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Implicit Understanding of Arithmetic with Rational Numbers: The Impact of Expertise

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Abstract

Recent work has shown that undergraduates at a major public university demonstrate implicit understanding of inverse relations between multiplication problems with fractions, as evidenced by the fact that solving one problem facilitates solving its inverse. The present study investigated whether such implicit understanding of mathematical relations is related to overall math ability. We found that low performers showed relational facilitation only when it was supported by perceptual similarity, whereas high performers showed relational facilitation on both perceptually similar and dissimilar problems. These findings are interpreted in terms of novice-expert differences in the representation of mathematical relations.

Keywords: mathematical reasoning, rational numbers, relational reasoning, expertise

Introduction

Research on expertise has highlighted differences between the mental representations of experts and novices. Experts and novices not only approach problems differently, but also differ in how they allocate attention and relate problems to one another (e.g., Chi, Feltovich & Glaser, 1981; Novick, 1988; Chase & Simon, 1973). These differences have been observed in a variety of problem-solving contexts, including chess, physics, and mathematics. One important consequence of differences between expert and novice processing involves transfer between problems or situations. For example, Chi et al. (1981) found that expert physicists tended to group certain problems together based on the physical laws involved in solving each problem. In contrast, novices grouped problems based on perceptual similarity, rather than on underlying principles. By attending to the relational structure governing the problems, expert physicists were able to transfer problem-solving strategies effectively between problems involving the same abstract principles. The perceptual similarities on which novices focus are much less effective in supporting transfer. It is often difficult for beginning students to ignore surface features and encode relational structure, a fact that likely contributes to the difficulty in obtaining transfer between problems that are analogically similar but perceptually dissimilar (e.g., Gick & Holyoak, 1980, 1983, Hayes & Simon, 1977; Holyoak & Koh, 1987; Ross, 1987). In general, expertise or deep understanding is characterized by mental representations that go beyond perceptual similarity.

In the current study we investigated the extent to which perceptual similarity between two relationally-similar math problems facilitates the performance of solvers who differ in their level of math expertise.

Expertise in Mathematics: The Case of Rational Numbers

The general pattern of differences between expert and novice understanding is found within the realm of mathematics (Novick, 1988; Schoenfeld & Hermann, 1982). Expert mathematicians (e.g., math professors, or those who achieve high scores on a math proficiency test) are more likely to judge problems embodying the same mathematical structure to be similar, are more likely to apply the same problem-solving strategies to relationally-related problems, and demonstrate greater transfer after a delay (Novick, 1988; Novick & Holyoak, 1991). Expertise in mathematics is often associated with more rapid solution times (e.g., Kellman, Massey & Son, 2010; Stevenson et al., 1990).

Understanding of rational numbers (fractions and decimals) provides a particularly interesting context for investigating differences in expert and novice understanding. Novice students often inappropriately transfer characteristics of whole numbers to fractions, and therefore expect fractions to be countable and discrete (Ni & Zhou, 2005; Stafylidou & Vosniadou, 2004). These misconceptions persist in adults with lower overall competence in mathematics. For example, based on oral explanations of fraction concepts, Stigler and colleagues (Givvin, Stigler & Thompson, 2011; Stigler, Givvin & Thompson, 2010) found that community-college students exhibit many of the same misconceptions about fraction magnitudes and fraction arithmetic as do middle-school students. In contrast, other studies conducted with students at highly competitive universities (e.g., Schneider & Siegler, 2010; DeWolf, Grounds, Bassok & Holyoak, 2014) have found that these students, who tend to have greater mathematical expertise, are able to represent fraction magnitudes with little difficulty.

Expert-novice differences have been observed in fraction arithmetic. In the realm of (positive) whole numbers, multiplication yields a larger result and division a smaller result, but this is not generally true of rational numbers, such as fractions or decimals less than 1. Siegler and Lortie-Forgues (in press) asked participants from a wide range of math backgrounds to perform a simple task involving fraction multiplication and division. Participants simply had to decide whether problems like 31/56 * 17/42 > 31/56 are true. The investigators found that while math and science students from a highly competitive university were consistently correct, pre-service teachers and middle-school students performed at below-chance levels.

A central issue is that beginning students (and many elementary-school teachers) have difficulty understanding mathematical operations and how they relate to one another (Ma, 1999; Siegler et al., 2011, 2013). For example, fraction addition and subtraction require finding common denominators, whereas fraction multiplication and division do not. Students are often unclear as to when and why it is necessary to find common denominators. This lack of understanding of procedures may reflect a lack of deep conceptual understanding of the relations between operators, and of how the division operation within a fraction relates to other operations in the problem.

Understanding Fraction Multiplication and Inverse Relations

This lack of understanding is especially evident in the case of the "invert and multiply" strategy in fraction division. Early on, students are taught that to complete a fraction division problem, all that is required is to invert the second fraction in the problem and then proceed with the fraction multiplication procedure. But understanding *why* this strategy works is not simple. Tirosh (2000) found that even pre-service teachers have little understanding of this strategy.

The reason why the invert-and-multiply strategy works involves the reciprocal relationships between the two factors in a multiplication problem and their relationship to a product. For example, 5 ÷ 10 is the same as 5/10, which is the same as 5 X 1/10 since 10 and 1/10 are reciprocals. In addition, students have to understand that the "bar" in the fraction expression denotes a division operation; and as such, it can be used to represent a relation within the multiplication operation itself. Thus, the same strategy applies when multiplying either whole numbers or fractions. Performing the operation "2 ÷ 3" is equivalent to "2/3" and also "2 X 1/3". Furthermore, "2 ÷ 3" represents the same proportional relation as "4 ÷ 6". A deep understanding of this type of relational structure would allow students to move flexibly between any of these equivalent notations.

Overview of Current Study

In the current study we sought to better understand the differences in understanding of multiplication with rational numbers and inverse relations across high- and low-performing math students. We tested whether adults from varying math backgrounds are sensitive to different types of similarities between multiplication problems. DeWolf and Holyoak (2014) found that participants from a highly competitive university showed facilitation in solving a multiplication problem when it was preceded by its inverse. For example, participants were faster to solve 3 X 4/3 = 4 if it was preceded by 4 X 3/4 = 3. College students were sensitive to the inverse relation between problems when the second multiplier was expressed as a fraction, but not as a decimal (e.g., the pair 3 X 1.33 = 4 and 4 X .75 = 3 yielded no facilitation of the second problem). Importantly, facilitation was found for fraction problems even when the inverse relation was less perceptually apparent (e.g., 4 X 6/8 = 3 preceded by 3 X 4/3 = 4). These college students thus showed implicit understanding of the inverse relation between fraction multiplication problems.

This apparent relational transfer might seem surprising in light of the evidence discussed above indicating that many students have difficulty in understanding fraction multiplication, let alone transferring relational knowledge between inverse problems. One hypothesis is that this type of implicit understanding of inverse relations only emerges for relatively expert (or in our study, relatively high-performing) students.. In order to assess transfer performance across a wide range of math ability that would span relatively novice and expert levels of performance, we recruited participants from two universities in one American city. We administered a general math ability test to obtain a measure of participants' overall math ability, and used this measure to separate students into high- and low-performing groups. Our sample included students who ranged widely in overall mathematical ability.

We varied the degree of perceptual similarity between inverse fraction problems. In the high-similarity condition, the relationship between the fraction problems was perceptually salient (e.g., 3 X 4/3 = 4; 4 X 3/4 = 3). In the low-similarity condition, the two fraction multipliers were perceptually different but still maintained their relational similarity (e.g., 3 X 4/3 = 4; 4 X 6/8 = 3, where 4/3 and 6/8 are reciprocals of one another). We hypothesized that low-performing participants may show facilitation in the high-similarity case, for which a perceptual strategy supports relational similarity, but not in the low-similarity case where reliance on perceptual similarity is not possible.

Method

Participants

A total of 89 undergraduates participated in the study for course credit. Thirty-four participants were undergraduates from California State University, Los Angeles (CSULA) (19 females) and 55 were undergraduates from University of California, Los Angeles (UCLA) (44 females).

Design, Materials, and Procedure

Speeded Multiplication Task. This task was an adaptation of the multiplication-priming paradigm used by DeWolf and Holyoak (2014), which demonstrated implicit relational transfer. Participants were shown a series of multiplication problems that were either true (correct) or false (incorrect). They were simply asked to verify whether the problems were true or false. A quarter of the problems were true "primed pairs" in which a prime problem was inversely related to the successive target problem (e.g., 3 X 4/3 = 4 primes 4 X 3/4 = 3). Participants were randomly assigned to one of three between-subjects conditions in this task: high-similarity fraction pairs (N = 30), low-similarity fraction pairs (N = 29), and decimal pairs (N = 30).

The high-similarity fractions were identical to the "matching fractions" used by DeWolf and Holyoak (2014), which afford a variety of perceptually-driven strategies. The low-similarity fractions were constructed by mixing the prime and target problems from the "matching fractions" and "non-matching fractions" conditions in Experiment 2 of DeWolf and Holyoak (2014). That is, these primed pairs included an expression in which the fraction and whole number components matched (e.g., 3 X 4/3 = 4), and an expression in which the fraction components did not match the whole numbers (e.g., 4 X 6/8 = 3). Priming in this condition thus depends on appreciating the inverse relation between the successive problems despite the low perceptual similarity between them. The prime and target assignments within the primed pairs were counterbalanced across participants. Importantly, the two fraction conditions were identical except for this difference in primed pairs.

As in similar previous studies of rational numbers (e.g., DeWolf, Bassok & Holyoak, 2015a), a decimal condition was included for comparison with fractions. The decimal condition used the same values as the fraction conditions, in that the second term in the multiplication problem was simply the fraction converted to its equivalent decimal rounded to the nearest hundredth (e.g., 3 X 1.33 = 4).

A total of 240 multiplication problems were used, half true and half false. Sixty of the 120 true problems were true primed problems (30 true primed pairs). Sixty of the 120 false problems were false primed problems (30 false primed pairs); these shared the inverse relation between successive problems, but were false. The remaining 120 problems were foil problems that were not related to each other in any way. These problems were designed to obscure the similarity between the primed problems. Besides the pairing of the problems within the primed pairs, the overall order of the problems was random for every participant.

The multiplication task was administered using Superlab 4.5 (Cedrus Corp., 2004), which was used to collect accuracy and response time data. Participants were told that they would see multiplication problems. They were told to press the "a" key if the problem was true or the "l" key if the problem was false. Participants were told that the answers were shown rounded to the nearest whole number. As we were interested in potentially subtle response time differences, participants were instructed to respond as quickly as possible while maintaining high accuracy. They were first given four practice trials that used only whole numbers. After the practice trials, participants were given a chance to ask questions before starting the test trials.

Explicit General Math Test A second task that participants completed was an explicit measure of general math knowledge, which was used to split the participants into relatively low- and high-performing groups. This task involved a total of 25 multiple-choice problems, and eight problems requiring a solution (either equations or word problems). The test comprised three subsections, each designed to assess a domain of mathematical understanding: algebra, fractions, and multiplicative understanding. The algebra questions (adapted from Booth et al., 2014) included basic equation solving questions, word problems, and evaluations of algebraic expressions. Fraction problems queried participants about equivalent fractions, ratio relationships, and the relation between the size of the numerator and denominator. Multiplicative questions asked about the greatest common factor of two numbers, the reciprocals of certain numbers, and lowest common multiples of two numbers. These problems were adapted from released questions from the 2008 California State Standards exam for Algebra I. Successful performance on this test would only require a level of understanding corresponding to basic high school math.

This test was administered with paper and pencil. Participants were randomly assigned to one of three different random orders. They were encouraged to use space on the page to write out their work, and were told not to use a calculator.

Results

Explicit General Math Test

Because the explicit test consisted of questions that were multiple choice, or questions for which there was only one correct answer, questions were scored on a 0, 1 basis. Final scores for subtests were averaged across questions in a relevant subset.

Overall accuracy on the test across all participants was 79% (SD = 15; minimum score = 36%, maximum score = 100%). For the subset of algebra problems, overall accuracy was 77% (SD = 17, minimum score = 41%, maximum score = 100%). For the subset of fraction questions, overall accuracy was 77% (SD = 23, minimum score = 13%, maximum score = 100%). Finally, for the subset of multiplicative questions, overall accuracy was 77% (*SD* = 17, minimum score = 50%, maximum score = 100%). The participants thus ranged considerably in overall math ability, providing a high-variance sample to examine differences in performance between low- and high-performing participants.

Relation Between Speeded Multiplication and Math Expertise

Accuracy on the speeded multiplication task was computed for true[1] prime and target trials after dividing participants into low- and high-performance groups, based on a median split performed separately for each of the three conditions. For the high-similarity fractions condition, the high-performance group achieved significantly higher accuracy than the low-performance group (.97 vs. .87; $t(24) = 2.53$, $p = .02$). For the low-similarity fraction condition, the high-performance group showed a non-significant trend for higher accuracy than the low-performance group (.91 vs. .84; $t(27) = 1.33$, $p = .19$). For the decimals condition, the high-performance group also showed a non-significant trend for higher accuracy relative to the low-performance group (.67 vs .54, $t(27) = 1.36$, $p = .18$). There was no evidence of a priming effect for any of the conditions on the accuracy measure, and no evidence of a difference in priming across low- and high-performing groups.

Figure 1 shows response times for true prime and target trials, separated into low- and high-performance groups based on the same median split of the explicit task used for the accuracy analysis. Response times for error trials and those more than three standard deviations from the mean of accurate trials were excluded from analyses. The change in RT from the average prime RT to the average target RT (prime – target) was calculated to assess the speed-up attributable to priming for each participant. For the high-similarity fractions, the average speed-up across participants in the low-performing group was significantly larger than that for the high-performing group (.73 s vs. .23 s, $t(24) = 3.18$, $p = .004$). Thus when pairs of mathematical problems were perceptually similar, as in the high-similarity condition, the priming effect held for both high- and low-performing students, and was actually largest for low-performing students.

The priming difference observed in the high-similarity fraction condition may in part be related to general differences in RT between the low- and high-performing groups. Average prime RT for the low-performing group was considerably slower than for the high-performing group. Also, average target RT for the low-performing group was slower than RT for either the prime or target in

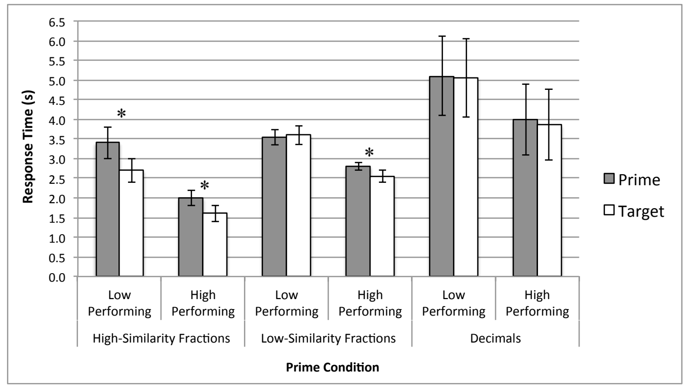

Figure 1. Average response times for true prime and target trials for each condition, separated by low- versus high-performing math students.

the high-performing group. Thus, although the speed-up for the low-performing group was much larger than that for the high-performing group, these differential gains may be related to the longer overall response times of the former group.

In stark contrast, for the low-similarity fraction condition, the high-performing group had a significantly greater average speed-up than did the low-performing group (.27 s vs. -.06 s, $t(27) = 2.33$, $p = .03$). In fact, the latter group showed no priming effect, and, their RTs were slower overall compared to those of the high-performing group.

For the decimals condition, neither high- nor low-performing students showed any evidence of priming. The average speed-up for the low-performing group did not differ reliably from that of the high-performing group (.03 s vs. .09 s, $t(27) = .13$, $p = .89$). For both groups, RTs on decimal trials were slower than on either type of fraction trials.

Correlations between Multiplication Task and Math Expertise In order to better understand how the subparts of the explicit math test related to performance on the speeded multiplication task, we correlated performance on each of the subparts of the explicit test (algebra, fraction, multiplicative) with overall accuracy and response times for each of the three conditions on the multiplication task, for all participants (i.e., combining high- and low-performing participants). Table 1 shows the correlations between each of the multiplication-task conditions and accuracy on the explicit math test (overall and for each subtest).

For the high-similarity fraction condition, both overall accuracy and RT on the multiplication task were significantly correlated with overall accuracy on the explicit test and with each of the subtests. A similar pattern of correlations was observed for the low-similarity fraction condition, except that correlations with the multiplicative subtest were not reliable. For the decimals condition, accuracy was significantly correlated with all but the multiplicative test, whereas RT was correlated only with the algebra subtest (and overall score).

[1] As in DeWolf and Holyoak (2014), we find that the priming effect only held for the true primed trials (high-similarity false primes: 4.5 s vs. 4.3 s, $t(25) = .91$, $p = .37$; low-similarity false primes: 4.9 s vs. 4.5 s, $t(28) = 1.93$, $p = .07$). The same pattern of results for false primed trials was found after conducting the median-split analysis.

Table 1: Correlations between performance on the speeded multiplication task (accuracy and RT) and score on explicit math test (overall and each subtest).

Multiplication Condition	Overall Test	Algebra Subtest	Fractions Subtest	Multipli-cative Subtest
High-similarity Fraction				
Accuracy	$.70^{***}$	$.60^{***}$	$.54^{**}$	$.75^{***}$
RT	$-.60^{***}$	$-.59^{**}$	$-.51^{**}$	$-.40^{*}$
Low-similarity Fraction				
Accuracy	$.53^{**}$	$.55^{**}$	$.50^{**}$	.25
RT	$-.54^{**}$	$-.47^{*}$	$-.57^{**}$	-.36
Decimals				
Accuracy	$.55^{**}$	$.54^{**}$	$.46^{*}$	.25
RT	$-.37^{*}$	$-.45^{*}$	-.15	-.19

* p<.05; ** p<.01; *** p<.001

The observed pattern of correlations between speeded multiplication and explicit math performance suggests that across all participants, multiplicative knowledge is not a reliable predictor when perceptual cues to relational structure are lacking (i.e., in the low-similarity fraction and decimal conditions). Especially in a speeded task, multiplicative strategies may not be employed without obvious perceptual supports (at least for students with lower math ability).

Discussion

The present study demonstrates clear differences in understanding of inverse relations and fraction multiplication between low- and high-performing math students. In the high-similarity fraction condition, perceptual cues guide attention to relational structure, as the numbers in the first problem are simply rearranged in the inverse manner in the second problem (e.g., 3 X 4/3 = 4; 4 X 3/4 = 3). For this pair type, the low-performing group showed *greater* facilitation on response time for target problems than did high-performing participants. Low-performing students were thus able to capitalize on perceptual similarity between problems to facilitate transfer.

Despite their large decrease in response time on target trials, low-performing participants were still slower in responding than high-performing participants. Moreover, the two groups performed very differently in the low-similarity fraction condition. In such problems the numbers in the successive fractions were different, even though they were reciprocals (e.g., 3 X 4/3 = 4; 4 X 6/8 = 3). For these low-similarity problems, for which perceptual cues did not strongly guide attention to relation structure, only high-performing students demonstrated facilitation on target problems.

The present findings are consistent with previous work on novice versus expert transfer in mathematics and other domains, but go beyond previous studies by using the speeded multiplication task to provide an implicit measure of relational transfer. This implicit measure of transfer indicates that individuals with greater math expertise process relationally-similar problems more effectively than do novices.

In general, low math performers were slower and less accurate on the speeded multiplication task. In addition, we found that while the magnitude of priming for the RT measure correlated with all of the subparts of the explicit math test, RT in the decimal condition only correlated with the algebra subset of questions. Because performance on decimal problems largely depends on correctly estimating the magnitude of the decimal and the resulting product (i.e., there is no possible simplification of the problem as there is for fraction problems), the decimal condition is largely an estimation task. Thus, the decimal version of the task is likely measuring something akin to decimal magnitude understanding. Previous work has shown that, at least for middle-school Algebra-I students, decimal magnitude understanding is a strong predictor of algebra performance (DeWolf, Bassok & Holyoak, 2015b).

Several potential mechanisms may contribute to the priming effect we observed in the speeded multiplication task. Perceptual similarity between problems accounts for part of the effect, as evidenced by the robust priming effect observed in the high-similarity fraction condition for both low- and high-performing participants. As observed in other relational tasks, reasoning is facilitated by salient semantic or perceptual cues that are correlated with more abstract relations (Bassok, 1996).

More interesting, perhaps, are the possible mechanisms by which relatively expert adults are able to exploit shared inverse relations between fraction problems that lack simple perceptual cues to the relational correspondences. Our findings suggest that low-performing participants understand the inverse relation at some level, but lack the abilities in pattern recognition or in simplifying fractions that are required to obtain priming in the low-similarity condition. It appears that experts have an advantage in recognizing equivalent fractions based on different constituent numbers, likely due to the greater fluency with which experts are able to reduce or simplify common fractions. These high-performing participants may in some sense "see" fractions differently. High-performing adults may have greater perceptual expertise with fractions, and in connecting alternative simplified or reduced forms of fractions (cf. Kellman et al, 2008). Thus, an expert (or high-performing student) may have a deeper appreciation for relevant relations and operations even at an implicit level. Low- and high-performing participants showed little difference in accuracy on the simple multiplication problems we tested; however, the large differences we observed in response times suggest that more expert adults benefit from more direct access to inverse relations, allowing them to make mappings between problems with greater ease.

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