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Dynamics of helicity transport and Taylor relaxation

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A simple model of the dynamics of Taylor relaxation is derived using symmetry principles alone. No statistical closure approximations are invoked or detailed plasma model properties assumed. Notably, the model predicts several classes of nondiffusive helicity transport phenomena, including traveling nonlinear waves and superdiffusive turbulent pulses. A universal expression for the scaling of the effective magnetic Reynolds number of a system undergoing Taylor relaxation is derived. Some basic properties of intermittency in helicity transport are examined. © *2003 American Institute of Physics*. [DOI: 10.1063/1.1576390]

I. INTRODUCTION

One of the most elegant ideas in plasma turbulence and self-organization theory is the Taylor conjecture on turbulent magnetic relaxation.[1,2] Loosely put, the Taylor conjecture states that the magnetic field configuration formed as the end-state of a turbulent relaxation process is one for which the magnetic energy is minimized subject to the constraint of conservation of *global* magnetic helicity $\int d^3x \underline{A}\cdot\underline{B}$. Global helicity is considered as the constraining quantity, as it is assumed that turbulence, magnetic field line stochasticity and the resulting coupling to small scale resistive damping together dissipate local magnetic helicity. The Taylor hypothesis is quite successful in predicting the magnetic configuration of reversed field pinch (RFP) plasmas, the profiles in some other magnetic confinement devices and of many astrophysical plasmas, and has also spawned the idea of helicity injection as a means for sustaining plasma currents. Indeed, the conservation of helicity in the RFP has been experimentally verified.[3] Taylor's proposal is referred to as a conjecture, since no rigorous justification of the comparative "ruggedness" of global helicity relative to energy has yet been given. However, there are at least three quite plausible physical arguments which support the Taylor hypothesis. These are:

(a) "Enhanced dissipation:" Turbulence and dissipation drive magnetic reconnection, which destroys domains of local magnetic helicity on all but the largest scale. Thus, time asymptotically, global magnetic helicity is the only surviving topological invariant.
(b) "Field line stochasticity:" During turbulent relaxation, magnetic field lines become stochastic, so global magnetic connections develop. All regions of the plasma are connected by any field line so, equivalently, there is only one field line. Since magnetic helicity is calculated by volume integration over a region enclosed by a magnetic surface, and all surfaces are destroyed, only global magnetic helicity is relevant to a stochastic state.
(c) "Selective decay:" In 3D magnetohydrodynamic (MHD) turbulence, energy cascades to small scale, while magnetic helicity inverse cascades to large scales. As a result, magnetic helicity accumulates on the largest scales of the system, with minimal coupling to dissipation. Thus, magnetic helicity is dynamically rugged, while energy decays.[4,5]

Taken together, (a), (b), and (c) constitute a plausible argument for the Taylor conjecture.

Stated in terms of profiles, the essence of the Taylor theory is that the radial profile of parallel current tends to flatten, i.e., $\underline{J}\cdot\underline{B}/B^2 \rightarrow$constant, so that the system "self-organizes" itself into a state with no available free energy for kink-tearing modes, which are current gradient driven. Note also that turbulence and magnetic field line stochasticity are fundamental to the Taylor relaxation process, both as means of justifying the comparative ruggedness of global magnetic helicity, and as mechanisms for cross-field transport of current by fluctuations, which is essential for "Tayloring" of the current gradient.

The Taylor state is, of course, an idealization of the actual final state of the relaxation process. Resistive magnetic diffusion leads to decay of the Taylor state, which in turn triggers the reappearance of MHD instabilities and turbulence. These then drive the system back toward the Taylor state by current profile modification. Alternatively put, departure from the Taylor state triggers a *dynamo*, which then returns the system to the Taylor configuration, thus turning itself off.[6] Note, in contrast to the solar dynamo, the dynamo associated with the Taylor relaxation process hovers near marginality, and is strongly coupled to fluctuation energetics. External energy input is thus necessary to sustain the dynamo, which keeps the system in the vicinity of the Taylor state. Indeed, the physical current profile is one determined by a self-organization process involving energy input, turbulence, dissipation, and transport, with eventual coupling to lossy boundaries. Thus, the actual physical (i.e., RFP plasma) configuration is *dynamic*, with finite turbulence levels maintaining the current profile against dissipation. Also, nonstationary cyclic or "bursty" states are possible system attractors. Thus, the profile achieved via a Taylor relaxation process may be loosely considered to be a type of *self-organized criticality*,[7,8] and the Taylor state can be thought of as an approximation to or idealization of the actual attractor for the system. From that perspective, then, one expects there

should be some universal properties, such as effective Reynolds number scalings, etc., which characterize the dynamics of the Taylor process. Finally, in the RFP, the fluctuations driving the dynamo are not just turbulent eddys, but include global $m=1$ tearing modes.[9,10] Nevertheless, strong nonlinear interaction, generation of higher m's, stochastization of field lines and current profile flattening on account of transport *do* occur in the vicinity of mode resonant surfaces (where $\underline{k}\cdot\underline{B}=0$) in the core of the pinch. In this region, then, the RFP relaxation process certainly exhibits the properties of a sand-pile or self-organized criticality alluded to above. This issue is discussed further in the conclusion.

Here, we develop a simple gedanken model based on a synergy between a general formulation of Taylor relaxation and continuum models of sandpile relaxation.[11,12] The basic idea exploits the observation that the simplest form of the helicity density flux which dissipates magnetic energy is diffusion of current, i.e., $\Gamma_H=-D\underline{\nabla}J_\parallel$.[13–15] Employing ideas from continuum models of avalanche dynamics, one can use simple symmetry constraints to constrain the form of Γ_H, and so derive an equation for deviations about the Taylor state, as well as the scalings of the effective magnetic Reynolds number, *without* postulating Fick's law for the structure of the flux. Moreover, no statistical closure theory is utilized. Thus, the model encompasses hyperresistive diffusion, but is far more general.

The remainder of this paper is organized as follows. In Sec. II, a simple model of helicity transport is derived using symmetry principles. Analogies with avalanche phenomena are discussed. In Sec. III, solutions in radius and time of the partial differential equation derived in Sec. II are analyzed. Both solitary pulse solutions and modulational wave solutions are found. In Sec. IV, the basic turbulence scaling properties of the system are discussed. The renormalized response function and self-similarity exponents are calculated. Superdiffusive pulse propagation is indicated. In Sec. V, basic properties of intermittency in helicity transport are discussed. Section VI contains a discussion and conclusions.

II. A SIMPLE MODEL OF TAYLOR RELAXATION DYNAMICS

Starting from Ohm's law,

$$\underline{E}+\frac{\underline{v}\times\underline{B}}{c}=\eta_0\underline{J}, \tag{1}$$

where η_0 is the collisional resistivity, and taking $\underline{B}/|\underline{B}|\cdot(\underline{E}+(\underline{v}\times\underline{B})/c=\eta_0\underline{J})$ we can write

$$-\frac{1}{c}\frac{\partial A_\parallel}{\partial t}-\hat{n}\cdot\underline{\nabla}\phi-\underline{\nabla}A_\parallel\times\hat{n}\cdot\nabla\phi=\eta_0 J_\parallel. \tag{2}$$

For simplicity, assume a locally strong $\underline{B}_0$ along $\hat{n}$, so $\nabla_\parallel=\hat{n}\cdot\nabla+\delta\underline{B}_\perp\cdot\underline{\nabla}/|\underline{B}_0|$ and $\delta B_\perp=\underline{\nabla}A_\parallel\times\hat{n}$. Further, take the mean inhomogeneity in the radial direction, orthogonal to $\underline{B}_0$. Thus, this gedanken problem corresponds loosely to that of determining the profile $J_\parallel(r)$ in a RFP and, more specifically, to the characterization of the deviation from the Taylor state (i.e., a flat current profile). As we seek the mean current profile, averaging Eq. (2) over poloidal and toroidal directions gives

$$-\frac{1}{c}\frac{\partial}{\partial t}\langle A_\parallel\rangle+\partial_r[\langle(\nabla_y\hat{\phi})\hat{A}_\parallel\rangle]=\eta_0\langle J_\parallel\rangle. \tag{3}$$

Here $\nabla_y=1/r(\partial/\partial\theta)$. Now, using Ampère's law and observing that in a strongly magnetized system, mean field magnetic helicity H is given by

$$H=\int d^3x\langle A_\parallel\rangle B_0, \tag{4}$$

the magnetic *helicity density* evolution equation is

$$\frac{\partial\langle\mathcal{H}\rangle}{\partial t}+\partial_r\left[\left\langle-\frac{c}{B_0}\nabla_y\hat{\phi}\mathcal{H}\right\rangle\right]=\frac{\partial}{\partial t}\langle\mathcal{H}\rangle+\partial_r\Gamma_H$$
$$=\eta B_0\partial_r^2\langle A_\parallel\rangle, \tag{5}$$

where $\eta=c^2\eta_0/4\pi$. Here $\Gamma_H=\langle-c/B_0\nabla_y\hat{\phi}\mathcal{H}\rangle$ is the magnetic helicity density flux. This equation obviously has the form of an advection equation for magnetic helicity density $\mathcal{H}=A_\parallel B_0$. Note that radial transport and turbulent mixing conserve magnetic helicity (up to the net flux thru the system boundary), so that resistive dissipation is the only helicity sink. Finally, one can short-circuit the derivation of this paragraph by noting that Eq. (5) follows directly (in reduced MHD)[16] from the flux equation.

Now, following Boozer,[17] one can ask what the form of the magnetic helicity density flux Γ_H must be to dissipate magnetic energy. Multiplying Eq. (3) by $\langle J_\parallel\rangle/B_0$ and integrating by parts gives

$$\frac{\partial}{\partial t}E_M-\int d^3x\frac{\Gamma_H}{B_0}\partial_r\langle J_\parallel\rangle=-\int d^3x\,\eta\langle J_\parallel\rangle^2, \tag{6}$$

where E_M is mean field magnetic energy. Thus, taking $\Gamma_H=-\mu\partial r\langle J_\parallel\rangle$ gives $\partial_t E_M<0$, corresponding to dissipation of magnetic energy. This observation is consistent with the notion of Taylor relaxation as being a process of current-gradient flattening, and supports the idea of a "hyperresistivity" (i.e., diffusion of current) as constituting a possible analytical representation of the Taylor relaxation process. It is interesting to note, however, that the hyperresistivity is just the smoothest of an infinite series of operators which dissipate E_M. For example, if one takes $\Gamma_H=\mu\partial_r^3\langle J_\parallel\rangle$, then

$$\frac{\partial}{\partial t}E_M=-\int d^3x\mu(\partial_r^2\langle J_\parallel\rangle)^2-\int\eta\langle J_\parallel\rangle^2d^3x. \tag{7}$$

Thus, any odd derivative of $\langle J_\parallel\rangle$ dissipates E_M, as does any combination of an odd derivative and an even power of $J_\parallel$, i.e., $\alpha\langle J_\parallel\rangle^2\partial r\langle J_\parallel\rangle$, etc. More generally, we can observe that since $\langle J_\parallel\rangle_{\text{Taylor}}=\text{const}$ (i.e., flat current profile), one can write the helicity flux which dissipates magnetic energy as

$$\Gamma_H=-D\nabla(\langle J_\parallel\rangle-\langle J_\parallel\rangle_T)=-D\partial_r\delta J_\parallel, \tag{8}$$

where $\delta J_\parallel$ is the excursion from the Taylor state. As MHD tearing type turbulence is expected to be the agent of relaxation, we in turn expect $D=D[\delta J_\parallel]$, i.e., *the strength of the*

current diffusion process should itself be proportional to the deviation from the Taylor state. Thus, we expect $\Gamma_H \sim \delta J_\parallel^2$, at least, *so that the actual magnetic helicity flux is nonlinear in* $\delta J_\parallel$. Indeed, we shall soon see that hyper-resistive diffusion is by no means the most general form of the flux.

From the above discussion, it seems that Taylor relaxation can profitably be viewed as similar to a running sandpile,[17,18] with local pile grain occupation density set by magnetic helicity density, which is a conserved order parameter (i.e., a locally conserved effective density). Writing $\delta\mathcal{H}$ for the deviation of the local magnetic helicity density from the self-organized state, conservation of helicity gives

$$\frac{\partial}{\partial t}\delta\mathcal{H} + \partial_r \Gamma_H[\delta\mathcal{H}] = \eta B_0 \partial_r^2 \delta A_\parallel + S', \tag{9a}$$

or, equivalently,

$$\frac{\partial}{\partial t}\delta A_\parallel + \frac{1}{B_0}\partial_r \Gamma_H[\delta\mathcal{H}] = \eta \partial_r^2 \delta A_\parallel + S. \tag{9b}$$

Here, Γ_H is the helicity density flux driven by deviations of the profile from the self-organized critical (SOC) state. It is important to note that the self-organized state is, in principle, *not necessarily equal to the Taylor state*, on account of resistive dissipation, the helicity source term S (which represents the external drive of the system) and boundary losses. Indeed, the departure from the Taylor state profile is most pronounced near the edge where the finite temperature gradient and large resistivity force a departure from a flat current profile. The key question is how to parametrize $\Gamma[\delta\mathcal{H}]$. Since the SOC state is one defined by the current profile, we ansatz $\Gamma_H[\delta\mathcal{H}] \to \Gamma_H[\delta J_\parallel]$, i.e., helicity flux as a function of the deviation of the current profile from the SOC state. *It is important to note that hereafter* $\delta J_\parallel$ *refers to the excursion from the self-organized profile, which is close, but not identical to, the Taylor state profile*. Note that by definition, $\Gamma_H \to 0$ for $\delta J_\parallel \to 0$. Thus, Eq. (9b) becomes

$$\frac{\partial}{\partial t}\delta A_\parallel + \frac{1}{B_0}\partial_r \Gamma_H[\delta J_\parallel] = \eta \partial_r^2 \delta A_\parallel + S. \tag{10}$$

Now, take the system to be bounded, so any helicity excess which reaches the edge ($r=a$) is lost, by expulsion thru the boundary. In practice, this implies a finite current gradient exists at the boundary, so that $\langle J_\parallel(r)\rangle$ deviates from Taylor there, and that a flux of helicity is transported thru the boundary. Hence, one can expect excesses ($\delta J_\parallel > 0$) to be expelled from the system, while deficits ($\delta J_\parallel < 0$) are absorbed, as depicted in Fig. 1. This is analogous to the case of the sandpile, where bumps, i.e., local excesses beyond the SOC profile, are expelled from the pile while local deficit regions (i.e., voids) are absorbed. Note that the expressions "expulsion/absorption of bumps/voids" are simply alternative ways to express the tendency of the current profile to regulate itself and relax to the self-organized state. Following Hwa and Kardar, one can require $\Gamma_H[\delta J_\parallel]$ to be joint reflection symmetric, i.e., Γ_H must remain invariant when $\delta J_\parallel \to -\delta J_\parallel$ and $x \to -x$, to ensure the expulsion of excess current and the absorption of current deficits.[19] This drastically simplifies the form of $\Gamma_H[\delta J_\parallel]$ by eliminating several classes of terms, so that

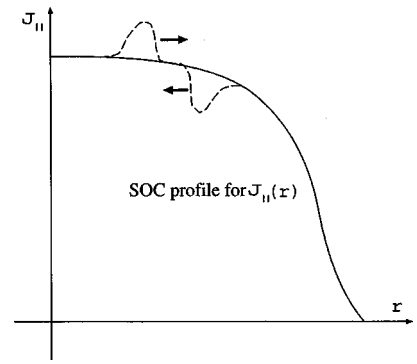

FIG. 1. Bumps or excesses ($\delta J_\parallel > 0$) above the SOC profile are expelled, while voids or defects ($\delta J_\parallel < 0$) are absorbed.

$$\Gamma_H[\delta J_\parallel] = \sum_i \lambda_{2i}(\delta J_\parallel)^{2i} + \sum_p D_{2p+1}(\partial_r \delta J_\parallel)^{2p+1} + \cdots, \tag{11a}$$

where the "smoothest" approximation (i.e., that which is dominant in the long wavelength, large scale limit) is

$$\Gamma_H[\delta J_\parallel] = \lambda(\delta J_\parallel)^2 + D\partial_r(\delta J_\parallel). \tag{11b}$$

Note that this corresponds to a constant diffusion of current and a flux quadratically nonlinear in $\delta J_\parallel$, as suggested earlier. Thus, in the hydrodynamic limit, the simple model equation for excursions from the Taylor state is

$$\frac{\partial}{\partial t}\delta A_\parallel + \frac{\partial}{\partial r}\left\{\frac{\lambda}{2}(\partial_r^2 \delta A_\parallel)^2 + D\partial_r(\partial_r^2 \delta A_\parallel)\right\} = \eta \partial_r^2 \delta A_\parallel + S. \tag{12}$$

Equation (12) constitutes a mesoscale model for fluctuations near the SOC/Taylor state induced by drive (S) and $\delta J_\parallel$-driven helicity flux. Here, "mesoscale" refers to the range of scales between the reconnection layer size and the system size. This range of scale sizes encompasses that of an $m=1$ tearing mode magnetic island width, in the case of the RFP. This is the scale over which the current gradient is irreversibly mixed. Note that Eq. (12) is manifestly scale invariant, and that the parameters λ and D can be chosen in accord with different plasma models. An interesting generic approach to determining them would be via Connor–Taylor scaling analysis.[20] Here the drive S can be taken to have the form of a colored noise. By colored noise, we refer to noise which is not white, i.e., noise which does not have a flat spectral density. There is no loss of generality in postulating a noisy source here, since Taylor relaxation tacitly presumes the release of local frozen-in constraints via reconnection and stochasticity. Indeed, one way of looking at the physical origin of the noise is that in the presence of the magnetic fluctuations which drive relaxation, the local toroidal electric field will naturally fluctuate as well.[21] Thus, the characterization of mesoscale dynamics near the Taylor state can be reduced to a problem of the "driven Burgers" genre.

Equation (12) may be viewed as an effective Ohm's law for Taylor relaxation, where the nonlinearity $\lambda\,\delta J_{\parallel}(\partial/\partial r)\,\delta J_{\parallel}$ is related to the familiar hyper-resistive diffusion flux with $D\approx\lambda\,\delta J_{\parallel}$, but more general. Noting that $\delta\mathcal{H}=B_0\,\delta A_{\parallel}$, it is also an equation for magnetic helicity density evolution and transport in radius and time. The obvious structural similarity between Eq. (12) and Burgers equation suggests that helicity transport during Taylor relaxation is strongly intermittent, and can exhibit nondiffusive scaling properties, symptomatic of front propagation, helicity density "avalanches," etc. Indeed, Eq. (12) is surely the minimal possible model for the study of intermittency in magnetic helicity transport, a subject of considerable interest in the context of nonlinear dynamo theory.

III. COHERENT NONLINEAR HELICITY TRANSPORT PHENOMENA

In this section, the unforced solutions of Eq. (12) are analyzed. The aim here is to understand the structure of the basic nonlinear solutions. As in Burgers turbulence, it is likely that the randomly forced state may be viewed as a "gas" of the coherent solutions, as Burgers turbulence may be thought of as a "gas" of shocks.[22,23] Particular attention is devoted to traveling wave solutions. These exist, and may be either solitary pulses or modulational waves. Here we consider the undriven limit of Eq. (12) by setting $S=0$. For the purposes of this section, it is convenient to rescale Eq. (12) in such a way as to make $D=\lambda/2=1$, so the rescaled magnetic diffusion coefficient η will be denoted as μ, $\delta A_{\parallel}$ as u, and r as x. The equation then reads

$$\frac{\partial u}{\partial t}+(u_{xx}^2)_x+u_{xxxx}=\mu u_{xx}. \tag{13}$$

First, let us consider the case in which the magnetic field perturbation u_x is spatially localized or periodic. More specific conditions will be given later. Suppose that u_x decays sufficiently rapidly as $|x|\to\infty$ so that the magnetic field perturbation u_x, as well as u_{xx}, $u_{xxx}\subset L^2$. The same results will be valid for the case of periodic u_x, i.e., $x \bmod 2\pi$. Differentiating Eq. (13) with respect to x, multiplying the result by u_x and integrating by parts we obtain

$$\frac{1}{2}\frac{\partial}{\partial t}\int u_x^2dx=-\int u_{xxx}^2dx-\mu\int u_{xx}^2dx, \tag{14}$$

where the integrals are taken between $-\infty$ and ∞ in the case $u_x\subset L^2$, or between 0 and 2π in the case of a 2π periodic function u_x. The right-hand side of this expression is negative definite and its absolute value can be shown to be larger than $\alpha\int dx u_x^2$, with some constant $\alpha>0$ that depends on the functional space under consideration. For example, in the case of $x \bmod 2\pi$, one can show $\alpha>4\pi^2/(4\pi^2+\mu)$. The perturbation of the magnetic field u_x thus decays exponentially as

$$\int u_x^2(t)dx=e^{-2\alpha t}\int u_x^2(t=0)dx. \tag{15}$$

Thus, *all spatially localized solutions decay asymptotically in time*.

Having demonstrated that all spatially localized solutions for the magnetic field perturbations must decay due to resistivity and hyper-resistivity, we now turn to the case in which the magnetic field perturbation does not vanish at infinity. For this purpose, it is convenient to work with the quantity proportional to the current density, $w=u_{xx}$ for which Eq. (13) takes the following form:

$$\frac{\partial w}{\partial t}+\frac{\partial^3}{\partial x^3}(w^2+w_x)=\mu w_{xx}. \tag{16}$$

An energy relation similar to Eq. (14) does not hold for w or w_x, so time-asymptotically nontrivial solutions are now possible. The simplest one can be obtained by setting $\partial_t=\mu=0$. From this equation we obtain

$$\frac{\partial w}{\partial x}+w^2=a^2 \tag{17}$$

with $a=\text{const}$. Hence,

$$w=w_0(x)\equiv a\tanh(ax). \tag{18}$$

Put physically, this solution describes the interface between two oppositely directed currents with strengths $\pm a$. The magnetic field perturbation is given by

$$b(x)\equiv u_x=ln\cosh(ax) \tag{19}$$

so that the behavior of $b(x)$ is linear at large $|x|$, i.e., as $b\sim|x|$. Of course, this is an inviscid solution, and the question of how it is affected by dissipation naturally arises. One simple way to understand how this solution is affected by the hyper-resistive term on the right-hand side (RHS) of Eq. (16) is to use an "adiabatic" approximation for sufficiently small μ, i.e., one can assume that the solution in Eq. (17) preserves its form $w_0(x,a)$ for $\mu\neq0$, but now a is replaced by $a(t)$. The time dependence of $a(t)$ captures the effect of hyper-resistive dissipation. By multiplying the x-derivative of Eq. (16) by w_{0x}, we obtain $\partial(\int_{-\infty}^{+\infty}w_{0x}^2dx)/\partial t=-2\mu\int_{-\infty}^{+\infty}w_{0xx}^2dx$, from which we formally obtain $a(t)=a_0/\sqrt{1+\sigma t}$. Here $\sigma=(16/15)a_0^2\mu$.

Traveling nonlinear wave solutions are of great interest, as they embody the structure of the fundamental nonlinear excitons of the system. In the analogous system described by Burgers equation, the traveling wave solutions steepen into shocks. Here, we will demonstrate the existence of both soliton-like pulse solutions and of nonlinear modulation waves. To study traveling wave solutions, it is convenient to rescale Eq. (16) to the variables in which the hyperdiffusion term appears with an arbitrary coefficient D, i.e.,

$$\frac{\partial w}{\partial t}+\frac{\partial^3}{\partial x^3}(w^2+Dw_x)=\mu w_{xx}. \tag{20}$$

Let us consider the nondissipative case $\mu=D=0$ and look for a traveling wave solution of the form $w=w(x-ct)$. Equation (20) can then be integrated to the following "energy conservation" form, i.e.,

$$\frac{1}{C}w_x^2-\frac{1}{3}w-\frac{a^3}{6w^2}=-\frac{a}{2}. \tag{21}$$

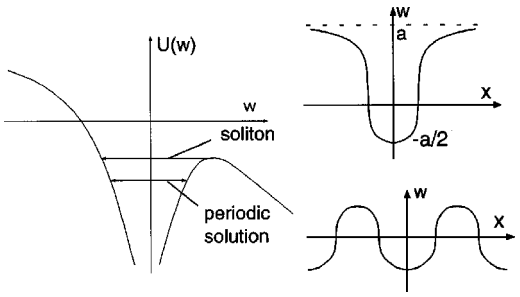

FIG. 2. The form of the potential $U(w)$ in Eq. (23) (left panel) and two types of bounded solutions (right panels). One solution is of the soliton type (upper right panel), corresponding to the highest possible "energy" level in the potential well. The family of periodic solutions corresponds to a continuum of all the remaining levels in the potential well.

Here a is an arbitrary integration constant which is the value of w at infinity, while the second integration constant that appears during integration has been chosen so that $w_x \to 0$ as $w \to a$. The solution of Eq. (22) can be most easily understood by examining Fig. 2, which shows the form of the potential well in Eq. (21), formed by the second and the third terms on the RHS. It starts from $w=a$ at $x=-\infty$, descends to the minimum $w_{\min}=-a/2$ and returns to $w=a$ at $x=\infty$. This solution can be written explicitly in an "inverse" form as $x(w)$, where

$$x(w)=\pm\sqrt{\frac{2a}{C}}\left[ln\left|\frac{\sqrt{2w/3a+1/3}+1}{\sqrt{2w/3a+1/3}-1}\right| -3\sqrt{2w/3a+1/3}\right]+x_{\pm}. \tag{22}$$

Here, the upper (lower) sign should be taken for $w>0(w<0)$. The integration constants $x_{\pm}$ are chosen in such a way as to make $x(w)$ continuous, namely $x_-=0$ and $x_+=2x(0-)$. This representation of the solution covers the half space $x>0$. For negative x, the solution is symmetric so $w(-x)=w(x)$. Note that w is a continuous function of x but $w'(x)$ is infinite where $w=0$, or more precisely $w(x) \simeq \pm\sqrt{|x-x(0)|}$ as $x \to x(0)$. It is thus clear that *dissipative terms* in Eq. (20) must be included to obtain smooth behavior at $w=0$.

For an arbitrary choice of integration constants, the energy integral [Eq. (21)] can be written in the following form:

$$\left(\frac{dw}{dx}\right)^2+U(w)=E, \tag{23}$$

where

$$U=-\frac{C}{3}w-\frac{B}{w^2}. \tag{24}$$

The solutions can be expressed in terms of elliptic functions. The particular solution considered above occurs when $C>0$ and $B>0$. For this case one can also construct periodic solutions corresponding to the lower levels in the potential well, with a singularity at $w=0$, as shown in Fig. 2. This solution behaves at $w=0$ in the same way as the one described above. In the case $C<0$ and $B<0$, there exist regular periodic solutions that do not cross the singular point $w=0$, as shown in Fig. 3. Note that other sign combinations of the constants B and C are equivalent to the previous, up to flipping the sign of w.

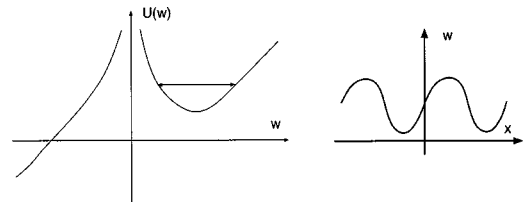

FIG. 3. Potential well corresponding to Eq. (23) in the case B, $C<0$ (left panel). The solution $w(x)$ is periodic and regular everywhere (right panel).

The physical properties of the traveling wave solutions to Eq. (12) merit some discussion. For the case where $C>0$ and $B<0$, two types of solution are possible (see Fig. 2). In the case corresponding to the highest possible energy in the potential well U, a "soliton" type solution is found. The solution is a localized pulse, as is typical of solitary waves. For lower energies in the well, a periodic, nonlinear wave train is the solution. This structure is somewhat reminiscent of a cnoidal wave, and may be thought of as arising from a modulation or corrugation of a flat current profile. Here a cnoidal wave is one with a wave form resembling that of the elliptic function cn.[24] It is interesting to speculate that such a modulational pattern may be related to the well known tendency of a current to form filaments. In the case $C>0$, $B<0$, as in Fig. 3, the solution is trapped in the well and thus is periodic and regular everywhere.

The inquiring reader may, at this point, be perplexed by the appearance of "soliton-like" solutions to a nondispersive equation. In such a system, the familiar scenario of pulse formation by the balance of nonlinear steepening with dispersion cannot be realized. However, the soliton-like solution obtained here is quite different from conventional solitons, such as those which occur in the Korteweg–de Vries (KdV) (Ref. 25) system. In addition to the singularity (already mentioned), the familiar relation of the amplitude and speed of the pulse is absent. However, the pulse width l is clearly related to its speed and amplitude. This is evident by setting $a=C=1$ in Eq. (21), so $w/a \to w$ and $\sqrt{C/a}x \to x$. Thus, $l \sim \sqrt{a/C}$, which is equivalent to KdV-type scaling for fixed a, only. For fixed C, it is reciprocal to the KdV scaling ($l \sim a^{-1/2}$). At the same time, the oft-quoted intuition that solitons form via the balance of steepening and dispersion may yet be applicable here. Indeed, counting w from its value at infinity and defining $\phi=a-w$, we obtain the equation

$$\frac{\partial\phi}{\partial t}+2a\frac{\partial^3\phi}{\partial x^3}-\frac{\partial^3\phi^2}{\partial x^3}=0.$$

Note that a small, localized perturbation $\phi \sim \epsilon < a$ spreads linearly due to dispersion ($\sim a$), just as for KdV. For larger ϵ, the nonlinear term can come into play to limit the spreading, thus allowing the formation of coherent, localized solutions. Under what conditions, and precisely how this happens, remain unclear. A numerical study is clearly required, and is ongoing. The results will be reported in a future pub-

lication. Of course, as in the KdV theory, the soliton solution cannot be obtained at any finite order in perturbation theory.

At a more general level, the considerations of this section clearly indicate that traveling waves of helicity density can develop during Taylor relaxation. Such waves transport helicity nondiffusively, at a speed C determined by the structure of $U(w)$. While the analysis here applies only to the limit where D, $\eta \to 0$, it nevertheless strongly suggests that such phenomena can be expected to occur in regimes of large effective magnetic Reynolds number.

IV. SCALING PROPERTIES OF TURBULENT HELICITY TRANSPORT

In this section, we determine the structure and scaling exponents of the turbulent helicity density flux and the effective "magnetic Reynolds number" of the relaxation process by applying standard methods of turbulence closure theory to Eq. (12). This study is analogous to those of the noisy Burgers equation for sandpile fluctuations. The aims here are to understand the structure of the turbulent dissipation (i.e., the effective eddy viscosity) which arises in Eq. (12), and to determine the scaling exponents of the turbulent response. To this end, we must explore the infrared behavior of the dissipation coefficient.[26] Writing Eq. (12) in Fourier variables gives

$$-i\omega A_{\substack{k\\ \omega}} + (\eta_0 k^2 + Dk^4) A_{\substack{k\\ \omega}} + ik\frac{\lambda}{2}\sum_{k',\omega'} k'^2 A_{\substack{-k'\\ -\omega'}} (k+k')^2 A_{\substack{k'+k\\ \omega'+\omega}} = S_{k,\omega}. \tag{25}$$

The δ and sub-$\|$ have been dropped for notational convenience, thus $A_{\substack{k\\ \omega}} = \delta A_{\substack{\|k\\ \omega}}$. Equation (25) is closed by extracting the phase coherent part of the nonlinearity via substitution for $A_{\substack{k+k'\\ \omega+\omega'}}$ as $A^{(2)}_{\substack{k+k'\\ \omega+\omega'}}$, where

$$A^{(2)}_{\substack{k+k'\\ \omega+\omega'}} = L_{\substack{k+k'\\ \omega+\omega'}}\left(-\frac{i\lambda}{2}(k+k')k'^2 A_{\substack{k'\\ \omega'}} k^2 A_{\substack{k\\ \omega}}\right), \tag{26a}$$

$$L^{-1}_{\substack{k+k'\\ \omega+\omega'}} = -i(\omega+\omega') + \eta_0(k+k')^2 + D(k+k')^4 + d_{\substack{k+k'\\ \omega+\omega'}}. \tag{26b}$$

Here $d_{\substack{k+k'\\ \omega+\omega'}}$ refers to the propagator renormalization. The renormalized $A_{\substack{k\\ \omega}}$ equation is then

$$-i\omega A_{\substack{k\\ \omega}} + k\frac{\lambda^2}{4}\sum_{\substack{k'\\ \omega'}} (k+k')^3 k'^2 |A_{\substack{k'\\ \omega'}}|^2 L_{\substack{k+k'\\ \omega+\omega'}} k^2 A_{\substack{k\\ \omega}} + (\eta_0 k^2 + Dk^4) A_{\substack{k\\ \omega}} = S_{k,\omega}. \tag{27a}$$

Taking the long wavelength, hydrodynamic limit, and noting parity forces cancellation of the k'^3 contribution then gives

$$-i\omega A_{\substack{k\\ \omega}} + k^4 D_T A_{\substack{k\\ \omega}} + (\eta_0 k^2 + Dk^4) A_{\substack{k\\ \omega}} = S_{\substack{k\\ \omega}}, \tag{27b}$$

where

$$D_T = \frac{3\lambda^2}{4}\sum_{\substack{k'\\ \omega'}} (k'^2)^3 |A_{\substack{k'\\ \omega'}}|^2 L_{\substack{k'\\ \omega'}} \tag{27c}$$

is the turbulent dissipation coefficient, itself a function of fluctuation level. As expected, here the turbulent dissipation has the form of hyper-resistive diffusion. *Note that here, D_T is derived using simple dynamical arguments and symmetry principles alone, and is a fundamentally model-independent result.*

The effective magnetic Reynolds number is simply the ratio of relaxation-induced helicity transport to collisional resistive dissipation, i.e.,

$$R_m \sim k^4 D_T / \eta_0 k^2 \sim k^2 D_T / \eta_0. \tag{28}$$

Following the convention of defining R_m at large scales (note this is a conservative estimate), we have $R_m \sim D_T / L^2 \eta_0$, where L is the system size. Now, using Eq. (27) to relate $A_{k,\omega}$ to $S_{k,\omega}$ (neglecting η_0 and D, the collisional transport coefficients), D_T may be written as

$$D_T = \frac{3}{4}\lambda^2 \sum_{k',\omega'} \frac{(k'^2)^3 |S_{k',\omega'}|^2 k'^4 D_T}{[\omega'^2 + (D_T k'^4)^2]^2}. \tag{29}$$

Assuming white noise for simplicity, the ω' integral may be performed, yielding

$$D_T^3 = \sum_{\underline{k}'} (3\pi/8)\lambda^2 S_0^2 / k'^4. \tag{30}$$

S_0^2 is the strength of the white noise. Here the remaining integral over k is manifestly divergent as $k \to 0$, and must be cut off at a scale corresponding to $k_{\min}$. Clearly, $k_{\min}^{-1}$ should be smaller than the scale of the phenomena being considered. Otherwise, the coarse graining inherent to the renormalization procedure is inappropriate. In this case, we may write

$$D_T \cong (\lambda^2 S_0^2)^{1/3} / k_{\min}. \tag{31}$$

The strong infrared divergence of D_T which appears in Eq. (31) is a "red flag" indicating the possibility of superdiffusive or ballistic transport dynamics. This follows from the implicit scale dependency of D_T (i.e., $D_T \sim l$), which appears on account of the infrared divergence. Indeed, taking $l^2 \sim D_T \tau$ and noting from Eq. (31) that $D_T \sim k_{\min}^{-1} \sim l$ gives $l \sim (\lambda^2 S_0^2)^{1/3} \tau$, namely ballistic transport scaling. Such scaling is also characteristic of transport in Burgers turbulence. Given that Eq. (12) has traveling wave solutions, with the form $A(x-ct)$, it is by no means surprising to find ballistic scaling appears in the turbulence analysis.

It is also possible to obtain a general scaling for the effective magnetic Reynolds number $R_m \sim D_T / L^2 \eta_0$. Taking $D_T \sim (\lambda^2 S_0^2)^{1/3} L$ (i.e., equivalent to assuming $k_{\min} \sim L^{-1}$) gives $R_m \sim (\lambda^2 S_0^2)^{1/3} / L \eta_0$. Note that R_m scales with $(\lambda^2 S_0^2)^{1/3}$ but *inversely* with L. This result gives a universal scaling relation for the effective magnetic Reynolds number in terms of system size L, coupling strength λ, excitation strength S_0^2, and resistivity η_0. This result contradicts the hypothesis of Colgate, which asserts that a single value of $R_m \sim 100$ is characteristic of most Taylor relaxation phenom-

ena in astrophysical plasmas.[27] The prediction given here is definitely testable by numerical and physical experiments.

V. DISCUSSION AND CONCLUSION

In this paper, we have considered the dynamics of helicity transport and Taylor relaxation. The principal results of this paper are listed below.

(i) A dynamical description of Taylor relaxation for magnetic configurations with two spatial symmetries (i.e., such as in a toroidal plasma) has been developed using helicity density flux invariance principles, alone. This approach subsumes and supercedes the prevailing picture of Taylor relaxation dynamics, based on hyper-resistive diffusion of the parallel current, and is applicable to a wide variety of plasma models.

(ii) This description of the relaxation process predicts fast, nondiffusive relaxation events, which correspond loosely to avalanches of magnetic helicity density. This phenomenon is manifested in the theory both by

(a) the prediction of coherent, soliton-like traveling wave solutions to the zero-forcing, zero-dissipation problem;
(b) the prediction of ballistic helicity density transport scalings for the forced, noisy problem; here, ballistic scaling arises from the infrared divergence of the turbulent hyper-resistivity.

(iii) A universal structure for the parameter scaling of the effective magnetic Reynolds number during Taylor relaxation (with white noise) has been derived. The scaling prediction is $R_M \sim (\lambda^2 S_0^2)^{1/3}/L\eta_0$, where λ is the coupling coefficient, S_0^2 is the noise strength, and L is the system size. This result contradicts certain recent assertions by Colgate.

(iv) More generally, this description suggests that Taylor relaxation is a generically strongly intermittent process, and that a statistical approach (i.e., PDF calculation) is necessary.

Certain aspects of these points are discussed further below.

One of the most striking results obtained here is the similarity between Eq. (12), which describes the Taylor relaxation of the current profile on mesoscales, and the familiar Burgers equation. It is well known that Burgers turbulence is strongly intermittent, a property which is a consequence of the fact that in the Burgers equation, negative slopes $[\partial x(V_x^2)<0]$ steepen to form shock fronts, while positive slope ramps $[\partial x(V_x^2)>0]$ smooth out.[28] A related (but more complicated) type of asymmetry is manifested in Eq. (12), namely that $\delta A_\parallel$ perturbations will be amplified in regions with $\lambda\partial_r(\delta J_\parallel)^2<0$ but will be reduced or smoothed where $\lambda\partial_r(\delta J_\parallel)^2>0$. Thus, Taylor relaxation is likely to exhibit intermittency rooted in the *local slope* of $\delta J_\parallel^2$, and thus be concentrated in localized structures, akin to shocks in Burgers turbulence. As in Burgers turbulence, the pdf of differences or, "jumps" in $\delta J_\parallel$ is likely to be strongly asymmetric. While validation of this speculation awaits numerical solution of Eq. (12) with noisy forcing, it seems clear that a statistical approach to the problem focused on computing the pdf of $\delta J_\parallel$, will ultimately be both necessary and illuminating.[29]

It is appropriate to comment on the relationship between the "sandpile model" of relaxation discussed here, and the well-known model of RFP relaxation based on the induced EMF driven by $m=1$ tearing modes, which has received extensive theoretical and computational scrutiny. First, it is important to emphasize that these two approaches are *not* in conflict or incompatible. This is because the $m=1$ modes are global, and thus produce a zone of reconnection in the core of the pinch, while at the same time driving ideal kink motions in the MHD exterior, beyond the reversal surface. In the core reconnection region, at the mode resonant surfaces (where $\underline{k}\cdot\underline{B}_0=0$) the mean current profile flattens due to reconnection-induced transport of current density. This process has been described by hyper-resistive diffusion of current density driven by the $m=1$ perturbations, and thus in principle, is indeed contained with the structure of the "sandpile model" discussed here. Note also that the magnetic stochasticity which follows from the island overlap and reconnection in the core naturally provides the elements of irreversibility and randomness tacitly assumed by our sandpile model. Moreover, the resonant nonlinear interaction of $m=1$ modes generates localized current perturbations corresponding to $m\geqslant 2$. Thus, the core of the RFP is in a state of strong MHD turbulence. On the other hand, the "kinking" in the exterior region of the tearing modes is intrinsically a *reversible* process, and thus is *not* contained within the sandpile model paradigm. It should be noted that since observed RFP current profiles often deviate markedly from the predictions of the Taylor theory beyond the reversal surface, it is far from clear what the role these exterior dynamics actually play in the relaxation process. Finally, we note that the "kinetic dynamo"[30] model based on the concept of microturbulence-induced diffusion of current, is clearly entirely consistent with the local flux picture adopted in the sandpile mode. Of course, it is possible that the kinetic dynamo and the $m=1$ driven dynamo can coexist. In that case, a unified description using the sandpile model is certainly feasible.

It is appropriate to mention some possible experimental observables of a SOC-like relaxation process in a RFP. First, one would naturally expect to observe propagating excesses ("blobs" or avalanches!) and voids in the radial profile of the parallel current. The PDF of current gradient fluctuations should be strongly non-Gaussian, as well. Second, it would be interesting to examine the relationship between large current transport events and sawtooth crashes observed in the RFP.[31,32] A possible precursor to large crashes might be an increase in the frequency of occurrence of small current avalanches.

At a practical level, the prediction of fast, non-diffusive relaxation events may have implications for feedback control of RFP current profiles, as in PPCD.[33] Such events evolve on a space–time trajectory different from that of a simple diffusion processes (i.e., due to hyper-resistivity). PPCD control loops should be designed with this possibility in mind, and

not be based solely on the presumption that current profile evolution is diffusive.

While this paper has discussed Taylor relaxation dynamics in the familiar context of the RFP, it should be noted that these ideas are potentially applicable to astrophysical plasma problems in general, and the heating of the solar corona, by relaxation and reconnection of coronal loops, in particular. In this vein, Lu and Hamilton have expanded Parker's original concepts of magnetic nonequilibria[34] and self-organization of nanoflare events[35] into a cascade model of coronal heating which is structurally similar to cellular automata models familiar from the study of sandpile models.[36,37] Quite recently, Liu *et al*. have advanced a continuum-limit version of the Lu–Hamilton cascade model.[38] The key effect of Liu *et al*. is nonlinear hyper-resistive diffusion, which is clearly related to the physics of both Taylor relaxation in general, and this theory in particular. Further detailed comparisons and contrasts between our theory and that of Liu *et al*. are ongoing and will be discussed in a future publication.

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