Title

Free resolutions, linkage, and representation theory

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Publication Date

2024

Peer reviewed|Thesis/dissertation

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by

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A dissertation submitted in partial satisfaction of the

requirements for the degree of

Doctor of Philosophy

in

Mathematics

in the

Graduate Division

of the

University of California, Berkeley

Committee in charge:

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Spring 2024

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Abstract

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Spanning two papers from 1989 and 2018, Weyman unearthed a fascinating connection between commutative algebra and representation theory in his study of generic free resolutions of length three. This thesis is devoted to analyzing this connection further. In the first half, we show that certain Kazhdan-Lusztig varieties provide generic examples of ideals in the linkage class of a complete intersection. For those of embedding codimension three, we also compute the free resolutions of their coordinate rings. We later show that these specialize to resolutions of all grade three licci ideals.

In the second half, we develop the machinery of higher structure maps originating from Weyman's generic ring. Using the free resolutions constructed previously, we disprove Hochster's conjecture on finite generation of generic rings. The two perspectives converge in the final chapter of the thesis, in which we develop an ADE correspondence to completely classify grade three perfect ideals with small type and deviation.

Contents

Acknowledgments

I am deeply grateful to David Eisenbud for his guidance throughout my six years at UC Berkeley. He encouraged me to explore my interests, and he would give me a much needed push in a lot of situations where I was indecisive or lost. He provided me with many opportunities that greatly shaped the trajectory of my studies, including the initial Macaulay2 project that eventually led to all the work in this thesis.

None of this work would have been possible without Jerzy Weyman and Lorenzo Guerrieri. Watching this project evolve through our ongoing work has been very exciting, and I always look forward to our meetings. I also thank them for their warm hospitality during my trip to Poland in Summer 2022. This trip was part of a longer stay in Europe, where I was at MPI Leipzig in the Nonlinear Algebra group headed by Bernd Sturmfels, whom I also thank. It was my first time traveling to Europe, and it was a wonderful experience.

I thank my friends both near and far, especially Andrew Gitlin, Marvin Castellon, Tahsin Saffat, and John Zhang. I would also like to thank everyone involved in the Spring 2024 commutative algebra program at SLMath/MSRI. I had many fun and insightful conversations with people I met at the institute. I also greatly appreciated the opportunity to give lectures on this material, which in turn has helped me organize this thesis.

Lastly, I thank my family for their support throughout this long journey. Even though I have often been irresponsibly aloof, they have constantly been understanding, and I am always happy to see them again. Of course, this includes our cats, who haven't forgotten me despite my prolonged absences from home.

Chapter 1

Introduction

Let R be a commutative local Noetherian ring, with maximal ideal $\mathfrak{m}$ and residue field k. We say that an ideal $I \subset R$ is *perfect* if $c := \operatorname{grade}(I)$, the maximal length of an regular sequence contained in I, is equal to the projective dimension $\operatorname{pdim} R/I$. The primary case of interest is when R is regular: in this situation the notions of grade and codimension coincide, and I being perfect is equivalent to R/I being Cohen-Macaulay by the Auslander-Buchsbaum formula. For simplicity of exposition we assume this to be the case for the rest of this introduction.

There has been an extensive amount of work analyzing the structure of perfect ideals $I \subset R$. For $c = 1$ the problem is trivial, as the ideal of a hypersurface is generated by a single equation. The first major result in this area dates back to [24], where Hilbert proved a structure theorem for ideals I in a polynomial ring with $\operatorname{pdim} R/I = 2$. This was generalized by Burch in 1968 to arbitrary commutative rings in [11]. Using the Hilbert-Burch theorem, one concludes that if $c = 2$, the ideal I is generated by the $(n-1) \times (n-1)$ minors of a $n \times (n-1)$ matrix, where n is the minimal number of generators of I.

The situation becomes more mysterious for $c \geq 3$, and many authors throughout the late 20th century have addressed various cases which we now survey. It is helpful to introduce two numerical quantities to assist in organizing the story:

- The *deviation* d of I is the quantity $n - c$. Note that $n \geq c$ always, with equality if I is a complete intersection. Hence the deviation is a measurement of how far an ideal is from being a complete intersection.

- The *type* t of R/I is the minimal number of generators of the canonical module $\operatorname{Ext}^c_R(R/I, R)$ of R/I. It is equal to the last Betti number b_c for a minimal free resolution of R/I. By abuse of terminology we will also refer to this as the type of I.

The ring R/I is Gorenstein exactly when $t = 1$, and we also refer to the ideal I itself as being Gorenstein in this case. In [7], Buchsbaum and Eisenbud characterized Gorenstein ideals of codimension $c = 3$. Explicitly, I is generated by the $(n-1) \times (n-1)$ pfaffians of a $n \times n$ skew matrix, where n is odd.

Given the ubiquity of Gorenstein ideals and their elegant characterization in codimension 3, it was a natural goal to try and develop an analogous structure theorem for Gorenstein ideals of codimension 4. The only known examples of such ideals of deviation 2 were hypersurface sections, and for a brief period, some believed that *all* codimension 4 Gorenstein ideals might just be hypersurface sections—i.e. that there was no more to their structure theory beyond what had already been discovered in codimension 3. This was gradually revealed to be drastically false. For instance, Kustin and Miller exhibited Gorenstein ideals of codimension 4 with any odd $d \geq 3$ in [33], contrasting with the codimension 3 case.

On the other hand, the observed behavior for $d = 2$ was later proven under mild hypotheses: see [23] and [47]. As Kunz showed Gorenstein ideals of arbitrary codimension cannot have $d = 1$ in [32], a natural question to ask was whether all such ideals with $d = 2$ are hypersurface sections, as observed for $c = 4$. But this is not the case: Huneke and Ulrich produced an interesting family of Gorenstein ideals of odd codimension $c \geq 5$ with $d = 2$ which are not hypersurface sections in [29]. Their construction also makes sense for $c = 3$, but it coincides with the Buchsbaum-Eisenbud example in that case.

The list of results given above is by no means exhaustive, but it is enough to illustrate the difficulty of the problem—in particular, there do not seem to be any obvious patterns as we vary the parameters (c, d, t), which we henceforth refer to as the triple of parameters associated to the perfect ideal I.

A recurring theme throughout many of these classical papers has been the technique of *liaison*, or linkage. In particular, in all situations where we have an a conclusive "structure theorem" for grade c perfect ideals with deviation d and type t, all such ideals are actually in the linkage class of a complete intersection (*licci*). For example, all perfect ideals associated to the triples $(2, d, t)$ and $(3, d, 1)$ are licci: see [41] and [48] respectively.

In these two examples, the licci property for each family of perfect ideals was deduced independently from the corresponding structure theorem. In retrospect, the connection may be explained as follows: Buchweitz showed that the deformation theory of reduced licci algebras is strongly unobstructed in his thesis [10], and Herzog showed in [22] how to consequently obtain explicit structure theorems. Specifically, one can impose an equivalence relation on reduced licci algebras so that each equivalence class (referred to in the literature as a *Herzog class*) has a generic example specializing to all members of the family. The generic example can then be interpreted as the "structure theorem" for that family.

It is moreover possible to translate these structure theorems between directly linked families. For instance, in the same paper where Buchsbaum and Eisenbud proved their structure theorem for codimension 3 Gorenstein ideals, they were able to use linkage to deduce a structure theorem for codimension 3 almost complete intersections as well. Continuing this idea, Brown and Sanchez found structure theorems for other families of grade 3 perfect ideals that were directly linked to almost complete intersections; see [4] and [45].

However, there are two major caveats to this approach. The first is that we can only hope to obtain structure theorems for licci ideals in this manner, whereas for $c \geq 3$ there exist[1] non-licci

[1] One area of active research aims to remedy this by replacing the notion of liaison with the more general notion of G-liaison, where links are done more generally with Gorenstein ideals rather than only complete intersections. In this direction, the main open question is whether all perfect ideals are in the Gorenstein linkage class of a complete intersection (*glicci*). While this has been proven for many families of perfect ideals, the general statement remains open even in codimension 3. Unfortunately, Gorenstein linkage does not preserve the deformation-theoretic properties exemplified by licci ideals, so an affirmative answer to this conjecture would not readily translate to structure theorems for perfect ideals. In this thesis we will only consider ordinary linkage.

perfect ideals. There are also examples of non-licci Gorenstein ideals for $c \geq 4$. This leads us to our first main question:

Question 1.1. For what triples (c, d, t) are all associated perfect ideals licci?

Even if we restrict to licci ideals, there is another caveat: there are generally multiple Herzog classes of licci ideals with the same parameters (c, d, t). The first example of this occurs when $(c, d, t) = (3, 2, 2)$: in [4] it was observed that there are at least two distinct Herzog classes of such licci ideals, and only one of these was characterized by the structure theorem given in that paper. This leads us to our second main question:

Question 1.2. Can we describe all Herzog classes of grade c licci ideals with deviation d and type t?

In view of the seemingly disparate examples given previously, it is perhaps doubtful whether such a question is even tractable. At least in theory, one can iteratively use linkage to produce new classes from old, until one no longer finds any new classes with the given parameters. For instance, Lopez used ideas along these lines in [37] to show that any licci ideal with $(c, d, t) = (5, 2, 1)$ is either a double hypersurface section of a grade 3 Gorenstein ideal on 5 generators, or a member of the family studied by Huneke and Ulrich in [29].

However, it is possible that this procedure does not terminate. And even when it does terminate, there are often far too many Herzog classes to work out by hand. To give a sense of scale, while there were only 2 for licci ideals with $(c, d, t) = (5, 2, 1)$, we will later see that there are 90 for $(3, 2, 4)$ and infinitely many for $(3, 3, 3)$.

In fact, there are conjectural answers to both questions above, which we will prove for $c = 3$ assuming that R has equicharacteristic zero. Implicit here is the claim that there actually *is* a pattern to all the examples given previously. It is one which comes from a rather unexpected source:

Definition 1.3. Let $c \geq 2$, $d \geq 0$, and $t \geq 1$ be integers. Let T be the T-shaped graph with arms of length $c - 2$, d, and t. We say that (c, d, t) is an *ADE triple* if T is a Dynkin diagram. This is equivalent to the following inequality:

$$\frac{1}{c-1} + \frac{1}{d+1} + \frac{1}{t+1} > 1.$$

With this definition, we can state the conjectural answer to Question 1.1.

Conjecture 1.4. *Let $c \geq 2$, $d \geq 0$, and $t \geq 1$ be integers. All perfect ideals associated to (c, d, t) are licci if and only if (c, d, t) is an ADE triple.*

The conjecture which addresses Question 1.2 requires more machinery to state precisely, and we defer it to Chapter 2 until after we have introduced the necessary background on representation theory and Schubert varieties.

While Conjecture 1.4 neatly unifies a whole assortment of classical results, we have given no hints as to why such a connection to representation theory should be present in the first place. One of the primary goals of this thesis is to demystify this connection. For now, we present just one example to give an informal sketch: the case of grade 3 Gorenstein ideals.

The graph T associated to $(c,d,t) = (3, n-3, 1)$ (in the sense of Definition 1.3) is the Dynkin diagram D_n. We analyze an excerpt of the proof in [7] to illustrate how the representation theory of the corresponding simple Lie algebra $\mathfrak{so}(2n)$ is subtly involved.

Let $I \subset R$ be a grade 3 Gorenstein ideal in a local Noetherian $\mathbb{C}$-algebra $(R, \mathfrak{m}, k)$. The minimal free resolution of R/I has the form

$$\mathbb{F}\colon 0 \to F_3 \to F_2 \to F_1 \to R.$$

One of the key ingredients in the proof is the existence of a graded-commutative differential graded algebra on $\mathbb{F}$ lifting the algebra structure on $\mathrm{Tor}_*(R/I, k)$. Fix such a structure; the assumption that $I \subset R$ is Gorenstein implies that the multiplication $F_1 \otimes F_2 \to F_3 \cong R$ induces an isomorphism $F_1 \cong F_2^*$. Using this isomorphism, we may rewrite our resolution in the form

$$0 \to R \xrightarrow{d_1^*} F_1^* \xrightarrow{d_2} F_1 \xrightarrow{d_1} R$$

so that the multiplication $F_1 \otimes F_1^* \to R$ is the evident pairing.

Let $e_1, \ldots, e_n$ be a basis of F_1 and $e_1', \ldots, e_n' \in F_1^*$ its dual basis. The graded Leibniz rule implies that

$$\begin{aligned} d_1^*((d_2 e_i') \cdot e_j' + e_i' \cdot (d_2 e_j')) &= (d_1 d_2 e_i') \cdot e_j' - (d_2 e_i') \cdot (d_2 e_j') + (d_2 e_i') \cdot (d_2 e_j') + e_i' \cdot (d_1 d_2 e_j') \\ &= 0. \end{aligned}$$

Since d_1^* is a monomorphism, we conclude $(d_2 e_i') \cdot e_j' + e_i' \cdot (d_2 e_j') = 0$, which equivalently says that $d_2 = d_2^*$. The conclusion is that every grade 3 Gorenstein ideal admits a minimal free resolution where the middle differential is represented by a skew matrix.

The above presentation is faithful to the original paper of Buchsbaum and Eisenbud, and we refer the reader there for more details. We propose a different way of interpreting the same argument. Using our choice of dg-algebra structure on the original resolution $\mathbb{F}$, assemble the $n \times 2n$ block matrix

$$w^{(2)} = \begin{array}{c c c} & F_1^* & F_1 \\ F_2^* & [\text{original } d_2^* & \text{multiplication}] \end{array}$$

In particular, the right block is invertible. As the whole matrix is surjective, it determines an R-point of the Grassmannian $\mathrm{Gr}(n, 2n)$. But we can say more: the Leibniz rule calculation shows that after performing row operations on this block matrix, which has no effect on the R-point of $\mathrm{Gr}(n, 2n)$ it describes, we obtain

$$\begin{array}{c c c} & F_1^* & F_1 \\ F_2^* & [\text{new skew } d_2^* & I_n] \end{array}.$$

Viewing $F_1 \oplus F_1^*$ as an orthogonal space with the evident quadratic form given by the pairing of the two factors, the above describes an n-plane which is isotropic for the quadratic form. In other words, $w^{(2)}$ determines an R-point of the *orthogonal Grassmannian* $\mathrm{OG}(n, 2n) \subset \mathrm{Gr}(n, 2n)$.

The orthogonal Grassmannian is a homogeneous space $\mathrm{SO}(2n)/P$ where $P \subset \mathrm{SO}(2n)$ is a certain maximal parabolic subgroup. Inside of $\mathrm{OG}(n, 2n)$, there is a Schubert cell C parametrized as

$$[\text{generic skew matrix } Y \quad I_n]$$

so $w^{(2)}$ determines a map $f\colon \operatorname{Spec} R \to C$. Moreover, there is a codimension 3 (opposite) Schubert variety $X \subset \mathrm{OG}(n, 2n)$ with the property that the ideal of $X \cap C \subset C$ is generated by the $(n-1)\times(n-1)$ pfaffians of the generic skew matrix Y. Hence the structure theorem of Buchsbaum and Eisenbud equivalently states that $f^{-1}X = \operatorname{Spec} R/I$. Further discussion relating grade 3 Gorenstein ideals to the orthogonal Grassmannian can be found in [15]. We will also revisit the various components of this perspective in the subsequent chapters.

The main goal of this thesis is to report on ongoing joint work with Lorenzo Guerrieri and Jerzy Weyman, giving answers to Questions 1.1 and 1.2 for $c = 3$ under the assumption that R has equicharacteristic zero[2]. We also hope that this thesis will provide an accessible introduction to the various aspects of this whole theory, and that it makes a compelling case that representation theory will be crucial in furthering our understanding of perfect ideals.

This project has evolved greatly over the years, and new developments have often recontextualized earlier work in more illuminating ways. Historically, the connection to representation theory originated in Weyman's study of generic free resolutions of length 3; c.f. [51] and [50]. The analysis of these generic free resolutions leads to a notion of *higher structure maps*. A thorough study of these higher structure maps then yields our desired statements regarding licci ideals. However, from an expository standpoint, this approach does not adequately motivate the representation theory—although it appears for good reason, the true extent of the connection is obscured by the heavy machinery surrounding it. Its importance is only made clear in retrospect, after everything has been proven.

To better motivate the story, we will instead take an approach where the connection to representation theory is transparent from the beginning, and the study of generic free resolutions is introduced at a later point to augment our understanding of the situation. Chapter 2 is devoted to the study of linked Schubert varieties. We exploit the symmetry of these Schubert varieties to prove that they give the generic examples of licci ideals. Using this, we state the conjectural answer to Question 1.2, and prove a portion of it. In particular, we show that if (c, d, t) is an ADE triple, then there are finitely many Herzog classes of associated licci ideals.

Afterwards, we specialize to $c = 3$. In Chapter 3, we generalize the well-known Buchsbaum-Eisenbud resolution of $(n-1)\times(n-1)$ pfaffians of a generic $n \times n$ skew matrix (n odd) by exhibiting it as one member of a family of resolutions constructed using Lie algebras. Some of these resolve the coordinate rings of certain Kazhdan-Lusztig varieties studied in Chapter 2, and we will later show that these give the generic resolutions of *all* grade 3 licci ideals.

However, the techniques of Chapter 2 require knowing *a priori* that $I \subset R$ is licci. In particular, they cannot be used to prove Conjecture 1.4. To analyze perfect ideals more generally, we pivot to generic free resolutions in Chapter 4, eventually leading to the definition of the previously mentioned higher structure maps. After thoroughly developing the necessary machinery for working with these maps, we show how this theory eventually converges with the material of earlier chapters, and we use it to disprove a conjecture of Hochster regarding finite generation of generic rings.

Finally, we apply this theory to establish our main results regarding linkage of perfect ideals in Chapter 5. We show that the transformation of higher structure maps under linkage is particularly elegant, and draw parallels to the geometric treatment of Chapter 2. We deduce various invariants

[2] Actually, we will simply work over $\mathbb{C}$ throughout, as this is the standard setting for much of the representation theory, but all the results we need are valid over $\mathbb{Q}$. We will comment on this throughout the text when needed.

of linkage for grade 3 perfect ideals, showing for instance that higher structure maps can be used to compute the non-licci locus. Then we use the machinery of Chapter 4 in full force to prove that this locus is empty for grade 3 perfect ideals associated to ADE triples, thereby proving Conjecture 1.4 for $c = 3$. We then conclude by sketching various consequences of this theory and possible directions for future study.

Chapter 2

Licci ideals and Schubert varieties

In this chapter, we introduce the necessary background on licci ideals and Schubert varieties. Then we analyze a very specific linked pair of Schubert varieties and prove that they yield generic examples of certain licci ideals. In the codimension 3 case, these Schubert varieties were studied by Sam and Weyman in [44], and a particular one was carefully examined in [12] and [34]. The original motivation for looking at these Schubert varieties comes from the study of generic free resolutions, but the treatment in this chapter will be independent of that theory.

Assumption 2.1. All rings considered in this thesis are assumed to be commutative $\mathbb{C}$-algebras. There is no particular reason to use $\mathbb{C}$ instead of any other algebraically closed field of characteristic zero, other than that it is standard for representation theory. Working over an algebraically closed field is convenient for geometric statements, but our results remain valid over $\mathbb{Q}$. All fields are assumed to have characteristic zero. We refer back to this with commentary as needed.

2.1 Linkage

To have an adequate theory of linkage, some assumptions are necessary either on the ambient ring R or the ideals under consideration. Taking the former approach, one typically assumes that R is local and Cohen-Macaulay. In this setting, one says that $I, J \subseteq R$ are *(directly) linked* if there exists a regular sequence $\alpha_1, \ldots, \alpha_c \in R$ such that

$$(\alpha_1, \ldots, \alpha_c) : I = J \text{ and } (\alpha_1, \ldots, \alpha_c) : J = I.$$

This guarantees that both ideals I, J are unmixed of height c, and also that $\alpha_1, \ldots, \alpha_c \in I \cap J$. However, many properties are not preserved under linkage at this level of generality: for instance, it is possible that R/I is Cohen-Macaulay whereas R/J is not. As shown in [41], this is remedied if one assumes R is moreover Gorenstein.

Rather than restrict the ring R, we will instead restrict the ideals under consideration, and exclusively consider the linkage of perfect ideals.

Definition 2.2. Let R be a Noetherian ring. An ideal $I \subset R$ is *perfect* if $\operatorname{pdim} R/I = \operatorname{grade} I$. Here $\operatorname{grade} I \coloneqq \min\{c : \operatorname{Ext}^c(R/I, R) \neq 0\} = \operatorname{depth}(I, R)$ is the maximal length of a regular sequence contained in I. An ideal $I \subset R$ is a *complete intersection* if it is generated by a regular sequence.

If I is a complete intersection, then R/I is resolved by a Koszul complex, so in particular I is perfect. The primary case of interest is when R is a regular local ring. In this situation, $\operatorname{pdim} R/I$ is equal to the codepth of R/I by the Auslander-Buchsbaum formula. On the other hand, $\operatorname{grade} I = \operatorname{codim} I$ since R is Cohen-Macaulay. Hence I is perfect if and only if $\operatorname{depth} R/I = \dim R/I$, i.e. R/I is Cohen-Macaulay.

In [17], Golod weakened some of the hypotheses in [41] and showed that linkage of perfect ideals is well-behaved even if the ambient ring R is only assumed to be Noetherian.

Definition 2.3. Let R be a Noetherian ring, $I \subset R$ a perfect ideal of grade c, and $\alpha_1, \ldots, \alpha_c \in I$ a regular sequence on R. In this situation we say that the ideal $J = (\alpha_1, \ldots, \alpha_c) : I$ is *(directly) linked* to I by the sequence $\alpha_1, \ldots, \alpha_c$, or by the complete intersection ideal $(\underline{\alpha}) = (\alpha_1, \ldots, \alpha_c)$. We write $I \sim J$, or $I \overset{(\underline{\alpha})}{\sim} J$ if we wish to emphasize the regular sequence.

We do not exclude the possibility that $J = (1)$, which necessarily means $I = (\underline{\alpha})$. As a matter of convention, we declare that $(\underline{\alpha}) \overset{(\underline{\alpha})}{\sim} (1)$ and $(1) \overset{(\underline{\alpha})}{\sim} (\underline{\alpha})$, even though (1) is not a grade c perfect ideal.

For perfect ideals it is not necessary to assume the symmetric condition that $I = (\alpha_1, \ldots, \alpha_c) : J$ in Definition 2.3, as it can be deduced as a consequence:

Theorem 2.4 ([41],[17])**.** *Suppose $I \subset R$ is a grade c perfect ideal and $I \sim J$. If $J \neq (1)$, then J is also a grade c perfect ideal and $J \sim I$.*

Furthermore, there is a systematic method for constructing a resolution for R/J starting from one for R/I. Specifically, let

$$\mathbb{F} \colon 0 \to P_c \to P_{c-1} \to \cdots \to R$$

be a minimal length projective resolution of R/I, and let $\mathbb{K}$ be the Koszul complex on the map $K \coloneqq R^c \xrightarrow{[\alpha_1, \ldots, \alpha_c]} R$ where $\alpha_1, \ldots, \alpha_c \in I$ is a regular sequence. Let $\psi \colon \mathbb{K} \to \mathbb{F}$ be any map of complexes covering the quotient $R/K \to R/I$:

$$\begin{array}{ccccccccccc} 0 & \longrightarrow & P_c & \longrightarrow & P_{c-1} & \longrightarrow & \cdots & \longrightarrow & P_1 & \longrightarrow & R \\ & & \uparrow{\scriptstyle\psi_c} & & \uparrow{\scriptstyle\psi_{c-1}} & & & & \uparrow{\scriptstyle\psi_1} & & \| \\ 0 & \longrightarrow & \bigwedge^c K & \longrightarrow & \bigwedge^{c-1} K & \longrightarrow & \cdots & \longrightarrow & K & \longrightarrow & R \end{array} \tag{2.1}$$

Finally, we take the dual of the mapping cone of ψ and twist by $\bigwedge^c K$ to obtain a complex

$$R \leftarrow (P_c^* \otimes \bigwedge^c K) \oplus K \leftarrow \cdots \leftarrow (P_2^* \otimes \bigwedge^c K) \oplus \bigwedge^{c-1} K \leftarrow P_1^* \otimes \bigwedge^c K \leftarrow 0.$$

(Note that we have cancelled the split part $R \to R$ at the right end.) The next result originally appeared in [41], where it is attributed to Ferrand. It was then extended in scope by Golod in [17], which is the generality we use here.

Theorem 2.5 ([41],[17])**.** *If I is a grade c perfect ideal in a Noetherian ring R, and $\alpha_1, \ldots, \alpha_c \in I$ is a regular sequence, then the complex constructed above is a resolution of $R/((\alpha_1, \ldots, \alpha_c) : I)$.*

Let $\varphi\colon R \to S$ be a homomorphism of Noetherian rings. If $I \subset R$ is a grade c perfect ideal, then $IS (:= \varphi(I)S)$ has grade at most c unless it is the unit ideal; see for instance [25]. If $\varphi(\alpha_1), \dots, \varphi(\alpha_c)$ is a regular sequence in S, then grade $IS = c$, and $\mathbb{F} \otimes S$ is a resolution of S/IS. Tensoring (2.1) with S yields the following corollary.

Corollary 2.6. *Let $\varphi\colon R \to S$ be a homomorphism of Noetherian rings, and let $I \overset{(\underline{\alpha})}{\sim} J$ be linked grade c perfect ideals in R, where $\underline{\alpha} = (\alpha_1, \dots, \alpha_c)$. If $\varphi(\alpha_1), \dots, \varphi(\alpha_c)$ is a regular sequence in S, then $IS \overset{(\underline{\alpha})S}{\sim} JS$.*

Although linkage is not transitive, we can consider the equivalence relation it generates.

Definition 2.7. Let R be a Noetherian ring, and let $I \subset R$ be a grade c perfect ideal. If there exists a sequence of links

$$I = I_0 \sim I_1 \sim \cdots \sim I_N = J,$$

where each I_j is either a grade c perfect ideal or the unit ideal, then we say that I and J are in the same *linkage class*. If N is even (resp. odd), we say J is in the *even linkage class* (resp. *odd linkage class*) of I.

The ideal I is *licci* if it is in the linkage class of a complete intersection.

Note that with our conventions, a perfect ideal $I \subset R$ is licci if and only if there exist a sequence of links $I \sim \cdots \sim (1)$.

Now we specialize to the setting where R is local and reintroduce some quantities from Chapter 1.

Definition 2.8. Let I be a grade c perfect ideal in a local Noetherian ring $(R, \mathfrak{m}, k)$, and let

$$0 \to F_c \to \cdots \to F_1 \to F_0$$

be a minimal free resolution of R/I. Let $b_i = \dim_k \operatorname{Tor}_i(R/I, k)$ denote the (ordinary) Betti numbers of R/I, so e.g $b_0 = 1$ and b_1 is the minimum number of generators of I.

- The *deviation* $d(I)$ of I is $b_1 - c$.
- The *type* $t(R/I)$ of R/I is b_c. By abuse of terminology, we will also refer to this as being the type of I.

Definition 2.9. Let $I \subset R$ be a grade c perfect ideal in a local Noetherian ring. If the regular sequence $\underline{\alpha}$ is part of a minimal generating set of I, then we say that $I \overset{(\underline{\alpha})}{\rightsquigarrow} (\underline{\alpha}) : I$ is a *minimal link*. Note that this condition is not symmetric, hence the usage of $\rightsquigarrow$ rather than $\sim$.

If $I \rightsquigarrow J$ is a minimal link, then the type of J is exactly the deviation of I, and the deviation of J is at most the type of I. More generally:

Lemma 2.10. *Let $I \sim J$ be linked grade c perfect ideals in a local Noetherian ring $(R, \mathfrak{m}, k)$, and let $\overline{K}$ be the subspace of $\operatorname{Tor}_1(R/I, k) = I/\mathfrak{m}I$ spanned by the images of $\alpha_1, \dots, \alpha_c$. Let $m\colon \bigwedge^{c-1} \overline{K} \subseteq \bigwedge^{c-1} \operatorname{Tor}_1(R/I, k) \to \operatorname{Tor}_{c-1}(R/I, k)$ denote the multiplication map in the Tor algebra. Then:*

1. $t(R/J) = d(I) + c - \dim \overline{K}$,

2. $d(J) = t(R/I) - \operatorname{rank} m$.

Proof. This is immediate from (2.1) and Theorem 2.5: take $\mathbb{F}$ to be a minimal free resolution of R/I, and use that $\operatorname{rank}(\psi_{c-1} \otimes k) = m$ and $\operatorname{rank}(\psi_1 \otimes k) = \dim \overline{K}$. □

In some situations we will want to perform links with control over the subspace $\overline{K}$, for which the next result is helpful.

Lemma 2.11. *Let I be a grade c ideal in a local Noetherian ring $(R, \mathfrak{m}, k)$ (over an infinite field by Assumption 2.1). Let $\overline{K} \subseteq I/\mathfrak{m}I$ be any subspace with $\dim \overline{K} \leq c$. Then there exists $K \subseteq I$ generated by a regular sequence such that the image of K in $I/\mathfrak{m}I$ is $\overline{K}$.*

Proof. Over an infinite field, it is well-known that if $\operatorname{grade}(h_1, \ldots, h_n) = c$ then c general linear combinations of $h_1, \ldots, h_n$ form a regular sequence. Pick $\alpha_1, \ldots, \alpha_c \in I$ so that their images generate the desired $\overline{K}$ in $I/\mathfrak{m}I$. We can simply apply the preceding fact to the ideal

$$I' = (\alpha_1, \ldots, \alpha_c) + \mathfrak{m}I$$

which also has grade c, e.g. because $I^2 \subseteq I' \subseteq I$. A general linear combination of generators for I' will be a regular sequence in I and have the same image $\overline{K}$. □

Proposition 2.12 ([28, Corollary 2.5])**.** *Let I be a licci ideal in a local ring R (with infinite residue field by Assumption 2.1). Then there exists a sequence of links*

$$I = I_0 \sim I_1 \sim \cdots \sim I_N \tag{2.2}$$

where all links $I_i \rightsquigarrow I_{i+1}$ are minimal, and I_N is a complete intersection.

The proof moreover shows that, to obtain a sequence of links such as (2.2), it suffices to use a regular sequence consisting of general linear combinations of minimal generators at every step.

Example 2.13. Let $R = \mathbb{C}[\![w, x, y, z]\!]$ and consider the perfect ideal given by

$$I = (xz - y^2, xy - wz, wy - x^2).$$

This ideal has grade $c = 2$, deviation $d = 1$, and type $t = 2$. It is directly linked to the complete intersection (w, x), and an additional link takes it to the unit ideal:

$$\begin{aligned} I &\sim (w, x) && \text{by } (xy - wz, wy - x^2) \\ &\sim (1) && \text{by } (w, x). \end{aligned}$$

The ideal I is generated by the 2×2 minors of a 2×3 matrix. A verbose restatement is as follows. Let $\operatorname{Gr}(2, 5)$ be the Grassmannian of 2-planes in $\mathbb{C}^5 = \operatorname{span}(e_1, \ldots, e_5)$. Points of $\operatorname{Gr}(2, 5)$ may be represented as 2×5 matrices, up to row operations. There is a Schubert cell C consisting of all 2-planes transverse to $\operatorname{span}(e_1, e_2, e_3)$, parametrized as

$$C = \left\{ \begin{bmatrix} a_{11} & a_{12} & a_{13} & 1 & 0 \\ a_{21} & a_{22} & a_{23} & 0 & 1 \end{bmatrix} \right\}$$

and a codimension 2 opposite Schubert variety X consisting of all 2-planes *not* transverse with $\operatorname{span}(e_4, e_5)$. On the cell C, the variety $X \cap C$ is defined by the 2×2 minors of (a_{ij}). This is an example of a matrix Schubert variety, which is in turn an example of a *Kazhdan-Lusztig variety*. If we consider

$$\begin{bmatrix} w & x & y & 1 & 0 \\ x & y & z & 0 & 1 \end{bmatrix}$$

as defining a morphism $f\colon \operatorname{Spec} R \to C$, then $f^{-1}X = \operatorname{Spec} R/I$. We will generalize this to many other licci ideals shortly, but the appropriate Kazhdan-Lusztig varieties reside in homogeneous spaces other than the Grassmannian, which we now introduce.

2.2 Representation theory

We summarize the necessary results on representation theory and Schubert varieties, working in the general setting of Kac-Moody Lie algebras where possible, since this will be needed for Chapters 3 through 5. At this level of generality, most of the material in this section can be found in [31]. Other general facts about Lie algebras, e.g. Baker-Campbell-Hausdorff, can be found in [26].

2.2.1 Lie algebras and representations

Construction

Fix integers $p, q, r \geq 1$, and let $T = T_{p,q,r}$ denote the graph

$$\begin{array}{ccccccccc} x_{p-1} & — & \cdots & — & x_1 & — & u & — & y_1 & — & \cdots & — & y_{q-1} \\ & & & & & & | & & & & & & \\ & & & & & & z_1 & & & & & & \\ & & & & & & | & & & & & & \\ & & & & & & \vdots & & & & & & \\ & & & & & & | & & & & & & \\ & & & & & & z_{r-1} & & & & & & \end{array}$$

Let $n = p + q + r - 2$ be the number of vertices. From the above graph, we construct an $n \times n$ matrix A, called the *Cartan matrix*, whose rows and columns are indexed by the nodes of T:

$$A = (a_{i,j})_{i,j \in T}, \quad a_{i,j} = \begin{cases} 2 & \text{if } i = j, \\ -1 & \text{if } i, j \in T \text{ are adjacent,} \\ 0 & \text{otherwise.} \end{cases}$$

T is a Dynkin diagram if and only if $1/p + 1/q + 1/r > 1$; in this case we say it is of *finite type*. (For the applications to §2.3, we will only consider this case.) We next describe how to construct the associated Lie algebra $\mathfrak{g}$.

Let $\mathfrak{h} = \mathbb{C}^{2n - \operatorname{rank} A}$, and pick independent sets $\Pi = \{\alpha_i\}_{i \in T} \subset \mathfrak{h}^*$ and $\Pi^\vee = \{\alpha_i^\vee\}_{i \in T} \subset \mathfrak{h}$ satisfying the condition

$$\langle \alpha_i^\vee, \alpha_j \rangle = a_{i,j}.$$

The α_i are the *simple roots* and the $\alpha_i^\vee$ are the *simple coroots*. If $1/p + 1/q + 1/r = 1$, then $T = E_{n-1}^{(1)}$ is of *affine type* and rank $A = n - 1$. Otherwise rank $A = n$, and $\Pi, \Pi^\vee$ are bases of $\mathfrak{h}^*, \mathfrak{h}$ respectively.

The Lie algebra $\mathfrak{g} \coloneqq \mathfrak{g}(T)$ is generated by $\mathfrak{h}$ together with elements e_i, f_i for $i \in T$, subject to the defining relations

$$[e_i, f_j] = \delta_{i,j}\alpha_i^\vee,$$
$$[h, e_i] = \langle h, \alpha_i\rangle e_i, [h, f_i] = -\langle h, \alpha_i\rangle f_i \text{ for } h \in \mathfrak{h},$$
$$[h, h'] = 0 \text{ for } h, h' \in \mathfrak{h},$$
$$\operatorname{ad}(e_i)^{1-a_{i,j}}(e_j) = \operatorname{ad}(f_i)^{1-a_{i,j}}(f_j) \text{ for } i \neq j.$$

Under the adjoint action of $\mathfrak{h}$, the Lie algebra $\mathfrak{g}$ decomposes into eigenspaces as $\mathfrak{g} = \bigoplus \mathfrak{g}_\alpha$, where

$$\mathfrak{g}_\alpha = \{x \in \mathfrak{g} : [h, x] = \alpha(h)x \text{ for all } h \in \mathfrak{h}\}.$$

This is the *root space decomposition* of $\mathfrak{g}$.

Assumption 2.14. For simplicity, we henceforth assume that T is *not* of affine type. Thus the Cartan matrix is invertible, and $\mathfrak{g}$ is generated by e_i and f_i for $i \in T$ since $\alpha_i^\vee$ is a basis of $\mathfrak{h}$. This is just for convenience of exposition; the theory we discuss remains valid in the affine case with minor adjustments.

Gradings on $\mathfrak{g}$

Let $Q \subset \mathfrak{h}^*$ be the root lattice $\bigoplus_{i \in T} \mathbb{Z}\alpha_i$. If $\mathfrak{g}_\alpha \neq 0$, then necessarily $\alpha \in Q$. If such an α is nonzero, we say it is a *root*, and denote the set of all roots by Δ. Hence the Lie algebra $\mathfrak{g}$ is Q-graded. By singling out a vertex $t \in T$, this Q-grading can be coarsened to a $\mathbb{Z}$-grading by considering only the coefficient of α_t. We refer to this as the t-grading. The sum of all t-gradings for $t \in T$ is called the *principal gradation* on $\mathfrak{g}$. The degree zero part in the principal gradation is the Cartan subalgebra $\mathfrak{h}$. For $\alpha \in \Delta \cup \{0\}$, we write:

- $\alpha >_t 0$ (resp. $\alpha \geq_t 0$) if the coefficient of α_t in α is positive (resp. nonnegative),
- $\alpha > 0$ (resp. $\alpha \geq 0$) if the coefficient of α_t in α is positive (resp. nonnegative) for some $t \in T$,

and similarly for $\alpha <_t 0, \alpha \leq_t 0, \alpha < 0, \alpha \leq 0$. We have $\Delta = \Delta^+ \sqcup \Delta^-$ where $\Delta^+ = \{\alpha \in \Delta : \alpha > 0\}$ and $\Delta^- = \{\alpha \in \Delta : \alpha < 0\}$ are the sets of *positive* and *negative* roots respectively.

Using these notions, we define a few important subalgebras of $\mathfrak{g}$:

$$\mathfrak{n}^+ = \bigoplus_{\alpha > 0} \mathfrak{g}_\alpha \qquad \mathfrak{n}^- = \bigoplus_{\alpha < 0} \mathfrak{g}_\alpha$$
$$\mathfrak{b}^+ = \bigoplus_{\alpha \geq 0} \mathfrak{g}_\alpha \qquad \mathfrak{b}^- = \bigoplus_{\alpha \leq 0} \mathfrak{g}_\alpha$$
$$\mathfrak{n}_t^+ = \bigoplus_{\alpha >_t 0} \mathfrak{g}_\alpha \qquad \mathfrak{n}_t^- = \bigoplus_{\alpha <_t 0} \mathfrak{g}_\alpha$$
$$\mathfrak{p}_t^+ = \bigoplus_{\alpha \geq_t 0} \mathfrak{g}_\alpha \qquad \mathfrak{p}_t^- = \bigoplus_{\alpha \leq_t 0} \mathfrak{g}_\alpha$$

Write $h_i \in \mathfrak{h}$ for the basis dual to the simple roots $\alpha_i \in \mathfrak{h}^*$. The degree zero part of $\mathfrak{g}$ in the t-grading is

$$\mathfrak{g}^{(t)} \times \mathbb{C}h_t$$

where $\mathfrak{g}^{(t)}$ is the subalgebra generated by $\{e_i, f_i\}_{i \neq t}$ and $\mathbb{C}h_t$ is the one-dimensional abelian Lie algebra spanned by h_t. The decomposition of $\mathfrak{g}$ into t-graded components is just its decomposition into eigenspaces for the adjoint action of h_t:

$$\mathfrak{g} = \bigoplus_{j \in \mathbb{Z}} \ker(\mathrm{ad}(h_t) - j).$$

Example 2.15. For the Dynkin diagram A_n, with vertices labeled as

$$1 \text{ --- } 2 \text{ --- } \cdots \text{ --- } n,$$

the associated Lie algebra is $\mathfrak{sl}_{n+1} \coloneqq \mathfrak{sl}(\mathbb{C}^{n+1})$. Let ϵ_{ij} denote the $(n+1) \times (n+1)$ matrix whose entries are all 0 except for a single 1 in the i-th row and j-th column. Then it is customary to use the Lie algebra generators $e_i = \epsilon_{i,i+1}$ and $f_i = \epsilon_{i+1,i}$ for $i = 0, \ldots, n$.

An ordered sequence of vertices $t_1, \ldots, t_n$ forming a subgraph $A_n \subset T$ yields an inclusion $\mathfrak{sl}_{n+1} \hookrightarrow \mathfrak{g}$ by sending the generators e_i, f_i of $\mathfrak{sl}_{n+1}$ to the corresponding elements $e_{t_i}, f_{t_i} \in \mathfrak{g}$.

Representations

Let V be a representation of $\mathfrak{g}$. For $\lambda \in \mathfrak{h}$, define the *λ-weight space of V* to be

$$V_\lambda = \{v \in V : hv = \lambda(h)v \text{ for all } h \in \mathfrak{h}\}.$$

If $V_\lambda \neq 0$, then we say λ is a *weight* of V. A nonzero vector $v \in V_\lambda$ is a *highest weight vector* if $\mathfrak{n}^+ v = 0$. If such a v generates V as a $\mathfrak{g}$-module, then we say V is a *highest weight module* with highest weight λ.

Let $\mathcal{U}$ denote the universal enveloping algebra functor. Representations of $\mathfrak{g}$ are equivalent to modules over $\mathcal{U}(\mathfrak{g})$. Given $\lambda \in \mathfrak{h}^*$, the *Verma module* $M(\lambda)$ is defined to be

$$M(\lambda) = \mathcal{U}(\mathfrak{g}) \otimes_{\mathcal{U}(\mathfrak{b}^+)} \mathbb{C}_\lambda.$$

Here $\mathbb{C}_\lambda$ is the $\mathfrak{b}^+$-module where $\mathfrak{h}$ acts by λ and $\mathfrak{n}^+$ acts trivially. All the weights of $M(\lambda)$ are in $\lambda + Q$. If $v \in V_\lambda$ is a highest weight vector, then there is a map $M(\lambda) \to V$ sending $1 \mapsto v$. If V is a highest weight module then this map is surjective.

Every Verma module $M(\lambda)$ has a unique maximal proper submodule $J(\lambda)$, namely the sum of all submodules which do not contain v. It follows that $L(\lambda) = M(\lambda)/J(\lambda)$ is an irreducible highest weight module with heighest weight λ, and any such module is isomorphic to $L(\lambda)$.

Let $\omega_i \in \mathfrak{h}^*$ be the basis dual to $\alpha_i^\vee \in \mathfrak{h}$. Explicitly, ω_i is the linear combination of α_i given by the i-th column of A^{-1}. These are the *fundamental weights*, and the representations $L(\omega_i)$ are called *fundamental representations*. Their nonnegative integral span is the collection of *dominant weights*.

One can alternatively work with lowest weights instead of highest weights, interchanging the roles of positive and negative parts of the Lie algebra in all of the preceding. The irreducible representation with lowest weight $-\lambda$ is $L(\lambda)^\vee$, where $(-)^\vee$ represents the "restricted" dual. That is, $V^\vee \coloneqq \bigoplus V_\lambda^*$ for a weight module V with finite-dimensional weight spaces. One has $V^\vee \subseteq V^*$, with equality when V is finite-dimensional.

Weight grading on representations

The decomposition of $L(\lambda)$ into weight spaces gives an $\mathfrak{h}^*$-grading on $L(\lambda)$. Moreover, all the weights of $L(\lambda)$ are in the translate $\lambda + \bigoplus_{i\in T} \mathbb{Z}\alpha_i$ of the root lattice.

In §2.2.1 it was described how singling out a vertex $t \in T$ allows us to impose a $\mathbb{Z}$-grading on $\mathfrak{g}$ by considering only the coefficient of α_t in the $\mathfrak{h}^*$-grading. This works for representations $L(\lambda)$ as well: if $v \in L(\lambda)$ is a highest weight vector then $h_t v = \langle h_t, \lambda\rangle v$ and the eigenvalues for the action of h_t on $L(\lambda)$ are $\langle h_t, \lambda\rangle, \langle h_t, \lambda\rangle - 1, \ldots$, terminating iff $L(\lambda)$ is finite-dimensional. The eigenspaces give the t-graded components. Each one is a representation of the subalgebra $\mathfrak{g}^{(t)} \times \mathbb{C}h_t \subset \mathfrak{g}$. In particular, v is a highest weight vector for the top graded component, thus this component is the representation of $\mathfrak{g}^{(t)}$ with highest weight $\sum_{i\neq t} c_i\omega_i$ if $\lambda = \sum_{i\in T} c_i\omega_i$.

Exponential action and Baker-Campbell-Hausdorff

Let $\bigoplus_{i>0} \mathbb{L}_i$ be a strictly positively graded Lie algebra, e.g. $\mathfrak{n}_t^{\pm}$ for some $t \in T$. Its bracket naturally extends to one on $\mathbf{L} = \prod_{i>0} \mathbb{L}_i$. Suppose R is an R_0-algebra on which elements $X \in \mathbb{L}_i$ act by locally nilpotent R_0-linear derivations. Here "locally nilpotent" means that for any Lie algebra element X and ring element $f \in R$, we have $X^N f = 0$ for $N \gg 0$. Then for any $X \in \mathbf{L}$, the exponential

$$\exp X = \mathrm{Id} + X + \frac{1}{2!}X^2 + \frac{1}{3!}X^3 + \cdots$$

defines an R_0-algebra automorphism of R.

Moreover, given Lie algebra elements $X, Y \in \mathbf{L}$, the Baker-Campbell-Hausdorff formula gives a well-defined element $Z \in \mathbf{L}$ such that $\exp Z = \exp X \exp Y$:

$$Z = X + Y + \frac{1}{2}[X,Y] + \frac{1}{12}[X,[X,Y]] - \frac{1}{12}[Y,[X,Y]] + \cdots.$$

We will not need the explicit expression for Z, only that such an expression exists in terms of iterated commutators.

2.2.2 G/P and Schubert varieties

If $\mathfrak{g}$ is of finite type, there is a unique simply connected Lie group G associated to the Lie algebra $\mathfrak{g}$, and the representations of G correspond to those of $\mathfrak{g}$. For a fundamental weight ω_t, the action of G on the highest weight line in $\mathbb{P}(L(\omega_t))$ has stabilizer P_t^+, the subgroup of G corresponding to the maximal parabolic subalgebra $\mathfrak{p}_t^+$ as defined in §2.2.1. Hence the orbit of this highest weight line can be identified with the homogeneous space G/P_t^+. For Dynkin type A_n with the standard labeling of vertices, this construction produces the Grassmannian $\mathrm{Gr}(t, n+1)$. Accordingly, the reader may think of G/P_t^+ as a "generalized Grassmannian."

This theory generalizes to the Kac-Moody setting, where G/P_t^+ is instead a projective ind-variety, and [31] is an excellent reference for this. To avoid having to define G, the subgroup P_t^+, or what it means to take a quotient, we will define everything purely algebraically. That being said, we will occasionally want to use legitimate group actions, but the group in each case will either be the special or general linear group, or the exponential of some Lie algebra whose elements have locally nilpotent actions (so that the exponential is algebraically well-defined).

Definition of the homogeneous space G/P

Pick a vertex $t \in T$, let ω_t be the corresponding fundamental weight, and $L(\omega_t)$ the irreducible representation with highest weight ω_t. Let $L(\omega_t)^\vee$ be its restricted dual, i.e. the irreducible representation with lowest weight $-\omega_t$. Let $\mathfrak{A} = \bigoplus_{n\geq 0} L(n\omega_t)^\vee$. Since $L((n+1)\omega_t)^\vee$ appears with multiplicity 1 inside of $L(n\omega_t)^\vee \otimes L(\omega_t)^\vee$, we may use this to define a $\mathfrak{g}$-equivariant multiplication on $\mathfrak{A}$ which makes it a graded $\mathbb{C}$-algebra generated by $L(\omega_t)^\vee$:

$$\bigoplus_{n\geq 0} L(n\omega_t)^\vee = (\operatorname{Sym} L(\omega_t)^\vee)/I_{\text{Plücker}}.$$

The ideal $I_{\text{Plücker}}$ is comprised of subrepresentations vanishing on a highest weight vector $v \in L(\omega_t)$.

We define $G/P_t^+ = \operatorname{Proj}\mathfrak{A}$ to be the corresponding projective ind-variety in $\mathbb{P}(L(\omega_t))$.

Weyl group and subgroups

Let W denote the Weyl group associated to T. It is generated by the *simple reflections* $\{s_i\}_{t\in T}$. Explicitly,

$$W = \langle \{s_i\}_{i\in T} \mid (s_i s_j)^{m_{ij}} = 1\rangle$$

where $m_{ij} = 1$ if $i = j$, $m_{ij} = 2$ if $i, j \in T$ are not adjacent, and $m_{ij} = 3$ if $i, j \in T$ are adjacent. The group W is finite if and only if T is a Dynkin diagram.

The Weyl group acts on $\mathfrak{h}^*$: the simple reflections act via

$$s_i(\lambda) = \lambda - \langle \alpha_i^\vee, \lambda\rangle \alpha_i.$$

For any $t \in T$, the expression

$$\exp(f_t)\exp(-e_t)\exp(f_t)$$

defines an automorphism of any representation on which the actions of e_t and f_t are locally nilpotent. This includes the adjoint representation of $\mathfrak{g}$ and $L(\lambda)$ for any dominant integral λ. By abuse of notation, we will also denote this automorphism by s_t, although it is really a (non-unique) lift thereof. For any element of the Weyl group, we may define an analogous automorphism by expressing σ as a product of simple reflections. The resulting automorphism is not unique, but this is not important for our purposes.

A *word* for σ is a sequence of simple reflections whose product is σ. It is *reduced* if there are no shorter words for σ. The *length* $\ell(\sigma)$ is the length of a reduced word for σ. There is a partial order, called the *(strong) Bruhat order* on W, defined so that $\sigma \geq \sigma'$ if a reduced word for σ contains a reduced word for σ' as a (not necessarily consecutive) substring. If this holds for some reduced word for σ, it in fact holds for all reduced words.

If $t \in T$, we let $W_{P_t} \subset W$ denote the subgroup generated by all simple reflections other than s_t. This is the stabilizer of the fundamental weight ω_t under the action of W. We write W^{P_t} for the set of minimal length representatives of W/W_{P_t}.

Plücker coordinates and Schubert cells

Let $v \in \mathbb{P}(L(\omega_t))$ denote the highest weight line. By abuse of notation we will sometimes use v to refer to a highest weight vector in $L(\omega_t)$ instead. The point v is the *Borel-fixed point* of G/P_t^+. For

$\sigma \in W^{P_t}$, the point σv is a *torus-fixed point* of G/P_t^+. For k a field, the k-points of G/P_t^+ decompose into a disjoint union of *Schubert cells*

$$(G/P_t^+)(k) = \coprod_{\sigma \in W^{P_t}} C_\sigma(k)$$

where $C_\sigma(k)$ is the orbit of the torus-fixed k-point σv under the action of $\exp(\mathfrak{n}^+ \otimes k)$.

The variables of $\mathfrak{A}$ are called *Plücker coordinates*. Let p_e denote a lowest weight vector of $L(\omega_t)^\vee$ and let $p_\sigma = \sigma p_e$. The set $\{p_\sigma : \sigma \in W^{P_t}\}$ is the set of *extremal Plücker coordinates*; they are defined up to scale. The representation $L(\omega_t)^\vee$ may have weights other than those in the W-orbit of $-\omega_t$, and thus there may be non-extremal Plücker coordinates. (The representation is *miniscule* if all weights belong to the same W-orbit, in which case all Plücker coordinates are extremal.)

The *Schubert variety* X_w is defined to be the closure of the Schubert cell C_w. The extremal Plücker coordinate p_σ vanishes on the Schubert cell C_w if and only if $\sigma \nleq w$ in the Bruhat order. The dimension of X_w is the dimension of C_w, which is $\ell(w)$.

The *opposite Schubert cell* C^w is the orbit of wv generated by the action of all one-parameter subgroups $\exp(\mathfrak{g}_\alpha)$ where $\alpha \in \Delta^-_{\text{re}} \coloneqq \Delta^- \cap W\Delta^+$. Its closure is the *opposite Schubert variety* X^w, which is an ind-variety of codimension $\ell(w)$ inside of G/P_t^+.

Unions of (opposite) Schubert varieties are set-theoretically cut out by the vanishing of extremal Plücker coordinates. To have this hold scheme-theoretically, it is necessary to use all Plücker coordinates rather than only extremal ones. Although the following is likely true in the Kac-Moody setting, we could only locate a reference for this in the setting where $\mathfrak{g}$ is of finite type, i.e. T is a Dynkin diagram.

Theorem 2.16 ([3])**.** *Assume that T is a Dynkin diagram. Let $X^{w_1}, \ldots, X^{w_n}$ be (opposite) Schubert varieties in $G/P_t \subset \mathbb{P}(L(\omega_t))$. Then in* $\operatorname{Sym} L(\omega_t)^\vee$*, the ideal of the union $X = \bigcup_i X^{w_i}$ is generated by $I_{\text{Plücker}}$ together with the linear forms in $L(\omega_t)^\vee$ which vanish on X.*

We accordingly say that unions of Schubert varieties are linearly defined.

The *Kazhdan-Lusztig variety* $\mathcal{N}^w_\sigma$ is defined to be the intersection $X^w \cap C_\sigma$ and it has codimension $\ell(w)$ inside of $C_\sigma \cong \mathbb{A}^{\ell(\sigma)}$.

2.3 Generic licci ideals

Assumption 2.17. Throughout this section, we fix an ADE triple (c, d, t) (c.f. Definition 1.3). Recall that this means $c \geq 2$, $d \geq 0$, and $t \geq 1$ are integers satisfying

$$\frac{1}{c-1} + \frac{1}{d+1} + \frac{1}{t+1} > 1$$

or equivalently that the graph $T = T_{c-1,d+1,t+1}$ below is a Dynkin diagram:

$$\begin{array}{ccccccccccc}
x_{c-2} & — & \cdots & — & u & — & y_1 & — & \cdots & — & y_d \\
 & & & & | & & & & & & \\
 & & & & z_1 & & & & & & \\
 & & & & | & & & & & & \\
 & & & & \vdots & & & & & & \\
 & & & & | & & & & & & \\
 & & & & z_t & & & & & &
\end{array}$$

Sometimes we will also use x_0 to refer to the middle vertex u. For brevity we will abbreviate $G/P^+_{x_{c-2}}$ as G/P. Using the notation of §2.2.2, we define two opposite Schubert varieties in G/P of particular interest, corresponding to

$$w = s_{z_1} s_u s_{x_1} \dots s_{x_{c-2}} \text{ and } w' = s_{y_1} s_u s_{x_1} \dots s_{x_{c-2}}. \tag{2.3}$$

We write $X = X^w$ and $X' = X^{w'}$.

To discuss these Schubert varieties, it is helpful to introduce some additional notation. Let[1]

$$F_1 = \mathbb{C}^{c+d}, \quad F_c = \mathbb{C}^t, \quad F_1' = \mathbb{C}^{c+t}, \quad F_c' = \mathbb{C}^d.$$

Following Example 2.15, we view the Lie algebras $\mathfrak{sl}(F_i)$ and $\mathfrak{sl}(F_i')$ as subalgebras of $\mathfrak{g}$ in the following manner:

- $\mathfrak{sl}(F_1)$ corresponds to the ordered sequence of vertices $y_d, \dots, y_1, u, x_1, \dots, x_{c-2}$,
- $\mathfrak{sl}(F_c)$ corresponds to the ordered sequence of vertices $z_2, \dots, z_t$,
- $\mathfrak{sl}(F_1')$ corresponds to the ordered sequence of vertices $z_t, \dots, z_1, u, x_1, \dots, x_{c-2}$, and
- $\mathfrak{sl}(F_c')$ corresponds to the ordered sequence of vertices $y_2, \dots, y_d$.

The Plücker coordinates on $G/P_{x_{c-2}} \hookrightarrow \mathbb{P}(L(\omega_{x_{c-2}}))$ reside in $L(\omega_{x_{c-2}})^\vee$. In the z_1-grading, each graded component is a representation of $\mathfrak{g}^{(z_1)} = \mathfrak{sl}(F_1) \times \mathfrak{sl}(F_c)$. Similarly, in the y_1-grading, each graded component is a representation of $\mathfrak{g}^{(y_1)} = \mathfrak{sl}(F_1') \times \mathfrak{sl}(F_c')$.

Lemma 2.18. *The bottom two graded components of $L(\omega_{x_{c-2}})^\vee$ in the z_1-grading are given by*

$$L(\omega_{x_{c-2}})^\vee = F_1 \oplus \bigwedge^c F_1 \otimes F_c^* \oplus \cdots$$

Proof. There are numerous ways of confirming this. One is to use the theory of crystals, which can be used to compute such decompositions ("branching formulas") in great generality. In fact, the full decompositions of these representations are tabulated in [36] using this method by computer. Alternatively, one could count the representations appearing in the first few terms of a parabolic BGG resolution for $L(\omega_{x_{c-2}})^\vee$; for $c = 3$ this sort of calculation is done in [50].

To avoid introducing more machinery than needed, we give a simple ad-hoc proof which is a simplified form of the latter argument. The bottom z_1-graded component of $L(\omega_{x_{c-2}})^\vee$ is simply $L(\omega_{x_{c-2}}, \mathfrak{g}^{(z_1)})^\vee = F_1$. The representation $L(\omega_{x_{c-2}})^\vee$ is generated by this bottom component as a $\mathfrak{n}^+_{z_1}$-representation. In particular, the degree 1 part of $\mathfrak{g}$ in the z_1-grading is $\bigwedge^{c-1} F_1 \otimes F_c^*$. Therefore the next component of $L(\omega_{x_{c-2}})^\vee$ must be a subrepresentation of

$$F_1 \otimes \bigwedge^{c-1} F_1 \otimes F_c^* = \bigwedge^c F_1 \otimes F_c' \oplus S_{2,1^{c-2}} F_1 \otimes F_c'.$$

Hence it suffices to show that the latter representation does not occur. Note that $w(-\omega_{x_{c-2}}) = \omega_{z_1} - \omega_{y_1} - \omega_{z_2}$ is the lowest weight of $\bigwedge^c F_1 \otimes F_c' \subset L(\omega_{x_{c-2}})^\vee$. The corresponding weight space is extremal (i.e. in the W-orbit of $-\omega_{x_{c-2}}$), thus one-dimensional. But this is also a weight of $S_{2,1^{c-2}} F_1 \otimes F_c'$, therefore this latter representation cannot appear. □

[1]The notation F_1 and F_c is motivated by the $c = 3$ case that will be the topic of later chapters.

Lemma 2.19. *In G/P, the Schubert variety X is scheme-theoretically cut out by the Plücker coordinates in the bottom z_1-graded component $F_1 \subset L(\omega_{x_{c-2}})^\vee$.*

Proof. Given Theorem 2.16, it suffices to determine which Plücker coordinates vanish on X, or equivalently those which vanish on the linear span of X. This linear span is a *Demazure module*, namely the $\mathfrak{n}^-$-representation generated by the extremal weight space corresponding to $w\omega_{x_{c-2}} = -\omega_{z_1} + \omega_{y_1} + \omega_{z_2}$. This is the highest weight space in the representation $\bigwedge^c F_1^* \oplus F_c \subset L(\omega_{x_{c-2}})$, so the Demazure module contains $\bigwedge^c F_1^* \oplus F_c$.

From Lemma 2.18, we conclude that this Demazure module is exactly the sum of all z_1-graded components except for $F_1^* \subset L(\omega_{x_{c-2}})$. The annihilator of this Demazure module in $L(\omega_{x_{c-2}})^\vee$ is exactly F_1, as desired. □

There is an action of $\mathrm{GL}(F_1) \times \mathrm{GL}(F_c)$ on $L(\omega_{x_{c-2}})^\vee$ and G/P extending the action of $\mathrm{SL}(F_1) \times \mathrm{SL}(F_c)$ so that Lemma 2.18 holds at the level of GL-representations. It suffices to describe the actions of scalars in $\mathfrak{gl}(F_1)$ and $\mathfrak{gl}(F_c)$: we have that $1 \in \mathfrak{gl}(F_1)$ acts by $(c-1)j$ and $1 \in \mathfrak{gl}(F_c)$ acts by $-j$ on the z_1-graded component in degree j, where we index so that F_1 is in degree $j = 0$. We similarly have an action of $\mathrm{GL}(F_1') \times \mathrm{GL}(F_c')$ on $L(\omega_{x_{c-2}})$.

Corollary 2.20. *The action of* $\mathrm{GL}(F_1) \times \mathrm{GL}(F_c)$ *on G/P preserves the Schubert variety X, and the action of* $\mathrm{GL}(F_1') \times \mathrm{GL}(F_c')$ *on G/P preserves the Schubert variety X'.*

Proof. This follows directly from Lemma 2.19. □

It is well-known that Schubert varieties are Cohen-Macaulay; see for instance [43]. Since G/P is locally isomorphic to affine space, it follows that the local defining ideals of Schubert varieties are perfect.

The varieties X and X' are irreducible and neither is contained in the other. Furthermore, their union is a complete intersection in G/P:

Lemma 2.21. *In G/P, the union $X \cup X'$ is scheme-theoretically cut out by the Plücker coordinates in $F_1 \cap F_1' = \mathbb{C}^c \subset L(\omega_{x_{c-2}})^\vee$.*

Proof. This follows from Theorem 2.16, since we have already determined the Plücker coordinates which vanish on X and X' individually. □

Thus in any local ring of G/P, the ideals of X and X' are linked.

Proposition 2.22. *Let R be a local Noetherian ring, and $f\colon \mathrm{Spec}\, R \to G/P$ a morphism. Up to scale, this determines a surjection $\gamma\colon L(\omega_{x_{c-2}})^\vee \otimes R \to R$ giving the projective coordinates of f. If $\gamma(K \otimes R)$ is a complete intersection, then $\gamma(F_1 \otimes R) \overset{\gamma(K\otimes R)}{\sim} \gamma(F_1' \otimes R)$.*

Proof. This follows from the preceding observations together with Corollary 2.6. □

Example 2.23. Let us apply this to the parameters $(c, d, t) = (2, 1, 2)$. Then $T = A_4$:

$$\begin{array}{ccc} 2 & — & 1 \\ | & & \\ 3 & & \\ | & & \\ 4 & & \end{array}$$

Here $z_1 = 3$ and in this degenerate case we have $x_0 = 2$. The homogeneous space SL_5/P_2 is the Grassmannian $\mathrm{Gr}(2,5)$, and the opposite Schubert varieties X and X' are the following closures:

$$X = \overline{\left\{\begin{bmatrix} 1 & * & * & 0 & * \\ 0 & 0 & 0 & 1 & * \end{bmatrix}\right\}}, \qquad X' = \overline{\left\{\begin{bmatrix} 0 & 1 & 0 & * & * \\ 0 & 0 & 1 & * & * \end{bmatrix}\right\}}.$$

For $1 \le i < j \le 5$, let p_{ij} be the function which sends a given (surjective) 2×5 matrix to its 2×2 minor involving columns i and j. These are the Plücker coordinates on $\mathrm{Gr}(2,5)$, and they reside in $L(\omega_2)^\vee = \bigwedge^2(\mathbb{C}^5)^*$. In the 2-grading, this representation decomposes as the sum of:

$$\begin{aligned} F_1 &= \mathrm{span}(p_{12}, p_{13}, p_{23}) \\ \bigwedge^2 F_1 \otimes F_2^* &= \mathrm{span}(p_{14}, p_{15}, p_{24}, p_{25}, p_{34}, p_{35}) \\ \bigwedge^3 F_1 \otimes \bigwedge^2 F_2^* &= \mathrm{span}(p_{45}) \end{aligned}$$

where $F_1 = \mathbb{C}^3$ and $F_2 = \mathbb{C}^2$. In the 1-grading it decomposes as the sum of

$$\begin{aligned} F_1' &= \mathrm{span}(p_{12}, p_{13}, p_{14}, p_{15}) \\ \bigwedge^2 F_1' \otimes F_2'^* &= \mathrm{span}(p_{23}, p_{24}, p_{25}, p_{34}, p_{35}, p_{45}) \end{aligned}$$

where $F_1' = \mathbb{C}^4$ and $F_2' = \mathbb{C}^1$. Indeed, X is cut out by p_{12}, p_{13}, p_{23}, and X' is cut out by $p_{12}, p_{13}, p_{14}, p_{15}$, confirming Lemma 2.19. Their union is cut out by $K = F_1 \cap F_1' = \mathrm{span}(p_{12}, p_{13})$.

Now let $R = \mathbb{C}[\![w,x,y,z]\!]$ and let $\varphi\colon \mathrm{Spec}\, R \to G/P$ be the morphism represented by the matrix

$$\begin{bmatrix} w & 1 & 0 & 0 & 0 \\ x & 0 & 1 & 0 & 0 \end{bmatrix}.$$

This determines a map $\gamma\colon \bigwedge^2(\mathbb{C}^5)^* \otimes R \to R$ where

$$\begin{aligned} \gamma(F_1 \otimes R) &= (-x, w, 1) \\ \gamma(K \otimes R) &= (-x, w) \\ \gamma(F_1' \otimes R) &= (-x, w, 0, 0) \end{aligned}$$

and Proposition 2.22 claims $(1) \overset{(x,w)}{\sim} (x,w)$, which is evidently true.

Now let

$$g' = \begin{bmatrix} x & y & 1 & 0 \\ y & z & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \in GL(F_1' \otimes R)$$

and act on γ by g' to obtain the R-point of G/P given by

$$\begin{bmatrix} w & x & y & 1 & 0 \\ x & y & z & 0 & 1 \end{bmatrix}.$$

The new Plücker coordinates γ' are such that

$$\gamma'(F_1 \otimes R) = (wy - x^2, wz - xy, wz - y^2)$$
$$\gamma'(K \otimes R) = (wy - x^2, wz - xy)$$
$$\gamma'(F_1' \otimes R) = (wy - x^2, wz - xy, -x, w)$$

where $\gamma'(F_1' \otimes R) = \gamma(F_1' \otimes R)$ since X' is preserved by the action of $GL(F_1')$. Now Proposition 2.22 claims $(x, w) \overset{(wy-x^2, wz-xy)}{\sim} (wy - x^2, wz - xy, wz - y^2)$, and we have thus recovered the links from Example 2.13 in this geometric manner.

Using the method in this preceding example as inspiration, we can prove the central result of this chapter.

Theorem 2.24. *Let R be a local Noetherian ring, and $I \subset R$ a grade c ideal. Then I is licci with $d(I) \leq d$ and $t(R/I) \leq t$ if and only if there exists a morphism $f\colon \operatorname{Spec} R \to G/P$ with $f^{-1}X = \operatorname{Spec} R/I$.*

Proof. First we prove the "only if" implication. By Proposition 2.12 there exists a sequence of minimal links

$$I = I_0 \overset{\kappa_0}{\rightsquigarrow} I_1 \overset{\kappa_1}{\rightsquigarrow} I_2 \rightsquigarrow \cdots \rightsquigarrow I_N = \kappa_N \overset{\kappa_N}{\rightsquigarrow} (1)$$

where the I_i are grade c perfect ideals and the κ_i are complete intersections.

We prove the statement by induction. If $I_1 = (1)$ then let $f'\colon \operatorname{Spec} R \to G/P$ be any morphism landing in the complement of X'; a constant map will suffice. If $I_1 \neq (1)$ then by Lemma 2.10 we have $d(I_1) \leq t(R/I) \leq t$ and $t(R/I_1) \leq d(I) \leq d$ because $I \rightsquigarrow I_1$ is minimal. So by induction we assume the existence of some map $f'\colon \operatorname{Spec} R \to G/P$ so that $f'^{-1}X' = \operatorname{Spec} R/I$. Let $\gamma'\colon L(\omega_{x_{c-2}})^\vee \otimes R \to R$ be the induced map on Plücker coordinates; the preceding equivalently says that $\gamma'(F_1' \otimes R) = I_1$. Since I_1/κ_0 is generated by no more than t elements, there exists some $g \in GL(F_1' \otimes R)$ such that the image of

$$K \otimes R \hookrightarrow F_1' \otimes R \xrightarrow{g} F_1' \otimes R \xrightarrow{\gamma'} R$$

is κ_0. Precompose γ' by the action of g on $L(\omega_{x_{c-2}})^\vee \otimes R$ to obtain a new map $\gamma\colon L(\omega_{x_{c-2}})^\vee \otimes R \to R$ describing some $f\colon \operatorname{Spec} R \to G/P$. By construction, we have $\gamma(F_1 \otimes R) = I_0$ from Proposition 2.22, finishing the proof.

For the "if" implication, let $\gamma\colon L(\omega_{x_{c-2}})^\vee \otimes R \to R$ be the induced map on Plücker coordinates. Then $\gamma \otimes k$ prescribes a k-point of G/P and is thus nonzero on some extremal weight space in $L(\omega_{x_{c-2}})^\vee$, say with weight $\sigma(-\omega_{x_{c-2}})$ where $\sigma \in W$. The simple reflections $s_{x_{c-2}}, \ldots, s_{x_1}, s_u, s_{y_1}, \ldots, s_{y_d}$ may be lifted to $GL(F_1)$, and the simple reflections $s_{x_{c-2}}, \ldots, s_{x_1}, s_u, s_{z_1}, \ldots, s_{z_t}$ may be lifted to $GL(F_1')$. Thus, by expressing σ as a product of simple reflections, we can construct a sequence $g_1, g_1', \ldots, g_N, g_N'$, where $g_i \in GL(F_1)$ and $g_i' \in GL(F_1')$ are permutation matrices, with the property that $(\gamma \otimes k) g_1 g_1' \ldots g_N g_N'$ is nonzero on the lowest weight space of $L(\omega_{x_{c-2}})^\vee$.

Next, we inductively choose lifts $\tilde{g}_i \in GL(F_1 \otimes R)$ for each $g_i \in GL(F_1 \otimes k)$, and similarly $\tilde{g}_i'$ for g_i', as follows. By assumption $\gamma(F_1 \otimes R) = I$ is a grade c ideal. Hence by Lemma 2.11, there is a lift $\tilde{g}_1$ of g_1 which makes $\gamma\tilde{g}_1(F_1 \otimes K)$ a complete intersection. From Proposition 2.22 we conclude that $I = \gamma\tilde{g}_1(F_1 \otimes R) \sim \gamma\tilde{g}_1(F_1' \otimes R)$. We then repeat this process replacing I with $I_1 = \gamma\tilde{g}_1(F_1' \otimes R)$ to choose $\tilde{g}_1'$, and so on.

Note that $\gamma\tilde{g}_1\tilde{g}_1'\cdots\tilde{g}_N\tilde{g}_N'(K\otimes R)=(1)$ by construction. We have therefore obtained a sequence of links

$$\begin{aligned} I &= \gamma\tilde{g}_1(F_1\otimes R)\\ &\sim \gamma\tilde{g}_1(F_1'\otimes R)\\ &= \gamma\tilde{g}_1\tilde{g}_1'(F_1'\otimes R)\\ &\sim \gamma\tilde{g}_1\tilde{g}_1'(F_1\otimes R)\\ &= \gamma\tilde{g}_1\tilde{g}_1'\tilde{g}_2(F_1\otimes R)\\ &\sim \cdots \sim (1) \end{aligned}$$

showing that I is licci. Moreover, $d(I)\le d$ since $\gamma(F_1\otimes R)=I$ where $F_1=\mathbb{C}^{c+d}$. Also, writing $\overline{K}$ for the image of $\gamma\tilde{g}_1(K\otimes R)$ in $I_1/\mathfrak{m}I_1$, we have $d(I_1)+c\le\dim\overline{K}+t$ because $F_1'=K\oplus\mathbb{C}^t$, so $t(R/I)\le t$ by Lemma 2.10 applied to the link $I_1\rightsquigarrow I$. □

Corollary 2.25. *Let w be as in* (2.3) *and suppose σ is a minimal length representative of $[\sigma]\in W/W_{P_{x_{c-2}}}$ with $\sigma\ge w$. Localizing at the torus-fixed point $\sigma v\in C_\sigma$, the ideal of the Kazhdan-Lusztig variety $\mathcal{N}_\sigma^w := X\cap C_\sigma$ is licci, with deviation at most d and type at most t.*

Proof. The variety $X\cap C_\sigma$ has codimension $\ell(w)=c$ inside of C_σ, with $\sigma v\in X\cap C_\sigma$. Simply apply the "if" implication of Theorem 2.24 to the morphism

$$\operatorname{Spec}\mathcal{O}_{C_\sigma,\sigma v}\to C_\sigma\to G/P.$$

□

Leveraging the symmetry of the Schubert variety X, we can refine Theorem 2.24 to show that licci ideals must be specializations of particular examples of the form in Corollary 2.25.

Proposition 2.26. *Let $(R,\mathfrak{m},k)$ be a local ring and $f\colon\operatorname{Spec}R\to G/P$ a morphism. Then there exists $\sigma\in W$ that is a minimal length representative of its double coset $[\sigma]\in W_{P_{z_1}}\backslash W/W_{P_{x_{c-2}}}$ and a morphism $f'\colon\operatorname{Spec}R\to C_\sigma$ with the properties that $f^{-1}X=f'^{-1}X$ and the k-point of C_σ determined by f' is σv.*

Proof. This essentially follows from the observations that, if we let $P_{z_1}^-$ be the maximal negative parabolic corresponding to $T-\{z_1\}$,

1. the k-point of G/P determined by f lies in some $P_{z_1}^-(k)$-orbit, which are indexed by double cosets $W_{P_{z_1}}\backslash W/W_{P_{x_{c_2}}}$,
2. the Schubert variety X is preserved under the action of $P_{z_1}^-$, and
3. The intersection of X with the big open cell σC^e is isomorphic to the product of $\mathcal{N}_\sigma^w$ with an affine space, c.f. [30, Lemma A.4].

To emphasize that this procedure is both algorithmic and not reliant on working over $\mathbb{C}$ (c.f. Assumption 2.1), we give an explicit algebraic proof following the above steps. Let $V=L(\omega_{x_{c-2}})^\vee$ and let $\gamma\colon V\otimes R\to R$ be the map induced by f. Then $\gamma\otimes k$ is nonzero on some extremal weight space in

$V \otimes k$; pick one which is minimal in the Bruhat order (in fact, there is a unique such weight space, corresponding to the Schubert cell containing the k-point $\gamma \otimes k$).

This extremal weight space belongs to some extremal representation $S_\lambda F_1 \otimes S_\mu F_c^*$ of $\mathrm{GL}(F_1) \times \mathrm{GL}(F_c)$ inside of V. There exists $g \in \mathrm{GL}(F_1 \otimes R) \times \mathrm{GL}(F_c \otimes R)$ such that $(\gamma g) \otimes k$ is nonzero on the lowest weight space V_ω of $S_\lambda F_1 \otimes S_\mu F_c^*$; indeed we may take g to be a pair of permutation matrices. We have $\gamma g(F_1 \otimes R) = \gamma(F_1 \otimes R)$ as ideals of R. Let $\sigma \in W$ be of minimal length so that $\sigma(-\omega_{x_{c-2}}) = \omega$; it is then a minimal length representative of its double coset $[\sigma] \in W_{P_{z_1}} \backslash W / W_{P_{x_{c-2}}}$. To ease notation slightly we write x for x_{c-2}.

The new γg describes an R-point of G/P which lands in the big open cell σC^e. As such, there exists a

$$Y \in \prod_{\sigma^{-1}\alpha <_x 0} (\mathfrak{g}_\alpha \otimes R)$$

such that $\gamma g = \pi_\omega(\exp Y)$ where $\pi_\omega \colon V \otimes R \to R$ is a projection onto the extremal weight space V_ω. Recall that $\sigma^{-1}\alpha <_x 0$ means that the coefficient of α_x in $\sigma^{-1}\alpha$ is negative, or equivalently that $\langle \sigma^{-1}\alpha, h_x \rangle < 0$.

Furthermore, using Baker-Campbell-Hausdorff, it is possible to solve for

$$Y^- \in \prod_{\alpha<0, \sigma^{-1}\alpha <_x 0} (\mathfrak{g}_\alpha \otimes R), \qquad Y^+ \in \prod_{\alpha>0, \sigma^{-1}\alpha <_x 0} (\mathfrak{g}_\alpha \otimes R)$$

with the property that $\exp Y = \exp(Y^+)\exp(Y^-)$. Notice that the action of $\exp(Y^-)$ on V maps F_1 isomorphically to itself, so

$$\gamma g(F_1 \otimes R) = \pi_\omega \exp(Y^+)\exp(Y^-)(F_1 \otimes R) = \pi_\omega \exp(Y^+)(F_1 \otimes R)$$

as ideals of R. We take f' to be the R-point of C_σ described by $\pi_\omega \exp(Y^+)$. This cancellation of $\exp(Y^-)$ corresponds to projection onto $\mathcal{N}_\sigma^w$ in point (3).

Finally, we have that $\gamma g \otimes k$ is zero on all weight spaces lower than V_ω by construction. Therefore $Y^+ \otimes k = 0$, and $\pi_\omega \exp(Y^+) \otimes k = \pi_\omega \otimes k$; i.e. the k-point of C_σ determined by f' is the torus-fixed point σv as desired. □

Theorem 2.27. *Let R be a local Noetherian ring and $I \subset R$ a grade c licci ideal with deviation $d(I) \le d$ and type $t(R/I) \le t$. Then there exists $\sigma \in W$ that is a minimal length representative of its double coset $[\sigma] \in W_{P_{z_1}} \backslash W / W_{P_{x_{c-2}}}$, and a local homomorphism $\varphi \colon \mathcal{O}_{C_\sigma, \sigma v} \to R$ specializing the local defining ideal of $\mathcal{N}_\sigma^w$ to I.*

Proof. Combine Theorem 2.24 with Proposition 2.26. □

We clarify the situation with an explicit example. The calculations in this example were carried out in the computer algebra system Macaulay2 [18].

Example 2.28. Let $R = \mathbb{C}[\![t_1, t_2, t_3]\!]$ and $I = (t_1t_2, t_2t_3, t_1^3, t_1^2t_3, t_1t_3^2 - t_2^3, t_3^3)$. The minimal free resolution of R/I has the form

$$0 \to R^2 \to R^7 \to R^6 \to R$$

so I has grade $c = 3$, deviation $d(I) = 3$, and type $t(R/I) = 2$. It is licci:

$$\begin{aligned}
I &\sim (t_1t_2 + t_2t_3, 2t_1^2 + t_1t_3, t_3^3, t_1t_3^2, t_2^3) && \text{by } (t_1t_2 + t_2t_3, t_1^3, -t_2^3 + t_1t_3^2 + t_3^3)\\
&\sim (2t_1t_3 - t_3^2, 4t_1^2 + t_3^2, t_2^2t_3, t_2^3, t_1t_2^2) && \text{by } (2t_1^2 + t_1t_3, t_3^3, t_2^3)\\
&\sim (t_2, t_3^2, t_1t_3, t_1^2) && \text{by } (2t_1t_3 - t_3^2, 4t_1^2 + t_3^2, t_2^3)\\
&\sim (t_3, t_2, t_1) && \text{by } (t_2, t_3^2, t_1^2)\\
&\sim (1) && \text{by } (t_3, t_2, t_1).
\end{aligned}$$

Now we simulate the proof of Theorem 2.27 for this ideal. The diagram associated to $(c, d, t) = (3, 3, 2)$ is $T = E_7$, whose vertices we label as

$$\begin{array}{ccccccccc}
2 & — & 4 & — & 5 & — & 6 & — & 7\\
 & & | & & & & & & \\
 & & 3 & & & & & & \\
 & & | & & & & & & \\
 & & 1 & & & & & &
\end{array}$$

following Bourbaki. Here $F_1 = \mathbb{C}^6$ and $F_1' = \mathbb{C}^5$. Let $V = L(\omega_2)^\vee$ be the 912-dimensional irreducible representation of E_7. We consider the homogeneous space $G/P = E_7/P_2$ embedded inside $\mathbb{P}(L(\omega_2))$ and apply the inductive procedure outlined in the proof of Theorem 2.24.

The starting point is the Borel-fixed point $v \in G/P$. Viewed as an R-point of G/P, this corresponds to the projection $\gamma_0 \colon V \otimes R \to R$ onto the lowest weight space. We display its Plücker coordinates as

$$\gamma_0 = \left[\begin{matrix} 1 & 0 & 0 & \begin{matrix} 0 & 0 & 0 \\ 0 & 0 & \end{matrix} & \cdots \end{matrix}\right]$$

where the top 6 coordinates come from $F_1 \subset V$ and the bottom 5 come from $F_1' \subset V$. In particular the left 3 come from the intersection $F_1 \cap F_1' = \mathbb{C}^3$.

Now we trace the sequence of links in reverse using the actions of $\mathrm{GL}(F_1 \otimes R)$ and $\mathrm{GL}(F_1' \otimes R)$. Precomposing γ_0 by the action of

$$g_1' = \begin{bmatrix} t_3 & t_2 & t_1 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \in \mathrm{GL}(F_1' \otimes R)$$

on V gives

$$\gamma_0 g_1' = \left[\begin{matrix} t_3 & t_2 & t_1 & \begin{matrix} 0 & 0 & 0 \\ 1 & 0 & \end{matrix} & \cdots \end{matrix}\right].$$

Further precomposing by the action of

$$g_1 = \begin{bmatrix} 0 & t_3 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & t_1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \in \mathrm{GL}(F_1 \otimes R)$$

yields

$$\gamma_0 g_1' g_1 = \begin{bmatrix} t_2 & t_3^2 & t_1^2 & \begin{matrix} t_1 \\ -t_1 t_3 \end{matrix} & \begin{matrix} t_3 \\ 0 \end{matrix} & \begin{matrix} 0 \\ \end{matrix} & \cdots \end{bmatrix}.$$

Further precomposing by the action of

$$g_2' = \begin{bmatrix} 0 & 0 & t_2^2 & 1 & 0 \\ -1 & 1 & 0 & 0 & 1 \\ 0 & 4 & 0 & 0 & 0 \\ -2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix} \in \mathrm{GL}(F_1' \otimes R)$$

yields

$$\gamma_0 g_1' g_1 g_2' = \begin{bmatrix} 2t_1t_3 - t_3^2 & 4t_1^2 + t_3^2 & t_2^3 & \begin{matrix} -4t_1t_2^2 - 2t_2^2 t_3 \\ t_2 \end{matrix} & \begin{matrix} 8t_1t_2^2 - 4t_2^2 t_3 \\ t_3^2 \end{matrix} & \begin{matrix} 0 \\ \end{matrix} & \cdots \end{bmatrix}.$$

Further precomposing by the action of

$$g_2 = \begin{bmatrix} 1/2 & -t_1 - t_3/2 & 0 & 1 & 0 & 0 \\ 1/2 & t_3/2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1/2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1/4 \\ 1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \in \mathrm{GL}(F_1 \otimes R)$$

yields

$$\gamma_0 g_1' g_1 g_2' g_2 = \begin{bmatrix} 2t_1^2 + t_1t_3 & t_3^3 & t_2^3 & \begin{matrix} 2t_1t_3 - t_3^2 \\ t_1t_2/2 + t_2t_3/2 \end{matrix} & \begin{matrix} 2t_1t_2^2 + t_2^2 t_3 \\ t_1t_3^2/2 + t_3^3/2 \end{matrix} & \begin{matrix} 2t_1t_2^2 - t_2^2 t_3 \\ \end{matrix} & \cdots \end{bmatrix}.$$

And finally, one last step where we precompose by the action of

$$g_3' = \begin{bmatrix} 0 & t_1/2 - t_3/4 & 0 & 1 & 0 \\ 0 & -1/4 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 1 \\ 2 & 0 & 0 & 0 & 0 \\ 0 & 1/2 & 2 & 0 & 0 \end{bmatrix} \in \mathrm{GL}(F_1' \otimes R)$$

brings us to $\gamma = \gamma_0 g_1' g_1 g_2' g_2 g_3'$ which is

$$\begin{bmatrix} t_1t_2 + t_2t_3 & t_1^3 & -t_2^3 + t_1t_3^2 + t_3^3 & \begin{matrix} -t_1t_2/2 + t_2t_3/2 \\ 2t_1^2 + t_1t_3 \end{matrix} & \begin{matrix} t_1^3 + t_1^2 t_3 \\ t_2^3 \end{matrix} & \begin{matrix} -t_1^2 t_3/2 + t_3^3/2 \\ \end{matrix} & \cdots \end{bmatrix}.$$

We have produced $\gamma\colon V \otimes R \to R$, describing an R-point of G/P, such that $\gamma(F_1 \otimes R) = I$.

The second half is Proposition 2.26, in which we fine-tune this R-point of G/P. Write k for $\mathbb{C}$ viewed as the residue field of R. We find the lowest $\mathrm{GL}(F_1) \times \mathrm{GL}(F_3)$-representation on which $\gamma \otimes k$ is nonzero, which ends up being

$$S_{2,2,2,1,1,1} F_1 \otimes S_{3,1} F_3^* \subset V.$$

The lowest weight of this representation is $\omega = \sigma(-\omega_2)$ where

$$\sigma = s_3s_4s_2s_5s_4s_6s_5s_7s_6s_3s_4s_5s_1s_3s_4s_2$$

is a minimal length representative of its double coset $[\sigma] \in W_{P_3}\backslash W/W_{P_2}$, with $\ell(\sigma) = 16$.

However, it turns out that $\gamma \otimes k$ is zero on the weight space V_ω. We can fix this using a permutation matrix in $\mathrm{GL}(F_1) \times \mathrm{GL}(F_3)$, which adjusts our γ to

$$\begin{bmatrix} t_1t_2 + t_2t_3 & -t_2^3 + t_1t_3^2 + t_3^3 & -t_1t_2/2 + t_2t_3/2 & t_1^3 & t_1^3 + t_1^2t_3 & -t_1^2t_3/2 + t_3^3/2 & \cdots \end{bmatrix}$$

where we have only displayed the 6 coordinates coming from $F_1 \subset V$. Now we have an R-point of the open cell $\sigma C^e \subset G/P$, which we wish to project onto C_σ. The coordinate ring of C_σ is

$$\begin{aligned}\mathbb{C}[C_\sigma] &= \mathrm{Sym}\left(\bigoplus_{\alpha>0,\sigma^{-1}\alpha<0} \mathfrak{g}_\alpha\right)^* \\ &= \mathbb{C}[x_{12,2}, x_{13,2}, x_{23,2}, x_{14,2}, x_{24,2}, x_{34,2}, x_{15,2}, x_{25,2}, x_{35,2}, x_{16,2}, x_{26,2}, x_{36,2} \\ &\qquad y_{1234,12}, y_{1235,12}, y_{1236,12}, z_{123456,2}]\end{aligned}$$

and the defining ideal of the Kazhdan-Lusztig variety $\mathcal{N}_\sigma^w \subset C_\sigma$ is

$$\begin{pmatrix} -x_{35,2}x_{26,2}y_{1234,12} + x_{25,2}x_{36,2}y_{1234,12} + x_{34,2}x_{26,2}y_{1235,12} \\ -x_{24,2}x_{36,2}y_{1235,12} - x_{34,2}x_{25,2}y_{1236,12} + x_{24,2}x_{35,2}y_{1236,12} - x_{23,2}z_{123456,2}, \\ x_{35,2}x_{16,2}y_{1234,12} - x_{15,2}x_{36,2}y_{1234,12} - x_{34,2}x_{16,2}y_{1235,12} \\ +x_{14,2}x_{36,2}y_{1235,12} + x_{34,2}x_{15,2}y_{1236,12} - x_{14,2}x_{35,2}y_{1236,12} + x_{13,2}z_{123456,2}, \\ -x_{25,2}x_{16,2}y_{1234,12} + x_{15,2}x_{26,2}y_{1234,12} + x_{24,2}x_{16,2}y_{1235,12} \\ -x_{14,2}x_{26,2}y_{1235,12} - x_{24,2}x_{15,2}y_{1236,12} + x_{14,2}x_{25,2}y_{1236,12} - x_{12,2}z_{123456,2}, \\ -x_{23,2}x_{16,2}y_{1235,12} + x_{13,2}x_{26,2}y_{1235,12} - x_{12,2}x_{36,2}y_{1235,12} \\ +x_{23,2}x_{15,2}y_{1236,12} - x_{13,2}x_{25,2}y_{1236,12} + x_{12,2}x_{35,2}y_{1236,12}, \\ x_{23,2}x_{16,2}y_{1234,12} - x_{13,2}x_{26,2}y_{1234,12} + x_{12,2}x_{36,2}y_{1234,12} \\ -x_{23,2}x_{14,2}y_{1236,12} + x_{13,2}x_{24,2}y_{1236,12} - x_{12,2}x_{34,2}y_{1236,12}, \\ -x_{23,2}x_{15,2}y_{1234,12} + x_{13,2}x_{25,2}y_{1234,12} - x_{12,2}x_{35,2}y_{1234,12} \\ +x_{23,2}x_{14,2}y_{1235,12} - x_{13,2}x_{24,2}y_{1235,12} + x_{12,2}x_{34,2}y_{1235,12} \end{pmatrix} \subset \mathbb{C}[C_\sigma] \tag{2.4}$$

Following the proof of Proposition 2.26, we parametrize the open cell using $\pi_\omega \exp(Y^+)\exp(Y^-)$ and equate to γ (after rescaling projective coordinates as needed), obtaining a specialization $\mathbb{C}[\sigma C^e] \to R$ where

$$\begin{aligned} x_{12,2} &\mapsto -t_1/2 + t_3/2 & y_{1234,12} &\mapsto t_1 + t_3 \\ x_{13,2} &\mapsto t_2^2 & y_{1235,12} &\mapsto -t_1 \\ x_{16,2} &\mapsto -t_3 & z_{123456,2} &\mapsto 2t_2 \\ x_{23,2} &\mapsto t_1 + t_3 & & \\ x_{35,2} &\mapsto 2t_3 & & \\ x_{36,2} &\mapsto 4t_1 + 2t_3 & & \end{aligned}$$

and all other coordinates on C_σ are sent to zero. The remaining coordinates on σC^e are the variables appearing in Y^-, which we discard in our projection to C_σ.

Performing the above substitutions on (2.4) yields

$$I = (-2t_1t_2 - 2t_2t_3, \quad 2t_2^3 - 2t_1t_3^2 - 2t_3^3, \quad t_1t_2 - t_2t_3, \quad -2t_1^3, \quad -2t_1^3 - 2t_1^2t_3, \quad t_1^2t_3 - t_3^3). \tag{2.5}$$

Remark 2.29. Our naming scheme for variables in $\mathbb{C}[C_\sigma]$ comes from

$$\bigoplus_{\alpha>0,\sigma^{-1}\alpha<0} \mathfrak{g}_\alpha \subseteq \mathfrak{n}_3^+ = (\bigwedge^2 F_1 \otimes F_3^*) \oplus (\bigwedge^4 F_1 \otimes \bigwedge^2 F_3^*) \oplus (\bigwedge^6 F_1 \otimes S_{2,1}F_3^*).$$

There is algebraic significance to the "fine-tuning" half of the procedure. In Example 2.28, the algebra structure on $\mathrm{Tor}_*(R/I,k)$ is of class $\mathbf{G}(3)$. Originally it was conjectured in [1] that the Tor algebra structures $\mathbf{G}(r)$ were only exhibited by Gorenstein ideals, but non-Gorenstein examples were found in [13]. We refer the reader to these references for further details.

There is a distinguished k^3 inside of $\mathrm{Tor}_1(R/I,k)$ given by the kernel of the multiplication map

$$\mathrm{Tor}_1(R/I,k) \to \mathrm{Tor}_2(R/I,k)^* \otimes \mathrm{Tor}_3(R/I,k).$$

With the choice of generators in (2.5), the k^3 is spanned by the images of the last three generators $-2t_1^3$, $-2t_1^3 - 2t_1^2t_3$, and $t_1^2t_3 - t_3^3$ in $I/\mathfrak{m}I \cong \mathrm{Tor}_1(R/I,k)$. Thus the procedure has isolated the more "special" generators in a sense.

Remark 2.30. In fact, we will see that all grade 3 licci ideals come from explicit generic examples in Theorem 5.11. The algebra structures on these generic examples can be inferred from §4.4; hence for licci ideals it is possible to reduce Avramov's realizability question of Tor algebra structures (c.f. [1, Question 3.8]) to combinatorics of Weyl groups. The technical point is then to prove that no new classes are introduced by non-licci perfect ideals. We hope to address these ideas in an upcoming paper.

On the other hand, for a grade 3 Gorenstein ideal minimally generated by n elements, any minimal generating set can be realized as the submaximal pfaffians of a skew matrix. This reflects that the adjustment portion of the procedure is unnecessary in this case, unlike Example 2.28. The representation $L(\omega_{x_1})^\vee$ is the half-spin representation

$$L(\omega_{x_1})^\vee = F_1 \oplus (\bigwedge^3 F_1 \otimes F_3^*) \oplus \cdots \oplus (\bigwedge^n F_1 \otimes S_{(n-1)/2}F_3^*)$$

of $\mathfrak{so}_{2n}$, and after producing an R-point γ of G/P following Theorem 2.24, the only $\mathrm{GL}(F_1)\times\mathrm{GL}(F_3)$-representation on which $\gamma \otimes k \neq 0$ is the highest weight line $\bigwedge^n F_1 \otimes S_{(n-1)/2}F_3^*$. Thus the desired conclusions of Proposition 2.26 are already satisfied without any additional work.

2.4 Herzog classes of licci ideals

For this part, we adopt the following strengthening of Assumption 2.1. We maintain Assumption 2.17.

Assumption 2.31. k is an algebraically closed field of characteristic zero and every ring is a quotient of a power series ring $R = k[\![\{X\}]\!]$ in finitely many variables $\{X\}$. By G/P, we mean the projective variety obtained after base-change to $\operatorname{Spec} k$.

Given two such k-algebras A_1 and A_2, write $A_1 \sim A_2$ when they admit a common deformation A, i.e. $A_1 \cong A/(\underline{\alpha})$ and $A_2 \cong A/(\underline{\alpha}')$ where $\underline{\alpha}$ and $\underline{\alpha}'$ are regular sequences in A. Herzog defined and analyzed this relation in [22] and showed that it is an equivalence relation on the class of strongly unobstructed algebras R. We will not define this condition; for our purposes it is enough to know that reduced licci algebras belong to this class by [10].

Definition 2.32. If $R = k[\![\{X\}]\!]$ and $I \subset R$ is a radical licci ideal, the *Herzog class* $[A]$ of $A = R/I$ is the set of all A' such that $A \sim A'$.

Corollary 2.33. *Let w be as in* (2.3) *and let σ be a minimal length representative of its double coset $[\sigma] \in W_{P_{z_1}} \backslash W / W_{P_{x_{c-2}}}$. Assume that $[\sigma] \neq [e]$ where $e \in W$ is the identity. Consider the function sending $[\sigma]$ to the Herzog class of the completed local ring $\widehat{\mathcal{O}}_{\mathcal{N}_\sigma^w,\sigma v}$ of the Kazhdan-Lusztig variety $\mathcal{N}_\sigma^w$ at its torus-fixed point σv. Then this defines a surjection*

$$W_{P_{z_1}} \backslash W / W_{P_{x_{c-2}}} - [e]$$
$$\downarrow$$
$$\left\{ \begin{matrix} \textit{Herzog classes of reduced licci algebras } A = k[\![\{X\}]\!]/I \textit{ with embedding} \\ \textit{codimension at most } c\textit{, deviation } d(I) \leq d\textit{, and type } t(A) \leq t \end{matrix} \right\}$$

The ring $\widehat{\mathcal{O}}_{C_\sigma,\sigma v}$ is isomorphic to a power series ring over k in $\ell(\sigma)$ variables.

Proof. This map is well-defined by Corollary 2.25, since $\sigma \geq w$ if $[\sigma] \neq [e]$.

To prove the map is surjective, suppose R is a power series ring over k and $I \subset R$ is a radical licci ideal with codimension $c' \leq c$, deviation $d(I) \leq d$, and type $t(R/I) \leq t$. By adjoining $c - c'$ variables to R and adding those variables to I, we increase the codimension to c while leaving R/I unchanged. Let σ be as in Theorem 2.27 applied to $I \subset R$. Then $\widehat{\mathcal{O}}_{\mathcal{N}_\sigma^w,\sigma v}$ is in the same Herzog class as R/I. To see this, let $A = \widehat{\mathcal{O}}_{\mathcal{N}_\sigma^w,\sigma v} \hat{\otimes} R$. This is a trivial deformation of $\widehat{\mathcal{O}}_{\mathcal{N}_\sigma^w,\sigma v}$ and the homomorphism $\varphi\colon \mathcal{O}_{C_\sigma,\sigma v} \to R$ from Theorem 2.27 determines a regular sequence consisting of $X - \varphi(X)$ for all $\ell(\sigma)$ variables X in $\mathcal{O}_{C_\sigma,\sigma v}$. The quotient of A by this regular sequence recovers R. □

So far everything is under Assumption 2.17. In particular, we conclude that there are finitely many Herzog classes, since the set $W_{P_{z_1}} \backslash W / W_{P_{x_{c-2}}}$ is finite. Now we are ready to state the conjectural answer to Question 1.2 in Chapter 1.

Conjecture 2.34. *The map is a bijection for all triples (c, d, t) with $c \geq 2$, $d \geq 0$, and $t \geq 1$.*

For each ADE triple (c, d, t), let

$$N(c, d, t) = |W_{P_{z_1}} \backslash W / W_{P_{x_{c-2}}}|$$

and define $N(c, d, 0) = N(c, -1, t) = 1$. Then Conjecture 2.34 would imply that there are

$$N(c, d, t) - N(c, d-1, t) - N(c, d, t-1) + N(c, d-1, t-1)$$

Herzog classes of licci algebras of embedding codimension at most c, deviation exactly d, and type exactly t. We list these counts below, which have been found by computer for the exceptional types.

- $(2, d, t)$: 1 if $t = d + 1 \geq 1$ and 0 otherwise.
- $(c, 0, t)$: 1 if $t = 1$ and 0 otherwise.
- $(3, d, 1)$: 1 if d even and 0 if d odd.
- $(3, 1, t)$: 1 if $t \geq 2$.
- $(c, 1, 1)$: 0
- $(3, 2, 2)$: 2
- $(3, 2, 3)$: 11
- $(3, 2, 4)$: 90
- $(3, 3, 2)$: 7
- $(3, 4, 2)$: 49
- $(4, 1, 2)$: 2
- $(4, 1, 3)$: 4
- $(4, 1, 4)$: 27
- $(4, 2, 1)$: 1
- $(4, 3, 1)$: 2
- $(4, 4, 1)$: 10
- $(5, 1, 2)$: 3
- $(5, 2, 1)$: 2
- $(6, 1, 2)$: 8
- $(6, 2, 1)$: 5

Furthermore, Conjecture 2.34 would imply that there are infinitely many Herzog classes when (c, d, t) is not on the list above.

The weaknesses of our current approach are as follows. For one, we have assumed throughout that T is of finite type. If Theorem 2.16 remains true in the general Kac-Moody setting, then these results readily generalize to the infinite case, with some additional care for dealing with ind-varieties.

Note that the proof of Theorem 2.24 requires choosing a sequence of links from I to a complete intersection. Consequently, it is inadequate for analyzing perfect ideals which we do not *a priori* know to be licci, and even for licci ideals we cannot determine whether the double coset $[\sigma]$ in

Theorem 2.27 is intrinsically determined, or potentially dependent on the choice of links used. This is the obstacle to proving injectivity of the map in Corollary 2.33.

Furthermore, the classical structure theory of grade 2 perfect ideals and grade 3 Gorenstein ideals includes descriptions of their free resolutions. On the other hand, Theorem 2.27 only describes the ideal generators for other families of licci ideals.

In the following chapters, we will show that all of the preceding can be remedied for $c = 3$. In the next chapter, we construct a family of free resolutions resolving the coordinate rings of $\mathcal{N}_\sigma^w$ for $c = 3$. Moreover, we will work in the general Kac-Moody setting without the assumption that $(3, d, t)$ is an ADE triple. Later we will see that these yield the generic free resolutions of all grade 3 licci ideals in Theorem 5.11.

After developing the machinery of Weyman's generic free resolutions of length 3 in Chapters 4 and 5, we prove that σ in Theorem 2.27 is uniquely determined by I and consequently Conjecture 2.34 is true. Finally, the proof of Conjecture 1.4 for ADE triples $(3, d, t)$ in Theorem 5.14 allows us to replace "licci" by "perfect" in Theorem 2.27 for those triples.

Chapter 3

A family of length three resolutions

For the remainder of this thesis, we will be concerned with resolutions of length 3. For $c = 3$, Sam and Weyman used linkage to infer properties about free resolutions of the coordinate ring $\mathbb{C}[\mathcal{N}_\sigma^w]$ in [44]. Here $\mathcal{N}_\sigma^w$ denotes the Kazhdan-Lusztig varieties considered in the preceding chapter. In [39] these free resolutions were revisited, and their differentials made more explicit. There, some resolutions of grade 4 Gorenstein ideals are also studied, but we will not discuss that portion here.

Both of these papers only discuss the case when $(3, d, t)$ is an ADE triple, and furthermore they take C_σ to be the big open cell $w_0 C^e$, where $w_0 \in W$ is the longest element. The particular interest in this case was more so a matter of perspective, and not the result of any serious technical limitation. We will work more generally without these assumptions.

We begin by giving the explicit construction of the differentials of $\mathbb{F}$ in §3.1. Then we discuss the multigrading on $\mathbb{F}$ in §3.2 corresponding to the torus action on $\mathcal{N}_\sigma^w$, and show how it can be coarsened to the $\mathbb{Z}$-grading found in [44]. Finally, we prove that these are acyclic complexes in §3.3 using the Buchsbaum-Eisenbud acyclicity criterion. The content of this chapter is mainly based on [39], but our treatment here aims to give a more general overview whereas the cited paper focused on making the ADE cases explicit. As in Chapter 2, the construction here was originally motivated by the study of Weyman's generic free resolutions of length three, but our treatment will be independent of it.

3.1 Construction of the resolutions

Let $r_1 \geq 1$, $r_2 \geq 2$, and $r_3 \geq 1$ be positive integers. Define $f_0 = r_1$, $f_1 = r_1 + r_2$, $f_2 = r_2 + r_3$, and $f_3 = r_3$. In this section we define the differentials of certain free resolutions $\mathbb{F}$ of the form

$$\mathbb{F}\colon 0 \to R^{f_3} \xrightarrow{d_3} R^{f_2} \xrightarrow{d_2} R^{f_1} \xrightarrow{d_1} R^{f_0}$$

over polynomial rings R. We defer the proofs that $d^2 = 0$ and that $\mathbb{F}$ is acyclic to §3.3. The base ring R and the differentials d_i depend on an additional parameter σ, which we now explain.

3.1.1 The coordinate ring R of C_σ

Fix the graph $T = T_{r_1+1, r_2-1, r_3+1}$ with labels

$$
\begin{array}{ccccccccccccc}
x_{r_1} & — & \cdots & — & x_1 & — & u & — & y_1 & — & \cdots & — & y_{r_2-2} \\
 & & & & & & | & & & & & & \\
 & & & & & & z_1 & & & & & & \\
 & & & & & & | & & & & & & \\
 & & & & & & \vdots & & & & & & \\
 & & & & & & | & & & & & & \\
 & & & & & & z_{r_3} & & & & & &
\end{array}
$$

and use the setup and notation from §2.2.2. Let σ be a minimal length representative of its double coset $[\sigma] \in W_{P_{z_1}} \backslash W / W_{P_{x_1}}$. The ring R will be the coordinate ring of the Schubert cell C_σ inside of $G/P_{x_1}^+$, which we henceforth abbreviate as G/P. Explicitly, if we let

$$\mathfrak{n}_\sigma = \bigoplus_{\alpha > 0, \sigma^{-1}\alpha < 0} \mathfrak{g}_\alpha$$

then $R = \mathrm{Sym}(\mathfrak{n}_\sigma)^*$. Note that even if $\mathfrak{g}$ is infinite-dimensional, in which case G/P is an ind-variety, this cell C_σ is a finite-dimensional affine space $\mathbb{A}^{\ell(\sigma)}$. The variables corresponding to coordinates on $\mathfrak{g}_\alpha$ are given multidegree $-\alpha$. In this manner, the ring R has a grading by the root lattice $Q = \bigoplus_{i \in T} \mathbb{Z}\alpha_i$, where each variable is "negatively graded" in the sense that all multidegrees are nonpositive.

We assumed σ to be a minimal length representative of its double coset, which guarantees that if $\alpha > 0$ and $\sigma^{-1}\alpha < 0$ for some root α, then $\alpha >_{z_1} 0$ and $\sigma^{-1}\alpha <_{x_1} 0$. The former implies that if we coarsen the multigrading to a $\mathbb{Z}$-grading via the projection $Q \to \mathbb{Z}\alpha_{z_1}$, then all variables have strictly negative degree.

Let Y denote a generic element of $\mathfrak{n}_\sigma$, with coefficients in R. In other words, if $\{a_i\}$ is a basis of $\mathfrak{n}_\sigma$ and $\{a_i'\}$ its dual basis,

$$Y = \sum a_i \otimes a_i' \in \mathfrak{n}_\sigma \otimes \mathfrak{n}_\sigma^* \subset \mathfrak{n}_\sigma \otimes R.$$

As usual, let $v \in G/P$ be the point P/P. Consider the Plücker embedding of G/P into $\mathbb{P}(L(\omega_{x_1}))$. The Plücker coordinates reside in $L(\omega_{x_1})^\vee$. On the cell C_σ, these Plücker coordinates restrict to polynomials in R via

$$L(\omega_{x_1})^\vee \otimes R \xrightarrow{(\exp(Y)\sigma)^{-1}} L(\omega_{x_1})^\vee \otimes R \to \mathbb{C} \otimes R$$

where the latter map is projection onto the lowest weight space of $L(\omega_{x_1})^\vee$. Note that the action of $(\exp(Y)\sigma)^{-1}$ on the dual $L(\omega_{x_1})^\vee$ is precomposition by $\exp(Y)\sigma$ on $L(\omega_{x_1})$. Taking the dual, we obtain

$$\mathbb{C} \otimes R \to L(\omega_{x_1}) \otimes R \xrightarrow{\exp(Y)\sigma} L(\omega_{x_1}) \otimes R$$

where the first map is the highest weight line of $L(\omega_{x_1})$ corresponding to $v \in G/P \hookrightarrow \mathbb{P}(L(\omega_{x_1}))$. Hence this is just the parametrization of C_σ as $\exp(\mathfrak{n})\sigma v$.

3.1.2 The differentials of $\mathbb{F}$

Let $F_j = \mathbb{C}^{f_j}$ for $0 \le j \le 3$. Following Example 2.15, we view the Lie algebras $\mathfrak{sl}(F_j)$ as subalgebras of $\mathfrak{g}$:

- $\mathfrak{sl}(F_0)$ corresponds to the ordered sequence of vertices $x_2, \dots, x_{r_1}$,
- $\mathfrak{sl}(F_1)$ corresponds to the ordered sequence of vertices $y_{r_2-2}, \dots, y_1, u, x_1, \dots, x_{r_1}$,
- $\mathfrak{sl}(F_2)$ corresponds to the ordered sequence of vertices $y_{r_2-2}, \dots, y_1, u, z_1, \dots, z_{r_3}$, and
- $\mathfrak{sl}(F_3)$ corresponds to the ordered sequence of vertices $z_2, \dots, z_{r_3}$.

In particular, $\mathfrak{g}^{(x_1)} = \mathfrak{sl}(F_0) \times \mathfrak{sl}(F_2)$ and $\mathfrak{g}^{(z_1)} = \mathfrak{sl}(F_1) \times \mathfrak{sl}(F_3)$.

Let $\omega = \sum_{i \in T} c_i \omega_i$ be a dominant integral weight of $\mathfrak{g}$ (i.e. $c_i \ge 0$ for all i) and let $L(\omega)$ be the associated irreducible representation with highest weight ω. For $t \in T$, the top graded component of $L(\omega)$ in the t-grading is the irreducible representation of $\mathfrak{g}^{(t)}$ with highest weight $\sum_{i \neq t} c_i \omega_i$. Applying this for $t \in \{x_1, z_1\}$ to the three fundamental representations associated with the extremal vertices x_{r_1}, y_{r_2-2}, and z_{r_3}, we obtain

$$\begin{aligned} L(\omega_{x_{r_1}}) &= \cdots \oplus F_0^* \text{ in the } x_1\text{-grading and} \\ &= \cdots \oplus F_1^* \text{ in the } z_1\text{-grading,} \\ L(\omega_{y_{r_2-2}}) &= \cdots \oplus F_2 \text{ in the } x_1\text{-grading and} \\ &= \cdots \oplus F_1 \text{ in the } z_1\text{-grading,} \\ L(\omega_{z_{r_3}}) &= \cdots \oplus F_2^* \text{ in the } x_1\text{-grading and} \\ &= \cdots \oplus F_3^* \text{ in the } z_1\text{-grading.} \end{aligned}$$

To be precise, the identification of each $\mathfrak{g}^{(t)}$-representation is only up to a nonzero scalar in $\mathbb{C}$. We fix such identifications; any other choice will only alter the subsequent definitions of d_i by scaling.

Using the above, we define the differential d_1 to be dual to

$$F_0^* \otimes R \to L(\omega_{x_{r_1}}) \otimes R \xrightarrow{\exp(Y)\sigma} L(\omega_{x_{r_1}}) \otimes R \to F_1^* \otimes R,$$

the differential d_2 to be

$$F_2 \otimes R \to L(\omega_{y_{r_2-2}}) \otimes R \xrightarrow{\exp(Y)\sigma} L(\omega_{y_{r_2-2}}) \otimes R \to F_1 \otimes R,$$

and the differential d_3 to be dual to

$$F_2^* \otimes R \to L(\omega_{z_{r_3}}) \otimes R \xrightarrow{\exp(Y)\sigma} L(\omega_{z_{r_3}}) \otimes R \to F_3^* \otimes R.$$

In each composite, the first map is the inclusion of the top z_1-graded component, and the last map is the projection onto the top x_1-graded component. We define $\mathbb{F}$ as

$$0 \to F_3 \otimes R \xrightarrow{d_3} F_2 \otimes R \xrightarrow{d_2} F_1 \otimes R \xrightarrow{d_1} F_0 \otimes R.$$

Example 3.1. If $\sigma = e \in W$ is the identity, then the Schubert cell C_σ is a point, $R = \mathbb{C}$, and $Y = 0$. The differentials d_1^*, d_2, and d_3^* are

$$F_0^* \hookrightarrow L(\omega_{x_{r_1}}) \twoheadrightarrow F_1^*$$
$$F_2 \hookrightarrow L(\omega_{y_{r_2-2}}) \twoheadrightarrow F_1$$
$$F_2^* \hookrightarrow L(\omega_{z_{r_3}}) \twoheadrightarrow F_3^*$$

and $\mathbb{F}$ is a split exact complex of $\mathbb{C}$-vector spaces.

Example 3.2. Let $n \geq 3$ be an odd integer. We apply this construction with the parameters $r_1 = 1$, $r_2 = n-1$, $r_3 = 1$, and demonstrate that it recovers the well-known Buchsbaum-Eisenbud resolution from [7] for the $(k-1) \times (k-1)$ pfaffians of a generic $k \times k$ matrix, where $k \leq n$ is odd. As we will see, the value of k depends on the choice of the parameter σ. In particular, $\sigma = e$ corresponds to $k = 1$ and we obtain a resolution of the unit ideal—i.e. a split complex—as was demonstrated in the previous example.

The diagram T is D_n, and we label the vertices as

$$\begin{array}{ccccccccc} n-1 & — & n-2 & — & n-3 & — & \cdots & — & 1 \\ & & | & & & & & & \\ & & n & & & & & & \end{array}$$

following Bourbaki.

The simple Lie algebra associated to this Dynkin diagram is $\mathfrak{so}_{2n}$. Using the vertex n, we may decompose the Lie algebra as

$$\mathfrak{so}_{2n} = \bigwedge^2 F_1^* \oplus \mathfrak{gl}(F_1) \oplus \bigwedge^2 F_1$$

where $F_1 = \mathbb{C}^n$. Concretely, elements of $\mathfrak{so}_{2n}$ are skew endomorphisms of the self-dual space $F_1 \oplus F_1^*$.

On the other hand, we could also use the vertex $n-1$ instead of the vertex n. With this perspective we get another decomposition of the standard representation as $F_2 \oplus F_2^*$ for another n-plane F_2, where $F_1 \cap F_2$ is $(n-1)$-dimensional (corresponding to the overlapping A_{n-2} diagram consisting of the vertices 1 through $n-2$). Let $e_1, \ldots, e_n$ be a basis for F_1 and $e_1', \ldots, e_n'$ its dual basis. Then one can arrange for $e_1, \ldots, e_{n-1}, e_n'$ to be a basis for F_2 and $e_1', \ldots, e_{n-1}', e_n$ its dual.

The space $F_2 \oplus F_2^*$ comes equipped with an evident quadratic form. The subspace F_2 is isotropic; its $\mathrm{SO}(2n)$-orbit in $\mathrm{Gr}(n, 2n)$ gives the *isotropic/orthogonal Grassmannian* $\mathrm{OG}(n, 2n)$. This is one of two isomorphic components comprising the entire set of isotropic n-planes. (Taking $n = 2$ as an example, there are two families of lines on a smooth quadric surface in $\mathbb{P}^3$.)

Here the Borel-fixed point $v \in \mathrm{OG}(n, 2n)$ is represented by F_2. Its stabilizer $P_{n-1}^+ \subset \mathrm{SO}(2n)$ corresponds to the subalgebra $\mathfrak{gl}(F_2) \oplus \bigwedge^2 F_2$, and consists of all automorphisms of $F_2 \oplus F_2^*$ of the form

$$(a, \varphi) \mapsto (g(a + f(\varphi)), \varphi \circ g^{-1})$$

where $g \in \mathrm{GL}(F_2)$ is arbitrary and $f \colon F_2^* \to F_2$ is skew.

We analogously have $P_n^- \subset SO(2n)$ consisting of all automorphisms of $F_1 \oplus F_1^*$ of the form

$$(a, \varphi) \mapsto (ga, (\varphi + f(a)) \circ g^{-1})$$

where $g \in GL(F_1)$ is arbitrary and $f\colon F_1 \to F_1^*$ is skew.

The orthogonal Grassmannian G/P_{n-1}^+ decomposes into $\lceil \frac{n}{2} \rceil$ orbits under the action of P_n^-. The orbit containing a given isotropic n-plane F is determined by $\dim(F \cap F_1^*)$, which can be any odd[1] number from 1 to n. The various choices of σ correspond to the following isotropic n-planes, which are representatives of each P_n^--orbit:

$$\begin{aligned} v &= \operatorname{span}(e_1, \dots, e_{n-1}, e_n') = F_2 \\ \sigma_1 v &= \operatorname{span}(e_1, \dots, e_{n-3}, e_n', e_{n-1}', e_{n-2}') \\ &\vdots \\ \sigma_{(n-1)/2} v &= \operatorname{span}(e_n', \dots, e_1') = F_1^* \end{aligned}$$

We have

$$G/P_{n-1}^+ = \coprod_{i=0}^{(n-1)/2} P_n^- \sigma_i v = P_n^- v \amalg X$$

where the complement of the open orbit $P_n^- v$ is a Schubert variety X consisting of all isotropic F such that $\dim(F \cap F_1^*) \geq 3$.

Using the ordered basis $e_1, e_2, \dots, e_n, e_n', \dots, e_2', e_1'$ for the ambient space, the cell C_{σ_i} can be parametrized as

$$C_{\sigma_i} = \left\{ \begin{bmatrix} I_{n-1-2i} & 0 & 0 & 0 \\ 0 & Y_{2i+1} & I_{2i+1} & 0 \end{bmatrix} \right\}$$

where I_k is a $k \times k$ identity matrix, and Y_{2i+1} is a generic $(2i+1) \times (2i+1)$ skew matrix, but written so that it is antisymmetric across the antidiagonal (instead of the diagonal).

Let's examine the specific example $\sigma = \sigma_{(n-1)/2}$, so that C_σ is parametrized as

$$C_\sigma = \left\{ \begin{bmatrix} Y_n & I_n \end{bmatrix} \right\}$$

where $Y = Y_n$ is a generic $n \times n$ skew matrix. The left half of this block matrix is the dual of

$$F_2 \otimes R \hookrightarrow L(\omega_1) \otimes R \xrightarrow{(\exp Y)\sigma} L(\omega_1) \otimes R \to F_1 \otimes R.$$

Next, consider the Plücker embedding $G/P_{n-1}^+ \hookrightarrow \mathbb{P}(L(\omega_{n-1}))$, where $L(\omega_{n-1})$ is one of the half-spin representations of $\mathfrak{so}_{2n}$. The coordinates in this projective space are given by $L(\omega_{n-1})^\vee$, which has a $\mathbb{Z}$-grading induced by $n \in D_n$ as follows:

$$L(\omega_{n-1})^\vee = F_1 \oplus \bigwedge^3 F_1 \oplus \bigwedge^5 F_1 \oplus \cdots$$

Its symmetric square $S_2 L(\omega_{n-1})$ contains the irreducible representation $L(2\omega_{n-1})$, which is also present in $\bigwedge^n L(\omega_1)$. This reflects the fact that the Plücker coordinates on $\mathbb{P}(L(\omega_{n-1}))$ are square-roots of certain maximal minors of $\begin{bmatrix} Y_n & I_n \end{bmatrix}$; namely the coordinates in $\bigwedge^j F_1$ are square-roots of minors involving $n-j$ columns from the left block and the complementary j columns from the right block.

[1] The other component of the moduli of isotropic n-planes contains those with even-dimensional intersection with F_1^*.

So for the differential d_1, whose dual is the composite

$$\mathbb{C} \otimes R \hookrightarrow L(\omega_{n-1}) \otimes R \xrightarrow{(\exp Y)\sigma} L(\omega_{n-1}) \otimes R \to F_1^* \otimes R,$$

we recover the $(n-1) \times (n-1)$ pfaffians of Y. A similar calculation shows that d_3 consists of these pfaffians as well.

3.2 Multigrading by the root lattice

From our setup, the ring R is graded by the root lattice $Q = \bigoplus_{t \in T} \mathbb{Z}\alpha_t$. We show that, by giving the free modules in $\mathbb{F}$ appropriate multidegrees, we can arrange for the differentials d_i to be homogeneous of degree zero. This grading can be inferred combinatorially without explicitly knowing $\exp Y$.

3.2.1 Q-grading on $\mathbb{F}$

From highest to lowest, the sequence of weights in F_0^* is obtained by applying the reflections $s_{x_{r_1}}, \ldots, s_{x_2}$ sequentially to $\omega_{x_{r_1}}$:

$$Q_0' \coloneqq (\omega_{x_{r_1}}, \omega_{x_{r_1-1}} - \omega_{x_{r_1}}, \ldots, \omega_{x_1} - \omega_{x_2}).$$

The decreasing sequence of weights in F_1^* is obtained by applying the reflections $s_{x_{r_1}}, \ldots, s_u, \ldots, s_{y_{r_2-2}}$ sequentially to $\omega_{x_{r_1}}$:

$$Q_1' \coloneqq (\omega_{x_{r_1}}, \omega_{x_{r_1-1}} - \omega_{x_{r_1}}, \ldots, \omega_{y_1} - \omega_{y_2}, \omega_u - \omega_{y_1}, \omega_{z_1} + \omega_{y_1} - \omega_u, \ldots, \omega_{z_1} - \omega_{y_{r_2-2}}).$$

Similarly the decreasing sequence of weights in $F_2 \subset L(\omega_{y_{r_2-2}})$ is

$$Q_2 \coloneqq (\omega_{y_{r_2-2}}, \omega_{y_{r_2-3}} - \omega_{y_{r_2-2}}, \ldots, \omega_u - \omega_{y_1}, \omega_{x_1} + \omega_{z_1} - \omega_u, \ldots, \omega_{x_1} - \omega_{z_{r_3}})$$

and the decreasing sequence of weights in $F_1 \subset L(\omega_{y_{r_2-2}})$,is

$$Q_1 \coloneqq (\omega_{y_{r_2-2}}, \omega_{y_{r_2-3}} - \omega_{y_{r_2-2}}, \ldots, \omega_u - \omega_{y_1}, \omega_{z_1} + \omega_{x_1} - \omega_u, \ldots, \omega_{z_1} - \omega_{x_{r_1}}).$$

Finally the decreasing sequence of weights in $F_2^* \subset L(\omega_{z_{r_3}})$ is

$$Q_2' \coloneqq (\omega_{z_{r_3}}, \omega_{z_{r_3-1}} - \omega_{z_{r_3}}, \ldots, \omega_{z_1} - \omega_{z_2}, \omega_u - \omega_{z_1}, \omega_{x_1} + \omega_{y_1} - \omega_u, \ldots, \omega_{x_1} - \omega_{y_{r_2-2}})$$

and the decreasing sequence of weights in $F_3^* \subset L(\omega_{z_{r_3}})$ is

$$Q_3' \coloneqq (\omega_{z_{r_3}}, \omega_{z_{r_3-1}} - \omega_{z_{r_3}}, \ldots, \omega_{z_1} - \omega_{z_2}).$$

With our grading on R and our definition of Y, the automorphism $\exp Y$ of each representation is homogeneous of degree zero by construction. It is the action of σ which does not respect the grading.

To work around this, we can alternatively view the maps d_1^*, d_2, and d_3^* as

$$\begin{aligned} \sigma F_0^* \otimes R \hookrightarrow L(\omega_{x_{r_1}}) \otimes R &\xrightarrow{\exp Y} L(\omega_{x_{r_1}}) \otimes R \twoheadrightarrow F_1^* \otimes R, \\ \sigma F_2 \otimes R \hookrightarrow L(\omega_{y_{r_2-2}}) \otimes R &\xrightarrow{\exp Y} L(\omega_{y_{r_2-2}}) \otimes R \twoheadrightarrow F_1 \otimes R, \\ \sigma F_2^* \otimes R \hookrightarrow L(\omega_{z_{r_3}}) \otimes R &\xrightarrow{\exp Y} L(\omega_{z_{r_3}}) \otimes R \twoheadrightarrow F_3^* \otimes R. \end{aligned} \tag{3.1}$$

Now all the maps in each composite are homogeneous of degree zero. In the following theorem, when we say to take F_i as being generated in a sequence of degrees $(\lambda_1, \ldots, \lambda_{f_i})$, we mean this in the decreasing weight order on F_i. So for instance, the highest weight vector of F_i is in degree λ_1 and the lowest weight vector of F_i is in degree λ_{f_i}.

In the following statement, if $\underline{\lambda}$ is a sequence, $\operatorname{rev}(\underline{\lambda})$ means its reverse, $\underline{\lambda} + \lambda$ means to add λ to each term, $\sigma\underline{\lambda}$ means to apply σ to each term, and $-\underline{\lambda}$ means to multiply each term by -1.

Theorem 3.3. *Take $F_0 \otimes R$ as being generated in degrees*

$$-\sigma \operatorname{rev}(Q_0') + \sigma\omega_{x_{r_1}},$$

$F_1 \otimes R$ generated in degrees

$$-\operatorname{rev}(Q_1') + \sigma\omega_{x_{r_1}}, \text{ or equivalently } Q_1 - \omega_{z_1} + \sigma\omega_{x_{r_1}},$$

$F_2 \otimes R$ generated in degrees

$$\sigma Q_2 - \omega_{z_1} + \sigma\omega_{x_{r_1}}, \text{ or equivalently } -\sigma \operatorname{rev}(Q_2') + \sigma\omega_{x_1} - \omega_{z_1} + \sigma\omega_{x_{r_1}},$$

and $F_3 \otimes R$ generated in degrees

$$-\sigma \operatorname{rev}(Q_3') + \sigma\omega_{x_1} - \omega_{z_1} + \sigma\omega_{x_{r_1}}.$$

Then the differentials of

$$\mathbb{F}\colon F_3 \otimes R \xrightarrow{d_3} F_2 \otimes R \xrightarrow{d_2} F_1 \otimes R \xrightarrow{d_1} F_0 \otimes R$$

as defined in §3.1.2 are homogeneous of degree zero.

Proof. This follows by stitching together the gradings in (3.1). The overall shift by $\sigma\omega_{x_{r_1}}$ is somewhat arbitrary; it is only for the purpose of having one generator of $F_0 \otimes R$ in multidegree zero. All weights in a given highest weight representation differ from one another by elements of the root lattice Q, so this overall shift ensures that the multidegrees lie in the root lattice Q instead of some affine translate thereof. Furthermore if $r_1 = 1$ we have $F_0 \otimes R = R$ in multidegree zero. □

To figure out the explicit representations of these weights in $Q = \bigoplus_{i \in T} \mathbb{Z}\alpha_i \cong \mathbb{Z}^n$, where $n = r_1 + r_2 + r_3 - 1$, we can use the Cartan matrix A. Since

$$\alpha_i = \sum_j A_{i,j}\omega_j,$$

writing out the multidegrees as linear combinations of the fundamental weights and then multiplying the coefficients by A^{-1} yields our desired $\mathbb{Z}^n$-multidegrees.

Remark 3.4. If T is one of the affine Dynkin diagrams $E_n^{(1)}$ then the Cartan matrix A is not invertible. However, we can simply enlarge the diagram by increasing one of the parameters r_i, keeping σ the same. Then we are no longer in affine type, and the output $\mathbb{F}$ is only altered by the addition of a split exact part corresponding to the parameter r_i that was increased.

3.2.2 Symmetry of x and z arms

The roles played by the left and bottom arms are symmetric in our construction. Given $\sigma \in W$ a minimal length representative of its double coset $[\sigma] \in W_{P_{z_1}} \backslash W / W_{P_{x_1}}$, its inverse $\sigma^{-1} \in W$ is then a minimal length representative of $[\sigma^{-1}] \in W_{P_{x_1}} \backslash W / W_{P_{z_1}}$. Hence we can do the same construction of §3.1 interchanging the roles of the x and z arms. Let

$$\mathfrak{n}_{\sigma^{-1}} = \bigoplus_{\alpha>0, \sigma\alpha<0} \mathfrak{g}_\alpha.$$

Let $R' = \mathrm{Sym}(\mathfrak{n}_{\sigma^{-1}})^*$, $Y' \in \mathfrak{n}_{\sigma^{-1}} \otimes R'$ be the generic element of $\mathfrak{n}_{\sigma'}$, and

$$\mathbb{F}' \colon 0 \to F_0 \otimes R \to F_1 \otimes R \to F_2 \otimes R \to F_3 \otimes R$$

be the sequence of differentials produced in this setting.

There is an involution τ of $\mathfrak{g}$, the *Cartan involution*, which interchanges the Lie algebra generators $e_i \leftrightarrow -f_i$ for $i \in T$ and acts by -1 on $\mathfrak{h}$ (here f_i refers to §2.2.1 in Chapter 2, not the integers f_i fixed throughout this chapter). We have an isomorphism of nilpotent subalgebras

$$\mathfrak{n}_{\sigma^{-1}} = \bigoplus_{\alpha>0, \sigma\alpha<0} \mathfrak{g}_\alpha \xrightarrow{\sigma\tau} \mathfrak{n}_\sigma = \bigoplus_{\alpha>0, \sigma^{-1}\alpha<0} \mathfrak{g}_\alpha$$

which dually induces an isomorphism $R \xrightarrow{\cong} R'$ and a map $\mathfrak{n}_\sigma \otimes R \to \mathfrak{n}_\sigma \otimes R'$ which sends $Y \mapsto \sigma\tau Y'$.

The formula $(Xf)(v) = f((-\tau X)v)$ for $X \in \mathfrak{g}$, $v \in L(\omega)$, and $f \in L(\omega)^\vee$ defines an action of $\mathfrak{g}$ on $L(\omega)^\vee$ that makes it isomorphic to $L(\omega)$. We fix identifications $L(\omega) \cong L(\omega)^\vee$, and by restriction we get identifications $F_i \cong F_i^*$ as vector spaces.

Proposition 3.5. *With the setup as above,* $\mathbb{F}' \cong \mathbb{F}^* \otimes R'$.

Proof. As an example, let us consider the differential d_1^* of $\mathbb{F}^*$. By construction it is

$$0 \to F_0^* \otimes R \to L(\omega_{x_{r_1}}) \otimes R \xrightarrow{\exp(Y)\sigma} L(\omega_{x_{r_1}}) \otimes R \to F_1^* \otimes R.$$

Base-change to R' amounts to replacing Y with $\sigma\tau Y'$, and

$$\exp(\sigma\tau Y')\sigma = \sigma \exp(\tau Y')\sigma^{-1}\sigma = (\exp(-\tau Y')\sigma^{-1})^{-1}.$$

We have the commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & F_0^* \otimes R' & \longrightarrow & L(\omega_{x_{r_1}}) \otimes R' & \xrightarrow{(\exp(-\tau Y')\sigma^{-1})^{-1}} & L(\omega_{x_{r_1}}) \otimes R' & \longrightarrow & F_1^* \otimes R' \\ & & \downarrow{\cong} & & \downarrow{\cong} & & \downarrow{\cong} & & \downarrow{\cong} \\ 0 & \longrightarrow & F_0 \otimes R' & \longrightarrow & L(\omega_{x_{r_1}})^\vee \otimes R' & \xrightarrow{(\exp(Y')\sigma^{-1})^{-1}} & L(\omega_{x_{r_1}})^\vee \otimes R' & \longrightarrow & F_1 \otimes R' \end{array}$$

where the vertical maps are induced by the involution τ. Note that the action of $(\exp(Y')\sigma^{-1})^{-1}$ on $L(\omega_{x_{r_1}})^\vee \otimes R'$ is dual to the action of $\exp(Y')\sigma^{-1}$ on $L(\omega_{x_{r_1}}) \otimes R'$, so the bottom row is the last differential of $\mathbb{F}'$ by definition.

The situation for the other two differentials is completely analogous, so we omit it. □

The isomorphism $R \xrightarrow{\cong} R'$ is not degree-preserving. By construction, we instead have:

Corollary 3.6. *The grading on $\mathbb{F}'$ is obtained by applying σ^{-1} to the grading on $\mathbb{F}$ and then multiplying by -1.*

Example 3.7. Let $r_1 = 2$, $r_2 = 4$, and $r_3 = 1$. The diagram is $T = E_6$ and we use Bourbaki numbering of the vertices so that $(1,2,3,4,5,6) = (z_2, x_1, z_1, u, y_1, y_2)$. Take

$$\sigma = s_3 s_4 s_2 s_5 s_6 s_1 s_4 s_5 s_3 s_4 s_2 \in W.$$

It is a minimal length representative of its double coset $[\sigma] \in W_{P_3} \backslash W / W_{P_2}$. The construction applied to σ^{-1} yields

$$\mathbb{F}' := 0 \to R' \to R'^6 \to R'^5 \to R'^2.$$

Taking its dual and shifting so that the last term is generated in degree zero, we obtain the following multigrading:

$$0 \to \bigoplus \begin{matrix} R'((1, & 5, & 3, & 5, & 2, & 1)) \\ R'((1, & 6, & 4, & 7, & 4, & 2)) \end{matrix} \to$$

$$\to \bigoplus \begin{matrix} R'((1, & 4, & 3, & 5, & 3, & 2)) \\ R'((1, & 4, & 3, & 5, & 3, & 1)) \\ R'((1, & 4, & 3, & 5, & 2, & 1)) \\ R'((1, & 4, & 3, & 4, & 2, & 1)) \\ R'((1, & 4, & 2, & 4, & 2, & 1)) \\ R'((0, & 4, & 2, & 4, & 2, & 1)) \end{matrix} \to$$

$$\to \bigoplus \begin{matrix} R'((0, & 2, & 1, & 2, & 1, & 0)) \\ R'((0, & 2, & 1, & 2, & 1, & 1)) \\ R'((1, & 3, & 2, & 3, & 2, & 1)) \\ R'((1, & 3, & 2, & 4, & 2, & 1)) \\ R'((1, & 3, & 3, & 4, & 2, & 1)) \end{matrix} \to$$

$$\to R'.$$

Here for example $R'((1,5,3,5,2,1))$ means a copy of R' generated in multidegree

$$-(\alpha_1 + 5\alpha_2 + 3\alpha_3 + 5\alpha_4 + 2\alpha_5 + \alpha_6) \in Q$$

As mentioned in §3.1.1, if we coarsen the multigrading to a $\mathbb{Z}$-grading by sending $\sum c_i \alpha_i$ to $-c_2$, the ring R' is a positively graded polynomial ring. The resolution has the grading

$$0 \to R'(-5) \oplus R'(-6) \to R'^6(-4) \to R'^2(-2) \oplus R'^3(-3) \to R'.$$

(In fact, this is none other than the resolution in [4, Proposition 3.6] with $n = 5$.)

3.3 Proof that $\mathbb{F}$ is a resolution

We now prove that the differentials d_i defined in §3.1 actually assemble into an acyclic complex.

Lemma 3.8. *The composites d_1d_2 and d_2d_3 are identically zero.*

Proof. In the following, $\otimes$ with no subscript means $\otimes_{\mathbb{C}}$. The composite d_1d_2 is adjoint to the composite $\epsilon_1(d_1^* \otimes_R d_2)$ where ϵ_1 is the contraction $F_1^* \otimes F_1 \to \mathbb{C}$ tensored with R. By construction the tensor product $d_1^* \otimes_R d_2$ is

$$F_0^* \otimes F_2 \otimes R \hookrightarrow L(\omega_{x_{r_1}}) \otimes L(\omega_{y_{r_2-2}}) \otimes R \xrightarrow{\exp(Y)\sigma} L(\omega_{x_{r_1}}) \otimes L(\omega_{y_{r_2-2}}) \otimes R \twoheadrightarrow F_1^* \otimes F_1 \otimes R.$$

The representation $F_1^* \otimes F_1$ is the sum $\mathfrak{sl}(F_1) \oplus \mathbb{C}$. A highest weight vector for $\mathfrak{sl}(F_1)$ is also a highest weight vector for $F_0^* \otimes F_2$, with weight $\omega_{x_{r_1}} + \omega_{y_{r_2-2}}$. In particular, $\mathfrak{sl}(F_1)$ and $F_0^* \otimes F_2$ belong to the same irreducible $\mathfrak{g}$-representation $L(\omega_{x_{r_1}} + \omega_{y_{r_2-2}})$ inside of the tensor product, whereas the $\mathbb{C}$ factor in $F_1^* \otimes F_1$ belongs to $L(\omega_{z_1})$. Thus the image of $d_1^* \otimes_R d_2$ lands in $\mathfrak{sl}(F_1) \otimes R$. As ϵ_1 is the projection onto the complementary $\mathbb{C}$ factor, the composite $\epsilon_1(d_1^* \otimes_R d_2)$ is zero as claimed.

Similarly, the composite d_2d_3 is adjoint to the composite $(d_2 \otimes_R d_3^*)\iota_2$ where ι_2 is the canonical map $\mathbb{C} \to F_2 \otimes F_2^*$ tensored with R. The tensor product $d_2 \otimes_R d_3^*$ is

$$F_2 \otimes F_2^* \otimes R \hookrightarrow L(\omega_{y_{r_2-2}}) \otimes L(\omega_{z_{r_3}}) \otimes R \xrightarrow{\exp(Y)\sigma} L(\omega_{y_{r_2-2}}) \otimes L(\omega_{z_{r_3}}) \otimes R \twoheadrightarrow F_1 \otimes F_3^* \otimes R.$$

The representation $F_2 \otimes F_2^*$ is the sum $\mathfrak{sl}(F_2) \oplus \mathbb{C}$. A highest weight vector for $\mathfrak{sl}(F_2)$ is also a highest weight vector for $F_1 \otimes F_3^*$, with weight $\omega_{y_{r_2-2}} + \omega_{r_3}$. Again we have that $\mathfrak{sl}(F_2)$ and $F_1 \otimes F_3^*$ belong to the same irreducible $\mathfrak{g}$-representation $L(\omega_{x_{r_1}} + \omega_{y_{r_2-2}})$ inside of the tensor product, whereas the $\mathbb{C}$ factor in $F_2 \otimes F_2^*$ belongs to $L(\omega_{x_1})$. So the tensor product $d_1^* \otimes_R d_2$ is zero on the image of ι_2. $\square$

To prove the acyclicity of $\mathbb{F}$, we will make use of the Buchsbaum-Eisenbud acyclicity criterion. The *rank* of a homomorphism d between free R-modules is the maximum value of r for which $\bigwedge^r d$ is nonzero, i.e. it is the size of the largest nonvanishing minor of d.

Definition 3.9. Let R be a ring, $I \subset R$ an ideal, and

$$\nu_n := \text{maximum length of an } R[x_1, \ldots, x_n]\text{-sequence in } IR[x_1, \ldots, x_n].$$

The *(true) grade* of I, introduced by Northcott in [40], is $\operatorname{grade} I := \sup_{n \geq 0} \nu_n$. If R is Noetherian, then this recovers the usual notion of grade.

Theorem 3.10 ([9])**.** *Let R be a ring. A complex*

$$0 \to F_n \xrightarrow{d_n} F_{n-1} \xrightarrow{d_{n-1}} \cdots \xrightarrow{d_2} F_2 \xrightarrow{d_1} F_0$$

of free R-modules is exact if and only if

$$\operatorname{rank} F_k = \operatorname{rank} d_k + \operatorname{rank} d_{k+1}$$

and

$$\operatorname{grade} I(d_k) \geq k$$

for $k = 1, \ldots, n$, where $I(d_k) := I_{\operatorname{rank} d_k}(d_k)$ is the ideal of $(\operatorname{rank} d_k) \times (\operatorname{rank} d_k)$ minors of d_k.

Remark 3.11. The original theorem statement in [9] assumed moreover that R is Noetherian, but Northcott showed in [40] that it holds without this assumption provided one uses the notion of true grade. This will not be important for us at present, since our base ring is a finitely generated polynomial ring throughout this chapter.

As the ring R is a polynomial ring, in particular Cohen-Macaulay, the grade of an ideal is the same as its codimension. Furthermore, since R is a domain, the fact that $\mathbb{F}$ is a complex implies $\operatorname{rank} d_i \le r_i$. Therefore it is sufficient to prove that $\operatorname{grade} I_{r_i}(d_i) \ge i$ for each i, from which $\operatorname{rank} d_i = r_i$ follows as a consequence. In fact, we will show that $\operatorname{grade} I_{r_i}(d_i) = 3$.

For this, let $G/P = G/P_{x_1}^+$ as in §3.1.1 and let $X^w \subset G/P$ be the codimension 3 opposite Schubert variety associated to $w = s_{z_1}s_u s_{x_1}$. For each cell C_σ meeting X^w, it is known that the codimension of $\mathcal{N}_\sigma^w = X^w \cap C_\sigma$ inside of C_σ is 3; see [31, Lemma 7.3.10].

Lemma 3.12. *The Plücker coordinates belonging to the bottom z_1-graded component $\bigwedge^{r_0} F_1 \subset L(\omega_{x_1})^\vee$ cut out X^w set-theoretically in G/P. Thus on the cell C_σ, the ideal generated by the entries of*

$$\bigwedge^{r_1} F_1 \otimes R \hookrightarrow L(\omega_{x_1})^\vee \otimes R \xrightarrow{(\exp(Y)\sigma)^{-1}} L(\omega_{x_1})^\vee \otimes R \twoheadrightarrow \mathbb{C} \otimes R \tag{3.2}$$

has codimension 3, where the first map is inclusion of the bottom z_1-graded component and the last map is projection onto the lowest weight space (equivalently the bottom x_1-graded component).

Proof. For $\rho \in W$ a minimal length representative of its coset $[\rho] \in W/W_{P_{x_1}}$, let $p_\rho = \rho p_e$ denote the corresponding extremal Plücker coordinate, where $p_e \in L(\omega_{x_1})^\vee$ is a lowest weight vector. Then p_ρ vanishes on X^w if and only if $\rho \ngeq w$, i.e. a reduced word for ρ does not contain (s_{z_1}, s_u, s_{x_1}) as a subword.

Let $\rho = s_{t_N} \cdots s_{t_1}$ where $(s_{t_N}, \dots, s_{t_1})$ is a reduced word. The assumption that ρ is a minimal length representative of $[\rho]$ means that for all $i = 1, \dots, N$, the weight $s_{t_{i-1}} s_{t_{i-2}} \cdots s_{t_1} \omega_{x_1}$ has a positive coefficient for ω_{t_i}. From this it is easy to see that $t_i = z_1$ for some i implies $t_j = u$ for some $j < i$. Since $t_1 = x_1$, we conclude $\rho \ge w$ if and only if $t_i = z_1$ for some i.

Thus the extremal Plücker coordinates vanishing on X^w are exactly the extremal Plücker coordinates belonging to the bottom z_1-graded component of $L(\omega_{x_1})^\vee$. This component is dual to the representation of $\mathfrak{g}^{(z_1)} = \mathfrak{sl}(F_1) \times \mathfrak{sl}(F_3)$ with highest weight ω_{x_1}, so it is $\bigwedge^{r_0} F_1$. All the weight spaces in $\bigwedge^{r_0} F_1$ are extremal (i.e. $\bigwedge^{r_0} F_1$ is *miniscule*), so this representation is equal to the span of the extremal Plücker coordinates vanishing on X^w. Since every Schubert variety is set-theoretically cut out by extremal Plücker coordinates, the claim follows.

The other part of the lemma statement is just reiterating the fact that

$$L(\omega_{x_1})^\vee \otimes R \xrightarrow{(\exp(Y)\sigma)^{-1}} L(\omega_{x_1})^\vee \otimes R \twoheadrightarrow \mathbb{C} \otimes R$$

gives the restriction of Plücker coordinates to the affine cell $C_\sigma = \exp(Y)\sigma v$. □

Using this lemma, we can prove the acyclicity of $\mathbb{F}$ via the Buchsbaum-Eisenbud acyclicity criterion. Before we do so, it is helpful to note a few representations appearing the z_1-graded decompositions of $L(\omega_{y_{r_2-2}})$ and $L(\omega_{z_{r_3}})$:

- Since
$$s_{z_1}s_u s_{y_1}\cdots s_{y_{r_2-2}}\omega_{y_{r_2-2}} = \omega_{x_1} + \omega_{z_2} - \omega_{z_1},$$
the $\mathfrak{g}^{(z_1)} = \mathfrak{sl}(F_1)\times\mathfrak{sl}(F_3)$-representation with highest weight $\omega_{x_1}+\omega_{z_2}$ appears as an extremal subrepresentation in $L(\omega_{y_{r_2-2}})$. This is $\bigwedge^{r_1} F_1^* \otimes F_3$.

- Since
$$s_{z_1}s_{z_2}\cdots s_{z_{r_3}}\omega_{z_{r_3}} = \omega_u - \omega_{z_1},$$
the $\mathfrak{g}^{(z_1)} = \mathfrak{sl}(F_1)\times\mathfrak{sl}(F_3)$-representation with highest weight ω_u appears as an extremal subrepresentation in $L(\omega_{z_{r_3}})$. This is $\bigwedge^{r_1+1} F_1^*$.

Theorem 3.13. *The complex $\mathbb{F}$ resolves a Cohen-Macaulay R-module supported on the Kazhdan-Lusztig variety $\mathcal{N}_\sigma^w = X^w \cap C_\sigma \subset C_\sigma = \operatorname{Spec} R$, where $w = s_{z_1}s_u s_{x_1}$.*

Proof. We will prove that powers of the Plücker coordinates (3.2) can be found in $I_{r_i}(d_i)$ for $i = 1,2,3$. From Lemma 3.12 it then follows that grade $I_{r_i}(d_i) = 3$. This means $\mathbb{F}^*$ is also acyclic, thus $\mathbb{F}$ resolves a perfect module, or equivalently a Cohen-Macaulay module since R is a polynomial ring.

The method for exhibiting powers of the extremal Plücker coordinates inside $I_{r_i}(d_i)$ was hinted at near the end of Example 3.2, where we demonstrated that $I_{n-1}(d_2)$ contained the squares of entries in d_1.

More generally, we will show the following. Let Ω denote the entries of the matrix (3.2), i.e. the extremal Plücker coordinates set-theoretically cutting out X^w on the affine cell C_σ. For each Plücker coordinate $p \in \Omega$, we will show that $p \in I_{r_1}(d_1)$, $p^{r_3+1} \in I_{r_2}(d_2)$, and $p^{r_2-1} \in I_{r_3}(d_3)$.

The situation for d_1 is relatively straightforward. The bottom x_1-graded component of $L(\omega_{x_{r_1}})^\vee$ is F_0, so the bottom x_1-graded component of $\bigwedge^{r_1} L(\omega_{x_{r_1}})^\vee$ is the one-dimensional representation $\bigwedge^{r_1} F_0$. Its weight is the sum of those in $F_0 \subset L(\omega_{x_{r_1}})^\vee$, which are given by $-Q_0'$ (c.f. §3.2.1). This sum is $-\omega_{x_1}$, so $\bigwedge^{r_1} F_0$ belongs to the irreducible subrepresentation $L(\omega_{x_1})^\vee$. Therefore $\bigwedge^{r_1} d_1$ factors through $L(\omega_{x_1})^\vee \otimes R$ and the maximal minors of d_1 are exactly the Plücker coordinates $p \in \Omega$. As such, the cokernel of d_1 is supported on $\mathcal{N}_\sigma^w$.

For d_2, we consider the (r_3+1)-th symmetric power of (3.2):

$$S_{r_3+1}\bigwedge^{r_2} F_1^* \otimes R \hookrightarrow S_{r_3+1}L(\omega_{x_1})^\vee \otimes R \xrightarrow{(\exp(Y)\sigma)^{-1}} S_{r_3+1}L(\omega_{x_1})^\vee \otimes R \twoheadrightarrow \mathbb{C}\otimes R. \tag{3.3}$$

Here the powers p^{r_3+1} for $p \in \Omega$ correspond to the weights in $S_{r_3+1}\bigwedge^{r_2} F_1^*$ in the $W_{P_{z_1}}$-orbit of the lowest weight $-(r_3+1)\omega_{x_1}$. So these powers p^{r_3+1} still appear when we restrict to the irreducible subrepresentation of this lowest weight, generated by the bottom x_1-graded component $\mathbb{C} \subset S_{r_3+1}L(\omega_{x_1})^\vee$:

$$S_{(r_3+1)^{r_2}}F_1^* \hookrightarrow L((r_3+1)\omega_{x_1})^\vee \otimes R \to L((r_3+1)\omega_{x_1})^\vee \otimes R \twoheadrightarrow \mathbb{C}\otimes R.$$

The crucial point is that $L((r_3+1)\omega_{x_1})^\vee$ also resides in $\bigwedge^{f_2} L(\omega_{y_{r_2-2}})^\vee$ as the irreducible representation generated by its bottom x_1-graded component $\bigwedge^{f_2} F_2^* \cong \mathbb{C}$. Again, this can be seen by adding the weights $-Q_2$ in F_2^* (c.f. §3.2.1).

Inside $\bigwedge^{f_2} L(\omega_{y_{r_2-2}})^\vee$, an analysis of weights shows that $\bigwedge^{f_2} F_2^*$ resides in the $\mathfrak{g}^{(z_1)} = \mathfrak{sl}(F_1)\times\mathfrak{sl}(F_3)$-representation

$$\bigwedge^{r_2} F_1^* \otimes \bigwedge^{r_3}(\bigwedge^{r_1} F_1 \otimes F_3^*) \subset \bigwedge^{f_2}(F_1 \oplus \bigwedge^{r_1} F_1 \otimes F_3^*) \subset \bigwedge^{f_2} L(\omega_{y_{r_2-2}})^\vee.$$

Hence by comparing (3.3) to

$$\bigwedge^{r_2} F_1^* \otimes \bigwedge^{r_3}(\bigwedge^{r_1} F_1 \otimes F_3^*) \otimes R \hookrightarrow \bigwedge^{f_2} L(\omega_{y_{r_2-2}})^\vee \otimes R \xrightarrow{(\exp(Y)\sigma)^{-1}} \bigwedge^{f_2} L(\omega_{y_{r_2-2}})^\vee \otimes R \twoheadrightarrow \bigwedge^{f_2} F_2^* \otimes R$$

we see that for each $p \in \Omega$, the power p^{r_3+1} appears as a maximal minor of

$$L(\omega_{y_{r_2-2}})^\vee \otimes R \xrightarrow{(\exp(Y)\sigma)^{-1}} L(\omega_{y_{r_2-2}})^\vee \otimes R \twoheadrightarrow F_2^* \otimes R$$

involving r_2 columns from $F_1^* \otimes R$. But the restriction of this map to $F_1^* \otimes R$ is none other than the dual of the differential d_2 by construction, and thus cofactor expansion of the determinant implies $p^{r_3+1} \in I_{r_2}(d_2)$ as claimed.

The situation for d_3 is very similar to that of d_2. Consider the (r_2-1)-th symmetric power of (3.2):

$$S_{r_2-1} \bigwedge^{r_1} F_1 \otimes R \hookrightarrow S_{r_2-1}L(\omega_{x_1})^\vee \otimes R \xrightarrow{(\exp(Y)\sigma)^{-1}} S_{r_2-1}L(\omega_{x_1})^\vee \otimes R \twoheadrightarrow \mathbb{C} \otimes R. \tag{3.4}$$

Again the powers p^{r_2-2} for $p \in \Omega$ correspond to the weights in $S_{r_2-2} \bigwedge^{r_1} F_1$ in the $W_{P_{z_1}}$-orbit of the lowest weight $-(r_2-1)\omega_{x_1}$. Restricting to the irreducible subrepresentation $L((r_2-1)\omega_{x_1})^\vee$ we have

$$S_{(r_2-1)^{r_1}} F_1 \hookrightarrow L((r_2-1)\omega_{x_1})^\vee \otimes R \to L((r_2-1)\omega_{x_1})^\vee \otimes R \twoheadrightarrow \mathbb{C} \otimes R.$$

The reason for considering this subrepresentation is that it also appears in $\bigwedge^{f_2} L(\omega_{z_{r_3}})^\vee$ as the irreducible representation generated by its bottom x_1-graded component $\bigwedge^{f_2} F_2 \cong \mathbb{C}$, seen by adding the weights $-Q_2'$ in F_2 (c.f. §3.2.1). An analysis of weights shows that $\bigwedge^{f_2} F_2$ is contained in

$$\bigwedge^{r_3} F_3 \otimes \bigwedge^{r_2}(\bigwedge^{r_1+1} F_1) \subset \bigwedge^{f_2}(F_3 \oplus \bigwedge^{r_1+1} F_1) \subset \bigwedge^{f_2} L(\omega_{z_{r_3}})^\vee$$

so comparing (3.4) to

$$\bigwedge^{r_3} F_3 \otimes \bigwedge^{r_2}(\bigwedge^{r_1+1} F_1) \otimes R \hookrightarrow \bigwedge^{f_2} L(\omega_{z_{r_3}})^\vee \otimes R \xrightarrow{(\exp(Y)\sigma)^{-1}} \bigwedge^{f_2} L(\omega_{z_{r_3}})^\vee \otimes R \twoheadrightarrow \bigwedge^{f_2} F_2 \otimes R$$

we see that for each $p \in \Omega$, the power p^{r_2-1} appears as a maximal minor of

$$L(\omega_{z_{r_3}})^\vee \otimes R \xrightarrow{(\exp(Y)\sigma)^{-1}} L(\omega_{z_{r_3}})^\vee \otimes R \twoheadrightarrow F_2 \otimes R$$

involving all r_3 columns from $F_3 \otimes R$. These r_3 columns exactly comprise the differential d_3 by definition, so cofactor expansion proves $p^{r_2-1} \in I_{r_3}(d_3)$.

Alternatively, one could deduce grade $I_{r_3}(d_3) \geq 3$ by applying the already established result that grade $I_{r_1}(d_1) \geq 3$ to the resolution $\mathbb{F}'$ discussed in §3.2.2 and then using Proposition 3.5. Geometrically, this amounts to showing that the maximal minors of d_3 can be interpreted as the extremal Plücker coordinates cutting out $\mathcal{N}_{\sigma^{-1}}^{w^{-1}}$ in $C_{\sigma^{-1}} \subset G/P_{z_1}^+$ as opposed to $\mathcal{N}_\sigma^w$ in $C_\sigma \subset G/P_{x_1}^+$. □

It is generally not true that the Plücker coordinates $p \in \Omega$ cut out X^w scheme-theoretically in G/P. However, this is true when $r_1 = 1$, in which case (3.2) coincides with the differential d_1. It is likely the case that Theorem 2.16 holds in the Kac-Moody setting—even knowing this statement for individual Schubert varieties (as opposed to unions thereof) would suffice to extend Lemma 2.19 to the Kac-Moody setting, from which the claim would follow.

However, as we could not locate a suitable reference for that theorem in the desired generality, we sketch an alternative argument. The homogeneity of $I(d_1)$ implies that it is sufficient to verify reducedness of $R/I(d_1)$ after localizing at the ideal of variables in R, corresponding to the torus-fixed point $\sigma v \in C_\sigma$. Later we will prove that, in this localization, $I(d_1)$ gives the generic example of a family of licci ideals. (In fact, the family of complexes $\mathbb{F}$ defined for $r_1 = 1$ can be interpreted as structure theorems for *all* grade 3 licci ideals.) By general facts regarding the deformation theory of licci ideals, these generic examples are necessarily reduced, c.f. [10]. The sketch of this argument is left brief because we will not use reducedness of $R/I(d_1)$ for $r_1 = 1$ anywhere in the subsequent chapters.

Chapter 4

Generic free resolutions

We start anew in this chapter and discuss the theory of generic free resolutions. This is a topic which, at first glance, has no relation to the representation theory of the Kac-Moody Lie algebras that were the focus of preceding chapters. But in fact, the original connection to representation theory was unearthed from this perspective, and many of the constructions from previous chapters were motivated by this study.

The first section §4.1 discusses the historical origins of this project and its various successes in studying free resolutions of length 2. In §4.2 we review the construction of generic free resolutions of length 3, following [51] and [50]. Here the connection to representation theory of the Kac-Moody Lie algebra $\mathfrak{g}(T)$ appears. In §4.3, we develop the theory of higher structure maps arising from our continued study of $\widehat{R}_{\text{gen}}$. These maps remain mysterious in general, but we show that they are relatively straightforward to analyze for split exact complexes. Since any finite free resolution is split on a dense open set, we can use this perspective to deduce many basic results on higher structure maps which are valid for arbitrary free resolutions of length three. We conclude in §4.4 by applying this theory to the family of free resolutions $\mathbb{F}$ constructed in the preceding chapter, demonstrating that a particular choice of higher structure maps for $\mathbb{F}$ is implicit in its construction.

The first two sections are expository. The remainder is partially based on [38] and ongoing joint work with Lorenzo Guerrieri and Jerzy Weyman that is not currently written elsewhere. Throughout, we maintain Assumption 2.1: R is a $\mathbb{C}$-algebra but our statements are more generally true over a field of characteristic zero. (For §4.1, the classical statements are true over $\mathbb{Z}$.)

4.1 A brief history of generic free resolutions

Hochster formally posed the question of searching for universal free resolutions in [25]. The motivation is that these universal examples should codify the "best possible" structure theorems for free resolutions in a certain sense. To demonstrate, let us recast the Hilbert-Burch theorem in this language. Let R be a local ring, and $I \subset R$ an ideal with $\operatorname{pdim} R/I = 2$. The minimal free resolution of R/I has the form

$$0 \to R^{n-1} \xrightarrow{d_2} R^n \xrightarrow{d_1} R \tag{4.1}$$

and the Hilbert-Burch theorem states that $I = aJ$ where $a \in R$ is a nonzerodivisor and $J = I_{n-1}(d_2)$ is generated by the maximal minors of d_2. Here is an equivalent way of stating the theorem:

Theorem 4.1 ([24], [11]). *Fix an integer $n \geq 2$ and let R_{univ} be a polynomial ring in the variables $\{x_{ij}\}_{1\leq i\leq n, 1\leq j\leq n-1}$ and one more variable a_1. Let $\mathbb{F}^{\text{univ}}$ denote the complex*

$$0 \to R_{\text{univ}}^{n-1} \xrightarrow{d_2} R_{\text{univ}}^{n} \xrightarrow{d_1} R_{\text{univ}}$$

in which d_2 is the generic matrix with entries x_{ij} and $d_1 = a_1 \bigwedge^{n-1} d_2^*$ for some fixed identification of $R_{\text{univ}}^n \cong \bigwedge^{n-1}(R_{\text{univ}}^n)^*$. Then if R is any ring and $\mathbb{F}$ is a complex of the form (4.1)*, then there exists a unique homomorphism $R_{\text{univ}} \to R$ specializing $\mathbb{F}^{\text{univ}}$ to $\mathbb{F}$.*

Furthermore, the complex $\mathbb{F}^{\text{univ}}$ over R_{univ} is itself acyclic. This explains the suggestive notation used in the theorem: $\mathbb{F}^{\text{univ}}$ is the *universal example* of a free resolution of the form (4.1).

Definition 4.2. Let R be a ring and $\mathbb{F}$ a complex of free R-modules

$$0 \to R^{f_n} \to \cdots \to R^{f_0}.$$

We say the sequence $\underline{f} = (f_0, \ldots, f_n)$ is the *format* of $\mathbb{F}$. Suppose that

1. $\mathbb{F}$ is acyclic, and
2. for any ring S and free resolution $\mathbb{G}$ over S with the same format $(f_0, \ldots, f_n)$, there exists a unique homomorphism $R \to S$ specializing $\mathbb{F}$ to $\mathbb{G}$.

Then we say $(R, \mathbb{F})$ is the *universal pair* for the format $\underline{f}$. We refer to $\mathbb{F}$ as a *universal free resolution* and R the associated *universal ring*.

The standard category theory argument regarding universal properties shows that, if a universal pair exists for a given format $\underline{f}$, then it is unique up to unique isomorphism.

Conceptually, condition (2) expresses that $(R, \mathbb{F})$ encodes an equational structure theorem for free resolutions of format $\underline{f}$, and condition (1) expresses that this structure theorem is optimal since any other theorem must factor through $(R, \mathbb{F})$. Here the adjective "equational" qualifies that these structure theorems assert unique solutions to a system of equations involving the entries of $\mathbb{G}$. The acyclicity criterion Theorem 3.10 is not equational, for instance.

4.1.1 Results for length 2

In [25], Hochster constructed the universal pair $(R_{\text{univ}}, \mathbb{F}^{\text{univ}})$ for length 2 formats $\underline{f} = (f_0, f_1, f_2)$. We assume $f_0 \geq r_0$, $f_1 = r_1 + r_2$, and $f_2 = r_2$ for positive integers r_1, r_2 to avoid degenerate cases.

Let d_1 and d_2 be matrices of indeterminates, where d_1 is $f_0 \times f_1$ and d_2 is $f_1 \times f_2$. The construction starts by taking the ring R_c of generic complexes of format $\underline{f}$, which is a polynomial ring in the aforementioned indeterminates, modulo the ideal $I_1(d_1d_2) + I_{r_1+1}(d_1) + I_{r_2+1}(d_2)$. This ring comes equipped with a tautological complex $\mathbb{F}^c$ whose differentials are d_i. The ideals $I_{r_1}(d_1)$ and $I_{r_2}(d_2)$ both have grade 1, so in view of Theorem 3.10, the goal is to increase grade $I_{r_2}(d_2)$ to 2.

This can be achieved by taking the *ideal transform* with respect to $I = I_{r_2}(d_2)$, defined as

$$R = \{h \in \operatorname{Frac} R_c : I^t h \subseteq R_c \text{ for some } t\}.$$

With $\mathbb{F} = \mathbb{F}^c \otimes R$, Hochster proves that the pair $(R, \mathbb{F})$ is the universal pair for $\underline{f}$, and that it essentially encodes the first structure theorem of Buchsbaum and Eisenbud, which we now recall.

Definition 4.3. A complex $\mathbb{F}$ over a ring R is *acyclic in grade c* if $\mathbb{F} \otimes R_\mathfrak{p}$ is acyclic for all primes $\mathfrak{p}$ with grade $\mathfrak{p} \leq c$.

Recall that by Theorem 3.10, acyclicity is equivalent to grade $I_{r_i}(d_i) \geq i$ for all i. By contrast, acyclicity in grade c amounts to the weaker requirement that grade $I_{r_i}(d_i) \geq \min(i, c+1)$ instead.

Theorem 4.4 ([8])**.** *Let $\mathbb{F}$ be a complex of free R-modules*

$$0 \to F_n \xrightarrow{d_n} F_{n-1} \xrightarrow{d_{n-1}} \cdots \xrightarrow{d_1} F_0$$

that is acyclic in grade 1. Let $f_i = \operatorname{rank} F_i$ and $r_i = \operatorname{rank} d_i$. Fix identifications $\bigwedge^{f_i} F_i \cong R$. Then there exist uniquely determined maps $a_i \colon R \to \bigwedge^{r_i} F_{i-1}$ ($i = 1, \ldots, n$) so that

$$\begin{array}{ccc} \bigwedge^{r_i} F_i & \xrightarrow{\bigwedge^{r_i} d_i} & \bigwedge^{r_i} F_{i-1} \\ & {\scriptstyle -\wedge a_{i+1}} \searrow \quad \nearrow {\scriptstyle a_i} & \\ & \bigwedge^{f_i} F_i \cong R & \end{array}$$

commutes, where we set a_{n+1} to be the identity $R \to \bigwedge^0 F_n = R$.

The original theorem was stated under the stronger assumption that $\mathbb{F}$ is acyclic. The weaker hypotheses here are due to Eagon and Northcott in [16]. The maps a_i are called *Buchsbaum-Eisenbud multipliers*.

After Hochster's construction of the universal pair $(R_{\text{univ}}, \mathbb{F}^{\text{univ}})$, many authors established various properties of the ring R_{univ}. Let R_a be the ring obtained by adjoining the Buchsbaum-Eisenbud multipliers to R_c. In Hochster's original treatment, the ring R_{univ} was shown to be the integral closure of R_a. Huneke later showed in [27] that the ring R_a is already integrally closed, so $R_{\text{univ}} = R_a$. In [42], Pragacz and Weyman analyzed the relations in R_{univ}. They also proved that it has rational singularities (this is the only statement so far that uses Assumption 2.1). Tchernev established numerous more properties of R_{univ} using Gröbner bases in [46], and Kustin determined a free resolution of R_{univ} as a quotient of a polynomial ring in [35].

4.1.2 Generic free resolutions beyond length 2

In [25], Hochster expressed doubt in the existence of universal free resolutions for formats of length ≥ 3, and proposed a weakening of universality in which the specialization is no longer required to be unique.

Definition 4.5. Let R be a ring and $\mathbb{F}$ an acyclic complex of free R-modules. Suppose that the format of $\mathbb{F}$ is $\underline{f}$, and that for any ring S and free resolution $\mathbb{G}$ over S with the same format $(f_0, \ldots, f_n)$, there exists a (not necessarily unique) homomorphism $R \to S$ specializing $\mathbb{F}$ to $\mathbb{G}$. Then we say $(R, \mathbb{F})$ is a *generic pair* for the format $\underline{f}$. We refer to $\mathbb{F}$ as a *generic free resolution* and R the associated *generic ring*.

Unfortunately, the removal of unique specialization means that such generic pairs are not unique for a fixed format. Indeed, one could simply adjoin indeterminates to an existing generic ring without affecting its genericity.

However, this is a necessary concession, as Bruns confirmed Hochster's doubt in [6]: universal free resolutions do not exist for formats of length ≥ 3. In the same paper, he also showed that for any nonnegative integers $r_0, r_1, \dots, r_n$, the format

$$\underline{f} = (f_0, \dots, f_n) = (r_0 + r_1, r_1 + r_2, \dots, r_{n-1} + r_n, r_n) \tag{4.2}$$

admits a generic pair (this condition on $\underline{f}$ is forced by linear algebra).

In fact, Theorem 4.4 is the best possible structure theorem if one requires unique specialization. To paraphrase Bruns's argument, fix a format $\underline{f}$ and suppose that $(R_{\text{univ}}, \mathbb{F}^{\text{univ}})$ is a universal pair for $\underline{f}$. Let R_a denote the Buchsbaum-Eisenbud multiplier ring for $\underline{f}$ and $\mathbb{F}$ the tautological complex over R_a. The ideal $I = \prod_{i=2}^{n} I_{r_i}(d_i)$ has grade 2; let $h_1, h_2 \in I$ be a regular sequence. Then for $i = 1, 2$, the localization $\mathbb{F} \otimes (R_a)_{h_i}$ is acyclic, so there exists a map $R_{\text{univ}} \to (R_a)_{h_i}$ specializing $\mathbb{F}^{\text{univ}}$ to $\mathbb{F} \otimes (R_a)_{h_i}$. Localizing further, we obtain two maps $R_{\text{univ}} \to (R_a)_{h_i h_j}$ specializing $\mathbb{F}^{\text{univ}}$ to $\mathbb{F} \otimes (R_a)_{h_i h_j}$, which therefore must agree by the universality property of $(R_{\text{univ}}, \mathbb{F}^{\text{univ}})$. Because h_1, h_2 is a regular sequence, our maps $R_{\text{univ}} \to (R_a)_{h_i}$ actually come from a map $R_{\text{univ}} \to R_a$ (e.g. by the Čech complex). The proof concludes by showing that for formats of length ≥ 3, there are resolutions $\mathbb{F}$ with cycles that cannot be in the image of $R_a \to R$ specializing $\mathbb{F}^a$ to $\mathbb{F}$.

Bruns's construction of generic free resolutions is via a procedure he calls "generic exactification." Essentially, if $\mathbb{F}$ is a complex over R and $H_i(\mathbb{F}) \neq 0$ for some $i > 0$, we can pick a map $Z \colon R^N \to F_i$ surjecting onto the cycles which are nonzero in homology. Then we adjoin a generic $f_{i+1} \times N$ matrix of variables $X = [x_{ij}]$ to R and quotient by the relation $d_{i+1} X = Z$ to obtain a new ring R', thereby killing the cycles in $H_i(\mathbb{F})$. While this process may introduce new cycles, we at least have that the induced map $H_i(\mathbb{F}) \to H_i(\mathbb{F} \otimes R')$ is zero. Thus, taking R_{gen} to be the direct limit of the rings produced iteratively using this procedure, the complex $\mathbb{F}^{\text{gen}} = R_{\text{gen}}$ is acyclic by construction.

As the homology is finitely generated at each step, the ring R_{gen} constructed in this manner is a countably generated algebra. Here is a conjecture mentioned by Hochster in [25] and again by Bruns in [6]:

Conjecture 4.6. *For each $\underline{f}$ of the form* (4.2)*, there exists a generic pair $(R_{\text{gen}}, \mathbb{F}^{\text{gen}})$ where R_{gen} is a finitely generated algebra.*

We will disprove this conjecture at the end of this chapter in Theorem 4.39, in which we show that any generic ring for the format $\underline{f} = (1, 6, 8, 3)$ is necessarily not Noetherian. In particular, the recursive procedure of killing cycles described in [6] does not terminate for this format.

The non-uniqueness of generic rings means that it is insufficient to exhibit one non-Noetherian example to prove this, as it does not rule out the possibility that there is some alternative construction of a Noetherian generic ring. Instead, we use the theory of higher structure maps developed in §4.3 combined with the examples of Chapter 3 to produce an infinite ascending chain that must be present in *any* generic ring for $(1, 6, 8, 3)$.

4.2 Weyman's generic free resolutions of length three

If $(R_{\text{univ}}, \mathbb{F}^{\text{univ}})$ is a universal pair for the format $\underline{f} = (f_0, f_1, \ldots, f_n)$, then the action of $\prod \mathrm{GL}(f_i)$ on $\mathbb{F}^{\text{univ}}$ induces an action of $\prod \mathrm{GL}(f_i)$ on R_{univ}.

Example 4.7. Let $\underline{f} = (f_0, f_1, f_2) = (1, n, n-1)$ and let $F_i = \mathbb{C}^{f_i}$. From Theorem 4.1 we see that, under the action of $\prod \mathrm{GL}(F_i)$, the ring R_{univ} is

$$R_{\text{univ}} = \mathrm{Sym}\big[(F_1^* \otimes F_2) \oplus (F_0^* \otimes \bigwedge^{f_1} F_1 \otimes \bigwedge^{f_2} F_2^*)\big]$$

where the variables x_{ij} span the representation $F_1^* \otimes F_2$ and the Buchsbaum-Eisenbud multiplier a_1 spans the one-dimensional representation $F_0^* \otimes \bigwedge^{f_1} F_1 \otimes \bigwedge^{f_2} F_2^*$.

More generally, the universality property of $(R_a, \mathbb{F}^a)$ yields an action of $\prod \mathrm{GL}(F_i)$ on R_a. The decomposition of R_a under this action is described in [42].

If $(R_{\text{gen}}, \mathbb{F}^{\text{gen}})$ is an abstract generic pair, there is no natural action of $\prod \mathrm{GL}(F_i)$ on R_{gen}. Despite this, we can at least try to construct a generic pair taking symmetry into account. Moreover, it turns out that this process can result in the presence of *more* symmetry, rather than less. This is the main theme for Weyman's construction of generic free resolutions of length 3. Before we discuss his construction, we demonstrate this principle on a toy example.

4.2.1 A motivating example

For this part, we fix a format $\underline{f} = (f_0, f_1, f_2, f_3)$ with $f_0 = 1$ and $f_2 + 1 = f_1 + f_3$. For $i = 1, 2, 3$ let d_i be a $f_i \times f_{i-1}$ matrix of indeterminates. Define A_0 to be the ring

$$\mathbb{C}[d_1, d_2, d_3]/(d_1 d_2, d_2 d_3)$$

and write $F_i = \mathbb{C}^{f_i}$.

The data of a map $A_0 \to R$ is then a complex of free R-modules

$$0 \to F_3 \otimes R \to F_2 \otimes R \to F_1 \otimes R \to R.$$

The groups $\mathrm{GL}(F_i)$ act on the set of such complexes, and thus on A_1 by the Yoneda lemma. Explicitly $g \in GL(F_i)$ sends the entries of d_{i+1} to those of $g d_{i+1}$ and the entries of d_i to those of $d_i g^{-1}$. With this action, the $\mathbb{C}$-linear span of the entries of d_i is a representation $F_i \otimes F_{i-1}^*$ in A_0.

Now define A_1 analogously so that the data of a map $A_1 \to R$ is a complex as above, together with a choice of map $w_1^{(3)}$ making the following diagram commute:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & F_3 \otimes R & \longrightarrow & F_2 \otimes R & \longrightarrow & F_1 \otimes R & \longrightarrow & R \\
 & & & & \nwarrow^{w_1^{(3)}} & & \uparrow{\scriptstyle e_1 \wedge e_2 \mapsto d(e_1)e_2 - d(e_2)e_1} & & \\
 & & & & & & \bigwedge^2 F_1 \otimes R & &
\end{array} \tag{4.3}$$

Note that the image of the vertical map lands in $\ker d_1$. So if for instance $\mathbb{F}$ were exact, such a lift $w_1^{(3)}$ would always exist.

Again $GL(F_i)$ acts on A_1. The entries of $w_1^{(3)}$ span a representation $F_2^* \otimes \bigwedge^2 F_1 \subset A_1$. But this time we also have the action of the additive group $F_3^* \otimes \bigwedge^2 F_1$ on A_1. An element $g \in F_3^* \otimes \bigwedge^2 F_1$ acts by sending the entries of $w_1^{(3)}$ to those of $w_1^{(3)} + d_3 g$. This is compatible with the action of $\mathrm{GL}(F_i)$ in the sense that altogether we obtain and action of the semidirect product $(\bigwedge^2 F_1 \otimes F_3^*) \rtimes \prod \mathrm{GL}(F_i)$ on A_1, where the semidirect product is formed using the evident action of $\prod \mathrm{GL}(F_i)$ on $\bigwedge^2 F_1 \otimes F_3^*$.

Let's do one more example, with some exercises. Define A_2 so that the data of a map $A_2 \to R$ is the data of a map $A_1 \to R$ (i.e. maps d_i and $w_1^{(3)}$ satisfying the required relations) together with a choice of map $w_2^{(3)}$ making the following diagram commute:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \bigwedge^2 F_3 \otimes R & \longrightarrow & F_3 \otimes F_2 \otimes R & \longrightarrow & S_2F_2 \otimes R & \xrightarrow{S_2 d_2} & S_2 F_1 \otimes R \\ & & & & & & \uparrow {\scriptstyle S_2 w_1^{(3)}} & & \\ & & & & & & S_2 \bigwedge^2 F_1 \otimes R & & \\ & & & & & & \uparrow & & \\ & & & & {\scriptstyle w_2^{(3)}} \dashleftarrow\!\!\!\!\!\!\! & & \bigwedge^4 F_1 \otimes R & & \end{array} \tag{4.4}$$

In the top row, the map $\bigwedge^2 F_3 \otimes R \to F_3 \otimes F_2 \otimes R$ is given by

$$e_1 \wedge e_2 \mapsto e_1 \otimes d_3(e_2) - e_2 \otimes d_3(e_1)$$

and the map $F_3 \otimes F_2 \otimes R \to S_2F_2 \otimes R$ is given by

$$e \otimes s \mapsto d_3(e)s.$$

The map $\bigwedge^4 F_1 \to S_2 \bigwedge^2 F_1$ is

$$e_1 \wedge e_2 \wedge e_3 \wedge e_4 \mapsto (e_1 \wedge e_2)(e_3 \wedge e_4) - (e_1 \wedge e_3)(e_2 \wedge e_4) + (e_1 \wedge e_4)(e_2 \wedge e_3).$$

It is a straightforward calculation to check that the vertical composite lands in the kernel of $S_2 d_2$. Furthermore, if the original $\mathbb{F}$ were exact, then so is the top row in this diagram by general theory of Schur complexes (see for instance [49]). So a lift $w_2^{(3)}$ always exists in that case.

We have the evident action of $\prod \mathrm{GL}(F_i)$ on A_2, as well as the action of $g \in \bigwedge^4 F_1^* \otimes \bigwedge^2 F_3$ by

$$\begin{aligned} d_i &\mapsto d_i \\ w_1^{(3)} &\mapsto w_1^{(3)} \\ w_2^{(3)} &\mapsto w_2^{(3)} + (\bigwedge^4 F_1 \xrightarrow{g} \bigwedge^2 F_3 \to F_3 \otimes F_2) \end{aligned} \tag{4.5}$$

(We are abusing notation here and writing F_i when we really mean $F_i \otimes R$.) But now there is a subtle point: since $w_2^{(3)}$ satisfies a relation with $w_1^{(3)}$, the action of $\bigwedge^2 F_1^* \otimes F_3$ on A_1 does not obviously extend to A_2.

It turns out that for $g \in \bigwedge^2 F_1^* \otimes F_3$ one can define an automorphism of A_2 as follows:

$$\begin{aligned} d_i &\mapsto d_i \\ w_1^{(3)} &\mapsto w_1^{(3)} + (\bigwedge^2 F_1 \xrightarrow{g} F_3 \xrightarrow{d_3} F_2) \\ w_2^{(3)} &\mapsto w_2^{(3)} + (\bigwedge^4 F_1 \to \bigwedge^2 F_1 \otimes \bigwedge^2 F_1 \xrightarrow{g\otimes w_1^{(3)}} F_3 \otimes F_2) \\ &\quad + \frac{1}{2}(\bigwedge^4 F_1 \to \bigwedge^2 F_1 \otimes \bigwedge^2 F_1 \xrightarrow{g\otimes d_3 g} F_3 \otimes F_2) \end{aligned} \tag{4.6}$$

where $\bigwedge^4 F_1 \mapsto \bigwedge^2 F_1 \otimes \bigwedge^2 F_1$ sends

$$\begin{aligned} e_1 \wedge e_2 \wedge e_3 \wedge e_4 \mapsto (e_1 \wedge e_2) \otimes (e_3 \wedge e_4) - (e_1 \wedge e_3) \otimes (e_2 \wedge e_4) + (e_1 \wedge e_4) \otimes (e_2 \wedge e_3) \\ + (e_2 \wedge e_3) \otimes (e_1 \wedge e_4) - (e_2 \wedge e_4) \otimes (e_1 \wedge e_3) + (e_3 \wedge e_4) \otimes (e_1 \wedge e_2) \end{aligned}$$

Making the substitutions (4.6) on (4.3) and (4.4), it is straightforward to verify that the diagrams still commute. Therefore this is a well-defined endomorphism of A_2.

On the other hand, if we let $g_1, g_2 \in \bigwedge^2 F_1^* \otimes F_3$, it turns out that acting by g_2 and then by g_1 is *not* the same as acting by g_1+g_2. Instead, it is the same as acting by $g_1+g_2 \in \bigwedge^2 F_1^* \otimes F_3$ and then by $\frac{1}{2}[g_1, g_2]$ for a certain element $[g_1, g_2] \in \bigwedge^4 F_1^* \otimes \bigwedge^2 F_3$. This bracket $[-,-]\colon \bigwedge^2(\bigwedge^2 F_1^* \otimes F_3) \to \bigwedge^4 F_1^* \otimes F_3$ turns

$$\mathbf{L} = \bigwedge^4 F_1^* \otimes \bigwedge^2 F_3 \oplus \bigwedge^2 F_1^* \otimes F_3$$

into a graded Lie algebra. We view the two pieces as living in degrees −2 and −1 respectively. The expressions (4.6) and (4.5) describe the exponential of an action of $\mathbb{L}$ on A_2 by derivations. In particular, the expression $g_1 + g_2 + \frac{1}{2}[g_1, g_2]$ comes from Baker-Campbell-Hausdorff.

Thus the following group acts on A_2:

$$\exp(\bigwedge^4 F_1^* \otimes \bigwedge^2 F_3 \oplus \bigwedge^2 F_1^* \otimes F_3) \rtimes \prod GL(F_i).$$

Note that for A_1, writing $\exp(\bigwedge^2 F_1 \otimes F_3^*)$ instead of $\bigwedge^2 F_1 \otimes F_3^*$ is inconsequential, as this Lie algebra is abelian.

4.2.2 Construction of $\widehat{R}_{\text{gen}}$

As in Chapter 3, we fix positive integers $r_1 \geq 1$, $r_2 \geq 2$, $r_3 \geq 1$ and define $f_0 = r_1$, $f_1 = r_1 + r_2$, $f_2 = r_2 + r_3$, $f_3 = r_3$. Let $\underline{f} = (f_0, f_1, f_2, f_3)$ and $F_i = \mathbb{C}^{f_i}$. Often we will abuse notation and write F_i when we really mean $F_i \otimes R$ for some ring R.

Weyman constructed a candidate generic pair $(\widehat{R}_{\text{gen}}(\underline{f}), \mathbb{F}^{\text{gen}}(\underline{f}))$ for the format $\underline{f}$ in [51], and verified its acyclicity in [50]. We will typically suppress $\underline{f}$ from the notation. As mentioned previously, generic pairs are not unique. Henceforth when we say *the* generic ring, we refer to the model $\widehat{R}_{\text{gen}}$ specifically.

Remark 4.8. One can more generally consider formats where $f_0 \geq r_1$, but we will not do so here. The main case of interest will actually be when $f_0 = r_1 = 1$.

We now briefly summarize the construction of $\widehat{R}_{\text{gen}}$. The starting point is Theorem 4.4, which we restate with two small adjustments: we state it only for $c = 3$ and we avoid identifying top exterior powers with the base ring in the interest of doing things $\mathrm{GL}(F_i)$-equivariantly.

Theorem 4.9. *Let $0 \to F_3 \xrightarrow{d_3} F_2 \xrightarrow{d_2} F_1 \xrightarrow{d_1} F_0$ be a complex of free modules, acyclic in grade 1, of format (f_0, f_1, f_2, f_3) over R. Then there are uniquely determined maps a_3, a_2, a_1, constructed as follows:*

- *a_3 is the top exterior power*
$$a_3\colon \bigwedge^{f_3} F_3 \to \bigwedge^{f_3} F_2.$$
- *a_2 is the unique map making the following diagram commute:*
$$\begin{array}{ccc} \bigwedge^{r_2} F_2 & \xrightarrow{\quad \bigwedge^{r_2} d_2 \quad} & \bigwedge^{r_2} F_1 \\ & {\scriptstyle -\wedge a_3} \searrow \qquad \nearrow {\scriptstyle a_2} & \\ & \bigwedge^{f_3} F_3^* \otimes \bigwedge^{f_2} F_2 & \end{array}$$
- *a_1 is the unique map making the following diagram commute:*
$$\begin{array}{ccc} \bigwedge^{f_0} F_1 & \xrightarrow{\quad \bigwedge^{r_1} d_1 \quad} & \bigwedge^{f_0} F_0 \\ & {\scriptstyle -\wedge a_2} \searrow \qquad \nearrow {\scriptstyle a_1} & \\ & \bigwedge^{f_3} F_3 \otimes \bigwedge^{f_2} F_2^* \otimes \bigwedge^{f_1} F_1 & \end{array}$$

Note that a_1 is just a scalar. If grade $I_{f_0}(d_1) \geq 2$, then a_1 is an isomorphism.

As mentioned in §4.1, Theorem 4.4 can be used to construct a free complex $\mathbb{F}^a$ of the given format $\underline{f}$ over a ring R_a, called the Buchsbaum-Eisenbud multiplier ring, such that $\mathbb{F}^a$ is acyclic in grade 1 and is universal with respect to this property. In particular, if $\mathbb{F}$ is any resolution of the same format over some ring R, then there is a unique map $R_a \to R$ specializing $\mathbb{F}^a$ to $\mathbb{F}$.

However, $\mathbb{F}^a$ is not acyclic: letting d_i denote the differentials of $\mathbb{F}^a$, we have

$$\operatorname{grade} I_{r_3}(d_3) = 2, \quad \operatorname{grade} I_{r_2}(d_2) = 2, \quad \operatorname{grade} I_{r_1}(d_1) = 1.$$

From the perspective of the acyclicity criterion Theorem 3.10, the failure of $\mathbb{F}$ to be acyclic can be attributed to the insufficient grade of $I_{r_3}(d_3)$.

Specifically we have $H_1(\mathbb{F}) \neq 0$, so one strategy to proceed would be to try and kill $H_1(\mathbb{F})$ following [6]. The first step would be to handle the Koszul cycles, as was done in our toy example in §4.2.1. However, it is not clear how to carry out this recursive procedure systematically in an explicit manner.

The alternative approach carried out in [51] is to look at the Koszul complex on $\bigwedge^{r_3} d_3$ instead of $\mathbb{F}$. Explicitly, writing $\mathcal{K} = \bigwedge^{f_3} F_3^* \otimes \bigwedge^{f_3} F_2$ so that $\bigwedge^{r_3} d_3$ can be viewed as a map $R_a \to \mathcal{K}$, the fact that grade $I_{r_3}(d_3) = 2$ implies the existence of nonzero H^2 in

$$0 \to \bigwedge^0 \mathcal{K} \to \bigwedge^1 \mathcal{K} \to \bigwedge^2 \mathcal{K} \to \bigwedge^3 \mathcal{K}.$$

Now the promised connection to representation theory gradually appears. First, in [51], the recursive procedure of killing H^2 in the complex above was carried out with the aid of a graded Lie algebra $\bigoplus_{i>0} \mathbb{L}_i$ where we view $\mathbb{L}_i$ as residing in degree $-i$. Let

$$\mathbb{L}^\vee := \bigoplus_{i>0} \mathbb{L}_i^*, \qquad \mathbf{L} := (\mathbb{L}^\vee)^* = \prod_{i>0} \mathbb{L}_i.$$

We call $\mathbf{L}$ the *defect Lie algebra*.

There is a diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \bigwedge^0 \mathcal{K} & \longrightarrow & \bigwedge^1 \mathcal{K} & \longrightarrow & \bigwedge^2 \mathcal{K} & \longrightarrow & \bigwedge^3 \mathcal{K} \\ & & & & {\scriptstyle p}\uparrow & & {\scriptstyle q}\uparrow & & \\ & & & & \mathbb{L}^\vee & \longrightarrow & \bigwedge^2 \mathbb{L}^\vee & & \end{array}$$

where the dual of the lower horizontal map is the bracket in $\mathbf{L}$. Let p_i denote the restriction of p to $\mathbb{L}_i^* \subset \mathbb{L}^\vee$ and similarly q_i the restriction of $q = \bigwedge^2 p$ to $(\bigwedge^2 \mathbb{L}^\vee)_i$. The map p_1 is defined to lift a cycle constructed using the second structure theorem of [8]. Since $\mathbb{L}^\vee$ is *strictly* positively graded, $(\bigwedge^2 \mathbb{L}^\vee)_i$ only involves $\mathbb{L}_j$ for $j < i$, which allows for recursive computation of the cycles q_m and their lifts p_m for $m \geq 2$.

For positive integers m, define R'_m to be the ring obtained from R_a by generically adjoining variables for the entries of $p_1, \ldots, p_m$ and quotienting by all relations they would satisfy on a split exact complex (see for instance [51, Lemma 2.4]). Let R_m be the ideal transform of R'_m with respect to $I_{r_2}(d_2)I_{r_3}(d_3)$. The ring $\widehat{R}_{\text{gen}}$ is defined to be the limit of the rings R_m, and $\mathbb{F}^{\text{gen}} := \mathbb{F}^a \otimes \widehat{R}_{\text{gen}}$.

The idea behind this construction is that adjoining the lifts p_i ought to kill H^2 in the Koszul complex, and the ideal transform with respect to $I_{r_2}(d_2)I_{r_3}(d_3)$ ensures that we do not generate homology elsewhere in our complex. The acyclicity of $\mathbb{F}^{\text{gen}}$ was reduced to the exactness of certain 3-term complexes (c.f. [51, Theorem 3.1]), and this was later proven in [50].

4.2.3 Exponential action of L

Given a free resolution $\mathbb{F}$ over some ring R, a choice of maps $p_1, \ldots, p_m$ making

$$\begin{array}{ccccccccc} 0 & \longrightarrow & R & \xrightarrow{\bigwedge^{r_3} d_3} & \mathcal{K} \otimes R & \longrightarrow & \bigwedge^2 \mathcal{K} \otimes R & \longrightarrow & \bigwedge^3 \mathcal{K} \otimes R \\ & & & & {\scriptstyle p_i}\uparrow & & {\scriptstyle q_i}\uparrow & & \\ & & & & \mathbb{L}_i^* \otimes R & \longrightarrow & (\bigwedge^2 \mathbb{L}^\vee)_i \otimes R & & \end{array} \tag{4.7}$$

commute determines a map $R'_m \to R$. This extends uniquely to the ideal transform R_m since the image of $I_{r_2}(d_2)I_{r_3}(d_3)$ in R has grade at least 2.

Lemma 4.10. *There is a natural bijection*

$$\{\textit{maps } w\colon \widehat{R}_{\text{gen}} \to R \textit{ specializing } \mathbb{F}^{\text{gen}} \textit{ to } \mathbb{F}\}$$
$$\updownarrow \simeq$$
$$\{\textit{choices of } \{p_i\}_{i>0} \textit{ making (4.7) commute}\}$$

Proof. This follows from the preceding discussion since $\widehat{R}_{\text{gen}} = \lim R_m$. □

Furthermore, having chosen p_i for $i < m$, the diagram (4.7) shows that the non-uniqueness of p_m lifting q_m is exactly $\operatorname{Hom}(\mathbb{L}_m^* \otimes R, R) = \mathbb{L}_m \otimes R$. In [51], the action of $\mathbf{L}$ on $\widehat{R}_{\text{gen}}$ by derivations is described. Specifically, elements $u \in \mathbb{L}_n$ act on $\widehat{R}_{\text{gen}}$ by R_{n-1}-linear derivations. It is sufficient to describe how they affect (the entries of) p_{n+k} for $k \geq 0$, and this is as follows: the derivation D_u sends p_n^* to

$$\mathcal{K}^* \xrightarrow{\bigwedge^{r_3} d_3^*} \widehat{R}_{\text{gen}} \xrightarrow{u} \mathbb{L}_n \otimes \widehat{R}_{\text{gen}}$$

and p_{n+k}^* to

$$\mathcal{K}^* \xrightarrow{p_k^*} \mathbb{L}_k \otimes \widehat{R}_{\text{gen}} \xrightarrow{[u,-]} \mathbb{L}_{n+k} \otimes \widehat{R}_{\text{gen}}.$$

These are just restatements of the formulas given in [51, Prop. 2.11] and [51, Thm. 2.12] respectively.

These formulas naturally extend to an arbitrary element $X \in \mathbb{L} = \prod_{i>0} \mathbb{L}_i$; the resulting derivation is well-defined because $\mathbb{L}_{>n}$ acts by zero on R_n. In a slight abuse of notation, we will also write X for the corresponding derivation. Homomorphisms $\widehat{R}_{\text{gen}} \to R$ correspond to R-algebra homomorphisms $\widehat{R}_{\text{gen}} \otimes R \to R$, and the Lie algebra $\mathbb{L} \otimes R$ acts on $\widehat{R}_{\text{gen}} \otimes R$.

For $X \in \mathbb{L} \otimes R$, the action of $\exp X \coloneqq \sum_{i \geq 0} \frac{1}{i!} X^i$ on $\widehat{R}_{\text{gen}} \otimes R$ is well-defined since every element of $\widehat{R}_{\text{gen}} \otimes R$ is killed by a sufficiently high power of X. Since X acts by an $(R_a \otimes R)$-linear derivation, it follows formally that $\exp X$ acts by an automorphism fixing $R_a \otimes R$. Such automorphisms completely describe the non-uniqueness of the map $\widehat{R}_{\text{gen}} \to R$ given a particular resolution $(R, \mathbb{F})$, as the following observation from [38] shows.

Theorem 4.11. *Let $\mathbb{F}$ be a resolution of length three over R and let $\widehat{R}_{\text{gen}}$ be the generic ring for the associated format. Fix a $\mathbb{C}$-algebra homomorphism $w\colon \widehat{R}_{\text{gen}} \to R$ specializing $\mathbb{F}^{\text{gen}}$ to $\mathbb{F}$. Then w determines a bijection*

$$\mathbf{L} \widehat{\otimes} R \coloneqq \prod_{i>0} (\mathbb{L}_i \otimes R) \simeq \{\mathbb{C}\text{-algebra homomorphisms } w'\colon \widehat{R}_{\text{gen}} \to R \text{ specializing } \mathbb{F}^{\text{gen}} \text{ to } \mathbb{F}\}.$$

Note that a $\mathbb{C}$-algebra homomorphism $\widehat{R}_{\text{gen}} \to R$ can be viewed as an R-algebra homomorphism $\widehat{R}_{\text{gen}} \otimes R \to R$. The correspondence above identifies $X \in \mathbf{L} \widehat{\otimes} R$ with the map $w \exp X$ obtained by precomposing w with the action of $\exp X$ on $\widehat{R}_{\text{gen}} \otimes R$.

Proof. By Lemma 4.10, the homomorphism $w\colon \widehat{R}_{\text{gen}} \otimes R \to R$ is completely determined by the choice of the structure maps p_i. For $X \in \mathbf{L} \widehat{\otimes} R$, let us write $X = \sum_{i>0} u_i$ where $u_i \in \mathbb{L}_i \otimes R$, and let $X_n = \sum_{i=1}^n u_i$ denote the partial sums.

Precomposing w by $\exp X$ or $\exp X_n$ has the same effect on the structure maps p_k for $k \leq n$. Acting by $\exp X$ on p_1, we get

$$p_1 + \Big(\bigwedge^{r_3} d_3\Big) u_1^*.$$

Here u_1^* means the dual of $R \xrightarrow{u_1} \mathbb{L}_1 \otimes R$. All possible choices of the structure map p_1 are obtained by lifting a particular map q_1 in the diagram (4.7), so it follows that choices of $u_1 \in \mathbb{L}_1 \otimes R$ correspond to choices for the structure map p_1.

Once X_{n-1} has been computed, $u_n \in \mathbb{L}_n \otimes R$ can be similarly determined by comparing p_n with p'_n. Acting by $\exp X$ on p_n gives

$$(p_n + p_{n-1}[u_1, -]^* + \cdots) + (\bigwedge^{r_3} d_3) u_n^*.$$

The first part consists of terms involving u_k for $k < n$, which have already been determined. Once again, (4.7) shows that there is a unique choice of $u_n \in \mathbb{L}_n \otimes R$ that makes the whole expression equal to p'_n.

Proceeding inductively in this fashion, we construct $X \in \mathbb{L} \otimes R$ with the desired property, and the uniqueness at each step is evident as well. □

If we view the generic pair as describing an equational structure theorem for $\mathbb{F}$ in the sense of §4.1, this tells us that the general solution to our system of equations is readily obtained once we have a particular solution.

The task of finding a particular solution (a map $w\colon \widehat{R}_{\text{gen}} \to R$ specializing $\mathbb{F}^{\text{gen}}$ to $\mathbb{F}$) may be difficult, but here is a helpful observation.

Proposition 4.12. *Let R be a ring and $\mathbb{F}$ a free resolution*

$$0 \to F_3 \otimes R \to F_2 \otimes R \to F_1 \otimes R \to F_0 \otimes R.$$

1. *Suppose a group G acts on the free modules $F_i \otimes R$ and the differentials of $\mathbb{F}$ are G-equivariant. Since $\prod \mathrm{GL}(F_i \otimes R)$ acts on $\widehat{R}_{\text{gen}} \otimes R$ and on $\mathbb{L}_m^*$, we have an induced action of G on $\widehat{R}_{\text{gen}} \otimes R$ and $\mathbb{L}_m^*$. If the maps p_m in (4.7) are chosen to be G-equivariant, then the induced map $w\colon \widehat{R}_{\text{gen}} \otimes R \to R$ is also G-equivariant.*

2. *Suppose R is graded and $\mathbb{F}$ is a graded free resolution where the differentials are homogeneous of degree zero. This induces a grading on $\widehat{R}_{\text{gen}} \otimes R$ and $\mathbb{L}_m^*$ for each m, and it is possible to choose all p_m in (4.7) to be homogeneous of degree zero. The corresponding $w\colon \widehat{R}_{\text{gen}} \otimes R \to R$ is then also homogeneous of degree zero.*

The construction of the cycle q_1 in terms of the differentials d_i and the construction of q_m in terms of $p_1, \ldots, p_{m-1}$ are both $\prod \mathrm{GL}(F_i)$-equivariant. So in the setting of Proposition 4.12, (1) the cycle q_m is G-equivariant provided $p_1, \ldots, p_{m-1}$ are, and similarly (2) q_m is homogeneous of degree zero provided $p_1, \ldots, p_{m-1}$ are.

Proof. This is evident from the construction of $\widehat{R}_{\text{gen}} = \lim R_m$ and Lemma 4.10. In the graded setting, it is always possible to recursively take p_m which is homogeneous of degree zero: simply take any lift of q_m (which is homogeneous of degree zero by induction) and discard the components which are not homogeneous of degree zero. □

4.2.4 The critical representations

Recall that we have fixed parameters $r_1 \geq 1$, $r_2 \geq 2$, $r_3 \geq 1$, from which our format $\underline{f}$ is defined as $(f_0, f_1, f_2, f_3) = (r_1, r_1 + r_2, r_2 + r_3, r_3)$. As in Chapter 3, consider the diagram $T = T_{r_1+1, r_2-1, r_3+1}$

$$\begin{array}{ccccccccccccc} x_{r_1} & — & \cdots & — & x_1 & — & u & — & y_1 & — & \cdots & — & y_{r_2-2} \\ & & & & & & | & & & & & & \\ & & & & & & z_1 & & & & & & \\ & & & & & & | & & & & & & \\ & & & & & & \vdots & & & & & & \\ & & & & & & | & & & & & & \\ & & & & & & z_{r_3} & & & & & & \end{array}$$

and let $\mathfrak{g}$ denote the associated Kac-Moody Lie algebra. We define subalgebras $\mathfrak{sl}(F_i)$ of $\mathfrak{g}$ as in Chapter 3, §3.1.2, so e.g. $\mathfrak{g}^{(z_1)} = \mathfrak{sl}(F_1) \times \mathfrak{sl}(F_3)$.

One of the main observations driving the results of [50] is that

$$\mathbb{L}^\vee = \mathfrak{n}_{z_1}^+ = \bigoplus_{\alpha >_{z_1} 0} \mathfrak{g}_\alpha, \qquad \mathbf{L} = \hat{\mathfrak{n}}_{z_1}^- = \prod_{\alpha <_{z_1} 0} \mathfrak{g}_\alpha,$$

and that the action of $\bigoplus \mathbb{L}_i$ on $\widehat{R}_{\text{gen}}$ extends to an action of $\mathfrak{g}$. We refer to Chapter 2, §2.2.1 for explanations regarding notation here. Using this connection, Weyman was able to

1. decompose $\widehat{R}_{\text{gen}}$ into representations of $\mathfrak{sl}(F_0) \times \mathfrak{sl}(F_2) \times \mathfrak{g}$, and

2. prove the needed exactness of certain complexes from [51] thereby proving $\mathbb{F}^{\text{gen}}$ is acyclic.

Interestingly, while (2) was the original goal, we have never used the acyclicity of $\mathbb{F}^{\text{gen}}$ at any point in this project—in particular, we will never use it in this thesis. This may come as a surprise, but notice that the classical Hilbert-Burch theorem does not actually claim the universal example to be acyclic; that is a separate result! Similarly, the first structure theorem of Buchsbaum and Eisenbud does not include the statement that $(R_a, \mathbb{F}^a)$ is universal for finite free complexes acyclic in grade 1. In essence, the fact that these theorems can be recast as "universal examples" is a certification that they are the *best possible* structure theorems for their respective objects. From this perspective, it is less surprising that $(\widehat{R}_{\text{gen}}, \mathbb{F}^{\text{gen}})$ has utility independent of the acyclicity of $\mathbb{F}^{\text{gen}}$.

On the other hand, (1) is essential to our results, as it allows us to define and analyze so-called "higher structure maps," which will be the topic of §4.3.

We also recall Assumption 2.1 that although we work over $\mathbb{C}$ throughout, the results remain valid over $\mathbb{Q}$. Indeed, the Lie algebra $\mathfrak{g}$ may be defined using the same generators and relations as in Chapter 2, §2.2.1 over $\mathbb{Q}$, which results in the *split form* of $\mathfrak{g}$. Its representation theory parallels the situation over $\mathbb{C}$, and the construction and decomposition of $\widehat{R}_{\text{gen}}$ remains valid.

Remark 4.13. There is one subtle point, which is that while the ring $\widehat{R}_{\text{gen}}$ certainly has an action of $\prod \mathfrak{gl}(F_i)$ by construction, this action does *not* come from an inclusion of $\prod \mathfrak{gl}(F_i)$ into $\mathfrak{sl}(F_0) \times \mathfrak{sl}(F_2) \times \mathfrak{g}$. Nor is it correct to say that $\mathfrak{gl}(F_0) \times \mathfrak{gl}(F_2) \times \mathfrak{g}$ acts on $\widehat{R}_{\text{gen}}$, since the action of $\mathfrak{gl}(F_2)$ does not commute with the action of $\mathfrak{g}$.

1. Rather, if we let

$$M = \bigwedge^{f_3} F_3 \otimes \bigwedge^{f_2} F_2^* \otimes \bigwedge^{f_1} F_1$$

then the $\prod \mathfrak{gl}(F_i)$-equivariant description of $\mathbb{L}_1$ from [51] is

$$\mathbb{L}_1 = \bigwedge^{r_1+1} F_1^* \otimes F_3 \otimes M.$$

Given an irreducible lowest weight representation $L(\omega)^\vee$ of $\mathfrak{g}$, the action of $\prod \mathfrak{gl}(F_i)$ on $L(\omega)^\vee$ can be inferred from its action on any z_1-graded component, e.g. the bottom one.

2. The Buchsbaum-Eisenbud multiplier a_1 from Theorem 4.9 is a map $M \to \bigwedge^{f_0} F_0$. If grade $I_{r_0}(d_0) \geq 2$, then a_1 is an isomorphism and we may instead view

$$\mathbb{L}_1 = \bigwedge^{r_1+1} F_1^* \otimes F_3 \otimes \bigwedge^{f_0} F_0$$

which is sometimes more convenient, especially when dealing with resolutions of cyclic modules.

Example 4.14. If $\underline{f} = (1,5,6,2)$, then the diagram is $T_{2,3,3} = E_6$. Writing $\mathfrak{g}_i$ for the component of $\mathfrak{g}$ in z_1-degree i (i.e. the sum of root spaces $\mathfrak{g}_\alpha$ where the coefficient of α_{z_1} in α is equal to i), we have

$$\begin{aligned}
\mathfrak{g}_2 &= \bigwedge^4 F_1 \otimes \bigwedge^2 F_3^* \otimes S_2 M^* \\
\mathfrak{g}_1 &= \bigwedge^2 F_1 \otimes F_3^* \otimes M^* \\
\mathfrak{g}_0 &= \mathfrak{sl}(F_1) \times \mathfrak{sl}(F_3) \times \mathbb{C} \\
\mathfrak{g}_{-1} &= \bigwedge^2 F_1^* \otimes F_3 \otimes M \\
\mathfrak{g}_{-2} &= \bigwedge^4 F_1^* \otimes \bigwedge^2 F_3 \otimes S_2 M
\end{aligned}$$

and $\mathbf{L} = \mathfrak{g}_{-2} \oplus \mathfrak{g}_{-1}$.

Example 4.15. If $\underline{f} = (1,6,8,3)$, then the diagram is $T_{2,4,4} = E_7^{(1)}$. Writing $\mathfrak{g}_i$ for the component of $\mathfrak{g}$

in z_1-degree i, we have

$$\begin{aligned}
&\vdots \\
\mathfrak{g}_4 &= S_{2^2,1^4}F_1 \otimes S_{2,1^2}F_3^* \otimes S_4 M^* \\
\mathfrak{g}_3 &= S_{2,1,1,1,1}F_1 \otimes S_{2,1}F_3^* \otimes S_3 M^* \\
\mathfrak{g}_2 &= \bigwedge^4 F_1 \otimes \bigwedge^2 F_3^* \otimes S_2 M^* \\
\mathfrak{g}_1 &= \bigwedge^2 F_1 \otimes F_3^* \otimes M^* \\
\mathfrak{g}_0 &= \mathfrak{sl}(F_1) \times \mathfrak{sl}(F_1) \times \mathbb{C}^2 \\
\mathfrak{g}_{-1} &= \bigwedge^2 F_1^* \otimes F_3 \otimes M \\
\mathfrak{g}_{-2} &= \bigwedge^4 F_1^* \otimes \bigwedge^2 F_3 \otimes S_2 M \\
\mathfrak{g}_{-3} &= S_{2,1,1,1,1}F_1^* \otimes S_{2,1}F_3 \otimes S_3 M \\
\mathfrak{g}_{-4} &= S_{2^2,1^4}F_1^* \otimes S_{2,1^2}F_3 \otimes S_4 M \\
&\vdots
\end{aligned}$$

and $\mathbf{L}$ is an infinite product in this case. Its decomposition is periodic, satisfying

$$\mathbb{L}_{i+3} \cong \mathbb{L}_i \otimes \bigwedge^6 F_1^* \otimes \bigwedge^3 F_3 \otimes S_3 M$$

for all $i > 0$, stemming from the fact that $\mathfrak{g}$ is an affine Lie algebra. This fact is also responsible for the extra copy of $\mathbb{C}$ appearing in $\mathfrak{g}_0$; c.f. Chapter 2, §2.2.1.

It is shown in [50] that the ring $\widehat{R}_{\text{gen}}$ is Noetherian if and only if the associated graph T is a Dynkin diagram. For example, it is not Noetherian for $(1,6,8,3)$. This is already strong evidence that Conjecture 4.6 is false, but we will need to develop more machinery to rule out the possibility of an alternative construction that results in a Noetherian generic ring.

The decomposition of $\widehat{R}_{\text{gen}}$ into representations for the product $\mathfrak{sl}(F_0) \times \mathfrak{sl}(F_2) \times \mathfrak{g}$ is detailed in [50]. Of these representations, there are a few of particular interest, which we call the *critical representations*—they are the ones generated by the entries of the differentials d_i and Buchsbaum-Eisenbud multipliers a_i for $\mathbb{F}^{\text{gen}}$. We denote these representations by $W(d_i)$ and $W(a_i)$ respectively. Let $L(\omega)$ be the irreducible representation with highest weight ω so that $L(\omega)^\vee$ is the irreducible representation with lowest weight $-\omega$. The aforementioned representations are

$$\begin{aligned}
W(d_3) &= F_2^* \otimes L(\omega_{z_{r-1}})^\vee \\
&= F_2^* \otimes [F_3 \oplus M^* \otimes \textstyle\bigwedge^{f_0+1} F_1 \oplus \cdots] \\
W(d_2) &= F_2 \otimes L(\omega_{y_{q-1}})^\vee \\
&= F_2 \otimes [F_1^* \oplus M^* \otimes F_3^* \otimes \textstyle\bigwedge^{f_0} F_1 \oplus \cdots] \\
W(d_1) &= F_0^* \otimes L(\omega_{x_{p-1}})^\vee \\
&= F_0^* \otimes [F_1 \oplus M^* \otimes F_3^* \otimes \textstyle\bigwedge^{f_0+2} F_1 \oplus \cdots] \\
W(a_3) &= \textstyle\bigwedge^{f_3} F_2^* \otimes L(\omega_{z_1})^\vee \\
&= \textstyle\bigwedge^{f_3} F_2^* \otimes [\textstyle\bigwedge^{f_3} F_3 \oplus \cdots] \\
W(a_2) &= \textstyle\bigwedge^{f_2} F_2 \otimes L(\omega_{x_1})^\vee \\
&= \textstyle\bigwedge^{f_2} F_2 \otimes [\textstyle\bigwedge^{r_2} F_1^* \otimes \textstyle\bigwedge^{f_3} F_3^* \oplus \cdots] \\
W(a_1) &= \textstyle\bigwedge^{f_0} F_0^* \otimes \textstyle\bigwedge^{f_1} F_1 \otimes \textstyle\bigwedge^{f_2} F_2^* \otimes \textstyle\bigwedge^{f_3} F_3
\end{aligned} \tag{4.8}$$

where the stated decompositions are into representations of $\prod \mathfrak{gl}(F_i)$, not just $\prod \mathfrak{sl}(F_i)$, following Remark 4.13. Implicit here is the claim that, for each representation $W(d_i)$, the bottom two components are both irreducible. This can be proven in a manner analogously to Lemma 2.18.

Remark 4.16. Although we will not use this fact (so we do not include a proof of it here), the ring $\widehat{R}_{\text{gen}}$ is generated by $W(d_3)$, $W(d_2)$, $W(d_1)$, $W(a_2)$, and $W(a_1)$. The representation $W(a_3)$ is not needed because it is contained in $S_{r_3} W(d_3)$:

$$W(a_3) = \textstyle\bigwedge^{f_3} F_2^* \otimes L(\omega_{z_1})^\vee \subseteq \textstyle\bigwedge^{f_3} F_2^* \otimes \textstyle\bigwedge^{f_3} L(\omega_{z_{r_3}})^\vee \subseteq S_{r_3} W(d_3).$$

4.3 Higher structure maps

Given a map $w\colon \widehat{R}_{\text{gen}} \to R$ for a complex $(R, \mathbb{F})$, we denote by $w^{(i)}$ the restriction of w to the representation $W(d_i) \subset R_{\text{gen}}$ and $w^{(a_i)}$ the restriction of w to the representation $W(a_i)$. We typically view these maps as being R-linear with source $L(\omega)^\vee \otimes R$, e.g. we think of $w^{(3)}$ as an R-linear map

$$w^{(3)}\colon L(\omega_{z_{r-1}})^\vee \otimes R = [F_3 \oplus M^* \otimes \textstyle\bigwedge^{f_0+1} F_1 \oplus \cdots] \otimes R \to F_2 \otimes R$$

as opposed to a $\mathbb{C}$-linear map $F_2 \otimes L(\omega_{z_{r-1}})^\vee \to R$.

We use $w_j^{(*)}$ to denote the restriction of $w^{(*)}$ to the j-th z_1-graded component of the representation, indexed so that $j = 0$ corresponds to the bottom graded piece. For instance, $w_0^{(i)} = d_i$ for $i = 1, 3$ and $w_0^{(2)} = d_2^*$. We call the maps $w_{>0}^{(*)}$ (a specific choice of) *higher structure maps* for $\mathbb{F}$. Let us demonstrate Theorem 4.11 from this perspective.

Example 4.17. Consider a free resolution $\mathbb{F}$ of format $(1, f_1, f_2, f_3)$ resolving R/I where depth $I \geq 2$, and make the identification described in Remark 4.13 (2). The structure maps $w_1^{(i)}$ give a choice of multiplicative structure on $\mathbb{F}$; see [36, Prop. 7.1]. Explicitly, such a resolution has the (non-unique) structure of a commutative differential graded algebra, and the non-uniqueness is evidently seen from the fact that the multiplication $\bigwedge^2 F_1 \to F_2$ may be chosen as any lift in the diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & F_3 & \longrightarrow & F_2 & \longrightarrow & F_1 & \longrightarrow & R \\ & & & & & \nwarrow & \uparrow & & \\ & & & & & & \bigwedge^2 F_1 & & \end{array} \tag{4.9}$$

where the map $\bigwedge^2 F_1 \to F_1$ is given by $e_1 \wedge e_2 \mapsto d_1(e_1)e_2 - d_1(e_2)e_1$. Indeed, we have that $\mathbb{L}_1 = F_3 \otimes \bigwedge^2 F_1^*$, which is exactly the non-uniqueness witnessed here.

Now suppose that $w\colon R_{\text{gen}} \to R$ (equivalently, $R \otimes R_{\text{gen}} \to R$) is one choice of higher structure maps for $\mathbb{F}$, and take an element $X = \sum_{i>0} u_i \in \mathbb{L} \otimes R$ using the same notation as before. Let $w' = w \exp(X)$, i.e.

$$w' = w\left(1 + u_1 + \left(\frac{1}{2}u_1^2 + u_2\right) + \cdots\right)$$

Note that u_k maps $W(d_i)_j$ to $W(d_i)_{j-k}$. If we restrict the above equation to the representation $W(d_3)$ and expand it degree-wise, we get

$$\begin{aligned} w_0'^{(3)} &= w_0^{(3)} \\ w_1'^{(3)} &= w_1^{(3)} + w_0^{(3)} u_1 \\ w_2'^{(3)} &= w_2^{(3)} + w_1^{(3)} u_1 + w_0^{(3)} \left(\frac{1}{2}u_1^2 + u_2\right) \\ &\vdots \end{aligned}$$

The first equation reflects that the underlying complex is still the same $\mathbb{F}$. The next equation shows that the new multiplication, viewed as a map $F_2^* \otimes \bigwedge^2 F_1 \to R$, was obtained from the old one by adding the composite

$$F_2^* \otimes \bigwedge^2 F_1 \xrightarrow{1 \otimes u_1} F_2^* \otimes F_3 \xrightarrow{d_3} R.$$

Here $u_1 \in \mathbb{L}_1 = F_3 \otimes \bigwedge^2 F_1^*$ could've been any map $\bigwedge^2 F_1 \to F_3$, and this exactly matches what we see in (4.9).

We next establish some straightforward representation theory lemmas, which will be used to highlight the importance of a particular subspace in $W(a_3) \subset \widehat{R}_{\text{gen}}$.

Lemma 4.18. *Let $b \in L(\omega_{z_1})^\vee$ be a lowest weight vector. The subspace*

$$V \coloneqq \{Xb : X \in \mathfrak{g}\} \subseteq L(\omega_{z_1})^\vee$$

is a representation of $\mathfrak{n}_{z_1}^-$, and thus a $\mathbf{L}$-representation.

Proof. Let $Y \in \mathfrak{n}_{z_1}^-$. Then
$$YXb = [Y,X]b - XYb = [Y,X]b$$
since $Yb = 0$. The representation $L(\omega_{z_1})^\vee$ is a lowest weight representation, so for any $Y' \in \hat{\mathfrak{n}}_{z_1}^-$, there exists a truncation $Y \in \mathfrak{n}_{z_1}^-$ of Y such that $Yb = Y'b$, so $\mathbf{L}$ also acts on V. □

Lemma 4.19. *The map*
$$\mathbb{C} \oplus \mathbb{L}^\vee \to V$$
sending $1 \in \mathbb{C}$ *to* b *and* $X \in \mathbb{L}^\vee = \mathfrak{n}_{z_1}^+$ *to* Xb *is an isomorphism of vector spaces.*

Proof. It is easy to see that the subspace
$$\mathfrak{p} := \{X \in \mathfrak{g} : Xb \in \mathbb{C}b\} \subset \mathfrak{g}$$
is a subalgebra. Moreover it contains the maximal parabolic $\mathfrak{p}_{z_1}^- = \bigoplus_{\alpha \leq_{z_1} 0} \mathfrak{g}_\alpha$. Since $L(\omega_{z_1})^\vee$ is not the trivial representation, we have $\mathfrak{p}_{z_1}^- \subseteq \mathfrak{p} \subsetneq \mathfrak{g}$. Therefore $\mathfrak{p} = \mathfrak{p}_{z_1}^-$, so $\mathbb{L}^\vee \cap \mathfrak{p} = 0$ and the map is injective.

Given any $X \in \mathfrak{g}$, we may express X as $X^+ + X^-$ where $X^+ \in \mathfrak{n}_{z_1}^+$ and $X^- \in \mathfrak{p}_{z_1}^-$. Then $Xb = X^+b + X^-b$ where $X^-b \in \mathbb{C}b$, showing surjectivity. □

Remark 4.20. The action of $\mathbb{L}_i$ on $\mathbb{C} \oplus \mathbb{L}^\vee$ looks very similar to its action on the entries of p_m as discussed around Theorem 4.11. In fact, we expect that the restriction of $w \colon \widehat{R}_{\text{gen}} \to R$ to
$$\bigwedge^{r_2} F_2^* \otimes \bigwedge^{r_3} F_3 \otimes \mathbb{L}_m^* \subset \bigwedge^{r_2} F_2^* \otimes \bigwedge^{r_3} F_3 \otimes [\mathbb{C} \oplus \mathbb{L}^\vee] \subseteq W(a_3)$$
should exactly recover the structure map p_m. While this should not be too difficult to prove, it would be somewhat technical and unnecessary for our purposes, so we leave it as a guess. Some of the results below should (in principle) be consequences of this statement, but of course we do not assume this statement in their proofs.

In the following, we will make some arguments using regular sequences. If the rings involved are not Noetherian, then it may be necessary to adjoin variables in view of Definition 3.9, but this has no effect on the conclusions.

Lemma 4.21. *Let* $h \in R$ *be a nonzerodivisor, let* $\pi \colon [\mathbb{C} \oplus \mathbb{L}^\vee] \otimes R \twoheadrightarrow \mathbb{C} \otimes R$ *be projection onto the first factor, and let* $\gamma \colon \mathbb{L}^\vee \otimes R \to \mathbb{C} \otimes R$ *be any map. Then*

- *there is a unique* $X \in \mathbf{L} \widehat{\otimes} R_h$ *such that* $(h\pi + \gamma) = h\pi \exp X$*, and*
- *if* S *is a ring containing* R *and* $X' \in \mathbf{L} \widehat{\otimes} S$ *satisfies* $(h\pi + \gamma) = h\pi \exp X'$*, then* X' *must belong to* $\mathbf{L} \widehat{\otimes} R_h$ *and thus equal* X.

Proof. For $X \in \mathbf{L} \widehat{\otimes} S$, we have $\pi X = 0$ only when $X = 0$. Thus, the precomposition action of $\exp(\mathbf{L} \widehat{\otimes} S)$ on π has trivial stabilizer, showing uniqueness. One can solve for X explicitly in a manner similar to the proof of Theorem 4.11, showing it must be an element of $\mathbf{L} \widehat{\otimes} R_h$. One should informally think of X as $\log(\pi + \gamma/h)$; we omit the details. □

The next result says that we do not lose any information by only considering the structure maps $w^{(i)}$, since they uniquely determine w. In fact, we only need the differentials and part of $w^{(a_3)}$, which we recall can be computed from $w^{(3)}$ (c.f. Remark 4.16).

Proposition 4.22. *Let w and w' be two maps $\widehat{R}_{\mathrm{gen}} \to R$ specializing $\mathbb{F}^{\mathrm{gen}}$ to the same resolution $\mathbb{F}$ over some ring R. Viewing them as maps $\widehat{R}_{\mathrm{gen}} \otimes R \to R$, write $\bar{w}, \bar{w}'$ for their restrictions to*

$$\bigwedge^{r_2} F_2^* \otimes \bigwedge^{r_3} F_3 \otimes [\mathbb{C} \oplus \mathbb{L}^\vee] \otimes R \subseteq W(a_3) \otimes R.$$

Then there is a unique element $X \in \mathbf{L} \widehat{\otimes} R$ such that $\bar{w}' = \bar{w} \exp X$.

In particular, if $\bar{w} = \bar{w}'$, then $X = 0$ and $w = w'$.

The existence of such an element X is already known by Theorem 4.11; the substance of this statement is that X is completely determined by comparing $\bar{w}$ and $\bar{w}'$. If Remark 4.20 were true, this would be immediate given Lemma 4.10.

Proof. Since $\mathbb{F}$ is acyclic, grade $I_{r_3}(d_3) = 3 \geq 1$ so there is some $e \in \bigwedge^{r_3} F_3 \otimes \bigwedge^{r_2} F_2^*$ such that $h = a_3(e) \in R$ is a nonzerodivisor. Let $\bar{w}_e$ and $\bar{w}'_e$ denote the restrictions of $\bar{w}$ and $\bar{w}'$ to $\mathbb{C}e \otimes [\mathbb{C} \oplus \mathbb{L}^\vee]$, viewed as maps

$$[\mathbb{C} \oplus \mathbb{L}^\vee] \otimes R \to (\mathbb{C}e)^* \otimes R \cong R$$

From Theorem 4.11, we know there exists $X \in \mathbf{L} \widehat{\otimes} R$ with the property that $\bar{w}_e = \bar{w}'_e \exp X$. The uniqueness follows from Lemma 4.21. ☐

It will often be convenient to manipulate higher structure maps $w^{(i)}$ over a larger ring containing R, for instance a localization in which $\mathbb{F}$ becomes split exact. The next result ensures that, even if we work in a larger ring, it is easy to tell when w factors through R.

Proposition 4.23. *Let $R \subset S$ be two rings, $\mathbb{F}$ a resolution over R with the property that $\mathbb{F} \otimes S$ is also acyclic, and $w' \colon \widehat{R}_{\mathrm{gen}} \to S$ a map specializing $\mathbb{F}^{\mathrm{gen}}$ to $\mathbb{F} \otimes S$.*

Suppose the restriction of w' to

$$\bigwedge^{r_2} F_2^* \otimes \bigwedge^{r_3} F_3 \otimes [\mathbb{C} \oplus \mathbb{L}^\vee] \subseteq W(a_3)$$

is R-valued. Then w' factors through R.

Proof. This proof is similar to the preceding one. This time we use that grade $I_{r_3}(d_3) = 3 \geq 2$ so there are $e_1, e_2 \in \bigwedge^{r_3} F_3 \otimes \bigwedge^{r_2} F_2^*$ such that $h_1 = a_3(e_1)$ and $h_2 = a_3(e_2)$ form a regular sequence.

Since $\mathbb{F}$ is acyclic, we may pick a $w \colon \widehat{R}_{\mathrm{gen}} \to R$ specializing $\mathbb{F}^{\mathrm{gen}}$ to $\mathbb{F}$. By Theorem 4.11, there exists $X \in \mathbf{L} \widehat{\otimes} S$ such that $w' = (w \otimes S) \exp X$.

Let $\bar{w}_e$ and $\bar{w}'_e$ denote the restrictions of $w \otimes S$ and w' to $(\mathbb{C}e_1 \oplus \mathbb{C}e_2) \otimes [\mathbb{C} \oplus \mathbb{L}^\vee] \subset W(a_3)$, viewed as maps

$$[\mathbb{C} \oplus \mathbb{L}^\vee] \otimes S \to (\mathbb{C}e_1 \oplus \mathbb{C}e_2)^* \otimes S \cong S^2.$$

We have $\bar{w}'_e = \bar{w}_e \exp X$. Applying Lemma 4.21 to the first row of $\bar{w}_e$ and $\bar{w}'_e$, we find that $X \in \mathbf{L} \widehat{\otimes} R_{h_1}$. Applying it to the second row, we find that $X \in \mathbf{L} \widehat{\otimes} R_{h_2}$. Since h_1, h_2 is a regular sequence, we have $R_{h_1} \cap R_{h_2} = R$ and $X \in \mathbf{L} \widehat{\otimes} R$, from which it follows that w' factors through R if viewed as a map from $\widehat{R}_{\mathrm{gen}}$. ☐

4.3.1 Computing particular higher structure maps

So far we have studied relationships between different choices of $w\colon \widehat{R}_{\mathrm{gen}} \to R$ specializing $\mathbb{F}^{\mathrm{gen}}$ to a given $\mathbb{F}$, but we have not discussed how to effectively compute a particular choice of w given a resolution $\mathbb{F}$ in the first place.

In general this is a difficult problem, but with the presence of some symmetry, Proposition 4.12 can often simplify the task, or at least allow us to deduce qualitative properties about particularly nice choices of w.

Example 4.24. Let $R = \mathbb{C}[t_1, t_2, t_3]$ with the standard $\mathbb{Z}$-grading and let $I = (t_1, t_2, t_3)^2$. The minimal graded free resolution of R/I is

$$\mathbb{F}\colon 0 \to R^3(-4) \to R^8(-3) \to R^6(-2) \to R.$$

As per Proposition 4.12, there is a choice of $w\colon \widehat{R}_{\mathrm{gen}} \otimes R \to R$ specializing $\mathbb{F}^{\mathrm{gen}}$ to $\mathbb{F}$ that is homogeneous of degree zero. We identify $M \cong R$ as in Remark 4.13. The structure maps have the form

$$\begin{aligned} w^{(1)}&\colon [F_1 \oplus \textstyle\bigwedge^3 F_1 \oplus \cdots] \otimes R \to R \\ w^{(2)}&\colon [F_1^* \oplus F_1 \otimes F_3^* \oplus \cdots] \otimes R \to F_2^* \otimes R \\ w^{(3)}&\colon [F_3 \oplus \textstyle\bigwedge^2 F_1 \oplus \cdots] \otimes R \to F_2 \otimes R \end{aligned}$$

In this example $\mathbb{L}_1 = \bigwedge^2 F_1^* \otimes F_3$ is concentrated in degree zero, causing all generators of each representation $L(\omega)^\vee$ to be in the same degree. So with a homogeneous choice of w, all entries of $w^{(1)}$ have degree 2, whereas all entries of $w^{(2)}$ and $w^{(3)}$ are linear.

Example 4.25. Let $\mathbb{F}$ be a resolution where d_3 is a split inclusion. After a change of coordinates, we assume it has the form

$$0 \to F_3 \otimes R \to (F_3 \oplus Z) \otimes R \to F_1 \otimes R \to F_0 \otimes R$$

where $Z = \mathbb{C}^{r_2}$ and d_3 maps $F_3 \otimes R$ identically to itself.

There is an action of $G = \mathrm{GL}(F_3 \otimes R)$ on this resolution, and the differentials are equivariant with respect to it. The Koszul complex on $\bigwedge^{r_3}(d_3)$ is just a split exact complex, so we can certainly pick lifts p_m which are G-equivariant, e.g. by using a G-equivariant splitting.

Let $w\colon \widehat{R}_{\mathrm{gen}} \otimes R \to R$ be the G-equivariant map obtained in this manner. The maps $w^{(i)}$ have the form

$$\begin{aligned} &[F_3 \oplus \textstyle\bigwedge^{f_0+1} F_1 \otimes M^* \oplus \cdots] \otimes R \to (F_3 \oplus Z) \otimes R \\ &[F_1^* \oplus F_3^* \otimes \textstyle\bigwedge^{f_0} F_1 \otimes M^* \oplus \cdots] \otimes R \to (F_3^* \oplus Z^*) \otimes R \\ &[F_1 \oplus F_3^* \otimes \textstyle\bigwedge^{f_0+2} F_1 \otimes M^* \oplus \cdots] \otimes R \to F_0 \otimes R \end{aligned}$$

Note that $M = \bigwedge^{f_3} F_3 \otimes \bigwedge^{f_2}(F_3^* \oplus Z^*) \otimes \bigwedge^{f_1} F_1$ is a trivial representation of G. Since $\mathbb{L}_1^* = \bigwedge^2 F_1 \otimes F_3^* \otimes M^*$, we accumulate an additional factor of F_3^* every time we go up in the z_1-grading in each representation. Thus by G-equivariance considerations the only components $w_j^{(i)}$ that have any chance of being nonzero are:

$$w_1^{(3)}\colon \bigwedge^{f_0+1} F_1 \otimes M^* \otimes R \to Z \otimes R \subset (F_3 \oplus Z) \otimes R$$
$$w_1^{(2)}\colon \bigwedge^{f_0} F_1 \otimes F_3^* \otimes M^* \otimes R \to F_3^* \otimes R \subset (F_3^* \oplus Z^*) \otimes R$$

in addition to the maps $w_0^{(i)}$ which are obviously nonzero since they give the differentials of the resolution.

We also note that for the map $w^{(a_2)}$

$$[\bigwedge^{r_2} F_1^* \otimes \bigwedge^{f_3} F_3^* \oplus \cdots] \otimes R \to \bigwedge^{r_2} Z^* \otimes \bigwedge^{r_3} F_3^* \otimes R$$

only the bottom component, namely a_2 itself, is nonzero by the same considerations.

Hence it would be beneficial to at least understand how to compute $w_1^{(3)}$ and $w_1^{(2)}$ explicitly. In [36, Prop. 7.1], it is described[1] how to compute these maps via a comparison map from a Buchsbaum-Rim complex, which we now recall. We write F_i to mean $F_i \otimes R$ in the following. Theorem 4.9 gives a factorization

$$\begin{array}{ccc} \bigwedge^{r_1} F_1 & \xrightarrow{\bigwedge^{r_1} d_1} & \bigwedge^{r_1} F_0 \\ & \searrow \quad \nearrow_{a_1} & \\ & M & \end{array}$$

in particular a map $\beta\colon M^* \otimes \bigwedge^{r_1} F_1 \to R$, which is essentially a_2^* after appropriate identifications. It is straightforward to check that the composite

$$M^* \otimes \bigwedge^{r_1+1} F_1 \to M^* \otimes \bigwedge^{r_1} F_1 \otimes F_1 \xrightarrow{\beta\otimes 1} F_1 \xrightarrow{d_1} F_0$$

is zero, thus we can lift through d_2 to obtain a map

$$w_1^{(3)}\colon M^* \otimes \bigwedge^{r_1+1} F_1 \to F_2.$$

The difference of the two maps

$$M^* \otimes \bigwedge^{r_1} F_1 \otimes F_2 \xrightarrow{\beta\otimes 1} F_2$$
$$M^* \otimes \bigwedge^{r_1} F_1 \otimes F_2 \xrightarrow{1\otimes d_2} M^* \otimes \bigwedge^{r_1} F_1 \otimes F_1 \to M^* \otimes \bigwedge^{r_1+1} F_1 \xrightarrow{w_1^{(3)}} F_2$$

has image landing in $\ker d_2$, and thus it can be lifted through d_3 to obtain

$$w_1^{(2)}\colon M^* \otimes \bigwedge^{r_1} F_1 \otimes F_2 \to F_3.$$

[1] In the referenced paper, it was assumed that $a_1\colon M \to \bigwedge^{f_0} F_0^*$ is an isomorphism.

In the case that $r_0 = 1$, these maps can be viewed as giving a choice of multiplication on the resolution

$$0 \to M^* \otimes F_3 \to M^* \otimes F_2 \to M^* \otimes F_1 \xrightarrow{\beta} R$$

recovering what was illustrated in Example 4.17.

A very important special case of Example 4.25 is when the entire complex $\mathbb{F}$ is split exact, e.g. we take $\mathbb{F}$ to be the split exact complex

$$\mathbb{F}^{\text{ssc}}\colon 0 \to F_3 \to F_3 \oplus Z \to F_0 \oplus Z \to F_0$$

of $\mathbb{C}$-vector spaces. Here $M = \bigwedge^{f_0} F_0$, and a direct computation shows that there is a unique $G = \mathrm{GL}(F_0) \times \mathrm{GL}(F_3) \times \mathrm{GL}(Z)$-equivariant choice of w. For this choice of w, direct computation with the explicit definitions of $w_1^{(3)}$ and $w_1^{(2)}$ above shows that

$$w_1^{(3)}\colon \bigwedge^{f_0+1} F_1 \otimes \bigwedge^{f_0} F_0^* = Z \oplus F_0^* \otimes \bigwedge^2 Z \oplus \cdots \to F_3 \oplus Z$$

maps Z identically to itself and is zero on all other factors by G-equivariance. Similarly

$$w_1^{(2)}\colon \bigwedge^{f_0} F_1 \otimes F_3^* \otimes \bigwedge^{f_0} F_0^* = (\mathbb{C} \oplus F_0^* \otimes Z \oplus \bigwedge^2 F_0^* \otimes \bigwedge^2 Z \oplus \cdots) \otimes F_3^* \to F_3^* \oplus Z^* = F_2^*$$

maps F_3^* identically to itself and is zero on all other factors.

Remark 4.26. This complex $\mathbb{F}^{\text{ssc}}$ already appeared in Example 3.1. In fact, the higher structure maps $w_{\text{ssc}}^{(i)}$ are implicit in that example, dual to the inclusions

$$\begin{aligned} F_0^* &\hookrightarrow L(\omega_{x_{r_1}}) \\ F_2 &\hookrightarrow L(\omega_{y_{r_2-2}}) \\ F_2^* &\hookrightarrow L(\omega_{z_{r_3}}). \end{aligned}$$

After we project onto F_1^*, F_1, and F_3^* (which is dual to the inclusion of the bottom z_1-graded components), we get $(w_0^{(1)})^* = d_1^*$, $(w_0^{(2)})^* = d_2$, and $(w_0^{(3)})^* = d_3^*$.

Here is an equivalent restatement of the preceding remark. View $\mathfrak{sl}(F_0)\times\mathfrak{sl}(F_2)$ as the subalgebra $\mathfrak{g}^{(x_1)}$ of $\mathfrak{g}$ (c.f. Chapter 3, §3.1.2).

Theorem 4.27. *There exists a $\mathbb{C}$-algebra homomorphism $w_{\text{ssc}}\colon \widehat{R}_{\text{gen}} \to \mathbb{C}$ so that*

$$\begin{aligned} w_{\text{ssc}}^{(1)}&\colon L(\omega_{x_{r_1}}) \twoheadrightarrow F_0 \\ w_{\text{ssc}}^{(2)}&\colon L(\omega_{y_{r_2-2}}) \twoheadrightarrow F_2^* \\ w_{\text{ssc}}^{(3)}&\colon L(\omega_{z_{r_3}}) \twoheadrightarrow F_2 \end{aligned}$$

are given by projection onto the bottom x_1-graded component. Note that this determines w_{ssc} completely by Proposition 4.22. Furthermore, $w^{(a_2)}$ is also projection onto its bottom x_1-graded component, which is its lowest weight space.

The map w_{ssc} specializes $\mathbb{F}^{\text{gen}}$ to the standard split complex *of $\mathbb{C}$-vector spaces*

$$\mathbb{F}^{\text{ssc}}\colon 0 \to F_3 \to F_2 \oplus Z \to F_0 \oplus Z \to F_0$$

where $Z = \mathbb{C}^{r_2}$.

The comment regarding $w^{(a_2)}$ in this theorem implies the following.

Corollary 4.28. *Let $w\colon \widehat{R}_{\text{gen}} \to R$ specialize $\mathbb{F}^{\text{gen}}$ to a resolution $\mathbb{F}$. Then there is a unique ring homomorphism*

$$\bigoplus_{n\geq 0} L(n\omega_{x_1})^\vee \to R$$

equal to $w^{(a_2)}$ in degree 1. Here the source is the homogeneous coordinate ring of $G/P = G/P_{x_1}$; c.f. Lemma 3.12.

If $w^{(a_2)}$ is surjective and nonzero on only finitely many weight spaces, then the above determines a map

$$\operatorname{Spec} R \to G/P_{x_1} \subset \mathbb{P}(L(\omega_{x_1})).$$

This map lands in the complement of $X^{s_{z_1}s_u s_{x_1}}$ if and only if a_2 generates the unit ideal.

Proof. The homogeneous coordinate ring of G/P is generated in degree 1 so the uniqueness is clear; we need to check that the map is well-defined. As noted in Theorem 4.27, this is certainly true for $w = w_{\text{ssc}}$, so the result for split $\mathbb{F}$ follows from Theorem 4.11 since $\mathrm{GL}(F_i)$ and $\exp \mathbf{L}$ both act on the homogeneous coordinate ring of G/P.

The ring $\bigoplus_{n\geq 0} L(n\omega_{x_1})^\vee$ is a quotient of $\operatorname{Sym} L(n\omega_{x_1})^\vee$ by Plücker relations. An arbitrary $\mathbb{F}$ is split after localization, and relations which hold over the localization must also hold over R.

For the other statement, we need to assume that $w^{(a_2)}$ is finitely supported because of the definition of G/P as an ind-variety. The statement about $X^{s_{z_1}s_u s_{x_1}}$ follows from Lemma 3.12. ☐

4.3.2 Addition of a split part

Let $\mathbb{F}$ be a resolution of format $\underline{f}$, and let $\mathbb{G}$ be a split exact complex. In this section, we study how one can deduce a choice of higher structure maps for $\mathbb{F}\oplus\mathbb{G}$ starting from a choice of higher structure maps for $\mathbb{F}$. Although this may seem like a peculiar question to consider, it is significant for a few reasons:

- We would like to use the theory of higher structure maps to define things which are intrinsic to the module resolved by $\mathbb{F}$. Currently Theorem 4.11 can only compare higher structure maps for resolutions of the same format.
- Later, we will study how higher structure maps evolve under linkage of perfect ideals as an extension of Theorem 2.5. The Betti numbers can increase or decrease during this process, so it is mandatory to know how to adjust higher structure maps appropriately.

We will find that the answer, although simple to state and prove, is surprisingly subtle. In addition to the parameters r_i, f_i already fixed, let $n_1, n_2, n_3 \geq 0$ be integers and let $N_i = \mathbb{C}^{n_i}$. Let $\mathbb{F}$ be a resolution of format $\underline{f}$ and let $\mathbb{F}'$ denote its direct sum with the split exact complex

$$0 \to N_3 \otimes R \to (N_3 \oplus N_2) \otimes R \to (N_2 \oplus N_1) \otimes R \to N_1 \otimes R.$$

Write $\underline{f}' = (f_0+n_1, f_1+n_1+n_2, f_2+n_2+n_3, f_3+n_3)$ for the format of $\mathbb{F}'$ and let T' be the corresponding enlarged diagram

$$\begin{array}{ccccccccccccccccc}
x_{r_1+n_1} & - & \cdots & - & x_{r_1+1} & - & x_{r_1} & - & \cdots & - & x_1 & — & u & — & y_1 & - & \cdots & - & y_{r_2-2} & - & y_{r_2-1} & - & \cdots & - & y_{r_2+n_2-2} \\
 & & & & & & & & & & & & | & & & & & & & & & & & & \\
 & & & & & & & & & & & & z_1 & & & & & & & & & & & & \\
 & & & & & & & & & & & & \vdots & & & & & & & & & & & & \\
 & & & & & & & & & & & & z_{r_3} & & & & & & & & & & & & \\
 & & & & & & & & & & & & | & & & & & & & & & & & & \\
 & & & & & & & & & & & & z_{r_3+1} & & & & & & & & & & & & \\
 & & & & & & & & & & & & \vdots & & & & & & & & & & & & \\
 & & & & & & & & & & & & z_{r_3+n_3} & & & & & & & & & & & &
\end{array}$$

which contains T as a subdiagram. Let $\mathfrak{g}'$ denote the Kac-Moody Lie algebra associated to T'; by deleting the vertices x_{r_1+1}, y_{r_2-1}, and z_{r_3+1} we observe

$$\mathfrak{g}'^{(x_{r_1+1}, y_{r_2-1}, z_{r_3+1})} = \mathfrak{g} \times \mathfrak{sl}(N_1) \times \mathfrak{sl}(N_2) \times \mathfrak{sl}(N_3).$$

For brevity we will call this $\mathfrak{g}'_0$. In the following, $L(\omega, \mathfrak{g}')$ denotes the irreducible highest weight representation of $\mathfrak{g}'$ with weight ω. We continue to use $L(\omega)$ to denote representations of $\mathfrak{g}$. The bottom x_1-graded components of the three extremal representations of $\mathfrak{g}'$ are:

$$\begin{aligned}
F'_0 &= F_0 \oplus N_1 \subset L(\omega_{x_{r_1+n_1}}, \mathfrak{g}')^\vee \\
F'^*_2 &= F_2^* \oplus N_2^* \oplus N_3^* \subset L(\omega_{y_{r_2+n_2-2}}, \mathfrak{g}')^\vee \\
F'_2 &= F_2 \oplus N_2 \oplus N_3 \subset L(\omega_{z_{r_3+n_3}}, \mathfrak{g}')^\vee.
\end{aligned}$$

By examining the weights, we can see which representations of $\mathfrak{g}'_0$ these components belong to, and these are all extremal:

$$\begin{aligned}
F_0 \oplus N_1 &\subset L(\omega_{x_{r_1}})^\vee \oplus N_1 \\
F_2^* \oplus N_2^* \oplus N_3^* &\subset L(\omega_{y_{r_2-2}})^\vee \oplus N_2^* \oplus (N_3^* \otimes L(\omega_{x_1})^\vee) \\
F_2 \oplus N_2 \oplus N_3 &\subset L(\omega_{z_{r_3}})^\vee \oplus (N_2 \otimes L(\omega_{x_1})^\vee) \oplus N_3.
\end{aligned} \tag{4.10}$$

For the upcoming Definition 4.31 and its later uses, it will be helpful to adjust from (4.8) and write $W(a_2)$ as

$$W(a_2) = M^* \otimes L(\omega_{x_1})^\vee = \bigwedge^{f_1} F_1^* \otimes \bigwedge^{f_2} F_2 \otimes \bigwedge^{f_3} F_3^* \otimes [\bigwedge^{r_1} F_1 \oplus \cdots]$$

i.e. we move a factor of $\bigwedge^{f_3} F_3^* \otimes \bigwedge^{f_1} F_1^*$ to the other side and think of $w^{(a_2)}$ as a map

$$[\bigwedge^{r_1} F_1 \oplus \cdots] \otimes R \to \bigwedge^{f_1} F_1 \otimes \bigwedge^{f_2} F_2^* \otimes \bigwedge^{f_3} F_3 \otimes R = M \otimes R.$$

Unlike the three extremal representations considered in (4.10), the one-dimensional bottom x_1-graded component of $L(\omega_{x_1}, \mathfrak{g}')^\vee$ is entirely contained in a single $\mathfrak{g}'_0$-representation, namely[2] $L(\omega_{x_1})^\vee \otimes \bigwedge^{n_1} N_1$. Analogously to (4.10) we have

$$\bigwedge^{f_1} F_1 \otimes \bigwedge^{f_2} F_2^* \otimes \bigwedge^{f_3} F_3 \otimes \bigwedge^{n_1} N_1 \subset L(\omega_{x_1})^\vee \otimes \bigwedge^{n_1} N_1 \subset L(\omega_{x_1}, \mathfrak{g}')^\vee. \tag{4.11}$$

[2] The $\bigwedge^{n_1} N_1$ factor is to keep everything $\prod \mathrm{GL}(F_i) \times \prod \mathrm{GL}(N_i)$-equivariant.

Theorem 4.29. *Let* $w\colon \widehat{R}_{\text{gen}} \to R$ *specialize* $\mathbb{F}^{\text{gen}}$ *to* $\mathbb{F}$. *Let* $(\widehat{R}'_{\text{gen}}, \mathbb{F}^{\text{gen}\prime})$ *denote the generic pair associated to the format* $\underline{f}'$. *Then there is a* $w'\colon \widehat{R}'_{\text{gen}} \to R$ *specializing* $\mathbb{F}^{\text{gen}\prime}$ *to* $\mathbb{F}'$ *such that:*

- $w'^{(1)}$ *is the composite*

$$\begin{array}{c} L(\omega_{x_{r_1+n_1}}, \mathfrak{g}')^\vee \otimes R \\ \big\downarrow \\ (L(\omega_{x_{r_1}})^\vee \oplus N_1) \otimes R \\ \big\downarrow {\scriptstyle (w^{(1)}, \mathrm{Id})} \\ (F_0 \oplus N_1) \otimes R \end{array}$$

- $w'^{(2)}$ *is the composite*

$$\begin{array}{c} L(\omega_{y_{r_2+n_2-2}}, \mathfrak{g}')^\vee \otimes R \\ \big\downarrow \\ (L(\omega_{y_{r_2-2}})^\vee \oplus N_2^* \oplus (N_3^* \otimes L(\omega_{x_1})^\vee)) \otimes R \\ \big\downarrow {\scriptstyle (w^{(2)}, \mathrm{Id}, \mathrm{Id}\otimes w^{(a_2)})} \\ (F_2^* \oplus N_2^* \oplus N_3^*) \otimes R \end{array}$$

- $w'^{(3)}$ *is the composite*

$$\begin{array}{c} L(\omega_{z_{r_3+n_3}}, \mathfrak{g}')^\vee \otimes R \\ \big\downarrow \\ (L(\omega_{z_{r_3}})^\vee \oplus (N_2 \otimes L(\omega_{x_1})^\vee) \oplus N_3) \otimes R \\ \big\downarrow {\scriptstyle (w^{(2)}, \mathrm{Id}\otimes w^{(a_2)}, \mathrm{Id})} \\ (F_2 \oplus N_2 \oplus N_3) \otimes R \end{array}$$

- $w'^{(a_2)}$ *is the composite*

$$\begin{array}{c} L(\omega_{x_1}, \mathfrak{g}')^\vee \otimes R \\ \big\downarrow \\ L(\omega_{x_1})^\vee \otimes \bigwedge^{n_1} N_1 \otimes R \\ \big\downarrow {\scriptstyle w^{(a_2)}\otimes \mathrm{Id}} \\ \bigwedge^{f_1} F_1 \otimes \bigwedge^{f_2} F_2^* \otimes \bigwedge^{f_3} F_3 \otimes \bigwedge^{n_1} N_1 \otimes R \end{array}$$

Each map denoted with a ↡ is given by projection onto the $\mathfrak{g}_0'$-representations identified in (4.10) *and* (4.11).

We include $w'^{(a_2)}$ not because it is necessary to describe w', but to point out that it remains essentially unchanged—only its source has been enlarged.

Proof. Since the proposed construction of w' from w is $\prod \mathrm{GL}(F_i)$-equivariant and $\mathfrak{g}_0'$-equivariant, it suffices to prove the statement for w_{ssc}. If $\mathbb{F}$ is arbitrary, then after localization it is isomorphic to a split exact complex, thus the result would follow from Theorem 4.11 and Proposition 4.23.

But for $w = w_{\mathrm{ssc}}$, this statement is immediate given Theorem 4.27, since by (4.10) this construction of w' simply yields $w'_{\mathrm{ssc}}\colon \widehat{R}'_{\mathrm{gen}} \to \mathbb{C}$ for the larger format $\underline{f}'$. □

Suppose that $\mathbb{F}$ resolves an R-module B. We are now finally ready to make some definitions which are intrinsic to B.

Lemma 4.30. *Let R be a ring and let $U \subset \widehat{R}_{\mathrm{gen}}$ be any subspace that is closed under the actions of $\prod \mathrm{GL}(F_i)$ and $\mathbf{L}$. Let B be an R-module and suppose $w\colon \widehat{R}_{\mathrm{gen}} \to R$ specializes $\mathbb{F}^{\mathrm{gen}}$ to $\mathbb{F}$ resolving B. Then the ideal $w(U)R$ depends only on B and not on the choice of w.*

Proof. Let $w'\colon \widehat{R}_{\mathrm{gen}} \to R$ be another map specializing $\mathbb{F}^{\mathrm{gen}}$ to a resolution $\mathbb{F}'$ of B. To show that $w(U)R = w'(U)R$, it suffices to check after localizing at each prime of R, so we reduce at once to the case that R is local.

In this situation, the resolutions $\mathbb{F}$ and $\mathbb{F}'$ of B must be isomorphic, hence related by the action of $\prod GL(F_i \otimes R)$. Different choices of $w\colon \widehat{R}_{\mathrm{gen}} \otimes R \to R$ specializing $\mathbb{F}^{\mathrm{gen}}$ to a fixed $\mathbb{F}$ are related by $\exp(\mathbf{L} \widehat{\otimes} R)$ by Theorem 4.11. As $U \otimes R$ is closed under both of these actions, the result follows. □

Note that in this lemma, we fix the format of $\mathbb{F}$. With the aid of Theorem 4.29, we can improve this, but first we make some definitions.

Definition 4.31. Let $V = S_\lambda F_1 \otimes S_\mu F_3^*$ be an irreducible representation in the z_1-graded decomposition of

$$L(\omega_{x_1})^\vee = \bigwedge^{r_1} F_1 \oplus \cdots.$$

The *degree* of V is $|\mu|$, and by our description of $\mathbb{L}_1^*$ we necessarily have $|\lambda| = r_1 + |\mu|(r_1 + 1)$. This degree gives the z_1-graded component in which V appears, where we index so that $\bigwedge^{r_1} F_1$ is in degree 0. If V is moreover an extremal representation, then it has multiplicity 1 inside of $L(\omega_{x_1})^\vee$, so it is well-defined to write $\mathbb{I}_{\le(\lambda,\mu)}$ for the $\mathbf{L}$-representation generated by

$$\bigwedge^{f_2} F_2^* \otimes \bigwedge^{f_3} F_3 \otimes V \subset \bigwedge^{f_2} F_2^* \otimes \bigwedge^{f_3} F_3 \otimes L(\omega_{x_1})^\vee = W(a_2)$$

in $\widehat{R}_{\mathrm{gen}}$, or equivalently the Demazure module generated by a highest weight vector of V.

Let $b \in L(\omega_{x_1})^\vee$ be a lowest weight vector for $\mathfrak{g}^{(z_1)}$. Then $b \in L(\omega_{x_1}, \mathfrak{g}')^\vee$ is also a lowest weight vector for $\mathfrak{g}'^{(z_1)}$ in $L(\omega_{x_1}, \mathfrak{g}')^\vee$. Explicitly, suppose b is a lowest weight vector for

$$V = S_\lambda F_1 \otimes S_\mu F_3^* \subset L(\omega_{x_1})^\vee.$$

If $\lambda = (\lambda_1, \ldots, \lambda_{f_1})$, define

$$\lambda' = (\underbrace{1 + |\mu|, \ldots, 1 + |\mu|}_{n_1 \text{ times}}, \lambda_1, \lambda_2, \ldots, \lambda_{f_1}).$$

Then an analysis of weights[3] shows that b is a lowest weight vector for

$$S_{\lambda'} F_1' \otimes S_\mu F_3'^* \subset L(\omega_{x_1}, \mathfrak{g}')^\vee.$$

Remark 4.32. In Chapter 5, we will always fix $r_1 = 1$, so the adjustment to λ will not be needed.

If b is extremal in $L(\omega_{x_1})^\vee$ then it is also extremal in $L(\omega_{x_1}, \mathfrak{g}')^\vee$. The extremal $\mathfrak{g}^{(z_1)}$-representations in $L(\omega_{x_1})^\vee$ are indexed by $W_{P_{z_1}} \backslash W / W_{P_{x_1}}$: if $\sigma \in W$ is a minimal length representative of its double coset and $b \in L(\omega_{x_1})^\vee$ is a lowest weight vector (for $\mathfrak{g}$), then σb is a lowest weight vector for an extremal $\mathfrak{g}^{(z_1)}$-representation.

Remark 4.33. Combinatorially, if we let W' denote the Weyl group for $\mathfrak{g}'$, we have a natural inclusion

$$W_{P_{z_1}} \backslash W / W_{P_{x_1}} \hookrightarrow W'_{P_{z_1}} \backslash W' / W'_{P_{x_1}}.$$

Consequently we may consider the union of all of these sets as we allow the arms of T to grow arbitrarily large:

$$\lim_{r_1, r_2, r_3 \to \infty} W(T)_{P_{z_1}} \backslash W(T) / W(T)_{P_{x_1}}$$

We will often treat σ as a minimal length representative of some element in this limit. The smallest T for which $[\sigma] \in W(T)_{P_{z_1}} \backslash W(T) / W(T)_{P_{x_1}}$ is just the smallest diagram that contains all the reflections needed to write a reduced word for σ.

Definition 4.34. Let R be a ring, and let B be an R-module which admits a free resolution

$$\mathbb{F} \colon 0 \to F_3 \to F_2 \to F_1 \to F_0$$

of some format $\underline{f}$ (with $f_i < \infty$). Let $(\widehat{R}_{\text{gen}}, \mathbb{F}^{\text{gen}})$ be the associated generic ring and choose a $w \colon \widehat{R}_{\text{gen}} \to R$ specializing $\mathbb{F}^{\text{gen}}$ to $\mathbb{F}$.

If $\sigma \in W$ is a minimal length representative of $[\sigma] \in W_{P_{z_1}} \backslash W / W_{P_{x_1}}$ and $S_\lambda F_1 \otimes S_\mu F_3^*$ is the corresponding extremal representation, we define the *higher structure ideal* $\mathrm{HSI}_\sigma(B) \coloneqq w(\mathbb{I}_{\le(\lambda,\mu)})R$.

Proposition 4.35. *The ideals* $\mathrm{HSI}_\sigma(B)$ *are well-defined.*

Proof. Let $\mathbb{F}$ and $\mathbb{F}'$ be two different free resolutions of B. By adding split complexes to both, we may assume that they have the same format. The crucial point is that Theorem 4.29 guarantees $\mathrm{HSI}_\sigma(B)$ does not change in this process.

Once they have the same format, the result follows from Lemma 4.30. The discussion preceding Definition 4.34 guarantees that the notation HSI_σ is unambiguous. □

[3] In particular, this computation requires taking the suppressed $S_{|\mu|} M^*$ factor into account.

Example 4.36. Suppose I is a grade 3 perfect ideal in a local Noetherian ring $(R,\mathfrak{m})$. Choose $w\colon \widehat{R}_{\text{gen}} \to R$ specializing $\mathbb{F}^{\text{gen}}$ to a minimal free resolution of R/I. Then the Buchsbaum-Eisenbud multiplier a_1 is an isomorphism, and we identify $W(a_2)$ with $W(d_1)$. For $e \in W$ the identity, $\mathrm{HSI}_e(R/I) = I$.

If I is not the unit ideal, then it is a complete intersection exactly when the multiplication $\bigwedge^3 F_1 \otimes R \to F_3 \otimes R$ is nonzero mod $\mathfrak{m}$. This multiplicative structure appears in the higher structure maps as

$$w_1^{(1)}\colon \bigwedge^3 F_1 \otimes F_3^* \otimes R \to R$$

where the extremal representation $\bigwedge^3 F_1 \otimes F_3^* \subset L(\omega_{x_1})^\vee$ corresponds to $\sigma = s_{z_1}s_u s_{x_1}$. Hence we see that $\mathrm{HSI}_\sigma(R/I)$ cuts out the *non-c.i. locus*, in the sense that for $\mathfrak{p} \in \operatorname{Spec} R$, $\mathrm{HSI}_\sigma(R/I) \subseteq \mathfrak{p}$ if and only if $I_\mathfrak{p} \subseteq R_\mathfrak{p}$ is a complete intersection (or the unit ideal).

In the next chapter, we will see that the sum $\sum_\sigma \mathrm{HSI}_\sigma(R/I)$, i.e. the image of $w^{(1)}$, is the *non-licci locus*.

Example 4.37. For $I = (t_1, t_2, t_3)^2 \subset \mathbb{C}[t_1, t_2, t_3]$ from Example 4.24, we see that

$$I = \mathrm{HSI}_e(R/I) \subseteq \mathrm{HSI}_\sigma(R/I) \subseteq I$$

since all entries of $w^{(1)}$ have degree 2. Thus $\mathrm{HSI}_\sigma(R/I) = I$ for all σ.

Proposition 4.38. *Let B be a module over a local ring $(R,\mathfrak{m},k)$ and suppose that $w^{(a_2)} \otimes k \neq 0$ for some choice of $w\colon \widehat{R}_{\text{gen}} \to R$ specializing $\mathbb{F}^{\text{gen}}$ to a resolution of B.*

Then there is a unique σ such that

$$\mathrm{HSI}_\rho(B) = (1) \iff \rho \geq \sigma.$$

Furthermore, there exists a choice of $w'\colon \widehat{R}_{\text{gen}} \to R$ specializing $\mathbb{F}^{\text{gen}}$ to a resolution $\mathbb{F}'$ isomorphic to $\mathbb{F}$ such that $w'^{(a_2)}$ determines a map

$$\operatorname{Spec} R \to C_\sigma \subset G/P_{x_1}$$

and $w'^{(a_2)} \otimes k$ gives the torus-fixed k-point of C_σ.

Proof. Both statements are purely representation-theoretic. In fact, this is really just a restatement of Proposition 2.26. The first statement follows from knowing that $w^{(a_2)}$ determines a ring homomorphism

$$\bigoplus_{n\geq 0} L(n\omega_{x_1})^\vee \to R$$

from Corollary 4.28. Just as in Proposition 2.26, the desired σ corresponds to the lowest $\mathfrak{g}^{(z_1)}$-representation on which $w^{(a_2)} \otimes k \neq 0$, which is necessarily an extremal representation. The remainder of the proof continues almost verbatim the same as Proposition 2.26, so we omit it. □

This strongly suggests that for resolutions of grade 3 perfect ideals, where we identify $w^{(a_2)}$ with $w^{(1)}$, the condition $w^{(1)} \otimes k \neq 0$ characterizes when such an ideal is licci. Indeed, if T is of finite type then this statement already follows from the results of Chapter 2, but we will prove something more general in the next chapter.

The main advantage of using higher structure maps over the approach in Chapter 2 is that $\mathrm{HSI}_\sigma(R/I)$ is intrinsic to I and makes sense for any grade 3 perfect ideal, whereas the σ resulting from Proposition 2.26 applied to Theorem 2.24 a priori depends on a choice of links.

4.4 Revisiting the complexes from Chapter 3

For this section, we adopt the notation of Chapter 3. In Remark 4.26, we saw that the construction of $\mathbb{F}$ for $\sigma = e \in W$ implicitly gave a choice of higher structure maps for $\mathbb{F}^{\text{ssc}}$. This is more generally true for *any* choice of σ: starting with $w_{\text{ssc}}\colon \widehat{R}_{\text{gen}} \to \mathbb{C}$, we first base-change to $R = \mathbb{C}[C_\sigma]$ to get $w\colon \widehat{R}_{\text{gen}} \otimes R \to R$, with

$$\begin{aligned} w^{(1)}\colon L(\omega_{x_{r_1}})^\vee \otimes R &\twoheadrightarrow F_0 \otimes R \\ w^{(2)}\colon L(\omega_{y_{r_2-2}})^\vee \otimes R &\twoheadrightarrow F_2^* \otimes R \\ w^{(3)}\colon L(\omega_{z_{r_3}})^\vee \otimes R &\twoheadrightarrow F_2 \otimes R. \end{aligned}$$

Next we precompose by the action of $(\exp(Y)\sigma)^{-1}$ on $\widehat{R}_{\text{gen}} \otimes R$, which is well-defined because $\widehat{R}_{\text{gen}}$ decomposes into integrable representations of $\mathfrak{g}$. This yields a map $w\colon \widehat{R}_{\text{gen}} \otimes R \to R$ with

$$\begin{aligned} w^{(1)}\colon L(\omega_{x_{r_1}})^\vee \otimes R &\xrightarrow{(\exp(Y)\sigma)^{-1}} L(\omega_{x_{r_1}})^\vee \otimes R \twoheadrightarrow F_0 \otimes R \\ w^{(2)}\colon L(\omega_{y_{r_2-2}})^\vee \otimes R &\xrightarrow{(\exp(Y)\sigma)^{-1}} L(\omega_{y_{r_2-2}})^\vee \otimes R \twoheadrightarrow F_2^* \otimes R \\ w^{(3)}\colon L(\omega_{z_{r_3}})^\vee \otimes R &\xrightarrow{(\exp(Y)\sigma)^{-1}} L(\omega_{z_{r_3}})^\vee \otimes R \twoheadrightarrow F_2 \otimes R. \end{aligned}$$

Now we restrict to the bottom z_1-graded components to get the differentials $w_0^{(1)} = d_1$, $w_0^{(2)} = d_2^*$, and $w_0^{(3)} = d_3$ for the complex $\mathbb{F}^{\text{gen}}$ specializes to:

$$\begin{aligned} d_1\colon F_1 \otimes R \hookrightarrow L(\omega_{x_{r_1}})^\vee \otimes R &\xrightarrow{(\exp(Y)\sigma)^{-1}} L(\omega_{x_{r_1}})^\vee \otimes R \twoheadrightarrow F_0 \otimes R \\ d_2^*\colon F_1^* \otimes R \hookrightarrow L(\omega_{y_{r_2-2}})^\vee \otimes R &\xrightarrow{(\exp(Y)\sigma)^{-1}} L(\omega_{y_{r_2-2}})^\vee \otimes R \twoheadrightarrow F_2^* \otimes R \\ d_3\colon F_3 \otimes R \hookrightarrow L(\omega_{z_{r_3}})^\vee \otimes R &\xrightarrow{(\exp(Y)\sigma)^{-1}} L(\omega_{z_{r_3}})^\vee \otimes R \twoheadrightarrow F_2 \otimes R. \end{aligned}$$

and we see that w specializes $\mathbb{F}^{\text{gen}}$ to the resolution $\mathbb{F}$ from Chapter 3!

Similarly, we have that $w^{(a_2)}$ is

$$L(\omega_{x_1})^\vee \otimes R \xrightarrow{(\exp(Y)\sigma)^{-1}} L(\omega_{x_1})^\vee \otimes R \twoheadrightarrow \bigwedge^{f_2} F_2^* \otimes \bigwedge^{f_3} F_3 \otimes R.$$

Let $\mathfrak{m} \subset R$ be the ideal of variables, and $k = R/\mathfrak{m} = \mathbb{C}$. Since $Y \otimes k = 0$, the map $w^{(a_2)} \otimes k$ is nonzero only on the extremal weight space corresponding to σ.

Thus, writing B for the module resolved by $\mathbb{F}$, $\text{HSI}_\rho(B) = (1)$ if and only if $\rho \geq \sigma$ in the Bruhat order. Although not important at present, we note that for this example, we may take $w' = w$ in Proposition 4.38 as $w^{(a_2)}$ already has the desired properties by construction. We are now equipped to disprove Conjecture 4.6.

Theorem 4.39. *Let*

$$\mathbb{F}^{\mathcal{R}}\colon 0 \to \mathcal{R}^3 \to \mathcal{R}^8 \to \mathcal{R}^6 \to \mathcal{R}$$

be a generic free resolution of format $(1, 6, 8, 3)$ *in the sense of Definition 4.5. Then the underlying ring* $\mathcal{R}$ *is not Noetherian.*

Proof. For this format, $T = E_7^{(1)}$ is not of finite type. In particular, there exists an infinite ascending chain in $W_{P_{z_1}} \backslash W / W_{P_{x_1}}$ in the Bruhat order, corresponding to extremal representations in $L(\omega_{x_1})^\vee$ in the z_1-grading. Write $\sigma_1 < \sigma_2 < \cdots$ for their minimal length representatives, and let $S_{\lambda^i} F_1 \otimes S_{\mu^i} F_3^*$ be the extremal representation corresponding to σ_i.

Fix $w \colon \widehat{R}_{\text{gen}} \to R$ specializing $\mathbb{F}^{\text{gen}}$ to $\mathbb{F}^{\mathcal{R}}$. We obtain a sequence of nested Demazure modules

$$\mathbb{I}_{\leq(\lambda^1,\mu^1)} \subsetneq \mathbb{I}_{\leq(\lambda^2,\mu^2)} \subsetneq \cdots \subset W(a_2).$$

Write B for the module resolved by $\mathbb{F}^{\mathcal{R}}$. For each i, let $J_i \subset \mathcal{R}$ be the ideal $\mathrm{HSI}_{\sigma_i}(B) = w(\mathbb{I}_{\leq(\lambda^i,\mu^i)})\mathcal{R}$, and consider the resolution $\mathbb{F}$ constructed in Chapter 3 for $\sigma = \sigma_i$. By genericity of $\mathbb{F}^{\mathcal{R}}$, there exists a map $\phi \colon \mathcal{R} \to R$ specializing $\mathbb{F}^{\mathcal{R}}$ to $\mathbb{F}$. In particular, the composite

$$\widehat{R}_{\text{gen}} \xrightarrow{w} \mathcal{R} \xrightarrow{\phi} R$$

specializes $\mathbb{F}^{\text{gen}}$ to $\mathbb{F}$, so

$$\phi w(\mathbb{I}_{\leq(\lambda^{i-1},\mu^{i-1})})R \subsetneq \phi w(\mathbb{I}_{\leq(\lambda^i,\mu^i)})R = (1)$$

by the discussion preceding this theorem. This says $\phi(J_{i-1})R \subsetneq \phi(J_i)R$, so obviously $J_{i-1} \subsetneq J_i$, and we have exhibited an infinite ascending chain of ideals in $\mathcal{R}$. □

Example 4.40. Informally, this proof says that there are "arbitrarily complicated" free resolutions of format $(1, 6, 8, 3)$, and that one can produce such a family using the representation-theoretic construction of Chapter 3. However, some readers may prefer to have a more explicit demonstration of this fact. Let $R = \mathbb{C}[t_1, t_2, t_3]$ viewed with the standard $\mathbb{Z}$-grading, and let

$$I_0 = (t_1 t_2^2 t_3^2,\ t_2^5 + t_2^3 t_3^2 + t_3^5,\ t_1^5 + t_2^2 t_3^3,\ t_1^4 t_3^3,\ t_1^4 t_2^3,\ t_2 t_3^7).$$

The minimal graded free resolution of R/I_0 is

$$0 \to R^2(-11) \oplus R(-13) \to R^8(-9) \to R^3(-5) \oplus R^2(-7) \oplus R(-8) \to R.$$

For $j \geq 1$, let $\alpha_1, \alpha_2, \alpha_3$ be minimal generators of I_{j-1} of *maximal* degree. Add a homogeneous linear combination of the other generators to each α_i as needed to make $\alpha_1, \alpha_2, \alpha_3$ a homogeneous regular sequence, and define $I_j \coloneqq (\alpha_1, \alpha_2, \alpha_3) : I_{j-1}$.

The starting ideal I_0 has been carefully constructed so that the ideals I_j require more and more links to reach a complete intersection as j increases. It would require a significant digression to prove this sequence of ideals actually has the claimed property, so we will not do so in this thesis, but see Example 5.13 for a sketch of the idea.

All of the ideals will have (ungraded) Betti numbers $(1, 6, 8, 3)$. On the other hand, we have already seen in Chapter 2 that if a licci ideal is associated to an ADE triple (c, d, t), the number of links can be uniformly bounded in terms of the triple—this follows from the proof of Theorem 2.24.

Chapter 5

Linkage of grade three perfect ideals

We return to studying the linkage of grade 3 perfect ideals, but now equipped with the theory of higher structure maps developed in the preceding chapter. The two main results are that

- $\sum_\sigma \mathrm{HSI}_\sigma(R/I)$ defines the non-licci locus, and

- this ideal is the unit ideal when the associated diagram T is Dynkin.

To prove the first result, we must first understand how higher structure maps behave under linkage. Although not coming from $\widehat{R}_{\mathrm{gen}}$, similar ideas have been used in the past, notably in the paper [2] of Avramov, Kustin, and Miller. In [21], the situation is explicitly worked out for some maps $w_j^{(i)}$ where j is small.

From the perspective of free resolutions, perfect ideals $I \subset R$ are characterized by the fact that if $\mathbb{F}$ resolves R/I, then $\mathbb{F}^*$ is also acyclic, resolving the canonical module of R/I. This is the primary reason for developing the theory in Chapter 4 without the assumption $r_1 = 1$: we will need to simultaneously apply it to $\mathbb{F}$ and $\mathbb{F}^*$ in §5.2. Using this, we are able to prove the technical heart of this whole story, which is Theorem 5.17.

The material discussed here is ongoing joint work with Lorenzo Guerrieri and Jerzy Weyman. The main references for this are [20] and [19]. As always, we maintain Assumption 2.1 regarding the base field. We continue in the setting of length three resolutions as in the previous two chapters, but now we will mainly restrict our attention to perfect ideals and make some simplifications accordingly.

Assumption 5.1. Let $r_1 = 1$, $r_2 \geq 3$, $r_3 \geq 1$, and

$$\underline{f} = (f_0, f_1, f_2, f_3) = (1, 1 + r_2, r_2 + r_3, r_3).$$

We will always identify $F_0 = R$, and consider resolutions of cyclic modules R/I. We assume that $\operatorname{grade} I = 3$, so in particular $a_1 \colon M \to \bigwedge^{f_0} F_0^* = R$ is an isomorphism and we make this identification throughout (c.f. Remark 4.13).

Sometimes it will be more convenient to use the parameters $(c, d, t) = (3, r_2 - 2, r_3)$ following Chapters 1 and 2.

Remark 5.2. The assumption $r_2 \geq 3$ is to avoid the degenerate case when the diagram T has no right arm. This is a harmless assumption because (1) we may always add a split part to a resolution to increase the format, and (2) if $\mathbb{F}$ resolves R/I for I perfect with $r_2 = 2$, then $d = 0$ and I must be a complete intersection.

Assumption 5.1 also allows us to identify

$$w^{(a_2)}\colon [\bigwedge^{r_1} F_1 \oplus \cdots] \otimes R \to \bigwedge^{f_1} F_1 \otimes \bigwedge^{f_2} F_2^* \otimes \bigwedge^{f_3} F_3 \otimes R = M \otimes R$$

with

$$w^{(1)}\colon [F_1 \oplus \cdots] \otimes R \to R.$$

Example 5.3. Let us revisit Remark 4.26 regarding the situation for the standard split complex $\mathbb{F}^{\mathrm{ssc}}$. In Theorem 4.27 we observed that $\mathbb{F}^{\mathrm{ssc}}$ admits a particularly simple choice of higher structure maps, where

$$\begin{aligned} w^{(1)}\colon L(\omega_{x_1})^\vee &\twoheadrightarrow \mathbb{C} \\ w^{(2)}\colon L(\omega_{y_{r_2-2}})^\vee &\twoheadrightarrow F_2^* \\ w^{(3)}\colon L(\omega_{z_{r_3}})^\vee &\twoheadrightarrow F_2 \end{aligned} \tag{5.1}$$

are just given by projection onto the bottom x_1-graded component. Here we make two amusing observations that highlight the "triality" of the diagram T:

1. Suppose we interchange the roles of the y and z arms, using the vertex y_1 in place of z_1 and defining F_1', F_2', and F_3' for the new diagram analogously to how F_1, F_2, and F_3 were defined for the original diagram. If the original diagram corresponded to the format $\underline{f} = (1, f_1, f_2, f_3) = (1, 3+d, 2+d+t, t)$, then the new diagram is for the format $\underline{f}' = (1, 3+t, 2+d+t, d)$. Theorem 4.27 tells us that

$$\begin{aligned} L(\omega_{x_{r_1}}) &\twoheadrightarrow \mathbb{C} \\ L(\omega_{z_{r_3}}) &\twoheadrightarrow F_2'^* \\ L(\omega_{y_{r_2-2}}) &\twoheadrightarrow F_2' \end{aligned}$$

   describes higher structure maps for the standard split complex $(\mathbb{F}^{\mathrm{ssc}})'$ of format $\underline{f}'$. But we have not changed the role of the vertex x_1, so if we identify $F_2^* \cong F_2'$ then these are the same maps as in (5.1), just with the roles of $w^{(2)}$ and $w^{(3)}$ interchanged! It is only *after* we restrict to the bottom graded components that we really violate the symmetry: the bottom y_1-graded components recover the differentials of $(\mathbb{F}^{\mathrm{ssc}})'$, whereas the bottom z_1-graded components recover the differentials of $\mathbb{F}'$.

   Hence the *same* maps from Theorem 4.27, decomposed and interpreted with respect to the vertex y_1 instead of z_1, describe structure maps for $(\mathbb{F}^{\mathrm{ssc}})'$.

2. Alternatively, we can exchange the roles of the x and z arms, in which case we get a split complex of format $(f_3, f_2, f_1, 1)$. If we use the involution τ from Chapter 3, §3.2.2 to identify

$L(\omega)$ with $L(\omega)^\vee$, Theorem 4.27 says the projections

$$\begin{aligned} L(\omega_{x_1}) &\twoheadrightarrow F_1^* \\ L(\omega_{y_{r_2-2}}) &\twoheadrightarrow F_1 \\ L(\omega_{z_{r_3}}) &\twoheadrightarrow F_3^* \end{aligned}$$

describe higher structure maps $w^{(3)}$, $w^{(2)}$, and $w^{(1)}$ for the dual $(\mathbb{F}^{\mathrm{ssc}})^*$. In this case the inclusions of F_0, F_2, and F_2^* pick out $w_0^{(3)} = d_1^*$, $w_0^{(2)} = d_2$, and $w_0^{(1)} = d_3^*$.

In pedestrian terms, this amounts to the observation that the differentials of $\mathbb{F}^{\mathrm{ssc}}$, viewed as the complex $\mathbb{F}$ constructed in Chapter 3 for $\sigma = e$, have the form

$$\begin{aligned} \mathbb{C} \hookrightarrow L(\omega_{x_1}) &\twoheadrightarrow F_1^* \\ F_2 \hookrightarrow L(\omega_{y_{r_2-2}}) &\twoheadrightarrow F_1 \\ F_2^* \hookrightarrow L(\omega_{z_{r_3}}) &\twoheadrightarrow F_3^* \end{aligned}$$

If we project first and then precompose by the inclusion, this interprets the differentials as $w_0^{(i)}$ for $(\mathbb{F}^{\mathrm{ssc}})^*$. If we include first and then postcompose by projection, this interprets the differentials as $w_0^{(i)}$ for $\mathbb{F}^{\mathrm{ssc}}$.

In a way, these two basic observations drive all of §5.1 and §5.2 respectively.

5.1 Higher structure maps and linkage

In this section, we will always assume 5.1. The program we carry out here should be viewed as an algebraic analogue of Chapter 2, §2.3. Specifically, we saw in Proposition 2.22 that, under suitable hypotheses, an R-point of G/P determines a *pair* of linked perfect ideals, both of which are necessarily licci by Theorem 2.24. Following Example 5.3, we demonstrate that an analogous phenomenon holds for higher structure maps, except now the perfect ideals involved need not be licci.

In the following, let $\underline{f} = (1, 3+d, 2+d+t, t)$ and $\underline{f}' = (1, 3+t, 2+d+t, d)$. We write $(\widehat{R}_{\mathrm{gen}}, \mathbb{F}^{\mathrm{gen}}) = (\widehat{R}_{\mathrm{gen}}(\underline{f}), \mathbb{F}^{\mathrm{gen}}(\underline{f}))$ and $(\widehat{R}'_{\mathrm{gen}}, \mathbb{F}^{\mathrm{gen}\prime}) = (\widehat{R}_{\mathrm{gen}}(\underline{f}'), \mathbb{F}^{\mathrm{gen}}(\underline{f}'))$. Just as how we defined $F_i, \mathbf{L}, \ldots$, we let $F_i', \mathbf{L}', \ldots$, be the analogous objects for $\underline{f}'$.

If $w\colon \widehat{R}_{\mathrm{gen}} \to R$ specializes $\mathbb{F}^{\mathrm{gen}}$ to some split complex $\mathbb{F}$ over R, then there exists

$$g \in \exp(\mathbf{L} \widehat{\otimes} R) \rtimes \prod \mathrm{GL}(F_i \otimes R)$$

such that $w_{\mathrm{ssc}} g = w$ as maps $\widehat{R}_{\mathrm{gen}} \otimes R \to R$, where $w_{\mathrm{ssc}} g$ means to precompose w_{ssc} by the action of g on $\operatorname{Spec} \widehat{R}_{\mathrm{gen}}$. This is a principle used throughout Chapter 4.

The idea is that, since we have identified w_{ssc} with w'_{ssc} in Example 5.3 (1), we perform the same action on w'_{ssc} to define $w' = w'_{\mathrm{ssc}} g$. This yields a map $\widehat{R}'_{\mathrm{gen}} \otimes R \to R$ with $w^{(i)} = w'^{(i)}$ by construction.

Here there is a slight subtlety. We already know that $\mathfrak{sl}(F_2) \times \mathfrak{g}$ acts on $\widehat{R}'_{\mathrm{gen}}$, simply because this ring comes equipped with an action of $\mathfrak{sl}(F_2') \times \mathfrak{g}$ and we already identified $F_2' \cong F_2^*$ to have w_{ssc} and w'_{ssc} match up in Example 5.3. However, it takes slightly more effort to explain why $\mathfrak{gl}(F_i)$ acts on $\widehat{R}'_{\mathrm{gen}}$. It suffices to show that this action can be seen using the actions of $\mathfrak{gl}(F_i')$ and $\mathfrak{g}$, which we already know to act on $\widehat{R}'_{\mathrm{gen}}$, and for this it is helpful to consider the (y_1, z_1)-bigrading.

5.1.1 Decomposing with respect to the (y_1, z_1)-bigrading

We let $K = \mathbb{C}^3$, and we view $\mathfrak{sl}(K)$ as the subalgebra of $\mathfrak{g}$ corresponding to the vertices u, x_1 in that order (c.f. Example 2.15). Given the decomposition of a $\mathfrak{g}$-representation into y_1 or z_1-graded components, we can further decompose into the (y_1, z_1)-bigrading, in which each component is a representation of

$$\mathfrak{g}^{(y_1,z_1)} = \mathfrak{sl}(K) \times \mathfrak{sl}(F_3) \times \mathfrak{sl}(F_3').$$

At the level of SL-representations, this amounts to writing $F_1 = F_3'^* \oplus K$ and $F_1' = F_3^* \oplus K$ (the duals are because of the order of vertices; see Chapter 3, §3.1.2). However, to correctly relate the actions of $\mathfrak{gl}(F_i)$ and $\mathfrak{gl}(F_i')$, we will instead use

$$\begin{aligned} F_1' &= (F_3^* \otimes \bigwedge^3 K) \oplus K \\ F_1 &= (F_3'^* \otimes \bigwedge^3 K) \oplus K \\ F_2' &= F_2^* \otimes \bigwedge^3 K. \end{aligned} \tag{5.2}$$

These formulas are motivated by Theorem 2.5. Using the above to decompose into representations of $\mathfrak{gl}(K) \times \mathfrak{gl}(F_3) \times \mathfrak{gl}(F_3')$, we find that we get the desired identifications

$$\begin{aligned} \widehat{R}_{\text{gen}} \supset W(d_1) = [F_1 \oplus \bigwedge^3 F_1 \oplus \cdots] = [F_1' \oplus \bigwedge^3 F_1' \oplus \cdots] = W(d_1)' \subset \widehat{R}'_{\text{gen}} \\ \widehat{R}_{\text{gen}} \supset W(d_2) = F_2 \otimes [F_1^* \oplus F_1 \otimes F_3^* \oplus \cdots] = F_2'^* \otimes [F_3' \oplus \bigwedge^2 F_1' \oplus \cdots] = W(d_3)' \subset \widehat{R}'_{\text{gen}} \\ \widehat{R}_{\text{gen}} \supset W(d_3) = F_2^* \otimes [F_3 \oplus \bigwedge^2 F_1 \oplus \cdots] = F_2' \otimes [F_1'^* \oplus F_1' \otimes F_3'^* \oplus \cdots] = W(d_2)' \subset \widehat{R}'_{\text{gen}} \end{aligned} \tag{5.3}$$

To prove this, it is sufficient to verify the statement for scalars in $\mathfrak{gl}(F_i)$ since the result holds automatically at the level of $\mathfrak{sl}(F_i)$-representations. In lieu of this, we display some of the (y_1, z_1)-bigraded components in the representations $W(d_i)$ as it clarifies the situation, and we will need it shortly.

We display the y_1-grading horizontally and the z_1-grading vertically. The decomposition of $W(d_1) = W(d_1)'$ is

$$\begin{array}{cc|cccc} & & & \iddots & \iddots & \\ \vdots & \vdots & & F_3'^* \otimes \wedge^2 F_3^* \otimes S_{3,2,2}K & \iddots & \iddots \\ F_3^* \otimes \wedge^3 F_1 & 1 & F_3^* \otimes \wedge^3 K & F_3'^* \otimes F_3^* \otimes S_{2,2,1} & \wedge^2 F_3'^* \otimes F_3^* \otimes S_{3,2,2}K & \iddots \\ F_1 & 0 & K & F_3'^* \otimes \wedge^3 K & & \\ \hline & & 0 & 1 & \cdots & \\ & & F_1' & F_3'^* \otimes \wedge^3 F_1' & \cdots & \end{array} \tag{5.4}$$

For $W(d_2) = W(d_3)'$ it is

$$
\begin{array}{c} \bigwedge^3 K \otimes F_2^* \\ (= F_2') \end{array} \otimes \left[\begin{array}{cc|cccc}
\vdots & \vdots & & & \iddots & \\
\bigwedge^3 K^* \otimes F_3^* \otimes \bigwedge^4 F_1 & 2 & & F_3'^* \otimes F_3^* \otimes \bigwedge^3 K & \iddots & \iddots \\
\bigwedge^3 K^* \otimes \bigwedge^2 F_1 & 1 & K^* & F_3'^* \otimes K & \bigwedge^2 F_3'^* \otimes \bigwedge^3 K & \\
\bigwedge^3 K^* \otimes F_3 & 0 & F_3 \otimes \bigwedge^3 K^* & & & \\
\hline
 & & 0 & 1 & 2 & \cdots \\
 & & F_1'^* & F_3'^* \otimes F_1' & \bigwedge^2 F_3'^* \otimes \bigwedge^3 F_1' & \cdots
\end{array} \right] \tag{5.5}
$$

Finally, for $W(d_3) = W(d_2)'$ it is

$$
\begin{array}{c} \bigwedge^3 K^* \otimes F_2 \\ (= F_2'^*) \end{array} \otimes \left[\begin{array}{cc|cccc}
\vdots & \vdots & & & \iddots & \\
\bigwedge^3 K \otimes \bigwedge^2 F_3^* \otimes \bigwedge^3 F_1 & 2 & & \bigwedge^2 F_3^* \otimes S_{2,2,2}K & \iddots & \iddots \\
\bigwedge^3 K \otimes F_3^* \otimes F_1 & 1 & & F_3^* \otimes S_{2,1,1}K & F_3'^* \otimes F_3^* \otimes S_{2,2,2}K & \\
\bigwedge^3 K \otimes F_1^* & 0 & F_3' & \bigwedge^2 K & & \\
\hline
 & & 0 & 1 & 2 & \cdots \\
 & & F_3' & \bigwedge^2 F_1' & F_3'^* \otimes \bigwedge^4 F_1' & \cdots
\end{array} \right]
$$

Using the above, we are ready to prove the algebraic analogue of Proposition 2.22.

Theorem 5.4. *Let $w: \widehat{R}_{\mathrm{gen}} \to R$ specialize $\mathbb{F}^{\mathrm{gen}}$ to a resolution $\mathbb{F}$ of R/I. Then the maps $w^{(i)}$, reinterpreted using* (5.3)*, define a map $w': \widehat{R}'_{\mathrm{gen}} \to R$ with the property that $w^{(i)} = w'^{(i)}$.*

Furthermore, if $w^{(1)}(K \otimes R) = (\alpha_1, \alpha_2, \alpha_3) \subset R$ is a complete intersection, where $K \subset L(\omega_{x_1})^\vee$ is the bottom (y_1, z_1)-bigraded component as in (5.4)*, then w' specializes $\mathbb{F}^{\mathrm{gen}\prime}$ to the resolution of $R/((\alpha_1, \alpha_2, \alpha_3) : I)$ described in Theorem 2.5.*

Proof. As the first statement is about w' satisfying the requisite relations in $\widehat{R}'_{\mathrm{gen}}$, we can perform the usual reduction to the split exact case, since any $\mathbb{F}$ is split on a dense open set. Now that we know $\mathfrak{gl}(F_i)$ acts on $\widehat{R}'_{\mathrm{gen}}$, the claim follows from the argument sketched at the beginning of this section.

We will abuse notation and write F_i to mean $F_i \otimes R$ below, and similarly for F_i'. For the other part of the theorem, w' specializes $\mathbb{F}^{\mathrm{gen}\prime}$ to some complex

$$0 \to F_3' \xrightarrow{d_3'} F_2' \xrightarrow{d_2'} F_1' \xrightarrow{d_1'} R.$$

To determine the differentials d_i', we need only look at the bottom y_1-graded components of each $w'^{(i)} = w^{(i)}$.

Examining the bigraded decompositions above, we find from (5.4) that the differential d_1' has two components in its source, being the sum of

$$K \hookrightarrow F_1 \xrightarrow{d_1} R$$
$$F_3^* \otimes \bigwedge^3 K \xrightarrow{w_1^{(1)}} R$$

The differential d_2' has those two components in its target:

$$F_2' = F_2^* \otimes \bigwedge^3 K \xrightarrow{d_3^*} F_3^* \otimes \bigwedge^3 K \hookrightarrow F_1'$$
$$F_2' = F_2^* \otimes \bigwedge^3 K \xrightarrow{(w_1^{(3)})^*} (\bigwedge^2 F_1)^* \otimes \bigwedge^3 K \to K \hookrightarrow F_1'.$$

and d_3' is just the part of d_2^* given by

$$F_3' \hookrightarrow F_1^* \otimes \bigwedge^3 K \xrightarrow{d_2^*} F_2^* \otimes \bigwedge^3 K = F_2'.$$

Compare this to the comparison map from the Koszul complex on $w^{(1)}(K \otimes R)$ to $\mathbb{F}$ induced by the multiplicative structure $w_1^{(i)}$:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & F_3 & \longrightarrow & F_2 & \longrightarrow & F_1 & \longrightarrow & R \\
 & & \uparrow{\scriptstyle w_1^{(1)}} & & \uparrow{\scriptstyle w_1^{(3)}} & & \uparrow & & \| \\
0 & \longrightarrow & \bigwedge^3 K & \longrightarrow & \bigwedge^2 K & \longrightarrow & K & \longrightarrow & R
\end{array}$$

By a miracle we have reconstructed the complex

$$0 \to F_3' \xrightarrow{d_3'} F_2^* \otimes \bigwedge^3 K \xrightarrow{d_2'} F_3^* \otimes \bigwedge^3 K \oplus K \xrightarrow{d_1'} R$$

described in Theorem 2.5!

The conscientious reader may worry that we have not carefully checked the coefficients for the two parts of d_1' and d_2' to ensure that $d_1'd_2' = 0$. But note that this is the specialization of $\mathbb{F}^{\text{gen}\prime}$ via w', so it is guaranteed to be a complex. One just needs to consistently identify $F_3^* \otimes \bigwedge^3 K \oplus K$ as the standard representation of $\mathfrak{gl}(F_1')$ in both the source of d_1' and the target of d_2'. □

5.1.2 Ranks of $w^{(i)} \otimes k$ and linkage

Throughout this subsection, we will work in a local Noetherian ring R.

Definition 5.5. If $I \subset R$ is a grade 3 perfect ideal, let $\mathrm{NL}(I) \coloneqq \sum_\sigma \mathrm{HSI}_\sigma(R/I)$. In other words, it is the image of the map $w^{(1)}: L(\omega_{x_1})^\vee \otimes R \to R$ for any choice of $w: \widehat{R}_{\text{gen}} \to R$ specializing $\mathbb{F}^{\text{gen}}$ to a resolution of R/I.

Theorem 5.6. *Let $(R, \mathfrak{m}, k)$ be a local Noetherian ring, $I \subset R$ a grade 3 perfect ideal, and $w: \widehat{R}_{\text{gen}} \to R$ a map specializing $\mathbb{F}^{\text{gen}}$ to a minimal resolution $\mathbb{F}$ of R/I.*

1. *The ideal* $\mathrm{NL}(I)$ *is invariant under linkage.*
2. *The ideal I is licci if and only if* $\mathrm{NL}(I) = (1)$.
3. *If* $w^{(3)} \otimes k \neq 0$ *then there exists* I' *in the even linkage class of I such that either* $t(R/I') < t(R/I)$ *or* $I' = (1)$.
4. *If* $w^{(2)} \otimes k \neq 0$ *then there exists* I' *in the even linkage class of I such that either* $d(I') < d(I)$ *or* $I' = (1)$.

Since $\mathrm{NL}(I)$ is defined using structure maps for a free resolution of R/I, it commutes with localization. So for R not necessarily local, point (2) allows us to interpret

$$V(\mathrm{NL}(I)) = \{\mathfrak{p} \in \operatorname{Spec} R : I_\mathfrak{p} \subset R_\mathfrak{p} \text{ is not licci or the unit ideal}\}$$

as the *non-licci locus* of I, explaining the notation $\mathrm{NL}(I)$.

Proof. Statement (1) follows immediately from Theorem 5.4. The "only if" implication of (2) is also immediate, since $I = (1)$ satisfies $\mathrm{NL}(I) = (1)$, and any licci ideal can be linked in some number of steps to the unit ideal. The "if" implication can be proved in the same fashion as Theorem 2.24 replacing γ by $w^{(1)}$: since $w^{(1)} \otimes k \neq 0$, there exist elements $g_1, g_1', \ldots, g_N, g_N'$, where $g_i \in \mathrm{GL}(F_1 \otimes k)$ and $g_i' \in \mathrm{GL}(F_1' \otimes k)$, such that $(w^{(1)} \otimes k) g_1 g_1' \cdots g_N g_N'$ is nonzero on the lowest weight space of $L(\omega_{x_1})^\vee$.

The existence of such elements can either be seen using the argument in Theorem 2.24. Alternatively, any weight space that is a lowest weight space for both $\mathrm{GL}(F_1)$ and $\mathrm{GL}(F_1')$ must be a lowest weight space for $\mathfrak{g}$ because the type A subdiagrams corresponding to $\mathrm{GL}(F_1)$ and $\mathrm{GL}(F_1')$ cover the whole diagram T. So it is always possible to use either the action of $\mathrm{GL}(F_1)$ or $\mathrm{GL}(F_1')$ to make $L(\omega_{x_1})^\vee \otimes k \to k$ nonzero on lower weight spaces, until it is nonzero on the lowest weight space.

Then we pick lifts $\tilde{g}_i \in \mathrm{GL}(F_1 \otimes R)$ of g_i and $\tilde{g}_i' \in \mathrm{GL}(F_1' \otimes R)$ of g_i' following the proof of Theorem 2.24, so that sequentially acting on $w^{(1)}$ by these elements realizes a sequence of links from I to the unit ideal.

Statements (3) and (4) can be proved in the same manner. If some row of $w^{(3)} \otimes k$ is nonzero, then the exact same argument applied to that row (instead of $w^{(1)}$) shows that there exists a sequence of links $I = I_0 \sim I_1 \cdots \sim I_{2N} = I'$ and a resolution $\mathbb{F}'$ of R/I' having the same format $(1, f_1, f_2, f_3)$ as the original resolution $\mathbb{F}$, but such that $w_0'^{(3)} \otimes k \neq 0$ for $\mathbb{F}'$. This means either $I' = (1)$ or $t(R/I') < f_3 = t(R/I)$ as desired. The proof of (4) is completely analogous. □

In Theorem 4.29, we saw that the addition of a split part to a resolution $\mathbb{F}$ causes copies of $w^{(a_2)}$ (here identified with $w^{(1)}$) to appear in the new maps $w^{(3)}$ and $w^{(2)}$. Combining this with Theorem 5.6, we see that there is a dichotomy in how the ranks of $w^{(3)}$ and $w^{(3)}$ behave, depending on whether I is licci.

Proposition 5.7. *If I is licci then* $w^{(3)}$ *and* $w^{(2)}$ *are surjective.*

Proof. This statement is true for the unit ideal by Theorem 4.27, preserved under linkage fixing a diagram T by Theorem 5.4, and preserved by addition of a split exact complex by Theorem 4.29. Note that this last point is reliant on $w^{(1)}$ being surjective. Technically we do not need to invoke Theorem 4.29 since I may be linked to the unit ideal using minimal links by Proposition 2.12, so we never need to change the diagram T. But it is reassuring to know that everything is consistent. □

The situation is quite different for non-licci perfect ideals:

Lemma 5.8. *Let I be a grade 3 perfect ideal in a local Noetherian ring R. Suppose that I is not licci. Let $w\colon \widehat{R}_{\text{gen}} \to R$ specialize $\mathbb{F}^{\text{gen}}$ to a resolution $\mathbb{F}$ of R/I, with format $\underline{f}$. Then the quantities*

$$f_3 - \operatorname{rank}(w^{(3)} \otimes k), \quad f_1 - 3 - \operatorname{rank}(w^{(2)} \otimes k)$$

are intrinsic to I, and are interchanged under linkage.

Proof. Fixing the format of $\mathbb{F}$, any two choices of w will have the same quantities $\operatorname{rank}(w^{(i)} \otimes k)$ by Lemma 4.30. Since I is not licci, $w^{(1)} \otimes k = 0$. Theorem 4.29 implies that the addition of a split part of format $(n_1, n_1 + n_2, n_2 + n_3, n_3)$ to $\mathbb{F}$ increases both f_3 and $\operatorname{rank}(w^{(3)} \otimes k)$ by n_3. Similarly it increases both f_1 and $\operatorname{rank}(w^{(2)} \otimes k)$ by n_2.

Theorem 5.4 shows that the quantities are interchanged by a link, since the new format is $(1, f_3 + 3, f_2, f_1 - 3)$. □

Theorem 5.9. *Assume the setup of Lemma 5.8. Then there is an ideal I' in the even linkage class of I such that*

$$t(R/I') = f_3 - \operatorname{rank}(w^{(3)} \otimes k), \quad d(I') = f_1 - 3 - \operatorname{rank}(w^{(2)} \otimes k).$$

In particular, if $\mathbb{F}'$ is a minimal free resolution of R/I' and w' is a choice of higher structure maps for $\mathbb{F}'$, then $w'^{(i)} \otimes k = 0$ for all i. The ideal I' has minimal deviation and type in its even linkage class.

Proof. This follows from repeated application of Theorem 5.6 and Lemma 5.8 until both $w^{(2)} \otimes k$ and $w^{(3)} \otimes k$ are zero. □

5.1.3 Classification of all grade 3 licci ideals

Fix a format $\underline{f}$ as in Assumption 5.1 and its corresponding diagram T. Let $(R_\sigma, \mathbb{F}^\sigma)$ be the resolution constructed in Chapter 3 for $\sigma \in W$ a minimal length representative of $W_{P_{z_1}} \backslash W / W_{P_{x_1}}$, so $R_\sigma = \mathbb{C}[C_\sigma]$ is a polynomial ring in $\ell(\sigma)$ variables. Write R_σ/I_σ for the module resolved by $\mathbb{F}^\sigma$. This is the coordinate ring of $\mathcal{N}_\sigma^w \subset C_\sigma$. Let $\mathfrak{m}_\sigma$ denote the maximal ideal generated by the variables of R_σ.

Proposition 5.10. *If $\sigma \neq e$, the ideal $(I_\sigma)_{\mathfrak{m}_\sigma}$ is a grade 3 licci ideal in $(R_\sigma)_{\mathfrak{m}_\sigma}$.*

Proof. If $\sigma = e$ then I_σ is the unit ideal. Otherwise it is a grade 3 perfect ideal as established in Chapter 3, §3.3. This proposition then follows from Theorem 5.6, and the observation that $\operatorname{HSI}_\sigma(R_\sigma/I_\sigma) = (1)$ made in Chapter 4, §4.4. □

Let I be a grade 3 perfect ideal in a local Noetherian ring $(R, \mathfrak{m}, k)$ and suppose R/I has a minimal resolution of format $\underline{f}$. If S is another local Noetherian ring, and $\phi\colon R \to S$ is a local homomorphism such that $J = \phi(I)S$ has grade 3 in S, then J is necessarily also perfect. Furthermore, the resolution of R/I specializes to one for S/J. So we have $\operatorname{HSI}_\rho(S/J) = (\operatorname{HSI}_\rho(R/I))S$ for all ρ. As we assumed ϕ to be a local homomorphism, $\operatorname{HSI}_\rho(S/J) = (1)$ if and only if $\operatorname{HSI}_\rho(R/I) = (1)$.

If I is licci, then Theorem 5.6 says

$$\operatorname{NL}(I) = \sum_\sigma \operatorname{HSI}_\sigma(R/I) = (1).$$

By Proposition 4.38, there exists a unique minimal σ for which $\mathrm{HSI}_\sigma(R/I) = (1)$. The above argument shows that this is preserved by local homomorphisms $\phi\colon R \to S$ as long as grade $\phi(I)S = 3$. In reverse, we see it is preserved under deformation.

Now we see the true significance of the resolutions constructed in Chapter 3: they yield the generic resolutions for all grade 3 licci ideals.

Theorem 5.11. *Let I be a grade 3 licci ideal in a local Noetherian ring $(R, \mathfrak{m}, k)$, and let σ be minimal such that $\mathrm{HSI}_\sigma(R/I) = (1)$. Let $(R_\sigma, \mathbb{F}^\sigma)$ be as above. Then there is a homomorphism $\phi\colon R_\sigma \to R$ with $\phi(\mathfrak{m}_\sigma) \subseteq \mathfrak{m}$ specializing $\mathbb{F}^\sigma$ to a resolution of R/I.*

To be precise, the definition of $\mathbb{F}^\sigma$ depends on the diagram T. We can simply take the smallest diagram T on which σ is defined (c.f. Remark 4.33); enlarging T only amounts to adding a split exact summand to $\mathbb{F}^\sigma$.

Proof. From Proposition 4.38, we get a map $w\colon \widehat{R}_{\mathrm{gen}} \to R$ such that

1. w specializes $\mathbb{F}^{\mathrm{gen}}$ to a minimal resolution of R/I,
2. $w^{(1)}$ describes an R-point of the Schubert cell C_σ, and
3. $w^{(1)} \otimes k$ is the torus-fixed k-point of C_σ, which was denoted σv in Chapters 2 and 3.

By (2), $w^{(1)}$ describes a homomorphism $\phi\colon R_\sigma \to R$. Furthermore, point (3) ensures that $\phi(\mathfrak{m}_\sigma) \subseteq \mathfrak{m}$. By construction

$$\phi(I_\sigma)R = w^{(1)}(F_1 \otimes R) = I$$

where $F_1 \otimes R \subset L(\omega_{x_1})^\vee \otimes R$ is the bottom z_1-graded component. Since I_σ is perfect and $\phi(I_\sigma)R$ has the same grade, it follows that the resolution $\mathbb{F}^\sigma$ of R_σ/I_σ specializes to a resolution of R/I. □

Note that we do *not* claim $w\colon \widehat{R}_{\mathrm{gen}} \to R$ and $\phi\colon R_\sigma \to R$ specialize $\mathbb{F}^{\mathrm{gen}}$ and $\mathbb{F}^\sigma$ to identical resolutions over R; by construction they have the same differential d_1 but the differentials d_2, d_3 may be different. Of course, they can be made to be equal after adjusting w further using the action of $\prod \mathrm{GL}(F_i \otimes R)$, but that is unnecessary for the proof.

Now we can address Conjecture 2.34 for $c = 3$. As in 2.31, we assume the ambient rings R and S in the following are power series rings in finitely many variables.

Theorem 5.12. *Let $T = T_{2,d+1,t+1}$ be the diagram associated to the format $(1, 3+d, 2+d+t, t)$. Consider the map*

$$\left\{\begin{matrix}\text{grade 3 licci ideals in } k[\![\{X\}]\!] \text{ with} \\ \text{deviation} \le d, \text{ and type} \le t\end{matrix}\right\} \xrightarrow{\ \Psi\ } W_{P_{z_1}} \backslash W / W_{P_{x_1}} - [e]$$

sending $I \subset R$ to the minimal σ for which $\mathrm{HSI}_\sigma(R/I) = (1)$.

1. *Ψ is surjective.*

2. *If $I \subset R$ and $J \subset S$ are grade 3 licci ideals with deviation $\leq d$ and type $\leq t$, then $\Psi(I) = \Psi(J)$ if and only if R/I and S/J admit a common deformation.*

Proof. The map is surjective by Proposition 5.10. The "if" part of (2) follows from the observation made above that $\Psi(I)$ is preserved under deformation.

For the "only if" part of (2), let $\sigma = \Psi(I) = \Psi(J)$ and let B be the completion of R_σ with respect to $\mathfrak{m}_\sigma$. Then $A = (B \widehat{\otimes} R \widehat{\otimes} S)/(I_\sigma)$ is a deformation of both R/I and S/J: Theorem 5.11 exhibits ϕ_I and ϕ_J specializing I_σ to I and J respectively, so we obtain R/I as the quotient of A by $X - \phi_I(X)$ and the variables of S, and similarly we obtain S/J as the quotient of A by $X - \phi_J(X)$ and the variables of R. ☐

Example 5.13. Consider the sequence of ideals I_j constructed in Example 4.40. Using $(b_1, \dots, b_n)$ to denote the product of simple reflections $s_{b_1} \cdots s_{b_n}$, one can show

$$\Psi(I_0) = (z_1, u, x_1, z_2, z_1, u, z_3, z_2, z_1, y_1, u, y_2, y_1, y_3, y_2, z_1, u, x_1, z_2, z_1, u, z_3, z_2, z_1, y_1, u, x_1).$$

Let χ be the function interchanging y_i and z_i, and write $\chi(b_1, \dots, b_n) := (\chi(b_1), \dots, \chi(b_n))$. Then for $j \geq 1$,

$$\Psi(I_j) = (z_1, u, x_1, z_2, z_1, u, z_3, z_2, z_1)\chi(\Psi(I_{j-1})).$$

Here the prefix sequence of reflections represents the permutation in $\mathfrak{S}_6$ sending $(1, 2, 3, 4, 5, 6)$ to $(4, 5, 6, 1, 2, 3)$. Each word obtained in this recursive manner is a reduced word for its respective $\Psi(I_j)$.

Combinatorially, a reduced word for $\Psi(I_j)$ contains a substring of the form $(z_1, y_1, z_1, y_1, \dots)$ that can be made arbitrarily long as we increase j, and this can be used to lower-bound the number of links required to link to a complete intersection, but we will not give the details here.

5.2 An ADE correspondence for grade 3 perfect ideals

We finally arrive at the proof of Conjecture 1.4 for $c = 3$. To be precise, we prove the "if" implication, as the existence of non-licci perfect ideals associated to non-ADE triples $(3, d, t)$ was already shown in [14].

Theorem 5.14. *Let I be a grade 3 perfect ideal in a local Noetherian ring R, with deviation $d = d(I)$ and type $t = t(R/I)$. Suppose that $(3, d, t)$ is an ADE triple (c.f. Definition 1.3), i.e. that one of the following holds:*

- $t \leq 1$,
- $d \leq 1$,
- $t \leq 2$ *and* $d \leq 4$, *or*
- $t \leq 4$ *and* $d \leq 2$.

Then I is licci, thus classified by Theorem 5.11.

Only the last two cases are new, but the proof uniformly handles all cases.

Proof. By Theorem 5.6, it is sufficient to show that $w^{(1)}$ is surjective. This is a special case of the upcoming Theorem 5.17. ☐

The reach of this theorem can be extended by coupling it with Theorem 5.9.

Corollary 5.15. *Let I be a grade 3 perfect ideal in a local Noetherian ring $(R, \mathfrak{m}, k)$, and let $w\colon \widehat{R}_{\text{gen}} \to R$ specialize $\mathbb{F}^{\text{gen}}$ to a resolution $\mathbb{F}$ of R/I, of format $(1, f_1, f_2, f_3)$. Suppose one of the following holds:*

- $f_3 - \operatorname{rank}(w^{(3)} \otimes k) \leq 1$,
- $f_1 - 3 - \operatorname{rank}(w^{(2)} \otimes k) \leq 1$,
- $f_3 - \operatorname{rank}(w^{(3)} \otimes k) \leq 2$ *and* $f_1 - 3 - \operatorname{rank}(w^{(2)} \otimes k) \leq 4$, *or*
- $f_3 - \operatorname{rank}(w^{(3)} \otimes k) \leq 4$ *and* $f_1 - 3 - \operatorname{rank}(w^{(2)} \otimes k) \leq 2$.

Then I is licci.

Proof. Suppose that I were not licci. Then Theorem 5.9 produces I' contradicting Theorem 5.14. ☐

Moreover, assuming one of the conditions in Corollary 5.15, then *a posteriori* the maps $w^{(2)} \otimes k$ and $w^{(3)} \otimes k$ have full rank f_2 by Proposition 5.7.

Example 5.16. This corollary generalizes perspectives which are present in many older works on the topic. For instance, [4] and [45] study grade 3 perfect ideals whose Tor algebra multiplication

$$\bigwedge^2 \operatorname{Tor}_1(R/I, k) \to \operatorname{Tor}_2(R/I, k)$$

has sufficiently high rank to guarantee that I is directly linked to an almost complete intersection, therefore licci. Taking w to be a choice of higher structure maps for a minimal free resolution of R/I, the above multiplication is none other than $w_1^{(3)} \otimes k$, and their assumptions amount to requiring that $f_3 - \operatorname{rank}(w_1^{(3)} \otimes k) \leq 1$. Specifically, [4] considers $f_3 = 2$ and $\operatorname{rank}(w_1^{(3)} \otimes k) \geq 1$, whereas [45] considers $f_3 = 3$ and $\operatorname{rank}(w_1^{(3)} \otimes k) \geq 2$. This falls under statement (1) of Corollary 5.15, since

$$f_3 - \operatorname{rank}(w^{(3)} \otimes k) \leq f_3 - \operatorname{rank}(w_1^{(3)} \otimes k) \leq 1.$$

Before we state and prove the main theorem, we give some informal motivation for the proof. In particular, we recall the geometric significance of desiring $w^{(1)}$ to be surjective. Assume that $\mathfrak{g}$ is of finite type. In Chapter 2, we saw that licci ideals come from R-points of G/P.

Now suppose we are presented with a grade 3 perfect ideal $I \subset R$ and we want to prove that it is licci. The candidate R-point of G/P is provided by the theory of higher structure maps; namely we fix $w\colon \widehat{R}_{\text{gen}}(\underline{f}) \to R$ specializing $\mathbb{F}^{\text{gen}}(\underline{f})$ to a resolution $\mathbb{F}$ of R/I and interpret the map $w^{(1)}\colon L(\omega_{x_1})^\vee \otimes R \to R$ as a homomorphism

$$\bigoplus_{n \geq 0} L(n\omega_{x_1})^\vee \to R$$

from the homogeneous coordinate ring of G/P as in Corollary 4.28. In order for this to define an R-point of G/P, we need $w^{(1)}$ to be surjective—otherwise, this only gives a map to the affine cone over G/P.

Our usual trick of working over a localization is ineffective here, since a map from an open set in $\operatorname{Spec} R$ to G/P need not extend to the whole of $\operatorname{Spec} R$. But in the proof of Theorem 2.24 for licci ideals, not only did we produce an R-point of G/P, we actually produced an R-point of G. Indeed, the links $I \sim \cdots \sim (1)$ were recast using the actions of $\mathrm{GL}(F_1)$ and $\mathrm{GL}(F_1')$, and using this we were able to produce an alternating product $g = g_1 g_1' \cdots g_N g_N' \in G$, where $g_i \in \mathrm{GL}(F_1 \otimes R)$ and $g_i' \in \mathrm{GL}(F_1' \otimes R)$, such that acting on the Borel-fixed point $v \in G/P$ by g yields our desired R-point of G/P.

So we ask the natural question: using the theory of higher structure maps in lieu of linkage, can we instead produce an R-point of G and use this to prove that $w^{(1)}$ truly is an R-point of G/P? At first glance, this may appear to be a harder problem. But there is a key difference between G and G/P: the former is affine! Therefore, since $\operatorname{grade} I = 3 \geq 2$, any map $\operatorname{Spec} R \to G$ defined on the complement of $V(I)$ must extend to all of $\operatorname{Spec} R$.

Hence we see that, for the purposes of exhibiting an R-point of G, we may work on the complement of $V(I)$ and reduce to the case of a split exact complex. The catch is that a single localization is no longer sufficient: we need to take a regular sequence $h_1, h_2 \in I$, define maps $\operatorname{Spec} R_{h_1} \to G$ and $\operatorname{Spec} R_{h_2} \to G$, and confirm that they agree on the overlap $\operatorname{Spec} R_{h_1 h_2}$.

There is too much indeterminacy trying to lift to G using w alone, and we cannot choose an arbitrary lift as that would cause incompatibility on the overlap. Now we use the hypothesis that I is perfect, which tells us that $\mathbb{F}^*$ is also acyclic. Thus we may choose some $w' \colon \widehat{R}_{\mathrm{gen}}(\underline{f}^*) \to R$ specializing $\mathbb{F}^{\mathrm{gen}}(\underline{f}^*)$ to $\mathbb{F}^*$, where $\underline{f}^* \coloneqq (f_3, f_2, f_1, 1)$ is the dual format. It turns out that the indeterminacy of lifting w to G is exactly $\exp(\mathfrak{n}_{x_1}^+)$, which parametrizes the non-uniqueness of w' specializing $\mathbb{F}^{\mathrm{gen}}(\underline{f}^*)$ to $\mathbb{F}^*$. Thus by fixing both w and w', we can construct $\operatorname{Spec} R_{h_i} \to G$ on each localization in a manner which glues and extends to yield a map $\operatorname{Spec} R \to G$.

All of this is informal motivation. In the actual proof, we will avoid looking at the group G because we have never even defined it. For the theorem, we drop the assumption that $r_1 = 1$, since it takes essentially no extra effort to prove it in this extra generality.

Theorem 5.17. *Suppose we have a free resolution $\mathbb{F}$ over a ring R whose format $\underline{f} = (f_0, f_1, f_2, f_3) = (r_1, r_1 + r_2, r_2 + r_3, r_3)$ corresponds to a diagram T which is Dynkin. Assume moreover that $\mathbb{F}^*$ is also acyclic. Then for any choice of $w \colon \widehat{R}_{\mathrm{gen}} \to R$ specializing $\mathbb{F}^{\mathrm{gen}}$ to $\mathbb{F}$, the structure maps $w^{(i)}$ are surjective.*

Remark 5.18. The prototypical example of Theorem 5.17 mentioned in Chapter 1 is $w^{(2)}$ for $\underline{f} = (1, n, n, 1)$. This is a $n \times 2n$ matrix consisting of the differential d_2 and an isomorphism $F_1 \cong F_2^*$ induced by a choice of multiplication on $\mathbb{F}$. The surjectivity of the matrix is evident from the presence of an invertible submatrix.

5.2.1 The setup of the proof

For the proof, we fix a map $w' \colon \widehat{R}_{\mathrm{gen}}(\underline{f}^*) \to R$ specializing $\mathbb{F}^{\mathrm{gen}}(\underline{f}^*)$ to $\mathbb{F}^*$. As usual, we will identify

$$\bigwedge^{f_3} F_3 \otimes \bigwedge^{f_2} F_2^* \otimes \bigwedge^{f_1} F_1 \cong \bigwedge^{f_0} F_0 \tag{5.6}$$

using a_1 following Remark 4.13. We also make the analogous identification

$$\bigwedge^{f_0} F_0^* \otimes \bigwedge^{f_1} F_1 \otimes \bigwedge^{f_2} F_2^* \cong \bigwedge^{f_3} F_3^* \tag{5.7}$$

for $\widehat{R}_{\text{gen}}(\underline{f}^*)$.

We will prove surjectivity of the maps $w^{(i)}$ by exhibiting them inside of larger invertible matrices. In this manner, we avoid having to define the group G by working with automorphisms of the representations $L(\omega)$ directly. We discuss $w^{(1)}$ as an example, but the situation for the other structure maps is completely analogous. Our strategy is to produce a map A_1 making the following diagram commute:

$$\begin{array}{ccccc} & & L(\omega_{x_{r_1}}) \otimes L(\omega_{x_{r_1}})^\vee & & \\ & \overset{\mathrm{Id}\otimes\iota}{\nearrow} & \Big\downarrow {\scriptstyle A_1} & \overset{\eta\otimes\mathrm{Id}}{\nwarrow} & \\ \widehat{R}_{\text{gen}}(\underline{f}^*) \supset L(\omega_{x_{r_1}}) \otimes F_1 & \xrightarrow{w'^{(3)}} & R & \xleftarrow{w^{(1)}} & F_0^* \otimes L(\omega_{x_{r_1}})^\vee \subset \widehat{R}_{\text{gen}}(\underline{f}) \\ & \underset{\eta\otimes\mathrm{Id}}{\nwarrow} & \Big\uparrow {\scriptstyle d_1} & \underset{\mathrm{Id}\otimes\iota}{\nearrow} & \\ & & F_0^* \otimes F_1 & & \end{array} \tag{5.8}$$

where ι is inclusion of the bottom z_1-graded component and η is inclusion of the top x_1-graded component. Here we are abusing notation somewhat: $w^{(1)}$ is by definition a map $L(\omega_{x_{r_1}})^\vee \otimes R \to F_0 \otimes R$ but we view it in the diagram as a restriction of $w\colon \widehat{R}_{\text{gen}} \to R$.

Obviously there are many choices of A_1 as stated, but the point is to construct it as an isomorphism when viewed as a map $L(\omega_{x_1})^\vee \otimes R \to L(\omega_{x_1})^\vee \otimes R$. (Morally, it should be viewed as the action of our desired R-point $g \in G$ on $L(\omega_{x_1})^\vee \otimes R$.) Note that in order to view A_1 in this manner, we are using the fact that T is a Dynkin diagram so $L(\omega_{x_{r_1}})^\vee = L(\omega_{x_{r_1}})^*$ since it is finite-dimensional.

5.2.2 The split exact case

Here is an equivalent way of formulating the observation made in part (2) of Example 5.3: if $R = \mathbb{C}$ and we take $A_1 = \mathrm{Id}$, corresponding in (5.8) to the evident pairing $L(\omega_{x_{r_1}}) \otimes L(\omega_{x_{r_1}})^\vee \to \mathbb{C}$, then its restriction to $L(\omega_{x_{r_1}}) \otimes F_1$ recovers $w'^{(3)}_{\text{ssc}}$ and its restriction to $F_0^* \otimes L(\omega_{x_{r_1}})^\vee$ recovers $w^{(1)}_{\text{ssc}}$. Restricting down further to $F_0^* \otimes F_1$ recovers the first differential of $\mathbb{F}^{\text{ssc}}$, viewed either as $(w'^{(3)}_{\text{ssc}})_0^*$ if we restrict down via the left half of the diagram, or as $(w^{(1)}_{\text{ssc}})_0$ if we restrict down the right half.

Hence $A_1 = \mathrm{Id}$ is a natural solution to the lifting problem (5.8) if $\mathbb{F} = \mathbb{F}^{\text{ssc}}$, $w = w_{\text{ssc}}$, and $w' = w'_{\text{ssc}}$. Now suppose $\mathbb{F}$ is a split exact complex over some ring R, with differentials d_i. Then one can choose an isomorphism $\mathbb{F}_{\text{ssc}} \otimes R \cong \mathbb{F}$, which amounts to picking $g_1 \in \mathrm{GL}(F_1 \otimes R)$ and $g_2 \in \mathrm{GL}(F_2 \otimes R)$:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & F_3 \otimes R & \longrightarrow & F_2 \otimes R & \longrightarrow & F_1 \otimes R & \longrightarrow & F_0 \otimes R \\ & & \Vert & & \Big\downarrow {\scriptstyle g_2} & & \Big\downarrow {\scriptstyle g_1} & & \Vert \\ 0 & \longrightarrow & F_3 \otimes R & \xrightarrow{d_3} & F_2 \otimes R & \xrightarrow{d_2} & F_1 \otimes R & \xrightarrow{d_1} & F_0 \otimes R \end{array}$$

Explicitly g_1, g_2 are such that

$$g_1^{-1} = \begin{array}{c} \\ {\scriptstyle F_0} \\ {\scriptstyle C} \end{array} \!\!\begin{array}{c} {\scriptstyle F_1} \\ \begin{bmatrix} d_1 \\ \gamma \end{bmatrix} \end{array} \qquad g_2 = \begin{array}{c} \\ {\scriptstyle F_2} \end{array} \!\!\begin{array}{cc} {\scriptstyle F_3} & {\scriptstyle C} \\ \multicolumn{2}{c}{\begin{bmatrix} d_3 & \beta \end{bmatrix}} \end{array}$$

with the property that the composite $F_1 \xrightarrow{\gamma} C \xrightarrow{\beta} F_2$ splits the differential d_2.

As discussed above, setting $A_1 = \mathrm{Id}$ in (5.8) works for $\mathbb{F}^{\mathrm{ssc}} \otimes R$. Precompose $w_{\mathrm{ssc}} \otimes R$ by the action of $g_2 g_1^{-1}$ on $\widehat{R}_{\mathrm{gen}}(\underline{f}) \otimes R$ to obtain a new map w_0, and similarly precompose $w'_{\mathrm{ssc}} \otimes R$ by the action of $g_2 g_1^{-1}$ on $\widehat{R}_{\mathrm{gen}}(\underline{f}^*) \otimes R$ to obtain a new map w'_0. By Theorem 4.11, $w = w_0 \exp Z^-$ for some $Z^- \in \mathbf{L} \widehat{\otimes} R$ where $\mathbf{L} = \hat{\mathfrak{n}}^-_{z_1}$, and similarly $w' = w'_0 \exp X^+$ for some $X^+ \in \mathbf{L}' \widehat{\otimes} R$ where $\mathbf{L}' = \hat{\mathfrak{n}}^+_{x_1}$. We remark that the completions in defining $\mathbf{L}$ and $\mathbf{L}'$ are not necessary here because $\mathfrak{g}$ is finite-dimensional.

With the identifications (5.6) and (5.7), $\mathrm{GL}(F_2)$ acts only on the left tensor factor in (5.8) whereas $\mathrm{GL}(F_1)$ only acts on the right tensor factor. So we act on the whole diagram (5.8) by $g_1 g_2$, thereby replacing $A_1 = \mathrm{Id}$ with

$$A_1 = \rho_2(g_2)\rho_1(g_1^{-1})$$

restricting down to our maps w_0 and w'_0. Here

$$\rho_1 \colon \mathrm{GL}(F_1) \to \mathrm{Aut}\, L(\omega_{x_{r_1}})^\vee$$
$$\rho_2 \colon \mathrm{GL}(F_2) \to \mathrm{Aut}\, L(\omega_{x_{r_1}})^\vee$$

denote the actions of $\mathrm{GL}(F_i)$ on the two tensor factors, viewed as the source and target respectively of A_1.

Finally, we act by $\exp X^+$ on the left factor and $\exp Z^-$ on the right factor in the diagram to obtain

$$A_1 = \exp(X^+)\rho_2(g_2)\rho_1(g_1^{-1})\exp(Z^-) \tag{5.9}$$

restricting down to w and w' by construction. This should be viewed as the action of our desired R-point of G, as discussed in the motivation at the beginning of our construction. In particular, A_1 is obviously invertible by construction, being the composite of invertible maps. Repeating this procedure for the other higher structure maps, we obtain

- A_1 lifting the pair $w^{(1)}$ and $w'^{(3)}$,
- A_2 lifting the pair $w^{(2)}$ and $w'^{(2)}$,
- A_3 lifting the pair $w^{(3)}$ and $w'^{(1)}$,

where A_2 and A_3 are defined using the same formula (5.9) but with the actions on $L(\omega_{y_{r_2-2}})^\vee$ and $L(\omega_{z_{r_3}})^\vee$ respectively. The reason for dealing with all $w^{(i)}$ simultaneously is explained by the following lemma.

Lemma 5.19. *Let $\widetilde{X}^+ \in \mathbf{L}' \widehat{\otimes} R$ and $\widetilde{Z}^- \in \mathbf{L} \widehat{\otimes} R$. Define $\widetilde{A}_i$ following* (5.9)*, replacing X^+ by $\widetilde{X}^+$ and Z^- by $\widetilde{Z}^-$.*

- *If $\widetilde{A}_1$ extends $w'^{(3)}$ in the sense of* (5.8)*, then $X^+ = \widetilde{X}^+$.*
- *If $\widetilde{A}_3$ extends $w^{(3)}$, then $Z^- = \widetilde{Z}^-$.*

Proof. The two statements are completely analogous, so we explain the second one. Observe that the action of $\exp(X^+)$ has no effect on the restriction of A_3 to $F_2 \otimes L(\omega_{z_{r_3}})^\vee$, which is $w^{(3)}$. Similarly, the action of $\exp(\widetilde{X}^+)$ has no effect on the restriction of $\widetilde{A}_3$ to $F_2 \otimes L(\omega_{z_{r_3}})^\vee$, which is also equal to $w^{(3)}$ by assumption. The statement then follows immediately from Proposition 4.22 since $W(a_3) \subset \bigwedge^{r_3} W(d_3)$ in $\widehat{R}_{\mathrm{gen}}$. □

5.2.3 Independence of choice of splitting

The construction of the matrices A_i was reliant on a choice of splitting $\mathbb{F}^{\mathrm{ssc}} \otimes R \cong \mathbb{F}$. This was the only step that required a choice; the elements Z^-, X^+ were uniquely determined afterwards by comparison of $w_{\mathrm{ssc}} \otimes R$ to w and $w'_{\mathrm{ssc}} \otimes R$ to w' using Theorem 4.11.

We now show that (5.9) is actually insensitive to our choice of g_1 and g_2. In the following we will often abuse notation and just write e.g. F_j when we mean $F_j \otimes R$.

Lemma 5.20. *Suppose that we pick a different isomorphism $\mathbb{F}_{\mathrm{ssc}} \otimes R \cong \mathbb{F}$, or equivalently, a different splitting $F_1 \xrightarrow{\gamma'} C \xrightarrow{\beta'} F_2$. Then there exist $\theta \in GL(C)$, $\eta_1 \in \mathrm{Hom}(F_0, C)$, and $\eta_2 \in \mathrm{Hom}(C, F_3)$ such that*

$$\gamma' = \theta\gamma + \eta_1 d_1, \quad \beta' = \beta\theta^{-1} + d_3\eta_2.$$

For the corresponding g_1', g_2', we can write this as

$$g_1'^{-1} = \theta(1 + \theta^{-1}\eta_1)g_1^{-1}, \quad g_2' = g_2(1 + \eta_2\theta)\theta^{-1}$$

recalling that $F_1 = F_0 \oplus C$ and $F_2 = F_3 \oplus C$.

Proof. Both γ, γ' must map $\ker d_1$ isomorphically onto C, so there exists an element $\theta \in GL(C)$ such that $\gamma' = \theta\gamma$ restricted to $\ker d_1$. The difference $\gamma' - \theta\gamma$ must then factor through d_1. This gives the first expression.

One similarly argues the existence of $\theta' \in GL(C)$ such that $\beta' = \beta\theta'$ modulo $\ker d_2 \subset F_2$. Note that if $s\colon F_1 \to F_2$ is a splitting, then

$$\ker d_1 \hookrightarrow F_1 \xrightarrow{s} F_2 \twoheadrightarrow F_2/(\ker d_2)$$

must be inverse to the map induced by d_2. In particular, $\beta\gamma$ and $\beta'\gamma'$ must agree as maps $(\ker d_1) \to F_2/(\ker d_2)$, which means $\theta' = \theta^{-1}$. The expression for β' thus follows. □

Let $\mathfrak{g}_{m,n}$ denote the part of the Lie algebra $\mathfrak{g}$ in (x_1, z_1)-bidegree (m, n). We fix a pair

$$(i, \omega) \in \{(1, \omega_{x_{r_1}}), (2, \omega_{y_{r_2-2}}), (3, \omega_{z_{r_3}})\}$$

and let ρ_1, ρ_2 denote the actions of $\mathrm{GL}(F_1), \mathrm{GL}(F_2)$ on $L(\omega)^\vee$.

Note that $1 + \theta^{-1}\eta_1 \in \mathrm{SL}(F_1)$. It can be written as $\exp(\theta^{-1}\eta_1)$, viewing $\theta^{-1}\eta_1 \in F_0^* \otimes C = \mathfrak{g}_{1,0}$. Similarly $1 + \eta_2\theta = \exp(\eta_2\theta)$ viewing $\eta_2\theta \in C^* \otimes F_3 = \mathfrak{g}_{0,-1}$.

If we go through the construction of §5.2.2 with g_1', g_2', we get

$$A_i' = \exp(X'^+)\rho_2(g_2')\rho_1(g_1'^{-1})\exp(Z'^-)$$

Expanding this using the above observations, we have

$$A_i' = \exp(X'^+)\rho_2(g_2)\exp(\eta_2\theta)\rho_2(\theta^{-1})\rho_1(\theta)\exp(\theta^{-1}\eta_1)\rho_1(g_1^{-1})\exp(Z'^-).$$

Now we use:

Lemma 5.21. *The map $\mathrm{GL}(C) \to \mathrm{GL}(F_1) \xrightarrow{\rho_1} \mathrm{Aut}\, L(\omega)^\vee$ agrees with $\mathrm{GL}(C) \to \mathrm{GL}(F_2) \xrightarrow{\rho_2} \mathrm{Aut}\, L(\omega)^\vee$.*

Proof. The statement is certainly true for $\mathrm{SL}(C)$ because both actions can be seen through $\mathfrak{sl}(C) \subset \mathfrak{g}_{0,0}$. So it is sufficient to check this statement for scalars, and we omit this. □

Hence $\rho_2(\theta^{-1})$ and $\rho_1(\theta)$ cancel, and we are left with

$$A_i' = \exp(X'^+)\rho_2(g_2)\exp(\eta_2\theta)\exp(\theta^{-1}\eta_1)\rho_1(g_1^{-1})\exp(Z'^-).$$

Elements of $\mathfrak{g}_{1,0}$ and $\mathfrak{g}_{0,-1}$ commute because $\mathfrak{g}_{1,-1} = 0$, so we can interchange the middle two terms. Note that

$$\begin{aligned}\theta^{-1}\eta_1 \in \mathfrak{g}_{1,0} &= F_0^* \otimes C \\ \subset \mathfrak{g}_{1,*} &= F_0^* \otimes \bigwedge^{f_3+1} F_2 \otimes \bigwedge^{f_3} F_3^*\end{aligned}$$

Applying g_2 to $\theta^{-1}\eta_1$ gives an element $X_1 \in \mathfrak{g}_{1,*}$ such that

$$\exp(X_1)\rho_2(g_2) = \rho_2(g_2)\exp(\theta^{-1}\eta_1).$$

Similarly, by applying g_1 to $\eta_2\theta$, we get $Z_1 \in \mathfrak{g}_{*,-1}$ such that

$$\rho_1(g_1^{-1})\exp(Z_1) = \exp(\eta_2\theta)\rho_1(g_1^{-1}),$$

allowing us to write

$$A_i' = \exp(X'^+)\exp(X_1)\rho_2(g_2)\rho_1(g_1^{-1})\exp(Z_1)\exp(Z'^-).$$

Baker-Campbell-Hausdorff yields elements $\widetilde{X}^+ \in \mathbf{L}'\widehat{\otimes} R$ and $\widetilde{Z}^- \in \mathbf{L}\widehat{\otimes} R$ such that

$$\exp(\widetilde{X}^+) = \exp(X'^+)\exp(X_1), \quad \exp(\widetilde{Z}^-) = \exp(Z_1)\exp(Z'^-),$$

and so

$$A_i' = \exp(\widetilde{X}^+)\rho_2(g_2)\rho_1(g_1)\exp(\widetilde{Z}^-).$$

However, compare this to

$$A_i = \exp(X^+)\rho_2(g_2)\rho_1(g_1)\exp(Z^-).$$

The hypotheses of Lemma 5.19 are met by construction, and we deduce that $X^+ = \widetilde{X}^+$ and $Z^- = \widetilde{Z}^-$. In particular $A_i = A_i'$ as desired.

Having established that the matrices A_i are independent of the choice of splitting, Theorem 5.17 readily follows.

Proof of Theorem 5.17. We fix w and w' as in §5.2.1. Let $h_1, h_2 \in I_{f_0}(d_1)$ be a regular sequence, which exists because $\operatorname{grade} I_{f_0}(d_1) = 3 \geq 2$. We perform the construction of §5.2.2 for the split exact complex $\mathbb{F} \otimes R_{h_1}$, obtaining matrices A_i invertible over R_{h_1}. We repeat the construction for $\mathbb{F} \otimes R_{h_2}$, obtaining matrices A_i' invertible over R_{h_2}. The results of this subsection then show that the constructions agree on the common localization $R_{h_1h_2}$. Hence the matrices $A_i = A_i'$ have entries in $R_{h_1} \cap R_{h_2} = R$. The same applies to their inverses $A_i^{-1} = A_i'^{-1}$, therefore the matrices A_i are invertible over R. Since $w^{(i)}$ consists of rows from A_i, we obtain that $w^{(i)}$ is surjective. □

5.3 Consequences and future directions

The construction of the matrices A_i in the proof of Theorem 5.17 is valid even without the hypothesis that T is a Dynkin diagram, with the caveat that since they came from maps $L(\omega) \otimes L(\omega)^\vee \to R$, they must be viewed as "infinite matrices" $L(\omega)^\vee \otimes R \to (L(\omega) \otimes R)^*$:

$$\begin{array}{c}
L(\omega)^\vee \otimes R \\
\big\downarrow {\scriptstyle \exp(Z^-)} \\
L(\omega)^\vee \otimes R \\
\big\downarrow {\scriptstyle \rho_1(g_1^{-1})} \\
L(\omega)^\vee \otimes R \\
\big\downarrow \\
(L(\omega) \otimes R)^* \\
\big\downarrow {\scriptstyle \rho_2(g_2)} \\
(L(\omega) \otimes R)^* \\
\big\downarrow {\scriptstyle \exp(X^+)} \\
(L(\omega) \otimes R)^*
\end{array}$$

In this composite, while the actions of $\mathrm{GL}(F_1 \otimes R)$ and $\mathrm{GL}(F_2 \otimes R)$ make sense on both $L(\omega)^\vee \otimes R$ and $(L(\omega) \otimes R)^*$, the action of $\exp(Z^-)$ is only well-defined on the source and the action of $\exp(X^+)$ is only well-defined on the target. Hence while we obtain infinite matrices A_i without the Dynkin hypothesis, we cannot view them as endomorphisms, let alone automorphisms.

We also point out that the theorem is not even true if one drops either hypothesis:

- In Example 4.24, all of the matrices $w^{(i)}$ have entries of positive degree, so they cannot be surjective. The ideal in that example is perfect, but the associated diagram is $T = E_7^{(1)}$, which is not a Dynkin diagram. So while we can still produce the maps A_i as in the proof of the theorem, we cannot discuss their invertibility.

- If $I = (t_1 t_2, t_2 t_3, t_3 t_4, t_4 t_1) \subset R = \mathbb{C}[t_1, t_2, t_3, t_4]$, then the minimal graded free resolution of R/I has the form
$$\mathbb{F}\colon 0 \to R(-4) \to R^4(-3) \to R^4(-2) \to R$$
and the associated diagram $T = D_4$ is Dynkin. However, the grade $I = 2$ so I is not perfect. Repeating the argument in Example 4.24, we see that there is a choice of $w\colon \widehat{R}_{\mathrm{gen}} \to R$ such that all of the matrices $w^{(i)}$ have entries of positive degree, hence again the conclusion of Theorem 5.17 fails. The proof breaks because we don't have higher structure maps w' for $\mathbb{F}^*$, so we cannot construct matrices A_i defined over R.

Beyond the Dynkin range, it remains unclear how to use representation theory to characterize non-licci perfect ideals. A concrete starting point would be to see whether one can produce some well-known examples of non-licci perfect ideals with Betti numbers $\underline{f} = (1, 6, 8, 3)$ directly from the representation theory of the affine Kac-Moody Lie algebra $\mathfrak{g} = E_7^{(1)}$ associated to this format. Two particularly simple examples of such perfect ideals are:

- the ideal of 2×2 minors of a generic 2×4 matrix (the ideal of $\mathbb{P}^1 \times \mathbb{P}^3 \subset \mathbb{P}^7$ in the Segre embedding), and
- the ideal of 2×2 minors of a generic 3×3 symmetric matrix (the ideal of $\mathbb{P}^2 \subset \mathbb{P}^5$ in the Veronese embedding).

There are evident avenues for future work given the discussion in Chapter 1. The natural first step would be to examine grade 4 Gorenstein ideals. We have already developed an analogous construction to Chapter 3 which produces resolutions of grade 4 Gorenstein ideals; see [39]. Hence it would be desirable to have a theory of higher structure maps that complements this construction. We expect that this theory should be compatible with Kustin's notion of "higher order products" for such ideals. Specifically, we hope to be able to recover and extend his construction, and to realize his conjectures as an appropriate analogue of Theorem 5.6.

Conjecture 1.4 predicts that grade 4 Gorenstein ideals with deviation $d \leq 4$ should be licci. This bound on d is known to be sharp: the "Tom and Jerry" examples of Reid and coauthors in [5] have deviation 5 and are not licci. Conjecture 1.4 also predicts that Gorenstein ideals with deviation 2 and grade $c \leq 6$ should be licci. But in this case we do not know whether this bound on c is sharp; to the author's knowledge there are no known examples of non-licci Gorenstein ideals of deviation 2.

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