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ABSTRACT

We have studied the solutions of a wave equation which describes a spin-zero particle in the Coulomb field of a nucleus. An interesting feature of this equation is that the kernel is not of the Fredholm type. The behavior of the momentum space wave function for large momentum is not determined solely by the angular momentum state but, as in the cases of the Dirac and Klein-Gordon equations, it depends on the electric charge as well. Our analysis of the asymptotic properties is based on a Mellin transformation of the momentum space equation. This leads to a singular integral equation with a Cauchy-type kernel which may be treated by standard methods. The equation is shown to have unique solutions.

I. INTRODUCTION

When we began a phenomenological analysis of pion alpha-particle scattering sometime ago we were faced with the problem of choosing a wave equation incorporating two-particle relativistic effects. We wished to describe both electromagnetic and strong interactions. At first, the Klein-Gordon equation appeared to be a likely possibility, but it has no probability interpretation so we were led to consider the problem of Coulomb scattering for two spin-zero particles from the field theoretic point of view.

This led us to the following wave equation for two free particles of mass m_1, m_2 and momenta $\underset{\sim}{p}_1$, $\underset{\sim}{p}_2$:

$$\left[(\underset{\sim}{p}_1^{\,2} + m_1^{\,2})^{1/2} + (\underset{\sim}{p}_2^{\,2} + m_2^{\,2})^{1/2}\right] \psi(\underset{\sim}{p}_1, \underset{\sim}{p}_2) = P_0\, \psi(\underset{\sim}{p}_1, \underset{\sim}{p}_2),$$

where P_0 is the total energy, and we choose $\hbar = c = 1$. If the Coulomb interaction is included, an additional term,

$$\int V(\underset{\sim}{p}_1, \underset{\sim}{p}_2, \underset{\sim}{p}'_1, \underset{\sim}{p}'_2)\, \psi(\underset{\sim}{p}'_1, \underset{\sim}{p}'_2) d^3\underset{\sim}{p}'_1\, d^3\underset{\sim}{p}'_2,$$

describing the Coulomb interaction, V, appears.

This equation is a member of the class derived on the basis of general relativistic principles by Bakamjian and Thomas[1] almost two decades ago. Such an equation has been known[2] even longer. The relationship between this equation and the Klein-Gordon equation has been discussed by Feshbach and Villars.[3] More recently, Zemach[4] has analyzed the relation between this equation and that for the two-body Green's function[5] defined by Schwinger.

The Bakamjian-Thomas equation has been studied by a number of authors[6] during the past few years. They have concerned themselves

with the case of short-range interactions. In the following pages we present the theory of the equation for the case of an interaction which is the time component of a vector field. In this case the resulting integral equation is not of the Fredholm type. Its solutions may be shown to behave as p^{-s} for large momentum where the specific value of s depends on the angular momentum state and the strength of the interaction. Because of this behavior a Mellin transformation of the momentum space wave function seems particularly appropriate. When such a transformation is carried out, the kernel of the new equation is found to have a Cauchy-type singularity. The choice of a contour of integration for the inverse Mellin transformation is made by demanding that the wave function be integrable for large momenta and that the transformed kernel be Hermitian. The integral equation may then be reduced to a Fredholm equation by a standard method which we describe in detail. We are thus able to prove the existence of unique solutions for both bound state and scattering wave functions.

II. THE BAKAMJIAN-THOMAS EQUATION FOR COULOMB SCATTERING

We consider two spin-zero fields ϕ, χ of masses m and M, respectively, interacting through a Coulomb interaction. The Schroedinger representation is employed. Free particle states are normalized by

$$\langle \underset{\sim}{p}' \mid \underset{\sim}{p} \rangle = p_0\, \delta(\underset{\sim}{p}' - \underset{\sim}{p}), \qquad (1)$$

where p_0 is the free particle energy and δ is the Dirac delta function. This implies commutation relations of the form

$$\left[a(\underset{\sim}{p}),\, a^+(\underset{\sim}{p}')\right] = p_0\, \delta(\underset{\sim}{p} - \underset{\sim}{p}') \qquad (2)$$

for the operators $a(\underset{\sim}{p})$, $b(\underset{\sim}{p})$ and $A(\underset{\sim}{p})$, $B(\underset{\sim}{p})$ associated with the fields ϕ and χ, respectively. For ϕ we write

$$\phi(\underset{\sim}{r}) = \frac{1}{(2\pi)^{3/2}\,(2)^{1/2}} \int \frac{d^3\underset{\sim}{p}}{p_0} \left[a(\underset{\sim}{p}) e^{i\underset{\sim}{p}\cdot\underset{\sim}{r}} + b^+(\underset{\sim}{p}) e^{-i\underset{\sim}{p}\cdot\underset{\sim}{r}} \right] \tag{3}$$

and correspondingly for $\chi(r)$. If $\psi(\underset{\sim}{p},\underset{\sim}{q})$ is a function of the variables $\underset{\sim}{p}$ and $\underset{\sim}{q}$ referring to two different particles, we may introduce a two-particle state vector $|\Psi\rangle$ by writing

$$|\Psi\rangle = \iint \frac{d^3\underset{\sim}{p}}{p_0} \frac{d^3\underset{\sim}{q}}{q_0} \, \psi(\underset{\sim}{p},\underset{\sim}{q}) \, a^+(\underset{\sim}{p}) \, A^+(\underset{\sim}{q}) \, |0\rangle , \tag{4}$$

where $|0\rangle$ is the vacuum state vector. One finds

$$\langle \Psi | \Psi \rangle = \iint \frac{d^3\underset{\sim}{p}}{p_0} \frac{d^3\underset{\sim}{q}}{q_0} \, | \psi(\underset{\sim}{p},\underset{\sim}{q}) |^2 . \tag{5}$$

The Schroedinger equation which we seek is just

$$\langle \underset{\sim}{p},\underset{\sim}{q} \,|\, H \,|\, \Psi \rangle = i \frac{\partial}{\partial t} \langle \underset{\sim}{p},\underset{\sim}{q} \,|\, \Psi \rangle , \tag{6}$$

where H is the Hamiltonian of the system. The noninteraction part of H contributes $\left[(\underset{\sim}{p}^2 + m^2)^{1/2} + (\underset{\sim}{q}^2 + M^2)^{1/2} \right] \psi(\underset{\sim}{p},\underset{\sim}{q})$ to the left-hand side of this equation, and the Coulomb interaction term is

$$H_c = Ze^2 \int \frac{\rho_m(\underset{\sim}{r}) \, \rho_M(\underset{\sim}{r}')}{|\underset{\sim}{r} - \underset{\sim}{r}'|} \, d^3\underset{\sim}{r} \, d^3\underset{\sim}{r}' , \tag{7}$$

where ρ_m and ρ_M are the charge densities of the two fields. In terms of the charges e and Ze for the ϕ and χ fields (the particles "a" and "A" have charges e and Ze, respectively)

$$\rho_m = i \, e(\phi^+ \pi^+ - \pi\phi) , \tag{8}$$

where π, π^+ are the fields canonically conjugate to ϕ, ϕ^+, respectively. The representation of the $\pi(r)$ field is:

$$\pi(\underset{\sim}{r}) = \frac{i}{(2\pi)^{3/2}} \frac{1}{2^{1/2}} \int d^3\underset{\sim}{p} \left[a^+(\underset{\sim}{p}) e^{-i\underset{\sim}{p}\cdot\underset{\sim}{r}} - b(\underset{\sim}{p}) e^{i\underset{\sim}{p}\cdot\underset{\sim}{r}} \right] . \tag{9}$$

Similar expressions describe ρ_M and the χ field. The evaluation of the Coulomb contribution is straightforward. One finds

$$\langle \underset{\sim}{p},\underset{\sim}{q} | H_c | \Psi \rangle = \frac{Ze^2}{8\pi^2} \int \frac{d^3\underset{\sim}{p}'}{p_0'} \frac{d^3\underset{\sim}{q}'}{q_0'} (p_0 + p_0')(q_0 + q_0') \, \delta \frac{(\underset{\sim}{p}+\underset{\sim}{q}-\underset{\sim}{p}'-\underset{\sim}{q}')}{(\underset{\sim}{p} - \underset{\sim}{p}')^2} \, \psi(\underset{\sim}{p}',\underset{\sim}{q}') . \tag{10}$$

Since we were immediately interested in pion-helium scattering where $m_\pi \ll m_{He}$ we were led to the approximate equation for the case of an infinitely massive field χ. If P_0 now denotes only the energy of the particle of finite mass we find

$$(\underset{\sim}{p}^2 + m^2)^{1/2} \psi(\underset{\sim}{p}) + \frac{Ze^2}{4\pi^2} \int \frac{d^3\underset{\sim}{p}'}{p_0'} \frac{(p_0 + p_0')}{(\underset{\sim}{p} - \underset{\sim}{p}')^2} \psi(\underset{\sim}{p}') = P_0 \, \psi(\underset{\sim}{p}) . \tag{11}$$

This is the equation we will study.

III. SOLUTIONS OF THE BAKAMJIAN-THOMAS COULOMB EQUATION

This section will be devoted to a study of the general properties of Eq. (11)[7] when the interaction is attractive. This restriction will be removed later. A partial-wave decomposition yields

$$(p^2 + m^2)^{1/2} \psi_\ell(p) + \frac{Ze^2}{2\pi} \int_0^\infty dp' \, \frac{(p_0 + p_0')}{p \, p_0'} \, p' Q_\ell\left(\frac{p^2 + p'^2}{2pp'} \right) \psi_\ell(p')$$

$$= P_0 \, \psi_\ell(p) , \tag{12}$$

where Q_ℓ is the Legendre function of the second kind and $\psi_\ell(p)$ is the new wave function. If we try to write Eq. (12) in standard integral equation form the resulting kernel is not symmetric. To find an equation with a symmetric kernel one may introduce ψ_ℓ^s by

$$p_0^{1/2}\,\psi_\ell^s = \left\{\left[P_0 - (p^2 + m^2)^{1/2}\right]\right\}^{1/2} p\psi_\ell(p) \; . \qquad (13)$$

The new equation is then

$$\psi_\ell^s(p) = \frac{Ze^2}{2\pi}\int_0^\infty dp' \, \frac{(p_0 + p_0')}{(p_0\, p_0')^{1/2}} \times \frac{Q_\ell\left(\frac{p^2 + p'^2}{2pp'}\right)}{\left[P_0 - (p^2 + m^2)^{1/2}\right]^{1/2}\left[P_0 - (p'^2 + m^2)^{1/2}\right]^{1/2}} \; . \qquad (14)$$

Equation (14) has a kernel which is not of the Fredholm type. To see this, we consider the integral of the square of the kernel (the Fredholm norm):

$$N^2 = \left(\frac{Ze^2}{2\pi}\right)^2 \int_0^\infty dp \int_0^\infty dp' \, \frac{(p_0 + p_0')^2}{p_0\, p_0'} \times \frac{Q_\ell^2\left(\frac{p^2 + p'^2}{2pp'}\right)}{\left|\left[P_0 - (p^2 + m^2)^{1/2}\right]\left[P_0 - (p'^2 + m^2)^{1/2}\right]\right|} \; . \qquad (15)$$

The kernel will be non-Fredholm if the energy, P_0 , is in the scattering region, because the energy denominators can then vanish.

This difficulty is common to scattering integral equations and can be readily removed,[8] so we will ignore it. If we set $p' = \alpha p$ and take account of the symmetry in p and p' we find

$$N^2 = 2\left(\frac{Ze^2}{2\pi}\right)^2 \int_0^\infty \frac{dp}{p}\int_0^1 d\alpha \times \frac{\left[\left(1 + \frac{m^2}{p^2}\right)^{\frac{1}{2}} + \left(\alpha^2 + \frac{m^2}{p^2}\right)^{\frac{1}{2}}\right] Q_\ell^2\left(\frac{1 + \alpha^2}{2\alpha}\right)}{\left[\left(1 + \frac{m^2}{p^2}\right)\left(\alpha^2 + \frac{m^2}{p^2}\right)\right]^{\frac{1}{2}} \left|\left[\frac{P_0}{p} - \left(1 + \frac{m^2}{p^2}\right)^{\frac{1}{2}}\right]\left[\frac{P_0}{p} - \left(\alpha^2 + \frac{m^2}{p^2}\right)^{\frac{1}{2}}\right]\right|} \; . \qquad (16)$$

As p tends to zero the integral is well-behaved; when p becomes large, however, the integral diverges logarithmically. The integrand is positive definite and for any nonzero region in α the logarithmic divergence is present. (Note that when $\alpha \to 1$ no trouble arises since Q_ℓ diverges only logarithmically and is therefore integrable.) It should be noted that if we had considered the interaction appropriate to the time component of a vector meson field of mass μ the argument of the function Q_ℓ would be replaced according to

$$\frac{p^2 + p'^2}{2pp'} \rightarrow \frac{p^2 + p'^2 + \mu^2}{2pp'} \; , \qquad (17)$$

but similar arguments to those above would show the kernel still not to be of the Fredholm type. Since the non-Fredholm nature of the kernel is related to its large momentum behavior, our next task is to study the behavior of the equation for large momenta. For this purpose it is convenient to use the unsymmetric form of the integral equation, Eq. (12).

Thus for $p_0 \gg P_0$,

$$\psi_\ell(p) \cong -\frac{Ze^2}{2\pi}\int_0^\infty dp' \, \frac{(p_0 + p_0')}{p_0 p_0'} \, \frac{p'}{p} \, Q_\ell\left(\frac{p^2 + p'^2}{2pp'}\right) \psi_\ell(p') \, . \qquad (18)$$

The integral representation for Q_ℓ, i.e.,

$$Q_\ell(z) = \frac{1}{2}\int_{-1}^{1} dt \, \frac{P_\ell(t)}{z - t} \, , \qquad (19)$$

may then be used to give

$$\psi_\ell(p) = -\frac{Ze^2}{2\pi}\int_{-1}^{1} dt \, P_\ell(t) \int_0^\infty dp' \, \frac{(p_0 + p_0')}{p_0 p_0'} \, \frac{p'^2 \, \psi_\ell(p')}{(p^2 + p'^2 - 2pp't)} \, . \qquad (20)$$

We now conjecture that solutions of this equation behave as $p^{-\xi}$ for large p. We are thus led to examine the two integrals

$$I_1 = -\frac{Ze^2}{2\pi p}\int_{-1}^{1} dt \, P_\ell(t) \int_0^\infty \frac{dp'(p')^{2-\xi}}{(p^2 + p'^2 - 2pp't)}$$

and (21)

$$I_2 = -\frac{Ze^2}{2\pi}\int_{-1}^{1} dt \, P_\ell(t) \int_0^\infty \frac{(p')^{2-\xi}}{p_0'(p^2 + p'^2 - 2pp't)} \, .$$

The integrals may be evaluated by standard contour integration methods. Both have branch points at the origin; I_2 has additional branch points at $\pm \, im$. We thus find that

$$I_1 = -\frac{Ze^2}{2\pi p} \, \frac{e^{i\pi\xi}}{2i \sin \pi \xi} \int_0^\pi \sin\theta \, d\theta \, P_\ell(\cos\theta) \int_C \frac{dp'(p')^{2-\xi}}{(p' - pe^{i\theta})(p' - pe^{-i\theta})}$$

and (22)

$$I_2 = -\frac{Ze^2}{2\pi} \, \frac{e^{i\pi\xi}}{2i \sin \pi \xi} \int_0^\pi \sin\theta \, d\theta \, P_\ell(\cos\theta) \int_C \frac{dp' \, (p')^{2-\xi}}{p_0'(p' - pe^{i\theta})(p' - pe^{-i\theta})} ,$$

where we have set

$$t = \cos\theta \, , \qquad (23)$$

and C is a contour from $+\infty$ to $+\infty$ taken around the origin in the counterclockwise direction below and above the branch cut which has been taken along the real axis from the origin to $+\infty$ as shown in Fig. 1. The integrals in Eq. (21) are well defined in the neighborhood of the origin and at ∞ if

$$1 < \xi < 2 \, . \qquad (24)$$

The integral I_1 may be evaluated by the method of residues. The integrand has poles at $\arg p' = \theta, \; 2\pi - \theta$ (the last value obtains since we may not pass through the branch cut along the real axis; see Fig. 1). One finds for I_1, if θ is less than $\pi/2$,

$$I_1 = \frac{Ze^2}{2} \, \frac{p^{-\xi}}{\sin \pi\xi} \int_0^\pi d\theta \, P_\ell(\cos\theta) \, \sin\left[(2 - \xi)(\pi - \theta)\right] . \qquad (25)$$

The integral I_2 may be treated in a similar manner except that account must be taken of the additional branch cuts from im to ∞ and $-im$ to $-\infty$. If θ is again assumed to be less than $\pi/2$ and if I_{2B}

represents the contribution from the branch cuts, one has, in the limit $p \gg m$,

$$I_2 = I_{2B} - \frac{Ze^2}{2}\frac{p^{-\xi}}{\sin \pi\xi}\int_0^\pi d\theta\, P_\ell(\cos\theta)\sin\left[(1-\xi)(\pi-\theta)\right] . \qquad (26)$$

Let us now consider the integral over the branch cuts, I_{2B}. Since we are dealing with a square root singularity, it follows that we need only integrate over the portions of the contour which lie in the left plane providing we double the result. Next let us consider an integral, $I_{2B'}$, of the integrand over the path C' in Fig. 1, which lies in the left half plane and which connects the branch points $\pm$ im. Clearly

$$I_{2B} + I_{2B'} = 0 . \qquad (27)$$

The contour for $I_{2B'}$ may be taken along the imaginary axis, except for a small indentation of radius ρ to the left of the origin which allows one to avoid the branch cut to the origin. We thus have

$$I_{2B'} = \frac{Ze^2}{2\pi \sin \pi\xi}\int_0^\pi \sin\theta\, d\theta\, P_\ell(\cos\theta)\left\{\int_\rho^m \left[\frac{y^{2-\xi}\,dy}{(m^2-y^2)^{\frac{1}{2}}}\right.\right.$$

$$\times \left.\frac{e^{i\pi\xi/2}}{(y - p\, e^{i(\theta-\pi/2)})(y + p\, e^{-i(\theta-\pi/2)})} + \text{complex conj}\right] - \rho^{3-\xi}\, e^{i\pi\xi}$$

$$\times \left.\int_{\pi/2}^{3\pi/2} \frac{e^{i(3-\xi)\alpha}\, d\alpha}{(m^2+\rho^2 e^{2i\alpha})^{1/2}\,(\rho\, e^{i\alpha} - p\, e^{i\theta})(\rho\, e^{i\alpha} - p\, e^{-i\theta})}\right\} . \qquad (28)$$

The last integral in Eq. (28) vanishes as $\rho \to 0$ for $\mathrm{Re}\,\xi < 3$. Thus one finds in the limit $p \to \infty$:

$$I_{2B'} \approx -\frac{Ze^2}{2\pi\, p^2 \sin\pi\xi}\int_0^\pi \sin\theta\, d\theta\, P_\ell(\cos\theta)\int_0^m \frac{y^{2-\xi}\,dy}{(m^2-y^2)^{1/2}}$$

$$\times \left\{\frac{e^{i\pi\xi/2}}{\left(1+\frac{y}{p}e^{i(\theta-\pi/2)}\right)\left(1-\frac{y}{p}e^{-i(\theta-\pi/2)}\right)} + \text{complex conj.}\right\} . \qquad (29)$$

Therefore I_{2B} behaves as p^{-2} so under the restrictions in Eq. (24) I_{2B} does not contribute to the asymptotic behavior. Thus

$$I_2 = -\frac{Ze^2\, p^{-\xi}}{2\sin\pi\xi}\int_0^\pi d\theta\, P_\ell(\cos\theta)\sin\left[(1-\xi)(\pi-\theta)\right] . \qquad (30)$$

If I_1 and I_2 are now combined, one finds that for self-consistency a solution whose asymptotic form is $p^{-\xi}$ requires that

$$\frac{Ze^2}{2\sin\pi\xi}\int_0^\pi d\theta\, P_\ell(\cos\theta)\left\{\sin\left[(2-\xi)(\pi-\theta)\right] - \sin\left[(1-\xi)(\pi-\theta)\right]\right\} = 1. \qquad (31)$$

This may also be written as

$$\frac{(-1)^\ell\, Ze^2}{\sin\pi\xi}\int_0^\pi d\theta\, P_\ell(\cos\theta)\sin\frac{\theta}{2}\cos(\xi-3/2)\theta = 1 . \qquad (32)$$

The integral vanishes as $\xi \to 2$, as it should, since otherwise there would be a pole in either I_1 or I_2 in contradiction to the condition (24).

The integrals in Eqs. (31) and (32) may be evaluated in a straightforward manner.[9] When ℓ is even, we find

$$\frac{Ze^2}{2\pi}\sum_{k=0}^{\ell}\binom{\ell}{k} B(\ell - k + \tfrac{1}{2},\ k + \tfrac{1}{2})\left(\frac{\tan \pi\xi/2}{\ell - 2k + 2 - \xi} - \frac{\cot \pi\xi/2}{\ell - 2k + 1 - \xi}\right) = 1\ , \qquad (33)$$

and when ℓ is odd,

$$\frac{Ze^2}{2\pi}\sum_{k=0}^{\ell}\binom{\ell}{k} B(\ell - k + \tfrac{1}{2},\ k + \tfrac{1}{2})\left(\frac{\cot \pi\xi/2}{\ell - 2k + 2 - \xi} - \frac{\tan \pi\xi/2}{\ell - 2k + 1 - \xi}\right) = 1\ . \qquad (34)$$

In these expressions the symbol $\binom{\ell}{k}$ is the usual binomial coefficient and $B(x,y)$ is the beta function of x,y .

The asymptotic behavior just developed strongly suggests that the Bakamjian-Thomas equation be studied by using a Mellin transformation.[10] We now turn to that task, which will verify rigorously that the asymptotic behavior is indeed given by a solution of Eq. (33) or Eq. (34), and, further, will lead to a method for obtaining a unique solution of the singular equation, Eq. (12).

IV. MELLIN TRANSFORMATION OF THE BAKAMJIAN-THOMAS EQUATION

The Mellin transformation and its inverse are defined by the equations[11]

$$\psi_\ell(s) = \int_0^\infty \psi_\ell(p)\, p^{s-1}\, dp \qquad (35)$$

and

$$\psi_\ell(p) = \frac{1}{2\pi i}\int_C \psi_\ell(s)\, p^{-s}\, ds\ , \qquad (36)$$

where C goes from $-i\infty$ to $i\infty$. The contour C must be chosen appropriately in order to effect a solution. We note that from Eq. (36) it follows that the asymptotic behavior of $\psi_\ell(p)$ as $p \to \infty$ is determined by the singularity in $\psi_\ell(s)$ with the smallest $\mathrm{Re}(s)$ to the right of C, while the behavior as $p \to 0$ is determined by the singularity with the largest $\mathrm{Re}(s)$ to the left of C. From these relations one finds the transformed integral equation:

$$\psi_\ell(s) = \frac{1}{2\pi i}\int_C K_\ell(s,\ s')\ \psi_\ell(s')\ ds'\ , \qquad (37)$$

where the kernel $K_\ell(s,\ s')$ is given by:

$$K_\ell(s,s') = \frac{Ze^2}{2\pi}\int_0^\infty dp \int_0^\infty dp'\ \frac{p^{s-1}(p_0 + p_0')p'}{(P_0 - p_0)\ p_0'\ p} \times Q_\ell\left(\frac{p^2 + p'^2}{2pp'}\right)(p')^{-s'}\ dp'\ . \qquad (38)$$

The conditions for the existence of $K_\ell(s,s')$ must now be examined. A consideration that the integrals over p and p' be convergent at both limits of integration gives the requirements:

$$\begin{aligned}
&\text{As} \quad p \to 0\ , && \mathrm{Re}(s) > -\ell\ . \\
&\text{As} \quad p \to \infty, && \mathrm{Re}(s) < \ell + 2. \\
&\text{As} \quad p' \to 0, && \mathrm{Re}(s') < \ell + 3. \\
&\text{As} \quad p' \to \infty, && \mathrm{Re}(s') > -\ell + 1.
\end{aligned} \qquad (39)$$

It may be noted that the factor $(P_0 - p_0)^{-1}$ may be expanded in an appropriate manner according to whether $|P_0/p_0|$ is greater or less than 1 and that such an expansion will not alter our conclusions about the domain of existence of K since each successive term is as well behaved at the origin and is better behaved at infinity than the one for which $P_0 = 0$.

Alternatively if the kernel, K, is divided into two parts, the first of which, $\bar{K}$, is obtained by setting $P_0 = 0$ in K, and the second is simply the difference between K and $\bar{K}$, by such an expansion argument for large p as has just been given one sees that the domain of existence for K is the same as that for $\bar{K}$. We find

$$\bar{K}_\ell(s,s') = -\frac{Ze^2}{2\pi}\int_0^\infty dp \int_0^\infty dp'\, p^{s-2}\, \frac{(p_0+p_0')}{p_0\, p_0'}\,(p')^{-s+1}\, Q_\ell\left(\frac{p^2+p'^2}{2pp'}\right). \tag{40}$$

To carry out the integrals we again replace Q_ℓ by its integral representation, Eq. (19), to get

$$\bar{K}_\ell(s,s') = -\frac{Ze^2}{2\pi}\int_0^\pi \sin\theta\, d\theta\, P_\ell(\cos\theta)\int_0^\infty dp\int_0^\infty dp'\, p^{s-1}$$

$$\times\left(\frac{1}{p_0}+\frac{1}{p_0'}\right)\frac{(p')^{-s'+2}}{(p'-p\,e^{i\theta})(p'-p\,e^{-i\theta})}. \tag{41}$$

Consider now the integral:

$$I_1(s,s') = \int_0^\infty dp\int_0^\infty dp'\, \frac{p^{s-1}\,(p')^{-s'+2}}{p_0(p'-p\,e^{i\theta})(p'-p\,e^{-i\theta})}$$

$$= \frac{e^{i\pi s'}}{2i\sin\pi s'}\int_0^\infty \frac{dp}{p_0}\int_C \frac{dp'\,(p')^{-s'+2}}{(p'-p\,e^{i\theta})(p'-p\,e^{-i\theta})}. \tag{42}$$

This is one of the terms in Eq. (41). The contour C is the same as that in Eq. (22). We integrate first over p' and then over p to avoid the branch cut associated with p_0 at the first integration. The other term in the integrand of Eq. (41) is treated by integrating first over p and then over p'. Denoting this second term by I_2, we have

$$I_2(s,s') = \int_0^\infty dp\int_0^\infty dp'\, \frac{(p')^{-s'+2}\,p^{s-1}}{p_0'(p-p'\,e^{i\theta})(p-p'\,e^{-i\theta})}$$

$$= \frac{e^{-is\pi}}{2i\sin\pi s}\int_0^\infty dp'\,\frac{(p')^{-s'+2}}{p_0'}\int_C dp\,\frac{p^{s-1}}{(p-p'e^{i\theta})(p-p'e^{-i\theta})}. \tag{43}$$

For I_1, we find

$$I_1(s,s') = -\frac{\pi}{\sin\pi s'}\,\frac{\sin[(2-s')(\pi-\theta)]}{\sin\theta}\int_0^\infty \frac{dp}{p_0}\,p^{s-s'}. \tag{44}$$

If this integral is to converge we see that C must be chosen so that

$$\mathrm{Re}(s) < \mathrm{Re}(s') < \mathrm{Re}(s+1). \tag{45}$$

The remaining integration can then be performed to give

$$I_1(s,s') = -\frac{\pi}{2\sin\pi s'}\,\frac{\sin[(2-s')(\pi-\theta)]}{\sin\theta}\left(\frac{m}{2}\right)^{s-s'} B\left(\frac{s'-s}{2},\, s'-s+1\right). \tag{46}$$

The second term, I_2, may be evaluated in the same way. The result is

$$I_2(s,s') = \frac{\pi}{2\sin\pi s}\,\frac{\sin[(1-s)(\pi-\theta)]}{\sin\theta}\left(\frac{m}{2}\right)^{s-s'} B\left(\frac{s'-s}{2},\, s-s'+1\right). \tag{47}$$

In the last two equations B denotes the beta function. Equations (46) and (47) may now be used to evaluate the expression for $\overline{K}_\ell(s,s')$:

$$\overline{K}_\ell(s,s') = \frac{Ze^2}{4}\left(\frac{m}{2}\right)^{s-s'} B\left(\frac{s'-s}{2}, s-s'+1\right)\int_0^\pi d\theta\, P_\ell(\cos\theta)$$

$$\times\left[\frac{\sin[(2-s')(\pi-\theta)]}{\sin\pi s'} - \frac{\sin[(1-s)(\pi-\theta)]}{\sin\pi s}\right]. \quad (48)$$

We thus find, when ℓ is even,

$$\overline{K}_\ell(s,s') = \frac{Ze^2}{4\pi}\left(\frac{m}{2}\right)^{s-s'} B\left(\frac{s'-s}{2}, s-s'+1\right)\sum_{k=0}^{\ell}\binom{\ell}{k} B(\ell-k+\tfrac{1}{2}, k+\tfrac{1}{2})$$

$$\times\left[\frac{\tan\pi s'/2}{\ell-2k+2-s'} - \frac{\cot\pi s/2}{\ell-2k+1-s}\right], \quad (49)$$

and when ℓ is odd,

$$\overline{K}_\ell(s,s') = \frac{Ze^2}{4\pi}\left(\frac{m}{2}\right)^{s-s'} B\left(\frac{s'-s}{2}, s-s'+1\right)\sum_{k=0}^{\ell}\binom{\ell}{k} B(\ell-k+\tfrac{1}{2}, k+\tfrac{1}{2})$$

$$\times\left[\frac{\cot\pi s'/2}{\ell-2k+2-s'} - \frac{\tan\pi s/2}{\ell-2k+1-s}\right]. \quad (50)$$

Poles of the beta functions relate to the conditions of Eq. (45). The reader may note that the even-odd alternative forms for $\overline{K}_\ell(s,s')$ have terms which produce poles for values of s or s' in the regions which are <u>not</u> excluded by the inequalities in Eq. (39). These poles are cancelled when the entire series in k is included. For example, when $\ell = 1$, we find:

$$\overline{K}_1(s,s') \propto \cot\pi s'/2\left[(3-s')^{-1} + (1-s')^{-1}\right]$$

$$- \tan\pi s/2\left[(2-s)^{-1} - s^{-1}\right]. \quad (51)$$

The poles at $s' = 2$ and $s = 1$ from the cotangent and tangent are thus cancelled by the zeroes in the brackets at these values. Hence $\overline{K}_\ell(s,s')$ for $\ell = 1$ is analytic for $-1 < \text{Re } s < 3$, and $0 < \text{Re } s' < 4$.

We are now in a position to begin a determination of the contour of integration, C. Firstly, the contour may be taken to run parallel to the imaginary axis from $-i\infty$ to $i\infty$. It is to be noted that the conditions for the existence of $\overline{K}_\ell(s,s')$ do not at first lead us to an integral equation of the usual type for $\psi_\ell(s)$, since we have derived an equation which relates $\psi_\ell(s)$ to values of $\psi_\ell(s')$ where the set of values of s is different from the set of s' values because of the requirement in Eq. (45). However, we may deform the s' contour by shifting it to the left so that it half encircles the pole contained in the beta function at $s' = s$ or we may increase Re(s) to Re(s'), again taking the contour to half encircle the pole at $s' = s$. In the neighborhood of this pole

$$\overline{K}_\ell(s,s') \simeq \frac{R_\ell(s)}{s'-s}, \quad (52)$$

where $R_\ell(s)$ is given by

$$R_\ell(s) = \frac{Ze^2}{2\pi}\sum_{k=0}^{\ell}\binom{\ell}{k} B(\ell-k+\tfrac{1}{2}, k+\tfrac{1}{2})$$

$$\times\left[\frac{\tan\pi s/2}{\ell-2k+2-s} - \frac{\cot\pi s/2}{\ell-2k+1-s}\right] \quad (53)$$

when ℓ is even, and by

$$R_\ell(s) = \frac{Ze^2}{2\pi} \sum_{k=0}^{\ell} \binom{\ell}{k} B(\ell - k + \tfrac{1}{2}, \ k + \tfrac{1}{2}) \times \left[\frac{\cot \pi s/2}{\ell - 2k + 2 - s} - \frac{\tan \pi s/2}{\ell - 2k + 1 - s} \right] \quad (54)$$

when ℓ is odd.

We may now write the kernel $\bar{K}_\ell(s,s')$ as

$$\bar{K}_\ell(s,s') = \frac{R_\ell(s)}{s' - s} + \left(\bar{K}_\ell(s,s') - \frac{R_\ell(s)}{s' - s}\right), \quad (55)$$

where the kernel

$$\bar{K}_{1\ell}(s,s') \equiv \bar{K}_\ell(s,s') - \frac{R_\ell(s)}{s' - s} \quad (56)$$

is not singular at $s' = s$.

This leads to the singular integral equation

$$\psi_\ell(s) = \frac{R_\ell(s)\,\psi_\ell(s)}{2} + \frac{1}{2\pi i} \int_{s-i\infty}^{s+i\infty} \bar{K}_{1\ell}(s,s')\, \psi_\ell(s')\, ds'$$

$$+ \frac{P}{2\pi i} \int_{s-i\infty}^{s+i\infty} \frac{R_\ell(s)}{s' - s}\, \psi_\ell(s')\, ds', \quad (57)$$

where P denotes a principal value integral. There is also an "associate" integral equation to Eq. (57):

$$\psi_\ell^{a}(s) = \frac{R_\ell(s)\psi_\ell^{a}(s)}{2} + \frac{1}{2\pi i} \int_{s-i\infty}^{s+i\infty} ds'\ \bar{K}_{1\ell}^{a}(s,s')\, \psi_\ell^{a}(s')$$

$$- \frac{P}{2\pi i} \int_{s-i\infty}^{s+i\infty} ds'\ \frac{R_\ell(s')}{s' - s}\, \psi_\ell^{a}(s'), \quad (58)$$

where

$$\bar{K}_{1\ell}^{a}(s,s') = \bar{K}_\ell(s',s) + \frac{R_\ell(s')}{s' - s}.$$

Our method of solution of Eq. (57) consists of first investigating solutions of a singular equation, the "dominant equation." We then derive a new integral equation for the problem which incorporates $\bar{K}_{1\ell}$. This new equation is of the Fredholm type; its development will be given later in this section.

We shall conclude this section with a qualitative discussion of the solutions of the Bakamjian-Thomas equation. For simplicity, we consider the case when $\ell = 0$. In this case, Eq. (53) then becomes (when we drop angular momentum subscripts)

$$R(s) = \frac{Ze^2}{2} \left[\frac{\tan \pi s/2}{2 - s} - \frac{\cot \pi s/2}{1 - s} \right]. \quad (59)$$

If we write $f(s)$ for the term involving $\bar{K}_1$ we have

$$\left(1 - \frac{R(s)}{2}\right) \psi(s) - \frac{R(s)\, P}{2\pi i} \int_{s-i\infty}^{s+i\infty} ds'\ \frac{\psi(s')}{s' - s} = f(s). \quad (60)$$

This equation can be written as $K^0\psi = f$, where K^0 is defined to be the dominant part of the original kernel K .

An equation of this form was first treated by Carlemann and is extensively discussed in the books by Muskhelishvilli[12] and Pogorzelski.[13] We follow the discussions given by these authors. First, we introduce the function

$$H(s) = \frac{1}{2\pi i} \int_C \frac{\psi(s')\, ds'}{s' - s}, \tag{61}$$

where the contour C goes from $-i\infty$ to $i\infty$. We can look at $H(s)$ as a single-valued function in the s plane, cut along C. If we denote the region to the left of the contour by S^+ and that to the right by S^-, we can obtain two functions, $H^\pm(s)$, analytic in $S^\pm$ respectively, according to whether s lies in S^+ or S^-. These two functions can then be analytically continued beyond the cut, C. Because of Cauchy's theorem the contour C can be varied without affecting $H^+(s)$ or $H^-(s)$ unless a singularity in the integrand is encountered on C: i.e., $\psi(s')$ is singular, or C passes through s. If C is chosen to pass through s, we have

$$\begin{aligned} \Big(H(s-\epsilon) - H(s+\epsilon)\Big)_{\epsilon\to 0} &\equiv H^+(s) - H^-(s) = \psi(s) \\ \Big(H(s-\epsilon) + H(s+\epsilon)\Big)_{\epsilon\to 0} &\equiv H^+(s) + H^-(s) = \frac{P}{\pi i}\int \frac{\psi(s')ds'}{s'-s}. \end{aligned} \tag{62}$$

These relations reduce Eq. (60) to an algebraic equation:[14]

$$\left(1 - \frac{R(s)}{2}\right)\left[H^+(s) - H^-(s)\right] = f(s) + \frac{R(s)}{2}\left[H^+(s) + H^-(s)\right]. \tag{63}$$

To solve this equation we begin by considering the solution of the equation with $f \equiv 0$ and denote the solutions by $H_0^\pm$. We have

$$\left[1 - R(s)\right] H_0^+(s) = H_0^-(s), \tag{64}$$

or

$$\frac{H_0^+(s)}{H_0^-(s)} = \frac{1}{1 - R(s)}. \tag{65}$$

Upon taking the logarithm of both sides one finds

$$\ln H_0^+(s) - \ln H_0^-(s) = -\ln\left[1 - R(s)\right]. \tag{66}$$

If one now introduces

$$\ln H_0(s) = -\frac{1}{2\pi i}\int_C \frac{\ln\left[1 - R(s')\right]}{s' - s}, \tag{67}$$

this effects a solution of the discontinuity Eq. (64) for the homogeneous equation.[15] From Eq. (67) one sees that $H_0^\pm(s)$ are neither singular nor zero in the regions $S^\pm$, respectively.

A solution of the inhomogeneous problem is achieved by using Eq. (64) to replace $1 - R$ in Eq. (63). Thus

$$\frac{H_0^-(s)}{H_0^+(s)} H^+(s) = H^-(s) + f(s),$$

or

$$\frac{H^+(s)}{H_0^+(s)} - \frac{H^-(s)}{H_0^-(s)} = \frac{f(s)}{H_0^-(s)}. \tag{68}$$

If we now introduce

$$H(s) \equiv F(s)\, H_0(s), \tag{69}$$

we obtain

$$F^+(s) - F^-(s) = \frac{f(s)}{H_0^-(s)}, \tag{70}$$

which can be formally solved by

$$F(s) = \frac{1}{2\pi i} \int_C \frac{f(s')\,ds'}{(s'-s)H_0^-(s)} , \tag{71}$$

and we see that $F^{\pm}(s)$ are regular in the regions $S^{\pm}$, respectively. The solution of our equation for ψ is then obtained using Eqs. (62) and (69).

We now continue consideration of the choice of contour for our problem. For the $\ell = 0$ case we have the conditions

$$0 < \mathrm{Re}(s) < 2 , \quad \mathrm{Re}(s) < \mathrm{Re}(s') < \mathrm{Re}(s+1), \quad 1 < \mathrm{Re}(s') < 3 . \tag{72}$$

At the various limiting values for s, s' there are singularities in $K_0(s,s')$, of which the pole at $s = s'$ has already been made explicit in the singular integral equation, Eq. (57). We note that $1 - R(s)$ can be given an infinite product representation[16] in the form

$$1 - R(s) = \prod_{n=-\infty}^{\infty} \left(\frac{s - \xi_n}{s - n} \right) , \tag{73}$$

since $R(s)$ has poles at all integers, $R(s) \to 1$ as $|\mathrm{Im}\, s| \to \infty$, and for each n there is an $s = \xi_n$ such that $R(\xi_n) = 1$.[17] Further, as $|n| \to \infty$, one finds

$$\xi_n - n \sim \frac{Ze^2}{\pi n} , \tag{74}$$

which guarantees that the infinite product converges. Finally, $R(s)$ is symmetric about $s = 3/2$, so that

$$R(\tfrac{3}{2} + t) = R(\tfrac{3}{2} - t) . \tag{75}$$

Thus the complex s plane shows a pattern of poles and zeroes as indicated in Fig. 2 (for $Ze^2 < 0$).

Let us now note the following facts: The function $H_0^+(s)$ is analytic and nonzero in S^+ , while $H_0^-(s)$ in analytic and nonzero in S^-. All three contours, C_0, C_1, and C_2, of Fig. 2 satisfy the conditions on s' .[18] We do not consider contours in which $\mathrm{Re}(s) < 3/2$, because such contours can either be distorted so that $\mathrm{Re}(s) > 3/2$ or there will be a singularity in $\psi(s)$ for $\mathrm{Re}\, s < 3/2$. The former case is of no interest, while the latter one would lead to a wave function in momentum space which is not square integrable, and is therefore excluded. If we consider C_0 , we then find from Eq. (67), that

$$H_0^{(0)+}(s) = \prod_{n=2}^{\infty} \left(\frac{s-n}{s-\xi_n} \right)$$

and (76)

$$H_0^{(0)-}(s) = \prod_{n=-\infty}^{1} \left(\frac{s-\xi_n}{s-n} \right) ,$$

while for C_2 ,

$$H_0^{(2)+}(s) = \prod_{n=3}^{\infty} \left(\frac{s-n}{s-\xi_n} \right)$$

and (77)

$$H_0^{(2)-}(s) = \prod_{n=-\infty}^{2} \left(\frac{s-\xi_n}{s-n} \right) .$$

On the other hand, for contour C_1 the integral in Eq. (67) is singular, since the phase of the logarithm does not go to zero as $\mathrm{Im}\, s \to \infty$, so $H_0(s)$ cannot be defined by Eq. (67). One could attempt to use

$$H_0^{(1)+}(s) = (s - 2) \prod_{n=3}^{\infty} \left(\frac{s - n}{s - \xi_n}\right)$$

and (78)

$$H_0^{(1)-}(s) = (s - \xi_2) \prod_{n=-\infty}^{1} \left(\frac{s - \xi_n}{s - n}\right),$$

since this separation of $1 - R(s)$ satisfies Eq. (65). In this case, if we consider the "solution" of the homogeneous equation for $\psi(s)$, we see that, as $|\text{Im } s| \to \infty$, $\psi(s) \to (\xi_2 - 2)$. But this asymptotic behavior is not allowed, since then the principle value integral in Eq. (60) is not well-defined; in fact, we will show that the contour C_1 is not acceptable. If $Ze^2 > 0$, the relative positions of the poles, n, and zeros, ξ_n, in $1 - R(s)$ are reversed. In this case we obtain a valid solution of the homogeneous equation using C_1, in which

$$H_0^{+}(s) = \frac{1}{(s - \xi_2)} \prod_{n=3}^{\infty} \left(\frac{s - n}{s - \xi_n}\right) \tag{79}$$

and

$$H_0^{-}(s) = \frac{1}{(s - 2)} \prod_{n=-\infty}^{1} \left(\frac{s - \xi_n}{s - n}\right). \tag{80}$$

Thus the solution for $\psi(s)$ will not be unique, because an arbitrary amount of the solution of the homogeneous equation can always be added to a particular solution. We will return to a further consideration of C_0, C_2 subsequently.

As was seen from Eq. (36), the behavior of $\psi(p)$ for $p \to \infty$ is determined by the properties of $\psi(s)$ in S^-. In this region

$H_0^-(s)$ and $F^-(s)$ are analytic, so it is convenient to express the solutions in terms of them. In S^- it is convenient to represent $H^+(s)$ as

$$H^+(s) = \frac{H_0^-(s)}{1 - R(s)} \left[F^-(s) + \frac{f(s)}{H_0^-(s)} \right]. \tag{81}$$

Thus if $1 - R(s)$ vanishes, $\psi(s)$ will have a pole; i.e., at points $s = \xi_n$. By construction we also know that $f(s)$ has a pole at $s = n$, but here $R(s)$ also has a pole which cancels the singularity, so that $\psi(s)$ is regular at n. Thus the asymptotic behavior of $\psi(p)$ will be dominated by the smallest ξ_n, $\xi_n^{\min}$, in the S^- region; i.e.,

$$\psi_\ell(p) \underset{p\to\infty}{\sim} p^{-\xi_n^{\min}}.$$

We now come to the decisive part of our investigation, the complete solution of Eq. (57). This depends on the existence of solutions of our equations,[19] which in turn can be determined using the Vekua theory of singular integral equations,[20] which we will now briefly recapitulate. Vekua's theorem states that, under certain conditions, each singular integral equation of the form

$$K\phi \equiv A(s)\,\phi(s) + \frac{P}{\pi i}\int_C ds'\,\frac{N(s,s')f(s')}{s' - s} = f(s) \tag{82}$$

is equivalent to a Fredholm equation with a completely continuous kernel.[21] The conditions which must be imposed are the Hölder relations:

$$|A(s) - A(s')| < \text{const. } |s - s'|^n ,$$

$$|f(s) - f(s')| < \text{const. } |s - s'|^n ,$$

and (83)

$$|K(s,s') - K(s'',s''')| < \text{const.} \left[|s - s''|^n + |s' - s'''|^n \right] ,$$

where $0 \leq n \leq 1$. Of central importance in the Vekua theory is the index κ,

$$\kappa \equiv \frac{1}{2\pi i} \Delta_C \ln \left(\frac{A(s) - B(s)}{A(s) + B(s)} \right) = \frac{1}{2\pi} \Delta_C \arg \left(\frac{A(s) - B(s)}{A(s) + B(s)} \right) . \tag{84}$$

Here,

$$B(s) \equiv N(s,s) , \tag{85}$$

and the notation Δ_C is meant to indicate the total change in phase of $(A - B)/(A + B)$ as we traverse the entire contour C. In our case, $A(s) = 1 - \frac{1}{2} R(s)$ and $B(s) = - \frac{1}{2} R(s)$, and the contours C_0 and C_2 give $\kappa = 0$, while C_1 gives $\kappa = \pm 1$ according to whether $Ze^2 \gtrless 0$. We have seen that only for $Ze^2 > 0$ and the contour C_1 is there a solution of the homogeneous equation, $K^0 \phi = 0$, in which $\phi(s) \rightarrow 0$ as $|\text{Im } s| \rightarrow \infty$. In order to effect the reduction of the singular equation to Fredholm form, the "dominant" operation

$$K^0 \phi = A(s)\, \phi(s) + \frac{B(s) P}{\pi i} \int_C ds' \, \frac{\phi(s')}{s' - s} \tag{86}$$

and its "associate" operation[22]

$$K^{0'} \phi = A(s)\, \phi(s) - \frac{P}{\pi i} \int_C ds' \, \frac{B(s')\, \phi(s')}{s' - s} \tag{87}$$

are introduced. If κ is the index of K^0, then κ is said to be the

index of the original equation. Since the sign of the imaginary unit has been changed in Eq. (87), $-\kappa$ is the index of $K^{0'}$. A theory of Eq. (82) was first developed by Carlemann.[23] If the index of Eq. (82) is κ and

$$\kappa > 0 , \tag{88}$$

there are κ linearly independent solutions of the homogeneous equation of the form

$$\bar{\phi}(s) = H_0(s)\, P_\kappa(s) , \tag{89}$$

where $P_\kappa(s)$ is a polynomial in s of degree κ. On the other hand, for $\kappa \geq 0$ the associate operation in Eq. (87) has a $\kappa \leq 0$, and there are then no nonvanishing solutions of $K^{0'} \phi = 0$ which tend to zero at infinity.

It can be shown by use of the Poincaré-Bertrand transformation[2] that $K^{0'}$ is a "regularizing" operator for the kernel K; that is, the kernel $K^{0'} K \left(\equiv \int_C K^{0'}(s,s'')\, K(s'',s')\, ds'' \right)$ is completely continuous, although K is not. Thus if $\kappa \geq 0$, solutions of the equation $K\phi = f$ can be sought via the regularized equation

$$K^{0'} K \phi = K^{0'} f \tag{90}$$

for which the usual Fredholm theorems apply. Since $K^{0'} \psi = 0$ has no nontrivial solutions, no extraneous solutions are introduced.

If, on the other hand, κ is negative, one may define

$$\phi = K^{0'} \psi \tag{91}$$

and form the equation

$$K K^{0'} \psi = f , \tag{92}$$

which may be shown to have a completely continuous kernel. The solution of the original Eq. (82) is then obtained by quadrature from the solution of this equation.

The relevance of the above theorem to our work depends on the following theorems. We first note that for any kernel K and its adjoint, if we have solutions ϕ, ψ such that

$$K \phi = f \tag{93}$$

and

$$K' \psi = 0 , \tag{94}$$

then the general relation

$$\int \psi K \phi \, ds \, ds' = \int \phi K' \psi \, ds \, ds' \tag{95}$$

requires that

$$\int \psi f \, ds = 0 . \tag{96}$$

This is, of course, just the generalization of the familiar property which is known from the theory of Fredholm operators; that is, a necessary and sufficient condition for the solution of an inhomogeneous Fredholm equation is that the driving term be orthogonal to the eigenfunctions of the transposed operator (or Hermitian conjugate operator if orthogonality includes complex conjugation). We now remark that, in analogy to the Fredholm case, the condition, Eq. (96), is also sufficient to guarantee a solution of Eq. (93): First, suppose that κ is positive or zero. We consider the solution, ω, of the Fredholm equation

$$(K^{0'} K)' \omega = 0 \tag{97}$$

or, equivalently,

$$K' K^0 \omega = 0 . \tag{98}$$

Since the solutions of Eq. (98) always satisfy Eq. (94), $K^0\omega$ must be a linear combination of the ψ's. According to the Fredholm theory, however, a necessary and sufficient condition that there be a solution of an inhomogeneous Fredholm equation is that the inhomogeneous term be orthogonal to all solutions, ω, of the homogeneous equation, with transposed kernel. Thus a sufficient condition for the solution of Eq. (90) is

$$\int \omega K^{0'} f \, ds = \int f K^0 \omega \, ds = \int f \sum_i a_i \psi_i \, ds . \tag{99}$$

Thus if Eq. (96) holds there is a solution of Eq. (90) and hence of Eq. (93), and sufficiency is proved.

If κ is negative, we introduce the solutions γ of the transposed Fredholm equation

$$(K K^{0'})' \gamma = 0 \tag{100}$$

or, equivalently,

$$K^0 K' \gamma = 0 . \tag{101}$$

The Fredholm theory shows here that if

$$\int f \gamma \, ds = 0 \tag{102}$$

one may find a solution of Eq. (92). This allows one to construct ϕ by quadrature [Eq. (91)] . Since the dominant equation for κ negative has no nontrivial solutions, those of Eq. (101) must be linear combinations of those of the homogeneous associate equation, Eq. (89). Thus the condition of Eq. (96) is sufficient in this case also.

A further theorem has been proved by Vekua: The difference between the number k of linearly independent solutions of the singular equation $K\phi = 0$ and the corresponding number, k', for $K'\psi = 0$ is equal to the index κ of the first equation. This can be shown as follows: We assume that $\kappa \geq 0$ without loss of generality, since if $\kappa < 0$ the roles of K and K' can simply be interchanged. Then we know that the equation

$$K\phi = 0 \tag{103}$$

is completely equivalent to

$$K^{0'} K \phi = 0 , \tag{104}$$

and therefore the latter also has k linearly independent solutions. From the Fredholm theory we know that then

$$K' K^{0} \psi = 0 \tag{105}$$

has k linearly independent solutions as well. Since $K^{0}\psi = 0$ has κ linearly independent solutions, it follows that $k' = k - \kappa$.

Let us now apply the foregoing analysis to our equation. As has been seen, the choice of contour C affects the resulting κ . We note that if κ is positive or zero, except for certain eigenvalues there are no nonzero solutions of the homogeneous adjoint equation. Hence there are no restrictions on the function f as indicated by Eq. (4). If κ is negative, however, f cannot be arbitrary. Thus the contour must be chosen so that κ is positive or zero. Thus, if $Ze^2 > 0$ the path C_1 must be excluded since there will not generally be a solution of the equation. On the other hand, if $Ze^2 < 0$, for physical reasons C_1 again is excluded since the solutions in this case would not be unique. Thus we are left with the possible contours C_0 and C_2.

Again let us consider $\ell = 0$. The generalization to arbitrary ℓ is simple. We have seen that the behavior of $\psi(p)$ as $p \to 0$ is determined by the highest singularity in S^+. Thus it is convenient to express the solution of Eq. (60), $\psi(s)$, in terms of $H_0^{\,+}$ and F^+:

$$\psi(s) = R(s)\, H_0^{\,+}(s)\, F^+(s) + f(s) . \tag{106}$$

The singularities of $\psi(s)$ in S^+ are then found either in $R(s)$ or $f(s)$, or in both. Thus there may be poles in $\psi(s)$ at all of the integers to the left of C. If Eq. (37) is used to continue $\psi(s)$, however, it is seen that only the singularity in s associated with $K(s,s')$ produces a singularity in $\psi(s)$, and hence if C_0 is chosen, there will be a pole in $\psi(s)$ at $s = 0$ (and at the negative even integers). On the other hand, if C_2 is chosen, it is convenient to first let $Re(s) \to Re(s')$ on C_2, since we have the analytic continuation explicity of $K(s,s')$ for $Re\ s \geq 2$, and then obtain a solution of the equation on C_2, and finally use Eq. (37) to continue back to $s = 2$. We thus find a pole at $s = 2$, and

$$\psi(p) \underset{p\to 0}{\sim} p^{-2} . \tag{107}$$

This behavior is not acceptable, however, since the wave function would not be normalizable, and so we exclude the path C_2 from further consideration, and we have a unique solution to the singular equation, Eq. (12). Since we are thus restricted to the path, C_0 , we may ask whether there is some especially appropriate path. It is shown in Appendix A that if $Re(s) = Re(s') = 3/2$, the kernel of the integral equation satisfies a hermiticity condition, and so this choice seems to be called for.

We shall close this section by noting that there is a maximum value for $-Ze^2$ for which a unique solution of Eq. (12) is possible. Again for $\ell = 0$, it is easily seen that $R(s)$ is real for $\mathrm{Im}\, s = 0$, and for $\mathrm{Re}\, s = 3/2$. In the latter case

$$R(s) = -\frac{Ze^2}{2}\left[\frac{1 - i \tanh \frac{\pi}{2} s_I}{1 + i \tanh \frac{\pi}{2} s_I} \cdot \frac{1}{\frac{1}{2} - i s_I} + \text{complex conj.}\right], \tag{108}$$

where $s = 3/2 + i s_I$. As $|s_I| \to \infty$,

$$R(s) \sim -\frac{Ze^2}{s_I}. \tag{109}$$

On the path along which $R(s)$ is real going from $s = 2$ to either $s = 3/2 \pm i\infty$, $|R(s)|$ is a monotonic decreasing function going from ∞ to 0, and if $Ze^2 < 0$, there will be a point ξ_2 on the path at which[25]

$$R(\xi_2) = 1. \tag{110}$$

This point ξ_2 will only have $\mathrm{Im}(s) = 0$ if

$$R(3/2) < 1, \tag{111}$$

that is,

$$|Ze^2| < 1/2. \tag{112}$$

If this condition is not satisfied, the points ξ_1 and ξ_2 become complex conjugate pairs and the contour C and solution $\psi(s)$ are not unique. The situation is completely analogous to that with the Dirac equation, for which there are too many acceptable solutions also if Ze^2 is too large.[26] The problem raised here is only of mathematical

interest, however, since a large Z nucleus would necessitate a form factor to describe its spatial extent and the potential for large p would be cut off, in contradistinction to the point particles dealt with here. Thus we will not pursue this case further.

The Dirac equation may also be dealt with using the Mellin transformation technique. In that case it is found that $R(s)$ is a quadratic function of s and there are only two possible ξ_i's. A brief account of the treatment of the Dirac equation is given in Appendix B.

V. MOMENTUM SPACE INTERPRETATION OF THE K_0 KERNEL

In the preceding section we have provided an analysis of the non-Fredholm Bakamjian-Thomas equation which leads to a unique solution. Although this provides a mathematically satisfactory solution, its significance is probably somewhat obscure. In the present section we will provide an alternative momentum space treatment which is closely related to the Carlemann approach in Mellin space, but which gives direct insight into the above results.

Since the non-Fredholm behavior of the equation is associated with high momenta, one might try to separate the kernel into an asymptotic part and a remainder. Thus we write Eq. (12) as

$$\psi_\ell(p) = -\frac{Ze^2}{2\pi}\int_0^\infty dp'\, \frac{(p+p')}{p^2}\, Q_\ell\left(\frac{p^2+p'^2}{2pp'}\right) U(p')\, \psi_\ell(p')$$

$$+ \frac{Ze^2}{2\pi}\int_0^\infty p'\,\frac{dp'}{p}\left[\frac{(p_0+p_0')}{p_0'(p_0-p_0)} + U(p')\,\frac{(p+p')}{pp'}\right] Q_\ell\left(\frac{p^2+p'^2}{2pp'}\right)\psi_\ell(p'), \tag{113}$$

where $U(p)$ is zero for $p < 1$, and is one for $p \geq 1$. Let us consider the kernel in the first integral to be $K_0(p,p')$, and the balance of the right-hand side as if it were an inhomogeneous term, $f(p)$. We thus look for solutions of the equation

$$\psi(p) = \int dp' \, K_0(p,p') \, \psi(p') + f(p) . \qquad (114)$$

The step function, U, must be introduced because otherwise there would be no solution of Eq. (114) because of the behavior as $p,p' \to 0$.

Clearly the kernel K_0 is of such form that we can write

$$\psi_\ell(p) = \int_1^\infty \frac{dp'}{p'} \, F\left(\frac{p}{p'}\right) \psi_\ell(p') + f(p) . \qquad (115)$$

This equation can be solved in two ways: If a new variable $x \equiv \ell n \, p$ is introduced, Eq. (115) is converted into a Wiener-Hopf equation which can be solved by known techniques.[27] On the other hand, if a Mellin transformation is carried out as in Sec. IV, we get

$$\psi_\ell(s) = \frac{1}{2\pi i} \int_C ds' \, \frac{F(s) \, \psi_\ell(s')}{s' - s} + f(s) , \qquad (116)$$

where $Re(s' - s) > 0$. This equation can be brought to the form of Eq. (57), and is easily seen to be identical to it. Thus the Carlemann solution of this equation corresponds to finding a solution of the inhomogeneous Eq. (114) to remove the non-Fredholm term, $K_0(p,p')$.

In obtaining Eq. (113) we essentially took the asymptotic form of the kernel for $p,p' \to \infty$, and then multiplied the kernel unsymmetrically by $U(p')$. If, on the other hand, we had multiplied by $U(p)$, the only change which would occur would be that in Eq. (116), $F(s')$ would appear instead of $F(s)$.

VI. CONCLUSION

In this paper we have given arguments which lead to an approximate wave equation for spin-zero particles. This equation has been studied for the case of an interaction which is the time component of a vector field. A simplification was afforded by assuming one of the particles to be infinitely massive. Because of the nature of the interaction the Schroedinger integral equation is singular so that the Fredholm theory does not immediately apply. We have given a simple discussion of the nature of the solutions to be expected for our equation and have then gone on to rigorously show that a unique solution may be achieved if the potential is repulsive or the coupling constant is not too large. We have also gone beyond the considerations of this paper to construct explicit numerical solutions for both bound state and scattering problems for the case when $\ell = 0$. Because of the length and complexity of this paper and the special techniques which are required to effect a numerical solution we will report these results elsewhere.

APPENDIX A. HERMITICITY OF THE MELLIN TRANSFORMED EQUATION

In this section we shall show that the kernel in the Mellin transformed integral equation is Hermitian if the contour C is taken to lie along $\mathrm{Re}(s') = 3/2$. For this development, we divide the kernel into two parts:

$$K_\ell(p,p') = \overline{K}_\ell(p,p') + K_\ell^R(p,p') , \qquad (A1)$$

where [see Eq. (40)]

$$\overline{K}_\ell(p,p') = -\frac{Ze^2}{2\pi}\,\frac{p'}{p}\,\frac{(p_0 + p_0')}{p_0\, p_0'}\, Q_\ell\left(\frac{p^2 + p'^2}{2pp'}\right) \qquad (A2)$$

and

$$K_\ell^R(p,p') = -\left[1 - \frac{(p_0\, p_0')^{\frac{1}{2}}}{(P_0 - p_0)^{\frac{1}{2}}(P_0 - p_0')^{\frac{1}{2}}}\right]\overline{K}_\ell(p,p') . \qquad (A3)$$

This division separates K_ℓ into a part which has a Mellin transform that is regular at $s = s'$, K_ℓ^R, and a singular part, $\overline{K}_\ell$, for which we already have the transform explicity. The singularity in $\overline{K}_\ell(s,s')$ for $s \to s'$ arises from the asymptotic behavior of $\overline{K}_\ell(p,p')$ for large p, p'. If we set $p' = \alpha p$ in Eq. (41), we find

$$\overline{K}_\ell(s,s') = -\frac{Ze^2}{2\pi}\int_0^\pi d\theta\,\sin\theta\, P_\ell(\cos\theta)\int_0^\infty d\alpha\,\frac{\alpha^{2-s'}}{(\alpha - e^{i\theta})(\alpha - e^{-i\theta})}$$

$$\times \int_0^\infty dp\, p^{s-s'}\left[(p^2 + m^2)^{-\frac{1}{2}} + (\alpha^2 p^2 + m^2)^{-\frac{1}{2}}\right] . \qquad (A4)$$

Thus, the integral over p diverges as $p \to \infty$, unless $\mathrm{Re}(s - s') < 0$. If we use the same approach to $K_\ell^R(s,s')$, however, as $p \to \infty$ the terms

involving p_0 and p_0' are now of order p^{-2}, and hence $K_\ell^R(s,s')$ is regular at $s = s'$.

To investigate the hermiticity condition for the kernel, we convert the transformed integral equation to one in real variables. Thus we set

$$s = s_R + is_I$$

and $\qquad (A5)$

$$s' = s_R + is_I' ,$$

so that the integral equation becomes (we do not explicitly exhibit the dependence on s_R, which now becomes a parameter in the equation):

$$\psi_\ell(s_I) = \frac{1}{2} R_\ell(is_I)\ \psi_\ell(s_I) + \frac{P}{2\pi i}\int_{-\infty}^{\infty} ds_I'\,\frac{R_\ell(is_I')\,\psi_\ell(s_I')}{s_I' - s_I}$$

$$+ \frac{1}{2\pi}\int_{-\infty}^{\infty} ds_I'\left[K_{1\ell}(is_I, is_I') + K_{1\ell}^R(is_I, is_I')\right]\psi_\ell(s_I') . \qquad (A6)$$

The $K_{1\ell}^R$ part of the kernel considered as a function of s_I, s_I' can easily be shown to be Hermitian for $s_R = 3/2$. We begin by noting that the transform can be written as

$$K_{1\ell}^R(s_R + is_I,\ s_R' + is_I') = \int_0^\infty dp\int_0^\infty dp'$$

$$\times\, p^{s_R + is_I - 1}\,(p')^{-s_R - is_I' + 2}\, F_\ell(pp') , \qquad (A7)$$

where

$$F_\ell(p,p') = F_\ell(p',p)^* . \tag{A8}$$

(We only consider bound states, so $P_0 < m$.) Then we find

$$K_1^R(s_R+is_I', s_R+is_I)^* = \int_0^\infty dp \int_0^\infty dp'\, p^{2-s_R+is_I} (p')^{s_R-is_I'-1} F_\ell(p,p') , \tag{A9}$$

in which Eqs. (A7) and (A8) have been used, and the dummy variables p, p' have been interchanged. Thus, if $s_R = 3/2$ we see that[28] as a function of the real variables s_I, s_I'

$$K_{1\ell}^R\left(\tfrac{3}{2} + is_I, \tfrac{3}{2} + is_I'\right) = K_{1\ell}^R\left(\tfrac{3}{2} + is_I', \tfrac{3}{2} + is_I\right)^* , \tag{A10}$$

so that $K_{1\ell}^R$ is Hermitian. We must now look more closely at the singular kernel. We consider the case in which ℓ is even. Choosing $s_R = 3/2$, we then find from Eq. (49);

$$\bar{K}_\ell(s_I, s_I') = \frac{Ze^2}{4\pi}\left(\frac{m}{2}\right)^{i(s_I-s_I')} B\left[\frac{i(s_I-s_I')}{2}, i(s_I - s_I') + 1\right]$$

$$\times \sum_{k=0}^{\ell} \binom{\ell}{k} B\left(\ell - k + \tfrac{1}{2}, k + \tfrac{1}{2}\right)\left[\frac{1 - i\tanh\frac{\pi s_I'}{2}}{1 + i\tanh\frac{\pi s_I'}{2}} \cdot \frac{1}{\ell - 2k + \frac{1}{2} - is_I'}\right.$$

$$\left. + \frac{1 + i\tanh\frac{\pi s_I}{2}}{1 - i\tanh\frac{\pi s_I}{2}} \cdot \frac{1}{\ell - 2k + \frac{1}{2} + is_I}\right] , \tag{A11}$$

in which the second term in the bracket is obtained from the $\cot \pi s/2$ term in Eq. (49) by interchanging k and $\ell - k$ in the summation over

k. From Eq. (116) one can easily see that

$$\bar{K}_\ell(s_I, s_I') = \bar{K}_\ell(s_I', s_I)^* \tag{A12}$$

and that $R_\ell(s_I)$, which is the residue of $\bar{K}_\ell$ in the pole of the beta function at $s' = s$ is real. Thus we have a Hermitian kernel for the equation. It may be mentioned here that the Fredholm kernels obtained in the Vekua theory do not satisfy the hermiticity requirement. This occurs because of the lack of symmetry in the choice of K_0 , for example, not from the singularity at $s' = s$.

APPENDIX B. MOMENTUM SPACE ANALYSIS OF THE DIRAC EQUATION

In this appendix we apply the Mellin transformation technique to the solution of the familiar Dirac equation for a spin-1/2 particle in a Coulomb field. The conventional discussion[29] is based on a study of the indicial equation of the differential equation for this problem in coordinate space and involves boundary conditions at the origin.

Let us now consider the nature of the solutions of this equation in momentum space:

$$(\underset{\sim}{\alpha}\cdot\underset{\sim}{p} + \beta m)\,\psi(\underset{\sim}{p}) + \frac{Ze^2}{2\pi^2}\int \frac{d^3p'}{(\underset{\sim}{p}-\underset{\sim}{p}')^2}\,\psi(\underset{\sim}{p}') = P_0\,\psi(\underset{\sim}{p}) . \qquad \text{(B1)}$$

The usual solutions of the Dirac equation involve the operator

$$k = \beta(\underset{\sim}{\sigma}\cdot\underset{\sim}{L} + 1) ,$$

which has the eigenvalues $\pm(j+\frac{1}{2})$, since it commutes with the Dirac Hamiltonian. If we write

$$\underset{\sim}{\alpha} = \rho_1\,\underset{\sim}{\sigma} ,$$

Eq. (B1) becomes

$$(\rho_1\,\underset{\sim}{\sigma}\cdot\underset{\sim}{p} + \beta m)\,\psi(\underset{\sim}{p}) + \frac{Ze^2}{2\pi^2}\int \frac{d^3\underset{\sim}{p}'}{(\underset{\sim}{p}-\underset{\sim}{p}')^2}\,\psi(\underset{\sim}{p}') = P_0\,\psi(\underset{\sim}{p}) . \qquad \text{(B2)}$$

We choose

$$\rho_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \beta = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \underset{\sim}{\sigma}\cdot\underset{\sim}{p} = \begin{pmatrix} \underset{\sim}{\sigma}\cdot\underset{\sim}{p} & 0 \\ 0 & \underset{\sim}{\sigma}\cdot\underset{\sim}{p} \end{pmatrix} \qquad \text{(B3)}$$

where each of the elements in a matrix is a 2 x 2 matrix, and the corresponding σ's are Pauli spin matrices.

An angular momentum decomposition can be achieved by setting

$$\psi_j^{\,m} = \begin{pmatrix} y^m_{j,j-\frac{1}{2}}(\hat{e}_p)\,F_j(p) \\ y^m_{j,j+\frac{1}{2}}(\hat{e}_p)\,G_j(p) \end{pmatrix} , \qquad \text{(B4)}$$

where $y^m_{j\ell}$ is an eigenfunction of total and orbital angular momentum of eigenvalues j and ℓ, respectively, and of $j_z = m$, and $\hat{e}_p$ is a unit vector in the direction of $\underset{\sim}{p}$. In the first place this function is an eigenfunction of k, since[30]

$$(\underset{\sim}{\sigma}\cdot\underset{\sim}{L} + 1)\,y^m_{j,j\pm\frac{1}{2}} = \mp(j+\tfrac{1}{2})\,y^m_{j,j\pm\frac{1}{2}} . \qquad \text{(B5)}$$

Now consider the effect of the operator $\underset{\sim}{\sigma}\cdot\underset{\sim}{p}$ on the state $y^m_{j,\ell}$. We set

$$Y \equiv \underset{\sim}{\sigma}\cdot\underset{\sim}{p}\; y^m_{j,\ell} . \qquad \text{(B6)}$$

To characterize this state we note that

$$\begin{aligned}(J^2, \underset{\sim}{\sigma}\cdot\underset{\sim}{p}) &= 0 , \\ (J_3, \underset{\sim}{\sigma}\cdot\underset{\sim}{p}) &= 0 , \\ (L^2, \underset{\sim}{\sigma}\cdot\underset{\sim}{p}) &= 2\,\underset{\sim}{\sigma}\cdot\underset{\sim}{p} + 2\,\underset{\sim}{\sigma}\cdot\underset{\sim}{p}\,\underset{\sim}{\sigma}\cdot\underset{\sim}{L} .\end{aligned} \qquad \text{(B7)}$$

Since

$$\underset{\sim}{\sigma}\cdot\underset{\sim}{L} = J^2 - L^2 - 3/4 , \qquad \text{(B8)}$$

we find that

$$L^2 Y = \left[2j(j+1) - \ell(\ell+1) + \tfrac{1}{2}\right] Y . \tag{B9}$$

If

$$\ell = j - \tfrac{1}{2} ,$$

the new eigenvalue is $j + \frac{1}{2}$, and when

$$\ell = j + \tfrac{1}{2} ,$$

the new one is

$$\ell = j - \tfrac{1}{2} .$$

Choosing the conventional Clebsch-Gordon coefficients, we may write

$$y^m_{jj+\frac{1}{2}} = \frac{\underset{\sim}{\sigma}\cdot\underset{\sim}{p}}{p}\, y^m_{jj-\frac{1}{2}}$$

and (B10)

$$y^m_{jj-\frac{1}{2}} = \frac{\underset{\sim}{\sigma}\cdot\underset{\sim}{p}}{p}\, y^m_{jj+\frac{1}{2}} .$$

If we now substitute Eq. (B4) into the Dirac equation (B2) we have, denoting a unit vector by $\hat{p}$,

$$pG_j(p)\, y^m_{jj-\frac{1}{2}}(\hat{p}) + mF_j(p) y^m_{jj-\frac{1}{2}}(\hat{p}) + \frac{Ze^2}{2\pi^2}\int \frac{d^3\underset{\sim}{p}'}{(\underset{\sim}{p}-\underset{\sim}{p}')^2} F_j(p')\, y^m_{jj-\frac{1}{2}}(\hat{p}')$$

$$= P_0\, F_j(p)\, y^m_{jj-\frac{1}{2}}(\hat{p})$$

and

$$pF_j(p)\, y^m_{jj+\frac{1}{2}}(\hat{p}) - mG_j(p)\, y^m_{jj+\frac{1}{2}}(\hat{p}) + \frac{Ze^2}{2\pi^2}\int \frac{d^3\underset{\sim}{p}'}{(\underset{\sim}{p}-\underset{\sim}{p}')^2} G_j(p')\, y^m_{jj+\frac{1}{2}}(\hat{p}')$$

$$= P_0\, G_j(p)\, y^m_{jj+\frac{1}{2}}(\hat{p}). \tag{B11}$$

To eliminate the angular functions we multiply by $y^m_{jj-\frac{1}{2}}(\hat{p})^*$ and integrate over solid angle. We use the relations

$$\frac{2\ell+1}{4\pi} P_\ell(\hat{p}'\cdot\hat{p}) = \sum_{jm} y^m_{j\ell}(\hat{p}')^*\, y^m_{j\ell}(\hat{p})$$

and (B12)

$$(z-t)^{-1} = \sum_{\ell=0}^{\infty} (2\ell+1)\, Q_\ell(z)\, P_\ell(t) ,$$

and the normalization condition. The interaction term in the first member of Eq. (B11) is

$$\int d\Omega_p\, y^m_{jj-\frac{1}{2}}(p)^* \frac{d^3\underset{\sim}{p}'}{(\underset{\sim}{p}-\underset{\sim}{p}')^2}\, y^m_{jj-\frac{1}{2}}(\hat{p}')\, F_j(p')$$

$$= \int_0^\infty \frac{p'}{p}\, dp'\, dr_p\, d\Omega_{p'}\, y^m_{jj-\frac{1}{2}}(\hat{p})^* \sum_{j'\ell'm'} Q_{\ell'}\left(\frac{p^2+p'^2}{2pp'}\right)$$

$$\times\, y^{m'}_{j'\ell'}(\hat{p})\, y^{m'}_{j'\ell'}(\hat{p}')^*\, y^m_{jj-\frac{1}{2}}(\hat{p}')\, F_j(p')$$

$$= 2\pi \int_0^\infty \frac{p'}{p}\, dp'\, Q_\ell\left(\frac{p^2+p'^2}{2pp'}\right) F_j(p') . \tag{B13}$$

We thus obtain the equations

$$pG_j(p) + m\,F_j(p) + \frac{Ze^2}{\pi}\int_0^\infty \frac{p'}{p}\, dp'\, Q_{j-\frac{1}{2}}\left(\frac{p^2+p'^2}{2pp'}\right) F_j(p') = P_0\, F_j(p)$$

and (B14)

$$pF_j(p) - m\,G_j(p) + \frac{Ze^2}{\pi}\int_0^\infty \frac{p'}{p}\, dp'\, Q_{j+\frac{1}{2}}\left(\frac{p^2+p'^2}{2pp'}\right) G_j(p') = P_0\, G_j(p) .$$

Equation (B14) may be rewritten as

$$F_j(p) = \frac{Ze^2}{\pi}\int_0^\infty \frac{p'\,dp'}{p(P_0^2 - p_0^2)}\left[(P_0 + m)Q_{j-\frac{1}{2}}F_j + pQ_{j+\frac{1}{2}}G_j\right]$$

and (B15)

$$G_j(p) = \frac{Ze^2}{\pi}\int_0^\infty \frac{p'\,dp'}{p(P^2 - p_0^2)}\left[pQ_{j-\frac{1}{2}}F_j + (P_0 - m)Q_{j+\frac{1}{2}}G_j\right] .$$

The arguments of the Legendre function have been suppressed.

We now make a Mellin transformation on Eq. (B15):

$$F_j(s) = \int_0^\infty p^{s-1}F_j(p)$$

and (B16)

$$G_j(s) = \int_0^\infty p^{s-1}G_j(p) .$$

For convenience we use the same symbols for the functions and their transforms. In matrix notation this leads to the integral equation

$$\psi(s) = \frac{1}{2\pi i}\int_C K(s,s')\,\psi(s')\,ds' , \qquad \text{(B17)}$$

where

$$K^j(s,s') = \frac{Ze^2}{\pi}\int_0^\infty dp\int_0^\infty dp'\,\frac{p^{s-2}(p')^{-s'+1}}{(P_0^2 - p_0^2)}\begin{vmatrix}(P_0+m)Q_{j-\frac{1}{2}} & pQ_{j+\frac{1}{2}}\\ pQ_{j-\frac{1}{2}} & (P_0-m)Q_{j+\frac{1}{2}}\end{vmatrix} .$$

(B18)

The integrals may be evaluated by the methods used in the foregoing paper. If K is conventionally labelled according to the scheme

$$K^j = \begin{vmatrix}K_{11}{}^j & K_{12}{}^j\\ K_{21}{}^j & K_{22}{}^j\end{vmatrix} , \qquad \text{(B19)}$$

we find the following: If $-1 < \mathrm{Re}(s - s') < 1$, and $-j + \frac{3}{2} < \mathrm{Re}\, s' < j + \frac{5}{2}$, we find

$$K_{11}{}^j(s,s') = \frac{Ze^2}{2}\,\frac{(P_0+m)\kappa^{s-s'-1}}{\cos\frac{\pi}{2}(s - s')}\,C_j(s')\,S_j(s') , \qquad \text{(B20)}$$

where we set

$$S_j(s') \equiv \sum_{k=0}^{j-\frac{1}{2}}\binom{j-\frac{1}{2}}{k}\frac{B(j-k,\ k+\frac{1}{2})}{(j - 2k + \frac{3}{2} - s')} , \qquad \text{(B21)}$$

$$\kappa = (m^2 - P_0^2)^{\frac{1}{2}}$$

and

$$C_j(s') = \frac{(-1)^{j-\frac{1}{2}}\left[1 - (-1)^{j-\frac{1}{2}}\cos \pi s'\right]}{\sin \pi s'} . \qquad \text{(B22)}$$

Thus, if $j - \frac{1}{2}$ is even, $C_j(s') = \tan \pi s'/2$, and if $j - \frac{1}{2}$ is odd, $C_j(s') = -\cot \pi s'/2$. The other elements of K are easily obtained from $K_{11}{}^j(s,s')$. Thus

$$K_{12}{}^{j}(s,s') = K_{11}^{j+1}(s+1,\ s')/(P_0 + m)\ ,$$

$$K_{21}{}^{j}(s,s') = K_{11}{}^{j}(s+1,\ s')/(P_0 + m)\ ,$$

and (B23)

$$K_{22}{}^{j}(s,s') = \left(\frac{P_0 - m}{P_0 + m}\right) K_{11}^{j+1}(s,s')\ .$$

It is seen that the integral equation is of the singular type since it has a pole at $s = s'$.

In analogy to the B-T equation, the integral equation can be written

$$\left[1 - \frac{R(s)}{2}\right]\psi(s) = \frac{P}{2\pi i}\int_C \frac{R(s)}{s' - s}\,\psi(s')ds' + \frac{1}{2\pi i}\int_C K_1(s,s')\psi(s')ds', \qquad \text{(B24)}$$

where K_1 is a regular 2×2 matrix, and $R(s)$ is the residue of K at $s' = s$. A singularity in $\psi(s)$ will now occur if the matrix $[1 - R(s)]$ is singular; i.e., it has a zero determinant. From Eqs. (B20) and (B23), we find that only K_{12} and K_{21} have poles at $s = s'$, so that

$$R_{12}(s) = \frac{Ze^2}{\pi}\ C_{j+1}(s) \sum_{k=0}^{j+\frac{1}{2}} \binom{j + \frac{1}{2}}{k} \frac{B(j - k + 1,\ k + \frac{1}{2})}{(j - 2k + \frac{5}{2} - s)} \qquad \text{(B25)}$$

and similarly for $R_{21}(s)$, where $j + 1 \rightarrow j$. The condition that $\det(1 - R) = 0$ is thus

$$1 = \left(\frac{Ze^2}{\pi}\right)^2 C_j(s)\ C_{j+1}(s)\ S_j(s)\ S_{j+1}(s)$$

$$= -\left(\frac{Ze^2}{\pi}\right)^2 S_j(s)\ S_{j+1}(s)\ . \qquad \text{(B26)}$$

We see that $S_j(s)$ has poles at $s = j + \frac{3}{2},\ \ j + \frac{1}{2}, \cdots,\ -j + \frac{5}{2}$, and hence S_jS_{j+1} can be expressed as a sum of poles times residues, which will now be evaluated.

We first consider the residue of the poles at $s = j + \frac{5}{2} - 2k$, where k is an integer. For this purpose we note that[31]

$$S_j(j + \frac{5}{2} - 2k) = (-1)^{j-\frac{1}{2}}\,\pi\left[1 - (-1)^{j-\frac{1}{2}}\cos\pi(j + \frac{5}{2} - 2k)\right]^{-1}$$

$$\times \int_0^{\pi} P_{j-\frac{1}{2}}(\cos\theta)\ \sin\left[(\pi - \theta)(2k - j - \tfrac{1}{2})\right] d\theta\ . \qquad \text{(B27)}$$

The factor in the square brackets is just 2, and the integral can be written as:

$$(-1)^{j+\frac{1}{2}} \int_0^{\pi} P_{j-\frac{1}{2}}(\cos\theta)\ \sin(j - 2k + \tfrac{1}{2})\theta\ d\theta\ . \qquad \text{(B28)}$$

Further,

$$\sin(j - 2k + \tfrac{1}{2})\theta = \sin\theta \sum_{m=0}^{[(j-2k+\frac{1}{2})/2]} a_m\ P_{j-\frac{1}{2}-2m}(\cos\theta),$$

so the integral vanishes unless $k = 0$ or $k = j + \frac{1}{2}$. Similarly, one gets zero for all of the residues associated with poles in C_j. Thus the product of $S_j(s)\ S_{j+1}(s)$ only has poles at $s = j + \frac{5}{2}$ and $s = \frac{3}{2} - j$. At $s = j + \frac{5}{2}$, we find

$$S_j(j + \tfrac{5}{2}) = -\frac{\pi}{2}\,\Gamma(j + \tfrac{1}{2})\ \Gamma(\tfrac{1}{2})/\Gamma(j + 1)\ , \qquad \text{(B29)}$$

and at $s = \frac{3}{2} - j$, the result is the same except for a change in sign. We finally obtain

$$1 = \frac{(Ze^2)^2}{(2j+1)} \left(\frac{1}{j + \frac{5}{2} - s} + \frac{1}{j - \frac{3}{2} + s} \right) . \tag{B30}$$

From this one easily finds the singular values in s :

$$s_{1,2} = 2 \pm \left[(j + \tfrac{1}{2})^2 - (Ze^2)^2 \right]^{\frac{1}{2}} . \tag{B31}$$

These are the analog of the well-known result in coordinate space for the indicial equation.

FOOTNOTES AND REFERENCES

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7. For simplicity, we shall deal with Eq. (11); the asymptotic properties of Eq. (10) are not affected by the additional factor $(q_0 + q_0')/q_0'$.

8. See: W. Hunziker, Helv. Phys. Acta 34, 593 (1961); A. Grossman and T. Wu, J. Math. Phys. 2, 710 (1961); K. Meetz, J. Math. Phys. 3, 690 (1962); or Scadron, Weinberg, and Wright, Phys. Rev. 135, B202 (1964).

9. In carrying out this integration we use the integral representation: $P_\ell(\cos\theta) = \pi^{-1}\int_0^\pi (\cos\theta + i\sin\theta\cos t)^\ell\, dt$ [see Higher Transcendental Functions, Bateman Manuscript Project (McGraw-Hill, New York, 1953), vol. I, Eq. (3.7.23)], and note that:

$$\cos\theta + i\sin\theta\cos t = \cos^2\frac{t}{2}\cdot e^{i\theta} + \sin^2\frac{t}{2}\cdot e^{-i\theta}\ .$$

10. E. C. Titchmarsh, Introduction to the Theory of Fourier Integrals (Oxford University Press, 1962), p. 60. It is also possible to make a Fourier transformation instead of the Mellin transformation. In this case one is led to a study of a Wiener-Hopf equation instead of Eq. (57).

11. In subsequent equations we will use the same symbol for a function as for its Mellin transform. The two will be distinguished by their arguments, p or s.

12. N. Muskhelishvili, Singular Integral Equations (P. Nordhoff N. V., Gröningen, Holland, 1953).

13. W. Pogorzelski, Integral Equations and their Applications (Pergamon Press, Long Island City, N. Y., 1966).

14. The problem of finding a function with a given discontinuity across a contour is called the "Hilbert problem."

15. In our particular case, $R(s) = O(1/|s|)$ as $|\text{Im } s| \to \infty$, and thus $\ln H_0(s) \to 0$ also. This implies that $H_0(s) \to 1$ as $|\text{Im } s| \to 0$. Thus, although we have a solution of the discontinuity equation, Eq. (64), $H_0(s)$ cannot be represented in the form of Eq. (61). We therefore do not have a solution of the original homogeneous equation.

16. E. T. Whitaker and G. N. Watson, Modern Analysis (Macmillan, New York, N. Y., 1943), p. 138.

17. For $\ell \neq 0$, as has been seen, $R_\ell(s)$ has no singularities for $-\ell + 1 < \text{Re}(s) < \ell + 2$. One also sees that, at least for very small Ze^2, there can be no ξ_n's near the integers in this region, either. For small Ze^2, the smallest ξ_n above $3/2$ will be near $\ell + 2$. Thus a representation of this form is also available for $\ell \neq 0$, except that some n values must be excluded.

18. For $\ell \neq 0$, C_1 and C_2 would be chosen between the lowest zero-pole pair in $R(s)$ for $\text{Re}(s) > 3/2$, and to the right of that pair, respectively. See Footnote 17.

19. It is to be noted that $H(s)$ must tend to zero at infinity if Eq. (61) is to hold.

20. See Ref. (12) or (13).

21. The latter kernel need not be bounded. It is only necessary that, for its kernel K, $\int_C\int_C |K(s,s')|^2\, ds\, ds'$ exists.

22. The associate to a kernel $K(s,s')$ is given by $K'(s,s') = K(s',s)$.

23. See Ref. (12) or (13).

24. See Ref. (13).

25. If $Ze^2 > 0$, the point ξ_2 lies at $\text{Re } s > 2$, and $R(s)$ is negative on the entire path discussed here so that for a repulsive potential there is no difficulty.

26. See K. M. Case, Phys. Rev. 80, 797 (1950).

27. See, e.g., Titchmarsh, loc. cit.

28. Throughout this analysis we have assumed that the integral equation has an integration weight factor of p'^2 so that the complete kernel is $p'^2 K(p,p')$, where $K(p,p')$ is Hermitian. This factor can be modified by a change in the wave function of the form $\psi'(p) = p^\alpha \psi(p)$, which then introduces a factor $(p'/p)^\alpha$ in the

kernel. If such a change is made, the hermiticity condition becomes $s_R = \frac{3}{2} + \alpha$, and at the same time the conditions of Eq. (39) are also shifted by α.

29. See, e.g., L. I. Schiff, Quantum Mechanics (McGraw-Hill Book Co., Inc., New York, N. Y., 1949), p. 322 ff.

30. We only consider the case in which the eigenvalue of k is $j + \frac{1}{2}$. The other case can be similarly treated.

31. This relation has already been used to obtain Eq. (B20), using the integral representation for Q_ℓ.

FIGURE CAPTIONS

Fig. 1. Structure of the p' plane together with contours used in evaluating integrals.

Fig. 2. Structure of $R(s)$; poles are represented by x, zeroes by o. At first sight C_0, C_1, C_2 represent possible contours of integration but only C_0 is really allowed.

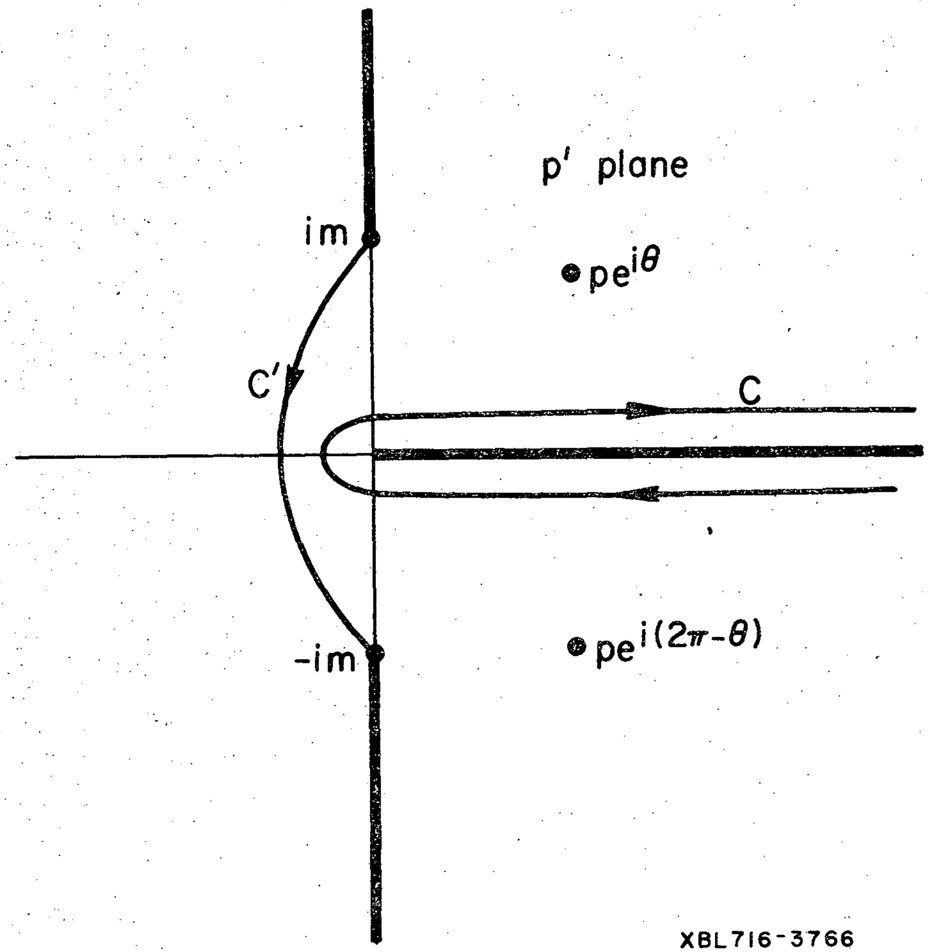

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Fig. 1

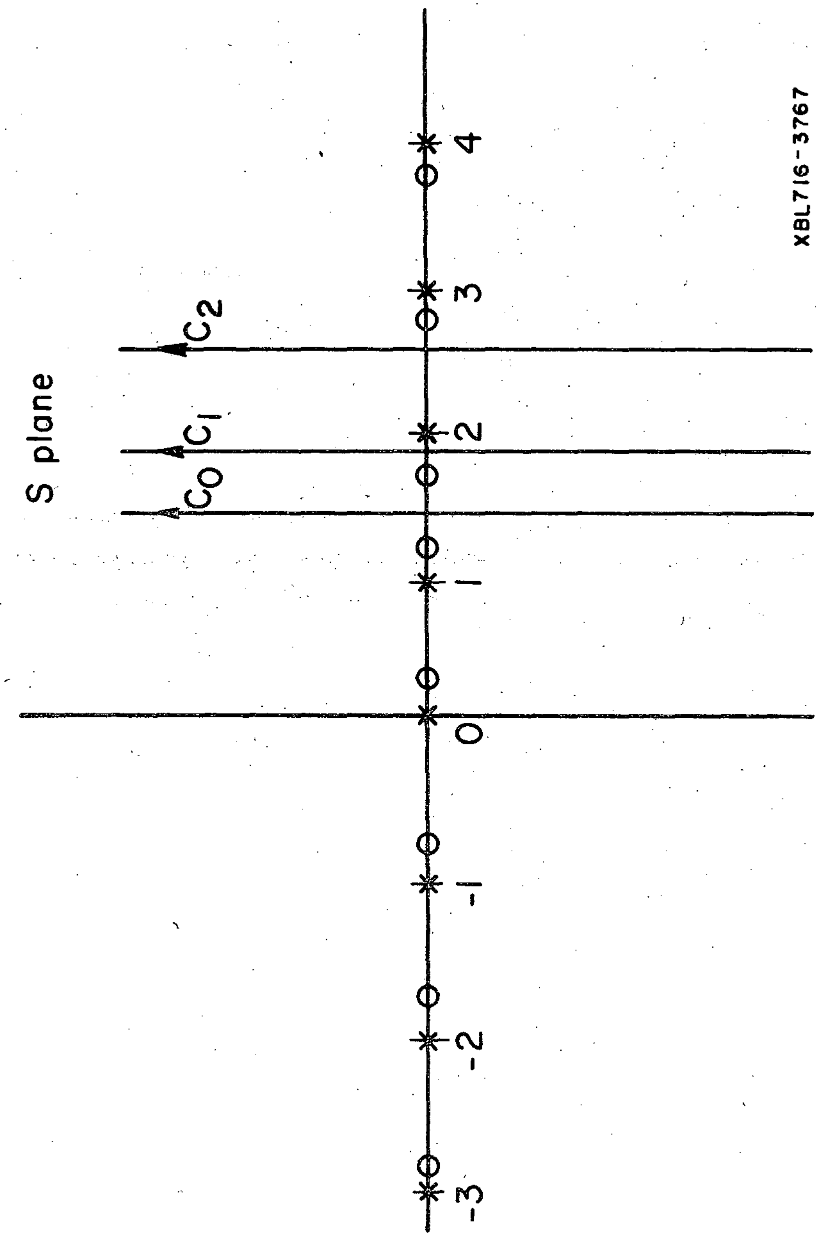

Fig. 2

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