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Large hypergraphs without tight cycles

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Abstract. An $r$-uniform tight cycle of length $\ell > r$ is a hypergraph with vertices $v_1, \ldots, v_\ell$ and edges $\{v_i, v_{i+1}, \ldots, v_{i+r-1}\}$ (for all $i$), with the indices taken modulo $\ell$. It was shown by Sudakov and Tomon that for each fixed $r \geq 3$, an $r$-uniform hypergraph on $n$ vertices which does not contain a tight cycle of any length has at most $n^{r-1+o(1)}$ hyperedges, but the best known construction (with the largest number of edges) only gives $\Omega(n^{r-1})$ edges. In this note we prove that, for each fixed $r \geq 3$, there are $r$-uniform hypergraphs with $\Omega(n^{r-1} \log n / \log \log n)$ edges which contain no tight cycles, showing that the $o(1)$ term in the exponent of the upper bound is necessary.

Mathematics Subject Classifications. 05C65, 05C38

1. Introduction

A well-known basic fact about graphs states that a graph on $n$ vertices containing no cycle of any length has at most $n - 1$ edges, with this upper bound being tight. To find generalisations of this result (and other results concerning cycles) for $r$-uniform hypergraphs with $r \geq 3$, we need a corresponding notion of cycles in hypergraphs. There are several types of hypergraph cycles for which Turán-type problems have been widely studied, including Berge cycles and loose cycles [1, 2, 3, 4, 6, 7]. In this note we will consider tight cycles, for which it appears to be rather difficult to obtain extremal results.

Given positive integers $r \geq 2$ and $\ell > r$, an $r$-uniform tight cycle of length $\ell$ is a hypergraph with vertices $v_1, \ldots, v_\ell$ and edges $\{v_i, v_{i+1}, \ldots, v_{i+r-1}\}$ for $i = 1, \ldots, \ell$, with the indices taken modulo $\ell$. Observe that for $r = 2$ a tight cycle of length $\ell$ is just a cycle of length $\ell$ in the usual sense. Let $f_r(n)$ denote the maximal number of edges that an $r$-uniform hypergraph on $n$ vertices can have if it has no subgraph isomorphic to a tight cycle of any length. So $f_2(n) = n - 1$. It is easy to see that the hypergraph obtained by taking all edges containing a certain point is tight-cycle-free, giving a lower bound $f_r(n) \geq \binom{n-1}{r-1}$. Sós and independently Verstraëte (see [10])

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raised the problem of estimating \( f_r(n) \), and asked whether the lower bound \( \binom{n-1}{r-1} \) is tight. This question was answered in the negative by Huang and Ma [5], who showed that for \( r \geq 3 \) there exists \( c_r > 0 \) such that if \( n \) is sufficiently large then \( f_r(n) \geq (1 + c_r) \binom{n-1}{r-1} \). Very recently, Sudakov and Tomon [9] showed that \( f_r(n) \leq n^{r-1+o(1)} \) for each fixed \( r \), and commented that it is widely believed that the correct order of magnitude is \( \Theta(n^{r-1}) \). The main result of this paper is the following theorem, which disproves this conjecture.

**Theorem 1.1.** For each fixed \( r \geq 3 \) we have \( f_r(n) = \Omega(n^{r-1} \log n / \log \log n) \). In particular, \( f_r(n)/n^{r-1} \to \infty \) as \( n \to \infty \).

The upper bound of Sudakov and Tomon [9] is \( n^{r-1}e^{c_r \sqrt{\log n}} \), although they remark that it might be possible to use their approach to get an upper bound of \( n^{r-1}(\log n)^{O(1)} \).

Concerning tight cycles of a given length, we mention the following interesting problem of Conlon (see [8]), which remains open.

**Question 1.2** (Conlon). Given \( r \geq 3 \), does there exist some \( c = c(r) \) constant such that whenever \( \ell > r \) and \( \ell \) is divisible by \( r \) then any \( r \)-uniform hypergraph on \( n \) vertices which does not contain a tight cycle of length \( \ell \) has at most \( O(n^{r-1+c/\ell}) \) edges?

Note that we need the assumption that \( \ell \) is divisible by \( r \), otherwise a complete \( r \)-uniform \( r \)-partite hypergraph has no tight cycle of length \( \ell \) and has \( \Theta(n^r) \) edges.

## 2. Proof of our result

The key observation for our construction is the following lemma.

**Lemma 2.1.** Assume that \( n, k, t \) are positive integers and \( G_1, \ldots, G_t \) are edge-disjoint subgraphs of \( K_{n,n} \) such that no \( G_i \) contains a cycle of length at most \( 2k \). Assume furthermore that \( kt \leq n \). Then there is a tight-cycle-free 3-partite 3-uniform hypergraph on at most \( 3n \) vertices having \( k \sum_{i=1}^t |E(G_i)| \) hyperedges.

**Proof.** Let the two vertex classes of \( K_{n,n} \) be \( X \) and \( Y \), and let \( Z = [t] \times [k] \). (As usual, \([m]\) denotes \( \{1, \ldots, m\} \).) Our 3-uniform hypergraph has vertex classes \( X, Y, Z \) and hyperedges \( \{\{x, y, z\} : x \in X, y \in Y, z \in Z, z = (i, s) \text{ for some } i \in [t] \text{ and } s \in [k] \}, \{\{x, y\} \in E(G_i)\} \).

In other words, for each \( G_i \) we add \( k \) new vertices (denoted \((i, s)\) for \( s = 1, \ldots, k \)), and we replace each edge of \( G_i \) by the \( k \) hyperedges obtained by adding one of the new vertices corresponding to \( G_i \) to the edge.

We need to show that our hypergraph contains no tight cycles. Since our hypergraph is 3-partite, it is easy to see that any tight cycle is of the form \( x_1y_1z_1x_2y_2z_2 \ldots \ell \ell y_\ell z_\ell \) (for some \( \ell \geq 2 \) positive integer) with \( x_i \in X, y_j \in Y, z_j \in Z \) for all \( j \). Assume that \( z_1 = (i, s_1) \). Then \( \{x_1, y_1\}, \{y_1, x_2\}, \{x_2, y_2\} \in E(G_i) \). But \( \{x_2, y_2\} \in E(G_i) \) implies that \( z_2 \) must be of the form \((i, s_2)\) for some \( s_2 \). Repeating this argument, we deduce that there are \( s_j \in [k] \) such that \( z_j = (i, s_j) \) for all \( j \), and \( x_jy_jy_{j+1} \in E(G_i) \) for all \( j \) (with the indices taken mod \( \ell \)). Hence \( x_1y_1x_2y_2 \ldots \ell y_\ell \) is a cycle in \( G_i \), giving \( \ell > k \). But the vertices \( z_j = (i, s_j) \) \((j = 1, \ldots, \ell)\) must all be distinct, and there are \( k \) possible values for the second coordinate, giving \( \ell \leq k \). We get a contradiction, giving the result. \( \square \)
We mention that Lemma 2.1 can be generalised to give \((r + r')\)-uniform tight-cycle-free hypergraphs if we have edge-disjoint \(r\)-uniform hypergraphs \(G_1, \ldots, G_t\) not containing tight cycles of length at most \(rk\) and edge-disjoint \(r'\)-uniform hypergraphs \(H_1, \ldots, H_t\) not containing tight cycles of length more than \(r'k\). Indeed, we can take all edges \(e \cup f\) with \(e \in E(G_i), f \in E(H_i)\) for some \(i\). (Then Lemma 2.1 may be viewed as the special case \(r = 2, r' = 1\).)

**Lemma 2.2.** There exists \(\alpha > 0\) such that whenever \(k \leq \alpha \log n/\log \log n\) then we can find edge-disjoint subgraphs \(G_1, \ldots, G_t\) of \(K_{n,n}\) with \(t = \lfloor n/k \rfloor\) such that no \(G_i\) contains a cycle of length at most \(2k\), and \(\sum_{i=1}^t |E(G_i)| = (1 - o(1))n^2\).

**Proof.** It is well-known (and can be proved by a standard probabilistic argument) that there are constants \(\beta, c > 0\) such that if \(n\) is sufficiently large and \(k \leq \beta \log n\) then there exists a subgraph \(H\) of \(K_{n,n}\) which has no cycle of length at most \(2k\) and has \(|E(H)| \geq n^{1+c/k}\). We randomly and independently pick copies \(H_1, \ldots, H_t\) of \(H\) in \(K_{n,n}\). Let \(G_1 = H_1\) and \(E(G_i) = E(H_i) \setminus \bigcup_{j=1}^{i-1} E(H_j)\) for \(i \geq 2\). Then certainly the \(G_i\) are edge-disjoint and no \(G_i\) contains a cycle of length at most \(2k\). Furthermore, the probability that a given edge is not contained in any \(H_i\) is

\[
(1 - |E(H)|/n^2)^t \leq \exp\left(-|E(H)|t/n^2\right) \\
\leq \exp\left(-n^{1+c/k}[n/k]/n^2\right) = \exp\left(-n^{c/k}/n(1+o(1))\right).
\]

This is \(o(1)\) as long as \(k \leq \alpha \log n/\log \log n\) for some constant \(\alpha > 0\). Therefore the expected value of \(\bigcup_{i=1}^t E(H_i)\) is \((1-o(1))n^2\). Since \(\sum_{i=1}^t |E(G_i)| = |\bigcup_{i=1}^t E(H_i)|\), the result follows. \(\square\)

**Proof of Theorem 1.1.** First consider the case \(r = 3\). Lemma 2.2 and Lemma 2.1 together show that if \(k \leq \alpha \log n/\log \log n\) then there is a tight-cycle-free 3-partite 3-uniform hypergraph on \(3n\) vertices with \((1 - o(1))kn^2\) edges. This shows \(f_3(n) = \Omega(n^2 \log n/\log \log n)\), as claimed.

For \(r \geq 4\), observe that \(f_r(2n) \geq f_{r-1}(n)n\). Indeed, if \(H\) is an \((r - 1)\)-uniform tight-cycle-free hypergraph on \(n\) vertices, then we can construct a tight-cycle-free \(r\)-uniform hypergraph \(H'\) on \(2n\) vertices with \(n|E(H)|\) edges as follows. The vertex set of \(H'\) is the disjoint union of \([n]\) and the vertex set \(V(H)\) of \(H\), and the edges are \(e \cup \{i\}\) with \(e \in E(H)\) and \(i \in [n]\). Then any tight cycle in \(H'\) must be of the form \(v_1 v_2 \ldots v_r \) with \(v_i \in V(H)\) if \(i\) is not a multiple of \(r\) and \(v_i \in [n]\) if \(i\) is a multiple of \(r\). But then we get a tight cycle \(v_1 v_2 \ldots v_{r-1} v_{r+1} v_{r+2} \ldots v_{2r-1} v_{2r+1} \ldots v_{tr-1}\) in \(H\) by removing each vertex from \([n]\) from this cycle. This is a contradiction, so \(H'\) contains no tight cycles. The result follows. \(\square\)

**References**


