, then we must assume that $c_1 = 0$. In this case, from (4.23) and (4.25) we must have

$$\begin{aligned} \rho_{z_1z_n\bar{z}_n^2}(0) &= \sum_{k=1}^{n-1} \rho_{z_k\bar{z}_n^2}(0)\rho_{z_1z_n\bar{z}_k}(0) \\ \rho_{z_1z_n^2\bar{z}_n}(0) &= \sum_{k=1}^{n-1} \rho_{\bar{z}_kz_n^2}(0)\rho_{z_1z_k\bar{z}_n}(0). \end{aligned} \tag{4.27}$$

as well as $\Delta_j = 0$ for $j = 2, \dots, n-1$.

Combining these results, we have the following theorem.

Theorem 4.4.1. *Let Ω, ρ, P, and Δ_j be as defined in (4.1), (4.2), (4.3), and (4.26). Assume that $\rho_{z_jz_n\bar{z}_n}(0) = 0$ for $j = 1, \dots, n$, $\Delta_j = 0$ for $j = 2, \dots, n-1$, and that $\rho_{z_n^2\bar{z}_n^2}(0) \neq \sum_{k=1}^{n-1} |\rho_{z_k\bar{z}_n^2}(0)|^2$.*

Assume further that one of the two following conditions holds:

(i) $P_\zeta(0) \in N(\mathcal{L})$ and $\Delta_1 = 0$.

(ii) $P_\zeta(0) = (b_1(0), 0, \dots, 0)$, $\rho_{z_1 z_n \bar{z}_n^2}(0) = \sum_{k=1}^{n-1} \rho_{z_k \bar{z}_n^2}(0)\rho_{z_1 z_n \bar{z}_k}(0)$, *and* $\rho_{z_1 z_n^2 \bar{z}_n}(0) = \sum_{k=1}^{n-1} \rho_{\bar{z}_k z_n^2}(0)\rho_{z_1 z_k \bar{z}_n}(0)$.

Then there are coordinates $(y, w) \in \mathbb{C}^n \times \mathbb{C}$ *such that* Ω *is given by* $\{\tilde{\rho}(y, w, \bar{y}, \bar{w}) < 0\}$, *where*

$$\tilde{\rho}(y, w, \bar{y}, \bar{w}) = -2Re\ w + \|y'\|^2 + |y_n|^4 + 2Re\ (\beta y_n \bar{y}_n^3) + o(\|y'\|^2) + O(5) \quad (4.28)$$

and $P(z, w, \partial)$ *transforms to* $\tilde{P}(y, w, \partial)$, *where*

$$\tilde{P}(y, w, \partial) = \tilde{a}(y, w)\Big(\frac{\partial}{\partial w}\Big)^m + \Big(\frac{\partial}{\partial w}\Big)^{m-1}\Big(\sum_{j=1}^{n} \tilde{b}_j(y, w)\frac{\partial}{\partial y_j}\Big) + \dots \quad (4.29)$$

where $\tilde{a}(0) = \tilde{b}_j(0) = 0$ *for* $j = 2, \dots, n-1$, *if* (i) *holds,* $\tilde{b}_1(0) = 0$ *and* $\tilde{b}_n(0) \neq 0$, *and if* (ii) *holds,* $\tilde{b}_1(0) \neq 0$ *and* $\tilde{b}_n(0) = 0$.

Henceforth we will assume that we have made such a change of coordinates, but will still use variables (z, w) instead of (y, w).

4.5 Theorems on Strong P-Convexity

Our goal in this section will be to provide necessary conditions on P and Ω that guarantee strong P-convexity when Ω is weakly pseudoconvex. We assume that the invariants discussed in the last chapter are zero, and that we have found coordinates in which the defining equation for Ω is particularly nice. Once this is done, we derive necessary conditions for the existence of solutions to $Pu = 0$ which do not extend holomorphically across the origin, both in the case when $P_\zeta(0) \notin N(\mathcal{L})$ and when $P_\zeta \in N(\mathcal{L})$. The method for each of the cases will be similar to that used in proving Theorem 2.2.1; we shall use the form of P to construct an everywhere characteristic analytic hypersurface S which stays outside of Ω.

When $P_\zeta(0) \notin N(\mathcal{L})$, if we are able to normalize the defining equation for Ω, the conditions that must be place on the coefficients of P are only slightly stronger from those in the strictly pseudoconvex case (Theorem 2.2.1). This makes sense, considering that while the Levi form of Ω is degenerate in some direction, it is not in the direction of the bicharacteristics of P, so along the direction of the principle bicharacteristic through p, (i.e. the vector $P_\zeta(0)$), Ω is in nondegenerate.

If, however, $P_\zeta(0) \in N(\mathcal{L})$, the situation is different. In this case, normalizing the defining function for Ω is a bit easier, but the direction of the principle bicharacteristic coincides with the direction in which Ω is degenerate. As we saw earlier, bicharacteristic convexity is necessary (though not sufficient) for the existence of solutions to $Pu = 0$ which do not extend across 0, and if Ω is flat to order 4 in the direction of the principle bicharacteristic, this will be much harder to achieve.

We first consider the case $P_\zeta(0) \notin N(\mathcal{L})$. Recall that $z' = (z_1, \ldots, z_{n-1})$, and $\|z'\| = \sum_1^{n-1} |z_j|^2$.

4.5.1 $P_\zeta(0) \notin N(\mathcal{L})$

Theorem 4.5.1. *Let $\Omega \subset \mathbb{C}^{n+1}$ be a hypersurface, $p \in \partial\Omega$ a point where Ω is weakly pseudoconvex, and let $P(Z, \partial)$ be a linear, analytic partial differential operator with coefficients holomorphic in a neighborhood of p such that $P_\zeta(p) \notin N(\mathcal{L})(p)$. Assume that, written in normal coordinates $Z = (z, w)$ centered at p, Ω can be given by $\{\rho(z, w, \bar{z}, \bar{w}) < 0\}$, where*

$$\rho(z, w, \bar{z}, \bar{w}) = -2\mathrm{Re}\ w + \sum_{j=1}^{n} |z_j|^2 + |z_n|^4 + 2\mathrm{Re}\ \beta \bar{z}_n z_n^3 + o(\|z'\|^2) + O(5). \quad (4.30)$$

In these coordinates, with $(\chi, \tau) \in \mathbb{C}^n \times \mathbb{C}$ we write the symbol of P as

$$P_m(z, w, \chi, \tau) = \tilde{a}(z, w)\tau^m + \tau^{m-1}\left(\sum_{j=1}^{n} \tilde{b}_j(z, w)\chi_j\right) + o(\|\chi\|^2) \quad (4.31)$$

where $\tilde{a}(0) = \tilde{b}_j(0) = 0$ for $j = 2, \ldots, n$, and $b_1(0) \neq 0$ and let

$$a(z, w) = \frac{\tilde{a}(z, w)}{\tilde{b}_1(z, w)}.$$

If $a_{z_n}(0) = a_{z_n^2}(0) = a_{z_n^3}(0) = 0$ and

$$\sum_{j=1}^{n-1} |a_{z_j}|^2 < 1 \quad (4.32)$$

then there is a solution u to $Pu = 0$ which is holomorphic in Ω near 0 but cannot be prolonged holomorphically across 0.

Proof. The proof is similar to that of Theorem 2.2.1; we will show that Ω is strongly P-convex at 0. As such, we construct an analytic hypersurface S through 0 which is everywhere characteristic with respect to P, and only intersects $\overline{\Omega}$ at 0. This is equivalent to finding a function $f : \mathbb{C}^n \to \mathbb{C}$, holomorphic near 0, such that $f(0) = 0$ and $P_m(z, f(z), \partial f(z), -1) \equiv 0$ and $\rho(z, f(z), \bar{z}, \overline{f(z)}) > 0$ for all z near 0, so that $S = \{(z, w) \,|\, w = f(z)\}$ is the desired hypersurface.

We first note that if a holomorphic function $c(z, w)$ does not vanish at 0, then $P_m(z, f(z), \partial f(z), -1) = 0$ exactly when $c(z, w)P_m(z, f(z), \partial f(z), -1) = 0$. Thus we may assume without loss of generality that we have divided by $b_1(z, w)$ in equation (4.43), and that P_m is given by

$$P_m(z, w, \chi, \tau) = a(z, w)\tau^m + \tau^{m-1}\left(\chi_1 + \sum_{j=2}^{n} b_j(z, w)\chi_j\right) + o(\|\chi\|^2). \tag{4.33}$$

To construct f, we note that $\frac{\partial P_m}{\partial \chi_1}(0, 0, 0, -1) = \pm 1 \neq 0$, so we may solve the equation

$$P_m(z, w, \chi, -1) = 0$$

for χ_1. Thus there is a holomorphic function $G(z, w, \chi_2, \ldots, \chi_n)$ such that

$$P_m(z, w, \chi, -1) = 0 \text{ if and only if } \chi_1 = G(z, w, \chi_2, \ldots, \chi_n).$$

To produce $f(z)$, we solve the noncharacteristic Cauchy problem

$$\begin{cases} f_{z_1}(z) = G(z, f(z), f_{z_2}(z), \ldots, f_{z_n}(z)) \\ f(0, z_2, \ldots, z_n) = g(z_2, \ldots, z_{n-1}) \end{cases} \tag{4.34}$$

where g is a quadratic function to be determined later. Then we see that $f(0) = 0$, $\partial f(0) = 0$, and $S = \{(z, w) \,|\, w = f(z)\}$ is everywhere characteristic with respect to P. In particular, we may relate derivatives of f to those of a with the identity

$$0 \equiv a(z, f(z)) - \left(f_{z_1}(z) + \sum_{j=2}^{n} b_j(z, f(z)) f_{z_j}(z)\right) + O(\|\partial f(z)\|^2). \tag{4.35}$$

In what follows, to simplify notation we omit the arguments of the functions, keeping in mind that f and its derivatives are evaluated at z, and a is evaluated at

$(z, f(z))$. We first differentiate equation (4.35) with respect to z_k for $k = 1, \ldots, n$. Since all the first derivatives of f vanish at 0, we have

$$a_{z_k} = f_{z_1 z_k} + O(1) \qquad \text{for } k = 1, \ldots, n \tag{4.36}$$

and hence

$$a_{z_k}(0) = f_{z_1 z_k}(0) \qquad \text{for } k = 1, \ldots, n. \tag{4.37}$$

In particular, we have that

$$f_{z_1 z_n}(0) = a_{z_n}(0) = 0$$

by assumption. We now wish to differentiate equation (4.35) several times with respect to z_n. By equation (4.34) we see that any order derivative of f which does not differentiate with respect to z_1 will be zero at the origin, and hence is $O(z_1)$. Thus if $c(z, w)$ is the coefficient of χ_1^2 in (4.33), then from (4.36) we have

$$a_{z_n} = f_{z_1 z_n} + c_{z_n}(f_{z_1})^2 + 2c f_{z_1} f_{z_1 z_n} + O(z_1).$$

Differentiating again with respect to z_n we have

$$a_{z_n^2} = f_{z_1 z_n^2} + 2c f_{z_1} f_{z_1 z_n^2} + O(z_1) + O(2). \tag{4.38}$$

and so we see that

$$f_{z_1 z_n^2}(0) = a_{z_n^2}(0) = 0.$$

Thus $2c f_{z_1} f_{z_1 z_n^2}$ is $O(2)$, and we may differentiate (4.38) once more to find

$$f_{z_1 z_n^3}(0) = a_{z_n^3}(0) = 0.$$

We now verify that S intersects $\overline{\Omega}$ only at 0. By the argument above, we have

$$\begin{aligned} \rho(z, f(z), \bar{z}, \overline{f(z)}) &= \sum_{j=1}^{n-1} |z_j|^2 + |z_n|^4 + 2\mathrm{Re}\ (\beta \bar{z}_n z_n^3) \\ - 2\mathrm{Re} &\left(\frac{1}{2} a_{z_1}(0) z_1^2 + \sum_{j=2}^{n-1} a_{z_j}(0) z_1 z_j + g(z_2, \ldots, z_{n-1}) \right) + o(\|z'\|^2) + O(5) \end{aligned} \tag{4.39}$$

We now choose g as given by Lemma 2.2.2 to make the quadratic form

$$H(z') = \sum_{j=1}^{n-1} |z_j|^2 - 2\mathrm{Re} \left(\frac{1}{2} a_{z_1}(0) z_1^2 + \sum_{j=2}^{n-1} a_{z_j}(0) z_1 z_j + g(z_2, \ldots, z_{n-1}) \right) \tag{4.40}$$

positive definite. Thus there is a positive constant k such that

$$H(z') \geq k\|z'\|^2$$

for all z', and we have

$$\rho(z, f(z), \bar{z}, \overline{f(z)}) \geq k\|z'\|^2\big(1 + O(1)\big) + |z_n|^4(1 - 2|\beta|) + O(5) > 0 \qquad (4.41)$$

for z near zero if $2|\beta| \leq 1$. But this follows directly from the pseudoconvexity of Ω, and thus Ω is strongly P-convex at 0. By Corollary 1.4.4, the theorem is proved. □

4.5.2 $P_\zeta \in N(\mathcal{L})$

We now prove an analogous theorem for the case in which $P_\zeta(0) \in N(\mathcal{L})$. Although we are more free with our change of coordinates in this case (see Theorem 4.4.1), it is actually somewhat more degenerate, and our assumptions on the coefficients of P must be much stronger.

Theorem 4.5.2. *Let $\Omega \subset \mathbb{C}^{n+1}$ be a hypersurface, $p \in \partial\Omega$ a point where Ω is weakly pseudoconvex, and let $P(Z, \partial)$ be a linear, analytic partial differential operator with coefficients holomorphic in a neighborhood of p such that $P_\zeta(p) \in N(\mathcal{L})(p)$. Assume that, written in normal coordinates $Z = (z, w)$ centered at p, Ω can be given by $\{\rho(z, w, \bar{z}, \bar{w}) < 0\}$, where*

$$\rho(z, w, \bar{z}, \bar{w}) = -2\mathrm{Re}\; w + \sum_{j=1}^{n} |z_j|^2 + |z_n|^4 + 2\mathrm{Re}\; \beta\bar{z}_n z_n^3 + o(\|z'\|^2) + O(5). \quad (4.42)$$

In these coordinates, with $(\chi, \tau) \in \mathbb{C}^n \times \mathbb{C}$ we write the symbol of P as

$$P_m(z, w, \chi, \tau) = \tilde{a}(z, w)\tau^m + \tau^{m-1}\left(\sum_{j=1}^{n} \tilde{b}_j(z, w)\chi_j\right) + o(\|\chi\|^2) \qquad (4.43)$$

where $\tilde{a}(0) = \tilde{b}_j(0) = 0$ for $j = 1, \ldots, n-1$, and $b_n(0) \neq 0$. Let

$$a(z, w) = \frac{\tilde{a}(z, w)}{\tilde{b}_n(z, w)}, \qquad b_j(z, w) = \frac{\tilde{b}_j(z, w)}{\tilde{b}_n(z, w)} \;\; \textit{for} \;\; j = 1, \ldots, n-1.$$

If

$$a_{z_j}(0) = a_{z_j z_n}(0) = 0 \qquad \text{for } \; j = 1, \ldots, n, \tag{4.44}$$

$$a_{z_j z_n^2}(0) = 2 \sum_{k=1}^{n-1} b_{j,z_n}(0) a_{z_j z_k}(0) \qquad \text{for } \; j = 1, \ldots, n-1, \tag{4.45}$$

and

$$\frac{1}{24} |a_{z_n^3}(0)| < 1 - 2|\beta|, \tag{4.46}$$

then there is a solution u *to* $Pu = 0$ *which is holomorphic in* Ω *near* 0*, but cannot be prolonged holomorphically across* 0*.*

Proof. We proceed as in Theorem 4.5.1. We solve $P_m(z, w, \chi, -1) = 0$ for χ_n, to get a holomorphic function $G(z, w, \chi_1, \ldots, \chi_{n-1})$ such that

$$P_M(z, w, \chi, -1) = 0 \;\text{ if and only if }\; \chi_n = G(z, w, \chi_1, \ldots, \chi_{n-1})$$

and solve the noncharacteristic Cauchy problem

$$\begin{cases} f_{z_n}(z) = G(z, f(z), f_{z_1}(z), \ldots, f_{z_{n-1}}(z)) \\ f(z_1, \ldots, z_{n-1}) = 0 \end{cases} \tag{4.47}$$

(nonzero initial data will not help us in this case). We again have that $f(0) = 0$ and $\partial f(0) = 0$, and obtain the identity

$$a(z, f(z)) = f_{z_n}(z) + \sum_{j=1}^{n-1} b_j(z, f(z)) f_{z_j}(z) + O(\|\partial f\|^2). \tag{4.48}$$

We will omit the arguments of the functions in what follows, with f and its derivatives evaluated at z, and the coefficients $a(z, w)$ and $b_j(z, w)$ evaluated at $(z, f(z))$. Differentiating (4.48) with respect to z_k gives us, for each $k = 1, \ldots, n$,

$$a_{z_k} = f_{z_k z_n} + \sum_{j=1}^{n-1} \left(b_j f_{z_j z_k} + b_{j,z_k} f_{z_j} + b_{j,w} f_{z_k} \right) + -a_w f_{z_k} + O(1) \tag{4.49}$$

Evaluating at zero, we find that

$$f_{z_k z_n}(0) = a_{z_k}(0) = 0 \tag{4.50}$$

by assumption (4.44). As any derivatives of f which do not differentiate with respect to z_n are also zero by (4.47), $\partial f(z)$ is $O(2)$, so we may write (4.49) as

$$a_{z_k} + a_w f_{z_k} = f_{z_k z_n} + \sum_{j=1}^{n-1} b_{j,z_k} f_{z_j} + b_j f_{z_j z_k} + O(3). \tag{4.51}$$

Differentiating with respect to z_l, $l = 1, \ldots, n$, gives

$$a_{z_k z_l} + a_w f_{z_k z_l} = f_{z_k z_l z_n} + \sum_{j=1}^{n-1} b_{j,z_k} f_{z_j z_l} + b_{j,z_l} f_{z_j z_k} + b_j f_{z_j z_k z_l} + O(2), \tag{4.52}$$

and thus

$$f_{z_k z_l z_n}(0) = a_{z_k z_l}(0) \qquad \text{for} \quad k, l = 1, \ldots, n. \tag{4.53}$$

We now set $l = n$ and differentiate once more with respect to z_n to find

$$a_{z_k z_n^2} = f_{z_k z_n^3} + \sum_{j=1}^{n-1} 2 b_{j,z_n} f_{z_j z_k z_n} + O(1), \tag{4.54}$$

and evaluating at zero, using (4.53) we have

$$f_{z_k z_n^3}(0) = a_{z_k z_n^2}(0) - 2\sum_{j=1}^{n-1} b_{j,z_n}(0) a_{z_j z_k}(0) = 0 \qquad \text{for} \quad k = 1, \ldots, n-1 \tag{4.55}$$

and

$$f_{z_n^4}(0) = a_{z_n^3}(0). \tag{4.56}$$

Combining these computations, we have that

$$f(z) = \frac{1}{24} a_{z_n^3}(0) z_n^4 + o(\|z\|^2) + O(5). \tag{4.57}$$

Now we only need to verify that $\rho(z, f(z), \bar{z}, \overline{f(z)}) > 0$ for small $z \neq 0$. By (4.57) and (4.46),

$$\begin{aligned} &\rho(z, f(z), \bar{z}, \overline{f(z)}) = \\ &\|z'\|^2 + |z_n|^4 + 2\mathrm{Re}\,(\beta z_n \bar{z}_n^3) - 2\mathrm{Re}\left(\frac{1}{24} a_{z_n^3}(0) z_n^4\right) + o(\|z'\|^2) + O(5) \\ &\geq \|z'\|^2(1 + O(1)) + |z_n|^4\left(1 - 2|\beta| - \frac{1}{24}|a_{z_n^3}(0)|\right) > 0 \end{aligned} \tag{4.58}$$

and so Ω is strongly P-convex at 0. By Corollary 1.4.4, we have the desired solution u to $Pu = 0$. $\square$

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