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UNIVERSITY OF CALIFORNIA, SAN DIEGO

**Classification of Convex Ancient Solutions to Curve Shortening Flow
on the Sphere**

A dissertation submitted in partial satisfaction of the
requirements for the degree
Doctor of Philosophy

in

Mathematics

by

Janelle Louie

Committee in charge:

Professor Bennett Chow, Chair
Professor Kenneth Intriligator
Professor Elizabeth Jenkins
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2014

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The dissertation of Janelle Louie is approved, and it is acceptable in quality and form for publication on microfilm and electronically:

Chair

University of California, San Diego

2014

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Chapters 2-6 include material from: P. Bryan, J. Louie, “Classification of Convex Ancient Solutions to Curve Shortening Flow on the Sphere”, *preprint*, *arXiv: 1408.5523*.

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PUBLICATIONS

P. Bryan, J. Louie, “Classification of Convex Ancient Solutions to Curve Shortening Flow on the Sphere”, *preprint, arXiv: 1408.5523*

ABSTRACT OF THE DISSERTATION

**Classification of Convex Ancient Solutions to Curve Shortening Flow
on the Sphere**

by

Janelle Louie

Doctor of Philosophy in Mathematics

University of California, San Diego, 2014

Professor Bennett Chow, Chair

We classify closed, convex, embedded ancient solutions to the curve shortening flow on the sphere, showing that the only such solutions are the family of shrinking round circles, starting at an equator and collapsing to a point, or the curve is a fixed equator for all time. A Harnack inequality for the curve shortening flow on the sphere and an application of Gauss-Bonnet allow us to obtain curvature bounds for convex ancient solutions, which lead to backwards smooth convergence to an equator. To complete the proof, we use an Aleksandrov reflection argument to show that maximal symmetry is preserved under the flow.

Chapter 1

Introduction

We consider solutions γ_t to the curve shortening flow, a geometric heat equation

$$\frac{\partial \Gamma}{\partial t} = kN$$

which moves each point Γ on a curve γ_t in the direction of its unit normal N with speed equal to its geodesic curvature k .

Let (M^2, g) be an oriented smooth surface, by which we mean a 2-dimensional manifold with metric g , and $I \subset \mathbb{R}$ an interval. Consider a family of simple, closed, embedded curves

$$\Gamma : S^1 \times I \rightarrow M$$

parametrized by $u \in S^1$, $t \in I$. We will write $\gamma_t = \Gamma(S^1, t)$. Denote the velocity and tangent vector fields, respectively, as

$$\frac{\partial \Gamma}{\partial t} = \Gamma_* \left(\frac{\partial}{\partial t} \right) \quad \text{and} \quad \frac{\partial \Gamma}{\partial u} = \Gamma_* \left(\frac{\partial}{\partial u} \right)$$

Let s be arc length so that $\left| \frac{\partial \Gamma}{\partial s} \right| \equiv 1$. The arc length element is

$$ds = \left| \frac{\partial \Gamma}{\partial u} \right| du$$

and we set $v = \left| \frac{\partial \Gamma}{\partial u} \right|$. The unit tangent vector T is given by

$$T = \frac{1}{v} \frac{\partial \Gamma}{\partial u} = \frac{\partial \Gamma}{\partial s}.$$

The Frenet formulas state that

$$\nabla_T T = kN \text{ and } \nabla_T N = -kT$$

where ∇ is covariant differentiation with respect to metric g . Define the geodesic curvature $k : S^1 \rightarrow \mathbb{R}$ to be

$$k = \langle \nabla_T T, N \rangle.$$

Definition 1.1. We say that $\Gamma : S^1 \times I \rightarrow M$ is a solution to the *curve shortening flow* if

$$\frac{\partial \Gamma}{\partial t}(u, t) = k(u, t)N(u, t). \quad (1.1)$$

1.1 Previous work and background

The curve shortening flow was studied by Mullins [Mul56] as a way to represent ideal grain boundary movement in two dimensions, where a grain boundary is the interface between two grains, or crystallites, in a polycrystalline material. There have since been numerous results for curve shortening flow in the plane $\mathbb{R}^2$. The behavior of embedded plane curves is developed in a series of papers. Gage and Hamilton [GH86] showed that under curve shortening flow in $\mathbb{R}^2$, the curvature of curve γ_t evolves according to

$$\partial_t k = \partial_{ss} k + k^3 \quad (1.2)$$

which is a strictly parabolic equation. For convex curves, one may use the angle θ of the tangent line as a parameter and the evolution equation can be written as

$$\partial_t k = k^2 \partial_{\theta\theta} k + k^3. \quad (1.3)$$

In [GH86], Gage and Hamilton proved that if γ_0 is a convex embedded curve, then the curve shortening flow shrinks γ_t to a point, with the curve remaining convex and becoming asymptotically circular close to its extinction time. Following this, Grayson [Gra87] studied the evolution of non-convex embedded curves under the curve shortening flow, and proved that if γ_0 is an embedded curve in $\mathbb{R}^2$, the

solution γ_t does not develop any singularities before it becomes strictly convex. Hamilton [Ham95b] gives a new and shortened proof of Grayson's theorem based on an isoperimetric estimate and singularity analysis (a singularity model is either a round shrinking circle or the Grim Reaper translating soliton). Isoperimetric estimates were also used by Huisken [Hui98] and Andrews and Bryan [AB11] to give proofs of Grayson's theorem.

Definition 1.2. A solution to the curve shortening flow is called an *ancient solution* if it is defined on an interval of the form $(-\infty, T)$ for $T \in \mathbb{R} \cup \infty$.

Remark 1.3. In the above definition, for $T < \infty$, we may assume $T = 0$, and will do so from this point on.

The study of ancient solutions to geometric flows is of interest since blow up limits (if they exist) of singularities are ancient solutions, and so their classification can help to understand the singularities of the flow (see [Ham95a]). An ancient solution is called Type I if

$$\lim_{t \rightarrow -\infty} \sup_{\gamma_t} |t|^{1/2} |k| < \infty$$

and Type II otherwise. In [DHS10], Daskalopoulos, Hamilton, and Sesum classified embedded compact ancient solutions to the curve shortening flow in the plane $\mathbb{R}^2$. The evolution of the family of curves γ_t is described by the evolution (1.3) of the curvature k , and they consider the quantity $p := k^2$ and its evolution to show that an ancient solution to (1.3) is either a family of contracting circles (a Type I ancient solution) defined by

$$p(\theta, t) = \frac{1}{2(-t)}$$

or a family of evolving Angenent ovals (a Type II ancient solution) defined by

$$p(\theta, t) = \lambda \left(\frac{1}{1 - e^{2\lambda t}} - \sin^2(\theta + \gamma) \right)$$

for two parameters $\lambda > 0$ and γ . Their argument uses a Harnack estimate for convex curves which gives $k_t \geq 0$ for convex ancient solutions and then examines

monotone quantities under the flow along with the fact that at its singular time, the solution becomes circular with sharp rates of convergence.

The study of curvature flow of closed curves in the plane was generalized to certain surfaces by Grayson [Gra89] and Gage [Gag90]. On surfaces, a smooth embedded curve collapses to a round point in finite time as in the plane case, or exists for all time, converging to a closed geodesic.

Aleksandrov [Ale56] proved that the only embedded closed surfaces with constant mean curvature in $\mathbb{R}^3$ are round spheres, and the technique he used became known as the Aleksandrov reflection technique. It involved reflecting the surface across a plane and using the maximum principle to conclude that the surface has a plane of symmetry; from the collection of planar symmetries, one concludes the surface is a sphere. In [CG96], [Cho97], [CG01] Chow and Gulliver employ the parabolic analogue of Aleksandrov's method of moving planes which inspired the ideas for this work.

1.2 Results and outline of the thesis

We prove the following theorem.

Theorem 1.4. *(Classification of convex ancient solutions) Let the family of curves γ_t be a closed, convex, embedded, ancient solution to the curve shortening flow on the sphere S^2 . Then γ_t must be either a fixed equator for $t \in (-\infty, \infty)$ or a family of shrinking round circles, existing for $t \in (-\infty, 0)$.*

The thesis is organized as follows: Chapter 2 collects some results about convex curves on S^2 and evolution of basic quantities under the curve shortening flow. Statements of the maximum principle and one of its applications are also given for reference. In Chapter 3, we derive a Harnack inequality for convex ancient solutions and use it to obtain that $k_t \geq 0$ under the flow. Thus for convex curves, k is monotonically increasing and hence bounded on $(-\infty, t_0]$; Chapter 4 describes a standard bootstrapping argument which then furnishes us with uniform bounds on all higher derivatives. In Chapter 5 we use the Gauss-Bonnet theorem to show

that $\int_{\gamma_t} k \rightarrow 0$ as $t \rightarrow -\infty$, then combine this result with the derivative bounds to show that γ_t converges to an equator $\gamma_{-\infty}$ as $t \rightarrow -\infty$. Finally in Chapter 6 we use a parabolic version of the Aleksandrov reflection technique, reflecting the curve γ_t across a tilted plane intersecting S^2 , to show that the curve γ_t is a circle for all t .

Chapter 2

Preliminaries

A closed embedded curve γ on S^2 divides S^2 into two open, disjoint regions. If one region has area less than 2π , we will call that the interior of γ and will say that this is the area enclosed by the curve γ .

2.1 Convex curves on S^2

For a Riemannian manifold M we will use the following definitions of convexity.

Definition 2.1. A subset $K \subset M$ is (geodesically) convex if any two points $x, y \in K$ can be joined by a length-minimizing geodesic (of M) entirely contained within K . Note that we don't require this geodesic to be unique, so a closed hemisphere of S^2 is (geodesically) convex.

Definition 2.2. A subset $K \subset M$ is weakly (geodesically) convex if any two points $x, y \in K$ can be joined by a length-minimizing geodesic (of K) entirely contained within K .

The difference between the two notions is that a length-minimizing geodesic in a weakly convex set $K \subset M$ need not be length-minimizing in M . On the sphere, we see that weakly convex sets are convex (Prop. 2.1). Since geodesics on S^2 are great circles and length-minimizing geodesics are half great circles, the length of any minimizing geodesic is at most π .

The following proposition characterizes convex regions $K \subset S^2$ whose boundary is a smooth embedded curve. The results are known but we could not find a good single reference, so give the details here. See [CE75], [San44], [Top06] for some material related to this section.

Proposition 2.3. *Let $K \subset S^2$ be a connected open set with boundary $\partial K = \gamma$ a smooth embedded curve. Let N be the interior unit normal vector field along γ and k the geodesic curvature with respect to N . Then the following are equivalent:*

1. K is convex
2. For every $x \in \gamma$, we have $K \subset H_x^+$ where H_x^+ is the open hemisphere with boundary the tangent great circle E_x to $\gamma(x)$ and interior normal $N(x)$
3. γ may be written as the graph over an equator of a smooth function $f = f(\theta)$, $\theta \in S^1$ such that

$$f'' + 2(\tan f)(f')^2 + \cos f \sin f \geq 0$$

4. The boundary curvature k with respect to N is non-negative

Proof. (1 $\Rightarrow$ 2): Let $x \in \gamma$ and E_x be the tangent great circle to $\gamma = \partial K$ at x . With $N(x)$ pointing interior to K , let $H^+(x)$ denote the hemisphere with boundary E_x and interior normal $N(x)$. Then $H_x^+ \cap K \neq \emptyset$. We want to show that $K \subset H_x^+$.

Suppose not. Then there exists $y \in K \cap H_x^-$ where $H_x^- = (H_x^+)^c$. Let σ be the unique (since $y \notin E_x$) length-minimizing geodesic joining x to y . We have that $\sigma(1) = y \in H_x^-$ but also does not intersect K in a neighborhood of x since it points exterior to K ($\langle \sigma'(x), N(x) \rangle < 0$). Since K is convex and $x, y \in K$ we must have $\sigma \subset K$, a contradiction.

(2 $\Rightarrow$ 3): Work on the sphere with spherical polar coordinates

$$(\cos \phi \cos \theta, \cos \phi \sin \theta, \sin \phi).$$

Note the equator is given by $\phi = 0$ (as opposed to the usual convention where $\phi = 0$ is the north pole). For graphs $(\theta, f(\theta))$, the curvature is given by

$$k = \frac{\cos f}{((f')^2 + \cos^2(f))^{3/2}}(f'' + 2(f')^2 \tan f + \sin f \cos f)$$

(see Appendix for calculation). We will show that (2) implies that γ may be written as a graph, and that $k \geq 0$.

First let us see that γ is a graph. Take any tangent great circle E_x with $K \subset H_x^+$, and let p be the center of H_x^+ so that $H_x^+ = \{y \in S^2 : d(p, y) \leq \pi/2\}$. If $p \notin K$, let $y \in \gamma$ be the closest point on $\overline{K}$ to p and let σ be the unique great circle passing through p and y . Then $\sigma \perp \gamma$ at y and $d(p, y) < \pi/2$. Letting E_y be the tangent great circle to γ at y and H_y^+ the hemisphere containing K , we have that K is a subset of the half hemisphere $H_x^+ \cap H_y^+$. Therefore we can rotate $E = E_x$ along σ until $p \in K$ while still maintaining that K is contained in the hemisphere determined by boundary E_x . Thus we may assume that $K \subset H$, H a hemisphere with the center $p \in K$ and boundary E .

Now for each $y \in E$, let σ_y be the minimizing segment of a great circle joining p to y . We claim σ_y intersects γ in precisely one point. Since $p \in K$ with K open and $K \subsetneq \overline{H}$, σ_y intersects γ in at least one point. Let $f(p)$ be the closest such point to p and let $E_{f(p)}$ be the tangent great circle to γ at $f(p)$. Then by assumption, K lies on one side of $E_{f(p)}$, necessarily the side containing p hence σ_y does not intersect γ at any other point. Thus we have expressed γ as the graph of f which is smooth.

$k \geq 0$: This follows since K lies on one side of every tangent great circle, so the local Taylor expansion of γ shows that the curvature vector is interior pointing everywhere.

(3 $\Rightarrow$ 4): As above the condition on f is precisely that $k \geq 0$.

(4 $\Rightarrow$ 1): We prove the contrapositive. Suppose K is not convex. We need to show that there is a $y \in \gamma$ such that $k(y) < 0$.

Since K is not convex, there is a length-minimizing geodesic α meeting $\overline{K}$ only at its endpoints. Moreover, we can choose α to have length strictly less than π , since if not, then any arbitrary great circle must intersect K^C in a connected arc of length at least π and so intersects K in a connected arc of length at most

π . Therefore any $x, y \in K$ may be connected by a length-minimizing geodesic, contradicting that K is not convex.

Let σ_z be the continuous family of geodesic rays of length π starting at $z \in \alpha$, perpendicular to α . There are precisely two such families, and we choose σ_z so that σ_z intersects γ at distance less than π for z near the endpoints of α . Note that on the sphere, for each $z \in \alpha$, we have $\sigma_z \cap \gamma \neq \emptyset$ since the endpoints of α (which also lie on γ) lie on either side of the equator containing σ_z (this is the part that fails in general, e.g. on the cylinder $\mathbb{R} \times S^1$ for a geodesic circle of radius $> \pi/2$). Let $y = y(z) \in \gamma$ be the first point where σ_z meets γ and let $\rho(z) = d(z, y)$.

Then ρ is continuous and attains its maximum at a point z_0 in the interior of α since $\rho = 0$ on the end points of α and by assumption $\rho(z) > 0$ for some $z \in \alpha$. Now we have ρ_{z_0} a geodesic meeting γ orthogonally at y_0 , hence we can solve for z as a function of y near y_0 . The second variation formula (varying y) shows that $k(y_0) \leq 0$. $\square$

From here on we will freely use the results of the above proposition without further comment, and by a convex curve we will mean a closed, embedded curve γ satisfying any of the above conditions. Finally let us note that for γ a convex curve, by approximating γ by convex polygons (with geodesic arcs), it is possible to show that the length of γ , $L(\gamma) \leq 2\pi$, a fact we will employ in section 5. See [Top06] for details.

2.2 Evolution of basic quantities

The following lemma collects basic results about the evolution of certain quantities under the curve shortening flow. These can be found, for instance, in [Gra89], [Gag90], [CZ01].

Lemma 2.4. *Let $\gamma_t = \Gamma(\cdot, t)$ be a solution to the curve shortening flow on an oriented smooth surface M . Denote the covariant derivatives along the map Γ in the space and time directions by ∇_s and ∇_t respectively. Then*

1. $v = \left| \frac{\partial \Gamma}{\partial u} \right|$ evolves by $v_t = -k^2 v$
2. $\left[\frac{\partial}{\partial t}, \frac{\partial}{\partial s} \right] = k^2 \frac{\partial}{\partial s}$
3. $\nabla_t T = k_s N$ and $\nabla_t N = -k_s T$
4. The evolution of curvature is $k_t = k_{ss} + k^3 + Kk$
5. $\frac{d}{dt} \int_{\gamma_t} ds = - \int_{\gamma_t} k^2 ds$
6. $\frac{d}{dt} \int_{\gamma_t} k ds = \int_{\gamma_t} K k ds$

where K is the Gauss curvature of the surface M .

Proof. For (1), we use the fact that $\frac{\partial}{\partial u}$ and $\frac{\partial}{\partial t}$ commute since u and t are independent variables, the Frenet equations, and the evolution equation to calculate

$$\begin{aligned} \frac{\partial}{\partial t}(v^2) &= \frac{\partial}{\partial t} \left\langle \frac{\partial \Gamma}{\partial u}, \frac{\partial \Gamma}{\partial u} \right\rangle = 2 \left\langle \frac{\partial \Gamma}{\partial u}, \frac{\partial}{\partial t} \left(\frac{\partial \Gamma}{\partial u} \right) \right\rangle = 2 \left\langle \frac{\partial \Gamma}{\partial u}, \frac{\partial}{\partial u} \left(\frac{\partial \Gamma}{\partial t} \right) \right\rangle \\ &= 2 \left\langle vT, \frac{\partial}{\partial u}(kN) \right\rangle = 2 \left\langle vT, \frac{\partial k}{\partial u} N - vk^2 T \right\rangle = -2v^2 k^2 \end{aligned}$$

and the result follows.

For (2), let $f : M^1 \times I \rightarrow \mathbb{R}$. Use $\frac{\partial}{\partial s} = \frac{1}{v} \frac{\partial}{\partial u}$.

$$\begin{aligned} \frac{\partial}{\partial t} \left(\frac{\partial f}{\partial s} \right) &= \frac{\partial}{\partial t} \left(\frac{1}{v} \frac{\partial f}{\partial u} \right) = -\frac{1}{v^2} \frac{\partial v}{\partial t} \frac{\partial f}{\partial u} + \frac{1}{v} \frac{\partial}{\partial t} \left(\frac{\partial f}{\partial u} \right) \\ &= -\frac{1}{v^2} (-k^2 v) \frac{\partial f}{\partial u} + \frac{1}{v} \frac{\partial}{\partial u} \left(\frac{\partial f}{\partial t} \right) = k^2 \frac{\partial f}{\partial s} + \frac{\partial}{\partial s} \left(\frac{\partial f}{\partial t} \right). \end{aligned}$$

For (3)

$$\begin{aligned} \nabla_t T &= \nabla_t \frac{\partial \Gamma}{\partial s} = \nabla_s \frac{\partial \Gamma}{\partial t} + k^2 \frac{\partial \Gamma}{\partial s} \\ &= \nabla_s(kN) + k^2 T = k_s N + k \nabla_s N + k^2 T. \end{aligned}$$

The last two terms cancel by the Frenet formulas. Then since $\langle T, N \rangle = 0$

$$\begin{aligned} 0 &= \frac{\partial}{\partial t} \langle T, N \rangle = \langle \nabla_t T, N \rangle + \langle T, \nabla_t N \rangle = \langle k_s N, N \rangle + \langle T, \nabla_t N \rangle \\ &\Rightarrow -k_s = \langle T, \nabla_t N \rangle \end{aligned}$$

so $\nabla_t N = -k_s T$.

For (4), we use the Frenet formulas, some of the above results, and the following equation relating covariant differentiation with respect to parameters s and t

$$\nabla_t \nabla_s = \nabla_s \nabla_t + k^2 \nabla_s + kR(T, N)$$

and compute

$$\begin{aligned} \frac{\partial}{\partial t} k &= \frac{\partial}{\partial t} \langle \nabla_s T, N \rangle = \langle \nabla_t \nabla_s T, N \rangle + \langle \nabla_s T, \nabla_t N \rangle \\ &= \langle \nabla_s \nabla_t T + k^2 \nabla_s T + kR(T, N)T, N \rangle + \langle kN, -k_s T \rangle \\ &= \langle \nabla_s (k_s N), N \rangle + k^3 + Kk \\ &= \langle k_{ss} N + k_s (-kT), N \rangle + k^3 + Kk \\ &= k_{ss} + k^3 + Kk. \end{aligned}$$

For (5) and (6) we use $(ds)_t = -k^2 ds$. (6) is calculated below; (5) is obtained immediately.

$$\begin{aligned} \frac{d}{dt} \int_{\gamma_t} k ds &= \int_{\gamma_t} (k_t ds + k(ds)_t) \\ &= \int_{\gamma_t} (k_{ss} + k^3 + Kk) ds + k(-k^2 ds) \\ &= \int_{\gamma_t} (k_{ss} + Kk) ds \\ &= \int_{\gamma_t} Kk ds \end{aligned}$$

since $\int_{\gamma_t} k_{ss} ds = 0$

□

In particular we record the evolution equations for the quantities (4) and (6) in Lemma 2.4 when $M = S^2$, whose Gauss curvature $K \equiv 1$.

$$k_t = k_{ss} + k^3 + k \quad (2.1)$$

$$\frac{d}{dt} \int_{\gamma_t} k ds = \int_{\gamma_t} k ds \quad (2.2)$$

2.3 Maximum principle

We cite a version of the maximum principle (see for example, Theorem 4.4 in [CK01]) that will be referenced in what follows as just *the (parabolic) maximum principle*. Consider the semilinear heat equation

$$\frac{\partial v}{\partial t} = \Delta_{g(t)} v + \langle X, \nabla v \rangle + F(v) \quad (2.3)$$

where $g(t)$ is a 1-parameter family of metrics, $X(t)$ is a 1-parameter family of vector fields, and $F : \mathbb{R} \rightarrow \mathbb{R}$ is a locally Lipschitz function. We say u is a *supersolution* of (2.3) if

$$\frac{\partial u}{\partial t} \geq \Delta_{g(t)} u + \langle X, \nabla u \rangle + F(u).$$

Theorem 2.5. (*Maximum principle; ODE gives pointwise bounds for PDE*) *Let $u : M^n \times [0, T) \rightarrow \mathbb{R}$ be a C^2 supersolution to (2.3) on a closed manifold. Suppose there exists $C_1 \in \mathbb{R}$ such that $u(x, 0) \geq C_1$ for all $x \in M^n$, and let φ_1 be the solution to the associated ordinary differential equation*

$$\frac{d\varphi_1}{dt} = F(\varphi_1)$$

satisfying $\varphi_1(0) = C_1$. Then

$$u(x, t) \geq \varphi_1(t)$$

for all $x \in M^n$ and $t \in [0, T)$ such that $\varphi_1(t)$ exists.

We will also use an application of the maximum principle (see for example [CZ01], [Cho97], [PW67]) in the final section of the argument so will state it here.

Theorem 2.6. *Let $F_1, F_2 : [a, b] \times [0, T) \rightarrow S^2$ be two compact solutions to the curve shortening flow which agree at their endpoints, are initially disjoint away from their endpoints, and which meet transversely at their endpoints. That is, $F_1(a, t) = F_2(a, t)$ and $F_1(b, t) = F_2(b, t)$ for $t \in [0, T)$, $F_1((a, b), 0) \cap F_2((a, b), 0) = \emptyset$, and there is some positive angle between tangents $F_1'(a, 0)$ and $F_2'(a, 0)$ and between $F_1'(b, 0)$ and $F_2'(b, 0)$. Then $F_1((a, b), t) \cap F_2((a, b), t) = \emptyset$ for all $t \in [0, T)$.*

Remark 2.7. The proof is a standard application of the maximum principle and the Hopf boundary point lemma. The former prevents the curves from touching away from the endpoints, and the latter ensures the curves remain separated near the endpoints.

2.4 The shrinking circles

We can write down the shrinking circles ancient solution of the curve shortening flow on the unit sphere explicitly. On S^2 we have that the evolution of curvature under the flow is given by

$$\partial_t k = \partial_{ss} k + k^3 + k.$$

Suppose we start with a circle of latitude (which has constant curvature so $\partial_s k = \partial_{ss} k = 0$) in the upper hemisphere. We can solve the separable ODE $\partial_t k = k^3 + k$ for curvature as a function of time as it rises towards the north pole to obtain an ancient solution described by its curvature

$$k(t) = \frac{1}{\sqrt{e^{-2t} - 1}}$$

which exists for $t \in (-\infty, 0)$.

We could also describe the shrinking circles solution as a curve on S^2 in coordinates (θ, ϕ) with where $\phi = 0$ is the equator and $\phi = \pi/2$ corresponds to the north pole. Write a circle of latitude in the upper hemisphere as a graph over the

equator as $(\theta, f(\theta, t))$ and solve the curve shortening flow equation

$$\partial_t(\theta, f(t)) = kN$$

for f as a function of time, i.e. solve $(0, \partial_t f(t)) = \tan(f(t))(0, 1)$. Here we used (A.2) and (A.4) with $f' = f'' = 0$ to obtain $k = \tan f$ and $N = (0, 1)$. We solve the separable ODE $\partial_t f = \tan f$ to get

$$f(t) = \arcsin(e^t)$$

for $t \in (-\infty, 0)$. We see that $f(t) = \arcsin(e^t) \rightarrow \pi/2$ as $t \rightarrow 0$ and $f(t) \rightarrow 0$ as $t \rightarrow -\infty$.

Chapter 2 includes material from: P. Bryan, J. Louie, “Classification of Convex Ancient Solutions to Curve Shortening Flow on the Sphere”, *preprint*, *arXiv: 1408.5523*.

Chapter 3

Harnack Inequality

As for ancient solutions to curve shortening flow in the plane, there is a Harnack inequality for convex, embedded, ancient solutions to the curve shortening flow on S^2 . This allows us to show that for such solutions, the curvature k satisfies $k_t \geq 0$. In this section subscripts denote differentiation.

Proposition 3.1. (*Harnack Inequality*) *For an immersed solution to curve shortening flow on the unit sphere S^2 with curvature $k > 0$ and defined on time interval $[t_0, T)$ we have, in terms of the arc length parameter s , that*

$$(\log k)_{ss} + k^2 + \frac{1}{2(t - t_0)} \geq 0 \quad (3.1)$$

Proof. Using the evolution equation for the curvature and the commutator formula from Lemma (2.4) we have

$$\begin{aligned} k_{st} &= k_{ts} + k^2 k_s = (k_{sss} + 3k^2 k_s + k_s) + k^2 k_s \\ &= k_{sss} + 4k^2 k_s + k_s \end{aligned} \quad (3.2)$$

$$\begin{aligned} k_{sst} &= k_{tss} + 2k k_s^2 + 2k^2 k_{ss} = (k_{ssss} + 3k^2 k_{ss} + 6k k_s^2 + k_{ss}) + 2k k_s^2 + 2k^2 k_{ss} \\ &= k_{ssss} + 5k^2 k_{ss} + 8k k_s^2 + k_{ss} \end{aligned} \quad (3.3)$$

Define the quantity Q by

$$Q = (\log k)_{ss} + k^2 \quad (3.4)$$

and take derivatives with respect to s to get

$$Q_s = 2kk_s + \frac{2k_s^3}{k^3} - \frac{3k_sk_{ss}}{k^2} + \frac{k_{sss}}{k} \quad (3.5)$$

$$Q_{ss} = 2k_s^2 - \frac{6k_s^4}{k^4} + 2kk_{ss} + \frac{12k_s^2k_{ss}}{k^3} - \frac{3k_{ss}^2}{k^2} - \frac{4k_sk_{sss}}{k^2} + \frac{k_{ssss}}{k}. \quad (3.6)$$

Next we compute Q_t . We have

$$\begin{aligned} Q_t &= -\frac{k_{ss}}{k^2}k_t + \frac{k_{sst}}{k} + 2k^{-3}k_tk_s^2 - k^{-2}(2k_sk_{st}) + 2kk_t \\ &= -\frac{k_{ss}^2}{k^2} + 6kk_{ss} + \frac{k_{ssss}}{k} + 2k_s^2 + \frac{2k_s^2k_{ss}}{k^3} - \frac{2k_sk_{sss}}{k^2} + 2k^4 + 2k^2 \end{aligned} \quad (3.7)$$

after using (2.1), (3.2), and (3.3) to replace k_t, k_{st}, k_{sst} in (3.7). We look for Q_s, Q_{ss} and $f(Q)$ terms, where $f(Q)$ is some function of Q :

$$\begin{aligned} Q_t &= \left(2k_s^2 - \frac{6k_s^4}{k^4} + 2kk_{ss} + \frac{12k_s^2k_{ss}}{k^3} - \frac{3k_{ss}^2}{k^2} - \frac{4k_sk_{sss}}{k^2} + \frac{k_{ssss}}{k} \right) \\ &\quad + \frac{2k_sk_{sss}}{k^2} + \frac{2k_{ss}^2}{k^2} - \frac{10k_s^2k_{ss}}{k^3} + 4kk_{ss} + \frac{6k_s^4}{k^4} + 2k^4 + 2k^2 \\ &= Q_{ss} + \frac{2k_sk_{sss}}{k^2} + \frac{2k_{ss}^2}{k^2} - \frac{10k_s^2k_{ss}}{k^3} + 4kk_{ss} + \frac{6k_s^4}{k^4} + 2k^4 + 2k^2 \\ &= Q_{ss} + \left(2k^4 - 4k_s^2 + \frac{2k_s^4}{k^4} + 4kk_{ss} - \frac{4k_s^2k_{ss}}{k^3} + \frac{2k_{ss}^2}{k^2} \right) \\ &\quad + 4k_s^2 + \frac{4k_s^4}{k^4} - \frac{6k_s^2k_{ss}}{k^3} + \frac{2k_sk_{sss}}{k^2} + 2k^2 \\ &= Q_{ss} + 2Q^2 + 4k_s^2 + \frac{4k_s^4}{k^4} - \frac{6k_s^2k_{ss}}{k^3} + \frac{2k_sk_{sss}}{k^2} + 2k^2 \\ &= Q_{ss} + 2Q^2 + \left(\frac{2k_s}{k} \right) \left(2kk_s + \frac{2k_s^3}{k^3} - \frac{3k_sk_{ss}}{k^2} + \frac{k_{sss}}{k} \right) + 2k^2 \\ &= Q_{ss} + \left(\frac{2k_s}{k} \right) Q_s + 2Q^2 + 2k^2. \end{aligned}$$

This implies

$$Q_t \geq Q_{ss} + \left(\frac{2k_s}{k} \right) Q_s + 2Q^2. \quad (3.8)$$

Let $q(t) = -\frac{1}{2(t-t_0)}$. Then $q(t)$ solves the associated ODE

$$\frac{dq}{dt} = 2q^2$$

with condition $\lim_{t \rightarrow t_0} q(t) = -\infty$ and so the maximum principle gives

$$Q(s, t) \geq q(t)$$

for $t \in [t_0, T)$. □

Corollary 3.2. (*Harnack for ancient solutions*) *For a convex ancient solution to curve shortening flow on the sphere, we have*

$$k_t \geq 0 \tag{3.9}$$

Proof. Since the solution is ancient, for any $t_0 \in (-\infty, 0)$, the solution exists on $[t_0, 0)$ so that (3.1) holds. That is,

$$(\log k)_{ss} + k^2 + \frac{1}{2(t - t_0)} = \frac{k_{ss}}{k} - \frac{k_s^2}{k^2} + k^2 + \frac{1}{2(t - t_0)} \geq 0$$

so

$$\frac{k_{ss}}{k} + k^2 + \frac{1}{2(t - t_0)} = \frac{k_{ss} + k^3}{k} + \frac{1}{2(t - t_0)} \geq 0 \tag{3.10}$$

which implies

$$\frac{k_{ss} + k^3 + k}{k} + \frac{1}{2(t - t_0)} \geq 0 \tag{3.11}$$

$$\Rightarrow \frac{k_t}{k} + \frac{1}{2(t - t_0)} \geq 0. \tag{3.12}$$

The corollary follows from taking the limit as $t_0 \rightarrow -\infty$. □

Chapter 3 includes material from: P. Bryan, J. Louie, “Classification of Convex Ancient Solutions to Curve Shortening Flow on the Sphere”, *preprint*, *arXiv: 1408.5523*.

Chapter 4

Derivative Estimates

Next we obtain derivative estimates for the curvature of a convex ancient solution of curve shortening flow on S^2 .

Proposition 4.1. (*Uniform derivative estimates*) *Suppose we have an embedded, convex ancient solution to the curve shortening flow on the unit sphere S^2 . Then for every integer $m \geq 0$ there exists a constant C_m such that*

$$|k^{(m)}| \leq C_m \text{ on } S^1 \times (-\infty, -1]. \quad (4.1)$$

Proof. Since $k > 0$ and the Harnack inequality gives $k_t \geq 0$, there exists $C_0 > 0$ such that

$$0 < k \leq C_0 \text{ on } S^1 \times (-\infty, -1]. \quad (4.2)$$

We obtain an estimate for $|k_s(s, t)|$ at $t_0 \leq -1$. Let $t \in [t_0 - 1, t_0]$, and let

$$\tilde{t} = t - (t_0 - 1)$$

so that $\tilde{t} \in [0, 1]$. Using the evolution equation (2.1) $k_t = k_{ss} + k^3 + k$ and equation (3.2) $k_{st} = (k_{sss} + 4k^2k_s + k_s)$, compute the following:

$$\begin{aligned} (k^2)_t &= 2kk_t = (k^2)_{ss} - 2k_s^2 + 2k^4 + 2k^2 \\ (k_s^2)_t &= 2k_sk_{st} = 2k_s(k_{sss} + 4k^2k_s + k_s) = 2k_sk_{sss} + 8k^2k_s^2 + 2k_s^2 \\ &= (k_s^2)_{ss} - 2k_{ss}^2 + 8k^2k_s^2 + 2k_s^2. \end{aligned}$$

Consider the quantity

$$\tilde{t}k_s^2 + Mk^2 = (t - (t_0 - 1))k_s^2 + Mk^2 \quad (4.3)$$

where M is a constant to be chosen. Compute

$$\begin{aligned} (\tilde{t}k_s^2 + Mk^2)_t &= k_s^2 + \tilde{t}(k_s^2)_t + M(k^2)_t \\ &= k_s^2 + \tilde{t}((k_s^2)_{ss} - 2k_{ss}^2 + 8k^2k_s^2 + 2k_s^2) + M((k^2)_{ss} - 2k_s^2 + 2k^4 + 2k^2) \\ &= (\tilde{t}k_s^2 + Mk^2)_{ss} + k_s^2 - 2\tilde{t}k_{ss}^2 + 8\tilde{t}k^2k_s^2 + 2\tilde{t}k_s^2 - 2Mk_s^2 + 2Mk^4 + 2Mk^2 \\ &= (\tilde{t}k_s^2 + Mk^2)_{ss} - 2\tilde{t}k_{ss}^2 + (8\tilde{t}k^2 + 2\tilde{t} + 1 - 2M)k_s^2 + 2Mk^4 + 2Mk^2. \end{aligned}$$

Then

$$\begin{aligned} (\tilde{t}k_s^2 + Mk^2)_t &= (\tilde{t}k_s^2 + Mk^2)_{ss} - 2\tilde{t}k_{ss}^2 + (8\tilde{t}k^2 + 2\tilde{t} + 1 - 2M)k_s^2 + 2Mk^4 + 2Mk^2 \\ &\leq (\tilde{t}k_s^2 + Mk^2)_{ss} + (8C_0^2 + 2 + 1 - 2M)k_s^2 + 2MC_0^4 + 2MC_0^2 \end{aligned}$$

since $\tilde{t} \in [0, 1]$ so $-2\tilde{t}k_{ss}^2 \leq 0$, and $|k| \leq C_0$ on $(-\infty, -1]$. Then choose M so large that $8C_0^2 + 3 - 2M \leq 0$ i.e. so that

$$M \geq (8C_0^2 + 3)/2.$$

Notice that M only depends on the uniform bound on k and not on t_0 . So we have

$$(\tilde{t}k_s^2 + Mk^2)_t \leq (\tilde{t}k_s^2 + Mk^2)_{ss} + 2MC_0^4 + 2MC_0^2$$

for $\tilde{t} \in [0, 1]$ or $t \in [t_0 - 1, t_0]$.

Now we wish to apply the maximum principle. Let

$$u(s, t) = \tilde{t}k_s(s, t)^2 + Mk(s, t)^2.$$

At $\tilde{t} = 0$ or $t = t_0 - 1$ we have $u = Mk^2 \leq MC_0^2$ since $|k| \leq C_0$ on $S^1 \times (-\infty, -1]$.

The associated ordinary differential equation is

$$\frac{dq}{dt} = 2MC_0^4 + 2MC_0^2$$

with condition $q(t_0 - 1) = MC_0^2$ so

$$q(t) = (2MC_0^4 + 2MC_0^2)t + MC_0^2 - (2MC_0^4 + 2MC_0^2)(t_0 - 1).$$

Then $u(s, t) \leq q(t)$ for $s \in S^1, t \in [t_0 - 1, t_0]$ i.e.

$$\begin{aligned} (t - (t_0 - 1))k_s^2 + Mk^2 &\leq (2MC_0^4 + 2MC_0^2)t + MC_0^2 - (2MC_0^4 + 2MC_0^2)(t_0 - 1) \\ \Rightarrow (t - (t_0 - 1))k_s^2 &\leq (2MC_0^4 + 2MC_0^2)t + MC_0^2 - (2MC_0^4 + 2MC_0^2)(t_0 - 1) \end{aligned}$$

so at $t = t_0$ we have

$$\begin{aligned} k_s(s, t_0)^2 &\leq (2MC_0^4 + 2MC_0^2)t_0 + MC_0^2 - (2MC_0^4 + 2MC_0^2)(t_0 - 1) \\ &= MC_0^2 + 2MC_0^4 + 2MC_0^2 = C_1^2 \end{aligned}$$

Thus, recalling $t_0 \leq -1$ was arbitrary, we have that for all $t_0 \leq -1$, $|k_s| \leq C_1$ as C_1 is a uniform bound, independent of t_0 .

Estimates for higher derivatives can be obtained by an induction argument by assuming that we have estimated $|k^{(j)}|$ for $1 \leq j < m$ (this is standard; see for example, [Ham95a]). $\square$

Chapter 4 includes material from: P. Bryan, J. Louie, “Classification of Convex Ancient Solutions to Curve Shortening Flow on the Sphere”, *preprint*, *arXiv: 1408.5523*.

Chapter 5

Backward Limit is an Equator

Using the estimates on curvature and its derivatives obtained in the previous section, we show that an ancient solution γ_t converges to an equator $\gamma_{-\infty}$ as $t \rightarrow -\infty$. First we see that the curvature and its derivatives go to 0 backwards in time.

5.1 Limit of curvature and its derivatives

Lemma 5.1. *Let γ_t be an ancient solution to the curve shortening flow. Then*

$$\lim_{t \rightarrow -\infty} \int_{\gamma_t} k ds = 0. \quad (5.1)$$

Proof. One way to see the result is to use Gauss-Bonnet. The Gauss-Bonnet formula for the area A enclosed by γ_t on S^2 gives

$$\int_{\gamma_t} k ds = 2\pi - A.$$

Taking the time derivative of the above equation and using that under curve shortening flow on the sphere, we have $k_t = k_{ss} + k^3 + k$ and $(ds)_t = -k^2 ds$ we have that

$$\begin{aligned} \frac{d}{dt} \int_{\gamma_t} k ds &= \int_{\gamma_t} k_t ds + \int_{\gamma_t} k(-k^2 ds) = \int_{\gamma_t} k_{ss} + k ds \\ &= \int_{\gamma_t} k ds = 2\pi - A \end{aligned}$$

where in the second line we used equation (2.2). Therefore the area enclosed by γ_t solves

$$\frac{dA}{dt} = A - 2\pi$$

and hence $A(t) = 2\pi + (A(0) - 2\pi)e^t$. In our case with condition $\lim_{t \rightarrow 0} A(t) = 0$ (under curve shortening flow, the curve shrinks to a point) the formula for the area enclosed by γ_t becomes

$$A(t) = 2\pi(1 - e^t) \text{ with } t \in (-\infty, 0). \quad (5.2)$$

We see that $\lim_{t \rightarrow -\infty} A(t) = 2\pi$. Then go back to the formula we had from Gauss-Bonnet $\int_{\gamma_t} k ds = 2\pi - A(t)$ and take the limit as $t \rightarrow -\infty$. $\square$

Remark 5.2. We note that we also could have shown (5.1) using equation (2.2) directly, without using Gauss-Bonnet, since (2.2) implies the quantity $\int_{\gamma_t} k ds$ is an exponential Ae^t which goes to 0 as $t \rightarrow -\infty$.

Combining the results so far, we get the following proposition.

Proposition 5.3. *For a convex ancient solution to curve shortening flow on the sphere, we have for every integer $m \geq 0$*

$$\max_{s \in S^1} |k^{(m)}(s, t)| \rightarrow 0 \text{ as } t \rightarrow -\infty. \quad (5.3)$$

Proof. First we show the case for $m = 0$, i.e. we show

$$\max_{s \in S^1} k(s, t) \rightarrow 0 \text{ as } t \rightarrow -\infty. \quad (5.4)$$

We argue by contradiction. Suppose the proposition is false. Then there exists $\epsilon > 0$ and a sequence $t_i \rightarrow -\infty$ such that $\max_{s \in S^1} k(s, t_i) \geq \epsilon$ for all i . So there exists s_i such that $k(s_i, t_i) \geq \epsilon$, and $|k_s| \leq C$ implies

$$k(s, t_i) \geq \epsilon - C|s - s_i| \geq \epsilon/2$$

for all s such that $|s - s_i| \leq \frac{\epsilon}{2C}$. Then

$$\int_{\gamma} k(s, t) ds \geq \int_{[s_i - \epsilon/2C, s_i + \epsilon/2C]} k(s, t) ds \geq \frac{\epsilon}{C} \frac{\epsilon}{2} = \frac{\epsilon^2}{2C}$$

a contradiction to Lemma 5.1.

Using the above result, we go back to the computations in the proof used to obtain the bounds on the derivatives of curvature to show the cases $m > 0$; the bounds get better now that we know $k \rightarrow 0$ as $t \rightarrow -\infty$. For the case $m = 1$, using (5.4) we have that for every $\epsilon > 0$, there is a $t_1 = t_1(\epsilon)$ such that

$$0 < k \leq \epsilon \text{ on } (-\infty, t_1]. \quad (5.5)$$

Pick $t_0 \leq t_1$. As in the argument to obtain derivative estimates on curvature, we estimate $|k_s(s, t)|$ at t_0 . Let $t \in [t_0 - 1, t_0]$, $\tilde{t} = t - (t - t_0)$, and consider the quantity (4.3) $(\tilde{t}k_s^2 + Mk^2 = (t - (t_0 - 1))k_s^2 + Mk^2)$ where

$$M \geq (8\epsilon^2 + 3)/2$$

The same calculation as before gives

$$\begin{aligned} (\tilde{t}k_s^2 + Mk^2)_t &= (\tilde{t}k_s^2 + Mk^2)_{ss} - 2\tilde{t}k_{ss}^2 + (8\tilde{t}k^2 + 2\tilde{t} + 1 - 2M)k_s^2 + 2Mk^4 + 2Mk^2 \\ &\leq (\tilde{t}k_s^2 + Mk^2)_{ss} + (8\epsilon^2 + 2 + 1 - 2M)k_s^2 + 2M\epsilon^4 + 2M\epsilon^2 \end{aligned}$$

since $\tilde{t} \in [0, 1]$ and (5.5). Thus

$$(\tilde{t}k_s^2 + Mk^2)_t \leq (\tilde{t}k_s^2 + Mk^2)_{ss} + 2M\epsilon^4 + 2M\epsilon^2 \quad (5.6)$$

After applying the maximum principle we get that at $t = t_0$

$$\begin{aligned} k_s(s, t_0)^2 &\leq (2M\epsilon^4 + 2M\epsilon^2)t_0 + M\epsilon^2 - (2M\epsilon^4 + 2M\epsilon^2)(t_0 - 1) \\ &= M\epsilon^2 + 2M\epsilon^4 + 2M\epsilon^2 \end{aligned}$$

so

$$|k_s| \leq \sqrt{(2M\epsilon^4 + 3M\epsilon^2)}$$

for any $t_0 \leq t_1 = t_1(\epsilon)$. Then $|k_s| \rightarrow 0$ as $t \rightarrow -\infty$ by taking $\epsilon \rightarrow 0$, as $t_1(\epsilon) \rightarrow -\infty$ as $\epsilon \rightarrow 0$. The results for higher derivatives are obtained similarly. $\square$

5.2 Convergence to equator

Now we may show that the backwards limit of γ_t is an equator. First, we show subsequential convergence.

Lemma 5.4. *Let γ_t be an embedded convex ancient solution to the curve shortening flow on S^2 . Then there is a sequence $t_k \rightarrow -\infty$ such that $\gamma_{t_k} \rightarrow \gamma_{-\infty}$ as $k \rightarrow \infty$, with $\gamma_{-\infty}$ a great circle.*

Proof. Notice that an upper bound on $L(\gamma_t) = \int |\gamma'_t| du$, the length of γ_t , gives us a bound on $|\gamma'_t|$. By [Top06], we have that $L(\gamma_t) \leq 2\pi$. Prop. 5.3, which shows the curvature and its derivatives converge to zero, gives that the second and higher derivatives of γ_t are uniformly bounded. The family γ_t is uniformly bounded as the curves lie in a compact region in S^2 . The Arzela-Ascoli theorem gives a sequence $t_k \rightarrow -\infty$ such that γ_{t_k} converges smoothly to a limit curve $\gamma_{-\infty}$.

Since $\frac{d}{dt}L(\gamma_t) = - \int k^2 ds < 0$, we have that $L(\gamma_t)$ is bounded below by $L(\gamma_{-1})$ for all $t \leq -1$, so that the curve remains immersed backwards in time. Thus $\gamma_{-\infty}$ is a closed immersed curve with zero curvature - an equator. $\square$

Next we show that the limit is unique and that the curve remains in a fixed hemisphere under the flow.

Corollary 5.5. *The equator $\gamma_{-\infty}$ is unique, and the solution γ_t lies in one of the hemispheres $H_{-\infty}^{\pm}$ defined by $\gamma_{-\infty}$ for all $t \in (-\infty, 0)$.*

Proof. Since $k > 0$, Gauss-Bonnet implies $A(t) < 2\pi$ for all t , so we may unambiguously define an interior to region Ω_t with area $A(t)$ enclosed by $\gamma_t = \partial\Omega_t$ and the curvature vector points towards the interior of Ω_t . Thus if $t_1 < t_2$, we have $\Omega_{t_2} \subsetneq \Omega_{t_1}$. For our sequence $t_k \rightarrow -\infty$, suppose that for some j , we have Ω_{t_j} is not contained in either hemisphere $H_{-\infty}^{\pm}$ determined by $\gamma_{-\infty}$, so that it contains points $x_j^{\pm} \in \Omega_{t_j}$ in both hemispheres: $x_j^{\pm} \in H_{-\infty}^{\pm}$. Since $\gamma_{t_k} \rightarrow \gamma_{-\infty}$ we may choose $t_k < t_j$ so γ_{t_k} is sufficiently close to $\gamma_{-\infty}$ so that x_j^+ and x_j^- lie on opposite sides of γ_{t_k} , contradicting that we must have $\Omega_{t_j} \subsetneq \Omega_{t_k}$. Thus Ω_{t_k} lies in either $H_{-\infty}^+$ or $H_{-\infty}^-$ for every k . Now for any $t \in (-\infty, 0)$, choose $t_k < t$ so that $\Omega_t \subsetneq \Omega_{t_k}$. The latter lies in a hemisphere, and hence so does Ω_t .

Lastly, suppose there is another sequence $t'_k \rightarrow -\infty$ with $\gamma_{t'_k}$ converging to a different equator. This equator has points in both hemispheres defined by $\gamma_{-\infty}$, and hence $\gamma_{t'_k}$ also has points in both hemispheres for time sufficiently negative, a contradiction. $\square$

Remark 5.6. Any closed, embedded curve on S^2 with $k > 0$ must lie in a hemisphere ([Top06]). The result above shows that under the curve shortening flow, an embedded convex ancient solution remains in a fixed hemisphere.

Next we extend subsequential convergence to full convergence.

Theorem 5.7. *An embedded convex ancient solution γ_t to the curve shortening flow on S^2 converges to a great circle $\gamma_{-\infty}$ as $t \rightarrow -\infty$. The convergence is in C^∞ .*

Proof. First we have C^0 convergence (uniform convergence of the functions γ_t). Given $\epsilon > 0$, choose t_k from our subsequence such that $\|\gamma_{t_k} - \gamma_{-\infty}\|_{C^0} < \epsilon$. Then for any $t < t_k$, both γ_{t_k} and γ_t lie in the same hemisphere with $\Omega_{t_k} \subsetneq \Omega_t$, so γ_t lies between γ_{t_k} and $\gamma_{-\infty}$ and so $\|\gamma_t - \gamma_{-\infty}\|_{C^0} < \epsilon$.

Next we have C^1 convergence. Since γ_{t_k} converges to equator $\gamma_{-\infty}$, $L(\gamma_{t_k}) \rightarrow 2\pi$ as $t_k \rightarrow -\infty$. $L(\gamma_t)$ is monotone increasing backwards in time as $\frac{d}{dt}L(\gamma_t) = -\int k^2 ds < 0$, so $L(\gamma_t) \rightarrow 2\pi$ as $t \rightarrow -\infty$. Parametrizing γ_t on $[0, 1]$ with constant speed gives $|\gamma'_t| = L(\gamma_t) \rightarrow 2\pi$ and so C^1 convergence up to diffeomorphism follows.

Smooth convergence up to diffeomorphism follows since curvature k (like γ'') and all its derivatives converge to 0. $\square$

Chapter 5 includes material from: P. Bryan, J. Louie, “Classification of Convex Ancient Solutions to Curve Shortening Flow on the Sphere”, *preprint*, *arXiv: 1408.5523*.

Chapter 6

Ancient Solutions are Shrinking Round Circles

In this chapter we use a parabolic version of the Aleksandrov reflection technique inspired by [Cho97], [CG96] to classify closed, convex, embedded ancient solutions to the curve shortening flow on S^2 as the shrinking round circles.

6.1 Aleksandrov reflection

The idea of Aleksandrov reflection is as follows: Suppose we have a nonlinear heat-type (i.e. second order parabolic) equation on a space with a lot of symmetry, such as Euclidean space or the unit sphere, and assume that the equation is invariant under reflections of the space, so that given a solution, its composition with a reflection is also a solution. If the solution on one side of a plane of reflection reflects above (defined for our problem later) the solution on the other side at a given time, then by using the maximum principle and the Hopf boundary point lemma, we obtain that the solution on one side reflects above the solution on the other side for all future times as long as the solution exists.

In the elliptic case, such as embedded constant mean curvature surfaces in $\mathbb{R}^3$, one proves that these surfaces satisfy exact symmetry in all directions. Hence, an embedded constant mean curvature surface in $\mathbb{R}^3$ must be a round sphere. This is a result due to Aleksandrov. Here, one moves planes with a fixed normal until

the reflected solution and solution on one side either touch at a point in their interior or have a zero angle at their common boundary. By the strong maximum principle and the Hopf boundary point lemma, one concludes that the solution and its reflection coincide, i.e., are identically equal. This is how one proves symmetry. In this elliptic case, since one moves the planes until symmetry, the method is also called the “method of moving planes”.

In contrast, in the parabolic case, the case we are studying, we consider fixed planes and prove that approximate symmetry, in the sense that the solution on one side reflects above the solution on that side, is preserved forward in time. Since the backward limit of our curve possesses exact symmetry, by applying this method starting at times arbitrarily far back, each with respect to a fixed plane, we can prove that exact symmetry is obtained.

Now we describe the set up for our argument. Let S^2 be the round unit sphere with the standard embedding in $\mathbb{R}^3$, and let $\gamma(t) = \gamma_t$ be a closed, convex, embedded ancient solution of the curve shortening flow on S^2 . By performing rotations (isometries) if necessary, we may assume that the equator

$$\gamma_{-\infty} := \{z \in S^2 : z = 0\}$$

is the backward limit as $t \rightarrow -\infty$ of γ_t .

Let P_ν be the plane through the origin with unit normal $\nu \in \mathbb{R}^3$ where $\nu \cdot (0, 0, 1) < 0$ so that ν points below the equator. Then P_ν intersects $\gamma_{-\infty}$ in two antipodal points. Let

$$\delta(\nu) = \angle(\nu, \{z = 0\}) \in (0, \pi/2)$$

be the angle between ν and $\{z = 0\}$. The set $\{\nu : \delta(\nu) = \delta_0\}$ for a fixed δ_0 is a compact set parametrized by S^1 with ν being rotated around the z -axis. Given ν , let

$$H_\nu^+ = \{X \in S^2 : X \cdot \nu < 0\}$$

$$H_\nu^- = \{X \in S^2 : X \cdot \nu > 0\}$$

so $H_\nu^\pm$ are hemispheres with boundary equator $S^2 \cap P_\nu$.

Figure 6.1: Plane of reflection P_ν passing through the origin with unit normal ν .

For a curve γ on S^2 let

$$\gamma_\nu^\pm = \gamma \cap H_\nu^\pm$$

so that γ_ν^+ is the portion of the curve on the side of P_ν where ν points. Reflection across P_ν is given by the map $R_\nu : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ with

$$R_\nu(X) = X - 2(X \cdot \nu)\nu$$

This is an isometry of $\mathbb{R}^3$ preserving S^2 and is hence also an isometry of S^2 . We note R_ν takes $H_\nu^\pm$ to $H_\nu^\mp$. When the choice of ν is clear we may drop it from the notation above.

Now now give an overview of our Aleksandrov reflection argument with the setup and notation above. Because γ_t converges to an equator $\gamma_{-\infty}$, one may go far back enough in time so that when γ_t^+ , the part of the solution curve which lies in H_ν^+ , is reflected across P_ν , the reflected portion $R(\gamma_t^+)$ lies above γ_t^- in H_ν^- for all earlier times. As reflection is an isometry, we have that both $R(\gamma_t)$ and γ_t are solutions of the curve shortening flow, and one may apply the maximum principle to conclude that $R(\gamma_t^+)$ lies above γ_t^- forward in time as well. If this property can be obtained for every ν as we let $\delta(\nu) \rightarrow 0$, symmetry will imply that the curve γ_t is a round circle.

Figure 6.2: Curves γ_t and $\gamma_{-\infty}$ and their reflections over P_ν .

Continuing the set up for arguments to follow regarding the comparison of two curves, we describe the nearest point projection to the equator $\gamma_{-\infty}$. For $x \in S^2$, let $\rho(x) = d(x, \gamma_{-\infty})$ denote the distance from x to $\gamma_{-\infty}$, and let $\pi(x)$ denote the set of nearest points on $\gamma_{-\infty}$ to x . For $x \in S^2 - \{(0, 0, 1), (0, 0, -1)\}$ i.e. x not the north or south pole, $\pi(x)$ is a single point. Otherwise, $\pi(x) = \gamma_{-\infty}$. For each point $y \in \pi(x)$ there is a unique length-minimizing geodesic joining x to y with length equal to $\rho(x)$. By the first variation formula, this geodesic meets the equator orthogonally and hence lies on the unique great circle passing through x and the north pole $(0, 0, 1)$.

Next we indicate explicitly what is meant by saying that one curve lies above another at a point x in equator $\gamma_{-\infty}$ or over a subset of $\gamma_{-\infty}$.

Definition 6.1. Let γ_α and γ_β be curves on S^2 and $x \in \gamma_{-\infty}$. We will write

$$\gamma_\alpha \geq_x \gamma_\beta$$

and say γ_α *lies above* γ_β at x if

$$\inf\{\rho(y) : y \in \pi^{-1}(x) \cap \gamma_\alpha\} \geq \sup\{\rho(y) : y \in \pi^{-1}(x) \cap \gamma_\beta\} \quad (6.1)$$

whenever both sets are nonempty. If the inequality in (6.1) is strict, we write $\gamma_\alpha >_x \gamma_\beta$. Similarly we say γ_α *lies above* γ_β on a set $W \subset \gamma_{-\infty}$ and write $\gamma_\alpha \geq_W \gamma_\beta$

if $\gamma_\alpha \geq_x \gamma_\beta$ for all $x \in W$. To say γ_α *lies strictly above* γ_β , replace $\geq$ with $>$. We may drop the W from the notation if the set in consideration is clear.

Remark 6.2. The inf and sup are in the above definition since a curve γ_α may not be a graph over the equator so $\pi^{-1}(x) \cap \gamma_\alpha$ could have multiple points. The relations $\geq_x$ and $\geq_W$ are not in general partial orders; $\gamma_\alpha \geq_x \gamma_\beta$ if and only if $\pi^{-1}(x) \cap \gamma_\alpha$ is a single point. Note that the relation $\geq$ is only a partial order when restricted to curves that are graphs over the equator.

For our argument, we find it convenient to parametrize the sphere using spherical polar coordinates

$$F(\theta, \phi) = (\cos \phi \cos \theta, \cos \phi \sin \theta, \sin \phi) \quad (6.2)$$

where $\phi \in (-\pi/2, \pi/2)$ measures the angle above the $\{z = 0\}$ plane (so $\phi = 0$ indicates the point is on the equator) and $\theta \in [-\pi, \pi]$ is the azimuthal angle. Our metric on S^2 is thus $g = \cos^2 \phi d\theta^2 + d\phi^2$. Take $\theta = 0$ to be the θ -coordinate of one of the points in $P_\nu \cap \gamma_{-\infty}$. The other then has θ -coordinate $\theta = \pi$.

Remark 6.3. In these coordinates, for $x \in S^2 - \{(0, 0, 1), (0, 0, -1)\}$ given by $x = F(\theta_0, \phi_0)$ the nearest-point projection is $\pi(x) = F(\theta_0, 0)$; if $y = (\theta_1, \phi_1)$, the statement $x \geq y$ is equivalent to $\phi_0 \geq \phi_1$.

Next we discuss the reflection of the equator $\gamma_{-\infty}$ across P_ν (see Fig. 6.3) and note some of its properties which we will use later.

Figure 6.3: Reflecting portion of the equator $\gamma_{-\infty}^+$ across P_ν .

If $\delta = \pi/4$, then $R_\nu(\gamma_{-\infty}^+)$ is reflected to a half great circle through the north pole (indicated by the dashed half great circle in Fig. 6.3 at $\theta = 0$). For $\delta < \pi/4$, we see that $R_\nu(\gamma_{-\infty}^+)$ gets reflected over $\gamma_{-\infty}^-$ in H_ν^- , i.e. we have $R_\nu(\gamma_{-\infty}^+) \geq_W \gamma_{-\infty}^-$ on the set $W = \gamma_{-\infty}^-$.

Notice that $R_\nu(\gamma_{-\infty})$ can be written as a graph

$$R_\nu(\gamma_{-\infty})(\theta) = (\theta, g_{-\infty}(\theta))$$

where $g_{-\infty} : [-\pi, \pi] \rightarrow \mathbb{R}$ is a smooth function, symmetric about $\theta = 0$ (see Fig. 6.4). As $R_\nu(\gamma_{-\infty})$ lies above the equator for $\theta \in (0, \pi)$, $R_\nu(\gamma_{-\infty})$ and lies below for $\theta \in (-\pi, 0)$, we see that $g_{-\infty}$ is odd about $\theta = 0$ and increasing near $\theta = 0$ so $g'_{-\infty}(0) > 0$ (and is actually equal to $\tan(2\delta)$ - see Appendix), and $g''_{-\infty}(0) = 0$.

6.2 Backwards approximate symmetry

From Prop. 2.3 and Corollary 5.4, we may assume that for $t \in (-\infty, 0)$, the curve γ_t lies in the upper hemisphere $\{z > 0\} \cap S^2$ and can be written as a graph in the θ, ϕ plane

$$\gamma_t(\theta) = (\theta, f_t(\theta))$$

with $f_t : [-\pi, \pi] \rightarrow \mathbb{R}$ a smooth family of positive smooth functions over the equator $\phi = 0$ (see Fig. 6.4).

Since γ_t converges uniformly to the equator $\gamma_{-\infty}$, we have that $\frac{\partial^m}{\partial \theta^m} f_t \rightarrow 0$ uniformly for each $m \geq 0$. As R_ν is an isometry, $R_\nu(\gamma_t)$ converges uniformly to the reflected equator $R_\nu(\gamma_{-\infty})$ as $t \rightarrow -\infty$. For any $\delta < \pi/4$, the latter can be written as a graph $(\theta, g_{-\infty}(\theta))$, so given a fixed $\delta_0 \in (0, \pi/4)$ we can go back far enough in time to find a $t_0 < 0$ where $R_\nu(\gamma_t)$ can be written as a graph

$$R_\nu(\gamma_t)(\theta) = (\theta, g_t(\theta))$$

for $t < t_0$ with $g_t \rightarrow g_{-\infty}$ as $t \rightarrow -\infty$.

We wish to prove the following:

Proposition 6.4. (*Backwards approximate symmetry*) *For every $\delta \in (0, \pi/4)$, there exists $t_\delta \in (-\infty, 0)$ such that for all directions ν with $\delta(\nu) = \delta$ and all $t \leq t_\delta$,*

Figure 6.4: Curves as graphs in θ, ϕ coordinates. $\phi = 0$ corresponds to equator $\gamma_{-\infty}$.

we have

$$R_\nu((\gamma_t)_\nu^+) \geq_W (\gamma_t)_\nu^- \quad (6.3)$$

where $W = \gamma_{-\infty}^-$.

Since $\gamma_t \rightarrow \gamma_{-\infty}$ in C^∞ , it is rather clear that Proposition 6.4 is true, but for completeness we give a detailed proof. To show that $R_\nu(\gamma_t^+) \geq \gamma_t^-$, it is enough to show that $g_t(\theta) \geq f_t(\theta)$ (see Remark 6.3). The proof that $g_t(\theta) \geq f_t(\theta)$ consists of estimates for interior points (points away from $\gamma_t \cap P_\nu$) and boundary points (points near to $\gamma_t \cap P_\nu$).

The interior estimates are obtained by using the fact that the $g_t \rightarrow g_{-\infty}$, with $g_{-\infty} > 0$ away from where $g_{-\infty}$ crosses the equator at $\theta = 0, \pi$, and $f_t \rightarrow 0$.

Lemma 6.5. (*Interior points*) *Given $\epsilon \in (0, \pi/4)$, there exists $t_1 = t_1(\epsilon)$ such that $g_t(\theta) > f_t(\theta)$ for $\theta \in (\epsilon, \pi - \epsilon)$ and $t < t_1$.*

Proof. For $\delta < \pi/4$, consider the portion of the reflected equator $R_\nu(\gamma_{-\infty}^+)$ given by $(\theta, g_{-\infty}(\theta))$ with $\theta \in [0, \pi]$. For $\theta \in [0, \pi]$, we have that $g_{-\infty}(\theta) \geq 0$ is a smooth, concave function, symmetric about $\pi/2$, and strictly positive for $\theta \in (0, \pi)$. Given $\epsilon \in (0, \pi/2)$, let $G = g_{-\infty}(\epsilon) = g_{-\infty}(\pi - \epsilon)$. Then $G < g_{-\infty}(\theta)$ for $\theta \in (\epsilon, \pi - \epsilon)$.

Figure 6.5: Showing $g_t > f_t$ for interior points.

Since $f_t \rightarrow 0$ and $g_t \rightarrow g_{-\infty}$ uniformly, we can find $t_1 < t_0$ such that for $\theta \in (\epsilon, \pi - \epsilon)$ and $t < t_1$, we have $f_t(\theta) < G/2$ and $|g_t(\theta) - g_{-\infty}(\theta)| < G/2$. Then

$$f_t(\theta) < \frac{G}{2} < G - G/2 < g_{-\infty}(\theta) - (g_{-\infty}(\theta) - g_t(\theta)) < g_t(\theta)$$

for $\theta \in (\epsilon, \pi - \epsilon)$ and $t < t_1$. □

Next we need to see that $g_t > f_t$ on an interval near where f_t and g_t intersect P_ν and estimate the size of that interval to ensure $g_t > f_t$ outside $\theta \in (\epsilon, \pi - \epsilon)$. Here we use that the derivatives of the curve and the reflected curve converge to the derivatives of the equator and reflected equator, respectively.

Choose an orientation for θ so that $P_\nu \cap \{z > 0\}$ corresponds to the region in the θ, ϕ plane with $\theta \in (-\pi, 0)$. Recalling that γ_t is a graph over the equator, define $\theta_t^*, \theta_t^{**} \in S^2$ to be the points

$$\gamma_t \cap P_\nu = \{\theta_t^*, \theta_t^{**}\}.$$

In the following arguments we may abuse notation and use $\theta_t^*, \theta_t^{**}$ to indicate just the θ -coordinate of these points; with this usage, we see that $\theta_t^* \in (-\pi/2, 0)$ and $\theta_t^{**} \in (-\pi, -\pi/2)$. Note that $\theta_t^* \rightarrow 0$ and $\theta_t^{**} \rightarrow -\pi$ as $t \rightarrow -\infty$.

Lemma 6.6. (*Boundary points*) *Given $\tilde{\epsilon} > 0$, there exists $t_2 = t_2(\tilde{\epsilon}) > -\infty$ such that $g_t(\theta) > f_t(\theta)$ on $(\theta_t^*, \tilde{\epsilon}) \cup (\pi - \tilde{\epsilon}, \theta_t^{**})$ for $t < t_2$.*

Figure 6.6: Showing $g_t > f_t$ for boundary points.

Proof. It is enough to show that $g_t(\theta) > f_t(\theta)$ for $\theta \in (\theta_t^*, \tilde{\epsilon})$, as the proof on $(\pi - \tilde{\epsilon}, \theta_t^{**})$ is similar. We first see that there exists $t_1 < t_0$ such that for all $t < t_1$, $f'_t(\theta_t^*) < g'_t(\theta_t^*)$ because $f'_t \rightarrow 0$ uniformly and $g'_t(\theta_t^*) \rightarrow g'_t(0) = \tan(2\delta) > 0$ as $t \rightarrow -\infty$. Now $f_t(\theta_t^*) = g_t(\theta_t^*)$ since θ_t^* is the point about which f_t is reflected across P_ν . So we have that

$$\begin{aligned} f_t(\theta_t^*) &= g_t(\theta_t^*) \\ f'_t(\theta_t^*) &< g'_t(\theta_t^*) \end{aligned}$$

and thus $g_t > f_t$ in a neighborhood of θ_t^* . We estimate the size of that neighborhood. Consider the quantity $g_t(\theta) - f_t(\theta)$ and write the Taylor expansion at θ_t^* with the Lagrange form of the remainder to get that $g_t(\theta) > f_t(\theta)$ if

$$\theta - \theta_t^* < 2 \frac{g'_t(\theta_t^*) - f'_t(\theta_t^*)}{\sup |g''_t - f''_t|} \quad (6.4)$$

with $\theta - \theta_t^* > 0$ and where $\sup |(g''_t - f''_t)(\theta)|$ is taken over (θ_t^*, θ) (for details see Appendix). The numerator of (6.4) converges to $\tan(2\delta)$ as $t \rightarrow -\infty$ while the denominator converges to 0 (recall that $g'_t(\theta_t^*) \rightarrow g''_t(0) = 0$) as $t \rightarrow -\infty$. Thus we can find a $t_2 = t_2(\tilde{\epsilon}) < t_1$ so $g_t > f_t$ for $\theta \in (\theta_t^*, \tilde{\epsilon})$ and $t < t_2$. $\square$

Combining the interior and boundary estimates, we can show backwards approximate symmetry.

Proof. (Proposition 6.4). Fix a δ . Fix $\epsilon > 0$. Use the interior estimates from Lemma 6.5 to obtain $g_t > f_t$ for $\theta \in (\epsilon, \pi - \epsilon)$ and $t < t_1$. Choose $\tilde{\epsilon} > \epsilon$ and use the boundary estimates from Lemma 6.6 to ensure $g_t > f_t$ for $(\theta_t^*, \tilde{\epsilon}) \cup (\pi - \tilde{\epsilon}, \theta_t^{**})$ and $t < t_2$. Finally let $t_\delta = \min\{t_1, t_2\}$ so that $g_t > f_t$ for all $\theta \in (\theta_t^*, \theta_t^{**})$ and $t < t_\delta$. $\square$

6.3 Approximate symmetry preserved

Figure 6.7: $R_\nu(\gamma_t^+)$ and γ_t^- disjoint except at their endpoints θ_t^* and θ_t^{**} .

Proposition 6.7. (*Approximate symmetry preserved*) *There exists $T \in (-\infty, 0)$ such that $R_\nu(\gamma_t^+) \geq_W \gamma_t^-$ for all $t \in (-\infty, T)$ and all $\delta \in (0, \pi/4)$, where $W = \gamma_{-\infty}^-$.*

Proof. Recall that for $t \in (-\infty, t_0)$, both γ_t^- and $R_\nu(\gamma_t^+)$ may be written as graphs over the equator. Also, both the equator $\gamma_{-\infty}$ and the reflected equator $R_\nu(\gamma_{-\infty})$ meet the reflection plane P_ν transversely. Since $\gamma_t \rightarrow \gamma_{-\infty}$ and $R_\nu(\gamma_t) \rightarrow R_\nu(\gamma_{-\infty})$ uniformly in C^∞ , there is a $t_1 = t_1(\delta) \in (-\infty, t_0)$ such that both γ_t^- and $R_\nu(\gamma_t^+)$ meet P_ν transversely for all $t \in (-\infty, t_1)$. So γ_t^- , $R_\nu(\gamma_t^+)$ are connected curves meeting P_ν transversely in precisely two points for each time t . Now we apply the maximum principle. Since R_ν is an isometry, $R_\nu(\gamma_t^+)$ evolves by curve shortening. $P_\nu \cap S^2$ is a great circle so it can be considered as a stationary solution to curve shortening flow. As both γ_t^- and $R_\nu(\gamma_t^+)$ meet P_ν transversely, the maximum

principle ensures that neither curve intersects P_ν at any other points and hence remain in H_ν^- for all $t \in (-\infty, t_1)$.

The curves γ_t^- and $R_\nu(\gamma_t^+)$ are connected, evolve by curve shortening flow, and agree at their endpoints $\theta_t^*, \theta_t^{**}$ which remain on P_ν . We also obtained that at the endpoints $\theta_t^*, \theta_t^{**}$, the angle $R_\nu(\gamma_t^+)$ makes with the $\{z = 0\}$ plane is strictly larger than the angle γ_t^- makes with the $\{z = 0\}$ plane (since $g'_{-\infty}(0) = \tan(2\delta) > 0 = f'_{-\infty}(0)$) so the curves meet transversely at the endpoints. By Theorem 2.6, γ_t^- and $R_\nu(\gamma_t^+)$ remain disjoint except for their endpoints under the flow.

This gives a T_δ for each δ such that $R_\nu(\gamma_t^+) \geq \gamma_t^-$ for $t \in (-\infty, T_\delta)$. Let

$$T = \inf\{T_\delta : \delta \in (0, \pi/4)\}.$$

We want to show that $T > -\infty$. To see this, observe that the above maximum principle argument is valid provided that both γ_t^- and $R_\nu(\gamma_t^+)$ satisfy the following for $t \in (-\infty, T)$

1. are graphs over the equator $\gamma_{-\infty}$
2. meet P_ν transversely
3. are non-empty

For (1): Recall $R_\nu(\gamma_{-\infty})$ is a graph $g_{-\infty}^\delta$ for each $\delta \in (0, \pi/4)$ with maximum derivative at the end points $\theta = 0, \pi$. As $\delta \rightarrow 0$, the derivative $(g_{-\infty}^\delta)'(0)$ monotonically decreases to 0 since $g_{-\infty}^\delta \rightarrow \gamma_{-\infty}$ (the equator) as $\delta \rightarrow 0$. For each fixed t then, $R_\nu(\gamma_t^+)$ becomes more horizontal as δ decreases. Hence if $R_\nu(\gamma_t^+)$ is a graph (so does not have a vertical tangent) for some δ_0 , then it remains a graph for every $\delta < \delta_0$. Whether γ_t^- is a graph or not is independent of δ , but by convexity we know γ_t is a graph for all $t \in (-\infty, 0)$.

For (2): The angle P_ν makes with the $\{z = 0\}$ plane increases monotonically (to $\pi/2$) as $\delta \searrow 0$. Thus if γ_t^- and $R_\nu(\gamma_t^+)$ meet P_ν transversely for some δ_0 then they continue to do so for $\delta < \delta_0$.

For (3): Provided γ_t^- lies below the maximum ϕ coordinate of $P_\nu \cap S^2$, both curves γ_t^- and $R_\nu(\gamma_t^+)$ are nonempty. Then observe that the maximum ϕ coordinate of $P_\nu \cap S^2$ increases to $\pi/2$ as $\delta \searrow 0$. $\square$

6.4 Exact symmetry

Next we characterize those curves with maximal approximate symmetry for every $\delta > 0$ as round circles and use this to complete the proof of the main theorem.

Proposition 6.8. (*Exact symmetry*) *Let γ be a smooth curve in the upper hemisphere $S^2 \cap \{z \geq 0\}$, and let $W = \gamma_{-\infty}^-$. If $R_\nu(\gamma_\nu^+) \geq_W \gamma_\nu^-$ for every ν such that $\langle \nu, e_z \rangle < 0$ and $\delta(\nu) \in (0, \pi/4)$, then γ is a round circle with center $(0, 0, z)$.*

Proof. Let $\nu_0 \in \mathbb{R}^3$ be such that $\langle \nu_0, e_z \rangle = 0$ (so $\delta(\nu_0) = 0$). Choose any $x \in W$. By assumption we have

$$R_\nu(\gamma_\nu^+) \geq_x \gamma_\nu^-$$

for ν in the plane spanned by e_z and ν_0 with $\langle \nu, e_z \rangle < 0$ and $\delta(\nu) \in (0, \pi/4)$. By continuity, letting $\nu \rightarrow \nu_0$ we obtain $R_{\nu_0}(\gamma_{\nu_0}^+) \geq_x \gamma_{\nu_0}^-$ for each $x \in \gamma_{-\infty}^-$ and hence

$$R_{\nu_0}(\gamma_{\nu_0}^+) \geq_x \gamma_{\nu_0}^- \tag{6.5}$$

for every ν_0 with $\langle \nu_0, e_z \rangle = 0$.

We have the following properties of reflection over P_{ν_0} where $\langle \nu_0, e_z \rangle = 0$.

1. $R_{\nu_0}^2 = R_{\nu_0} \circ R_{\nu_0} = \text{id}$
2. $\gamma_\alpha \geq \gamma_\beta \Rightarrow R_{\nu_0}(\gamma_\alpha) \geq R_{\nu_0}(\gamma_\beta)$
3. $R_{\nu_0} = R_{-\nu_0}$
4. $\gamma_{\nu_0}^\pm = \gamma_{-\nu_0}^\mp$

Using (6.5) and the above properties of reflection over P_{ν_0} , we get

$$\begin{aligned} \gamma_{\nu_0}^+ &= R_{\nu_0}(R_{\nu_0}(\gamma_{\nu_0}^+)) \\ &\geq R_{\nu_0}(\gamma_{\nu_0}^-) = R_{-\nu_0}(\gamma_{-\nu_0}^+) \\ &\geq \gamma_{-\nu_0}^- = \gamma_{\nu_0}^+. \end{aligned}$$

We must have equality all the way through. Hence

$$R_{-\nu_0}(\gamma_{-\nu_0}^+) = \gamma_{-\nu_0}^- \quad (6.6)$$

for any ν_0 . This implies γ must have a horizontal (no e_z component) tangent at $P_{-\nu_0} \cap \gamma$. Every point of γ lies on P_{ν_0} for some ν_0 , hence γ has a horizontal tangent everywhere and is so is a round circle. $\square$

From the results above, we have our classification.

Theorem 6.9. *Let γ_t be a convex, embedded ancient solution to the curve shortening flow on S^2 . Then γ_t is the unique (up to isometry of S^2) family of shrinking circles or a fixed equator for all time.*

Proof. The approximate symmetry preserved in Proposition 6.7 implies that for every ν with $\delta(\nu) \in (0, \pi/4)$, we have $R_{\nu_0}(\gamma_{\nu}^+) = \gamma_{\nu}^-$ for $t \in (-\infty, T)$. The exact symmetry in Prop 6.8 applies at each t , so that γ_t is a round circle for $t \in (-\infty, T)$. Uniqueness of solutions of the curve shortening flow then gives that γ_t is a round circle for every $t \in (-\infty, 0)$. $\square$

Chapter 6 includes material from: P. Bryan, J. Louie, “Classification of Convex Ancient Solutions to Curve Shortening Flow on the Sphere”, *preprint, arXiv: 1408.5523*.

Appendix A

Additional Calculations

Parametrize the sphere using spherical polar coordinates

$$F(\theta, \phi) = (\cos \phi \cos \theta, \cos \phi \sin \theta, \sin \phi) \quad (\text{A.1})$$

where $\phi \in (-\pi/2, \pi/2)$ measures the angle above the $\{z = 0\}$ plane (so $\phi = 0$ indicates the point is on the equator) and θ is the azimuthal angle. Then

$$\begin{aligned} F_\theta &= (-\cos \phi \sin \theta, \cos \phi \cos \theta, 0) \\ F_\phi &= (-\sin \phi \cos \theta, -\sin \phi \sin \theta, \cos \phi) \end{aligned}$$

and our round metric coefficients are

$$\begin{aligned} g_{\theta\theta} &= \langle F_\theta, F_\theta \rangle = \cos^2 \phi \\ g_{\theta\phi} &= \langle F_\theta, F_\phi \rangle = 0 \\ g_{\phi\phi} &= \langle F_\phi, F_\phi \rangle = 1. \end{aligned}$$

The only nonzero Christoffel symbols for our metric $g = d\phi^2 + \cos^2 \phi d\theta^2$ are

$$\begin{aligned} \Gamma_{\theta\theta}^\phi &= \cos \phi \sin \phi \\ \Gamma_{\phi\theta}^\theta &= \Gamma_{\theta\phi}^\theta = -\tan \phi. \end{aligned}$$

Lemma A.1. *For a curve γ which can be written as a graph $\gamma(\theta) = (\theta, f(\theta))$ over the equator, the curvature is given by*

$$k = \frac{\cos f}{((f')^2 + \cos^2(f))^{3/2}} (f'' + 2(f')^2 \tan f + \sin f \cos f) \quad (\text{A.2})$$

Proof. To see this we compute $k = \langle \nabla_T T, N \rangle$. We can write $\gamma' = \partial_\theta + f' \partial_\phi$ or $\gamma' = (1, f')$ so the unit tangent is

$$T = \frac{\gamma'}{|\gamma'|} = \frac{1}{\sqrt{(f')^2 + \cos^2 f}} (1, f') \quad (\text{A.3})$$

where $|\cdot|$ is computed with respect to metric g . Using the Christoffel symbols we get

$$\nabla_{\gamma'} \gamma' = (-2f' \tan f) \partial_\theta + (f'' + \cos f \sin f) \partial_\phi.$$

Find the unit normal to the curve $N = N^\theta \partial_\theta + N^\phi \partial_\phi$ using $\langle \gamma', N \rangle = 0$ and $|N| = 1$, with $\gamma' = (1, f')$. Note that as we are looking for the inward pointing normal we want N^ϕ to be positive. Solving for N^θ and N^ϕ we get

$$N = \left(\frac{-f'}{\cos f \sqrt{(f')^2 + \cos^2 f}} \right) \partial_\theta + \left(\frac{\cos f}{\sqrt{(f')^2 + \cos^2 f}} \right) \partial_\phi. \quad (\text{A.4})$$

Compute

$$\begin{aligned} \langle \nabla_{\gamma'} \gamma', N \rangle &= (-2f' \tan f) \left(\frac{-f'}{\cos f \sqrt{(f')^2 + \cos^2 f}} \right) (\cos^2 f) \\ &\quad + (f'' + \cos f \sin f) \left(\frac{\cos f}{\sqrt{(f')^2 \sin f + \cos^2 f}} \right) \\ &= \frac{1}{|\gamma'|} (f'' \cos f + 2(f')^2 \sin f + \cos^2 f \sin f). \end{aligned}$$

Then

$$\begin{aligned} k &= \langle \nabla_T T, N \rangle = \left\langle \nabla_{\frac{\gamma'}{|\gamma'|}} \left(\frac{\gamma'}{|\gamma'|} \right), N \right\rangle = \frac{1}{|\gamma'|^2} \langle \nabla_{\gamma'} \gamma', N \rangle \\ &= \frac{\cos f}{((f')^2 + \cos^2 f)^{3/2}} (f'' + 2(f')^2 \tan f + \sin f \cos f) \end{aligned}$$

as was to be shown. $\square$

Example A.2. As an example of reflecting a point on the equator across P_ν with $\delta(\nu) = \delta$ set up as in Chapter 6, let $\nu = (\nu_x, \nu_y, \nu_z)$, and $X = (-1, 0, 0) \in \gamma_{-\infty}^+$ viewed in $\mathbb{R}^3$. We have $X \cdot \nu = -\nu_x$, and $\delta = \cos^{-1}(X \cdot \nu) = \cos^{-1}(-\nu_x)$. Then

$$R_\nu(X) = (-1, 0, 0) - 2(-\nu_x)(\nu_x, \nu_y, \nu_z) = (-1 + 2\nu_x^2, 2\nu_x \nu_y, 2\nu_x \nu_z)$$

and we have that

$$\begin{aligned}
\angle(R_\nu(X), (1, 0, 0)) &= \cos^{-1}(R_\nu(X), (1, 0, 0)) = \cos^{-1}(-1 + 2\nu_x^2) \\
&= \cos^{-1}(-1 + 2(\cos \delta)^2) \\
&= \cos^{-1}(\cos(2\delta)) \\
&= 2\delta.
\end{aligned}$$

Remark A.3. If $R_\nu(\gamma_{-\infty}^+)$ is written as a graph $(\theta, g_{-\infty}(\theta))$, we have that

$$g'_{-\infty}(0) = \tan(2\delta). \quad (\text{A.5})$$

One way to see this is to note that we are working at $g_{-\infty}(0) = 0$, where the equator meets P_ν , the plane of reflection, so we can just use the Euclidean space calculation. We can assume $g_{-\infty}$ is a straight line in the $x - z$ plane and see that its slope is $\tan(2\delta)$.

Lemma A.4. (*Details of Taylor expansion in Eq. 6.4*) We have

$$\begin{aligned}
f_t(\theta_t^*) &= g_t(\theta_t^*) \\
f'_t(\theta_t^*) &< g'_t(\theta_t^*)
\end{aligned}$$

and wish to show that $g_t(\theta) > f_t(\theta)$ when

$$\theta - \theta_t^* < 2 \frac{g'_t(\theta_t^*) - f'_t(\theta_t^*)}{\sup |g''_t - f''_t|} \quad (\text{A.6})$$

Proof. Let $h(\theta) = g_t(\theta) - f_t(\theta)$ and write the Taylor expansion at θ_t^* with the Lagrange form of the remainder:

$$\begin{aligned}
h(\theta) &= h(\theta_t^*) + h'(\theta_t^*)(\theta - \theta_t^*) + \frac{h''(c(\theta))}{2}(\theta - \theta_t^*)^2 \text{ for } c(\theta) \in (\theta_t^*, \theta) \\
&= h'(\theta_t^*)(\theta - \theta_t^*) + \frac{h''(c(\theta))}{2}(\theta - \theta_t^*)^2
\end{aligned}$$

Then

$$|h(\theta) - h'(\theta_t^*)(\theta - \theta_t^*)| \leq \frac{\sup |h''|}{2}(\theta - \theta_t^*)^2$$

where $\sup |h''|$ is taken over (θ_t^*, θ) . This implies

$$-\frac{\sup |h''|}{2}(\theta - \theta_t^*)^2 \leq h(\theta) - h'(\theta_t^*)(\theta - \theta_t^*)$$

so

$$h(\theta) \geq h'(\theta_t^*)(\theta - \theta_t^*) - \frac{\sup |h''|}{2}(\theta - \theta_t^*)^2$$

Thus we have $h(\theta) = g_t(\theta) - f_t(\theta) > 0$ when

$$h'(\theta_t^*)(\theta - \theta_t^*) - \frac{1}{2} \sup |h''|(\theta - \theta_t^*)^2 > 0$$

i.e. when

$$\theta - \theta_t^* < 2 \frac{h'(\theta_t^*)}{\sup |h''|} = 2 \frac{g'_t(\theta_t^*) - f'_t(\theta_t^*)}{\sup |g''_t - f''_t|}$$

with $\theta - \theta_t^* > 0$ and $\sup |h''(\theta)|$ taken over (θ_t^*, θ) . □

Bibliography

- [AB11] Ben Andrews and Paul Bryan. Curvature bound for curve shortening flow via distance comparison and a direct proof of Grayson's theorem. *J. Reine Angew. Math.*, 2011.653:179–187, 2011.
- [Ale56] A.D. Aleksandrov. Uniqueness theorems for surfaces in the large i. *Vesnik Leningrad Univ.*, 11:5–17, 1956.
- [CE75] Jeff Cheeger and David G. Ebin. *Comparison theorems in Riemannian geometry*. AMS, 1975.
- [CG96] B. Chow and R. Gulliver. Aleksandrov reflection and nonlinear evolution equations i. the n-sphere and the n-ball. *Calc. Var. Partial Differential Equations*, 4.3:389–409, 1996.
- [CG01] B. Chow and R. Gulliver. Aleksandrov reflection and the geometric evolution of hypersurfaces. *Comm. in Analysis and Geometry*, 9:261–280, 2001.
- [Cho97] B. Chow. Geometric aspects of Aleksandrov reflection and gradient estimates for parabolic equations. *Comm. Anal. Geom.*, 5.3:389–409, 1997.
- [CK01] Bennett Chow and Dan Knopf. *The Ricci Flow: An Introduction*. AMS, 2001.
- [CZ01] K. S. Chou and X. P. Zhu. *The Curve Shortening Problem*. Chapman and Hall/CRC, 2001.
- [DHS10] P. Daskalopoulos, R. Hamilton, and N. Sesum. Classification of compact ancient solutions to the curve shortening flow. *J. Differential Geometry*, 84:455–464, 2010.
- [Gag90] M. Gage. Curve shortening on surfaces. *Ann. Scientifiques de l'E.N.S.*, 23:229–256, 1990.
- [GH86] M. Gage and R. Hamilton. The heat equation shrinking convex plane curves. *J. Differential Geometry*, 23:69–96, 1986.

- [Gra87] M. Grayson. The heat equation shrinks embedded plane curves to round points. *J. Differential Geometry*, 26:285–314, 1987.
- [Gra89] M. Grayson. Shortening embedded curves. *Ann. of Math.*, 129:71–111, 1989.
- [Ham95a] Richard S. Hamilton. The formation of singularities in the Ricci flow. *Surveys in Differential Geometry*, 2:23–26, 1995.
- [Ham95b] Richard S. Hamilton. Isoperimetric estimates for the curve shrinking flow in the plane. In *Modern methods in complex analysis (Princeton, NJ, 1992)*, volume 137 of *Ann. of Math. Stud.*, pages 201–222. Princeton Univ. Press, Princeton, NJ, 1995.
- [Hui98] Gerhard Huisken. A distance comparison principle for evolving curves. *Asian J. Math.*, 2.1:127–133, 1998.
- [Mul56] W.W. Mullins. Two-dimensional motion of idealized grain boundaries. *Journal of Applied Physics*, 2.8:900–904, 1956.
- [PW67] M. Protter and H. Weinberger. *Maximum principles in differential equations*. Prentice Hall, 1967.
- [San44] L.A. Santalo. Note on convex spherical curves. *Bull. Amer. Math. Soc.*, 50:528–534, 1944.
- [Top06] Victor Andreevich Topogonov. *Differential Geometry of Curves and Surfaces: A Concise Guide*. Birkhauser Boston, 2006. With editorial assistance of Rovenski, Vladimir Y.