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Fluid Limit for a Multi-Server, Multiclass Random Order of Service Queue with Reneging and Tracking of Residual Patience Times

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Fluid Limit for a Multi-Server, Multiclass Random Order of Service Queue with Reneging and Tracking of Residual Patience Times

A dissertation submitted in partial satisfaction of the requirements for the degree Doctor of Philosophy

in

Mathematics

by

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2024

The Dissertation of Eva Horne Loeser is approved, and it is acceptable in quality and form for publication on microfilm and electronically.

University of California San Diego

2024

DEDICATION

I'd like to dedicate this dissertation first to my husband and parents, who have supported me and pushed me to become the person I am today; the many friends I have made here who have made graduate school and the young adult portion of my growing up so rich and full of joy; and, lastly, the many wonderful students I had during my time here, who were daily reminders of the wonders of mathematics, learning, growth, and the value of the academic path.

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We would like to thank Reza Aghajani for access to an early version of [2], which uses a different model for a random order of service queue with reneging (tracking ages rather than residual times). His formulation and results derived for it inspired us to consider an alternative model tracking residual times and with some different assumptions. Furthermore, the techniques he used led us to study his past published work, and ultimately his paper in collaboration with Kavita Ramanan [3], all of which motivated us to develop the martingale arguments used in §7.2. In our text, at several specific points, we acknowledge where we used a similar construction or argument to [2].

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ABSTRACT OF THE DISSERTATION

Fluid Limit for a Multi-Server, Multiclass Random Order of Service Queue with Reneging and Tracking of Residual Patience Times

by

Eva Horne Loeser

Doctor of Philosophy in Mathematics

University of California San Diego, 2024

Professor Ruth Williams, Chair

In this thesis, we consider a multi-server, multiclass queue with reneging operating under the random order of service discipline. Interarrival times, service times, and patience times are assumed to be generally distributed. Under mild conditions, we establish a fluid limit theorem for a measure-valued process that keeps track of the remaining patience time for each job in the queue. We prove uniqueness for fluid model solutions in all but one case and study the asymptotic behavior of fluid model solutions as time goes to infinity.

Chapter 1

Introduction

In this thesis, we establish a fluid limit for a multi-server, multiclass random order of service queue with reneging under various traffic load parameters. We also study the behavior of solutions of the resulting fluid model.

1.1 Background and Relevant Literature

Random order of service queues have been a subject of interest within the field of queueing theory for some time (see, e.g., [9, 6, 20] and references therein), especially motivated by their applications in operations management. In recent work [22, 4], it was shown that in some computer systems with redundancy, random order of service has better performance with respect to certain performance measures than other well-known service disciplines like processor sharing and first come, first served. Furthermore, for certain service distributions, there are strong connections between random order of service and processor sharing [6], which is often used as a model in telecommunications applications [11]. An emerging area of application for random order of service queues is modeling competition for processing resources in systems biology [17, 8, 18, 21]. However, many current results for random order of service queues are only valid under restrictive assumptions, such as exponentially distributed interarrival or service times, and they often do not allow for reneging. Incorporating reneging is important for applications such as computing and enzymatic processing. The latter is a major motivation for our work. In this direction, in the earlier work [17], a random order of service model with reneging was analyzed and explicit formulas were established for correlations between concentrations of molecules of different types processed by a pool of common enzymes. Experiments inspired by these results yielded consistent observations [8]. However, this exact analysis assumed exponential distributions for interarrival, service, and reneging times. There are important examples of enzymatic processing, such as RNA polymerase transcribing DNA to mRNA and translation of mRNA by ribosomes into protein, where it is expected that the enzymatic processing times will not be exponentially distributed. A generally distributed model would be more realistic for these applications.

Some work has been done on random order of service models with generally distributed interarrival, patience, and service times. In particular, [23] considers heavy traffic asymptotics for a random order of service queue with general distributions for service times and interarrival times. However, the paper deals only with the heavy traffic case, does not allow for reneging, and the queue has only one server and class. Another paper looks at a generally distributed random order of service model with reneging [2]. However, that work tracks age in system, and it assumes the stochastic primitives have densities and that the patience and service times satisfy the additional assumption of having bounded hazard rates. In contrast, our model tracks residual times and is more general in that it does not require stochastic primitives to have densities. However, we assume there is a uniform bound on the second moments of our service times, while [2] does not have this assumption. The model in [2] is also for a single server, while our model allows for multiple servers. Throughout this thesis, we will indicate specific places where there is a relationship with what is in [2].

Many-server queues with reneging, in which the number of servers goes to infinity, have generated significant interest in recent years [15, 1, 19], in connection with applications to call centers. However, these works assume a head-of-the-line service discipline. Random order of service is not a head-of-the-line service discipline. Furthermore, because these works track age in system rather than residual times, for some of the results it is assumed that the underlying distributions for the stochastic primitives have densities, and some further assumptions (such as boundedness) may be required for the associated hazard rates. We do not have such restrictions. Lastly, these models take a limit as the number of servers goes to infinity. We use a different rescaling of time and space, and the number of servers is held constant.

1.2 Results in this thesis

In this thesis, we introduce a measure-valued process that keeps track of the remaining patience time of each job in a random order of service queue with reneging, where there are multiple classes of jobs and multiple servers. Under mild conditions, we establish tightness for a sequence of fluid scaled models and prove that subsequential limits satisfy certain fluid model equations. We prove that these subsequential limits, referred to as fluid model solutions, are unique in all but one case, and use this to establish convergence for the original sequence. We also characterize the invariant state for the fluid model and prove that fluid model solutions converge to the invariant state as time goes to infinity, uniformly for suitable initial conditions.

This thesis is organized as follows. In §2 and §3, we introduce the sequence of fluid scaled models. In §4, we introduce the notion of a fluid model solution. In §5, we summarize our main results. Uniqueness of fluid model solutions and continuous dependence on initial conditions is proved in §6. C-tightness and convergence to fluid model solutions is proved in §7 through §9. Convergence of fluid model solutions to the invariant state is proved in §10. Proofs related to some additional results for the case of exponentially distributed patience times are in §11.

1.3 Notation

We shall use the following notation throughout the thesis. Let $\mathbb{N}$ denote the set of strictly positive integers, $\{1, 2, \ldots\}$, and let $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. Let $\mathbb{R}_+ = [0, \infty)$, and consider it with the Borel σ-algebra, $\mathscr{B}(\mathbb{R}_+)$. For a positive integer $n \geq 1$, $\mathbb{R}^n$ is the n-dimensional Euclidean space, and we denote the zero vector there by $\mathbf{0}$. If $n = 1$, we also use $\mathbb{R}$ for $\mathbb{R}^1$ and 0 for $\mathbf{0}$. We denote the set of nonnegative, finite measures on $(\mathbb{R}_+, \mathscr{B}(\mathbb{R}_+))$ by $\mathbf{M}$, and endow it with the topology of weak convergence of measures. We note that this is a complete separable metric space, also known as a Polish space, when metrized with the following metric[1]:

$$d(\xi, \gamma) = \inf\{\epsilon > 0 : F_\xi((x - \epsilon)^+) - \epsilon \leq F_\gamma(x) \leq F_\xi(x + \epsilon) + \epsilon \ \forall \ x \in \mathbb{R}_+\} \tag{1.1}$$

for $\xi, \gamma \in \mathbf{M}$, where $F_\xi(\cdot) := \xi([0, \cdot])$. We denote the set of continuous measures in $\mathbf{M}$ by

$$\mathbf{K} := \{\xi \in \mathbf{M} : \xi(\{x\}) = 0 \ \forall \ x \in \mathbb{R}_+\}.$$

If $\xi \in \mathbf{M}$ and f is a Borel measurable function on $\mathbb{R}_+$ that is integrable with respect to ξ, we let $\langle f, \xi \rangle := \int_{\mathbb{R}_+} f d\xi$. For $x \in \mathbb{R}$, we denote the positive part of x by $x^+ := x \vee 0$. For a measure $\xi \in \mathbf{M}$, we denote the (closed) support of the measure by $\text{supp}(\xi)$. For a finite set $A \subset \mathbb{R}_+$, we denote the ith smallest element of A by $A_{\{i\}}$. For a positive integer n, we consider the product space $\mathbf{M}^n$, which is a Polish space with the metric

$$d_n(\boldsymbol{\xi}, \boldsymbol{\gamma}) = \sum_{i=1}^{n} d(\xi_i, \gamma_i) \tag{1.2}$$

[1] This is an extension of the metric that Lévy introduced for probability measures to finite, nonnegative measures. It induces the same topology as the Prokhorov metric. See Lévy metric in Encyclopedia of Mathematics. https://encyclopediaofmath.org/wiki/Levy_metric for more details.

for $\boldsymbol{\xi} = (\xi_1, \ldots, \xi_n)$ and $\boldsymbol{\gamma} = (\gamma_1, \ldots, \gamma_n)$ in $\mathbf{M}^n$. We denote the n-dimensional vector of zero measures $(0, \ldots, 0) \in \mathbf{M}^n$ by $\mathbf{0}$.

Let $\chi(x) := x$ for $x \geq 0$. For a vector $\boldsymbol{x} \in \mathbb{R}_+^n$, we write $\boldsymbol{x} > \mathbf{0}$ if and only if $x_i > 0$ for $i = 1, \ldots, n$. For $X = \mathbb{R}$ or $X = \mathbb{R}_+$, we denote the set of bounded continuous functions defined on X and taking values in $\mathbb{R}$ by $\mathbf{C}_b(X)$. The set of functions in $\mathbf{C}_b(X)$ that have bounded continuous derivatives up to order $n \geq 1$ is denoted by $\mathbf{C}_b^n(X)$. For $T \geq 0$ and a bounded function $f : \mathbb{R}_+ \to \mathbb{R}$, we write $\|f\|_T$ for $\sup_{t \in [0,T]} |f(t)|$. In the context of $\mathbb{R}_+$, we take $\sup \emptyset$ to be 0 and $\inf \emptyset$ to be $+\infty$.

We denote the space of functions from $[0, \infty)$ to $\mathbb{R}^n$ that are right continuous with finite left limits by $D([0, \infty), \mathbb{R}^n)$. We endow $D([0, \infty), \mathbb{R}^n)$ with the Skorokhod-J_1 topology, under which it is a Polish space. Similarly, we denote the space of functions from $[0, \infty)$ to $\mathbf{M}^n$ that are right continuous with finite left limits by $D([0, \infty), \mathbf{M}^n)$. This is also endowed with the Skorokhod-J_1 topology, under which it is also a Polish space. The mode of convergence for our fluid scaled models will be convergence in distribution of random elements taking values in $D([0, \infty), \mathbf{M}^n)$. In general, we will denote convergence in distribution for processes and random variables with $\Rightarrow$. The spaces of continuous functions from $[0, \infty)$ to $\mathbb{R}^n$ and $\mathbf{M}^n$ are closed subsets of $D([0, \infty), \mathbb{R}^n)$ and $D([0, \infty), \mathbf{M}^n)$, respectively. The Skorokhod-J_1 topology induced on these spaces is equivalent to that induced by uniform convergence on compact time intervals. For $T > 0$, a function of bounded variation $G : [0, T] \to \mathbb{R}$, and a function $f : [0, T] \to \mathbb{R}$ that is integrable on $[0, T]$ with respect to the Lebesgue–Stieltjes measure, μ_G, induced by $G(\cdot)$, we denote the Lebesgue–Stieltjes integral $\int_{(s,t]} f(x)\mu_G(dx)$ by $\int_s^t f(x)dG(x)$ for $0 \leq s \leq t \leq T$.

Chapter 2

Multi-Server, Multiclass Random Order of Service Queue

In this chapter, we introduce a model for a random order of service queue which uses a measure-valued process. The work of Aghajani [2] also uses a measure-valued process to describe such a queue. However, our description differs from that in [2] in that our description keeps track of the remaining patience time of each job in queue, remaining interarrival time of each class of job, and remaining service time of each job in service, whereas [2] tracks time since each job arrived to the queue, time since the last arrival for each class of job, time in service of the job currently in service, and class of the job currently in service. In other words, we use residual descriptions whereas [2] used age-based descriptions. Our approach enables us to adopt some more general assumptions than in [2]. In particular, we do not require the interarrival time, service time, and patience time distributions to have densities, whereas [2] does. However, we do require a uniform bound on the second moment of the service times, which [2] does not require. On the other hand, [2] assumes that the patience time and service time distributions have bounded hazard rates. Our model also allows for multiple servers, while [2] has a single server.

We will be working over a complete probability space denoted by $(\Omega, \mathscr{F}, P)$, and expectation will be denoted by $E[\cdot]$. We shall refer to the entities processed in the system as jobs, although in particular applications they may be customers, manufacturing or computer jobs, or molecules, for example. There will be J classes of jobs, indexed by $\mathcal{J} = \{1, \ldots, J\}$, where J is a finite positive integer. There will be K identical servers for processing jobs, indexed by $\mathcal{K} = \{1, \ldots, K\}$, where K is a finite positive integer.

2.1 Informal Description of the System

We begin with an informal description of our model dynamics. In many applications for this model, such as the example of enzymes breaking down different types of molecules, all of the jobs in the system will be in the same physical space. However, it does not change the dynamics to imagine that each class has its own (virtual) queue, as depicted in Figure 2.1. This representation will help us describe and visualize system behavior.

Jobs of each class will arrive to the system at random intervals according to a delayed renewal process. If there is an available server in the server bank when a job arrives, the job will immediately enter service and spend no time in any queue. If not, it will wait in the queue for its class until either its patience time expires and it reneges from the system or it is chosen for service. Once chosen for service, a job leaves its queue and cannot renege. If there are jobs waiting in the queues and a server becomes available, the server will choose a job to serve randomly from all waiting jobs. We allow for some jobs to be more likely to be chosen than others based on their class. Within each class, all jobs are equally likely to be chosen.

2.2 Stochastic Primitives

2.2.1 Arrivals

For each $j \in \mathcal{J}$, jobs of class j arrive to the system according to a delayed renewal process, which we denote by $A_j(\cdot)$. To make this precise, let $\{u_i^j\}_{i=1}^{\infty}$ be a sequence of i.i.d., strictly positive random variables having finite mean. For $i = 1, 2, \ldots$, the time between the arrival of the ith job of class j and the $(i+1)$th job of class j is u_i^j. Let u_0^j be a strictly

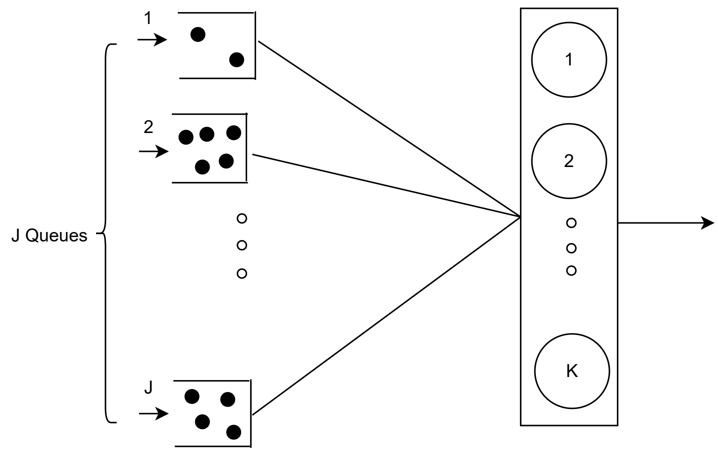

Figure 2.1. Depiction of a Random Order of Service queue with J classes of jobs and K servers.

positive random variable that is independent from $\{u_i^j\}_{i=1}^{\infty}$ but need not be identically distributed to $\{u_i^j\}_{i=1}^{\infty}$. Then u_0^j is the time until the arrival of the first job to the class j queue. The time at which the ith arrival to the class j queue occurs is given by the random variable

$$U_i^j := \sum_{l=0}^{i-1} u_l^j, \qquad i = 1, 2, \ldots.$$

We define $U_0^j := 0$ for convenience. The delayed renewal arrival process for class j, which counts the number of arrivals up to time t, is given by

$$A_j(t) := \sup\{i \in \mathbb{N} : U_i^j \leq t\}, \qquad t \geq 0.$$

We assume also that the sequences $\{u_0^j, u_1^j, u_2^j, \ldots\}$ for $j \in \mathcal{J}$ are mutually independent. Define the vector-valued arrival process

$$\boldsymbol{A}(\cdot) := (A_1(\cdot), \ldots A_J(\cdot)) \in \mathbb{N}_0^J, \qquad t \geq 0.$$

The average arrival rate for class j is defined by

$$\alpha_j := (E[u_1^j])^{-1}, \qquad j \in \mathcal{J},$$

and the average arrival rate vector is $\boldsymbol{\alpha} := (\alpha_1, \ldots, \alpha_J)$. For each $j \in \mathcal{J}$ and $t \geq 0$, let $a_j(t) := U_{A_j(t)+1} - t$, the time until the next job of class j arrives after time t. Let $\boldsymbol{a}(\cdot) = (a_1(\cdot), \ldots, a_J(\cdot))$. It follows that at time zero, $\boldsymbol{a}(0) = (u_0^1, \ldots, u_0^J)$. We make the further assumption that the underlying probability distributions for u_0^j, u_1^j, have no atoms for each $j \in \mathcal{J}$, and that simultaneous arrivals do not occur, which can be achieved without loss of generality by removing a simple null set.

2.2.2 Patience Times

For $j \in \mathcal{J}$ and $i = 1, 2, \ldots$, the ith class j job to arrive to the system after time zero is assigned a "patience time" ℓ_i^j. This is the maximum amount of time the job will wait in the queue for its class, in the sense that if the job is still in the queue for its class at time $U_i^j + \ell_i^j$, then it leaves the system at that time, i.e., it reneges. We assume that for each $j \in \mathcal{J}$, $\{\ell_i^j\}_{i=1}^{\infty}$ is a sequence of strictly positive, i.i.d., random variables having finite mean. We denote the distribution of ℓ_1^j by ϑ_j, a probability measure on $[0, \infty)$ with $\vartheta_j(\{0\}) = 0$, and let $\gamma_j := (E[\ell_1^j])^{-1}$. Let $\boldsymbol{\vartheta} := (\vartheta_1, \ldots, \vartheta_J)$ and $\boldsymbol{\gamma} := (\gamma_1, \ldots, \gamma_J)$. We assume that the sequences $\{\ell_i^j\}_{i=1}^{\infty}$ for $j \in \mathcal{J}$ are mutually independent. For each $j \in \mathcal{J}, i = 1, 2, \ldots$, let

$$\ell_i^j(t) := \ell_i^j + U_i^j - t, \qquad t \geq 0. \tag{2.1}$$

If the ith job that arrives to the class j queue is in the class j queue at time t, then $\ell_i^j(t)$ is the remaining patience time of that job, i.e., the amount of time remaining after t until it would renege if still in the queue for its class at that time.

2.2.3 Service Discipline

The service discipline is a weighted random order of service discipline. If there is an available server and there are no jobs in the queues, the next arriving job will go directly into service. For specificity, it will be served by the server with the lowest index out of those available at the last service completion time. If there are jobs in any of the queues and a server becomes available (and no arrival occurs at this time), the server will choose a new job at random from the jobs waiting for service in queues according to the following weights. For each class $j \in \mathcal{J}$, there is a weight $p_j \in (0,1)$ such that

$$\sum_{j=1}^{J} p_j = 1. \tag{2.2}$$

We let $\mathbf{p} := (p_1, \ldots, p_J)$. If z_j is the number of jobs of class j waiting in the queue for class j when a server becomes available, then the server selects class j to be served next with probability

$$\frac{p_j z_j}{\sum_{i=1}^{J} p_i z_i}.$$

Given that class j is chosen, then any specific job in the queue for class j is chosen to be served next with probability $\frac{1}{z_j}$. When a job enters service, it leaves its queue. Jobs cannot renege once they are in service. On the probability zero event where an arrival and a service completion occur simultaneously, for definiteness, we have the arrival immediately enter service. There cannot be more than one arrival at such a time, as we have ruled out simultaneous arrivals.

We now introduce a structure to specify this random order of service selection process. This selection process will be similar to that introduced in [2]. To specify the service process precisely, if $\boldsymbol{z} = (z_1, \ldots, z_J) \in \mathbb{N}_0^J \setminus \{\mathbf{0}\}$ and $j \in \mathcal{J}$, define

$$I_j(\boldsymbol{z}) := \left[\frac{\sum_{l=1}^{j-1} p_l z_l}{\sum_{l=1}^{J} p_l z_l}, \frac{\sum_{l=1}^{j} p_l z_l}{\sum_{l=1}^{J} p_l z_l} \right), \tag{2.3}$$

and define subintervals of $I_j(\boldsymbol{z})$ via

$$I_{j,m}(\boldsymbol{z}) := \left[\frac{\sum_{l=1}^{j-1} p_l z_l + p_j(m-1)}{\sum_{l=1}^{J} p_l z_l}, \frac{\sum_{l=1}^{j-1} p_l z_l + p_j m}{\sum_{l=1}^{J} p_l z_l}\right), \tag{2.4}$$

for $m = 1, \ldots, z_j$. Let $\{\kappa_i\}_{i\in\mathbb{N}}$ be i.i.d., uniformly distributed on $(0,1)$, and independent of all other stochastic primitives (interarrival times, service times, and patience times). Then, the ith job to enter service from the queues is the job from class j with the mth smallest remaining patience time if and only if $\kappa_i \in I_{j,m}(\boldsymbol{z})$.

2.2.4 Service Times

For each $j \in \mathcal{J}$, let $\{v_i^j\}_{i\in\mathbb{N}}$ and $\{v_i^{J+j}\}_{i\in\mathbb{N}}$ be sequences of strictly positive, i.i.d., random variables having finite means, where v_i^j and v_i^{J+j} have the same distribution. For $i = 1, 2, \ldots$, if the ith job to enter service from the queues is of class j, then it is assigned service time v_i^j. For $i = 1, 2, \ldots$, if the ith job of class j to arrive to the system arrives to an empty system with an idle server, it goes directly into service (i.e. it does *not* enter any queue), and it is assigned service time v_i^{J+j}. We define the average service rate for class j to be

$$\mu_j := (E[v_1^j])^{-1}$$

and let $\boldsymbol{\mu} := (\mu_1, \ldots, \mu_J)$. We assume that $0 < \mu_j < \infty$ for each $j \in \mathcal{J}$ and that the sequences $\{v_i^j\}_{i\in\mathbb{N}}, \{v_i^{J+j}\}_{i\in\mathbb{N}}$, $j \in \mathcal{J}$ are all mutually independent. We make the further assumption that the underlying probability distribution for v_i^j has no atoms for each $j \in \mathcal{J}$.

2.3 Initial Condition

For each $j \in \mathcal{J}$, we initialize the system with $Z_{0,j}$ jobs of class j in the queue for class j, where $Z_{0,j}$ is a random variable that takes values in $\mathbb{N}_0$ and has finite expectation.

For clarity, we will index the jobs that are initially in the system with negative indices. For example, the ith job of class j that is initially in the system will be referred to as job $-i$ for class j. Following this negative indexing scheme, at time 0 this job is assumed to have remaining patience time $\tilde{\ell}^j_{-i}$, taken from a sequence $\{\tilde{\ell}^j_{-i}\}_{i\in\mathbb{N}}$ of potential residual patience times. For $i \in \{-1, \dots, -Z_{0,j}\}$, let

$$\ell^j_{-i}(t) := \tilde{\ell}^j_{-i} - t, \qquad t \geq 0. \tag{2.5}$$

We assume that $\boldsymbol{Z}_0 = (Z_{0,1}, \dots, Z_{0,J})$, $\{\tilde{\ell}^j_{-i}\}_{i\in\mathbb{N}, j\in\mathcal{J}}$ are independent of $\{v^j_i\}_{i\in\mathbb{N}}$, $\{v^{J+j}_i\}_{i\in\mathbb{N}}$, $\{u^j_i\}_{i\in\mathbb{N}}$, $\{\ell^j_i\}_{i\in\mathbb{N}}$, $j \in \mathcal{J}$, and $\{\kappa_i\}_{i\in\mathbb{N}}$. For each $k \in \mathcal{K}$, let s^k_0 be a nonnegative random variable independent of $\{v^j_i\}_{i\in\mathbb{N}}$, $\{v^{J+j}_i\}_{i\in\mathbb{N}}$, $\{u^j_i\}_{i\in\mathbb{N}}$, $\{\ell^j_i\}_{i\in\mathbb{N}}$, $\{\tilde{\ell}^j_{-i}\}_{i\in\mathbb{N}}$ $j \in \mathcal{J}$, and $\{\kappa_i\}_{i\in\mathbb{N}}$. If $s^k_0 > 0$, it represents the amount of time required to complete the service of the job that is in service at server k at time 0. If $s^k_0 = 0$, there is no job in service at server k at time 0 and all queues must be empty. We make the further assumption that $u^j_0 \neq s^k_0$ and when $s^k_0 > 0$, $s^k_0 \neq s^l_0$ for all $l \neq k$, for each $j \in \mathcal{J}, k \in \mathcal{K}$, and $E[Z_{0,j}] < \infty$ for each $j \in \mathcal{J}$.

2.4 Simultaneous and Non-Simultaneous Events

Despite the fact that our model is very general, we have chosen to assume that the underlying distributions for interarrival times, residual interarrival times for each class at time $t = 0$, service times, and residual service times of jobs in service at time $t = 0$, do not have atoms. It follows from this mild assumption and our independence assumptions that on a set of probability one, times at which jobs are chosen from the queues for service are distinct from arrival times and reneging times, aside from the possibility of simultaneous service completion of a job in service at time $t = 0$ and reneging of a job that was in any queue at time $t = 0$. In the case that the $-i$th job in the queue for class j at time $t = 0$ has initial patience time $\tilde{\ell}^j_{-i} = s^k_0$, and that job is still in the queues at time s^k_0, it will be among the jobs available for server k to choose from at s^k_0 for server k's next service task. In this case, if the job $-i$ is picked for service by server k at that time, s_0^k, then the job will enter service, and it will not renege; otherwise, it will renege at that time, $s_0^k = \tilde{\ell}_{-i}^j$. We conjecture that a fluid limit theorem will still hold without these restrictions, but this would involve a significantly more complicated model description and more complex analysis, which we leave for future investigation. These assumptions, which avoid simultaneous events in most cases, were inspired by a similar observation made in [2].

2.5 Some Descriptive Processes

Having defined our stochastic primitives, we now define some descriptive processes that are functions of our primitives and track important quantities in our system. For $t \geq 0, k \in \mathcal{K}$, let $s^k(t)$ be the time until the next service completion after time t for server k, with the convention that if no job is in service or joins service at server k at time t then $s^k(t) = 0$. We let $s^k(0) = s_0^k$ for each $k \in \mathcal{K}$. For $t \geq 0, j \in \mathcal{J}$, define $S_j(t)$ to be the number of jobs of class j that have entered service (from the jth queue or directly from arrivals) at or before time t. We denote the total service process by $S(t) := \sum_{j=1}^{J} S_j(t)$ for each $t \geq 0$.

2.6 Adjusted Arrival and Service Processes

In this section we define additional descriptive processes based on an important observation. Namely, jobs that arrive to the system when there is an available server do not enter the queues. Instead, they go straight into service upon arrival to the system. Therefore, it will be helpful to introduce adjusted arrival processes that almost surely only count jobs that *arrive to the queues* and an adjusted service process that almost surely only counts jobs that *enter service from the queues.* With this in mind, we define the adjusted arrival processes

$$\mathcal{A}_j(t) := A_j(t) - \int_0^t 1_{\{s^k(r-)=0 \text{ for some } k\in\mathcal{K}\}} dA_j(r), \qquad t \geq 0, j \in \mathcal{J}, \tag{2.6}$$

and the adjusted service process

$$\mathcal{S}(t) := S(t) - \sum_{j=1}^{J} \int_0^t 1_{\{s^k(r-)=0 \text{ for some } k\in\mathcal{K}\}} dA_j(r). \tag{2.7}$$

Since simultaneous arrivals do not occur, the subtracted processes in (2.6) and (2.7) count arrivals to class j that immediately enter service and arrivals to the system that immediately enter service, respectively.

2.7 State Descriptor

We shall use a measure-valued process to keep track of the residual patience times of jobs in the queues. Let δ_x^+ be a unit point mass at $x \in \mathbb{R}$ if $x > 0$ and the zero measure otherwise. Then, for $j \in \mathcal{J}, t \geq 0$, let $\mathcal{Z}_j(t)$ be the measure on $\mathbb{R}_+$ that puts a unit point mass at the remaining patience time of each job in the jth queue at time t. For $j \in \mathcal{J}, t \geq 0$, we introduce the notation $Z_j(t)$ for the number of jobs in the queue for class j at time t, and $\boldsymbol{Z}(t) = (Z_1(t), \ldots, Z_J(t))$. Note that $Z_j(t) = \langle 1, \mathcal{Z}_j(t) \rangle$ for $t \geq 0, j \in \mathcal{J}$. Then we have that

$$\mathcal{Z}_j(t) := \sum_{i=1}^{Z_{0,j}} \delta^+_{\tilde{\ell}^j_{-i}-t} + \sum_{i=1}^{A_j(t)} 1_{\left\{s^k(U_i^j-)\neq 0 \quad \forall k\in\mathcal{K}\right\}} \delta^+_{U_i^j+\ell_i^j-t} - \sum_{\eta_i\in(0,t]} \delta^+_{T_{i,j}-t+\eta_i}, \qquad t \geq 0, \tag{2.8}$$

where for $i \in \mathbb{N}$, η_i is the time at which the ith job to enter service from the queues actually enters service (where $\eta_i = \infty$ if there is no ith job that enters service from the queues), and if the job is of class j, $T_{i,j}$ is the residual patience time of that job at time η_i, and

$T_{i,j} = 0$ for other $j \in \mathcal{J}$, so that on $\{\eta_i < \infty\}$

$$T_{i,j} := \sum_{l=1}^{Z_j(\eta_i-)} 1_{\{\kappa_i \in I_{j,l}(\boldsymbol{Z}(\eta_i-))\}}((\mathrm{supp}(\mathcal{Z}_j(\eta_i-)))_{\{l\}}), \quad j \in \mathcal{J},$$

and $T_{i,j} = 0$ on $\{\eta_i = \infty\}$. We note that because $\eta_i < \infty$ is a time at which a job is taken from a queue, $\boldsymbol{Z}(\eta_i-) \neq \mathbf{0}$, and so the above expression makes sense. Because $T_{i,j}$ depends only on the state of the system up until the ith service entry from the queues, (2.8) is well-defined. We remind the reader of the convention that if a job arrives to the system and there is an available server, it immediately enters service and does not enter any queue, which is why there is an indicator function in the term in (2.8) that adds in arriving jobs. We will refer to such a service entry as a service entry from arrivals, rather than a service entry from the queues. We define the $\mathbf{M}^J$-valued process $\boldsymbol{\mathcal{Z}}(\cdot) := (\mathcal{Z}_1(\cdot), \ldots, \mathcal{Z}_J(\cdot))$. Our state descriptor will be

$$(\boldsymbol{\mathcal{Z}}(t), \boldsymbol{a}(t), \boldsymbol{s}(t)), \qquad t \geq 0, \tag{2.9}$$

where $\boldsymbol{a}(0) = (u_0^1, \ldots, u_0^J)$ and $\boldsymbol{s}(0) = (s_0^1, \ldots, s_0^K)$.

Chapter 3

Sequence of Fluid Scaled Models

Informally, we will implement a fluid scaling (indexed by a parameter m) that will involve speeding up time by a factor of m, scaling up patience times by a factor of m, and making the "mass" of each atom in the measure-valued state descriptor be $\frac{1}{m}$ instead of 1.

Formally, define a sequence of queueing systems as described in §2, indexed by the parameter $m \in \mathbb{N}$.[1] We shall append a superscript of m to relevant quantities to indicate their dependence on m. For example, the sequence of interarrival times for the jth class in the mth system will be denoted by $\{u_i^{j,m}\}_{i=1}^{\infty}$. Some primitives will vary with m, while others are held fixed independent of m. In particular, for $j \in \mathcal{J}$, the weights p_j, the sequences $\{\ell_i^j\}_{i=1}^{\infty}$, and the probability distributions ϑ_j, as well as the sequence $\{\kappa_i\}_{i=1}^{\infty}$ are held fixed independent of m. However, in the mth system, we scale up the patience times the same amount that we scale up time. In particular, for the mth system we use patience times $\ell_i^{j,m} := m\ell_i^j$ for each $i \in \mathbb{N}, j \in \mathcal{J}$. The interarrival times $\{u_i^{j,m}\}_{i=1}^{\infty}$, service times, $\{v_i^{j,m}\}_{i=1}^{\infty}, \{v_i^{J+j,m}\}_{i=1}^{\infty}$, and initial data, $Z_{0,j}^m$, $\{\tilde{\ell}_{-i}^{j,m}\}_{i=1}^{\infty}$ for $j \in \mathcal{J}$, as well as $\boldsymbol{a}^m(0)$ and $\boldsymbol{s}^m(0)$, may vary with m. Then the fluid scaled state descriptor will be defined such that for each Borel set $B \subseteq \mathbb{R}_+$,

$$\bar{\mathcal{Z}}_j^m(t)(B) := \frac{1}{m}\mathcal{Z}_j^m(mt)(mB), \qquad t \geq 0, \tag{3.1}$$

[1] One can use any sequence $r \to \infty$ for the scaling parameter, but we use the natural numbers here for simplicity.

or equivalently, for each bounded Borel measurable function $f : \mathbb{R}_+ \to \mathbb{R}$,

$$\langle f, \bar{\mathcal{Z}}_j^m(t)\rangle = \frac{1}{m}\left\langle f\left(\frac{1}{m}\cdot\right), \mathcal{Z}_j^m(mt)\right\rangle, \qquad t \geq 0. \tag{3.2}$$

Here patience times are scaled up in proportion to the speed at which the system is running. Without this scaling, reneging would become the dominant way that jobs leave the system, and we would not capture the dynamics we are hoping to study. We denote a fluid scaled process associated with the mth system with a bar. In particular, we write

$$\bar{A}_j^m(t) = \frac{1}{m}A_j^m(mt) \quad \text{and} \quad \bar{\mathcal{A}}_j^m(t) = \frac{1}{m}\mathcal{A}_j^m(mt), \qquad t \geq 0, \tag{3.3}$$

$$\bar{S}^m(t) = \frac{1}{m}S^m(mt) \quad \text{and} \quad \bar{\mathcal{S}}^m(t) = \frac{1}{m}\mathcal{S}^m(mt), \qquad t \geq 0, \tag{3.4}$$

$$\bar{Z}_j^m(t) = \frac{1}{m}Z_j^m(mt), \qquad t \geq 0, \tag{3.5}$$

and so on.

We now introduce some assumptions on the sequence of fluid scaled models.

Assumption 1. *We assume the following conditions henceforth.*

(i) For each $j \in \mathcal{J}, m \in \mathbb{N}$, the service rate μ_j^m, reneging rate γ_j^m, and arrival rate α_j^m are all positive and finite, the expected initial number of class j jobs in the queue for class j, $E[Z_{j,0}^m]$, is finite, and the underlying probability distributions for $u_0^{j,m}$, $u_1^{j,m}$, $v_1^{j,m}$, and $v_1^{J+j,m}$ have no atoms. We assume that for each $k \in \mathcal{K}$, the underlying probability distribution of $s_0^{k,m}$ has no atoms. We also assume that for each $t \geq 0, j \in \mathcal{J}$, $\sup_{m\in\mathbb{N}} E[\bar{A}_j^m(t)] < \infty$.

(ii) For each $m \in \mathbb{N}$, the sequences $\{u_i^{j,m}\}_{i=1}^\infty$, $\{v_i^{j,m}\}_{i=1}^\infty$, $\{v_i^{J+j,m}\}_{i=1}^\infty$, $\{\ell_i^j\}_{i=1}^\infty$, $\{\kappa_i\}_{i=1}^\infty$ are mutually independent and independent of $(\boldsymbol{\mathcal{Z}}^m(0), \boldsymbol{a}^m(0), \boldsymbol{s}^m(0))$, $\{\tilde{\ell}_{-i}^j\}_{i=1}^\infty$.

(iii) There is some $\boldsymbol{\alpha} > 0, \boldsymbol{\mu} > 0$ such that $\boldsymbol{\alpha}^m \to \boldsymbol{\alpha}$ and $\boldsymbol{\mu}^m \to \boldsymbol{\mu}$, as $m \to \infty$ and that

$\check{\mathfrak{u}} := \sup_{m\in\mathbb{N}}(E[\min_{j\in\mathcal{J}} v_1^{j,m}])^{-1} < \infty$.

(iv) For each $j \in \mathcal{J}, k \in \mathcal{K}$, $E[u_0^{j,m}]/m$ *and* $E[s_0^{k,m}]/m$ *converge to* 0 *as* $m \to \infty$.

(v) For each $j \in \mathcal{J}$, $E[u_1^{j,m}; u_1^{j,m} > m]$, $E[v_1^{j,m}; v_1^{j,m} > m]$, *and* $E[v_1^{J+j,m}; v_1^{J+j,m} > m]$ *converge to* 0 *as* $m \to \infty$. *Furthermore, for each* $j \in \mathcal{J}$, $\sup_{m\in\mathbb{N}} E[(v_1^{j,m})^2] < \infty$.

(vi) There exists a random measure $\bar{\boldsymbol{\mathcal{Z}}}_0$ *taking values in* $\mathbf{K}^J$ *such that* $\langle \chi, \bar{\mathcal{Z}}_{0,j}\rangle < \infty$ *for* $j \in \mathcal{J}$, *and for* $\bar{\boldsymbol{\mathcal{Z}}}^m(0) := (\bar{\mathcal{Z}}_1^m(0), \ldots, \bar{\mathcal{Z}}_J^m(0))$, $\langle \chi, \bar{\boldsymbol{\mathcal{Z}}}^m(0)\rangle := (\langle \chi, \bar{\mathcal{Z}}_1^m(0)\rangle, \ldots, \langle \chi, \bar{\mathcal{Z}}_J^m(0)\rangle)$, *we have*

$$(\bar{\boldsymbol{\mathcal{Z}}}^m(0), \langle \chi, \bar{\boldsymbol{\mathcal{Z}}}^m(0)\rangle) \Rightarrow (\bar{\boldsymbol{\mathcal{Z}}}_0, \langle \chi, \bar{\boldsymbol{\mathcal{Z}}}_0\rangle)$$

as $m \to \infty$.

These assumptions, particularly (iv) and (v), together with our independence and distributional assumptions about stochastic primitives, give functional laws of large numbers for certain fundamental processes related to our sequence of fluid scaled models (see, e.g., Lemma A.2 in [11] for more details).

Definition 3.0.1 (Fluid model parameters)**.** A vector $(\boldsymbol{\alpha}, \boldsymbol{\mu}, \boldsymbol{p}, \boldsymbol{\vartheta}) \in \mathbb{R}_+^J \times \mathbb{R}_+^J \times (0,1)^J \times \mathbf{M}^J$ is a set of fluid model parameters if $\boldsymbol{\alpha} > 0$, $\boldsymbol{\mu} > 0$, $\sum_{j=1}^J p_j = 1$, and ϑ_j is a probability measure with $\vartheta_j(\{0\}) = 0$ for each $j \in \mathcal{J}$.

We note that, under Assumption 1 and the assumptions given in the original model setup in §2, limits of parameters for a sequence of fluid scaled models are always fluid model parameters.

Chapter 4

Fluid Model

In order to discuss subsequential limits of our fluid scaled processes, we define *fluid model solutions*. These will be solutions of a fluid model that describes the dynamics of a measure-valued function of time by specifying equations satisfied when the measures are integrated against a suitable class of functions and by specifying several extra conditions (see Definition 4.0.1). We define the class of functions,

$$\mathscr{C} := \{f \in \mathbf{C}_b^1(\mathbb{R}_+) : f(0) = 0\}. \tag{4.1}$$

In the following, $(\boldsymbol{\alpha}, \boldsymbol{\mu}, \boldsymbol{p}, \boldsymbol{\vartheta}) \in \mathbb{R}_+^J \times \mathbb{R}_+^J \times (0,1)^J \times \mathbf{M}^J$ are fluid model parameters satisfying Definition 3.0.1. At times in our analysis, we will characterize measures on $\mathbb{R}_+$ by specifying their values on certain intervals. Accordingly, for a path $\boldsymbol{\zeta}(\cdot) = (\zeta_1(\cdot), \ldots, \zeta_J(\cdot)) \in D([0,\infty), \mathbf{M}^J)$, define

$$M_j(t,x) := \langle 1_{[0,x]}, \zeta_j(t) \rangle, \qquad t, x \geq 0, \;\; j \in \mathcal{J},$$

$$z_j(t) := \langle 1_{[0,\infty)}, \zeta_j(t) \rangle, \qquad t \geq 0, \;\; j \in \mathcal{J},$$

$$N_j(x) := \langle 1_{[0,x]}, \vartheta_j \rangle, \qquad x \geq 0, \;\; j \in \mathcal{J},$$

and complementary functions

$$M_j^c(t,x) := \langle 1_{(x,\infty)}, \zeta_j(t)\rangle = z_j(t) - M_j(t,x), \qquad t, x \geq 0,\ \ j \in \mathcal{J},$$

$$N_j^c(x) := \langle 1_{(x,\infty)}, \vartheta_j\rangle = 1 - N_j(x), \qquad x \geq 0,\ \ j \in \mathcal{J}.$$

Given $\boldsymbol{z}(\cdot) \in D([0,\infty), \mathbb{R}_+^J)$, define a weighted mass at time t to be given by

$$L(t) := \sum_{j=1}^{J} p_j z_j(t), \qquad t \geq 0, \tag{4.2}$$

and an adjusted weighted mass at time t to be given by

$$\mathcal{L}(t) := \sum_{j=1}^{J} \frac{p_j}{\mu_j} z_j(t), \qquad t \geq 0. \tag{4.3}$$

Lastly, we define the nominal load parameter to be

$$\varrho := \sum_{j=1}^{J} \frac{\alpha_j}{K\mu_j}. \tag{4.4}$$

We refer to the case where $\varrho < 1$ as underloaded, the case where $\varrho = 1$ as critically (or heavily) loaded, and the case where $\varrho > 1$ as overloaded.

Definition 4.0.1 (Fluid Model Solution)**.** Let $\boldsymbol{\zeta} : [0,\infty) \to \mathbf{M}^J$ be a continuous function. Then we say that $\boldsymbol{\zeta}$ is a fluid model solution for fluid model parameters $(\boldsymbol{\alpha}, \boldsymbol{\mu}, \boldsymbol{p}, \boldsymbol{\vartheta})$ satisfying Definition 3.0.1 and initial condition $\boldsymbol{\zeta}_0 \in \mathbf{K}^J$ if

(i) $\boldsymbol{\zeta}(0) = \boldsymbol{\zeta}_0$,

(ii) $\langle 1_{\{0\}}, \zeta_j(t)\rangle = 0$ for each $t \geq 0, j \in \mathcal{J}$,

(iii) for each $f \in \mathscr{C}, j \in \mathcal{J}, t \geq 0$,

$$\langle f, \zeta_j(t)\rangle = \langle f, \zeta_j(0)\rangle - \int_0^t \langle f', \zeta_j(s)\rangle \, ds \quad - \int_0^t K 1_{\{\mathcal{L}(s)\neq 0\}} \frac{p_j\langle f, \zeta_j(s)\rangle}{\mathcal{L}(s)} ds$$
$$+ \alpha_j \langle f, \vartheta_j\rangle \int_0^t 1_{\{\mathcal{L}(s)\neq 0\}} ds, \tag{4.5}$$

(iv) and when $\varrho > 1$, $\mathcal{L}(t) > 0$ for each $t > 0$.

It will be helpful at this point to give an alternative equation in place of (4.5) that will be easier to work with in parts of the analysis.

Lemma 4.0.1. *Given fluid model parameters $(\boldsymbol{\alpha}, \boldsymbol{\mu}, \boldsymbol{p}, \boldsymbol{\vartheta})$ satisfying Definition 3.0.1 and $\boldsymbol{\zeta}_0 \in \mathbf{K}^J$, let $\boldsymbol{\zeta} : [0, \infty) \to \mathbf{M}^J$ be a continuous function. Let $\boldsymbol{\zeta}(\cdot)$ satisfy (4.5) for each $f \in \mathscr{C}, j \in \mathcal{J}, t \geq 0$. Then, for any $0 \leq u < v < \infty$ such that $\mathcal{L}(\cdot) > 0$ on $[u, v]$, we have for each $j \in \mathcal{J}$, $t \in [u, v], x \geq 0$,*

$$M_j^c(t, x) = M_j^c(u, x + t - u) + \int_u^t \left(\alpha_j N_j^c(x + t - s) - \frac{K p_j M_j^c(s, x + t - s)}{\mathcal{L}(s)} \right) ds. \tag{4.6}$$

Proof. This proof follows from the methods in the proof of Lemma 4.3 in [11]. Specifically, if one goes through the steps of the proof of the supporting lemma, Lemma 4.1 in [11], using our fluid model equations instead of their fluid model equations and the interval $[u, v]$ instead of the interval $[0, t^*)$, one obtains for $0 \leq u < v < \infty$ such that $\mathcal{L}(\cdot) > 0$ on $[u, v]$, for $t \in [u, v]$,

$$\langle f(t, \cdot), \zeta_j(t)\rangle = \langle f(u, \cdot), \zeta_j(u)\rangle + \int_u^t \left\langle \frac{\partial f}{\partial s}(s, \cdot), \zeta_j(s) \right\rangle ds - \int_u^t \left\langle \frac{\partial f}{\partial x}(s, \cdot), \zeta_j(s) \right\rangle ds$$
$$- \int_u^t \frac{K p_j \langle f(s, \cdot), \zeta_j(s)\rangle}{\mathcal{L}(s)} ds + \int_u^t \alpha_j \langle f(s, \cdot), \vartheta_j\rangle ds \tag{4.7}$$

for each $f \in \tilde{\mathscr{C}} := \{f \in \mathbf{C}_b^1([u, t] \times \mathbb{R}_+) : f(s, 0) = 0, \ \forall s \in [u, t]\}$. Next, we follow the proof of Lemma 4.3 in [11] but use a different time shift. In particular, let $g \in \boldsymbol{C}_b^1(\mathbb{R})$ such that $g(y) = 0$ for $y \leq 0$. Then setting $f(s, y) := g(y - t + s)$, $s \in [u, t], y \in \mathbb{R}_+$, we have $f(s, 0) = g(s - t) = 0$ for $s \in [u, t]$. Substituting this into 4.7, we have

$$\langle g(\cdot), \zeta_j(t) \rangle = \langle g(\cdot - t + u), \zeta_j(u) \rangle - \int_u^t \frac{K p_j \langle g(\cdot - t + s), \zeta_j(s) \rangle}{\mathcal{L}(s)} ds + \int_u^t \alpha_j \langle g(\cdot - t + s), \vartheta_j \rangle ds.$$

For $x \geq 0$ fixed, the final result (4.6) comes from approximating $1_{(x,\infty)}$ from below with nonnegative functions $g_n \in \boldsymbol{C}_b^1(\mathbb{R})$ satisfying $g_n(x) = 0$ for $x \leq 0$, $g_n \uparrow 1_{(x,\infty)}$ as $n \to \infty$ and applying the monotone convergence theorem as in Lemma 4.3 of [11]. $\square$

In formulating Definition 4.0.1, we have chosen to include only the most essential properties of the subsequential limits of our fluid scaled models. Nevertheless, we will see that aside from the exceptional case in which $\varrho > 1$ and $\boldsymbol{\zeta}_0 = \mathbf{0}$, we can prove that fluid model solutions are unique, and the set of fluid model solutions is exactly the set of subsequential limits of fluid scaled models. However, fluid model solutions do have some intrinsic properties besides the properties listed in Definition 4.0.1. In the next lemma, we state and prove some such additional properties.

Lemma 4.0.2. *Given fluid model parameters* $(\boldsymbol{\alpha}, \boldsymbol{\mu}, \boldsymbol{p}, \boldsymbol{\vartheta})$ *satisfying Definition 3.0.1 and* $\boldsymbol{\zeta}_0 \in \mathbf{K}^J$, *the following are properties of a fluid model solution* $\boldsymbol{\zeta}(\cdot)$ *for these parameters and initial condition.*

(i) If $t_0 \geq 0$, *then the translation* $\boldsymbol{\zeta}_{t_0}(\cdot) := \boldsymbol{\zeta}(\cdot + t_0)$ *is also a fluid model solution for the initial condition* $\boldsymbol{\zeta}(t_0)$.

(ii) For each $t \geq 0, j \in \mathcal{J}$, $\zeta_j(t)$ *has no atoms.*

(iii) Suppose $\varrho \leq 1$. *Define the following linear combination of the component-level total mass processes for the fluid model solution:*

$$B(t) := \sum_{j=1}^{J} \frac{1}{\mu_j} z_j(t), \qquad t \geq 0. \tag{4.8}$$

Then $B(\cdot)$ is nonincreasing on closed intervals where $\boldsymbol{\zeta}(\cdot) \neq \mathbf{0}$. In particular, if a fluid model solution hits $\mathbf{0}$, then it remains there forever after when $\varrho \leq 1$.

Proof. We begin by proving property (ii). Fix $j \in \mathcal{J}, t \geq 0$. If $\zeta_j(t)$ is the zero measure, then $\zeta_j(t)$ has no atoms. If $x = 0$, then $\zeta_j(t)(\{x\}) = 0$ by condition (ii) of Definition 4.0.1. Therefore, we only need to examine the case where $\zeta_j(t) \neq 0$ and $x > 0$. Let $t_0 = \sup\{s \in [0,t] : \zeta_j(s) = 0\}$. Then $\zeta_j(s) \neq 0$ for $s \in (t_0, t]$ and, by continuity of fluid model solutions, $t_0 < t$. By applying (4.6), for $t_0 < u < v = t$, $0 < h < x$, we have

$$\begin{aligned} M_j^c(t,x) - M_j^c(t,x-h) = M_j^c(u, x+t-u) - M_j^c(u, x-h+t-u) \\ + \int_u^t \big(\alpha_j N_j^c(x+t-s) - \alpha_j N_j^c(x-h+t-s)\big)\, ds \\ - \int_u^t \left(\frac{Kp_j(M_j^c(s,x+t-s) - M_j^c(s,x-h+t-s))}{\mathcal{L}(s)}\right) ds. \end{aligned}$$

By letting $h \to 0$ and using the bounded convergence theorem, we obtain

$$\langle 1_{\{x\}}, \zeta_j(t)\rangle = \langle 1_{\{x+t-u\}}, \zeta_j(u)\rangle + \int_u^t \alpha_j \langle 1_{\{x+t-s\}}, \vartheta_j\rangle ds - \int_u^t \frac{Kp_j\langle 1_{\{x+t-s\}}, \zeta_j(s)\rangle}{\mathcal{L}(s)} ds. \quad (4.9)$$

The first integral on the right (involving ϑ_j) is zero because the set of $s \in [u,t]$ where $\vartheta_j(\{x+t-s\}) \neq 0$ is at most countable, and so it is of Lebesgue measure zero. If $\zeta_j(t_0) = 0$, $\langle 1_{\{x+t-u\}}, \zeta_j(u)\rangle \leq z_j(u)$, where the right side tends to zero as $u \downarrow t_0$ by the continuity of fluid model solutions in time. Then, taking the limit $u \downarrow t_0$ in (4.9) yields

$$\langle 1_{\{x\}}, \zeta_j(t)\rangle = -\int_{t_0}^t \frac{Kp_j\langle 1_{\{x+t-s\}}, \zeta_j(s)\rangle}{\mathcal{L}(s)} ds.$$

Because the right-hand side and left-hand side have opposite signs, they both must be zero. On the other hand, if $\zeta_j(t_0) \neq 0$, then $t_0 = 0$. Assuming this, in the next calculation, for each $\epsilon \in \left(0, \frac{x}{2}\right)$ and f_ϵ a continuous function such that $1_{[x+t-\epsilon, x+t]} \leq f_\epsilon \leq 1_{[x+t-2\epsilon, x+t+\epsilon]}$, we have from (4.9) that for $0 < u < \epsilon \wedge t$,

$$\langle 1_{\{x\}}, \zeta_j(t)\rangle \leq \langle f_\epsilon, \zeta_j(u)\rangle - \int_u^t \frac{Kp_j\langle 1_{\{x+t-s\}}, \zeta_j(s)\rangle}{\mathcal{L}(s)} ds.$$

Letting $u \downarrow t_0 = 0$, we obtain by the continuity of fluid model solutions

$$\langle 1_{\{x\}}, \zeta_j(t)\rangle \leq \langle f_\epsilon, \zeta_j(0)\rangle - \int_0^t \frac{Kp_j\langle 1_{\{x+t-s\}}, \zeta_j(s)\rangle}{\mathcal{L}(s)} ds.$$

Finally, letting $\epsilon \downarrow 0$ yields

$$\langle 1_{\{x\}}, \zeta_j(t)\rangle \leq \langle 1_{\{x+t\}}, \zeta_j(0)\rangle - \int_0^t \frac{Kp_j\langle 1_{\{x+t-s\}}, \zeta_j(s)\rangle}{\mathcal{L}(s)} ds.$$

Since $\zeta_j(0) \in \mathbf{K}$, the first term on the right hand side is 0. Because the right hand side is thus nonpositive and the left hand side is nonnegative, we conclude that both must be 0.

For property (i), it follows from (ii) that for $t_0 \geq 0$, $\boldsymbol{\zeta}(t_0) \in \mathbf{K}^J$, and is thus a valid initial condition for a fluid model solution. It is then straightforward to verify that conditions (i), (ii), (iii), and (iv) of Definition 4.0.1 hold for the translated fluid model solution $\boldsymbol{\zeta}_{t_0}$.

Lastly, we prove property (iii). The intuition behind this proof is that, when $\varrho \leq 1$ and we normalize by the individual service rates, the total arriving mass over an interval of time minus the total mass serviced during that period of time will be nonpositive. This is because, in aggregate, the system is not overloaded. Let $[a, b]$ be an interval on which $\boldsymbol{\zeta}(\cdot) \neq 0$, and therefore $\mathcal{L}(\cdot) \neq 0$. We will show that for each $u, v \in [a, b]$ with $u < v$, we have $B(v) - B(u) \leq 0$. This implies that $B(\cdot)$ is nonincreasing on $[a, b]$. Consider a sequence of functions $\{f_n\}_{n=1}^\infty \subset \mathscr{C}$ such that $0 \leq f_n \uparrow 1_{(0,\infty)}$ as $n \to \infty$ and $f_n' \geq 0$ for each $n \in \mathbb{N}$. Then from (4.5), we have that for $j \in \mathcal{J}$,

$$\begin{aligned}\langle f_n, \zeta_j(v)\rangle - \langle f_n, \zeta_j(u)\rangle &= -\int_u^v \langle f_n', \zeta_j(s)\rangle ds - K\int_u^v \frac{p_j\langle f_n, \zeta_j(s)\rangle}{\mathcal{L}(s)} ds + \alpha_j\langle f_n, \vartheta_j\rangle(v-u) \\ &\leq -K\int_u^v \frac{p_j\langle f_n, \zeta_j(s)\rangle}{\mathcal{L}(s)} ds + \alpha_j\langle f_n, \vartheta_j\rangle(v-u)\end{aligned}$$

Letting $n \to \infty$ in the above and using property (ii) of Definition 4.0.1 as well as the fact that $\vartheta_j(\{0\}) = 0$, we obtain

$$\langle 1, \zeta_j(v)\rangle - \langle 1, \zeta_j(u)\rangle \leq -K\int_u^v \frac{p_j\langle 1, \zeta_j(s)\rangle}{\mathcal{L}(s)} ds + \alpha_j\langle 1, \vartheta_j\rangle(v-u).$$

Multiplying through by $\frac{1}{\mu_j}$ and summing over $j \in \mathcal{J}$ yields

$$\begin{aligned}B(v) - B(u) &\leq -K\int_u^v \frac{\sum_{j=1}^J \frac{p_j}{\mu_j} z_j(s)}{\mathcal{L}(s)} ds + \sum_{j=1}^J \frac{\alpha_j}{\mu_j}\langle 1, \vartheta_j\rangle(v-u) \\ &= K(\varrho - 1)(v-u) \qquad (4.10) \\ &\leq 0,\end{aligned}$$

since $\varrho \leq 1$. We can then conclude that because $B(\cdot)$ is zero if and only if $\boldsymbol{\zeta}(\cdot) = \mathbf{0}$, and is continuous, nonincreasing on intervals where it is nonzero, and nonnegative, once it hits zero it must remain there, and hence once $\boldsymbol{\zeta}(\cdot)$ reaches 0 it remains there when $\varrho \leq 1$.

□

Lastly, we prove a special property for the overloaded case that strengthens (iv) of Definition 4.0.1.

Lemma 4.0.3. *Let* $(\boldsymbol{\alpha}, \boldsymbol{\mu}, \boldsymbol{p}, \boldsymbol{\vartheta})$ *be a set of fluid model parameters satisfying Definition 3.0.1 with* $\varrho > 1$. *Then there exists* $\epsilon > 0$ *such that for any fluid model solution with parameters* $(\boldsymbol{\alpha}, \boldsymbol{\mu}, \boldsymbol{p}, \boldsymbol{\vartheta})$ *and initial condition* $\boldsymbol{\zeta}_0 \in \mathbf{K}^J$, *we have* $B(t) \geq \frac{\varrho-1}{2}K(\epsilon \wedge t)$ *for all*

$t \geq 0$, where ϵ depends only on $(\boldsymbol{\alpha}, \boldsymbol{\mu}, \boldsymbol{p}, \boldsymbol{\vartheta})$. It follows that there exists $t_0 \geq 0, \mathcal{L}_{min} > 0$ such that for any fluid model solution with the given parameters, $\mathcal{L}(t) \geq \mathcal{L}_{min}$ for each $t \geq t_0$, where t_0 and $\mathcal{L}_{min}$ depend only on $(\boldsymbol{\alpha}, \boldsymbol{\mu}, \boldsymbol{p}, \boldsymbol{\vartheta})$.

Proof. Fix $t > 0$. Then by (iv) of Definition 4.0.1, $\mathcal{L}(r) \neq 0$ for each $r \in [s, t]$ when $0 < s \leq t$. Therefore, we may apply (4.6) with s in place of u and $x = 0$ to obtain

$$\begin{aligned}
\sum_{j=1}^{J} \frac{1}{\mu_j} z_j(t) &= \sum_{j=1}^{J} \frac{1}{\mu_j} M_j^c(s, t-s) + \int_s^t \sum_{j=1}^{J} \frac{\alpha_j}{\mu_j} N_j^c(t-r) dr - \int_s^t \frac{K \sum_{j=1}^{J} \frac{p_j}{\mu_j} M_j^c(r, t-r)}{\mathcal{L}(r)} dr \\
&\geq \int_s^t \sum_{j=1}^{J} \frac{\alpha_j}{\mu_j} N_j^c(t-r) dr - \int_s^t \frac{K \sum_{j=1}^{J} \frac{p_j}{\mu_j} M_j^c(r, 0)}{\mathcal{L}(r)} dr \\
&= \int_0^{t-s} \sum_{j=1}^{J} \frac{\alpha_j}{\mu_j} N_j^c(r) dr - K(t-s),
\end{aligned}$$

where we have used (4.3) to obtain the last line and the fact that $M^c(r, \cdot)$ is non-negative and decreasing for each $r \in [s, t]$ to obtain the inequality. For the last equality, we used property (ii) of Definition 4.0.1 and a change of variables $r \to t - r$. Now, using the fact that $\vartheta_j(\{0\}) = 0$ for each $j \in \mathcal{J}$ we may choose ϵ such that $N_j^c(r) \geq \frac{\varrho+1}{2\varrho}$ for each $r \leq \epsilon, j \in \mathcal{J}$. Using (4.4) and the calculation above, we conclude that

$$\begin{aligned}
\sum_{j=1}^{J} \frac{1}{\mu_j} z_j(t) &\geq \int_0^{t-s} \sum_{j=1}^{J} \frac{\alpha_j}{\mu_j} N_j^c(r) dr - K(t-s) \\
&\geq \frac{\varrho+1}{2\varrho}(t-s) K \varrho - K(t-s) \\
&= \frac{\varrho-1}{2} K(t-s)
\end{aligned}$$

when $s \in ((t-\epsilon)^+, t)$. Letting $s \downarrow (t-\epsilon)^+$, we obtain $B(t) \geq \frac{\varrho-1}{2} K(\epsilon \wedge t)$ for $t > 0$. We note that this also holds for $t = 0$. Choosing $t_0 = \frac{\epsilon}{2}$, we obtain $B(t) \geq \frac{\epsilon(\varrho-1)}{4} K$ for each $t \geq t_0$. Then $\mathcal{L}(t) \geq \min_{j \in \mathcal{J}} p_j B(t) \geq \min_{j \in \mathcal{J}} p_j \frac{\epsilon(\varrho-1)}{4} K =: \mathcal{L}_{\min}$ for each $t \geq t_0$. □

Chapter 5

Main Results

5.1 Results on Uniqueness and Continuous Dependence on Initial Conditions for Fluid Model Solutions

Theorem 5.1.1 (Uniqueness of Solutions)**.** *Fix fluid model parameters* $(\boldsymbol{\alpha}, \boldsymbol{\mu}, \boldsymbol{p}, \boldsymbol{\vartheta})$ *satisfying Definition 3.0.1 and* $\boldsymbol{\zeta}_0 \in \mathbf{K}^J$*. Then, if either* $\varrho \leq 1$ *or* $\boldsymbol{\zeta}_0 \neq \mathbf{0}$*, any fluid model solution for these parameters and initial condition is unique.*

Theorem 5.1.2 (Continuous Dependence on Initial Conditions)**.** *Fix fluid model parameters* $(\boldsymbol{\alpha}, \boldsymbol{\mu}, \boldsymbol{p}, \boldsymbol{\vartheta})$ *satisfying Definition 3.0.1 and* $\boldsymbol{\zeta}_0 \in \mathbf{K}^J$*. Furthermore, assume that* $\boldsymbol{\zeta}_0 \neq \mathbf{0}$ *if* $\varrho > 1$*. Suppose that* $\{\boldsymbol{\zeta}_0^n\}_{n=1}^{\infty}$ *is a sequence in* $\mathbf{K}^J$ *which converges iweakly to* $\boldsymbol{\zeta}_0$*. Assume there is an associated sequence of fluid model solutions* $\{\boldsymbol{\zeta}^n(\cdot)\}_{n=1}^{\infty}$ *with fluid model parameters* $(\boldsymbol{\alpha}, \boldsymbol{\mu}, \boldsymbol{p}, \boldsymbol{\vartheta})$ *satisfying Definition 3.0.1 and initial conditions* $\{\boldsymbol{\zeta}_0^n\}_{n=1}^{\infty}$*. Then* $\{\boldsymbol{\zeta}^n(\cdot)\}_{n=1}^{\infty}$ *converges in* $C([0,\infty), \mathbf{M}^J)$ *to* $\boldsymbol{\zeta}(\cdot)$*, the unique fluid model solution associated with the parameters* $(\boldsymbol{\alpha}, \boldsymbol{\mu}, \boldsymbol{p}, \boldsymbol{\vartheta})$ *and initial condition* $\boldsymbol{\zeta}_0 \in \mathbf{K}^J$*.*

The proofs of Theorems 5.1.1 and 5.1.2 are in §6.

5.2 Fluid Limit Results

Theorem 5.2.1. *Let* $\{\bar{\boldsymbol{\mathcal{Z}}}^m(\cdot)\}_{m=1}^{\infty}$ *be a sequence of fluid scaled state descriptors, as described in §3, for which Assumption 1 holds. Then* $\{\bar{\boldsymbol{\mathcal{Z}}}^m(\cdot)\}_{m=1}^{\infty}$ *is C-tight. Suppose that*

$\bar{\mathcal{Z}}(\cdot)$ *is a limit in distribution along a subsequence of* $\{\bar{\mathcal{Z}}^m(\cdot)\}_{m=1}^\infty$*. Then, almost surely,* $\bar{\mathcal{Z}}(\cdot)$ *is a fluid model solution for the parameters* $(\boldsymbol{\alpha}, \boldsymbol{\mu}, \boldsymbol{p}, \boldsymbol{\vartheta})$ *and initial condition* $\bar{\mathcal{Z}}_0$*.*

Proofs for Theorem 5.2.1 are in §7 through §9.

Corollary 5.2.1. *For each set of fluid model parameters* $(\boldsymbol{\alpha}, \boldsymbol{\mu}, \boldsymbol{p}, \boldsymbol{\vartheta})$ *and initial condition* $\boldsymbol{\zeta}_0 \in \mathbf{K}^J$*, there exists a fluid model solution for these parameters and initial condition. Furthermore, if either* $\varrho \leq 1$ *or* $\boldsymbol{\zeta}_0 \neq 0$*, then that fluid model solution is unique.*

Proof. If $\boldsymbol{\zeta}_0$ satisfies $\langle \chi, \zeta_{0,j} \rangle < \infty$ for each $j \in \mathcal{J}$, then the existence follows by Theorem 5.2.1 with $\bar{\mathcal{Z}}_0 = \boldsymbol{\zeta}_0$. If $\boldsymbol{\zeta}_0$ does not satisfy this condition, one can define a sequence $\{\boldsymbol{\zeta}_0^n\}_{n=1}^\infty$ in $\mathbf{K}^J$ converging weakly to $\boldsymbol{\zeta}_0$ where each $\boldsymbol{\zeta}_0^n$ satisfies $\langle \chi, \boldsymbol{\zeta}_{0,j}^n \rangle < \infty$ for each $j \in \mathcal{J}$. Then there is a fluid model solution $\boldsymbol{\zeta}^n(\cdot)$ with initial condition $\boldsymbol{\zeta}_0^n$, for each n, and since $\boldsymbol{\zeta}_0 \neq 0$, by Theorem 5.1.2 $\{\boldsymbol{\zeta}^n(\cdot)\}_{n=1}^\infty$ converges to a fluid model solution $\boldsymbol{\zeta}(\cdot)$ with initial condition $\boldsymbol{\zeta}_0$. Thus existence holds for any $\boldsymbol{\zeta}_0 \in \mathbf{K}^J$. The uniqueness follows from Theorem 5.1.1. □

Corollary 5.2.2. *Let* $\{\bar{\mathcal{Z}}^m(\cdot)\}_{m=1}^\infty$ *be a sequence of fluid scaled state descriptors, as described in §3, for which Assumption 1 holds. Then, if either* $\varrho \leq 1$ *or* $\bar{\mathcal{Z}}_0 \neq \mathbf{0}$ *almost surely, the sequence converges in distribution to a process* $\bar{\mathcal{Z}}(\cdot)$ *which almost surely is the unique fluid model solution associated to the fluid model parameters* $(\boldsymbol{\alpha}, \boldsymbol{\mu}, \boldsymbol{p}, \boldsymbol{\vartheta})$ *satisfying Definition 3.0.1 and initial condition* $\bar{\mathcal{Z}}_0$*.*

Proof. If $\rho \leq 1$, for each $\omega \in \Omega$, let $\bar{\mathcal{Z}}(\cdot, \omega)$ be the unique fluid model solution for $(\boldsymbol{\alpha}, \boldsymbol{\mu}, \boldsymbol{p}, \boldsymbol{\vartheta})$ and initial condition $\bar{\mathcal{Z}}_0(\omega)$. If $\rho > 1$, for each $\omega \in \Omega \setminus \{\bar{\mathcal{Z}}_0 = \mathbf{0}\}$, let $\bar{\mathcal{Z}}(\cdot, \omega)$ be the unique fluid model solution for $(\boldsymbol{\alpha}, \boldsymbol{\mu}, \boldsymbol{p}, \boldsymbol{\vartheta})$ with initial condition $\bar{\mathcal{Z}}_0(\omega)$, and for $\omega \in \{\bar{\mathcal{Z}}_0 = \mathbf{0}\}$ let $\bar{\mathcal{Z}}(\cdot, \omega) = \mathbf{0}$. The existence and uniqueness of these solutions follows from Corollary 5.2.1, and by Theorem 5.1.2, $\bar{\mathcal{Z}} : \Omega \to D([0, \infty), \mathbf{M}^J)$ is measurable and $\bar{\mathcal{Z}}$ is a well-defined process. By Theorems 5.1.1 and 5.2.1, $\{\bar{\mathcal{Z}}^m(\cdot)\}_{m=1}^\infty$ is C-tight, and each subsequence that converges in distribution has a limit that is almost surely equal to $\bar{\mathcal{Z}}(\cdot)$. □

5.3 Invariant State and Asymptotic Behavior of Fluid Model Solutions

Before beginning our discussion of invariant states for our fluid model, we discuss a relevant fixed point equation.

Lemma 5.3.1. *Fix fluid model parameters* $(\boldsymbol{\alpha}, \boldsymbol{\mu}, \boldsymbol{p}, \boldsymbol{\vartheta})$ *satisfying Definition 3.0.1 such that* $\varrho > 1$*. Then the fixed point equation for* $\mathcal{L}$*:*

$$\mathcal{L} = \sum_{j=1}^{J} \int_0^\infty \frac{\alpha_j p_j}{\mu_j} N_j^c(s) \exp\left(\frac{-K p_j s}{\mathcal{L}}\right) ds, \tag{5.1}$$

has a unique positive real-valued solution, which we denote by $\mathcal{L}^*$*.*

Proof. Define $g(x) := \sum_{j=1}^{J} \int_0^\infty \frac{\alpha_j p_j}{\mu_j} N_j^c(s) \exp\left(\frac{-K p_j s}{x}\right) ds$ for $x > 0$. Then using the change of variables formula, we can re-write $g(x) = x \sum_{j=1}^{J} \int_0^\infty \frac{\alpha_j p_j}{\mu_j} N_j^c(xv) \exp(-K p_j v) dv$. Then, $x > 0$ is a fixed point of g if and only if $x = g(x)$, which is equivalent to

$$1 = \sum_{j=1}^{J} \int_0^\infty \frac{\alpha_j p_j}{\mu_j} N_j^c(xv) \exp(-K p_j v) dv. \tag{5.2}$$

By the dominated convergence theorem, we have

$$\lim_{x \to 0} \sum_{j=1}^{J} \int_0^\infty \frac{\alpha_j p_j}{\mu_j} N_j^c(xv) \exp(-K p_j v) dv = \sum_{j=1}^{J} \frac{\alpha_j}{K \mu_j} = \varrho > 1,$$

and

$$\lim_{x \to \infty} \sum_{j=1}^{J} \int_0^\infty \frac{\alpha_j p_j}{\mu_j} N_j^c(xv) \exp(-K p_j v) dv = 0.$$

Then, since $x \to \sum_{j=1}^{J} \int_0^\infty \frac{\alpha_j p_j}{\mu_j} N_j^c(xv) \exp(-K p_j v) dv$ is continuous and strictly decreasing on $(0, \infty)$, there must be a unique $x \in (0, \infty)$ such that (5.2) holds.

□

Definition 5.3.1. A vector of measures $\boldsymbol{\nu} \in \mathbf{K}^J$ is an invariant state for the fluid model if $\boldsymbol{\zeta}$, defined by $\boldsymbol{\zeta}(t) = \boldsymbol{\nu}$ for $t \geq 0$, is a fluid model solution.

Theorem 5.3.1. *Fix fluid model parameters* $(\boldsymbol{\alpha}, \boldsymbol{\mu}, \boldsymbol{p}, \boldsymbol{\vartheta})$ *satisfying Definition 3.0.1. Then there is a unique invariant state* $\boldsymbol{\nu} \in \mathbf{K}^J$ *for the fluid model with parameters* $(\boldsymbol{\alpha}, \boldsymbol{\mu}, \boldsymbol{p}, \boldsymbol{\vartheta})$. *If* $\varrho \leq 1$, $\boldsymbol{\nu} = \mathbf{0}$. *If* $\varrho > 1$, $\boldsymbol{\nu} := (\nu_1, \ldots, \nu_J)$ *is the unique element of* $\mathbf{K}^J$ *such that*

$$\langle 1_{(x,\infty)}, \nu_j \rangle := \alpha_j \int_0^\infty N_j^c(s+x) \exp\left(\frac{-Kp_j s}{\mathcal{L}^*} \right) ds, \qquad x \geq 0, j \in \mathcal{J}. \tag{5.3}$$

Furthermore, if we define sets in $\mathbf{K}^J$ *and* $C([0,\infty), \mathbf{M}^J)$ *as follows for* $c > 0$*:*

$$\mathbf{K}_c^J := \{\boldsymbol{\xi} \in \mathbf{K}^J : \langle 1, \xi_j \rangle \leq c \ \ \forall j \in \mathcal{J}\}$$

and $\mathscr{M}_c^J$ *be the set of* $\boldsymbol{\zeta} \in C([0,\infty), \mathbf{M}^J)$ *such that* $\boldsymbol{\zeta}$ *is a fluid model solution for parameters* $(\boldsymbol{\alpha}, \boldsymbol{\mu}, \boldsymbol{p}, \boldsymbol{\vartheta})$*and initial condition* $\boldsymbol{\zeta}_0 \in \mathbf{K}_c^J$. *Then, using the metric on* $\mathbf{M}^J$ *given in* (1.2), *we have for each* $c > 0$,

$$\lim_{t \to \infty} \sup\{d_J(\boldsymbol{\zeta}(t), \boldsymbol{\nu}) : \boldsymbol{\zeta} \in \mathscr{M}_c^J\} = 0.$$

Theorem 5.3.1 is proved in §10. For the remainder of this thesis, we let $F_j^c(x) := \langle 1_{(x,\infty)}, \nu_j \rangle$ for $x \geq 0, j \in \mathcal{J}$ and $z_j^* := \langle 1, \nu_j \rangle$ for $j \in \mathcal{J}$.

5.4 An Illustrative Example: Exponential Patience Times

When one assumes exponentially distributed patience times and initial conditions, then provided $\boldsymbol{\zeta}_0 \neq \mathbf{0}$ if $\varrho > 1$, the measure-valued description of fluid model solutions reduces to one involving queue length, which satisfies a J-dimensional system of nonlinear ordinary differential equations. In particular, if for each $j \in \mathcal{J}$, the patience time distribution ϑ_j and shape of the initial condition $\zeta_{0,j}$ are exponentially distributed, we have a simpler characterization of fluid model solutions, given in Theorem 5.4.1, and we see that the shape of a fluid model solution is constant.

Theorem 5.4.1. *Let $(\boldsymbol{\alpha}, \boldsymbol{\mu}, \boldsymbol{\vartheta}, \boldsymbol{p})$ be a set of fluid model parameters satisfying Definition 3.0.1 such that ϑ_j is the exponential distribution with mean $1/\gamma_j$ for each $j \in \mathcal{J}$. Let $\zeta_{0,j} = z_{0,j}\vartheta_j$ for each $j \in \mathcal{J}$ for some vector $\boldsymbol{z}_0 = (z_{0,1}, \dots, z_{0,J}) \in \mathbb{R}_+^J$, with the added assumption that $\boldsymbol{z}_0 \neq \mathbf{0}$ if $\varrho > 1$. Let $\boldsymbol{\zeta}(\cdot)$ be the fluid model solution for the parameters $(\boldsymbol{\alpha}, \boldsymbol{\mu}, \boldsymbol{\vartheta}, \boldsymbol{p})$ and initial condition $\boldsymbol{\zeta}_0 = (\zeta_{0,1}, \dots, \zeta_{0,J})$, with $\boldsymbol{z}(\cdot) = (z_1(\cdot), \dots, z_J(\cdot))$ defined as in §4. Then*

$$\zeta_j(t) = z_j(t)\vartheta_j, \quad t \geq 0, \tag{5.4}$$

where $\boldsymbol{z}(\cdot) = \boldsymbol{x}(\cdot)$ and $\boldsymbol{x} : \mathbb{R}_+ \to \mathbb{R}_+^J$ is the unique continuous path such that, setting $t^* := \inf\{t \geq 0 : \boldsymbol{x}(t) = \mathbf{0}\}$, then $(x_1(t), \dots, x_J(t)) = \mathbf{0}$ for each $t \geq t^*$, and the equations

$$x_j(t) = z_{j,0} + \alpha_j t - \int_0^t \left(\gamma_j + \frac{Kp_j}{\mathcal{L}(s)} \right) x_j(s) ds, \quad j \in \mathcal{J}, \tag{5.5}$$

where

$$\mathcal{L}(s) = \sum_{j=1}^{J} \frac{p_j}{\mu_j} x_j(s) \tag{5.6}$$

hold for each $t \in [0, t^*)$.

Furthermore, the following is a straightforward consequence of (5.1) and (5.3).

Theorem 5.4.2. *Let $(\boldsymbol{\alpha}, \boldsymbol{\mu}, \boldsymbol{\vartheta}, \boldsymbol{p})$ be a set of fluid model parameters satisfying Definiton 3.0.1 such that $\varrho > 1$. Let ϑ_j be the exponential distribution with mean $1/\gamma_j$ for each $j \in \mathcal{J}$. Then the invariant state for the associated fluid model is*

$$\left(\frac{\alpha_1 \mathcal{L}^*}{\gamma_1 \mathcal{L}^* + Kp_1} \vartheta_1, \dots, \frac{\alpha_J \mathcal{L}^*}{\gamma_J \mathcal{L}^* + Kp_J} \vartheta_J \right), \tag{5.7}$$

where $\mathcal{L} = \mathcal{L}^*$ is the unique solution to the equation

$$1 = \sum_{j=1}^{J} \frac{\alpha_j p_j}{\mu_j(\gamma_j \mathcal{L} + K p_j)}. \tag{5.8}$$

We also have the following additional property that can be proved using our results for fluid model solutions and which highlights a special role for exponentially distributed patience times in characterizing invariant shape behavior. In particular, the only conditions under which the shape of a measure-valued fluid model solution $\boldsymbol{\zeta}(\cdot)$ is constant are when patience times are exponentially distributed or the fluid model solution is started in its invariant state. In essence, if patience times are not exponentially distributed, then non-constant fluid model solutions do not have a constant shape.

Theorem 5.4.3. *Let $\boldsymbol{\zeta}(\cdot)$ be a fluid model solution with fluid model parameters $(\boldsymbol{\alpha}, \boldsymbol{\mu}, \boldsymbol{\vartheta}, \boldsymbol{p})$ satisfying Definition 3.0.1 and where $\boldsymbol{\zeta}_0 \neq \mathbf{0}$ if $\varrho > 1$. Let $\boldsymbol{\zeta}(\cdot)$ be of the form*

$$\zeta_j(\cdot) = z_j(\cdot)\sigma^j, \quad j \in \mathcal{J},$$

for some probability measures σ^j, $j \in \mathcal{J}$, on $\mathbb{R}_+$. Then either $\zeta_j(\cdot) \equiv \nu_j$, the invariant state for class j, for each $j \in \mathcal{J}$, or there exists $\gamma_j > 0$ such that $\sigma_j = \vartheta_j$ and ϑ_j is the exponential distribution with mean $1/\gamma_j$ for each $j \in \mathcal{J}$.

The proofs of Theorems 5.4.1, 5.4.2, and 5.4.3 are given in §11.

Chapter 6

Proofs of Uniqueness and Continuous Dependence on Initial Conditions for Fluid Model Solutions

This chapter is devoted to proving Theorems 5.1.1 and 5.1.2.

6.1 Analysis of (4.6) as a System of Integral Equations

In this section, we analyze (4.6) as a system of integral equations in two variables, time and space (t and x respectively), that each fluid model solution must satisfy. Because most of the analysis will happen up until a fluid model solution hits zero, given a fluid model solution $\boldsymbol{\zeta}(\cdot)$, we define

$$t^* := \inf\{t \geq 0 : \mathcal{L}(t) = 0\}, \tag{6.1}$$

which is the time at which the fluid model solution first equals $\mathbf{0}$.

Lemma 6.1.1. *Let $\boldsymbol{\zeta}(\cdot)$ be a fluid model solution for fluid model parameters $(\boldsymbol{\alpha}, \boldsymbol{\mu}, \boldsymbol{p}, \boldsymbol{\vartheta})$ satisfying Definition 3.0.1 and with initial condition $\boldsymbol{\zeta}_0 \in \mathbf{K}^J$ with $\boldsymbol{\zeta}_0 \neq 0$. Define the function*

$$G(t) := \int_0^t \frac{1}{\mathcal{L}(r)} dr, \qquad 0 \leq t < t^*. \tag{6.2}$$

Then for each $j \in \mathcal{J}$, $M_j^c(\cdot,\cdot)$ satisfies the equation

$$M_j^c(t,x) = \exp(-Kp_jG(t))M_j^c(0,t+x) + \int_0^t \exp(-Kp_j(G(t)-G(t-r)))\alpha_j N_j^c(r+x)dr \tag{6.3}$$

for $t \in [0,t^*), x \geq 0$. In particular, for each $j \in \mathcal{J}$, $z_j(\cdot)$ satisfies the equation

$$z_j(t) = \exp(-Kp_jG(t))M_j^c(0,t) + \int_0^t \exp(-Kp_j(G(t)-G(t-r)))\alpha_j N_j^c(r)dr \tag{6.4}$$

for $t \in [0,t^*)$.

Proof. Fix $j \in \mathcal{J}$. By (4.6) with $u = 0$ we have

$$M_j^c(t,x) = M_j^c(0,t+x) + \int_0^t \left(\alpha_j N_j^c(t+x-r) - \frac{Kp_j M_j^c(r,t+x-r)}{\mathcal{L}(r)}\right) dr \tag{6.5}$$

for $t \in [0,t^*), x \geq 0$. Fix $t \in [0,t^*), x \geq 0$, and define $H_{t,x}^j(s) := M_j^c(s,t+x-s)$ for $s \in [0,t]$. Then by equation (6.5) with $t+x-s$ in place of x and s in place of t, we see that $H_{t,x}^j(\cdot)$ satisfies the equation

$$H_{t,x}^j(s) = M_j^c(0,t+x) + \int_0^s \left(\alpha_j N_j^c(t+x-r) - \frac{Kp_j H_{t,x}^j(r)}{\mathcal{L}(r)}\right) dr \tag{6.6}$$

for $s \in [0,t]$. Given $\mathcal{L}$, this is a linear integral equation for $H_{t,x}^j(\cdot)$. Using the integrating factor $\exp(-Kp_jG(\cdot))$, we can solve (6.6) to obtain

$$H_{t,x}^j(s) = M_j^c(0,t+x)\exp(-Kp_jG(s)) + \int_0^s \exp(-(Kp_jG(s)-Kp_jG(r)))\alpha_j N_j^c(t+x-r)dr \tag{6.7}$$

for $s \in [0,t]$, $x \geq 0$. On setting $s = t$ and then replacing r by $t-r$ in the integral, we obtain for $t \in [0, t^*), x \geq 0$,

$$M_j^c(t,x) = M_j^c(0, t+x)\exp(-Kp_jG(t)) + \int_0^t \exp(-Kp_j(G(t) - G(t-r)))\alpha_j N_j^c(x+r)dr,$$

which is (6.3). Equation (6.4) follows immediately on setting $x = 0$ and using the fact that $\langle 1_{\{0\}}, \zeta_j(0)\rangle = 0$ for each $t \geq 0$. □

6.2 Uniqueness of $\mathcal{L}$

We begin this section by proving a useful lemma about the total mass.

Lemma 6.2.1. *Let $\boldsymbol{\zeta}(\cdot)$ be a fluid model solution for fluid model parameters $(\boldsymbol{\alpha}, \boldsymbol{\mu}, \boldsymbol{p}, \boldsymbol{\vartheta})$ satisfying Definition 3.0.1 and $\boldsymbol{\zeta}_0 \in \mathbf{K}^J$. Then, for each $j \in \mathcal{J}, t \geq 0$,*

$$z_j(t) \leq z_j(0) + \alpha_j\langle \chi, \vartheta_j\rangle. \tag{6.8}$$

Proof. We first prove the case in which $\boldsymbol{\zeta}_0 \neq \mathbf{0}$ or $\varrho \leq 1$. From (6.4) in Lemma 6.1.1 and using the fact that for $t^* > 0$, $G(0) = 0$, and G is an increasing function on $[0, t^*)$, we have for $t \in [0, t^*)$,

$$\begin{aligned} z_j(t) &= \exp(-Kp_jG(t))M_j^c(0,t) + \int_0^t \exp(-Kp_j(G(t) - G(t-r)))\alpha_j N_j^c(r)dr \\ &\leq z_j(0) + \alpha_j \int_0^\infty N_j^c(r)dr \\ &= z_j(0) + \alpha_j\langle \chi, \vartheta_j\rangle. \end{aligned}$$

If $\varrho > 1$, then since $\boldsymbol{\zeta}_0 \neq 0$, by property (iv) of Definition 4.0.1, $t^* = \infty$ and so (6.8) holds for all $t \geq 0$. If $\varrho \leq 1$, then by Lemma (4.0.2) (iii), $\boldsymbol{z}(t) = 0$ for all $t \geq t^*$, and so, combining with the above, we see that (6.8) holds for all $t \geq 0$.

Now, we extend to the case where $\varrho > 1$ and $\boldsymbol{\zeta}_0 = 0$. Let $\boldsymbol{\zeta}(\cdot)$ be a fluid model solution with $\varrho > 1$ and $\boldsymbol{\zeta}_0 = 0$. We know from Definition 4.0.1 (iv) that, for $\delta > 0$, $\mathcal{L}(\delta) > 0$ in this case. Applying Lemma 4.0.2 (i) and the proof in the $\boldsymbol{\zeta}_0 \neq 0$ case, we see that $\boldsymbol{\zeta}_\delta(\cdot) := \boldsymbol{\zeta}(\cdot + \delta)$ is a fluid model solution such that $z_{j,\delta}(t) \leq z_{j,\delta}(0) + \alpha_j \langle \chi, \vartheta_j \rangle$ for each $j \in \mathcal{J}$, where $z_{j,\delta}(\cdot)$ is the total mass of the jth component of the fluid model solution shifted by δ. It follows that for each $\delta > 0$, in the original fluid model solution we have $z_j(t) \leq z_j(\delta) + \alpha_j \langle \chi, \vartheta_j \rangle$ for each $t \geq \delta$. Taking $\delta \to 0$ and using the fact that fluid model solutions are continuous, we achieve the bound $z_j(t) \leq z_j(0) + \alpha_j \langle \chi, \vartheta_j \rangle$ for each $t \geq 0$, $j \in \mathcal{J}$. □

Lemma 6.2.2. *Fix fluid model parameters* $(\boldsymbol{\alpha}, \boldsymbol{\mu}, \boldsymbol{p}, \boldsymbol{\vartheta})$ *satisfying Definition 3.0.1 and* $\boldsymbol{\zeta}_0 \in \mathbf{K}^J$. *Assume that either* $\varrho \leq 1$ *or* $\boldsymbol{\zeta}_0 \neq \mathbf{0}$. *Let* $z_{0,j} = \langle 1, \zeta_{0,j} \rangle$ *for each* $j \in \mathcal{J}$. *Then we say that a continuous path* $(z_1, \ldots, z_J) : \mathbb{R}_+ \to \mathbb{R}_+^J$ *is a solution to the system of equations given by* (4.3), (6.2), *and* (6.4) *and the initial condition* $\boldsymbol{\zeta}_0$ *if*

(i) For each $j \in \mathcal{J}$, $z_j(0) = z_{0,j}$,

(ii) setting $t^* := \inf\{t \geq 0 : \boldsymbol{z}(t) = \mathbf{0}\}$, *then* $(z_1(t), \ldots, z_J(t)) = \mathbf{0}$ *for each* $t \geq t^*$,

(iii) and the equations (4.3), (6.2), *and* (6.4) *hold on* $[0, t^*)$.

Then solutions to this system are unique. It follows that all fluid model solutions for these parameters and initial condition have the same associated t^* *and* $\boldsymbol{z}(\cdot)$ *function.*

Proof. First, note that if $\boldsymbol{\zeta}_0 = \mathbf{0}$, then by (ii), $t^* = 0$ and the zero solution is the unique solution for this initial condition. We turn now to the the $\boldsymbol{\zeta}_0 \neq \mathbf{0}$ case. Let $(z_1, \ldots, z_J)$ and $(\tilde{z}_1, \ldots, \tilde{z}_J)$ be two solutions to (4.3), (6.2), and (6.4) with the initial condition $\boldsymbol{\zeta}_0$. Let t^* and $\tilde{t}^*$ be as defined in (ii). Let $G(t) := \int_0^t \frac{1}{\mathcal{L}(s)} ds$ for $t \in [0, t^*)$ and $\tilde{G}(t) := \int_0^t \frac{1}{\tilde{\mathcal{L}}(s)} ds$ for $t \in [0, \tilde{t}^*)$. For $\epsilon > 0$, let $t^*_\epsilon := \inf\{t \geq 0 : \mathcal{L}(s) \leq \epsilon\}$ and $\tilde{t}^*_\epsilon := \inf\{t \geq 0 : \tilde{\mathcal{L}}(s) \leq \epsilon\}$. Then $\mathcal{L}$ and $\tilde{\mathcal{L}}$ are greater than or equal to ϵ on $[0, t^*_\epsilon \wedge \tilde{t}^*_\epsilon)$. It suffices to prove that $G = \tilde{G}$

on $[0, t_\epsilon^* \wedge \tilde{t}_\epsilon^*]$ when $t_\epsilon^* \wedge \tilde{t}_\epsilon^* > 0$, for each $\epsilon > 0$. Fix $\epsilon > 0$ and assume $t_\epsilon^* \wedge \tilde{t}_\epsilon^* > 0$. For $t \in [0, t_\epsilon^* \wedge \tilde{t}_\epsilon^*]$, let $u(t) = G(t) - \tilde{G}(t)$. Then

$$u'(t) = G'(t) - \tilde{G}'(t) = \frac{1}{\mathcal{L}(t)} - \frac{1}{\tilde{\mathcal{L}}(t)} = \frac{\tilde{\mathcal{L}}(t) - \mathcal{L}(t)}{\mathcal{L}(t)\tilde{\mathcal{L}}(t)} = \frac{\sum_{j=1}^J \frac{p_j}{\mu_j}(\tilde{z}_j(t) - z_j(t))}{\mathcal{L}(t)\tilde{\mathcal{L}}(t)}. \tag{6.9}$$

Applying (6.4), we see that

$$\begin{aligned} u'(t) = &\frac{\sum_{j=1}^J \frac{p_j}{\mu_j} \Big(\exp(-Kp_j\tilde{G}(t)) - \exp(-Kp_jG(t))\Big) M_j^c(0,t)}{\mathcal{L}(t)\tilde{\mathcal{L}}(t)} \\ &+ \frac{\sum_{j=1}^J \frac{p_j}{\mu_j} \int_0^t \Big(\exp(-Kp_j(\tilde{G}(t) - \tilde{G}(t-r))) - \exp(-Kp_j(G(t) - G(t-r)))\Big) \alpha_j N_j^c(r)dr}{\mathcal{L}(t)\tilde{\mathcal{L}}(t)} \end{aligned} \tag{6.10}$$

The functions $x \to \exp(-Kp_jx)$, $j \in \mathcal{J}$, are uniformly Lipschitz continuous on $[0, \infty)$, and so there is a constant $C_1 > 0$ such that $|\exp(-Kp_jx) - \exp(-Kp_jy)| \leq C_1|x - y|$ for each $x, y \in [0, \infty)$ and $j \in \mathcal{J}$. Then from (6.10), recalling the lower bound on $\mathcal{L}$ and $\tilde{\mathcal{L}}$ and that $N_j^c(\cdot) \leq 1$, we have

$$\begin{aligned} |u'(t)| &\leq \frac{\sum_{j=1}^J \frac{p_j}{\mu_j} C_1 M_j^c(0,t)}{\epsilon^2}|u(t)| + \frac{\sum_{j=1}^J \frac{p_j\alpha_j}{\mu_j} C_1}{\epsilon^2} \int_0^t |u(t-r) - u(t)|dr \\ &\leq g_\epsilon(t)|u(t)| + C_\epsilon \int_0^t |u(r)|dr \end{aligned}$$

where $C_\epsilon = \frac{1}{\epsilon^2}\sum_{j=1}^J \frac{\alpha_j p_j}{\mu_j} C_1$, $g_\epsilon(t) = \frac{1}{\epsilon^2}\sum_{j=1}^J \frac{p_j}{\mu_j} C_1 M_j^c(0,t) + C_\epsilon t$, and we used a change of variables $t - r \to r$ in the integral. Hence, for $t \in [0, t_\epsilon^* \wedge \tilde{t}_\epsilon^*]$, noting that $u(0) = 0$ and using Fubini's theorem, we have

$$\begin{aligned}
|u(t)| &\le |u(0)| + \int_0^t |u'(s)|ds \\
&\le \int_0^t g_\epsilon(s)|u(s)|ds + C_\epsilon \int_0^t \int_0^s |u(r)|drds \\
&= \int_0^t g_\epsilon(s)|u(s)|ds + C_\epsilon \int_0^t |u(r)|(t-r)dr \\
&\le \int_0^t (g_\epsilon(s) + C_\epsilon t)|u(s)|ds.
\end{aligned}$$

It follows from Gronwall's inequality that $u \equiv 0$ on $[0, t^*_\epsilon \wedge \tilde{t}^*_\epsilon]$, as desired. Taking $\epsilon \to 0$, it follows using the continuity of $(z_1, \ldots, z_J)$ and $(\tilde{z}_1, \ldots, \tilde{z}_J)$ that $t^* = \tilde{t}^*$ and $G(t) = \tilde{G}(t)$ for $t \in [0, t^*)$. Applying (6.4), we conclude that $(z_1(t), \ldots, z_J(t)) = (\tilde{z}_1(t), \ldots, \tilde{z}_J(t))$ for $t \in [0, t^*)$. It follows from Definition 4.0.1 (iv), Lemma 4.0.2 (iii), and Lemma 6.1.1 that for any fluid model solution with initial condition $\boldsymbol{\zeta}_0$, the associated $(z_1, \ldots, z_J)$ is a solution to (4.3), (6.2), and (6.4). We conclude that all fluid model solutions for these parameters and initial condition have the same associated t^* and $\boldsymbol{z}(\cdot)$ function. □

Proof of Theorem 5.1.1. Applying (6.3), we see that $\mathcal{L}(\cdot)$ determines $M^c_j(t, x)$ for each $t \in [0, t^*), x \ge 0, j \in \mathcal{J}$. Therefore, $\mathcal{L}$ specifies the fluid model solution on $[0, t^*)$. By Lemma 6.2.2, $\mathcal{L}$ is unique, and so we conclude that there is a unique fluid model solution until the time t^*. For $\varrho > 1$, since $\boldsymbol{\zeta}_0 \neq \mathbf{0}$, $t^* = \infty$ by Definition 4.0.1 (iv), the proof is complete in this case. For $\varrho \le 1$, $\boldsymbol{z}(\cdot) = \mathbf{0}$ on $[t^*, \infty)$ by Lemma 4.0.2 (iii), and so the solution is unique for each $t \ge 0$. □

6.3 Continuous Dependence on Initial Conditions

We now prove Theorem 5.1.2.

Proof. We outline the proof before we begin. Without loss of generality, we will assume throughout the proof that no member of the sequence $\{\boldsymbol{\zeta}_0^n\}_{n=1}^\infty$ is equal to $\mathbf{0}$ in the $\rho > 1$

case. This is valid because, when $\varrho > 1$, the limit of the sequence $\boldsymbol{\zeta}_0 \neq \mathbf{0}$, and so for n sufficiently large, $\boldsymbol{\zeta}_0^n \neq \mathbf{0}$. We first show that $\{\zeta_j^n(\cdot)\}_{n=1}^{\infty}$ is precompact in $C([0,\infty),\mathbf{M})$ for each $j \in \mathcal{J}$. This implies precompactness of $\{\boldsymbol{\zeta}^n(\cdot)\}_{n=1}^{\infty}$ in $C([0,\infty),\mathbf{M}^J)$. Next, we show that subsequential limits satisfy the definition of a fluid model solution. Because fluid model solutions are unique by Theorem 5.1.1 since we have assumed that $\boldsymbol{\zeta}_0 \neq 0$ if $\varrho > 1$, this implies that the subsequential limits in $C([0,\infty),\mathbf{M}^J)$ of $\{\boldsymbol{\zeta}^n(\cdot)\}_{n=1}^{\infty}$ are all equal to $\boldsymbol{\zeta}(\cdot)$, the fluid model solution with initial condition $\boldsymbol{\zeta}_0$. It then follows that the whole sequence $\{\boldsymbol{\zeta}^n(\cdot)\}_{n=1}^{\infty}$ converges to $\boldsymbol{\zeta}(\cdot)$ in $C([0,\infty),\mathbf{M}^J)$.

To execute the precompactness proof, we apply Theorem 4.6 from [13]. Note that within the closed subspaces $C([0,\infty),\mathbb{R})$ of $D([0,\infty),\mathbb{R})$ and $C([0,\infty),\mathbf{M})$ of $D([0,\infty),\mathbf{M})$, the topology of uniform convergence on compact time intervals is equivalent to the subspace topology induced by the J_1-topology on $D([0,\infty),\mathbb{R})$ and $D([0,\infty),\mathbf{M})$, respectively. Therefore, precompactness of $\{\zeta_j^n(\cdot)\}_{n=1}^{\infty}$ in $D([0,\infty),\mathbf{M})$ is equivalent to precompactness in $C([0,\infty),\mathbf{M})$ and precompactness of $\{\langle f,\zeta_j^n(\cdot)\rangle\}_{n=1}^{\infty}$ for $f \in \boldsymbol{C}_b^1(\mathbb{R}_+)$ in $D([0,\infty),\mathbb{R})$ is equivalent to precompactness in $C([0,\infty),\mathbb{R})$. In particular, when we apply Theorem 4.6 from [13], the following two conditions imply $\{\zeta_j^n(\cdot)\}_{n=1}^{\infty}$ is precompact in $D([0,\infty),\mathbf{M})$, and hence $C([0,\infty),\mathbf{M})$, for each $j \in \mathcal{J}$:

(i) For each $T > 0$, there exists a compact set $A \in \mathbf{M}$ such that $\zeta_j^n(t) \in A$ for each $t \in [0,T], n \in \mathbb{N}$.

(ii) For each $f \in \mathbf{C}_b^1(\mathbb{R}_+)$, $\{\langle f,\zeta_j^n(\cdot)\rangle\}_{n=1}^{\infty}$ is a precompact set in the space $C([0,\infty),\mathbb{R})$.

Note that we were able to simplify the conditions from [13] because the paths we are considering in $D([0,\infty),\mathbf{M})$ are deterministic and continuous. Therefore the associated measures used in (i) are point measures. Lastly, observe that $\mathbf{C}_b^1(\mathbb{R}_+)$ separates points and is closed under addition, so it is a suitable class of functions for Theorem 4.6 from [13].

We first verify (i) for each $j \in \mathcal{J}$. We define

$$M^c_{j,n}(t,x) := \langle 1_{(x,\infty)}, \zeta^n_j(t)\rangle, \qquad t, x \geq 0, j \in \mathcal{J},$$

and

$$t^*_n := \inf\{t \geq 0 : \boldsymbol{z}^n(t) = \mathbf{0}\},$$

where $z^n_j(t) = \langle 1, \zeta^n_j(t)\rangle$ for $t \geq 0$. Fix $T > 0$. By Theorem 15.7.5 in [14], to show (i), it suffices to show that

(a) $\sup_{t\in[0,T]} \sup_{n\in\mathbb{N}} z^n_j(t) < \infty$, and

(b) $\lim_{x\to\infty} \sup_{t\in[0,T]} \sup_{n\in\mathbb{N}} M^c_{j,n}(t,x) = 0$ for each $j \in \mathcal{J}$.

Note that since $\zeta^n_{0,j} \to \zeta_{0,j}$ in $\mathbf{M}$ for each $j \in \mathcal{J}$, the associated initial total masses $z^n_j(0)$ converge to $z_j(0)$ as $n \to \infty$. In particular, they are uniformly bounded. Therefore, applying Lemma 6.2.1, we can uniformly bound the total masses for each $t \geq 0, n \in \mathbb{N}, j \in \mathcal{J}$:

$$z^n_j(t) \leq C_j := \sup_{n\in\mathbb{N}} z^n_j(0) + \alpha_j\langle\chi, \vartheta_j\rangle, \tag{6.11}$$

and so (a) holds for each $j \in \mathcal{J}$. It follows from (6.11) that

$$\mathcal{L}^n(t) = \sum_{j=1}^{J} \frac{p_j z^n_j(t)}{\mu_j} \leq C := \sum_{j=1}^{J} \frac{p_j C_j}{\mu_j}, \quad t \geq 0. \tag{6.12}$$

Now for (b), fix $\epsilon > 0$. Since $\{\boldsymbol{\zeta}^n_0\}_{n=1}^\infty$ converges to $\boldsymbol{\zeta}_0$, we can choose $x^\epsilon_1 > 0$ such that for each $n \in \mathbb{N}, j \in \mathcal{J}$,

$$M^c_{j,n}(0, x^\epsilon_1) < \frac{\epsilon}{2}. \tag{6.13}$$

By the dominated convergence theorem, there exists $x^\epsilon_2 > 0$ such that for each $j \in \mathcal{J}$,

$$\int_0^\infty \exp\left(\frac{-Kp_j r}{C}\right) \alpha_j N^c_j(x^\epsilon_2 + r)dr < \frac{\epsilon}{2}.$$

Choose $x^\epsilon = \max\{x_1^\epsilon, x_2^\epsilon\}$. Then, for each $n \in \mathbb{N}, t \geq 0, j \in \mathcal{J}$, and $x \geq x^\epsilon$, we prove the inequality $M_{j,n}^c(t,x) < \epsilon$. We will do this in two cases, case (i) being $t \geq t_n^*$ and case (ii) being when $t < t_n^*$. In case (i), if $t \geq t_n^*$, using Definition 4.0.1 (iv) and our assumption that $\boldsymbol{\zeta}_0^n \neq \mathbf{0}$, we see that $\rho \leq 1$. Applying Lemma 4.0.2 (iii), we see that $M_{j,n}^c(t,x) = 0 < \epsilon$ for each $x > x^\epsilon$, $j \in \mathcal{J}$. In case (ii), because $t < t_n^*$, we can apply (6.3), (6.12), and the fact that $N_j^c(\cdot)$ and $M_j^c(t,\cdot)$ are non-increasing to obtain that for each $x > x^\epsilon, j \in \mathcal{J}$,

$$\begin{aligned} M_{j,n}^c(t,x) &\leq \int_0^t \exp\left(\frac{-Kp_j r}{C}\right) \alpha_j N_j^c(x^\epsilon + r)dr + \exp(-Kp_j G^n(t)) M_{j,n}^c(0, t + x^\epsilon) \\ &\leq \int_0^t \exp\left(\frac{-Kp_j r}{C}\right) \alpha_j N_j^c(x^\epsilon + r)dr + M_{j,n}^c(0, x^\epsilon) \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon. \end{aligned}$$

Since $\epsilon > 0$ was arbitrary, (b) holds, and so (i) is proved.

Next, we verify (ii). Fix $f \in \boldsymbol{C}_b^1(\mathbb{R}_+)$, $j \in \mathcal{J}$. By the Arzela–Ascoli theorem and a standard diagonalization argument (see, e.g., Theorem 4.44 in [10]), we need to prove that for each $T > 0$, $\{\langle f, \zeta_j^n(t)\rangle\}_{n=1}^\infty$ is bounded for each $t \in [0,T]$ and $\{\langle f, \zeta_j^n(\cdot)\rangle\}_{n=1}^\infty$ is an equicontinuous sequence on $[0,T]$. Since for each $n \in \mathbb{N}$, $t \in [0,T]$, $|\langle f, \zeta_j^n(t)\rangle| \leq ||f||_\infty z_j^n(t) \leq ||f||_\infty C_j$, where C_j is given in (6.11), $\{\langle f, \zeta_j^n(t)\rangle\}_{n=1}^\infty$ is bounded for each $t \in [0,T]$. For the equicontinuity, note that because $f \in \mathbf{C}_b^1(\mathbb{R}_+)$, we have $f(\cdot) - f(0) \in \mathscr{C}$. Therefore, applying (4.5), we see that for each $t \in [0,T], h \geq 0$,

$$\begin{aligned} |\langle f, \zeta_j^n(t+h)\rangle - \langle f, \zeta_j^n(t)\rangle| &\leq |\langle f(\cdot) - f(0), \zeta_j^n(t+h)\rangle - \langle f(\cdot) - f(0), \zeta_j^n(t)\rangle| \\ &\quad + |f(0)||z_j^n(t+h) - z_j^n(t)| \\ &\leq h(||f'||_\infty C_j + ||f(\cdot) - f(0)||_\infty(\mu_j K + \alpha_j)) \\ &\quad + |f(0)||z_j^n(t+h) - z_j^n(t)|, \end{aligned}$$

where we have used (6.11). Then, it suffices for the equicontinuity to show that

$$\lim_{\delta\to 0}\sup_{n\in\mathbb{N}}\left(\sup_{t\in[0,T-\delta]}\sup_{h\in[0,\delta]}|z_j^n(t+h)-z_j^n(t)|\right)=0.$$

To this end, fix $n\in\mathbb{N}$. We see that if $t\geq t_n^*$, then $|z_j^n(t+h)-z_j^n(t)|=0$. For the $t<t_n^*$ case, applying (4.6) for $0=u\leq v'\leq v<t_n^*$ we obtain the bound

$$|z_j^n(v)-z_j^n(v')|\leq|M_{j,n}^c(0,v)-M_{j,n}^c(0,v')|+(\alpha_j+\mu_jK)|v-v'|.$$

By continuity of $z_j^n(\cdot)$ and $M_{j,n}^c(0,\cdot)$, we see that this also holds with $v=t_n^*$. Thus, if $t+h\leq t_n^*$, then $|z_j^n(t+h)-z_j^n(t)|\leq|M_{j,n}^c(0,t+h)-M_{j,n}^c(0,t)|+(\alpha_j+\mu_jK)h$. On the other hand, if $t<t_n^*<t+h$, we have

$$\begin{aligned}|z_j^n(t+h)-z_j^n(t)|&\leq|z_j^n(t+h)-z_j^n(t_n^*)|+|z_j^n(t_n^*)-z(t)|\\&\leq 0+|M_{j,n}^c(0,t_n^*)-M_{j,n}^c(0,t)|+(\alpha_j+\mu_jK)h.\end{aligned}$$

Hence, for all $t\geq 0,h\geq 0$,

$$|z_j^n(t+h)-z_j^n(t)|\leq|M_{j,n}^c(0,t+h)-M_{j,n}^c(0,t)|+(\alpha_j+\mu_jK)h. \tag{6.14}$$

Because $M_{j,n}^c(0,\cdot)\to M_j^c(0,\cdot)$ as $n\to\infty$ and $M_{j,n}^c(0,\cdot),M_j^c(0,\cdot)$ are decreasing and continuous (by assumption), it follows that $\{M_{j,n}^c(0,\cdot)\}_{n=1}^\infty$ converges uniformly to $M_j^c(0,\cdot)$ on $[0,T]$. Furthermore, because $M_j^c(0,\cdot)$ is continuous, it is uniformly continuous on $[0,T]$. Therefore, we can conclude that $\{M_{j,n}^c(0,\cdot)\}_{n=1}^\infty$ is uniformly equicontinuous on $[0,T]$. In other words,

$$\lim_{\delta\to 0}\sup_{n\in\mathbb{N}}\left(\sup_{t\in[0,T-\delta]}\sup_{h\in[0,\delta]}|M_{j,n}^c(0,t+h)-M_{j,n}^c(0,t)|\right)=0.$$

Applying (6.14), we have that

$$
\begin{aligned}
&\lim_{\delta\to 0}\sup_{n\in\mathbb{N}}\left(\sup_{t\in[0,T-\delta]}\sup_{h\in[0,\delta]}|z_j^n(t+h)-z_j^n(t)|\right)\\
&\quad\le \lim_{\delta\to 0}\sup_{n\in\mathbb{N}}\left(\sup_{t\in[0,T-\delta]}\sup_{h\in[0,\delta]}|M_{j,n}^c(0,t+h)-M_{j,n}^c(0,t)|\right)\\
&\quad+\lim_{\delta\to 0}\sup_{n\in\mathbb{N}}\sup_{h\in[0,\delta]}|\alpha_j+\mu_j K|h\\
&\quad=0.
\end{aligned}
$$

Now that precompactness has been established, we must show that subsequential limits are fluid model solutions. Let $\boldsymbol{\xi}(\cdot)$ be a subsequential limit of $\{\boldsymbol{\zeta}^n(\cdot)\}_{n=1}^{\infty}$ in $C([0,\infty),\mathbf{M}^J)$. We prove that $\boldsymbol{\xi}(\cdot)$ satisfies properties (i)-(iv) of Definition 4.0.1 for the parameters $(\boldsymbol{\alpha},\boldsymbol{\mu},\boldsymbol{p},\boldsymbol{\vartheta})$ and initial condition $\boldsymbol{\zeta}_0$. For ease of notation, we denote the converging subsequence $\{\boldsymbol{\zeta}^n(\cdot)\}_{n=1}^{\infty}$ again. Because $\boldsymbol{\zeta}_0^n\to\boldsymbol{\zeta}_0$, the limit $\boldsymbol{\xi}$ has the property that $\boldsymbol{\xi}(0)=\boldsymbol{\zeta}_0$, and thus $\boldsymbol{\xi}$ satisfies (i). For (ii), we note that because $\boldsymbol{\zeta}^n(\cdot)$ converges to $\boldsymbol{\xi}(\cdot)$ uniformly on compact sets, it converges pointwise. Fixing $t\ge 0$ and applying the Portmanteau theorem, we see that for each $j\in\mathcal{J}$,

$$
\lim_{h\to 0}\liminf_{n\to\infty}\zeta_j^n(t)([0,h))\ge \xi_j(t)(\{0\}). \tag{6.15}
$$

However, using (4.6) and the fact that $\zeta_j^n(t)([0,h))=0$ for each $h>0$ when $t\ge t_n^*$, for each $j \in \mathcal{J}, t \geq 0, h > 0$,

$$\begin{aligned}
&\liminf_{n\to\infty} \zeta_j^n(t)\left([0,h)\right) \\
&\qquad = \liminf_{n\to\infty} \left(z_j^n(t) - M_{j,n}^c(t,h)\right) \\
&\qquad \leq \liminf_{n\to\infty} \left(M_{j,n}^c(0,t) - M_{j,n}^c(0,t+h) + \int_0^t \alpha_j \left(N_j^c(s) - N_j^c(s+h)\right) ds\right) \\
&\qquad = \liminf_{n\to\infty} \left(\zeta_{0,j}^n((t,t+h]) + \int_0^t \alpha_j \left(N_j^c(s) - N_j^c(s+h)\right) ds\right) \\
&\qquad \leq \zeta_{0,j}([t,t+h]) + \int_0^t \alpha_j \left(N_j^c(s) - N_j^c(s+h)\right) ds
\end{aligned}$$

applying the Portmanteau theorem again, but with the closed set $[t, t+h]$, on the last line. Because $\zeta_{0,j} \in \mathbf{K}$ and by the right continuity of $N_j^c(\cdot)$, we conclude that $\lim_{h\to 0} \liminf_{n\to\infty} \zeta_j^n(t)([0,h)) = 0$, and thus, by (6.15), $\xi_j(t)(\{0\}) = 0$ for each $t \geq 0, j \in \mathcal{J}$. Next, we verify (iv). When $\varrho > 1$, by Lemma 4.0.3 we have that $B^n(\cdot) = \sum_{j=1}^J \frac{1}{\mu_j}\langle 1, \zeta_j^n(\cdot)\rangle$ is bounded below by $\frac{\varrho-1}{2}K(\epsilon\wedge\cdot)$ for some $\epsilon > 0$. Applying the continuous mapping theorem, it follows that the associated $B(\cdot) = \sum_{j=1}^J \frac{1}{\mu_j}\langle 1, \xi_j(\cdot)\rangle \geq \frac{\varrho-1}{2}K(\epsilon \wedge \cdot)$. We conclude that $\boldsymbol{\xi}(t) \neq \mathbf{0}$ for all $t > 0$, and hence (iv) holds for $\boldsymbol{\xi}$.

Next, we show that $\boldsymbol{\xi}$ satisfies (iii). Fix $f \in \mathscr{C}$, $T \geq 0$. Similar to (6.1), for $\epsilon > 0$, define $t^*_{\boldsymbol{\xi},\epsilon} := \inf\{t \geq 0 : \sum_{j=1}^J \frac{p_j}{\mu_j}\langle 1, \xi_j(t)\rangle \leq \epsilon\}$. Because $\boldsymbol{\zeta}^n(\cdot)$ converges uniformly to $\boldsymbol{\xi}(\cdot)$ on $[0,T]$, it follows that for sufficiently large n, $\boldsymbol{\zeta}^n(\cdot)$ is not equal to $\mathbf{0}$ on $[0, t^*_{\boldsymbol{\xi},\epsilon} \wedge T)$, and thus one may remove the indicator functions in (4.5) with $\boldsymbol{\zeta}^n$ in place of $\boldsymbol{\zeta}$ and $t \in [0, t^*_{\boldsymbol{\xi},\epsilon})$. Taking the limit on both sides of such versions of (4.5) as $n \to \infty$ and applying bounded convergence, we see that $\boldsymbol{\xi}(\cdot)$ must satisfy (4.5) on $[0, t^*_{\boldsymbol{\xi},\epsilon} \wedge T)$. Taking $\epsilon \to 0$, we have that $\boldsymbol{\xi}(\cdot)$ satisfies (4.5) on $[0, t^*_{\boldsymbol{\xi}} \wedge T)$, where $t^*_{\boldsymbol{\xi}} := \inf\{t \geq 0 : \sum_{j=1}^J \frac{p_j}{\mu_j}\langle 1, \xi_j(t)\rangle = 0\}$. Because we have assumed that when $\varrho > 1$, $\boldsymbol{\zeta}_0 = \boldsymbol{\xi}(0)$ is nonzero, it follows from (iv) that $t^*_{\boldsymbol{\xi}} = \infty$ for $\varrho > 1$. It follows that if $t^*_{\boldsymbol{\xi}} \leq T$, then $\varrho \leq 1$. Therefore, applying Lemma 4.0.2(iii), in order to verify (4.5) on $[t^*_{\boldsymbol{\xi}}, T]$, we simply need to check that $\boldsymbol{\xi}(t) = 0$ for $t \geq t^*_{\boldsymbol{\xi}}$. Because $\boldsymbol{\zeta}^n(\cdot)$ converges uniformly to $\boldsymbol{\xi}(\cdot)$ on $[0,T]$, it follows that for sufficiently large n, $\mathcal{L}^n(t^*_{\boldsymbol{\xi},\epsilon}) \leq 2\epsilon$. Using the fact that for each fluid model solution $\boldsymbol{\zeta}^n(\cdot)$ and $t \geq 0$, $B^n(t) \leq C\mathcal{L}^n(t)$, where $B^n(\cdot)$ is as defined in Lemma 4.0.2 (iii), for $C = (\min_{j\in\mathcal{J}} p_j)^{-1}$, we conclude that $B^n(t^*_{\boldsymbol{\xi},\epsilon}) \leq C2\epsilon$ for all sufficiently large n. Because $B^n(\cdot)$ is nonincreasing on nonzero intervals and stays zero after the fluid model solution $\boldsymbol{\zeta}^n(\cdot)$ hits zero, by Lemma 4.0.2 (iii), it follows that for all sufficiently large n, $B^n(t) \leq C2\epsilon$ for each $t \geq t^*_{\boldsymbol{\xi},\epsilon}$. Therefore, letting $n \to \infty$, the associated $B(t) = \sum_{j=1}^{J} \frac{1}{\mu_j}\langle 1, \xi_j(t)\rangle \leq C2\epsilon$ for each $t \geq t^*_{\boldsymbol{\xi},\epsilon}$. Taking $\epsilon \to 0$, we conclude that $\boldsymbol{\xi}(\cdot) = \mathbf{0}$ on $[t^*_{\boldsymbol{\xi}}, T]$. We conclude that $\boldsymbol{\xi}(\cdot)$ satisfies (4.5) on $[0,T]$ for each $T \geq 0$. Now we have verified (i)-(iv) from Definition 4.0.1 for $\boldsymbol{\xi}(\cdot)$. It follows that $\boldsymbol{\xi}(\cdot) = \boldsymbol{\zeta}(\cdot)$ by uniqueness of fluid model solutions. This gives convergence of $\{\boldsymbol{\zeta}^n(\cdot)\}_{n=1}^{\infty}$ to $\boldsymbol{\zeta}(\cdot)$ in $C([0,\infty), \mathbf{M}^J)$.

□

Chapter 7

A Difference Equation for $\mathcal{Z}(\cdot)$

In this chapter, we introduce a difference equation for the measure-valued component of the state space descriptor, $\mathcal{Z}(\cdot)$, as defined in (2.8). In later chapters, we will apply this difference equation representation to each member of the sequence of systems introduced in §3 and apply fluid scaling to the equation to obtain a prelimit equation similar to (4.5). To ease notation, it is convenient to not introduce superscripts of m associated with the sequence until those later chapters.

7.1 Separating $\mathcal{Z}(\cdot)$ into its Component Parts

Much of this section was inspired by the analogous section in [2].

Lemma 7.1.1. *Let $f \in \boldsymbol{C}_b(\mathbb{R}_+)$. For $j \in \mathcal{J}$, let $r_i^j := U_i^j + \ell_i^j$ for $i \in \mathbb{N}$ and $r_{-i}^j = \tilde{\ell}_{-i}^j$ for $i \in \{1, 2, \dots, Z_{0,j}\}$ be the times that the ith class j arrival and the $-i$th class j job initially in the system, respectively, would renege if not already chosen for service. Let b_i^j be the time that the ith class j arrival enters service and b_{-i}^j be the time that the $-i$th job initially in the system enters service. We take b_i^j to be infinity if the ith class j arrival reneges before it enters service and b_{-i}^j to be infinity if the $-i$th job of class j initially in* *the system reneges before it enters service. Then for each $\epsilon > 0, t \geq 0, j \in \mathcal{J}$*

$$\begin{aligned}\langle f, \mathcal{Z}_j(t+\epsilon)\rangle - \langle f, \mathcal{Z}_j(t)\rangle = \langle f((\cdot - \epsilon)^+) - f, \mathcal{Z}_j(t)\rangle - \sum_{\eta_i \in (t,t+\epsilon]} \left\langle f, \delta^+_{T_{i,j}-t-\epsilon+\eta_i} \right\rangle \\ + \sum_{i=A_j(t)+1}^{A_j(t+\epsilon)} 1_{\left\{s^k(U_i^j-)\neq 0 \quad \forall k \in \mathcal{K}\right\}} f((\ell_i^j(t+\epsilon))^+) \\ - X^j_{t,t+\epsilon}(f) - \sum_{\substack{i=1 \\ b^j_{-i}>t}}^{Z_{0,j}} 1_{\{t<\tilde{\ell}^j_{-i}\leq t+\epsilon\}} f(0) \qquad (7.1)\end{aligned}$$

where for $\epsilon > 0, j \in \mathcal{J}, t \geq 0$,

$$X^j_{t,t+\epsilon}(f) := f(0) \sum_{\substack{i=1 \\ b^j_i>t}}^{A_j(t+\epsilon)} 1_{\{s^k(U_i^j-)\neq 0 \quad \forall k \in \mathcal{K}\}} 1_{\{t<U_i^j+\ell_i^j\leq t+\epsilon\}}. \qquad (7.2)$$

Proof. To ease the notation, let $\mathcal{Q}_j(t)$ be the set of indices of the class j jobs in the class j queue at time t. That is, $i \in \mathcal{Q}_j(t)$ if and only if the ith job to arrive to class j is in the class j queue at time t, and $-i \in \mathcal{Q}_j(t)$ if and only if the $-i$th class j job present at time 0 is in the class j queue at time t. Then for each $f \in \mathbf{C}_b(\mathbb{R}_+)$ and $\epsilon > 0$,

$$\langle f, \mathcal{Z}_j(t+\epsilon)\rangle - \langle f, \mathcal{Z}_j(t)\rangle = \sum_{i\in\mathcal{Q}_j(t+\epsilon)} f(\ell_i^j(t+\epsilon)) - \sum_{i\in\mathcal{Q}_j(t)} f(\ell_i^j(t)), \qquad (7.3)$$

where $\ell_i^j(\cdot)$ is as defined in (2.1) and (2.5). We have the following decomposition:

$$\mathcal{Q}_j(t+\epsilon) = (\mathcal{Q}_j(t) \cup \mathcal{V}_j) \setminus \{i \in \mathbb{N} \cup \{-1,-2,\ldots,-Z_{0,j}\} : t < b_i^j \leq t+\epsilon \text{ or } t < r_i^j \leq t+\epsilon\}$$

where $\mathcal{V}_j = \left\{i \in \mathbb{N} : U_i^j \in (t, t+\epsilon] \text{ and } s^k(U_i^j-) \neq 0 \ \ \forall k \in \mathcal{K}\right\}$, the set of indices of jobs of class j that arrived in the interval $(t, t+\epsilon]$ and entered the class j queue rather than immediately entering service. Therefore, (7.3) can be re-written as

$$
\begin{aligned}
&\langle f, \mathcal{Z}_j(t+\epsilon)\rangle - \langle f, \mathcal{Z}_j(t)\rangle = \\
&\qquad \sum_{i \in \mathcal{Q}_j(t)} f((\ell_i^j(t) - \epsilon)^+) + \sum_{i=A_j(t)+1}^{A_j(t+\epsilon)} 1_{\left\{s^k(U_i^j-) \neq 0 \quad \forall k \in \mathcal{K}\right\}} f((\ell_i^j(t+\epsilon))^+) \\
&\qquad - \sum_{\eta_i \in (t, t+\epsilon]} \left\langle f, \delta^+_{\{T_{i,j} - t - \epsilon + \eta_i\}} \right\rangle - \sum_{\substack{b_i^j > t \\ r_i^j \in (t, t+\epsilon]}} 1_{\{s^k(U_i^j-) \neq 0 \quad \forall k \in \mathcal{K}\}} f(0) \\
&\qquad - \sum_{i \in \mathcal{Q}_j(t)} f(\ell_i^j(t)).
\end{aligned}
$$

Note here that we have only removed a job that would have reneged in the time interval $(t, t+\epsilon]$ if it had not already entered service by time t ($b_i^j > t$) and if it entered the class j queue when it arrived to the system ($s^k(U_i^j-) \neq 0 \;\; \forall k \in \mathcal{K}$). We avoid double removals when $b_i^j \in (t, t+\epsilon]$ and $r_i^j \in (t, t+\epsilon]$ by the use of δ^+ in the sum over the η_i. Using the definition of the r_i^j, we obtain

$$
\begin{aligned}
\langle f, \mathcal{Z}_j(t+\epsilon)\rangle - \langle f, \mathcal{Z}_j(t)\rangle &= \langle f((\cdot - \epsilon)^+) - f, \mathcal{Z}_j(t)\rangle \\
&\quad + \sum_{i=A_j(t)+1}^{A_j(t+\epsilon)} 1_{\left\{s^k(U_i^j-) \neq 0 \quad \forall k \in \mathcal{K}\right\}} f((\ell_i^j(t+\epsilon))^+) \\
&\quad - \sum_{\eta_i \in (t, t+\epsilon]} \left\langle f, \delta^+_{\{T_{i,j} - t - \epsilon + \eta_i\}} \right\rangle \\
&\quad - \sum_{\substack{i=1 \\ b_i^j > t}}^{A_j(t+\epsilon)} 1_{\{s^k(U_i^j-) \neq 0 \quad \forall k \in \mathcal{K}\}} 1_{\left\{t < U_i^j + \ell_i^j \leq t+\epsilon\right\}} f(0) \\
&\quad - \sum_{\substack{i=1 \\ b_{-i}^j > t}}^{Z_{0,j}} 1_{\{t < \tilde{\ell}_{-i}^j \leq t+\epsilon\}} f(0).
\end{aligned}
$$

□

Now, we use (7.1) to get an equation more similar to (4.5).

Lemma 7.1.2. *Let $f \in \mathscr{C}$. Then almost surely, for each $t \geq 0, j \in \mathcal{J}$,*

$$\langle f, \mathcal{Z}_j(t)\rangle = \langle f, \mathcal{Z}_j(0)\rangle - \int_0^t \langle f', \mathcal{Z}_j(s)\rangle ds + \sum_{i=1}^{A_j(t)} 1_{\left\{s^k(U_i^j-)\neq 0 \quad \forall k\in\mathcal{K}\right\}} f(\ell_i^j)$$
$$- \sum_{\eta_l\in(0,t]} \sum_{i=1}^{Z_j(\eta_l-)} 1_{\{\kappa_l\in I_{j,i}(\boldsymbol{Z}(\eta_l-))\}} f\left(\text{supp}(\mathcal{Z}_j(\eta_l-))_{\{i\}}\right). \tag{7.4}$$

We note that the last term is well defined because if η_l is a time at which a job enters service from the queues, then $\boldsymbol{Z}(\eta_l-) \neq 0$.

Proof. Since both sides of (7.4) are right continuous it suffices to show that (7.4) holds almost surely for each fixed $t \geq 0$, $j \in \mathcal{J}$. Consequently, for the following analysis, we fix $t \geq 0$, $j \in \mathcal{J}$. For fixed $n \in \mathbb{N}$, define $t_m^n = \frac{tm}{n}$ for $m = 0, 1, \ldots, n-1$. Then by (7.1), we see that

$$\langle f, \mathcal{Z}_j(t)\rangle - \langle f, \mathcal{Z}_j(0)\rangle = \tag{7.5}$$
$$\lim_{n\to\infty} \sum_{m=0}^{n-1} \left(\left\langle f, \mathcal{Z}_j\left(t_m^n + \frac{t}{n}\right)\right\rangle - \langle f, \mathcal{Z}_j(t_m^n)\rangle\right) \tag{7.6}$$
$$= \lim_{n\to\infty} \sum_{m=0}^{n-1} \left\langle f\left(\left(\cdot - \frac{t}{n}\right)^+\right) - f, \mathcal{Z}_j(t_m^n)\right\rangle \tag{7.7}$$
$$- \lim_{n\to\infty} \sum_{m=0}^{n-1} \sum_{\eta_i\in(t_m^n, t_m^n+t/n]} \left\langle f, \delta^+_{\{T_{i,j}-t_m^n-t/n+\eta_i\}}\right\rangle \tag{7.8}$$
$$+ \lim_{n\to\infty} \sum_{m=0}^{n-1} \sum_{i=A_j(t_m^n)+1}^{A_j(t_m^n+t/n)} 1_{\left\{s^k(U_i^j-)\neq 0 \quad \forall k\in\mathcal{K}\right\}} f\left(\left(\ell_i^j + U_i^j - t_m^n - \frac{t}{n}\right)^+\right), \tag{7.9}$$

provided that all of the limits exist and using the fact that $f(0) = 0$ for all $f \in \mathscr{C}$. The first term to examine is in (7.7). We will first prove that the limit in (7.7) equals $\int_0^t \langle f', \mathcal{Z}_j(s)\rangle ds$ for a subset of $f \in \mathscr{C}$. Then, we will extend to all of $\mathscr{C}$. For the subset, fix $f \in \mathscr{C}$ with compact support and $f'(0)=0$. It follows from the mean value theorem that for $x \geq 0$,

$$f(x) - f\left(\left(x - \frac{t}{n}\right)^+\right) = \int_{(x-t/n)^+}^{x} f'(r)dr$$
$$= f'(r_x^n)(x - (x-t/n)^+) \quad (7.10)$$

for some $r_x^n \in [(x-t/n)^+, x]$. Because f' is continuous and compactly supported, it is uniformly continuous. Therefore, if we fix $\epsilon > 0$, there is n_ϵ such that for all $n \geq n_\epsilon$,

$$\sup_{x\geq 0}\sup_{h\leq t/n} |f'(x+h) - f'(x)| < \epsilon.$$

Therefore, for r_x^n as in (7.10), we see that $|f'(x) - f'(r_x^n)| < \epsilon$ for all $n \geq n_\epsilon, x \geq 0$. It follows that for $x \geq \frac{t}{n}, n \geq n_\epsilon$

$$\left|f(x) - f\left(\left(x - \frac{t}{n}\right)^+\right) - f'(x)\frac{t}{n}\right| = \left|f'(r_x^n)\frac{t}{n} - f'(x)\frac{t}{n}\right|$$
$$\leq \epsilon\frac{t}{n}.$$

For $n \geq n_\epsilon$ $0 \leq x < \frac{t}{n}$, we have

$$\left|f(x) - f\left(\left(x - \frac{t}{n}\right)^+\right) - f'(x)\frac{t}{n}\right| = \left|f'(r_x^n)x - f'(x)\frac{t}{n}\right|$$
$$\leq |f'(r_x^n)x - f'(x)x| + \left|f'(x)x - f'(x)\frac{t}{n}\right|$$
$$\leq x\epsilon + |f'(x)|\frac{t}{n}$$
$$\leq \frac{t}{n}\left(\epsilon + |f'(x)|\right).$$

We conclude that for $n \geq n_\epsilon, x \geq 0$,

$$\left| f(x) - f\left(\left(x - \frac{t}{n}\right)^+\right) - f'(x)\frac{t}{n} \right| \leq 1_{\{x < \frac{t}{n}\}} |f'(x)| \frac{t}{n} + \epsilon \frac{t}{n}. \tag{7.11}$$

Then for $g_n(x) = f(x) - f\left(\left(x - \frac{t}{n}\right)^+\right) - f'(x)\frac{t}{n}$,

$$\sum_{m=0}^{n-1} \left\langle f\left(\left(\cdot - \frac{t}{n}\right)^+\right) - f, \mathcal{Z}_j(t_m^n) \right\rangle = -\sum_{m=0}^{n-1} \frac{t}{n} \left\langle f', \mathcal{Z}_j(t_m^n) \right\rangle \tag{7.12}$$

$$- \sum_{m=0}^{n-1} \left\langle g_n, \mathcal{Z}_j(t_m^n) \right\rangle, \tag{7.13}$$

where the error term (7.13) has the property that $|g_n(x)| \leq 1_{\{x < \frac{t}{n}\}} |f'(x)| \frac{t}{n} + \epsilon \frac{t}{n}$ for each $x \geq 0, n \geq n_\epsilon$. Because the function $\langle f', \mathcal{Z}_j(\cdot) \rangle$ is right continuous, it is Riemann integrable on $[0, t]$, and so the right side of (7.12) converges to $-\int_0^t \langle f', \mathcal{Z}_j(s) \rangle ds$ as $n \to \infty$. Furthermore, using the given bound on $g_n(\cdot)$ for $n \geq n_\epsilon$ we have

$$\begin{aligned} 0 \leq \left| \sum_{m=0}^{n-1} \langle g_n, \mathcal{Z}_j(t_m^n) \rangle \right| &\leq \sum_{m=0}^{n-1} \frac{t}{n} \left(\sup_{x \leq t/n} |f'(x)| + \epsilon \right) Z_j(t_m^n) \\ &\leq t \left(\sup_{x \leq t/n} |f'(x)| + \epsilon \right) \sup_{s \in [0,t]} Z_j(s) \\ &\leq t \left(\sup_{x \leq t/n} |f'(x)| + \epsilon \right) (A_j(t) + Z_{0,j}) \\ &\to t\epsilon(A_j(t) + Z_{0,j}) \end{aligned}$$

as $n \to \infty$ because we have assumed that $f'(0) = 0$. We note that $A_j(t) + Z_{0,j}$ is almost surely finite. Because ϵ can be chosen to be arbitrarily small, we conclude that the limit as $n \to \infty$ of the sum in (7.13) is zero almost surely. Therefore, almost surely, the limit in (7.7) exists and is equal to $-\int_0^t \langle f', \mathcal{Z}_j(s) \rangle ds$. By the uniform continuity of f and the expression for $T_{i,j}$, the limits in (7.8) and (7.9) exist and are given by the corresponding terms in (7.4). Now we extend to all $f \in \mathscr{C}$. Let $f \in \mathscr{C}$. We will approximate f with a sequence of functions for which we know that (7.4) holds, and show that each term in (7.4) converges appropriately. Choose a sequence of functions in $\mathscr{C}$, $\{f_n\}_{n=1}^{\infty}$, such that $f_n \to f$ and $f_n' \to f'$ pointwise on $(0,\infty)$ as $n \to \infty$, $\{f_n\}_{n=1}^{\infty}$ and $\{f_n'\}_{n=1}^{\infty}$ are uniformly bounded, and for each n, f_n has compact support and $f_n'(0) = 0$. Such a sequence could be constructed by taking f_n such that $f_n = f$ on $[1/n, n]$ and $f_n = 0$, $[0, 1/2n] \cup [2n, \infty)$ and suitably interpolating between. Using the fact that $\mathcal{Z}_j(s)$ does not charge the origin, we see that $\langle f_n, \mathcal{Z}_j(t)\rangle \to \langle f, \mathcal{Z}_j(t)\rangle, \langle f_n, \mathcal{Z}_j(0)\rangle \to \langle f, \mathcal{Z}_j(0)\rangle, \langle f_n', \mathcal{Z}_j(s)\rangle \to \langle f', \mathcal{Z}_j(s)\rangle$ for all $s \in [0,t]$ as $n \to \infty$ by bounded convergence. Since

$$\begin{aligned}\sup_{n\in\mathbb{N}} \sup_{s\in[0,t]} |\langle f_n', \mathcal{Z}_j(s)\rangle| &\leq \sup_{n\in\mathbb{N}} ||f_n'||_\infty \sup_{s\in[0,t]} |Z_j(s)| \\ &\leq \sup_{n\in\mathbb{N}} ||f_n'||_\infty (Z_{0,j} + A_j(t)) < \infty\end{aligned}$$

almost surely, this gives convergence almost surely of the first three terms in (7.4) with f_n in place of f to those with f. The convergence for the last two terms in (7.4) follows from the fact that $f_n \to f$ pointwise on $(0,\infty)$. □

7.2 Decompositions of Component Parts Involving Martingales

The goal of this section is to obtain a prelimit equation that is more clearly analogous to (4.5). We note that, while our argument looks quite different, the inspiration for the decompositions in this section came from [3], which we came to look at after examining [2]. We will prove the following two lemmas at the end of this section, specifically in §7.2.2. In these lemmas, the martingale properties will be with respect to the filtration $\{\mathscr{F}_{\mathcal{S}(t)}\}_{t\geq 0}$, where $\{\mathscr{F}_q : q \in \mathbb{N}_0 \cup \{\infty\}\}$ is defined in Lemma 7.2.3 and $\mathcal{S}(\cdot)$ is defined in (2.7). For $j \in \mathcal{J}$, a martingale property for $\mathscr{X}_j(\cdot)$ with respect to $\{\mathscr{G}^j_{A_j(t)}\}_{t\geq 0}$ will also hold, where $\{\mathscr{G}^j_q : q \in N_0 \cup \{\infty\}\}$ is defined in Lemma 7.2.4 and $A_j(t)$ is defined in §2.2.1.

Lemma 7.2.1. *Let $f \in \mathbf{C}_b^1(\mathbb{R}_+)$ and $j \in \mathcal{J}$. Then for this f and $t \geq 0$, the last term of* (7.4) *can be decomposed as follows:*

$$\sum_{\eta_l \in (0,t]} \sum_{i=1}^{Z_j(\eta_l-)} 1_{\{\kappa_l \in I_{j,i}(\boldsymbol{Z}(\eta_l-))\}} f\left(\operatorname{supp}(\mathcal{Z}_j(\eta_l-))_{\{i\}}\right) = H_t^j(f) + Y_t^j(f) \tag{7.14}$$

where

$$H_t^j(f) := \int_0^t 1_{\{\mathcal{L}(s-)\neq 0\}} \frac{p_j\langle f, \mathcal{Z}_j(s-)\rangle}{L(s-)} dS(s), \qquad t \geq 0,$$

$$Y_t^j(f) := \sum_{\eta_l \in (0,t]} \sum_{i=1}^{Z_j(\eta_l-)} 1_{\{\kappa_l \in I_{j,i}(\boldsymbol{Z}(\eta_l-))\}} f\left(\operatorname{supp}(\mathcal{Z}_j(\eta_l-))_{\{i\}}\right) - H_t^j(f), \qquad t \geq 0,$$

and $\{Y_\cdot^j(f), t \geq 0\}$ is a square-integrable martingale. We note that $H_t^j(f)$ is well-defined because $\mathcal{L}(s-) \neq 0$ if and only if $L(s-) \neq 0$. Then, for each $f \in \mathscr{C}$, almost surely for each $j \in \mathcal{J}, t \geq 0$,

$$\begin{aligned}\langle f, \mathcal{Z}_j(t)\rangle = \langle f, \mathcal{Z}_j(0)\rangle - \int_0^t \langle f', \mathcal{Z}_j(s)\rangle ds + \sum_{i=1}^{A_j(t)} 1_{\{s^k(U_i^j-)\neq 0) \quad \forall k \in \mathcal{K}\}} f(\ell_i^j) \\ - H_t^j(f) - Y_t^j(f). \end{aligned} \tag{7.15}$$

Note: In these definitions, S is as defined in §2.5 and $dS(\cdot)$ represents integration against the Lebesgue-Stieltjes measure associated to the bounded variation process S. Furthermore, L is given by (4.2).

At this point, it is also useful to define processes that track total service time given up to time t and a similar martingale decomposition for the portion of that time that goes to jobs that entered service from the queues. For $t \geq 0$, let

$$\mathscr{T}(t) := \sum_{k=1}^{K} \int_0^t 1_{\{s^k(r)\neq 0\}} dr = \sum_{k=1}^{K} \int_0^t 1_{\{s^k(r-)\neq 0\}} dr, \tag{7.16}$$

where the last two are equal because the set of times at which $s^k(r-) \neq s^k(r)$ for some

$k \in \mathcal{K}$ is countable and thus of Lebesgue measure zero. This will be the process which tracks the total amount of service time spent on jobs (across all servers) up to time t. Almost surely, for each $t \geq 0$, this is equal to

$$\sum_{\eta_i \in (0,t]} \sum_{j=1}^{J} 1_{\{\kappa_i \in I_j(\boldsymbol{Z}(\eta_i-))\}} v_i^j + \sum_{j=1}^{J} \sum_{i=1}^{A_j(t)} 1_{\{s^k(U_i^j-)=0 \text{ for some } k \in \mathcal{K}\}} v_i^{J+j} + \sum_{k=1}^{K} s^k(0) - \sum_{k=1}^{K} s^k(t). \tag{7.17}$$

The first sum is the sum of the service times for all of the jobs that have entered service from the queues; the second sum is the sum of the service times for all of the jobs that have entered service immediately from arrivals (without ever entering any queue), where the almost surely is needed to exclude simultaneous arrivals and service completions; the third sum is of the remaining service time at each server at time 0, and the subtracted fourth sum is the sum of the remaining service time at each server at time t. We give a similar decomposition involving a martingale for the total service time of jobs that enter service from any of the queues. Recalling that $\mathcal{L}(t) = 0$ if and only if $L(t) = 0$ for $t \geq 0$, define

$$\mathcal{Y}_t := \sum_{\eta_i \in (0,t]} \left(\sum_{j=1}^{J} 1_{\{\kappa_i \in I_j(\boldsymbol{Z}(\eta_i-))\}} v_i^j - 1_{\{\mathcal{L}(\eta_i-) \neq 0)\}} \frac{\mathcal{L}(\eta_i-)}{L(\eta_i-)} \right), \qquad t \geq 0, \tag{7.18}$$

the centering process

$$\mathcal{H}(t) = \int_0^t 1_{\{\mathcal{L}(r-) \neq 0\}} \frac{\mathcal{L}(r-)}{L(r-)} dS(r), \qquad t \geq 0, \tag{7.19}$$

and

$$\Upsilon_1(t) = \sum_{j=1}^{J} \int_0^t 1_{\{s^k(r-)=0 \text{ for some } k \in \mathcal{K}\}} d\mathscr{V}_j(r), \qquad t \geq 0, \tag{7.20}$$

where $\mathscr{V}_j(t) = \sum_{i=1}^{A_j(t)} v_i^{J+j}$ for each $t \geq 0$. Note that we are re-writing some sums up to our jump processes as Lebesgue-Stieltjes integrals against those processes or related processes. Let

$$\Upsilon_2(t) := \sum_{k=1}^{K} s^k(0) - \sum_{k=1}^{K} s^k(t), \qquad t \geq 0. \tag{7.21}$$

Then it follows from (7.17) that for $t \geq 0$,

$$\mathscr{T}(t) = \mathcal{Y}_t + \mathcal{H}(t) + \Upsilon_1(t) + \Upsilon_2(t). \tag{7.22}$$

We also give a second decomposition of $\mathscr{T}(\cdot)$ that will be useful for one part of the proof of Proposition 9.3.1. In particular, for $t \geq 0$,

$$\mathscr{T}(t) = \mathscr{Y}_t + \sum_{j=1}^{J} \mathscr{X}_j(t) + \sum_{j=1}^{J} \frac{1}{\mu_j} S_j(t) + \Upsilon_2(t) \tag{7.23}$$

where

$$\mathscr{Y}_t := \sum_{\eta_i \in (0,t]} \sum_{j=1}^{J} 1_{\{\kappa_i \in I_j(\boldsymbol{Z}(\eta_i -))\}} \left(v_i^j - \frac{1}{\mu_j} \right), \tag{7.24}$$

and

$$\mathscr{X}_j(t) := \sum_{i=1}^{A_j(t)} 1_{\{s^k(U_i^j -)=0 \text{ for some } k \in \mathcal{K}\}} \left(v_i^{J+j} - \frac{1}{\mu_j} \right). \tag{7.25}$$

We will prove the following lemma in §7.2.2.

Lemma 7.2.2. *The processes $\mathcal{Y}_\cdot$ and $\mathscr{Y}_\cdot$ given in* (7.18) *and* (7.24) *are square-integrable martingales.*

We will prove later, in Lemma 7.2.4, that, for each $j \in \mathcal{J}$, $\mathscr{X}_j(\cdot)$ is a martingale with respect to a suitable filtration.

7.2.1 Constructing Key Martingales in Discrete Time

It will be useful to construct some key martingales that can be thought of as discrete-time analogues of $Y_\cdot^j(f)$, $\mathcal{Y}_\cdot$, and $\mathscr{Y}_\cdot$. Fix $f \in \boldsymbol{C}_b^1(\mathbb{R}_+)$. We will construct discrete-time martingales $\{\tilde{Y}^j_q(f)\}_{q=0}^{\infty}, j \in \mathcal{J}, \{\tilde{\mathcal{Y}}_q\}_{q=0}^{\infty}$, and $\{\tilde{\mathscr{Y}}_q\}_{q=0}^{\infty}$ such that $\tilde{Y}^j_{\mathcal{S}(t)}(f) = Y^j_t(f)$, $j \in \mathcal{J}$, $\tilde{\mathcal{Y}}_{\mathcal{S}(t)} = \mathcal{Y}_t$, and $\tilde{\mathscr{Y}}_{\mathcal{S}(t)} = \mathscr{Y}_t$ for each $t \geq 0$. We will show that for each $t \geq 0$, $\mathcal{S}(t)$ is a stopping time with respect to a filtration to which $\tilde{Y}^j_\cdot(f), j \in \mathcal{J}, \tilde{\mathcal{Y}}_\cdot$, and $\tilde{\mathscr{Y}}_\cdot$ are adapted. It will follow from the optional sampling theorem and uniform integrability of $\tilde{Y}^j_{\cdot \wedge \mathcal{S}(T)}(f), j \in \mathcal{J}, \tilde{\mathcal{Y}}_{\cdot \wedge \mathcal{S}(T)}$, and $\tilde{\mathscr{Y}}_{\cdot \wedge \mathcal{S}(T)}$ for each fixed $T > 0$ that $Y^j_\cdot(f), \mathcal{Y}_\cdot$, and $\mathscr{Y}_\cdot$ are martingales. Before proving the martingale properties for $\tilde{Y}^j_\cdot(f), j \in \mathcal{J}, \tilde{\mathcal{Y}}_\cdot$, and $\tilde{\mathscr{Y}}_\cdot$, we prove the following lemma.

Lemma 7.2.3. *For each integer $q \geq 0$, define*

$$\mathscr{F}_q := \sigma(\{v^j_i\}_{j \in \mathcal{J}, 1 \leq i \leq q}, \{\kappa_i\}_{1 \leq i \leq q}, \{u^j_i\}_{j \in \mathcal{J}, i \in \mathbb{N}_0}, \{v^{J+j}_i\}_{j \in \mathcal{J}, i \in \mathbb{N}}, \{\ell^j_i\}_{j \in \mathcal{J}, i \in \mathbb{N}}, \boldsymbol{\mathcal{Z}}(0), \{s^k_0\}_{k \in \mathcal{K}}) \vee P_0$$

where P_0 denotes the set of P-null sets in the complete probability space $(\Omega, \mathscr{F}, P)$. Define $\mathscr{F}_\infty = \bigvee_{q \geq 0} \mathscr{F}_q$. Then for each $q \in \mathbb{N}_0$, η_{q+1} is $\mathscr{F}_q$-measurable and for each $q \in \mathbb{N}$, $\boldsymbol{T}_q$ is $\mathscr{F}_q$-measurable. Furthermore, one can use the rules for the model outlined in §2 to construct, for each $q \in \mathbb{N}_0$, $\mathscr{F}_q$-measurable processes $\boldsymbol{Z}_q(\cdot), \boldsymbol{\mathcal{Z}}_q(\cdot)$ such that $\boldsymbol{\mathcal{Z}}_q(\cdot \wedge \eta_q) = \boldsymbol{\mathcal{Z}}(\cdot \wedge \eta_q)$, $\boldsymbol{Z}_q(\cdot \wedge \eta_q) = \langle 1, \boldsymbol{\mathcal{Z}}_q(\cdot \wedge \eta_q) \rangle = \boldsymbol{Z}(\cdot \wedge \eta_q)$, and $\boldsymbol{\mathcal{Z}}_{q+1}(\eta_i -) 1_{\{\eta_i < \infty\}}$ is $\mathscr{F}_q$-measurable for each $i \leq q+1$. Lastly, using these variables and processes we can define discrete-time processes $\{\tilde{Y}^j_q(f) : q = 0, \ldots, \infty\}, j \in \mathcal{J}, \{\tilde{\mathcal{Y}}_q : q = 0, \ldots, \infty\}$, and $\{\tilde{\mathscr{Y}}_q : q = 0, \ldots, \infty\}$ that are adapted to $\{\mathscr{F}_q : q = 0, \ldots, \infty\}$ such that almost surely, for all $t \geq 0$, $\tilde{Y}^j_{\mathcal{S}(t)}(f) = Y^j_t(f), j \in \mathcal{J}$, $\tilde{\mathcal{Y}}_{\mathcal{S}(t)} = \mathcal{Y}_t$, and $\tilde{\mathscr{Y}}_{\mathcal{S}(t)} = \mathscr{Y}_t$.

Proof. We will begin by defining some $\{\mathscr{F}_q : q = 0, \ldots, \infty\}$-adapted analogues for relevant primitive and descriptive processes from our model, some of which are described in the statement of the lemma, but others of which are not. These will be needed in order to define $\{\tilde{Y}^j_q(f)\}_{q=0}^{\infty}, j \in \mathcal{J}, \{\tilde{\mathcal{Y}}_q\}_{q=0}^{\infty}$, and $\{\tilde{\mathscr{Y}}_q\}_{q=0}^{\infty}$. Let $\mathcal{Z}_{j,0}(t) = \mathcal{Z}_j(0) + \sum_{i=1}^{A_j(t)} \delta^+_{U^j_i + \ell^j_i - t}$ for $t \geq 0$, $\eta_0 = 0$, $\boldsymbol{c}_0 = \boldsymbol{0}$, $\boldsymbol{s}_0 = (s^1_0, \ldots, s^K_0)$, and $\mathscr{I}^j_0 = \emptyset$ for $j \in \mathcal{J}$. For $q \in \mathbb{N}, j \in \mathcal{J}$, $\mathscr{I}^j_q$ will represent the index set of the jobs of class j that arrived before the qth service entry from the queues but did not enter any queue because there was an available server when they arrived. In particular, if the ith job of class j arrives to the system before the qth entry to service from the queues and does *not* enter any queue because a server is available at its arrival time U_i^j, then $i \in \mathscr{I}_q^j$. Note that $Z_{j,0}(t) = \langle 1, \mathcal{Z}_{j,0}(t)\rangle$ is $\mathscr{F}_0$-measurable for each $t \geq 0, j \in \mathcal{J}$, and so are $\boldsymbol{c}_0$, $\boldsymbol{s}_0$, $\mathscr{I}_0^j$, and η_0. Then we can inductively define the following variables and processes: $\eta_{q+1}, \boldsymbol{c}_{q+1}, \mathscr{I}_{q+1}^j, j \in \mathcal{J}, \boldsymbol{s}_{q+1}, \boldsymbol{T}_{q+1}, \boldsymbol{Z}_{q+1}(\cdot), \boldsymbol{\mathcal{Z}}_{q+1}(\cdot)$, such that for $q \geq 0$, $\eta_{q+1}, \mathscr{I}_{q+1}^j, j \in \mathcal{J}$, $\boldsymbol{\mathcal{Z}}_{q+1}(\eta_i -)1_{\{\eta_i < \infty\}}$ for $i \leq q+1$, are $\mathscr{F}_q$-measurable and $\boldsymbol{c}_{q+1}, \boldsymbol{s}_{q+1}, \boldsymbol{T}_{q+1}, \boldsymbol{Z}_{q+1}(\cdot), \boldsymbol{\mathcal{Z}}_{q+1}(\cdot)$ are $\mathscr{F}_{q+1}$-measurable. Now, for each $q \in \mathbb{N}_0$, we use the variables in $\mathscr{F}_q$ to identify the time at which the $(q+1)$th entry to service from the queues occurs if such an event occurs. Then, we will use κ_{q+1} and $v_{q+1}^1, \ldots, v_{q+1}^J$ to determine which job is chosen from the queues at that time and the service time it is assigned.

Note that $\eta_0, \boldsymbol{c}_0, \boldsymbol{s}_0, \boldsymbol{T}_0, \boldsymbol{Z}_0(\cdot), \boldsymbol{\mathcal{Z}}_0(\cdot), \mathscr{I}_0^j$ for $j \in \mathcal{J}$, are $\mathscr{F}_0$-measurable. Assume for some $q \geq 0$ that $\eta_q, \mathscr{I}_q^j$, for $j \in \mathcal{J}$, are $\mathscr{F}_{(q-1)^+}$-measurable and $\boldsymbol{c}_q, \boldsymbol{s}_q, \boldsymbol{T}_q, \boldsymbol{Z}_q(\cdot), \boldsymbol{\mathcal{Z}}_q(\cdot)$, are $\mathscr{F}_q$-measurable.

On $\{\eta_q = \infty\}$, set $\eta_{q+1} = \infty, n_{q+1} = \infty, h_{q+1} = 0, \boldsymbol{c}_{q+1} = \boldsymbol{0}, \mathscr{I}_{q+1}^j = \mathscr{I}_q^j$ for $j \in \mathcal{J}$, $\boldsymbol{s}_{q+1} = \boldsymbol{0}, \boldsymbol{T}_{q+1} = \boldsymbol{0}, \boldsymbol{Z}_{q+1}(\cdot) = \boldsymbol{Z}_q(\cdot), \boldsymbol{\mathcal{Z}}_{q+1}(\cdot) = \boldsymbol{\mathcal{Z}}_q(\cdot)$.

On $\{\eta_q < \infty\}$, for each $n \in \mathbb{N}_0$, we define the following variables related to jobs that enter service upon arrival and do not enter any queue. The variables with superscript n will only be used in the case that n jobs enter service from arrivals between the qth and $(q+1)$th time that a job enters service from the queues. However, it is important to define all of the variables below so that we may know the times that jobs will *potentially* enter service upon arrival after the qth service entry from the queues, even if this event does not occur because all servers are busy at that arrival time. Set $\tilde{b}_q^0 = \eta_q$, $\tilde{s}_q^{k,0} = s_q^k$, and $\tilde{x}_q^0 = \min\{\tilde{s}_q^{k,0} : k \in \mathcal{K}\}$. These variables will be used for the case in which 0 jobs enter service from arrivals in between the qth and $(q+1)$th entries to service from the queues. Now, we use induction to define $\tilde{x}_q^n$ for $n \in \mathbb{N}_0$, which will be the time until a server will be free after n immediate entries to service from arrivals have occurred after the qth entry to service from the queues, when no other entries to service have occured in that time.

Assuming that $\tilde{b}_q^n$, $\tilde{s}_q^{k,n}, \tilde{x}_q^n$ have been defined, define $\tilde{h}_q^{n+1} := \min\{k \in \mathcal{K} : \tilde{s}_q^{k,n} = \tilde{x}_q^n\}$, interpreted as the server that will take the $(n+1)$th job. Then, define

$$\tilde{b}_q^{n+1} := \min\{U_i^j : U_i^j \geq \tilde{b}_q^n + \tilde{x}_q^n, j \in \mathcal{J}, i = 1, 2, \ldots\}, \tag{7.26}$$

interpreted as the next time that a job would potentially enter service from arrivals. Then the index and class of that job are $(i_q^{n+1}, j_q^{n+1}) = \{(i,j) \in \mathbb{N} \times \mathcal{J} : U_i^j = \tilde{b}_q^{n+1}\}$. For $k \neq \tilde{h}_q^{n+1}$ we update the remaining service time for server k as follows

$$\tilde{s}_q^{k,n+1} := (\tilde{s}_q^{k,n} + \tilde{b}_q^n - \tilde{b}_q^{n+1})^+$$

and for server $k = \tilde{h}_q^{n+1}$ we set

$$\tilde{s}_q^{k,n+1} := v_{i_q^{n+1}}^{J+j_q^{n+1}}. \tag{7.27}$$

Lastly, we set

$$\tilde{x}_q^{n+1} = \min\{\tilde{s}_q^{k,n+1} : k \in \mathcal{K}\}.$$

Now that these variables have been defined inductively for each $n \in \mathbb{N}_0$, we see that the number of immediate service entries from arrivals after the qth service entry from the queues and before the $(q+1)$th entry to service from the queues is

$$n_{q+1} := \begin{cases} 0 & \text{if } \boldsymbol{Z}_q((\eta_q + \tilde{x}_q^0)-) \neq \mathbf{0} \\ \inf\left\{n \in \mathbb{N} : \sum_{j=1}^J \sum_{i=1}^\infty 1_{\{\tilde{b}_q^n < U_i^j < \tilde{b}_q^n + \tilde{x}_q^n\}} 1_{\{\ell_i^j(\tilde{b}_q^n + \tilde{x}_q^n) > 0\}} > 0\right\} & \text{otherwise.} \end{cases}$$

For the above, if a queue is nonempty at the time just before $\eta_q + \tilde{x}_q^0$, which is the next time a server is available after η_q, then there will be a service entry from the queues at that time, and $n_{q+1} = 0$. Otherwise, we find the smallest n such that a job arrives when all servers are busy, which will happen only if in between the time at which the last job entered service, $\tilde{b}_q^n$, and the time at which a server will next be available after that, $\tilde{b}_q^n + \tilde{x}_q^n$, a job arrives, and the job that arrived in that time did not renege before a server became available. The last step of this part of the proof is to update the index set of jobs that have entered service from class j arrivals before the $(q+1)$th service entry from the queues:

$$\mathscr{I}^j_{q+1} = \mathscr{I}^j_q \cup \left\{ i \in \mathbb{N} : U^j_i = \tilde{b}^n_q \text{ for some } 1 \le n \le n_{q+1} \right\}, j \in \mathcal{J}.$$

On the set where $\{\eta_q < \infty, n_{q+1} = \infty\}$, define $\eta_{q+1} = \infty, h_{q+1} = 0, \boldsymbol{c}_{q+1} = \boldsymbol{0}$, $\boldsymbol{s}_{q+1} = \boldsymbol{0}, \boldsymbol{T}_{q+1} = \boldsymbol{0}, \boldsymbol{Z}_{q+1}(\cdot) = \boldsymbol{Z}_q(\cdot), \boldsymbol{\mathcal{Z}}_{q+1}(\cdot) = \boldsymbol{\mathcal{Z}}_q(\cdot)$.

On the other hand, on the set $\{\eta_q < \infty, n_{q+1} < \infty\}$, define

$$h_{q+1} := \tilde{h}^{n_{q+1}+1}_q,$$

and define the next time after η_q that a job enters service from the queues to be

$$\eta_{q+1} := \tilde{b}^{n_{q+1}}_q + \tilde{x}^{n_{q+1}}_q.$$

Combining the definitions on $\{\eta_q = \infty\} \cup \{n_{q+1} = \infty\}$ and $\{\eta_q < \infty, n_{q+1} < \infty\}$, and using the induction assumption, we see that $\mathscr{I}^j_{q+1}$ for $j \in \mathcal{J}$, h_{q+1}, and η_{q+1} are $\mathscr{F}_q$-measurable. In the next part we will use some stochastic primitives that are in the set of variables that generates $\mathscr{F}_{q+1}$ but not in the set of variables that generates $\mathscr{F}_q$.

On the $\mathscr{F}_q$-measurable set $\{\eta_q < \infty, n_{q+1} < \infty\}$, we define $\boldsymbol{c}_{q+1}$, $\boldsymbol{s}_{q+1}$, $\boldsymbol{T}_{q+1}$, $\boldsymbol{Z}_{q+1}, \boldsymbol{\mathcal{Z}}_{q+1}$ as follows. To describe the class of job that enters service at time η_{q+1} we have

$$\boldsymbol{c}_{q+1} := \sum_{j=1}^{J} \boldsymbol{e}_j 1_{\{\kappa_{q+1} \in I_j(\boldsymbol{Z}_q(\eta_{q+1}-))\}},$$

Then for $k \neq h_{q+1}$, we set

$$s^k_{q+1} := (\tilde{s}^{k,n_{q+1}}_q + \tilde{b}^{n_{q+1}}_q - \eta_{q+1})^+$$

and

$$s_{q+1}^{h_{q+1}} := \boldsymbol{c}_{q+1} \cdot (v_{q+1}^1, \dots, v_{q+1}^J).$$

For $j \in \mathcal{J}$, let

$$\mathcal{Z}_{j,q+1}(t) := \mathcal{Z}_j(0) + \sum_{i=1}^{A_j(t)} 1_{\{i \notin \mathscr{I}_{q+1}^j\}} \delta^+_{U_i^j + \ell_i^j - t} - \sum_{i=1}^{q+1} 1_{\{t \geq \eta_i\}} \delta^+_{T_{i,j} - t + \eta_i}, \quad t \geq 0, \tag{7.28}$$

the jth component of $\boldsymbol{\mathcal{Z}}_{q+1}(\cdot)$, and

$$Z_{j,q+1}(t) = \langle 1, \mathcal{Z}_{j,q+1}(t) \rangle, \quad t \geq 0.$$

For the remaining patience time of the $(q+1)$th job that enters service from the queues, for each $j \in \mathcal{J}$, we have

$$T_{q+1,j} := \sum_{l=1}^{Z_{j,q+1}((\eta_{q+1})-)} 1_{\{\kappa_{q+1} \in I_{j,l}(\boldsymbol{Z}_{q+1}(\eta_{q+1}-))\}} (\mathrm{supp}(\mathcal{Z}_{j,q+1}(\eta_{q+1}-))_{\{l\}}).$$

Note that for $j \in \mathcal{J}, l \leq q+1$,

$$\mathcal{Z}_{j,q+1}(\eta_l-) = \mathcal{Z}_j(0) + \sum_{i=1}^{A_j(\eta_l-)} 1_{\{i \notin \mathscr{I}_{q+1}^j\}} \delta^*_{U_i^j + \ell_i^j - \eta_l} - \sum_{i=1}^{l-1} \delta^*_{T_{i,j} - \eta_l + \eta_i}$$

where, for $x \in \mathbb{R}$, δ^*_x is the measure with unit mass at x for $x \geq 0$ and the zero measure for $x < 0$.

Combining the definitions on $\{\eta_{q+1} = \infty\} \cup \{n_{q+1} = \infty\}$ and $\{\eta_{q+1} < \infty, n_{q+1} < \infty\}$ with the induction assumption and the $\mathscr{F}_q$-measurability of $\eta_{q+1}, h_{q+1}, n_{q+1}$, we see that $\boldsymbol{\mathcal{Z}}_q(\eta_l-)1_{\{\eta_l < \infty\}}$ for $l \leq q+1$ is $\mathscr{F}_q$-measurable and $\boldsymbol{c}_{q+1}$, $\boldsymbol{s}_{q+1}$, $\boldsymbol{T}_{q+1}$, $\boldsymbol{\mathcal{Z}}_{q+1}(\cdot)$, $\boldsymbol{Z}_{q+1}(\cdot)$ are $\mathscr{F}_{q+1}$-measurable, which completes the induction step.

Now, for $j \in \mathcal{J}$, let $\tilde{Y}_0^j(f) = \tilde{Y}_\infty^j(f) = 0, \tilde{\mathcal{Y}}_0 = \tilde{\mathcal{Y}}_\infty = 0$, $\tilde{\mathscr{Y}}_0 = \tilde{\mathscr{Y}}_\infty = 0$ and for

$1 \le q < \infty$, let

$$\tilde{Y}^j_{q+1}(f) := \sum_{i=1}^{q+1} 1_{\{\eta_i<\infty\}} \sum_{l=1}^{Z_{j,q+1}(\eta_i-)} \left(1_{\{\kappa_i \in I_{j,l}(\boldsymbol{Z}_{q+1}(\eta_i-))\}} - \frac{p_j}{L_{q+1}(\eta_i-)}\right) f(\mathrm{supp}(\mathcal{Z}_{j,q+1}(\eta_i-))_{\{l\}}),$$

$$\tilde{\mathcal{Y}}_{q+1} := \sum_{i=1}^{q+1} 1_{\{\eta_i<\infty\}} \left(\sum_{j=1}^{J} 1_{\{\kappa_i \in I_j(\boldsymbol{Z}_{q+1}(\eta_i-))\}} v_i^j - 1_{\{\mathcal{L}_{q+1}(\eta_i-)\neq 0)\}} \frac{\mathcal{L}_{q+1}(\eta_i-)}{L_{q+1}(\eta_i-)}\right),$$

and

$$\tilde{\mathscr{Y}}_{q+1} := \sum_{i=1}^{q+1} 1_{\{\eta_i<\infty\}} \sum_{j=1}^{J} 1_{\{\kappa_i \in I_j(\boldsymbol{Z}_{q+1}(\eta_i-))\}} \left(v_i^j - \frac{1}{\mu_j}\right),$$

where $L_{q+1}(t) = \sum_{j=1}^J p_j Z_{j,q+1}(t)$ and $\mathcal{L}_{q+1}(t) = \sum_{j=1}^J \frac{p_j}{\mu_j} Z_{j,q+1}(t)$ for $t \ge 0$. Then, by the $\mathscr{F}_{q+1}$-measurability already proved, $\tilde{Y}^j_{q+1}(f), j \in \mathcal{J}, \tilde{\mathcal{Y}}_{q+1}, \tilde{\mathscr{Y}}^j_{q+1}$ are $\mathscr{F}_{q+1}$-measurable. Lastly, we note that because $\mu_j > 0$ for each $j \in \mathcal{J}$ as described in §2.2.4, almost surely, for all $t \ge 0$, $\mathcal{S}(t)$ is finite. Hence, because $\boldsymbol{\mathcal{Z}}_q(\cdot \wedge \eta_q) = \boldsymbol{\mathcal{Z}}(\cdot \wedge \eta_q)$ for all $q \ge 0$ by construction, it follows that almost surely, for all $t \ge 0$, the equalities $\tilde{Y}^j_{\mathcal{S}(t)}(f) = Y^j_t(f), j \in \mathcal{J}$, $\tilde{\mathcal{Y}}_{\mathcal{S}(t)} = \mathcal{Y}_t$, and $\tilde{\mathscr{Y}}_{\mathcal{S}(t)} = \mathscr{Y}_t$ hold. □

The first part of the proof of the following lemma uses a line of argument similar to Lemma 7.2.3, although with an additional index r. Due to this similarity, the proof is briefer where similar constructions are already in the proof of Lemma 7.2.3.

Lemma 7.2.4. *Fix $j \in \mathcal{J}$. For each $r \in \mathbb{N}_0$, let $\mathscr{G}^j_r$ be the σ-algebra generated by $\{v^l_i\}_{l\in\mathcal{J},i\in\mathbb{N}}$, $\{\kappa_i\}_{i\in\mathbb{N}}$, $\{u^l_i\}_{l\in\mathcal{J}\setminus\{j\},i\in\mathbb{N}_0}$, $\{u^j_i\}_{0\le i\le r}$, $\{v^{J+l}_i\}_{l\in\mathcal{J}\setminus\{j\},i\in\mathbb{N}}$, $\{v^{J+j}_i\}_{1\le i\le r}$, $\{\ell^l_i\}_{l\in\mathcal{J},i\in\mathbb{N}}$, $\boldsymbol{\mathcal{Z}}(0), \{s^k_0\}_{k\in\mathcal{K}}$, P_0, where P_0 denotes the set of P-null sets in the complete probability space $(\Omega, \mathscr{F}, P)$. Define $\mathscr{G}^j_\infty = \bigvee_{r\ge 0} \mathscr{G}^j_r$. Then for each $r \in \mathbb{N}$, $1_{\{s^k(U^j_r-)=0 \text{ for some } k\in\mathcal{K}\}}$ is $\mathscr{G}^j_{r-1}$-measurable. If we define $\tilde{\mathscr{X}}_j(0) = \tilde{\mathscr{X}}_j(\infty) = 0$ and for $1 \le r < \infty$, let*

$$\tilde{\mathscr{X}}_j(r) := \sum_{i=1}^{r} 1_{\{s^k(U^j_i-)=0 \text{ for some } k\in\mathcal{K}\}} \left(v_i^{J+j} - \frac{1}{\mu_j}\right),$$

then the discrete-time process $\{\tilde{\mathscr{X}}_j(r) : r = 0, \ldots, \infty\}$ is adapted to $\{\mathscr{G}^j_r : r = 0, \ldots, \infty\}$ and almost surely, for all $t \ge 0$, $\tilde{\mathscr{X}}_j(A_j(t)) = \mathscr{X}_j(t)$. Furthermore, $\{\tilde{\mathscr{X}}_j(r) : r \in \mathbb{N}_0\}$ is a *martingale with respect to* $\{\mathscr{G}_r^j : r \in \mathbb{N}_0\}$ *and* $\tilde{\mathscr{X}}_j(A_j(\cdot))$ *is a martingale with respect to* $\{\mathscr{G}_{A_j(t)}^j, 0 \le t < \infty\}$.

Proof. We begin by defining some $\{\mathscr{G}_{(r-1)^+}^j : r \in \mathbb{N}_0\}$-adapted analogues for relevant primitive and descriptive processes from our model. These will be needed in order to show the $\mathscr{G}_{(r-1)^+}^j$-measurability of $1_{\{s^k(U_r^j-)=0 \text{ for some } k\in\mathcal{K}\}}$ for $r > 0$. For $r \in \mathbb{N}_0, l \neq j$, let $\mathcal{Z}_{l,0}^r(t) = \mathcal{Z}_l(0) + \sum_{i=1}^{A_l(t)} \delta^+_{U_i^l+\ell_i^l-t}$ for $t \ge 0$, and let $\eta_0^r = 0$, $\boldsymbol{c}_0^r = \mathbf{0}$, $\boldsymbol{s}_0^r = (s_0^1, \dots, s_0^K)$, $\boldsymbol{T}_0^r = \mathbf{0}$ and $\mathscr{I}_0^{l,r} = \emptyset, l \in \mathcal{J}$. For $r \in \mathbb{N}_0, q \in \mathbb{N}$, $l \neq j$, $\mathscr{I}_q^{l,r}$ will represent the index set of the jobs of class l that arrived before the qth service entry from any of the queues and that did not enter the class l queue because there was an available server when they arrived. In particular, if the ith arrival to class l arrives to the system before the qth entry to service from the queues, and it does *not* enter any queue because a server is available at its arrival time U_i^l, then $i \in \mathscr{I}_q^{l,r}$. For $r \in \mathbb{N}_0$, define $\mathcal{Z}_{j,0}^r(t) = \mathcal{Z}_j(0) + \sum_{i=1}^{A_j(t)\wedge r} \delta^+_{U_i^j+\ell_i^j-t}$ for $t \ge 0$. Note that for $r \in \mathbb{N}_0$, $A_j(t) \wedge r$ is $\mathscr{G}_{(r-1)^+}^j$-measurable, since $\{A_j(t) \wedge r = s\} = \{U_s^j \le t < U_{s+1}^j\} \in \mathscr{G}_s^j$ for $s < r$ and $\{A_j(t) \wedge r = r\} = \{A_j(t) \wedge r \le r-1\}^c \in \mathscr{G}_{(r-1)^+}^j$. Following along the same lines, for $r \in \mathbb{N}_0, q \in \mathbb{N}$, $\mathscr{I}_q^{j,r}$ will represent the index set of the jobs of class j that arrived before the qth service entry from any of the queues and that did not enter any queue because there was an available server when they arrived, intersected with $\{1, \dots, r\}$.

We observe that for $r \in \mathbb{N}_0$, $\eta_0^r, \boldsymbol{c}_0^r$, $\boldsymbol{s}_0^r$, $\boldsymbol{T}_0^r, \boldsymbol{Z}_0^r(\cdot)$, $\boldsymbol{\mathcal{Z}}_0^r(\cdot), \mathscr{I}_0^{l,r}$ for $l \in \mathcal{J}$, are $\mathscr{G}_{(r-1)^+}^j$-measurable. Then, for $r \in \mathbb{N}_0$ fixed, using an induction on q, we define and prove $\mathscr{G}_{(r-1)^+}^j$-measurability of the following variables and processes: $\eta_q^r, \boldsymbol{c}_q^r, \mathscr{I}_q^{r,l}, l \in \mathcal{J}$, $\boldsymbol{s}_q^r, \boldsymbol{T}_q^r, \boldsymbol{Z}_q^r(\cdot), \boldsymbol{\mathcal{Z}}_q^r(\cdot)$, $q \in \mathbb{N}$. For the induction on q hypothesis, fix $q \in \mathbb{N}_0$ and assume that $\eta_q^r, \boldsymbol{c}_q^r, \mathscr{I}_q^{r,l}$ for $l \in \mathcal{J}$, $\boldsymbol{s}_q^r$, $\boldsymbol{T}_q^r$, $\boldsymbol{Z}_q^r(\cdot)$, $\boldsymbol{\mathcal{Z}}_q^r(\cdot)$ have been defined and shown to be $\mathscr{G}_{(r-1)^+}^j$-measurable. Note that this has already been proven to hold for $q = 0$. On $\{\eta_q^r = \infty\}$, set $\eta_{q+1}^r = \infty, n_{q+1}^r = \infty, h_{q+1}^r = 0, \boldsymbol{c}_{q+1}^r = \mathbf{0}, \mathscr{I}_{q+1}^{r,l} = \mathscr{I}_q^{r,l}$, $l \in \mathcal{J}$, $\boldsymbol{s}_{q+1}^r = \mathbf{0}, \boldsymbol{T}_{q+1}^r = \mathbf{0}, \boldsymbol{Z}_{q+1}^r(\cdot) = \boldsymbol{Z}_q^r(\cdot), \boldsymbol{\mathcal{Z}}_{q+1}^r(\cdot) = \boldsymbol{\mathcal{Z}}_q^r(\cdot)$.

On $\{\eta_q^r < \infty\}$, besides the induction on q, we have some intermediate entities defined using a further induction on n. For this, set $\tilde{b}_q^{0,r} = \eta_q^r$, $\tilde{s}_q^{k,0,r} = s_q^{k,r}$, and $\tilde{x}_q^{0,r} = \min\{\tilde{s}_q^{k,0,r} : k \in \mathcal{K}\}$, and assuming that $\tilde{b}_q^{n,r}$, $\tilde{s}_q^{k,n,r}, \tilde{x}_q^{n,r}$ for some $n \in \mathbb{N}_0$ have been defined, define $\tilde{h}_q^{n+1,r} := \min\{k \in \mathcal{K} : \tilde{s}_q^{k,n,r} = \tilde{x}_q^{n,r}\}$. Then, define

$$\tilde{b}_q^{n+1,r} := \min\{U_i^l : U_i^l \geq \tilde{b}_q^{n,r} + \tilde{x}_q^{n,r}, (i,l) \in (\mathbb{N} \times \mathcal{J}) \setminus (\{r+1, r+2, \ldots\} \times \{j\})\}, \quad (7.29)$$

interpreted as the next time that a job would potentially enter service from arrivals aside from jobs that arrive to class j after the rth job to arrive to class j. Then the index and class of that job are $(i_q^{n+1,r}, l_q^{n+1,r}) = \{(i,l) \in (\mathbb{N} \times \mathcal{J}) \setminus (\{r+1, r+2, \ldots\} \times \{j\}) : U_i^l = \tilde{b}_q^{n+1,r}\}$. For $k \neq \tilde{h}_q^{n+1,r}$ we update the remaining service time for server k as follows

$$\tilde{s}_q^{k,n+1,r} := (\tilde{s}_q^{k,n,r} + \tilde{b}_q^{n,r} - \tilde{b}_q^{n+1,r})^+$$

and for server $k = \tilde{h}_q^{n+1,r}$ we set

$$\tilde{s}_q^{k,n+1,r} := v_{i_q^{n+1,r}}^{J + l_q^{n+1,r}} \quad (7.30)$$

if $i_q^{n+1,r} < r$ or $l \neq j$ and

$$\tilde{s}_q^{k,n+1,r} := 1 \quad (7.31)$$

otherwise. We do this in order to put a placeholder of 1 in for the service time that would be used if the rth job of class j entered service upon arrival because for $r > 0$, that service time is not $\mathscr{G}_{(r-1)^+}^j$-measurable. The service on that job will occur after time U_r^j, so the first property in the lemma, which only involves behavior up to the time U_r^j, will still hold even if we put in another service time at that point. Lastly, we set

$$\tilde{x}_q^{n+1,r} = \min\{\tilde{s}_q^{k,n+1,r} : k \in \mathcal{K}\}.$$

Let

$$n_{q+1}^r := \begin{cases} 0 & \text{if } \boldsymbol{Z}_q^r((\eta_q^r + \tilde{x}_q^{0,r})-) \neq \mathbf{0} \\ \inf\left\{n \in \mathbb{N} : \sum_{l=1}^J \sum_{i=1}^\infty 1_{\{\tilde{b}_q^{n,r} < U_i^l < \tilde{b}_q^{n,r} + \tilde{x}_q^{n,r}\}} 1_{\{\ell_i^l(\tilde{b}_q^{n,r} + \tilde{x}_q^{n,r}) > 0\}} > 0\right\} & \text{otherwise,} \end{cases}$$

and update the index set of jobs that have entered service from class l arrivals before the $(q+1)$th service entry from the queues:

$$\mathscr{I}_{q+1}^{l,r} = \mathscr{I}_q^{l,r} \cup \left\{i \in \mathbb{N} : U_i^l = \tilde{b}_q^{n,r} \text{ for some } 1 \leq n \leq n_{q+1}^r\right\}, l \in \mathcal{J}.$$

On $\{\eta_q^r < \infty, n_{q+1}^r = \infty\}$, set $\eta_{q+1}^r = \infty, h_{q+1}^r = 0, \boldsymbol{c}_{q+1}^r = \mathbf{0}$, $\boldsymbol{s}_{q+1}^r = \mathbf{0}, \boldsymbol{T}_{q+1}^r = \mathbf{0}, \boldsymbol{Z}_{q+1}^r(\cdot) = \boldsymbol{Z}_q^r(\cdot), \boldsymbol{\mathcal{Z}}_{q+1}^r(\cdot) = \boldsymbol{\mathcal{Z}}_q^r(\cdot)$.

On the other hand, on the set $\{\eta_q^r < \infty, n_{q+1}^r < \infty\}$, define

$$h_{q+1}^r := \tilde{h}_q^{n_{q+1}^r+1,r} \quad \text{and} \quad \eta_{q+1}^r := \tilde{b}_q^{n_{q+1}^r,r} + \tilde{x}_q^{n_{q+1}^r,r},$$

$$\boldsymbol{c}_{q+1}^r := \sum_{i=1}^J \boldsymbol{e}_i 1_{\{\kappa_{q+1} \in I_i(\boldsymbol{Z}_q^r(\eta_{q+1}^r-))\}},$$

$$s_{q+1}^{k,r} := (\tilde{s}_q^{k,n_{q+1},r} + \tilde{b}_q^{n_{q+1},r} - \eta_{q+1}^r)^+, \quad k \neq h_{q+1}^r, \quad \text{and} \quad s_{q+1}^{h_{q+1}^r,r} := \boldsymbol{c}_{q+1}^r \cdot (v_{q+1}^1, \ldots, v_{q+1}^J),$$

$$\mathcal{Z}_{l,q+1}^r(t) := \mathcal{Z}_l(0) + \sum_{i=1}^{A_l(t)} 1_{\{i \notin \mathscr{I}_{q+1}^{l,r}\}} \delta^+_{U_i^l + \ell_i^l - t} - \sum_{i=1}^{q+1} 1_{\{t \geq \eta_i^r\}} \delta^+_{T_{i,l}^r - t + \eta_i^r}, \qquad t \geq 0, l \neq j,$$

$$\mathcal{Z}_{j,q+1}^r(t) := \mathcal{Z}_j(0) + \sum_{i=1}^{A_j(t) \wedge r} 1_{\{i \notin \mathscr{I}_{q+1}^{j,r}\}} \delta^+_{U_i^j + \ell_i^j - t} - \sum_{i=1}^{q+1} 1_{\{t \geq \eta_i^r\}} \delta^+_{T_{i,j}^r - t + \eta_i^r}, \qquad t \geq 0,$$

$$Z_{l,q+1}^r(t) = \langle 1, \mathcal{Z}_{l,q+1}^r(t) \rangle, \qquad t \geq 0, l \in \mathcal{J},$$

$$T_{q+1,l}^r := \sum_{i=1}^{Z_{l,q+1}^r((\eta_{q+1}^r)-)} 1_{\{\kappa_{q+1} \in I_{l,i}(\boldsymbol{Z}_{q+1}^r(\eta_{q+1}^r-))\}} (\mathrm{supp}(\mathcal{Z}_{l,q+1}^r(\eta_{q+1}^r-))_{\{i\}}), \qquad l \in \mathcal{J}.$$

Combining the definitions on $\{\eta_q^r = \infty\} \cup \{n_{q+1}^r = \infty\}$ with those on $\{\eta_q^r <$

$\infty, n^r_{q+1} < \infty\}$, we see that η^r_{q+1}, h^r_{q+1}, $\boldsymbol{c}^r_{q+1}$, $\boldsymbol{s}^r_{q+1}$, $\boldsymbol{T}^r_{q+1}$, $\boldsymbol{Z}^r_{q+1}$, $\boldsymbol{\mathcal{Z}}^r_{q+1}$, $\mathscr{I}^{l,r}_{q+1}$ for $l \in \mathcal{J}$ are $\mathscr{G}^j_{(r-1)^+}$-measurable. This completes the induction on q step.

Hence, since

$$1_{\{s^k(U^j_r-)=0 \text{ for some } k\in\mathcal{K}\}} = 1_{\{r\in\cup_{q\geq 0}\mathscr{I}^{j,r}_q\}} \tag{7.32}$$

almost surely, and $\mathscr{G}^j_{(r-1)^+}$ contains P_0, it follows that the left side of (7.32) is $\mathscr{G}^j_{(r-1)}$-measurable for each $r > 0$. Since $r \in \mathbb{N}_0$ was arbitrary, it follows that $\{\tilde{\mathscr{X}}_j(r) : r = 0, \ldots, \infty\}$ is adapted to $\{\mathscr{G}^j_r : r = 0, \ldots, \infty\}$. Also, because $\alpha_j > 0$ for each $j \in \mathcal{J}$, then almost surely, for all $t \geq 0$, $A_j(t)$ is finite. Hence, it follows that almost surely, for all $t \geq 0$ the equality $\tilde{\mathscr{X}}_j(A_j(t)) = \mathscr{X}_j(t)$ holds.

Lastly, we prove the martingale property. For each $r \in \mathbb{N}_0$, $\tilde{\mathscr{X}}_j(r)$ is $\mathscr{G}^j_r$-measurable and $E[|\tilde{\mathscr{X}}_j(r)|] \leq \frac{2r}{\mu_j}$, so $\tilde{\mathscr{X}}_j(r)$ is integrable. Furthermore, for $r \geq 1$,

$$\begin{aligned} E\left[\tilde{\mathscr{X}}_j(r) - \tilde{\mathscr{X}}_j(r-1)|\mathscr{G}^j_{r-1}\right] &= E\left[1_{\{s^k(U^j_r-)=0 \text{ for some } k\in\mathcal{K}\}}\left(v^{J+j}_r - \frac{1}{\mu_j}\right)\Bigg|\mathscr{G}^j_{r-1}\right] \\ &= 1_{\{s^k(U^j_r-)=0 \text{ for some } k\in\mathcal{K}\}} E\left[\left(v^{J+j}_r - \frac{1}{\mu_j}\right)\Bigg|\mathscr{G}^j_{r-1}\right] \\ &= 0, \end{aligned}$$

using the fact that $1_{\{s^k(U^j_r-)=0 \text{ for some } k\in\mathcal{K}\}}$ is $\mathscr{G}^j_{r-1}$-measurable and $\left(v^{J+j}_r - \frac{1}{\mu_j}\right)$ is independent of $\mathscr{G}^j_{r-1}$. Hence, $\{\tilde{\mathscr{X}}_j(r) : r \in \mathbb{N}_0\}$ is a martingale with respect to $\{\mathscr{G}^r_j : r \in \mathbb{N}_0\}$. Next, note that for any $T > 0, r \geq 0$, $|\tilde{\mathscr{X}}_j(r \wedge A_j(T))|$ is dominated by the random variable

$$\sum_{i=1}^{A_j(T)} v^{J+j}_i + \frac{1}{\mu_j} A_j(T). \tag{7.33}$$

Using Wald's inequality and the independence of $\{v^{J+j}_i\}^\infty_{i=1}$ from $\boldsymbol{A}(\cdot)$, we see that the expectation of (7.33) is dominated by

$$\frac{2}{\mu_j} E[A_j(T)] < \infty.$$

Thus, $\{\tilde{\mathscr{X}}_j(r\wedge A_j(T)) : r \in \mathbb{N}_0\}$ is uniformly integrable. Next, we note that, by construction of $\{\mathscr{G}^j_r : r \in \mathbb{N}_0\}$, in particular the fact that U^j_{r+1} is $\mathscr{G}^j_r$-measurable, we have for any $t \in [0, T]$ and $r \in \mathbb{N}_0$,

$$\{A_j(t) = r\} = \{U^j_r \le t < U^j_{r+1}\} \in \mathscr{G}^j_r.$$

It follows that $\{A_j(t)\}_{0\le t\le T}$ is an increasing family of stopping times for the filtration $\{\mathscr{G}^j_r : r = 0, 1, \ldots, \infty\}$. Then, $\{\tilde{\mathscr{X}}_j(r \wedge A_j(T)) : r = 0, 1, \ldots\}$ is a uniformly integrable martingale (see, e.g., Corollary 1.7 in [7]), and by Doob's Optional Sampling Theorem (see, e.g., Theorem 1.6 in [7]), $\{\tilde{\mathscr{X}}_j(A_j(t)) : 0 \le t \le T\}$ is a martingale with respect to $\{\mathscr{G}^j_{A_j(t)} : 0 \le t \le T\}$, for each $T > 0$. □

7.2.2 Proofs of Lemmas 7.2.1 and 7.2.2

Proof of Lemma 7.2.1. We simply need to prove that $Y^j_\cdot(f)$ is a square integrable martingale.

We begin by showing that $\tilde{Y}^j_\cdot(f)$ is a martingale with respect to the filtration $\{\mathscr{F}_q : q \in \mathbb{N}\}$. By Lemma 7.2.3, $\tilde{Y}^j_\cdot(f)$ is adapted to the filtration $\{\mathscr{F}_q : q \in \mathbb{N}\}$. Furthermore, for each q, $\tilde{Y}^j_q(f) \le 2q||f||_\infty$ so it is integrable. For the martingale property, by Lemma 7.2.3, for each $q \ge 0$, η_{q+1}, $\boldsymbol{\mathcal{Z}}_{q+1}(\eta_{q+1}-)1_{\{\eta_{q+1}<\infty\}}$, are $\mathscr{F}_q$-measurable, and so

$$\begin{aligned}
&E[\tilde{Y}^j_{q+1}(f) - \tilde{Y}^j_q(f)|\mathscr{F}_q]\\
&= E\left[1_{\{\eta_{q+1}<\infty\}} \sum_{l=1}^{Z_{j,q+1}(\eta_{q+1}-)} \left(1_{\{\kappa_{q+1}\in I_{j,l}(\boldsymbol{Z}_{q+1}(\eta_{q+1}-))\}} - \frac{p_j}{L_{q+1}(\eta_{q+1}-)}\right) f(\mathrm{supp}(\mathcal{Z}_{j,q+1}(\eta_{q+1}-))_{\{l\}}) \middle| \mathscr{F}_q\right]\\
&= 1_{\{\eta_{q+1}<\infty\}} \sum_{\boldsymbol{i}\in\mathbb{N}_0^J} 1_{\{\boldsymbol{Z}_{q+1}(\eta_{q+1}-)=\boldsymbol{i}\}} \sum_{l=1}^{i_j} f(\mathrm{supp}(\mathcal{Z}_{j,q+1}(\eta_{q+1}-))_{\{l\}}) E\left[\left(1_{\{\kappa_{q+1}\in I_{j,l}(\boldsymbol{i})\}} - |I_{j,l}(\boldsymbol{i})|\right)|\mathscr{F}_q\right]\\
&= 0
\end{aligned}$$

where the notation $|I_{j,l}(\boldsymbol{i})|$ refers to the length of the interval $I_{j,l}(\boldsymbol{i})$, which equals $\frac{p_j}{\sum_{m=1}^J p_m i_m}$. Hence, $\{\tilde{Y}_q(f), \mathscr{F}_q, q \in \mathbb{N}_0\}$ is a martingale. Next, note that for any $T > 0$, $|\tilde{Y}^j_{q\wedge\mathcal{S}(T)}(f)|$ is dominated by the random variable $2||f||_\infty \mathcal{S}(T)$. Because the number of jobs served up until time T is at most the number of jobs that were in the system at time 0 plus the number of jobs that have arrived during the time interval $[0,T]$, it follows that

$$0 \leq \mathcal{S}(T) \leq \sum_{j=1}^{J} (Z_{0,j} + A_j(T)), \tag{7.34}$$

which is integrable. Therefore $\{\tilde{Y}^j_{q\wedge\mathcal{S}(T)}(f) : q \geq 0\}$ is uniformly integrable. Next, we note that for each $q \geq 0$,

$$\{\mathcal{S}(t) = q\} = \{\eta_q \leq t < \eta_{q+1}\} \in \mathscr{F}_q,$$

again applying the facts that $\eta_0 = 0$ and η_{q+1} is $\mathscr{F}_q$-measurable for each $q \geq 0$. It follows that $\{\mathcal{S}(t)\}_{0\leq t\leq T}$ is an increasing family of stopping times for the filtration $\{\mathscr{F}_q\}$. This allows us to apply Doob's Optional Sampling Theorem (Chung and Williams [7], Theorem 1.6), to conclude that $\{\tilde{Y}^j_{\mathcal{S}(t)}(f)\}_{0\leq t\leq T}$ is a martingale with respect to $\{\mathscr{F}_{\mathcal{S}(t)}, 0 \leq t \leq T\}$. By Lemma 7.2.3, $\{Y^j_t(f)\}_{0\leq t\leq T}$ is equal to $\{\tilde{Y}^j_{\mathcal{S}(t)}(f)\}_{0\leq t\leq T}$ almost surely. Hence, since T was arbitrary, $\{Y^j_t(f), \mathscr{F}_{\mathcal{S}(t)}, t \geq 0\}$ is a martingale.

The last step in our proof is to check that the martingale $\{Y^j_t(f), \mathscr{F}_{\mathcal{S}(t)}, t \geq 0\}$ is square-integrable. We do this by bounding the quadratic variation:

$$\begin{aligned}
\langle Y^j(f)\rangle_t &= \sum_{\eta_l\in[0,t]} (Y^j_{\eta_l}(f) - Y^j_{\eta_{l-1}}(f))^2 \\
&= \sum_{\eta_l\in(0,t]} \left(\sum_{i=1}^{Z_j(\eta_l-)} \left(1_{\{\kappa_l\in I_{j,i}(\boldsymbol{Z}(\eta_l-))\}} - \frac{p_j}{L(\eta_l-)} \right) f\left(\mathrm{supp}(\mathcal{Z}_j(\eta_l-))_{\{i\}}\right) \right)^2 \\
&\leq ||f||^2_\infty \sum_{\eta_l\in(0,t]} \left(\sum_{i=1}^{Z_j(\eta_l-)} \left| 1_{\{\kappa_l\in I_{j,i}((\boldsymbol{Z}(\eta_l-))\}} - \frac{p_j}{L(\eta_l-)} \right| \right)^2 \\
&\leq 4\mathcal{S}(t)||f||^2_\infty
\end{aligned} \tag{7.35}$$

where we used the fact that $\sum_{i=1}^{Z_j(\eta_l-)} \frac{p_j}{L(\eta_l-)}$ and $\sum_{i=1}^{Z_j(\eta_l-)} 1_{\{\kappa_l\in I_{j,i}(\boldsymbol{Z}(\eta_l-))\}}$ are both in the interval $[0,1]$ to conclude that their sum is less than or equal to 2 and so its square is less than or equal to 4. However, invoking once again that

$$\mathcal{S}(t) \leq \sum_{j=1}^{J}(Z_{0,j} + A_j(t)),$$

it follows that

$$4E[\mathcal{S}(t)||f||_\infty^2] \leq 4||f||_\infty^2 E\left[\sum_{j=1}^{J}(Z_{0,j} + A_j(t))\right] < \infty.$$

By Theorem 7.35 in [16], this means that $Y_\cdot^j(f)$ is a square integrable martingale. We see that (7.15) follows from (7.4). □

Using similar methods, we can also prove that $\mathcal{Y}_\cdot$ and $\mathscr{Y}_\cdot$ are square integrable martingales with respect to $\{\mathscr{F}_{\mathcal{S}(t)}, t \geq 0\}$, as we now show.

Proof of Lemma 7.2.2. This proof will be quite similar to the proof of Lemma 7.2.1. We begin by showing that $\tilde{\mathcal{Y}}_\cdot$ and $\tilde{\mathscr{Y}}_\cdot$ are martingales with respect to the filtration $\{\mathscr{F}_q : q \in \mathbb{N}_0\}$. By Lemma 7.2.3, $\tilde{\mathcal{Y}}_\cdot, \tilde{\mathscr{Y}}_\cdot$ are adapted to the filtration $\{\mathscr{F}_q : q \in \mathbb{N}_0\}$. Furthermore, for each $q \geq 0$, $E[|\tilde{\mathscr{Y}}_q|], E[|\tilde{\mathcal{Y}}_q|] \leq q \sum_{j=1}^{J}\left(E[v_1^j] + \max_{j\in\mathcal{J}} \frac{1}{\mu_j}\right)$, so $\tilde{\mathcal{Y}}_q$ and $\tilde{\mathscr{Y}}_q$ both are integrable. For the martingale property, we observe that because η_{q+1} is $\mathscr{F}_q$-measurable for each $q \geq 0$, using the same logic as in the proof of Lemma 7.2.1 we have

$$\begin{aligned}
&E[\tilde{\mathcal{Y}}_{q+1} - \tilde{\mathcal{Y}}_q|\mathscr{F}_q] \\
&= 1_{\{\eta_{q+1}<\infty\}} \sum_{\boldsymbol{i}\in\mathbb{N}_0^J} 1_{\{\boldsymbol{Z}_{q+1}(\eta_{q+1}-)=\boldsymbol{i}\}} E\left[\left(\sum_{j=1}^{J} 1_{\{\kappa_{q+1}\in I_j(\boldsymbol{i})\}} v_{q+1}^j - 1_{\{\mathcal{L}_{q+1}(\boldsymbol{i})\neq 0\}} \frac{\mathcal{L}_{q+1}(\boldsymbol{i})}{L_{q+1}(\boldsymbol{i})}\right) \middle| \mathscr{F}_q\right] \\
&= 1_{\{\eta_{q+1}<\infty\}} \sum_{\boldsymbol{i}\in\mathbb{N}_0^J} 1_{\{\boldsymbol{Z}_{q+1}(\eta_{q+1}-)=\boldsymbol{i}\}} \left(\sum_{j=1}^{J} |I_j(\boldsymbol{i})| \frac{1}{\mu_j} - 1_{\{\boldsymbol{i}\neq\boldsymbol{0}\}} \frac{\sum_{j=1}^{J} \frac{p_j}{\mu_j} \boldsymbol{i}_j}{\sum_{j=1}^{J} p_j \boldsymbol{i}_j}\right) \\
&= 0
\end{aligned}$$

since $|I_j(\boldsymbol{i})| = 1_{\{\boldsymbol{i}\neq 0\}} \frac{p_j \boldsymbol{i}_j}{\sum_{l=1}^J p_l \boldsymbol{i}_l}$. Similarly,

$$
\begin{aligned}
&E[\tilde{\mathscr{Y}}_{q+1} - \tilde{\mathscr{Y}}_q | \mathscr{F}_q] \\
&= 1_{\{\eta_{q+1}<\infty\}} \sum_{\boldsymbol{i}\in\mathbb{N}_0^J} 1_{\{\boldsymbol{Z}_{q+1}(\eta_{q+1}-)=\boldsymbol{i}\}} E\left[\sum_{j=1}^J 1_{\{\kappa_{q+1}\in I_j(\boldsymbol{i})\}} \left(v_{q+1}^j - \frac{1}{\mu_j} \right) \middle| \mathscr{F}_q \right] \\
&= 1_{\{\eta_{q+1}<\infty\}} \sum_{\boldsymbol{i}\in\mathbb{N}_0^J} 1_{\{\boldsymbol{Z}_{q+1}(\eta_{q+1}-)=\boldsymbol{i}\}} \sum_{j=1}^J \left(E\left[\left(v_{q+1}^j - \frac{1}{\mu_j} \right) \right] E\left[1_{\{\kappa_{q+1}\in I_j(\boldsymbol{i})\}} \middle| \mathscr{F}_q \right] \right) \\
&= 0,
\end{aligned}
$$

because $\left(v_{q+1}^j - \frac{1}{\mu_j} \right)$ is independent of both $\mathscr{F}_q$ and $1_{\{\kappa_{q+1}\in I_j(\boldsymbol{i})\}}$.

Next, note that for any $T > 0$, $|\tilde{\mathcal{Y}}_{q\wedge\mathcal{S}(T)}|$ and $|\tilde{\mathscr{Y}}_{q\wedge\mathcal{S}(T)}|$ are dominated by the random variable

$$
J \sum_{i=1}^{\mathcal{S}(T)} \left(\max_{j\in\mathcal{J}} v_i^j + \max_{j\in\mathcal{J}} \frac{1}{\mu_j} \right) \leq J \sum_{i=1}^{Z_{0,j}+A_j(t)} \left(\max_{j\in\mathcal{J}} v_i^j + \max_{j\in\mathcal{J}} \frac{1}{\mu_j} \right), \tag{7.36}
$$

by (7.34). Using Wald's inequality and the independence of $\{v_i^j\}_{i=1}^\infty$, $j \in \mathcal{J}$, from $\boldsymbol{Z}_0$ and $\boldsymbol{A}(\cdot)$, we see that the expectation of (7.36) is dominated by

$$
JE\left[\left(\max_{j\in\mathcal{J}} v_1^j + \max_{j\in\mathcal{J}} \frac{1}{\mu_j} \right) \right] \sum_{j\in\mathcal{J}} E[Z_{0,j} + A_j(T)] < \infty.
$$

Thus, $\{\tilde{\mathcal{Y}}_{q\wedge\mathcal{S}(T)} : q \in \mathbb{N}_0\}, \{\tilde{\mathscr{Y}}_{q\wedge\mathcal{S}(T)} : q \in \mathbb{N}_0\}$ are uniformly integrable. As noted previously, $\{\mathcal{S}(t)\}_{t\leq T}$ is an increasing family of stopping times for the filtration $\{\mathscr{F}_q : q \in \mathbb{N}_0\}$. This allows us to apply Doob's Optional Sampling Theorem (Chung and Williams [7], Corollary 1.7), to conclude that $\{\tilde{\mathcal{Y}}_{\mathcal{S}(t)}\}_{t\leq T}$ and $\{\tilde{\mathscr{Y}}_{\mathcal{S}(t)}\}_{t\leq T}$ are martingales with respect to $\{\mathscr{F}_{\mathcal{S}(t)}, 0 \leq t \leq T\}$ (Chung and Williams [7], Theorem 1.6). Since $\mathcal{Y}_t = \tilde{\mathcal{Y}}_{\mathcal{S}(t)}$, $\mathscr{Y}_t = \tilde{\mathscr{Y}}_{\mathcal{S}(t)}$ almost surely, and $T \geq 0$ was arbitrary, it follows that $\{\mathcal{Y}_t\}_{t\geq 0}$ and $\{\mathscr{Y}_t\}_{t\geq 0}$ are martingales with respect to $\{\mathscr{F}_{\mathcal{S}(t)}, t \geq 0\}$.

The last step in our proof is to check that the martingales are square-integrable. We do this by bounding the quadratic variation:

$$\begin{aligned}\langle \mathcal{Y} \rangle_t &= \sum_{\eta_i \in (0,t]} \left(\sum_{j=1}^{J} 1_{\{\kappa_i \in I_j(\boldsymbol{Z}(\eta_i -))\}} v_i^j - 1_{\{\mathcal{L}(\eta_i -) \neq 0)\}} \frac{\mathcal{L}(\eta_i -)}{L(\eta_i -)} \right)^2 \\ &\leq \sum_{\eta_i \in (0,t]} 2 \left(\left(\max_{j \in \mathcal{J}} v_i^j \right)^2 + \left(\max_{j \in \mathcal{J}} \frac{1}{\mu_j} \right)^2 \right). \end{aligned} \tag{7.37}$$

However, invoking once again that

$$\mathcal{S}(t) \leq \mathcal{C}(t) := \sum_{j=1}^{J} (Z_{0,j} + A_j(t)), \tag{7.38}$$

it follows that for $t \geq 0$,

$$\begin{aligned} E[\langle \mathcal{Y} \rangle_t] &\leq 2E \left[\sum_{i=1}^{\mathcal{C}(t)} \left(\left(\max_{j \in \mathcal{J}} v_i^j \right)^2 + \left(\max_{j \in \mathcal{J}} \frac{1}{\mu_j} \right)^2 \right) \right] \\ &\leq 2E[\mathcal{C}(t)] \left(E\left[\left(\max_{j \in \mathcal{J}} v_1^j \right)^2 \right] + \left(\max_{j \in \mathcal{J}} \frac{1}{\mu_j} \right)^2 \right) < \infty, \end{aligned} \tag{7.39}$$

where the last inequality is achieved by Wald's inequality using the independence of $\boldsymbol{Z}_0$, $\boldsymbol{A}$, and $\{v_i^j\}_{i=1}^{\infty}$ for all $j \in \mathcal{J}$. By Theorem 7.35 in [16], this means that $\mathcal{Y}_.$ is a square integrable martingale. The same argument, but with the constant J in front of the initial bound, will work for $\mathscr{Y}_.$.

□

Chapter 8

C-Tightness

This chapter will be devoted to proving the C-tightness portion of Theorem 5.2.1.

8.1 Fluid Scaled Difference Equation

Applying the fluid scaling given in (3.2) to (7.15), we see that for each $f \in \mathscr{C}$, $j \in \mathcal{J}$, and $t \geq 0$,

$$\langle f, \bar{\mathcal{Z}}_j^m(t)\rangle = \langle f, \bar{\mathcal{Z}}_j^m(0)\rangle - \bar{R}_t^{j,m}(f) + \bar{\mathcal{A}}_t^{j,m}(f) - \bar{H}_t^{j,m}(f) - \bar{Y}_t^{j,m}(f), \tag{8.1}$$

where, using the fact that $\frac{d}{dx} f\left(\frac{x}{m}\right) = \frac{1}{m} f'\left(\frac{x}{m}\right)$,

$$\bar{R}_t^{j,m}(f) := \frac{1}{m}\int_0^{mt} \left\langle \frac{1}{m} f'\left(\frac{1}{m}\cdot\right), \mathcal{Z}_j^m(s)\right\rangle ds = \int_0^t \langle f', \bar{\mathcal{Z}}_j^m(s)\rangle ds, \tag{8.2}$$

$$\bar{\mathcal{A}}_t^{j,m}(f) = \int_0^t 1_{\left\{\bar{s}^{k,m}(r-)\neq 0 \quad \forall k\in\mathcal{K}\right\}} d\bar{A}_r^{j,m}(f), \tag{8.3}$$

where $\bar{s}^{k,m}(t) = \frac{1}{m}s^{k,m}(mt)$ and $\bar{A}^{j,m}_t(f) := \frac{1}{m}\sum_{i=1}^{m\bar{A}^{j,m}(t)} f(\frac{1}{m}\ell_i^{j,m}) = \frac{1}{m}\sum_{i=1}^{m\bar{A}^{j,m}(t)} f(\ell_i^j)$ for $t \geq 0$,

$$
\begin{aligned}
\bar{H}^{j,m}_t(f) :&= \frac{1}{m}H^{j,m}_{mt}\left(f\left(\frac{1}{m}\cdot\right)\right) \\
&= \frac{1}{m}\int_0^{mt} 1_{\{\mathcal{L}^m(s-)\neq 0\}}\frac{p_j\langle f\left(\frac{1}{m}\cdot\right), \mathcal{Z}^m_j(s-)\rangle}{L^m(s-)}dS^m(s) \\
&= \frac{1}{m}\int_0^{t} 1_{\{\mathcal{L}^m(ms-)\neq 0\}}\frac{p_j\langle f\left(\frac{1}{m}\cdot\right), \mathcal{Z}^m_j(ms-)\rangle}{L^m(ms-)}dS^m(ms) \\
&= \int_0^{t} 1_{\{\bar{\mathcal{L}}^m(s-)\neq 0\}}\frac{mp_j\langle f(\cdot), \bar{\mathcal{Z}}^m_j(s-)\rangle}{m\bar{L}^m(s-)}d\bar{S}^m(s) \\
&= \int_0^{t} 1_{\{\bar{\mathcal{L}}^m(s-)\neq 0\}}\frac{p_j\langle f(\cdot), \bar{\mathcal{Z}}^m_j(s-)\rangle}{\bar{L}^m(s-)}d\bar{S}^m(s), \qquad t\geq 0,
\end{aligned} \tag{8.4}
$$

$$
\begin{aligned}
\bar{Y}^{j,m}_t(f) &:= \frac{1}{m}Y^{j,m}_{mt}\left(f\left(\frac{1}{m}\cdot\right)\right) \\
&= \frac{1}{m}\sum_{\eta_l^m\in(0,mt]}\sum_{i=1}^{Z^m_j(\eta_l^m-)}\left(1_{\{\kappa_l\in I_{j,i}(\boldsymbol{Z}^m(\eta_l^m-))\}} - \frac{p_j}{L^m(\eta_l^m-)}\right) f\left(\frac{1}{m}\operatorname{supp}(\mathcal{Z}^m_j(\eta_l^m-))_{\{i\}}\right) \\
&= \frac{1}{m}\sum_{\bar{\eta}_l^m\in(0,t]}\sum_{i=1}^{m\bar{Z}^m_j(\bar{\eta}_l^m-)}\left(1_{\{\kappa_l\in I_{j,i}(\bar{\boldsymbol{Z}}^m(\bar{\eta}_l^m-))\}} - \frac{p_j}{m\bar{L}^m(\bar{\eta}_l^m-)}\right) f\left(\operatorname{supp}(\bar{\mathcal{Z}}^m_j(\bar{\eta}_l^m-))_{\{i\}}\right).
\end{aligned} \tag{8.5}
$$

where $\bar{\eta}_l^m = \frac{\eta_l^m}{m}$, and the last line uses (3.1), which implies that the ith element of the support of $\bar{\mathcal{Z}}^m_j(\bar{\eta}_l^m-)$ is $\frac{1}{m}$ times the ith element of the support of $\mathcal{Z}^m_j(\eta_l^m-)$. We note that, by (2.4), $I_{j,i}(\bar{\boldsymbol{Z}}^m(\bar{\eta}_l^m-)) = I_{j,i}(m\bar{\boldsymbol{Z}}^m(\bar{\eta}_l^m-))$ for each $m\in\mathbb{N}$. We also define

$$
\bar{\mathcal{Y}}^m_t := \frac{1}{m}\mathcal{Y}^m_{mt} = \frac{1}{m}\sum_{\bar{\eta}_l^m\in(0,t]}\left(\sum_{j=1}^J 1_{\left\{\kappa_l\in I_j(\bar{\boldsymbol{Z}}^m(\bar{\eta}_l^m-))\right\}}v_l^{j,m} - 1_{\{\bar{\mathcal{L}}^m(\bar{\eta}_l^m-)\neq 0)\}}\frac{\bar{\mathcal{L}}^m(\bar{\eta}_l-)}{\bar{L}^m(\bar{\eta}_l-)}\right), t\geq 0, \tag{8.6}
$$

and

$$
\bar{\mathcal{H}}^m(t) = \frac{1}{m}\mathcal{H}^m(mt) = \int_0^t 1_{\{\bar{\mathcal{L}}^m(s-)\neq 0\}}\frac{\bar{\mathcal{L}}^m(s-)}{\bar{L}^m(s-)}d\bar{S}^m(s) \qquad t\geq 0, \tag{8.7}
$$

where (8.7) is arrived at by a change of variables similar to what was done to obtain (8.4). Here, similar to (4.2) and (4.3), $\bar{L}^m(t) := \sum_{j=1}^J p_j \bar{Z}_j^m(t)$ and $\bar{\mathcal{L}}^m(t) := \sum_{j=1}^J \frac{p_j}{\mu_j^m} \bar{Z}_j^m(t)$.

8.2 Continuity

We begin this section by defining a family of sets that varies over $\epsilon, \delta, T > 0$,

$$A_{\epsilon,T}^{\delta} := \left\{ g \in D([0,\infty),\mathbb{R}) : \sup_{t\in[0,T-\delta]} \sup_{h\in[0,\delta]} |g(t+h) - g(t)| \le \epsilon \right\}. \tag{8.8}$$

We note that, using the Portmanteau theorem, if $\liminf_{m\to\infty} P\left\{\langle f, \bar{\mathcal{Z}}_j^m(\cdot)\rangle \in A_{\epsilon,T}^{\delta}\right\} = 1$, then when $\bar{\mathcal{Z}}_j(\cdot)$ is a subsequential limit in distribution of $\{\bar{\mathcal{Z}}_j^m(\cdot)\}_{m=1}^{\infty}$, we have that $\langle f, \bar{\mathcal{Z}}_j(\cdot)\rangle \in A_{\epsilon,T}^{\delta}$ almost surely. This section will be devoted to proving the following lemma.

Lemma 8.2.1. *Fix a sequence of models satisfying Assumption 1 with state descriptors* $\{\bar{\boldsymbol{\mathcal{Z}}}^m(\cdot)\}_{m=1}^{\infty}$*. Then for each* $j \in \mathcal{J}, \epsilon, T > 0$, $0 < \eta < 1$, $f \in \boldsymbol{C}_b^1(\mathbb{R}_+)$*, there exists* $\delta, M > 0$ *such that for each* $m > M$*,*

$$P\left\{\langle f, \bar{\mathcal{Z}}_j^m(\cdot)\rangle \in A_{\epsilon,T}^{\delta}\right\} > 1 - \eta.$$

It follows that any subsequential limit in distribution, $\bar{\boldsymbol{\mathcal{Z}}}(\cdot)$*, of* $\{\bar{\boldsymbol{\mathcal{Z}}}^m(\cdot)\}_{m=1}^{\infty}$ *is continuous almost surely. Furthermore, for each* $T > 0$, $j \in \mathcal{J}$, $f \in \mathscr{C}$, $\langle f, \bar{\mathcal{Z}}_j(\cdot)\rangle$ *is almost surely Lipschitz continuous on* $[0,T)$ *with Lipschitz constant*

$$L_{f,T}^j := 2(\bar{Z}_{0,j} + \alpha_j T)||f'||_\infty + 2\alpha_j||f||_\infty + 2K\check{\mathfrak{u}}||f||_\infty.$$

This lemma may at first appear odd because without proving tightness we cannot be sure that such limits exist. However, the analysis we do here will contribute to the proof of C-tightness in §8.3, so we put this lemma first. We prove Lemma 8.2.1 at the end of this section. However, we first need some preliminary results.

Lemma 8.2.2. *For each $j \in \mathcal{J}$, define*

$$\bar{X}_j^m(\cdot) := \frac{1}{m} \sum_{i=1}^{m\bar{A}_j^m(\cdot)} 1_{\{\ell_i^j \leq \cdot - U_i^{j,m}/m\}}. \tag{8.9}$$

This process is a bound for $\frac{\bar{X}_{0,t}^{j,m}(f)}{f(0)}$, as in (7.2), for each $f \in \boldsymbol{C}_b^1(\mathbb{R}_+), t \geq 0$ with $f(0) \neq 0$. Then

$$\bar{X}_j^m(\cdot) \to \alpha_j \int_0^{\cdot} N_j(x)dx$$

as $m \to \infty$, where the convergence is in probability, uniformly on compact time intervals.

Proof. Fix $\epsilon, \eta, T > 0$ with $\epsilon < \max_{j\in\mathcal{J}} \alpha_j T$, and $j \in \mathcal{J}$. For each $n \in \mathbb{N}$, define a partition of $[0, T)$ using intervals of the form $[t_k^n, t_{k+1}^n)$, $k = 0, 1, \ldots, n-1$, where $t_k^n = \frac{kT}{n}$, $k = 0, 1, \ldots, n$. Define the functions

$$f_{j,-}^n(t) := \sum_{k=0}^{n-1} 1_{\{t_{k+1}^n \leq t\}} \frac{\alpha_j T}{n} N_j(t_k^n) \text{ and } f_{+,j}^n(t) := \sum_{k=0}^{n-1} 1_{\{t_k^n \leq t\}} \frac{\alpha_j T}{n} N_j(t_{k+1}^n), \quad t \in [0, T],$$

to be the lower and upper Darboux approximations for the integrals $\alpha_j \int_0^{\lfloor nt/T \rfloor (T/n)} N_j(x)dx$ and
$\alpha_j \int_0^{\lfloor nt/T+1 \rfloor (T/n)} N_j(x)dx$, respectively. Because $N_j(\cdot)$ is right continuous, it is Riemann integrable on the interval $[0, T]$. It follows that $f_{j,-}^n(\cdot), f_{j,+}^n(\cdot) \to \alpha_j \int_0^{\cdot} N_j(x)dx$ uniformly on $[0, T]$ as $n \to \infty$. Therefore, there is some $M_1 > 0$, depending on ϵ, such that for each $n > M_1$, $t \in [0, T]$,

$$f_{j,+}^n(t) - \frac{\epsilon}{2} \leq \alpha_j \int_0^t N_j(x)dx \leq f_-^n(t) + \frac{\epsilon}{2}$$

for all $t \in [0, T]$. Fix such an $n > M_1$. Also, for $t \in [0, T]$, $m \in \mathbb{N}$,

$$\tilde{X}_{j,-}^{n,m}(t) \leq \bar{X}_j^m(t) \leq \tilde{X}_{j,+}^{n,m}(t), \tag{8.10}$$

where

$$\tilde{X}^{n,m}_{j,-}(t) = \sum_{k=0}^{n-1} 1_{\{t^n_{k+1}\le t\}} \frac{1}{m} \sum_{i=m\bar{A}^m_j(t-t^n_{k+1})+1}^{m\bar{A}^m_j(t-t^n_k)} 1_{\{\ell^j_i \le t^n_k\}}, \quad t\in[0,T], \tag{8.11}$$

$$\tilde{X}^{n,m}_{j,+}(t) = \sum_{k=0}^{n-1} 1_{\{t^n_{k}\le t\}} \frac{1}{m} \sum_{i=m\bar{A}^m_j((t-t^n_{k+1})^+)+1}^{m\bar{A}^m_j(t-t^n_k)} 1_{\{\ell^j_i \le t^n_{k+1}\}}, \quad t\in[0,T]. \tag{8.12}$$

In (8.10), the term on the left hand side covers all but an initial segment of $[0,t]$ with disjoint intervals of the form $(t-t^n_{k+1}, t-t^n_k]$, and counts the number of arrivals in such an interval which have patience times $\ell^j_i \le t^n_k$, which implies $\ell^j_i \le t - U^{j,m}_i/m$ since the associated $U^{j,m}_i/m \le t - t^n_k$. This yields the first inequality in (8.10). The second inequality has one more interval and uses $\ell^j_i \le t - U^{j,m}_i/m$ implies $\ell^j_i \le t^n_{k+1}$. Applying the functional weak law of large numbers for renewal processes (see e.g., Lemma A.2. in [11], with the function $g(x)=1$ for each $x\ge 0$) we see that $\bar{A}^m_j(\cdot)\to\alpha_j(\cdot)$ in probability, uniformly on $[0,T]$, as $m\to\infty$. We note that in [11] only convergence in distribution is proved, but when the limit is deterministic and continuous one can get convergence in probability uniformly on compact sets. Here, $\alpha_j(t)=\alpha_j t$ for $t\ge 0$. It follows that there exists M_2 such that for each $m>M_2$,

$$P\left\{\sup_{t\in[0,T]}\left|\bar{A}^m_j(t)-\alpha_j(t)\right|<\delta\right\} > 1-\frac{\eta}{4},$$

where $\delta = \min_{0\le k\le n}\left\{\frac{\epsilon}{4(N(t^n_k)+1)n}\right\}$. Next, note that by the functional weak law of large numbers again (see, e.g., the beginning of the proof of Theorem 14.6 in [5]), for each $0\le k\le n$, $\frac{1}{m}\sum_{i=1}^{\lfloor m\cdot\rfloor} 1_{\{\ell^j_i\le t^n_k\}} \to N(t^n_k)(\cdot)$, where $N(t^n_k)(t)=N(t^n_k)t$, $t\ge 0$, and the convergence is in probability, uniformly on compact time intervals. Hence, there exists M_3 such that for each $m>M_3$,

$$P\left\{\sup_{0\le k\le n}\ \sup_{0\le s\le t\le \alpha_j T+\delta}\left|\frac{1}{m}\sum_{i=\lfloor ms\rfloor+1}^{\lfloor mt\rfloor} 1_{\{\ell^j_i\le t^n_k\}} - N(t^n_k)(t-s)\right| < \frac{\epsilon}{4n}\right\} > 1-\frac{\eta}{4}.$$

It follows that for $m > \max\{M_2, M_3\}, t \in [0,T]$, and the δ we have chosen, with probability greater than $1 - \frac{\eta}{2}$,

$$\begin{aligned} f_{j,-}^n(t) - \frac{\epsilon}{2} &\le \sum_{k=0}^{n-1} 1_{\{t_{k+1}^n \le t\}} N_j(t_k^n) \left(\alpha_j \frac{T}{n} - 2\delta \right) - \frac{\epsilon}{4} \\ &\le \sum_{k=0}^{n-1} 1_{\{t_{k+1}^n \le t\}} \frac{1}{m} \sum_{i=\lfloor m(\alpha_j(t-t_{k+1}^n)+\delta)\rfloor+1}^{\lfloor m(\alpha_j(t-t_k^n)-\delta)\rfloor} 1_{\{\ell_i^j \le t_k^n\}} \le \tilde{X}_{j,-}^{n,m}(t). \end{aligned}$$

Similar arguments for (8.12) yield that for sufficiently large m and $t \in [0,T]$, with probability greater than $1 - \frac{\eta}{2}$,

$$\begin{aligned} \tilde{X}_{j,+}^{n,m}(t) &\le \sum_{k=0}^{n-1} 1_{\{t_k^n \le t\}} \frac{1}{m} \sum_{i=\lfloor m(\alpha_j(t-t_{k+1}^n)-\delta)^+\rfloor+1}^{\lfloor m(\alpha_j(t-t_k^n)+\delta)\rfloor} 1_{\{\ell_i^j \le t_{k+1}^n\}} \\ &\le \sum_{k=0}^{n-1} 1_{\{t_k^n \le t\}} N_j(t_{k+1}^n) \left(\alpha_j \frac{T}{n} + 2\delta \right) + \frac{\epsilon}{4} \\ &\le f_{j,+}^n(t) + \frac{\epsilon}{2}. \end{aligned}$$

Therefore, for sufficiently large m, with probability at least $1 - \eta$,

$$\alpha_j \int_0^t N_j(x)dx - \epsilon \le f_{j,-}^n(t) - \frac{\epsilon}{2} \le \tilde{X}_{j,-}^{n,m}(t) \le \bar{X}_j^m(t) \le \tilde{X}_{j,+}^{n,m}(t) \le f_{j,+}^n(t) + \frac{\epsilon}{2} \le \alpha_j \int_0^t N_j(x)dx + \epsilon,$$

for each $t \in [0,T]$. The desired result follows. □

Next, we prove C-tightness of a sequence of fluid scaled counting processes that count the number of service entries from the queues.

Lemma 8.2.3. *Let $\{\bar{\mathcal{S}}^m(\cdot)\}_{m=1}^\infty$ be the sequence of fluid-scaled processes that tracks the number of service entries from the queues up until time $t \ge 0$ as described in* (2.7) *and*

(3.4). *In particular,*

$$\bar{\mathcal{S}}^m(t) := \bar{S}^m(t) - \sum_{j=1}^{J} \int_0^t 1_{\{\bar{s}^{k,m}(r-)=0 \text{ for some } k \in \mathcal{K}\}} d\bar{A}_j^m(r).$$

Then $\{\bar{\mathcal{S}}^m(\cdot)\}_{m=1}^{\infty}$ *is C-tight. Furthermore, for each* $\eta, T > 0, \epsilon \in (0,T]$, *there exists* $M > 0$ *such that for each* $m > M$,

$$P\left\{\bar{\mathcal{S}}^m(\cdot) \in A^{\epsilon}_{2K\epsilon\check{\mathfrak{u}},T}\right\} > 1 - \eta,$$

where $A^{\epsilon}_{2K\epsilon\check{\mathfrak{u}},T}$ *is as defined in* (8.8), *with* δ, ϵ *replaced by* $\epsilon, 2\epsilon K\check{\mathfrak{u}}$, *respectively.*

Proof. Let $\eta, T > 0, \epsilon \in (0,T]$. Because $\bar{\mathcal{S}}^m(\cdot) \leq \sum_{j=1}^{J}(\bar{Z}_j^m(0) + \bar{A}_j^m(\cdot))$, and we have Assumption 1 (i) and (vi), it satisfies the usual conditions for compact containment (see, e.g., Theorem 3.21 in [12]). Now, we turn to the controlled oscillations condition for tightness. For ease of notation, for each $i, m \in \mathbb{N}$ we define the random variable $j_i^m \in \mathcal{J}$ to be j if the ith service entry from the queues in the mth fluid scaled system is of class j. Now, observe that for $t \geq 0, h > 0$, in the interval of time $(t, t+h]$ the K servers can provide a total of at most Kh units of service time. It follows that the total amount of service time assigned to the jobs that enter service from the queues during the interval $(t, t+h]$ minus the total remaining service time of the jobs in service at time $t+h$ must be less than or equal Kh. Writing this out formally, we have

$$\frac{1}{m} \sum_{i=m\bar{\mathcal{S}}^m(t)+1}^{m\bar{\mathcal{S}}^m(t+h)} v_i^{j_i^m, m} - \frac{1}{m} \sum_{k=1}^{K} s^{k,m}(m(t+h)) \leq Kh. \tag{8.13}$$

For the error term above, we let $\epsilon^m(s) := \frac{1}{m}\sum_{k=1}^{K} s^{k,m}(m(s))$ for $s \geq 0$. Now, we bound the first sum in (8.13) from below using an array of i.i.d random variables. For each m, define the sequence of i.i.d. random variables

$$\tilde{v}_i^m := \min_{1 \le j \le J} v_i^{j,m}, \qquad i = 1, 2, \ldots. \tag{8.14}$$

Because each $v_i^{j_i^m,m} \ge \tilde{v}_i^m$, we obtain the inequality

$$\frac{1}{m} \sum_{i=m\bar{\mathcal{S}}^m(t)+1}^{m\bar{\mathcal{S}}^m(t+h)} \tilde{v}_i^m - \epsilon^m(t+h) \le Kh. \tag{8.15}$$

Because $\bar{\mathcal{S}}^m(\cdot)$ and $\bar{A}_j^m(\cdot), j \in \mathcal{J}$, satisfy a compact containment condition, there exists $N \in \mathbb{N}$, such that for all $m \in \mathbb{N}$,

$$P\left\{ \sup_{t\in[0,T]} \max\{\bar{A}_1^m(t), \ldots, \bar{A}_J^m(t), \bar{\mathcal{S}}^m(t)\} \le N \right\} \ge 1 - \frac{\eta}{3}. \tag{8.16}$$

Next, we show that the error term $\epsilon^m(\cdot) \to 0$ as $m \to \infty$ in probability, uniformly on $[0, T]$. Because $s^{k,m}(t)$ is the remaining service time of a job in service at time t, we have

$$\sup_{t\in[0,T]} \epsilon^m(t) \le \frac{K}{m} \left(\max_{1\le j\le 2J} \max_{1\le i\le m\bar{\mathcal{S}}^m(T)\vee m\bar{A}_1^m(T)\ldots\vee m\bar{A}_J^m(T)} v_i^{j,m} \vee \max_{1\le k\le K} s_0^{k,m} \right). \tag{8.17}$$

Therefore, it suffices to show that the right member of (8.17) converges to 0 in probability as $m \to \infty$. We will partially bound this using processes of the form

$$\bar{V}_j^m(t) := \frac{1}{m} \sum_{i=1}^{\lfloor mt \rfloor} v_i^{j,m}, \qquad t \ge 0, j = 1, \ldots, 2J. \tag{8.18}$$

By Assumption 1, $\bar{V}_j^m(\cdot), 1 \le j \le 2J$, converge in probability to continuous deterministic limit processes of the form $V_j(\cdot) = E[v_i^j](\cdot)$, where $E[v_i^j](t) = E[v_i^j]t$ for each $t \ge 0$ (see Lemma A.2 of [11]). This convergence implies that the jumps of the process $\bar{V}_j^m(\cdot)$ on

$[0,N]$ must converge uniformly to 0 in probability as $m \to \infty$, that is, for each $\delta > 0$,

$$P\left\{\max_{1\le j\le 2J}\max_{1\le i\le mN} v_i^{j,m}/m \ge \delta\right\} \to 0$$

as $m \to \infty$. Using (8.16), (8.17), and Assumption 1 (iv), we conclude that for each $\delta > 0$

$$\begin{aligned}
P\left\{\sup_{t\in[0,T]} \epsilon^m(t) \ge \delta\right\} &\le P\left\{\frac{K}{m}\left(\max_{1\le j\le 2J}\max_{1\le i\le m\bar{S}^m(T)\vee m\bar{A}_1^m(T)\ldots\vee m\bar{A}_J^m(T)} v_i^{j,m}\right) \ge \delta\right\} \\
&\quad + P\left\{\frac{K}{m}\max_{1\le k\le K} s_0^{k,m} \ge \delta\right\} \\
&\le P\left\{K\left(\max_{1\le j\le 2J}\max_{1\le i\le mN} v_i^{j,m}/m\right) \ge \delta\right\} \\
&\quad + P\left\{\frac{K}{m}\max_{1\le k\le K} s_0^{k,m} \ge \delta\right\} + \frac{\eta}{3} \\
&\le \eta
\end{aligned}$$

for sufficiently large m. Hence, $\{\epsilon^m(\cdot)\}_{m=1}^{\infty}$ converges in probability to the zero process, uniformly on $[0,T]$. By the functional weak law of large numbers (see Lemma A.2 of [11]), we see that there exists $M_1 > 0$ such that for all $m > M_1$,

$$P\left\{\sup_{0\le s\le t\le N}\left|\frac{1}{m}\sum_{i=\lfloor ms\rfloor+1}^{\lfloor mt\rfloor} \tilde{v}_i^m - E[\tilde{v}_i^m](t-s)\right| \le \frac{K\epsilon}{3}\right\} \ge 1 - \frac{\eta}{3}. \tag{8.19}$$

We conclude, using (8.16), that for all $m > M_1$,

$$P\left\{\sup_{t\in[0,T-\epsilon]}\sup_{h\in[0,\epsilon]}\left|\frac{1}{m}\sum_{i=\lfloor m\bar{S}^m(t)\rfloor+1}^{\lfloor m\bar{S}^m(t+h)\rfloor} \tilde{v}_i^m - E[\tilde{v}_i^m](\bar{S}^m(t+h)-\bar{S}^m(t))\right| \le \frac{K\epsilon}{3}\right\} \ge 1 - \frac{2\eta}{3}. \tag{8.20}$$

Applying (8.15), we conclude that for all $m > M_1$

$$P\left\{\sup_{t\in[0,T-\epsilon]}\sup_{h\in[0,\epsilon]} E[\tilde{v}_i^m](\bar{S}^m(t+h)-\bar{S}^m(t)) \le \frac{4}{3}K\epsilon + \sup_{t\in[0,T]}\epsilon^m(t)\right\} \ge 1 - \frac{2\eta}{3}. \tag{8.21}$$

Using the fact that $\sup_{t\in[0,T]} \epsilon^m(t) \to 0$ in probability as $m \to \infty$, we conclude that there exists $M_2 > 0$ such that for each $m > M_2$,

$$P\left\{ \sup_{t\in[0,T-\epsilon]} \sup_{h\in[0,\epsilon]} \left(\bar{\mathcal{S}}^m(t+h) - \bar{\mathcal{S}}^m(t)\right) \leq \frac{2K\epsilon}{E[\tilde{v}_1^m]} \right\} \geq 1-\eta. \tag{8.22}$$

We note that by Assumption 1 (iii), $\frac{1}{E[\tilde{v}_1^m]} \leq \check{\mathfrak{u}}$ for all $m \in \mathbb{N}$, and thus the result follows. $\square$

Now that we have examined these individual processes, we are equipped to prove Lemma 8.2.1.

Proof of Lemma 8.2.1. Fix $f \in \mathbf{C}_b^1(\mathbb{R}_+), T > 0, j \in \mathcal{J}$, $\epsilon > 0$, and $0 < \eta < 1$. To make the following calculations more concise, we define $\bar{A}_j^m(t,h) := |\bar{A}_j^m(t+h) - \bar{A}_j^m(t)|$, $\bar{\mathcal{S}}_j^m(t,h) := |\bar{\mathcal{S}}_j^m(t+h) - \bar{\mathcal{S}}_j^m(t)|$, and $\bar{X}_j^m(t,h) := |\bar{X}_j^m(t+h) - \bar{X}_j^m(t)|$ for $t \in [0,T]$ and $h \in [0, T-t]$. Then, applying (7.1), (7.2) and (8.9), for $t \in [0,T]$ and $h \in [0,T-t]$, we have

$$\begin{aligned} |\langle f, \bar{\mathcal{Z}}_j^m(t+h)\rangle - \langle f, \bar{\mathcal{Z}}_j^m(t)\rangle| &\leq ||f'||_\infty h \bar{Z}_j^m(t) \\ &\quad + ||f||_\infty (\bar{\mathcal{Z}}_j^m(0)((t,t+h]) + \bar{X}_j^m(t,h)) \\ &\quad + ||f||_\infty (\bar{A}_j^m(t,h) + \bar{\mathcal{S}}^m(t,h)) \\ &\leq ||f'||_\infty h (\bar{Z}_j^m(0) + \bar{A}_j^m(t)) \\ &\quad + ||f||_\infty (\bar{\mathcal{Z}}_j^m(0)((t,t+h]) + \bar{X}_j^m(t,h)) \\ &\quad + ||f||_\infty (\bar{A}_j^m(t,h) + \bar{\mathcal{S}}^m(t,h)) \end{aligned} \tag{8.23}$$

using the fact that $\bar{Z}_j^m(t) \leq \bar{Z}_j^m(0) + \bar{A}_j^m(t)$. Now, $\{\bar{X}_j^m(\cdot)\}_{m=1}^\infty$, $j \in \mathcal{J}$, and $\{\bar{\mathcal{S}}^m(\cdot)\}_{m=1}^\infty$ are C-tight by Lemmas 8.2.2 and 8.2.3, and $\{\bar{A}_j^m(\cdot)\}_{m=1}^\infty$ is C-tight by the functional law of large numbers for renewal processes (see, e.g., Lemma A.2 in [11]). Then it follows (see, e.g., Theorem 7.3 in [5]) that

$$\lim_{\delta\to 0}\liminf_{m\to\infty} P\left\{\sup_{t\in[0,T-\delta]}\sup_{h\in[0,\delta]}\left|\bar{X}^m(t,h)+\bar{\mathcal{S}}^m(t,h)+\bar{A}^m_j(t,h)\right|\le\frac{\epsilon}{3(||f||_\infty+1)}\right\}=1. \tag{8.24}$$

Next, note that by the Portmanteau theorem, $\bar{\mathcal{Z}}^m_j(0)\Rightarrow\bar{\mathcal{Z}}_j(0)$ implies that for each $\delta>0$, $\liminf_{m\to\infty} P(d(\bar{\mathcal{Z}}^m_j(0),\bar{\mathcal{Z}}_j(0))<\delta)=1$. Therefore, applying the definition of the metric on $\mathbf{M}$, (1.1), we conclude that for each $\delta>0$,

$$\liminf_{m\to\infty} P\{\bar{\mathcal{Z}}^m_j(0)((t,t+\delta])\le\bar{\mathcal{Z}}_j(0)(((t-\delta)^+,t+2\delta])+2\delta \text{ for each } t\ge 0\}=1. \tag{8.25}$$

Because $\bar{\mathcal{Z}}_j(0)$ has no atoms, for any $a>0$,

$$\lim_{\delta\to 0} P\left\{\sup_{x\in\mathbb{R}_+}\langle 1_{[x,x+\delta]},\bar{\mathcal{Z}}_j(0)\rangle<a\right\}=1. \tag{8.26}$$

(For a proof of the equivalence of this with the no atoms condition, see Lemma A.1. in [11].) Combining (8.25) and (8.26), we see that

$$\lim_{\delta\to 0}\liminf_{m\to\infty} P\left\{\sup_{t\ge 0}\sup_{h\in[0,\delta]}\bar{\mathcal{Z}}^m_j(0)((t,t+h])\le\frac{\epsilon}{3(||f||_\infty+1)}\right\}=1 \tag{8.27}$$

Because $\{\bar{Z}^m_j(0)+\bar{A}^m_j(\cdot)\}_{m=1}^\infty$ is a tight family of processes, it follows that

$$\lim_{\delta\to 0}\liminf_{m\to\infty} P\left\{\sup_{t\in[0,T-\delta]}\sup_{h\in[0,\delta]}||f'||_\infty h(\bar{Z}^m_j(0)+\bar{A}^m_j(t))\le\frac{\epsilon}{3}\right\}=1 \tag{8.28}$$

Combining (8.23), (8.24), (8.27), and (8.28), we conclude that

$$\lim_{\delta\to 0}\liminf_{m\to\infty} P\left\{\langle f,\bar{\mathcal{Z}}^m_j(\cdot)\rangle\in A^\delta_{\epsilon,T}\right\}=1.$$

This is equivalent to the first part of Lemma 8.2.1 (see, e.g., Theorem 7.3 in [5]).

For the Lipschitz continuity, we observe that if we restrict to $f \in \mathscr{C}$, (7.4) holds, and then

$$|\langle f, \bar{\mathcal{Z}}_j^m(t+h)\rangle - \langle f, \bar{\mathcal{Z}}_j^m(t)\rangle| \leq h||f'||_\infty \sup_{0\leq t\leq T} \bar{Z}_j^m(t) + ||f||_\infty(\bar{A}_j^m(t,h) + \bar{\mathcal{S}}^m(t,h)) \quad (8.29)$$

can be used in place of (8.23). Applying Lemma 8.2.3 and the convergence we have established in the proof of Lemma 8.2.2 for $\{\bar{A}_j^m(\cdot)\}_{m=1}^\infty$ to $\alpha_j(\cdot)$, we see that for each $\epsilon \in (0,T]$,

$$\liminf_{m\to\infty} P\left\{\sup_{t\in[0,T-\epsilon]}\sup_{h\in[0,\epsilon]}(\bar{A}_j^m(t,h) + \bar{\mathcal{S}}^m(t,h)) \leq 2\epsilon(\alpha_j + K\check{\mathfrak{u}})\right\} = 1.$$

Using the fact that $\sup_{t\in[0,T]} \bar{Z}_j^m(t) \leq \bar{Z}_j^m(0) + \bar{A}_j^m(T)$ and the convergence $\bar{Z}_j^m(0) + \bar{A}_j^m(T) \to \bar{Z}_{0,j} + \alpha_j T$ in in distribution as $m \to \infty$, we conclude that $\liminf_{m\to\infty} P\{\bar{Z}_j^m(0) + \bar{A}_j^m(T) < 2(\bar{Z}_{0,j} + \alpha_j T)\} = 1$.

Applying (8.29), we conclude that

$$\liminf_{m\to\infty} P\left\{\sup_{t\in[0,T-\epsilon]}\sup_{h\in[0,\epsilon]}|\langle f, \bar{\mathcal{Z}}_j^m(t+h)\rangle - \langle f, \bar{\mathcal{Z}}_j^m(t)\rangle| \leq 2\epsilon\left((\bar{Z}_{j,0} + \alpha_j T)||f'||_\infty + \alpha_j||f||_\infty + 2K\check{\mathfrak{u}}||f||_\infty\right)\right\} = 1.$$

One can show the desired result for almost sure Lipschitz continuity of a limit $\bar{\mathcal{Z}}_j(\cdot)$ on $[0,T)$ using this fact. □

8.3 Proof of C-Tightness

Proof of the first part of Theorem 5.2.1. By Lemma 8.2.1, it suffices to prove tightness of $\{\bar{\boldsymbol{\mathcal{Z}}}^m(\cdot)\}_{m=1}^\infty$. By Jakubowski's Criterion (See, e.g., Theorem 3.1 of [13]), it suffices to prove:

(i) For each $j \in \mathcal{J}$, $\delta > 0$, $T > 0$ there exists a compact set $C_{\delta,T}$ in $\mathbf{M}$ such that

$$\inf_{m\in\mathbb{N}} P\{\bar{\mathcal{Z}}_j^m(t) \in C_{\delta,T} \ \ \forall t \in [0,T]\} \geq 1-\delta, \text{ and}$$

(ii) For each $f \in \mathbf{C}_b^1(\mathbb{R}_+)$, $j \in \mathcal{J}$, $\{\langle f, \bar{\mathcal{Z}}_j^m(\cdot)\rangle\}_{m=1}^\infty$ is tight.

We show (i) by bounding the first moment and total mass. Using (7.4), the Monotone Convergence Theorem, and a standard truncation argument, we obtain for $T > 0$, $t \in [0,T]$,

$$\begin{aligned}\langle \chi, \bar{\mathcal{Z}}_j^m(t)\rangle &= \langle \chi, \bar{\mathcal{Z}}_j^m(0)\rangle - \int_0^t \langle 1, \bar{\mathcal{Z}}_j^m(s)\rangle ds + \frac{1}{m}\sum_{i=1}^{m\bar{A}_j^m(t)} 1_{\{\bar{s}^{k,m}(U_i^{j,m}/m-)\neq 0 \quad \forall k\in\mathcal{K}\}}\ell_i^j \\ &\quad - \sum_{\bar{\eta}_l^m\in(0,t]}\sum_{i=1}^{m\bar{Z}_j^m(\bar{\eta}_l^m-)} 1_{\left\{\kappa_l\in I_{j,i}(\bar{\boldsymbol{Z}}(\bar{\eta}_l^m-))\right\}}\operatorname{supp}(\bar{\mathcal{Z}}_j^m(\bar{\eta}_l^m-))_{\{i\}} \\ &\leq \langle \chi, \bar{\mathcal{Z}}_j^m(0)\rangle + \frac{1}{m}\sum_{i=1}^{m\bar{A}_j^m(t)} \ell_i^j.\end{aligned}$$

By Lemma A.2 of [11], $\frac{1}{m}\sum_{i=1}^{m\bar{A}_j^m(\cdot)} \ell_i^j$ converges in distribution to $\alpha_j E[\vartheta_j](\cdot)$, where $\alpha_j E[\vartheta_j](t) = \alpha_j E[\vartheta_j]t$ for each $t \geq 0$. Because $\{\langle \chi, \bar{\mathcal{Z}}_j^m(0)\rangle\}_{m=1}^\infty$ is tight by Assumption 1 (vi), this gives compact containment of $\{\langle \chi, \bar{\mathcal{Z}}_j^m(\cdot)\rangle\}_{m\in\mathbb{N}}$ on $[0,T]$. Furthermore, because $\bar{Z}_j^m(\cdot) \leq \bar{Z}_j^m(0) + \bar{A}_j^m(\cdot)$, we also have compact containment for $\{\bar{Z}_j^m(\cdot)\}_{m=1}^\infty$ on $[0,T]$ because $\{\bar{Z}_j^m(0)\}_{m=1}^\infty$ and $\{\bar{A}_j^m(\cdot)\}_{m=1}^\infty$ are tight. In particular, this implies that for each $\delta > 0$, there is $M_\delta > 0$ such that

$$\inf_{m\in\mathbb{N}} P\left\{\sup_{0\leq t\leq T}\langle \chi, \bar{\mathcal{Z}}_j^m(t)\rangle \vee \sup_{0\leq t\leq T} \bar{Z}_j^m(t) \leq M_\delta\right\} \geq 1-\delta. \tag{8.30}$$

Setting $C_{\delta,T} := \{\sigma \in \mathbf{M} : \langle \chi, \sigma\rangle, \langle 1, \sigma\rangle \leq M_\delta\}$, we have (i). For (ii), note that for

$f \in \boldsymbol{C}_b^1(\mathbb{R}_+)$

$$\langle f, \bar{\mathcal{Z}}_j^m(t)\rangle \leq ||f||_\infty \bar{Z}_j^m(t), \quad t \in [0,T].$$

Using (8.30), this implies compact containment for $\langle f, \bar{\mathcal{Z}}_j^m(\cdot)\rangle$. For controlled oscillations, we note that by Lemma 8.2.1, for each $T > 0, \epsilon > 0, \eta > 0$, there exists $M, \delta > 0$ such that for $m > M$, $P\{\langle f, \bar{\mathcal{Z}}_j^m(\cdot)\rangle \in A_{\epsilon,T}^\delta\} > 1 - \eta$. For each $N > 0$, all paths $x(\cdot) \in A_{\epsilon,N}^\delta$ have modulus of continuity $w_N'(x, \delta) \leq \epsilon$, where the modulus of continuity w_N' is as defined in Chapter 3 of [12]. Therefore, $\{\langle f, \bar{\mathcal{Z}}_j^m(\cdot)\rangle\}_{m=1}^\infty$ satisfies the controlled oscillations condition given in Proposition 3.26 of [12].

□

Chapter 9

Fluid Limit Properties

In this chapter, we prove the second part of Theorem 5.2.1, namely that a subsequential limit in distribution of $\{\bar{\boldsymbol{\mathcal{Z}}}^m(\cdot)\}_{m=1}^{\infty}$ is almost surely a fluid model solution. First we give a brief outline of the proof. By Lemma 8.2.1, for each $j \in \mathcal{J}$ and $f \in \mathscr{C}$, $\langle f, \bar{\mathcal{Z}}_j(\cdot)\rangle$ is almost surely Lipschitz continuous and hence absolutely continuous (as a function of time). It follows that, almost surely, $\langle f, \bar{\mathcal{Z}}_j(\cdot)\rangle$ can be recovered from its almost everywhere defined time derivative and its initial value. Note that, by considering left and right derivatives, a nonnegative, absolutely continuous function $x : \mathbb{R}_+ \to \mathbb{R}$ has a derivative of zero at any time $t > 0$ where it is differentiable and takes the value zero. Because $\bar{\mathcal{L}}(t) = 0$ implies that $\bar{Z}_j(t) = 0$ for each j, this implies that, almost surely, for each $j \in \mathcal{J}$, the time derivative of $\langle f, \bar{\mathcal{Z}}_j(\cdot)\rangle$ is zero at almost every $t > 0$ such that $\langle f, \bar{\mathcal{Z}}_j(\cdot)\rangle$ is differentiable at t and $\bar{\mathcal{L}}(t) = 0$ (see (9.28)). Therefore, almost surely, to determine the time derivative of $\langle f, \bar{\mathcal{Z}}_j(\cdot)\rangle$ wherever it is defined on $(0, \infty)$, we need only determine the derivative of $\langle f, \bar{\mathcal{Z}}_j(\cdot)\rangle$ on the open intervals where $\bar{\mathcal{L}}(\cdot) \neq 0$. Finding the derivative on those intervals is the main part of the proof.

Chapter 9 will be divided into three sections. In §9.1, we will take the limits in distribution of terms in (8.1) for which it will be helpful to the proof to be on the original probability space. Then, in §9.2, we will take a Skorokhod representation in order to use almost sure convergence to find the limits of the remaining terms. In §9.3, we will use the limits of all of the terms in (8.1) in order to show that (4.5) holds almost surely with $\zeta_j(\cdot) = \bar{\mathcal{Z}}_j(\cdot)$.

9.1 Limits of Some Terms in (8.1) on the Original Probability Space

Lemma 9.1.1. *Fix $T > 0$, $f \in \boldsymbol{C}_b^1(\mathbb{R}_+)$. Then $\bar{Y}_\cdot^{j,m}(f), j \in \mathcal{J}, \bar{\mathfrak{Y}}_\cdot^m, \bar{\mathscr{Y}}_\cdot^m, \bar{\Upsilon}_2^m(\cdot) \to 0$ in probability, uniformly on $[0,T]$, as $m \to \infty$.*

Proof. Because $\Upsilon_2^m(\cdot)$ is extremely similar to the error term $\epsilon^m(\cdot)$ from Lemma 8.2.3, it follows from a similar argument to the argument in the proof of that lemma that shows that $\epsilon^m(\cdot) \to 0$ in probability, uniformly on compact sets, that $\Upsilon_2^m(\cdot)$ does as well. We note that for each m, j, $\bar{Y}_\cdot^{j,m}(f), \bar{\mathfrak{Y}}_\cdot^m, \bar{\mathscr{Y}}_\cdot^m$, as defined in (8.5), (8.6), and (unscaled in) (7.24), are square-integrable martingales with respect to the filtration $\{\mathscr{F}^m_{\bar{\mathcal{S}}^m(t)}, t \geq 0\}$ by Lemmas 7.2.1 and 7.2.2. For these martingale terms, we will use Doob's inequality, similar to what was done in [2]. By scaling and (7.35), the quadratic variation of $\bar{Y}_\cdot^{j,m}$ is given by

$$\begin{aligned} \langle \bar{Y}^{j,m}(f) \rangle_t &= \frac{1}{m^2} \left\langle Y^{j,m}\left(f\left(\frac{1}{m}\cdot\right)\right)\right\rangle_{mt} \\ &\leq \frac{4}{m} ||f||_\infty^2 \bar{\mathcal{S}}^m(t) \\ &\leq \frac{4}{m} ||f||_\infty^2 \bar{\mathcal{C}}^m(t), \end{aligned} \tag{9.1}$$

where, as in the proof of Lemma 7.2.2, $\bar{\mathcal{C}}^m(t) = \sum_{j=1}^J (\bar{Z}_j^m(0) + \bar{A}_j^m(t))$. Using a similar argument for $\bar{\mathfrak{Y}}_\cdot^m$, but instead with the quantities and initial calculations used to obtain (7.39), we obtain

$$\langle \bar{\mathfrak{Y}}^m \rangle_t \leq \frac{2}{m^2} \sum_{i=1}^{m\bar{\mathcal{C}}^m(t)} \left(\left(\max_{j \in \mathcal{J}} v_i^{j,m} \right)^2 + \left(\max_{j \in \mathcal{J}} \frac{1}{\mu_j^m} \right)^2 \right). \tag{9.2}$$

By Doob's inequality, (9.1) and (9.2), and since $\bar{Y}_\cdot^{j,m}(f), \bar{\mathfrak{Y}}_\cdot^m$ are square-integrable martingales, we have for each $\epsilon > 0$,

$$P\left\{\sup_{0\le t\le T}|\bar{Y}_t^{j,m}(f)|\ge\epsilon\right\}\le\frac{E[\langle\bar{Y}^{j,m}(f)\rangle_T]}{\epsilon^2}\le\frac{4(\sup_{m\in\mathbb{N}}E[\bar{\mathcal{C}}^m(T)])||f||_\infty^2}{m\epsilon^2},\qquad(9.3)$$

and

$$\begin{aligned}P\left\{\sup_{0\le t\le T}|\bar{\mathcal{Y}}_t^m|\ge\epsilon\right\}&\le\frac{E[\langle\bar{\mathcal{Y}}^m\rangle_T]}{\epsilon^2}\\&\le\frac{2\sup_{m\in\mathbb{N}}E[\bar{\mathcal{C}}^m(T)]\left(E\left[\left(\max_{j\in\mathcal{J}}v_1^{j,m}\right)^2\right]+\left(\max_{j\in\mathcal{J}}\frac{1}{\mu_j^m}\right)^2\right)}{m\epsilon^2},\end{aligned}$$

where, analogous to what was done in the proof of Lemma 7.2.2, Wald's inequality was used. As in the proof of Lemma 7.2.2, a similar argument will yield the same estimate for $E[\langle\bar{\mathscr{Y}}^m\rangle_t]$ with an additional multiplier of J. Furthermore, because $\bar{\mathcal{C}}^m(t)=\sum_{j=1}^J\bar{Z}_j^m(0)+\bar{A}_j(t)$, it follows from Assumption 1 (i), (vi) that $\sup_{m\in\mathbb{N}}E[\bar{\mathcal{C}}^m(T)]<\infty$. Therefore, it follows from Assumption 1 that $\bar{Y}_\cdot^{j,m}(f),\bar{\mathcal{Y}}_\cdot^m,\bar{\mathscr{Y}}_\cdot^m\to 0$ in probability, uniformly on $[0,T]$, as $m\to\infty$. □

Lemma 9.1.2. *For each $T>0$, $j\in\mathcal{J}$, $\bar{\mathscr{X}}_j^m(\cdot)\to 0$ in probability, uniformly on $[0,T]$, as $m\to\infty$, where the unscaled $\mathscr{X}_j(\cdot)$ is defined in (7.25).*

Proof. Fix $j\in\mathcal{J}$. This argument will be very similar to the one in the proof of Lemma 9.1.1. By Lemma 7.2.4, $\mathscr{X}_j^m(\cdot)=\tilde{\mathscr{X}}_j^m(A_j^m(\cdot))$ almost surely, and this is a martingale with respect to $\{\mathscr{G}^j_{A_j^m(t)}:0\le t<\infty\}$ for each $m\in\mathbb{N}$.

The main step in our proof is to fluid-scale and show that the rescaled martingales go to zero in probability, uniformly on $[0,T]$. This will be as in Lemma 9.1.1, so we will keep the proof brief. We do this by bounding the quadratic variation:

$$\begin{aligned}\langle \bar{\mathscr{X}}_j^m\rangle_t &= \sum_{i=1}^{m\bar{A}_j^m(t)} \left(\frac{1}{m}1_{\{\bar{s}^{k,m}(U_i^{j,m}/m-)=0 \text{ for some } k\in\mathcal{K}\}}\left(v_i^{J+j,m} - \frac{1}{\mu_j^m}\right)\right)^2 \\ &\le \frac{2}{m^2}\sum_{i=1}^{m\bar{A}_j^m(t)}\left((v_i^{J+j,m})^2 + \left(\frac{1}{\mu_j^m}\right)^2\right).\end{aligned}$$

Using a similar argument to that in Lemma 9.1.1 with Doob's inequality for $\bar{\mathcal{Y}}_\cdot$, we obtain the desired result. In particular, we see using Doob's and Wald's inequality that

$$P\left\{\sup_{0\le t\le T}|\bar{\mathscr{X}}_j^m(\cdot)|\ge\epsilon\right\}\le\frac{E[\langle\bar{\mathscr{X}}_j^m\rangle_T]}{\epsilon^2}\le\frac{2\sup_{m\in\mathbb{N}}\left(E[\bar{A}_j^m(T)]\left(E\left[(v_1^{J+j,m})^2\right]+\frac{1}{(\mu_j^m)^2}\right)\right)}{m\epsilon^2}, \tag{9.4}$$

the right hand side of which converges to zero by Assumption (1) (i),(iii), (v), and the fact that $v_1^{J+j,m}$ has the same distribution as $v_1^{j,m}$ so that $\sup_{m\in\mathbb{N}}E[(v_1^{J+j,m})^2]<\infty$. □

For his model, Aghajani [2] has a result similar to the following. While our proof for the derivative of $\bar{S}$ proceeds by also finding upper and lower bounds for an approximate derivative, it is different in details and gives the additional inequality (9.6).

Lemma 9.1.3. *Fix $j\in\mathcal{J}$. Then, $\{(\bar{S}^m(\cdot),\bar{\boldsymbol{\mathcal{Z}}}^m(\cdot))\}_{m=1}^\infty$ is C-tight. Let $(\bar{S}(\cdot),\bar{\boldsymbol{\mathcal{Z}}}(\cdot))$ be a subsequential limit in distribution of $\{(\bar{S}^m(\cdot),\bar{\boldsymbol{\mathcal{Z}}}^m(\cdot))\}_{m=1}^\infty$. Then almost surely, for any $t>0$ such that $\bar{\mathcal{L}}(t)>0$, $\bar{S}(\cdot)$ is differentiable at t and $\frac{d}{dt}\bar{S}(t)=\frac{K\bar{L}(t)}{\bar{\mathcal{L}}(t)}$, where $\bar{L}(t)=\sum_{j=1}^J p_j\bar{Z}_j(t)$ and $\bar{\mathcal{L}}(t)=\sum_{j=1}^J\frac{p_j}{\mu_j}\bar{Z}_j(t)$.*

Proof. We first verify the C-tightness of $\{\bar{S}^m(\cdot)\}_{m=1}^\infty$. By (2.7), for each $m\in\mathbb{N}$, we have

$$\bar{S}^m(\cdot)=\bar{\mathcal{S}}^m(\cdot)+\sum_{j=1}^J\int_0^\cdot 1_{\{\bar{s}^{k,m}(r-)=0 \text{ for some } k\in\mathcal{K}\}}d\bar{A}_j^m(r). \tag{9.5}$$

From Lemma 8.2.3, we have that $\{\bar{\mathcal{S}}^m(\cdot)\}_{m=1}^\infty$ is C-tight. Next, since the increments of the second term on the right side of (9.5) are bounded by the increments of the C-tight sequence of processes $\{\sum_{j=1}^J \bar{A}_j^m(\cdot)\}_{m=1}^\infty$, it follows that the sequence of processes defined by the second term on the right side of (9.5) is C-tight. Hence, $\{\bar{S}^m(\cdot)\}_{m=1}^\infty$, the sum of two sequences of C-tight processes is C-tight. Combining this with the first part of Theorem 5.2.1 (C-tightness of $\{\bar{\boldsymbol{\mathcal{Z}}}^m(\cdot)\}_{m=1}^\infty$), we conclude that $\{(\bar{S}^m(\cdot), \bar{\boldsymbol{\mathcal{Z}}}^m(\cdot))\}_{m=1}^\infty$ is C-tight. Furthermore, by Lemma 9.1.1, we know that $\{\bar{\Upsilon}_2^m(\cdot)\}_{m=1}^\infty$ and $\{\bar{\mathcal{Y}}_\cdot^m\}_{m=1}^\infty$ converge in distribution to processes that are identically zero. Also, by Assumption 1 and Lemma A.2 of [11], we have that for each $j \in \mathcal{J}$, $\{\bar{\mathcal{V}}_j^m(\cdot)\}_{m=1}^\infty$ converges in distribution to the continuous deterministic process $\frac{\alpha_j}{\mu_j}(\cdot)$, where $\frac{\alpha_j}{\mu_j}(t) = \frac{\alpha_j t}{\mu_j}$ for $t \geq 0$. Thus, if $(\bar{S}(\cdot), \bar{\boldsymbol{\mathcal{Z}}}(\cdot))$ is a subsequential limit in distribution of $\{(\bar{S}^m(\cdot), \bar{\boldsymbol{\mathcal{Z}}}^m(\cdot))\}_{m=1}^\infty$, for the properties we want to prove, we may further assume (possibly by passing to a further subsequence) that $\{(\bar{\Upsilon}_2^m(\cdot), \bar{\mathcal{Y}}_\cdot^m, \{\bar{\mathcal{V}}_j^m(\cdot)\}_{j\in\mathcal{J}})\}_{m=1}^\infty$ is converging in distribution jointly with such a subsequence. We shall slightly abuse notation below and use the original index m for the index of such a converging subsequence. Thus for the following we assume that $\{(\bar{S}^m(\cdot), \bar{\boldsymbol{\mathcal{Z}}}^m(\cdot), \bar{\Upsilon}_2^m(\cdot), \bar{\mathcal{Y}}_\cdot^m, \{\bar{\mathcal{V}}_j^m(\cdot)\}_{j\in\mathcal{J}})\}_{m=1}^\infty$ converges in distribution to $(\bar{S}(\cdot), \bar{\boldsymbol{\mathcal{Z}}}(\cdot), \mathbf{0}, \mathbf{0}, \{\frac{\alpha_j}{\mu_j}(\cdot)\}_{j\in\mathcal{J}})$, and we will prove the desired differentiability of $\bar{S}(\cdot)$, as described in the lemma.

The key inequality, which we prove below, is that almost surely, for each $t > 0$ and $\delta \in (-t, \infty)$, with $O_{t,\delta} = (t, t+\delta)$ if $\delta \geq 0$ and $O_{t,\delta} = (t+\delta, t)$ if $\delta < 0$, we have

$$
\begin{aligned}
&K \int_{O_{t,\delta}} 1_{\{\bar{\mathcal{L}}(r)\neq 0\}} dr - \sum_{j=1}^J \frac{\alpha_j}{\mu_j} \int_{O_{t,\delta}} 1_{\{\bar{\mathcal{L}}(r)=0\}} dr - \breve{\mu} \int_{O_{t,\delta}} 1_{\{\bar{\mathcal{L}}(r)=0\}} d\bar{S}(r) \\
&\leq \int_{O_{t,\delta}} 1_{\{\bar{\mathcal{L}}(r)\neq 0\}} \frac{\bar{\mathcal{L}}(r)}{\bar{L}(r)} d\bar{S}(r) \leq K|\delta|, \qquad (9.6)
\end{aligned}
$$

where

$$
\breve{\mu} := \sup_{m\in\mathbb{N}} \max_{j\in\mathcal{J}} \frac{1}{\mu_j^m}. \qquad (9.7)
$$

Note that $\breve{\mu} < \infty$, since for each $j \in \mathcal{J}$, $\mu_j^m > 0$ for each $m \in \mathbb{N}$ and $\mu_j^m \to \mu_j > 0$ as $m \to \infty$. It follows immediately from (9.6) that, almost surely, for each $t > 0$ such that

$\bar{\mathcal{L}}(t) > 0$, for all $\delta > -t$ sufficiently small that $\bar{\mathcal{L}}(\cdot) > 0$ on $O_{t,\delta}$, we have

$$K|\delta| \leq \int_{O_{t,\delta}} \frac{\bar{\mathcal{L}}(r)}{\bar{L}(r)} d\bar{S}(r) \leq K|\delta|. \tag{9.8}$$

Then, by the continuity of $\frac{\bar{\mathcal{L}}(\cdot)}{\bar{L}(\cdot)}$ on such $O_{t,\delta}$, it follows that $\bar{S}(\cdot)$ is differentiable at t with derivative $\frac{d}{dt}\bar{S}(t) = \frac{K\bar{L}(t)}{\bar{\mathcal{L}}(t)}$. Thus, to prove the desired result, we only need to prove (9.6). We now turn to proving this.

By (7.22) with fluid scaling, we see that for each $m \in \mathbb{N}$, $t > 0$ and $\delta \in (-t, \infty)$,

$$\begin{aligned}
&\bar{\mathcal{H}}^m(t+\delta) - \bar{\mathcal{H}}^m(t) \\
&= \bar{\mathscr{T}}^m(t+\delta) - \bar{\mathscr{T}}^m(t) - \bar{\Upsilon}_1^m(t+\delta) + \bar{\Upsilon}_1^m(t) - \bar{\mathcal{Y}}_{t+\delta}^m + \bar{\mathcal{Y}}_t^m - \bar{\Upsilon}_2^m(t+\delta) + \bar{\Upsilon}_2^m(t).
\end{aligned} \tag{9.9}$$

We would like to let $m \to \infty$ in (9.9) to obtain upper and lower bounds, as in (9.6), for increments of

$$\bar{\mathcal{H}}(t) = \int_0^t 1_{\{\bar{\mathcal{L}}(r)\neq 0\}} \frac{\bar{\mathcal{L}}(r)}{\bar{L}(r)} d\bar{S}(r), \quad t \geq 0. \tag{9.10}$$

However, the integrands in the integrals defining $\bar{\mathcal{H}}^m$, $\bar{\mathscr{T}}^m$ and $\bar{\Upsilon}_1^m$ have discontinuities, and so we will approximate them first. For this, let $\{g_n\}_{n=1}^\infty$ be an increasing sequence of continuous functions defined on $\mathbb{R}_+$ and taking values in $[0, 1]$, such that $0 \leq g_n(\cdot) \uparrow 1_{\{\cdot\neq 0\}}$ as $n \to \infty$, $g_n(x) = 0$ for $0 \leq x \leq \frac{1}{2n}$, and $g_n(x) = 1$ for $x \geq \frac{1}{n}$.

Towards proving the second inequality in (9.6), for each $n, m \in \mathbb{N}$, $t \geq 0$, define

$$\bar{\mathcal{H}}_{-,n}^m(t) = \int_0^t g_n(\bar{\mathcal{L}}^m(r-)) \frac{\bar{\mathcal{L}}^m(r-)}{\bar{L}^m(r-)} d\bar{S}^m(r), \qquad \bar{\mathcal{H}}_{-,n}(t) = \int_0^t g_n(\bar{\mathcal{L}}(r)) \frac{\bar{\mathcal{L}}(r)}{\bar{L}(r)} d\bar{S}(r). \tag{9.11}$$

Using the fact that $\bar{\Upsilon}_1^m(\cdot)$ is nondecreasing, we have from (9.9) that for each $n, m \in \mathbb{N}$,

$t > 0$ and $\delta \geq 0$,

$$\bar{\mathcal{H}}^m_{-,n}(t+\delta) - \bar{\mathcal{H}}^m_{-,n}(t) \leq \bar{\mathcal{H}}^m(t+\delta) - \bar{\mathcal{H}}^m(t) \leq K\delta - \bar{\mathcal{Y}}^m_{t+\delta} + \bar{\mathcal{Y}}^m_t - \bar{\Upsilon}^m_2(t+\delta) + \bar{\Upsilon}^m_2(t). \tag{9.12}$$

For each fixed $n \in \mathbb{N}$, $r > 0$, $g_n(\bar{\mathcal{L}}^m(r-))$ is zero whenever $\bar{L}^m(r-) \leq \frac{1}{2n\breve{\mu}}$, since then $\bar{\mathcal{L}}^m(r-) \leq \breve{\mu}\bar{L}^m(r-) \leq \frac{1}{2n}$. Thus, for fixed $n \in \mathbb{N}$, at each $r > 0$, the integrand in the definition (see (9.11)) of $\bar{\mathcal{H}}^m_{-,n}(\cdot)$ is a fixed continuous function applied to $(\bar{\boldsymbol{\mathcal{Z}}}^m(r-), \boldsymbol{\mu}^m)$. It follows that for each $n \in \mathbb{N}$, as $m \to \infty$, the integrand process in $\bar{\mathcal{H}}^m_{-,n}(\cdot)$ converges in distribution to the continuous process $g_n(\bar{\mathcal{L}}(\cdot))\frac{\bar{\mathcal{L}}(\cdot)}{\bar{L}(\cdot)}$, jointly with the convergence in distribution of $\{(\bar{S}^m(\cdot),\, \bar{\boldsymbol{\mathcal{Z}}}^m(\cdot),\, \bar{\Upsilon}^m_2(\cdot),\, \bar{\mathcal{Y}}^m_\cdot,\, \{\bar{\mathcal{V}}^m_j(\cdot)\}_{j\in\mathcal{J}})\}_{m=1}^\infty$ to the continuous process $(\bar{S}(\cdot), \bar{\boldsymbol{\mathcal{Z}}}(\cdot), \mathbf{0}, \mathbf{0}, \{\frac{\alpha_j}{\mu_j}(\cdot)\}_{j\in\mathcal{J}})$. Furthermore, the integrator $\bar{S}^m(\cdot)$ is a nondecreasing process and $\bar{S}(\cdot)$ is a continuous, nondecreasing process. It follows, for example by using the Skorokhod representation theorem and a real analysis argument for integrals where $d\bar{S}^m(\cdot)$ converges weakly to $d\bar{S}(\cdot)$ and the integrands converge uniformly on compact time intervals, that the sequence of integral processes $\{\bar{\mathcal{H}}^m_{-,n}(\cdot)\}_{m=1}^\infty$ converges in distribution to $\bar{\mathcal{H}}_{-,n}(\cdot)$, jointly with the other processes that are converging in distribution. Then, for each $n \in \mathbb{N}$, $t > 0$ and $\delta \geq 0$, on taking distributional limits in the left and right members of (9.12), we obtain almost surely,

$$\bar{\mathcal{H}}_{-,n}(t+\delta) - \bar{\mathcal{H}}_{-,n}(t) \leq K\delta, \tag{9.13}$$

where we have used the fact that $\{(\bar{\mathcal{Y}}^m_\cdot, \bar{\Upsilon}^m_2(\cdot))\}_{m=1}^\infty$ converges to $(\mathbf{0}, \mathbf{0})$. Since the left and right members of (9.13) are continuous functions of $t > 0$ and $\delta \geq 0$, and n ranges over a countable set, it follows that (9.13) holds almost surely for all $n \in N$, $t > 0$ and $\delta \geq 0$, i.e., the exceptional null set on which it may not hold can be chosen independent of $t > 0, \delta \geq 0$ and $n \in \mathbb{N}$. Using monotone convergence, on letting $n \to \infty$ in (9.13), we obtain that almost surely, for all $t > 0$, $\delta \geq 0$,

$$\bar{\mathcal{H}}(t+\delta) - \bar{\mathcal{H}}(t) \leq K\delta. \tag{9.14}$$

For $t > 0$ and $\delta \in (-t, 0)$, by replacing t by $t - |\delta|$ and δ by $|\delta|$ in the above inequality, we obtain that almost surely, for all $t > 0$, $\delta \in (-t, 0)$, $\bar{\mathcal{H}}(t) - \bar{\mathcal{H}}(t+\delta) \leq K|\delta|$. It follows from this and (9.14) that almost surely, the second inequality in (9.6) holds for all $t > 0$ and $\delta \in (-t, \infty)$.

We now prove that the first inequality in (9.6) holds almost surely for all $t > 0$ and $\delta \in (-t, \infty)$. Towards this end, for each $n, m \in \mathbb{N}$, $t \geq 0$, define

$$\bar{\mathscr{T}}_n^m(t) = K\int_0^t g_n(\bar{\mathcal{L}}^m(r-))dr \quad \text{and} \quad \bar{\mathscr{T}}_n(t) = K\int_0^t g_n(\bar{\mathcal{L}}(r))dr, \tag{9.15}$$

$$\bar{\Upsilon}_{1,n}^m(t) = \sum_{j=1}^J \int_0^t (1 - g_n(\bar{\mathcal{L}}^m(r-))d\bar{\mathscr{V}}_j^m(r) \quad \text{and} \quad \bar{\Upsilon}_{1,n}(t) = \sum_{j=1}^J \frac{\alpha_j}{\mu_j}\int_0^t (1 - g_n(\bar{\mathcal{L}}(r))dr, \tag{9.16}$$

and

$$\begin{aligned} \bar{\mathcal{H}}_{+,n}^m(t) &= \bar{\mathcal{H}}_{-,n}^m(t) + \breve{\mu}\int_0^t (1 - g_n(\bar{\mathcal{L}}^m(r-))d\bar{S}^m(r), \\ \bar{\mathcal{H}}_{+,n}(t) &= \bar{\mathcal{H}}_{-,n}(t) + \breve{\mu}\int_0^t (1 - g_n(\bar{\mathcal{L}}(r))d\bar{S}(r). \end{aligned} \tag{9.17}$$

We first note that for each $k \in \mathcal{K}$, for each $r > 0$, except possibly for the countable set of r where a service completion occurs (and $\bar{s}^{k,m}(r-) = 0$ for some $k \in \mathcal{K}$), we have for each $n, m \in \mathbb{N}$,

$$g_n(\bar{\mathcal{L}}^m(r-)) \leq 1_{\left\{\bar{\mathcal{L}}^m(r-)\neq 0\right\}} \leq 1_{\left\{\bar{s}^{\tilde{k},m}(r-)\neq 0 \ \ \forall \tilde{k}\in\mathcal{K}\right\}} \leq 1_{\left\{\bar{s}^{k,m}(r-)\neq 0\right\}}. \tag{9.18}$$

The second inequality follows, for $r > 0$ other than the service completion times, from the fact that if there are jobs in any queue at such a time, then none of the servers is idle. It follows from the definition of $\bar{\mathscr{T}}^m(\cdot)$ and (9.18), where the countable set of exceptional r has Lebesgue measure zero, that for each $n, m \in \mathbb{N}$, $t > 0$ and $\delta \geq 0$,

$$\bar{\mathscr{T}}^m(t+\delta) - \bar{\mathscr{T}}^m(t) \geq \bar{\mathscr{T}}_n^m(t+\delta) - \bar{\mathscr{T}}_n^m(t). \tag{9.19}$$

Also, since almost surely, service completions and arrivals do not occur simultaneously, we have that almost surely, (9.18) holds at all r where $\bar{\mathscr{V}}_j^m(\cdot)$ increases, and so from the definition of $\bar{\Upsilon}_1^m(\cdot)$, we have almost surely, for each $n, m \in \mathbb{N}$, $t > 0, \delta \geq 0$,

$$\bar{\Upsilon}_{1,n}^m(t+\delta) - \bar{\Upsilon}_{1,n}^m(t) \geq \bar{\Upsilon}_1^m(t+\delta) - \bar{\Upsilon}_1^m(t). \tag{9.20}$$

Now, using the fact that $\bar{\mathcal{L}}^m(r-) \leq \breve{\mu}\bar{L}^m(r-)$ for any $r > 0$, and (9.9), we have for all $n, m \in \mathbb{N}$, $t > 0$ and $\delta \geq 0$,

$$\begin{aligned}
\bar{\mathcal{H}}_{+,n}^m(t+\delta) - \bar{\mathcal{H}}_{+,n}^m(t) &= \int_t^{t+\delta} \left(g_n(\bar{\mathcal{L}}^m(r-)) \frac{\bar{\mathcal{L}}^m(r-)}{\bar{L}^m(r-)} + (1 - g_n(\bar{\mathcal{L}}^m(r-))\breve{\mu} \right) d\bar{S}^m(r) \\
&\geq \int_t^{t+\delta} 1_{\{\bar{\mathcal{L}}^m(r-)\neq 0\}} \frac{\bar{\mathcal{L}}^m(r-)}{\bar{L}^m(r-)} d\bar{S}^m(r) \\
&= \bar{\mathcal{H}}^m(t+\delta) - \bar{\mathcal{H}}^m(t) \\
&= \bar{\mathscr{T}}^m(t+\delta) - \bar{\mathscr{T}}^m(t) - \bar{\Upsilon}_1^m(t+\delta) + \bar{\Upsilon}_1^m(t) \\
&\quad -\bar{\mathcal{Y}}_{t+\delta}^m + \bar{\mathcal{Y}}_t^m - \bar{\Upsilon}_2^m(t+\delta) + \bar{\Upsilon}_2^m(t).
\end{aligned} \tag{9.21}$$

By (9.19) and (9.20), the expression after the last equals sign in (9.21) is almost surely greater than or equal to the following for all $n, m \in \mathbb{N}$, $t > 0$ and $\delta \geq 0$:

$$\bar{\mathscr{T}}_n^m(t+\delta) - \bar{\mathscr{T}}_n^m(t) - \bar{\Upsilon}_{1,n}^m(t+\delta) + \bar{\Upsilon}_{1,n}^m(t) - \bar{\mathcal{Y}}_{t+\delta}^m + \bar{\mathcal{Y}}_t^m - \bar{\Upsilon}_2^m(t+\delta) + \bar{\Upsilon}_2^m(t). \tag{9.22}$$

In a similar manner to the justification given for $\bar{\mathcal{H}}_{-,n}^m(\cdot)$ converging in distribution as $m \to \infty$, we have that for each $n \in \mathbb{N}$, $t > 0$, $\delta \geq 0$, $\{(\bar{\mathcal{H}}^m_{+,n}(\cdot), \bar{\mathscr{T}}^m_n(\cdot), \bar{\Upsilon}^m_{1,n}(\cdot))\}_{m=1}^\infty$ converges in distribution as $m \to \infty$ to $(\bar{\mathcal{H}}_{+,n}(\cdot), \bar{\mathscr{T}}_n(\cdot), \bar{\Upsilon}_{1,n}(\cdot))$, jointly with the convergence in distribution of

$$\{(\bar{S}^m(\cdot), \bar{\boldsymbol{\mathcal{Z}}}^m(\cdot), \bar{\Upsilon}^m_2(\cdot), \bar{\mathcal{Y}}^m_\cdot, \{\bar{\mathscr{V}}^m_j(\cdot)\}_{j \in \mathcal{J}})\}_{m=1}^\infty$$

to the continuous process

$$\left(\bar{S}(\cdot), \bar{\boldsymbol{\mathcal{Z}}}(\cdot), \mathbf{0}, \mathbf{0}, \frac{\alpha_j}{\mu_j}(\cdot) : j \in \mathcal{J}\right).$$

Thus, on taking distributional limits as $m \to \infty$ in the combination of (9.21) and (9.22), we have for each $n \in \mathbb{N}$, $t > 0$ and $\delta \geq 0$, almost surely,

$$\bar{\mathcal{H}}_{+,n}(t+\delta) - \bar{\mathcal{H}}_{+,n}(t) \geq \bar{\mathscr{T}}_n(t+\delta) - \bar{\mathscr{T}}_n(t) - \bar{\Upsilon}_{1,n}(t+\delta) + \bar{\Upsilon}_{1,n}(t). \tag{9.23}$$

Since the left and right members of (9.23) are continuous functions of $t > 0$ and $\delta \geq 0$, and n ranges over a countable set, it follows that (9.23) holds almost surely for all $n \in N$, $t > 0$ and $\delta \geq 0$, i.e., the exceptional null set on which it may not hold can be chosen independent of $t > 0, \delta \geq 0$ and $n \in \mathbb{N}$. Now, on letting $n \to \infty$ in (9.23), using monotone convergence, we obtain almost surely for each $t > 0$ and $\delta \geq 0$,

$$\begin{aligned} K \int_t^{t+\delta} 1_{\{\bar{\mathcal{L}}(r) \neq 0\}} \frac{\bar{\mathcal{L}}(r)}{\bar{L}(r)} d\bar{S}(r) + \breve{\mu} \int_t^{t+\delta} 1_{\{\bar{\mathcal{L}}(r)=0\}} d\bar{S}(r) \\ \geq K \int_t^{t+\delta} 1_{\{\bar{\mathcal{L}}(r) \neq 0\}} dr - \sum_{j=1}^J \frac{\alpha_j}{\mu_j} \int_t^{t+\delta} 1_{\{\bar{\mathcal{L}}(r)=0\}} dr. \end{aligned} \tag{9.24}$$

This yields the first inequality in (9.6) for $O_{t,\delta} = (t, t+\delta)$ and $t > 0$, $\delta \geq 0$. The result for $\delta \in (-t, 0)$ follows by replacing t by $t - |\delta|$ and δ by $|\delta|$ in the above. This completes the proof that (9.6) holds almost surely, for all $t > 0$, $\delta \in (-t, \infty)$.

□

9.2 Limits of Remaining Terms Under a Skorokhod Representation

For proving Theorem 5.2.1, we may and do assume that we have already passed to a subsequence along which $\{\bar{\boldsymbol{\mathcal{Z}}}^m(\cdot)\}_{m=1}^\infty$ converges in distribution to $\bar{\boldsymbol{\mathcal{Z}}}(\cdot)$. Fix $f \in \mathscr{C}$. A main tool used in this section is to use the Skorokhod representation theorem so that we can give an equivalent distributional representation for a sequence of processes converging in distribution where the convergence is almost sure.

By Assumption 1 and Lemma A.2 of [11], for each $j \in \mathcal{J}$, $\{\bar{A}^{j,m}_\cdot(f)\}_{m=1}^\infty$ (defined in §8.1) converges in distribution to $\alpha_j(\cdot)\langle f, \vartheta_j\rangle$, where $\alpha_j(t) = \alpha_j t$ for each $t \geq 0$. We also have that $\{\bar{A}^m(\cdot)\}_{m=1}^\infty$ converges in distribution to $\alpha_j(\cdot)$ for each $j \in \mathcal{J}$, $\{\bar{\boldsymbol{\mathcal{Z}}}^m(\cdot)\}_{m=1}^\infty$ converges in distribution to $\bar{\boldsymbol{\mathcal{Z}}}(\cdot)$, and by Lemma 9.1.1, $\{\boldsymbol{Y}^m_\cdot(f)\}_{m=1}^\infty$ converges in distribution to the identically zero process. By Lemma 8.2.2, $\{\bar{X}^m_j(\cdot)\}_{m=1}^\infty$ converges in probability, uniformly on compact intervals, to $\alpha_j \int_0^\cdot N_j(x)dx$, for each $j \in \mathcal{J}$, and by Lemma 9.1.3, $\{\bar{S}^m(\cdot)\}_{m=1}^\infty$ is C-tight. For each $j \in \mathcal{J}$, the C-tightness of $\{\bar{\mathcal{Z}}^m_j(\cdot)\}_{m=1}^\infty, \{\bar{A}^{j,m}_\cdot(f)\}_{m=1}^\infty$, $\{\bar{S}^m(\cdot)\}_{m=1}^\infty$ and $\{\bar{X}^m_j(\cdot)\}_{m=1}^\infty$, implies C-tightness of $\{\bar{R}^{j,m}_\cdot(f)\}_{m=1}^\infty$, $\{\bar{S}^m_j(\cdot)\}_{m=1}^\infty$ and $\{\bar{\mathcal{A}}^{j,m}_\cdot(f)\}_{m=1}^\infty$. Lastly, note that for $0 \leq s \leq t < \infty, j \in \mathcal{J}, m \in \mathbb{N}$,

$$\begin{aligned}
|\bar{H}^{j,m}_t(f) - \bar{H}^{j,m}_s(f)| &= \left| \int_s^t 1_{\{\bar{\mathcal{L}}^m(r-)\neq 0\}} \frac{p_j\langle f, \bar{\mathcal{Z}}^m_j(r-)\rangle}{\bar{L}^m(r-)} d\bar{S}^m(r) \right| \\
&\leq \int_s^t \left| 1_{\{\bar{\mathcal{L}}^m(r-)\neq 0\}} \frac{p_j ||f||_\infty \langle 1, \bar{\mathcal{Z}}^m_j(r-)\rangle}{\bar{L}^m(r-)} \right| d\bar{S}^m(r) \\
&\leq ||f||_\infty |\bar{S}^m(t) - \bar{S}^m(s)|.
\end{aligned}$$

It follows that $\{\bar{H}^{j,m}_\cdot(f)\}_{m=1}^\infty$ inherits C-tightness from $\{\bar{S}^m(\cdot)\}_{m=1}^\infty$. Thus, the probability measures induced on $\mathbf{M}^J \times D([0,\infty), \mathbf{M}^J \times \mathbb{R}^{8J+1})$ by the laws of the sequence $\{(\bar{\boldsymbol{\mathcal{Z}}}^m(0)$, $\bar{\boldsymbol{\mathcal{Z}}}^m(\cdot)$, $\bar{\boldsymbol{A}}^m(\cdot)$, $\bar{\boldsymbol{A}}^m_\cdot(f)$, $\{\bar{\mathcal{A}}^{j,m}_\cdot(f)\}_{j\in\mathcal{J}}, \bar{S}^m(\cdot)$, $\{\bar{S}^m_j(\cdot)\}_{j\in\mathcal{J},}$, $\{\bar{R}^{j,m}_\cdot(f)\}_{j\in\mathcal{J}}$, $\{\bar{H}^{j,m}_\cdot(f)\}_{j\in\mathcal{J}}$, $\{\bar{Y}^{j,m}_\cdot(f)\}_{j\in\mathcal{J}}$, $\{\bar{X}^m_j(\cdot)\}_{j\in\mathcal{J}})\}_{m=1}^\infty$ are tight, and so along a subsequence there is convergence in distribution to a limit:

$$(\bar{\boldsymbol{\mathcal{Z}}}(0), \bar{\boldsymbol{\mathcal{Z}}}(\cdot), \bar{\boldsymbol{A}}(\cdot), \bar{\boldsymbol{A}}_{\cdot}(f), \{\bar{\mathcal{A}}^j_{\cdot}(f)\}_{j\in\mathcal{J}}, \bar{S}(\cdot), \{\bar{S}_j(\cdot)\}_{j\in\mathcal{J}}, \{\bar{R}^j_{\cdot}(f)\}_{j\in\mathcal{J}}, \{\bar{H}^j_{\cdot}(f)\}_{j\in\mathcal{J}}, \{\bar{Y}^j_{\cdot}(f)\}_{j\in\mathcal{J}}, \{\bar{X}_j(\cdot)\}_{j\in\mathcal{J}}), \tag{9.25}$$

where $\bar{A}_j(\cdot) = \alpha_j(\cdot), \bar{A}^j_{\cdot}(f) = \alpha_j(\cdot)\langle f, \vartheta_j\rangle$, $\bar{X}_j(\cdot) = \alpha_j \int_0^{\cdot} N_j(x)dx$, $\bar{Y}^j_{\cdot}(f) = 0$ for $j \in \mathcal{J}$, and all the processes in the limit are continuous. To ease the notation, we will index this subsequence with m as well. By the Skorokhod representation theorem, and since the limit processes have continuous paths, there is a sequence that is equal to this convergent sequence in distribution (possibly defined on a different probability space) that converges almost surely where the process convergence is uniform on compact time intervals. Because we are only interested in the distributions of limiting quantities, we continue using that representative sequence for the remainder of this section. For ease of notation, we will continue to denote it with the same notation and its limit by (9.25) as well. We denote by Ω_0 a set of probability one on which the uniform convergence on compact intervals occurs and the properties of the limit as described in Lemma 9.1.3 all hold. In this section, we shall usually fix $\omega \in \Omega_0$ and consider processes for this realization. To ease the notation, we shall suppress the explicit dependence on ω in these manipulations.

Lemma 9.2.1. *Fix $\omega \in \Omega_0, j \in \mathcal{J}, T \geq 0$. Then as $m \to \infty$, for this ω,*

$$\bar{R}^{j,m}_{\cdot}(f) = \int_0^{\cdot} \langle f', \bar{\mathcal{Z}}^m_j(r)\rangle dr \to \int_0^{\cdot} \langle f', \bar{\mathcal{Z}}_j(r)\rangle dr,$$

uniformly on $[0, T]$.

Proof. Because f' is a bounded continuous function and $\bar{\mathcal{Z}}^m_j(\cdot)$ converges uniformly to $\bar{\mathcal{Z}}_j(\cdot)$ on $[0, T]$, uniform convergence of the integral follows. □

Lemma 9.2.2. *Fix $\omega \in \Omega_0$. Let $[u, v]$ be an interval on which $\bar{\mathcal{L}}(\cdot) \neq 0$ for this ω. Then* *for this* ω, *for* $u \le s < t \le v$, $j \in \mathcal{J}$,

$$\bar{H}^{j,m}_t(f) - \bar{H}^{j,m}_s(f) \to \int_s^t \frac{Kp_j\langle f, \bar{\mathcal{Z}}_j(r)\rangle}{\bar{\mathcal{L}}(r)} dr, \qquad \text{as } m \to \infty.$$

Proof. Since $\{\bar{\boldsymbol{\mathcal{Z}}}^m(\cdot)\}_{m=1}^{\infty}$ converges to the continuous $\bar{\boldsymbol{\mathcal{Z}}}(\cdot)$ as $m \to \infty$, and $1, f$ are bounded continuous functions on $\mathbb{R}_+$, $\langle f, \bar{\mathcal{Z}}^m_j(\cdot)\rangle$, $j \in \mathcal{J}$, and $\bar{\mathcal{L}}^m(\cdot)$ converge uniformly on $[u, v]$ to $\langle f, \bar{\mathcal{Z}}_j(\cdot)\rangle, j \in \mathcal{J}$, and $\bar{\mathcal{L}}(\cdot)$, respectively. Because $\bar{\mathcal{L}}(\cdot)$ is nonzero on the compact interval $[u, v]$, it must be uniformly bounded below by some $\epsilon > 0$. Thus, for sufficiently large m, $\bar{\mathcal{L}}^m(\cdot) \neq 0$ on $[u, v]$. This implies that $\bar{H}^{j,m}_t(f) - \bar{H}^{j,m}_s(f) = \int_s^t \frac{p_j\langle f, \bar{\mathcal{Z}}^m_j(r-)\rangle}{\bar{L}^m(r-)} d\bar{S}^m(r)$ for $u \le s < t \le v, j \in \mathcal{J}$, for those m. Using the limit property in Lemma 9.1.3 and the uniform convergence of $\bar{S}^m$ to $\bar{S}$ on the compact interval $[u, v]$, we have that the Lebesgue-Stieltjes measure $d\bar{S}^m(r)$ converges weakly to $d\bar{S}(r) = \frac{K\bar{L}(r)}{\bar{\mathcal{L}}(r)} dr$ on $[u, v]$. Then,

$$\begin{aligned}
\left|\bar{H}^{j,m}_t(f) - \bar{H}^{j,m}_s(f) - \int_s^t \frac{Kp_j\langle f, \bar{\mathcal{Z}}_j(r)\rangle}{\bar{\mathcal{L}}(r)} dr\right| = \\
&\left|\int_s^t \frac{p_j\langle f, \bar{\mathcal{Z}}^m_j(r-)\rangle}{\bar{L}^m(r-)} d\bar{S}^m(r) - \int_s^t \frac{Kp_j\langle f, \bar{\mathcal{Z}}_j(r)\rangle}{\bar{\mathcal{L}}(r)} dr\right| \\
&\le \left|\int_s^t \left(\frac{p_j\langle f, \bar{\mathcal{Z}}^m_j(r-)\rangle}{\bar{L}^m(r-)} - \frac{p_j\langle f, \bar{\mathcal{Z}}_j(r)\rangle}{\bar{L}(r)}\right) d\bar{S}^m(r)\right| \\
&+ \left|\int_s^t \frac{p_j\langle f, \bar{\mathcal{Z}}_j(r)\rangle}{\bar{L}(r)} d\bar{S}^m(r) - \int_s^t \frac{p_j\langle f, \bar{\mathcal{Z}}_j(r)\rangle}{\bar{L}(r)} d\bar{S}(r)\right|,
\end{aligned}$$

where the first term on the right hand side converges to 0 as $m \to \infty$ because the integrand converges uniformly to zero and the integrator has total mass that is bounded on $[s, t]$ for all $m \in \mathbb{N}$ and the second term on the right hand side converges to 0 as $m \to \infty$ by the weak convergence of $d\bar{S}^m$ to $d\bar{S}$ and continuity of the integrand on $[s, t]$. □

Lemma 9.2.3. *Fix* $\omega \in \Omega_0$. *Let* $[u, v]$ *be an interval in* $\mathbb{R}_+$ *on which* $\bar{\mathcal{L}}(\cdot) \neq 0$ *for this* ω.

Then for this ω and $u \le s < t \le v$, $j \in \mathcal{J}$, as $m \to \infty$,

$$\bar{\mathcal{A}}_t^{j,m}(f) - \bar{\mathcal{A}}_s^{j,m}(f) \to \alpha_j \langle f, \vartheta_j \rangle (t-s).$$

Proof. It follows from the same argument as in the proof of Lemma 9.2.2 that for sufficiently large m, $\bar{\mathcal{L}}^m(\cdot) \neq 0$ on $[u,v]$. This implies that for $u \le s < t \le v$,

$$\bar{\mathcal{A}}_t^{j,m}(f) - \bar{\mathcal{A}}_s^{j,m}(f) = \bar{A}_t^{j,m}(f) - \bar{A}_s^{j,m}(f).$$

The result is immediate from the limit already given for $\{\bar{A}_\cdot^{j,m}(f)\}_{m=1}^\infty$ in (9.25). □

9.3 Proof that Fluid Limits Satisfy Definition 4.0.1

Proposition 9.3.1. *Let $\bar{\boldsymbol{\mathcal{Z}}}(\cdot)$ be as in the statement of Theorem 5.2.1. Then, $\bar{\boldsymbol{\mathcal{Z}}}(\cdot)$ satisfies Definition 4.0.1 almost surely.*

Proof. Property (i) of Definition (4.0.1) is immediate from Assumption 1 (vi).

For property (ii), for each $m \in \mathbb{N}, j \in \mathcal{J}, h > 0, t \ge 0$, combining (2.8), (3.1) with (8.9) we have

$$\bar{\mathcal{Z}}_j^m(t)([0,h)) \le \bar{\mathcal{Z}}_j^m(0)([t,t+h]) + \bar{X}_j^m(t+h) - \bar{X}_j^m(t).$$

Taking the limit as $m \to \infty$, applying Lemma 8.2.2 and the Portmanteau Theorem, we obtain almost surely

$$\bar{\mathcal{Z}}_j(t)([0,h)) \le \bar{\mathcal{Z}}_j(0)([t,t+h]) + \int_t^{t+h} \alpha_j N_j(r) dr, \tag{9.26}$$

for $t \ge 0, h > 0$. Since the left member of (9.26) is lower semi-continuous in t and left continuous in h, and, applying the fact that $\bar{\mathcal{Z}}_j(0)$ is a continuous measure, the right member of (9.26) is continuous in t and h, the almost sure set can be chosen independent of t, h. Taking $h \to 0$ and again applying the fact that $\bar{\mathcal{Z}}_j(0)$ is a continuous measure, we obtain $\bar{\mathcal{Z}}_j(t)(\{0\}) = 0$, for all $t \geq 0$, almost surely.

Furthermore, for property (iii), there is a countable set of functions $\tilde{\mathscr{C}} \subset \mathscr{C}$ such that for each $f \in \mathscr{C}$, there is a sequence $\{f_n\}_{n=1}^{\infty} \subset \tilde{\mathscr{C}}$ such that $f_n \to f, f_n' \to f'$ pointwise and boundedly on $\mathbb{R}_+$. Then, since $\tilde{\mathscr{C}}$ is countable, it suffices to prove that for each $T > 0, j \in \mathcal{J}$, $f \in \tilde{\mathscr{C}}$, (4.5) holds almost surely for all $t \in [0, T]$ with $\boldsymbol{\zeta}(\cdot) = \bar{\boldsymbol{\mathcal{Z}}}(\cdot)$. To prove this, fix $f \in \tilde{\mathscr{C}}, T > 0, j \in \mathcal{J}$. Note that there exists a nonnegative $g \in \mathscr{C}$ such that $|f(x)| \leq g(x)$ for all $x \geq 0$. For instance, take g to be $x \to \int_0^x |f'(y)|dy$, modified for x sufficiently large if necessary to make it bounded. Then by Lemma 8.2.1, with T replaced by $T + 1$, almost surely, $\langle f, \bar{\mathcal{Z}}_j(\cdot)\rangle$ and $\langle g, \bar{\mathcal{Z}}_j(\cdot)\rangle$ are Lipschitz continuous, and thus absolutely continuous, on $[0, T]$. It will be convenient to represent the convergence using the Skorokhod representation theorem, similar to §9.2. We can choose Ω_0 so that for all $\omega \in \Omega_0$, $\langle f, \bar{\mathcal{Z}}_j(\cdot)\rangle$ and $\langle g, \bar{\mathcal{Z}}_j(\cdot)\rangle$ are absolutely continuous on $[0, T]$, (8.1), and the limits in Lemmas 9.1.1, 9.2.1, 9.2.2, and 9.2.3 hold surely. Then for $\omega \in \Omega_0$ fixed, we see that if $\bar{\mathcal{L}}(\cdot) \neq 0$ on $[u, w]$ and $0 \leq u < s < t < w \leq T$, then

$$\begin{aligned}
\langle f, \bar{\mathcal{Z}}_j(t)\rangle - \langle f, \bar{\mathcal{Z}}_j(s)\rangle &= \lim_{m\to\infty} (\langle f, \bar{\mathcal{Z}}_j^m(t)\rangle - \langle f, \bar{\mathcal{Z}}_j^m(s)\rangle) \\
&= \lim_{m\to\infty} (-\bar{R}_t^{j,m}(f) + \bar{R}_s^{j,m}(f) + \bar{\mathcal{A}}_t^{j,m}(f) - \bar{\mathcal{A}}_s^{j,m}(f) \\
&\quad - \bar{H}_t^{j,m}(f) + \bar{H}_s^{j,m}(f) - \bar{Y}_t^{j,m}(f) + \bar{Y}_s^{j,m}(f)) \\
&= -\int_s^t \langle f', \bar{\mathcal{Z}}_j(r)\rangle dr + \alpha_j \langle f, \vartheta_j\rangle (t - s) - \int_s^t \frac{Kp_j \langle f, \bar{\mathcal{Z}}_j(r)\rangle}{\bar{\mathcal{L}}(r)} dr.
\end{aligned}$$

It follows that for any $t \in (0, T)$ at which $\langle f, \bar{\mathcal{Z}}_j(\cdot)\rangle$ is differentiable and $\bar{\mathcal{L}}(t) \neq 0$, we have

$$\frac{d}{dt}\langle f, \bar{\mathcal{Z}}_j(t)\rangle = -\langle f', \bar{\mathcal{Z}}_j(t)\rangle + \alpha_j \langle f, \vartheta_j\rangle - \frac{Kp_j \langle f, \bar{\mathcal{Z}}_j(t)\rangle}{\bar{\mathcal{L}}(t)}. \tag{9.27}$$

For the case in which $\bar{\mathcal{L}}(t) = 0$, for any $t \in (0, T)$ at which $\langle f, \bar{\mathcal{Z}}_j(\cdot)\rangle$ and $\langle g, \bar{\mathcal{Z}}_j(\cdot)\rangle$ are differentiable (which is Lebesgue almost everywhere) and $\bar{\mathcal{L}}(t)=0$, we have

$$\left|\frac{d}{dt}\langle f, \bar{\mathcal{Z}}_j(t)\rangle\right| = \left|\lim_{h\to 0}\frac{\langle f, \bar{\mathcal{Z}}_j(t+h)\rangle}{h}\right| \leq \left|\lim_{h\to 0}\frac{\langle g, \bar{\mathcal{Z}}_j(t+h)\rangle}{h}\right| = 0 \qquad (9.28)$$

because $\bar{\mathcal{Z}}_j(t)=0$ and since the left and right derivatives of $\langle g, \bar{\mathcal{Z}}_j(\cdot)\rangle$ at t, being of opposite signs, must both be zero. Since $\langle f, \bar{\mathcal{Z}}_j(\cdot)\rangle$ is absolutely continuous, we can recover it from its almost everywhere defined derivative, and so combining the above we have for all $t \in [0,T]$,

$$\begin{aligned}\langle f, \bar{\mathcal{Z}}_j(t)\rangle = \langle f, \bar{\mathcal{Z}}_j(0)\rangle &- \int_0^t \left\langle f', \bar{\mathcal{Z}}_j(s)\right\rangle ds \\ &- \int_0^t 1_{\{\bar{\mathcal{L}}(s)\neq 0\}}\frac{Kp_j\langle f, \bar{\mathcal{Z}}_j(s)\rangle}{\bar{\mathcal{L}}(s)}ds + \alpha_j\langle f, \vartheta_j\rangle\int_0^t 1_{\{\bar{\mathcal{L}}(s)\neq 0\}}ds, \end{aligned} \qquad (9.29)$$

where we used the fact that $\langle f', \bar{\mathcal{Z}}_j(s)\rangle = 0$ if $\bar{\mathcal{L}}(s)=0$ because then $\bar{\mathcal{Z}}_j(s)=0$. This completes the proof of property (iii).

Lastly, we prove property (iv) of Definition (4.0.1). Suppose $\varrho > 1$. Similar to the proof of Lemma 4.0.2 (iii), we will show that when $\rho > 1$, almost surely, $B(t) > 0$ for each $t > 0$. Note that in the prelimit, by construction, for $0 < s < t$,

$$\bar{Z}_j^m(t) - \bar{Z}_j^m(s) = \bar{A}_j^m(t) - \bar{A}_j^m(s) - \bar{S}_j^m(t) + \bar{S}_j^m(s) - \bar{R}_j^m(t) + \bar{R}_j^m(s), \qquad m \in \mathbb{N}, \quad (9.30)$$

where $\bar{R}_j^m(\cdot) = \frac{R_j^m(m\cdot)}{m}$ and $R_j^m(r)$ is the number of jobs that have reneged from class j up until time r for $r \geq 0$ in the mth system. We note that for $0 \leq s < t$, using (2.8) and (3.1), we have

$$\bar{R}_j^m(t) - \bar{R}_j^m(s) \leq \bar{Z}_j^m(s) + \frac{1}{m}\sum_{i=m\bar{A}_j^m(s)+1}^{m\bar{A}_j^m(t)} 1_{\{\ell_i^j \leq t-s\}}, \qquad m \in \mathbb{N}.$$

Combining the above, we conclude that for $0 \le s < t$,

$$\bar{Z}_j^m(t) \ge \bar{A}_j^m(t) - \bar{A}_j^m(s) - \bar{S}_j^m(t) + \bar{S}_j^m(s) - \frac{1}{m} \sum_{i=m\bar{A}_j^m(s)+1}^{m\bar{A}_j^m(t)} 1_{\{\ell_i^j \le t-s\}}. \tag{9.31}$$

Fixing $0 \le s < t$, and letting $m \to \infty$, passing to a subsequence if necessary to get joint convergence of $(\bar{\boldsymbol{Z}}^m(\cdot), \bar{\boldsymbol{A}}^m(\cdot), \bar{\boldsymbol{S}}^m(\cdot))$ along with the last term in (9.31), using $\frac{1}{m}\sum_{i=1}^{m\cdot} 1_{\{\ell_i^j \le t-s\}} \Rightarrow N_j(t-s)(\cdot)$, where $N_j(t-s)(r) = N_j(t-s)r$ for each $r \ge 0$, convergence of $\bar{A}_j^m(t), \bar{A}_j^m(s)$, and independence of interarrival and patience times, for the last term on the right hand side, we have for fixed $0 \le s < t$, almost surely,

$$\bar{Z}_j(t) \ge \alpha_j(t-s) - \bar{S}_j(t) + \bar{S}_j(s) - \alpha_j(t-s)\langle 1_{[0,t-s]}, \vartheta_j \rangle,$$

Let $B(\cdot) := \sum_{j=1}^J \frac{1}{\mu_j} \bar{Z}_j(\cdot)$ as in Lemma 4.0.2. It follows from (7.23), Lemmas 9.1.1 and 9.1.2, and the fact that $\mathscr{T}(t) - \mathscr{T}(s) \le K(t-s)$ that $\sum_{j=1}^J \frac{\bar{S}_j(t)-\bar{S}_j(s)}{\mu_j} \le K(t-s)$. Therefore we obtain for fixed $0 \le s < t$, almost surely,

$$\begin{aligned} B(t) &\ge \sum_{j=1}^J \frac{1}{\mu_j}\left(\alpha_j(t-s) - (\bar{S}_j(t) - \bar{S}_j(s)) - \alpha_j(t-s)\langle 1_{[0,t-s]}, \vartheta_j\rangle\right) \\ &\ge K(\varrho - 1)(t-s) - \sum_{j=1}^J \frac{1}{\mu_j}\alpha_j(t-s)\langle 1_{[0,t-s]}, \vartheta_j\rangle. \end{aligned} \tag{9.32}$$

Now, the left member of (9.32) is a continuous function of t, and the right member of (9.32) is a right continuous function of $t - s > 0$, and so we have that, almost surely, (9.32) holds for all $0 \le s < t$ simultaneously, that is, the exceptional null set can be chosen not to depend on s, t. Since patience times are strictly positive, there exists $\epsilon > 0$ such that $\langle 1_{[0,\epsilon]}, \vartheta_j\rangle \le \frac{\varrho-1}{2\varrho}$ for each $j \in \mathcal{J}$. Then setting $s = (t-\epsilon)^+$, we have that almost surely for all $t > 0$,

$$\begin{aligned} B(t) &\geq K(\varrho - 1)(t \wedge \epsilon) - \sum_{j=1}^{J} \frac{\alpha_j}{\mu_j}(t \wedge \epsilon)\langle 1_{[0,t\wedge\epsilon]}, \vartheta_j \rangle \\ &\geq (t \wedge \epsilon) K \left(\frac{\varrho - 1}{2} \right) > 0 \end{aligned}$$

since $\varrho > 1$.

$\square$

Chapter 10

Proof of Results for the Invariant State

We shall prove Theorem 5.3.1 in this chapter by proving the various pieces of its statement.

10.1 The Invariant State

We first prove the first part of Theorem 5.3.1, namely the characterization of the invariant state.

Theorem 10.1.1. *Fix fluid model parameters* $(\boldsymbol{\alpha}, \boldsymbol{\mu}, \boldsymbol{p}, \boldsymbol{\vartheta})$ *satisfying Definition 3.0.1. Then there exists a unique invariant state for the associated fluid model. When* $\varrho \leq 1$*, the invariant state is* $\boldsymbol{\nu} = \mathbf{0}$*. In the case where* $\varrho > 1$*, the invariant state is* $\boldsymbol{\nu}$ *as defined in Theorem 5.3.1.*

Proof. A vector of J measures $\boldsymbol{\zeta}^\dagger \in \mathbf{K}^J$ is an invariant state for the fluid model if and only if $\boldsymbol{\zeta}(t) = \boldsymbol{\zeta}^\dagger$ for each $t \geq 0$ satisfies Definition 4.0.1. For the remainder of this proof, we indicate quantities associated with such a $\boldsymbol{\zeta}^\dagger$ by appending a dagger. For example, we will write $\zeta_j^\dagger$, $M_j^{c,\dagger}(x)$, or $z_j^\dagger$.

We begin by considering the $\varrho \leq 1$ case. It is straightforward to see that $\boldsymbol{\zeta}(t) = \mathbf{0}$ for all $t \geq 0$ satisfies Definition 4.0.1. So $\boldsymbol{\zeta}^\dagger = \mathbf{0}$ is an invariant state. Now, suppose that $\boldsymbol{\zeta}^\dagger \neq \mathbf{0}$ is an invariant state with complementary functions $M_j^{c,\dagger}(x) := \langle 1_{(x,\infty)}, \zeta_j^\dagger \rangle$ for each

$x \geq 0, j \in \mathcal{J}$ and component masses $z_j^\dagger = \langle 1, \zeta_j^\dagger \rangle$, $j \in \mathcal{J}$. Applying (4.6) with $x = 0$, we see upon multiplying by $\frac{1}{\mu_j}$, summing over $j \in \mathcal{J}$, using the change of variables $t - s \to s$ in the second term on the right hand side, and using the identities $N_j^c(\cdot) = 1 - N_j(\cdot)$ and $M_j^{c,\dagger}(x) = z_j^\dagger - M_j^\dagger(x)$, $x \geq 0$, that for each $t \geq 0$,

$$B^\dagger = \sum_{j=1}^{J} \frac{z_j^\dagger}{\mu_j} = \sum_{j=1}^{J} \frac{1}{\mu_j} M_j^{c,\dagger}(t) + \int_0^t \sum_{j=1}^{J} \frac{\alpha_j}{\mu_j} N_j^c(s) ds - \int_0^t \frac{\sum_{j=1}^{J} \frac{K p_j}{\mu_j} M_j^{c,\dagger}(t-s)}{\mathcal{L}^\dagger} ds$$

$$\Longrightarrow \sum_{j=1}^{J} \frac{1}{\mu_j} M_j^\dagger(t) = K(\varrho - 1)t + \int_0^t \left[\frac{\sum_{j=1}^{J} \frac{K p_j}{\mu_j} M_j^\dagger(s)}{\mathcal{L}^\dagger} - \sum_{j=1}^{J} \frac{\alpha_j}{\mu_j} N_j(s) \right] ds.$$

Because $\varrho \leq 1$, this implies that for all $x \geq 0$,

$$0 \leq \sum_{j=1}^{J} \frac{1}{\mu_j} M_j^\dagger(x) \leq \frac{K}{\mathcal{L}^\dagger} \int_0^x \sum_{j=1}^{J} \frac{1}{\mu_j} M_j^\dagger(s) ds.$$

Applying Gronwall's inequality, we see that this implies that $\sum_{j=1}^{J} \frac{1}{\mu_j} M_j^\dagger(x) = 0$ for all $x \geq 0$. Because each $M_j^\dagger(\cdot)$ is nonnegative, this implies that $M_j^\dagger(x) = 0$ for each $j \in \mathcal{J}, x \geq 0$. This contradiction of $\boldsymbol{\zeta}^\dagger \neq \mathbf{0}$ implies that the zero measure is the unique invariant state for the $\varrho \leq 1$ case.

In the $\varrho > 1$ case, assuming $\boldsymbol{\zeta}^\dagger$ is an invariant state, by Definition 4.0.1 (iv), $\boldsymbol{\zeta}^\dagger \neq \mathbf{0}$, and so $t^{*,\dagger} = \infty$. Then by (6.2), for each $t \geq 0$, $G^\dagger(t) = \frac{t}{\mathcal{L}^\dagger}$, and so by (6.4),

$$z_j^\dagger = \int_0^t \alpha_j N_j^c(r) \exp\left(-\frac{K p_j r}{\mathcal{L}^\dagger}\right) dr + \exp\left(-\frac{K p_j t}{\mathcal{L}^\dagger}\right) M_j^{c,\dagger}(t), \quad t \geq 0.$$

Therefore, for each $t \geq 0$ we have that

$$\mathcal{L}^\dagger = \sum_{j=1}^{J} \frac{p_j z_j^\dagger}{\mu_j} = \sum_{j=1}^{J} \int_0^t \frac{\alpha_j p_j}{\mu_j} N_j^c(r) \exp\left(-\frac{K p_j r}{\mathcal{L}^\dagger}\right) dr + \sum_{j=1}^{J} \frac{p_j}{\mu_j} \exp\left(-\frac{K p_j t}{\mathcal{L}^\dagger}\right) M_j^{c,\dagger}(t).$$

Taking the limit as $t \to \infty$, we see that $\mathcal{L} = \mathcal{L}^\dagger$ satisfies (5.1). By Lemma 5.3.1, (5.1) has a unique positive solution, and so $\mathcal{L}^\dagger = \mathcal{L}^*$. Applying (6.3), we see that this implies that for $t \geq 0$, $x \geq 0$,

$$M_j^{c,\dagger}(x) = \int_0^t \exp\left(-\frac{Kp_j r}{\mathcal{L}^*}\right) \alpha_j N_j^c(r+x)dr + \exp\left(-\frac{Kp_j t}{\mathcal{L}^*}\right) M_j^{c,\dagger}(t+x).$$

Taking the limit as $t \to \infty$ again, we see that any invariant state $\boldsymbol{\zeta}^\dagger$ satisfies (5.3) with $\boldsymbol{\zeta}^\dagger$ in place of $\boldsymbol{\nu}$, and hence is unique.

□

10.2 Asymptotic Properties of Total Mass

In this chapter, we show that for any fluid model solution $\boldsymbol{\zeta}(\cdot)$ in the set $\mathscr{M}_c^J$, the associated function $\mathcal{L}(\cdot)$ converges to $\mathcal{L}^*$ (as defined in Lemma 5.3.1), as its argument goes to infinity, uniformly for all $\boldsymbol{\zeta}(\cdot) \in \mathscr{M}_c^J$. We will first prove the statement for the overloaded case and then for the other cases.

Lemma 10.2.1. *Let $(\boldsymbol{\alpha}, \boldsymbol{\mu}, \boldsymbol{p}, \boldsymbol{\vartheta})$ be a set of fluid model parameters satisfying Definition 3.0.1. For each $\boldsymbol{\xi} \in \mathbf{K}^J \setminus \{\mathbf{0}\}$, let $\boldsymbol{\zeta}^{\boldsymbol{\xi}}(\cdot)$ be the unique fluid model solution with initial condition $\boldsymbol{\xi}$. Then, denote the associated $\mathcal{L}_{\boldsymbol{\xi}}(\cdot) := \sum_{j=1}^J \frac{p_j}{\mu_j}\langle 1, \zeta_j^{\boldsymbol{\xi}}(\cdot)\rangle$. For each $M > 0$, define*

$$\mathcal{L}_{\boldsymbol{\xi}}^{n,M} := \sup\{\mathcal{L}_{\boldsymbol{\xi}}(t) : nM \leq t \leq (n+1)M\}$$

and

$$\mathcal{L}_{n,\boldsymbol{\xi}}^{M} := \inf\{\mathcal{L}_{\boldsymbol{\xi}}(t) : nM \leq t \leq (n+1)M\},$$

for $n = 0, 1, 2, 3, \ldots$. For each $c > 0$, define

$$C_c := \sum_{j=1}^J \frac{p_j}{\mu_j}\left(c + \alpha_j \langle \chi, \vartheta_j \rangle\right). \tag{10.1}$$

Then for all $\boldsymbol{\xi} \in \mathbf{K}_c^J \setminus \{\mathbf{0}\}$ *and* $n \in \mathbb{N}$ *satisfying* $n < \frac{t^*_{\boldsymbol{\xi}}}{M} - 1$, *we have*

$$\frac{\mathcal{L}^{n,M}_{\boldsymbol{\xi}}}{\max\{\mathcal{L}^{n,M}_{\boldsymbol{\xi}}, \mathcal{L}^{n-1,M}_{\boldsymbol{\xi}}\}} \leq g(\max\{\mathcal{L}^{n,M}_{\boldsymbol{\xi}}, \mathcal{L}^{n-1,M}_{\boldsymbol{\xi}}\}) + \frac{\epsilon_1(M)}{\max\{\mathcal{L}^{n,M}_{\boldsymbol{\xi}}, \mathcal{L}^{n-1,M}_{\boldsymbol{\xi}}\}} + \frac{\epsilon_2(M)}{\max\{\mathcal{L}^{n,M}_{\boldsymbol{\xi}}, \mathcal{L}^{n-1,M}_{\boldsymbol{\xi}}\}}, \tag{10.2}$$

and

$$\frac{\mathcal{L}^{M}_{n,\boldsymbol{\xi}}}{\min\{\mathcal{L}^{M}_{n-1,\boldsymbol{\xi}}, \mathcal{L}^{M}_{n,\boldsymbol{\xi}}\}} \geq g(\min\{\mathcal{L}^{M}_{n-1,\boldsymbol{\xi}}, \mathcal{L}^{M}_{n,\boldsymbol{\xi}}\}) - \frac{\epsilon_1(M)}{\min\{\mathcal{L}^{M}_{n-1,\boldsymbol{\xi}}, \mathcal{L}^{M}_{n,\boldsymbol{\xi}}\}}, \tag{10.3}$$

where $t^*_{\boldsymbol{\xi}} = \inf\{t \geq 0 : \mathcal{L}_{\boldsymbol{\xi}}(t) = 0\}$,

$$\epsilon_1(t) := \int_t^\infty \sum_{j=1}^J \frac{p_j\alpha_j}{\mu_j} \exp\left(\frac{-Kp_js}{C_c}\right) N^c_j(s)ds, \qquad \epsilon_2(t) = \sum_{j=1}^J \frac{p_j}{\mu_j} c \exp\left(\frac{-Kp_jt}{C_c}\right), \tag{10.4}$$

$g(x) := \int_0^\infty \sum_{j=1}^J \frac{p_j\alpha_j}{\mu_j} \exp(-Kp_js)N^c_j(sx)ds$, $x \geq 0$.

Proof. Fix $c > 0$. First, applying Lemma 6.2.1, we see that for C_c as defined in (10.1), $\mathcal{L}_{\boldsymbol{\xi}}(t) \leq C_c$ for all $t \geq 0$ and $G_{\boldsymbol{\xi}}(t) = \int_0^t \frac{1}{\mathcal{L}_{\boldsymbol{\xi}}(s)}ds \geq \frac{t}{C_c}$ for all $\boldsymbol{\xi} \in \mathbf{K}_c^J$, $t < t^*_{\boldsymbol{\xi}}$. Then, for $\boldsymbol{\xi} \in \mathbf{K}_c^J \setminus \{\mathbf{0}\}$ and $t < t^*_{\boldsymbol{\xi}}$, using (6.4), we have for $M > 0$, $1 \leq n < \frac{t^*_{\boldsymbol{\xi}}}{M} - 1$, $t \in [nM, (n+1)M]$,

$$\begin{aligned}
\mathcal{L}_{\boldsymbol{\xi}}(t) &= \int_0^t \sum_{j=1}^J \frac{p_j\alpha_j}{\mu_j} \exp(-Kp_j(G_{\boldsymbol{\xi}}(t) - G_{\boldsymbol{\xi}}(t-s)))N^c_j(s)ds \\
&\quad + \sum_{j=1}^J \frac{p_j}{\mu_j} \exp(-Kp_jG_{\boldsymbol{\xi}}(t))M^c_j(0,t) \\
&\leq \int_0^M \sum_{j=1}^J \frac{p_j\alpha_j}{\mu_j} \exp(-Kp_j(G_{\boldsymbol{\xi}}(t) - G_{\boldsymbol{\xi}}(t-s)))N^c_j(s)ds + \epsilon_1(M) + \epsilon_2(M) \\
&\leq \int_0^M \sum_{j=1}^J \frac{p_j\alpha_j}{\mu_j} \exp\left(\frac{-Kp_js}{\max\{\mathcal{L}^{n-1,M}_{\boldsymbol{\xi}}, \mathcal{L}^{n,M}_{\boldsymbol{\xi}}\}}\right) N^c_j(s)ds + \epsilon_1(M) + \epsilon_2(M), \qquad (10.5)
\end{aligned}$$

where the last inequality follows from the fact that for each $s \in [0, M]$,

$$G_{\boldsymbol{\xi}}(t) - G_{\boldsymbol{\xi}}(t-s) \geq \frac{s}{\sup_{r\in[t-s,t]} \mathcal{L}_{\boldsymbol{\xi}}(r)}.$$

Because the bound holds for each $t \in [nM, (n+1)M]$, we have

$$\mathcal{L}_{\boldsymbol{\xi}}^{n,M} \leq \int_0^\infty \sum_{j=1}^J \frac{p_j \alpha_j}{\mu_j} \exp\left(\frac{-Kp_j s}{\max\{\mathcal{L}_{\boldsymbol{\xi}}^{n-1,M}, \mathcal{L}_{\boldsymbol{\xi}}^{n,M}\}}\right) N_j^c(s)ds + \epsilon_1(M) + \epsilon_2(M),$$

and using a change of variables, we obtain

$$\begin{aligned}
\mathcal{L}_{\boldsymbol{\xi}}^{n,M} &\leq \max\{\mathcal{L}_{\boldsymbol{\xi}}^{n,M}, \mathcal{L}_{\boldsymbol{\xi}}^{n-1,M}\} g(\max\{\mathcal{L}_{\boldsymbol{\xi}}^{n,M}, \mathcal{L}_{\boldsymbol{\xi}}^{n-1,M}\}) \\
&\quad + \epsilon_1(M) + \epsilon_2(M) \\
\implies \frac{\mathcal{L}_{\boldsymbol{\xi}}^{n,M}}{\max\{\mathcal{L}_{\boldsymbol{\xi}}^{n,M}, \mathcal{L}_{\boldsymbol{\xi}}^{n-1,M}\}} &\leq g(\max\{\mathcal{L}_{\boldsymbol{\xi}}^{n,M}, \mathcal{L}_{\boldsymbol{\xi}}^{n-1,M}\}) \\
&\quad + \frac{\epsilon_1(M)}{\max\{\mathcal{L}_{\boldsymbol{\xi}}^{n,M}, \mathcal{L}_{\boldsymbol{\xi}}^{n-1,M}\}} + \frac{\epsilon_2(M)}{\max\{\mathcal{L}_{\boldsymbol{\xi}}^{n,M}, \mathcal{L}_{\boldsymbol{\xi}}^{n-1,M}\}},
\end{aligned}$$

as desired for (10.2). For the inequality (10.3), we do similar steps on the other side, to obtain from (10.5), that for each $\boldsymbol{\xi} \in \mathbf{K}_c^J \backslash \{\mathbf{0}\}$, $M > 0$, $1 \leq n < \frac{t^*_{\boldsymbol{\xi}}}{M} - 1$, $t \in [nM, (n+1)M]$,

$$\mathcal{L}_{\boldsymbol{\xi}}(t) \geq \int_0^M \sum_{j=1}^J \frac{p_j \alpha_j}{\mu_j} \exp\left(\frac{-Kp_j s}{\min\{\mathcal{L}_{n-1,\boldsymbol{\xi}}^M, \mathcal{L}_{n,\boldsymbol{\xi}}^M\}}\right) N_j^c(s)ds.$$

Taking an infimum on the left hand side and doing the same change of variables as before, we obtain

$$\begin{aligned}
\mathcal{L}_{n,\boldsymbol{\xi}}^M &\geq \int_0^\infty \sum_{j=1}^J \frac{p_j \alpha_j}{\mu_j} \exp\left(\frac{-Kp_j s}{\min\{\mathcal{L}_{n-1,\boldsymbol{\xi}}^M, \mathcal{L}_{n,\boldsymbol{\xi}}^M\}}\right) N_j^c(s)ds - \epsilon_1(M) \\
\implies \frac{\mathcal{L}_{n,\boldsymbol{\xi}}^M}{\min\{\mathcal{L}_{n-1,\boldsymbol{\xi}}^M, \mathcal{L}_{n,\boldsymbol{\xi}}^M\}} &\geq g(\min\{\mathcal{L}_{n-1,\boldsymbol{\xi}}^M, \mathcal{L}_{n,\boldsymbol{\xi}}^M\}) - \frac{\epsilon_1(M)}{\min\{\mathcal{L}_{n-1,\boldsymbol{\xi}}^M, \mathcal{L}_{n,\boldsymbol{\xi}}^M\}}.
\end{aligned}$$

□

Theorem 10.2.1. *Let $(\boldsymbol{\alpha}, \boldsymbol{\mu}, \boldsymbol{p}, \boldsymbol{\vartheta})$ be a set of fluid model parameters satisfying Definition 3.0.1 with $\varrho > 1$. Fix $c > 0$ and let*

$$\mathscr{L}_c^J := \left\{ \mathcal{L}(\cdot) : \mathcal{L}(\cdot) = \sum_{j=1}^{J} \frac{p_j}{\mu_j} \langle 1, \zeta_j(\cdot) \rangle \text{ for some } \boldsymbol{\zeta} \in \mathscr{M}_c^J \right\},$$

where $\mathscr{M}_c^J$ is as in Theorem 5.3.1. Then $\mathcal{L}(t)$ converges to $\mathcal{L}^*$ as $t \to \infty$, uniformly for all $\mathcal{L}$ in $\mathscr{L}_c^J$. Here, $\mathcal{L}^*$ is as defined in Lemma 5.3.1.

Proof. Let $\epsilon > 0$. We want to show that there exists $T > 0$ such that $\sup_{\mathcal{L}(\cdot) \in \mathscr{L}_c^J} \sup_{t \geq T} |\mathcal{L}(t) - \mathcal{L}^*| < \epsilon$. Using the result of Lemma 4.0.3 and shifting the time origin to t_0, we can reduce to the case where $\mathcal{L}(t) \geq \mathcal{L}_{min}$ for each $t \geq 0$, $\boldsymbol{\zeta}_0 = \boldsymbol{\xi} \in \mathbf{K}_c^J \setminus \{\mathbf{0}\}$, and $t_{\boldsymbol{\xi}}^* = +\infty$. By Lemma 6.2.1, with C_c as in Lemma 10.2.1, we have $\mathcal{L}_{\boldsymbol{\xi}}(t) \leq C_c$ for all $t \geq 0, \boldsymbol{\xi} \in \mathbf{K}_c^J$. In order to use the bound obtained in Lemma 10.2.1, we examine the properties of the function $g(\cdot)$. Notice that $g(\cdot)$ is continuous, strictly decreasing, positive, and tends to 0 at infinity. Therefore, it has an inverse $f(\cdot)$ that is also strictly decreasing and continuous on $(0, g(0)]$. Note also that by Lemma 5.3.1 and a change of variables in the integral defining g that $g(\mathcal{L}^*) = 1$ and so $f(1) = \mathcal{L}^*$. Furthermore, on the compact set $[g(C_c)/2, g(0)]$, $f(\cdot)$ is uniformly continuous. Hence, there is a modulus of continuity function $h(\cdot)$ such that $h(\cdot)$ is increasing, $|f(x) - f(y)| < h(|x - y|)$ for all $x, y \in [g(C_c)/2, g(0)]$, and $\lim_{\delta \to 0} h(\delta) = 0$. Let $\delta > 0$ be sufficiently small that $2\delta < \min\{1 - g(C_c)/2, g(0) - 1\}$, $h(2\delta) < \epsilon$. Choose M sufficiently large that $\frac{\epsilon_1(M)}{\mathcal{L}_{\min}}, \frac{\epsilon_2(M)}{\mathcal{L}_{min}} < \min\left(\delta/2, \frac{g(0) - g(\mathcal{L}_{min})}{2}\right)$. Consider $\boldsymbol{\xi} \in \mathbf{K}_c^J \setminus \{\mathbf{0}\}$. We examine three subcases: (i) $\mathcal{L}_{\boldsymbol{\xi}}^{n,M} \geq \mathcal{L}_{\boldsymbol{\xi}}^{n-1,M}$, (ii) $\mathcal{L}_{\boldsymbol{\xi}}^{n,M} < \mathcal{L}_{\boldsymbol{\xi}}^{n-1,M}$ and $g(\mathcal{L}_{\boldsymbol{\xi}}^{n-1,M}) + \frac{\epsilon_1(M)}{\mathcal{L}_{min}} + \frac{\epsilon_2(M)}{\mathcal{L}_{min}} \geq 1 - \delta$, and (iii) $\mathcal{L}_{\boldsymbol{\xi}}^{n,M} < \mathcal{L}_{\boldsymbol{\xi}}^{n-1,M}$ and $g(\mathcal{L}_{\boldsymbol{\xi}}^{n-1,M}) + \frac{\epsilon_1(M)}{\mathcal{L}_{\min}} + \frac{\epsilon_2(M)}{\mathcal{L}_{\min}} < 1 - \delta$. In case (i), we see that (10.2) gives $1 \leq g(\mathcal{L}_{\boldsymbol{\xi}}^{n,M}) + \frac{\epsilon_1(M)}{\mathcal{L}_{min}} + \frac{\epsilon_2(M)}{\mathcal{L}_{min}}$. Applying f to both sides and using the fact that $f(1) = \mathcal{L}^*$, we see that we have

$$\begin{aligned} f(1) &\geq f\left(g(\mathcal{L}_{\boldsymbol{\xi}}^{n,M}) + \frac{\epsilon_1(M)}{\mathcal{L}_{min}} + \frac{\epsilon_2(M)}{\mathcal{L}_{min}}\right) \\ &\implies \mathcal{L}^* \geq f(g(\mathcal{L}_{\boldsymbol{\xi}}^{n,M})) - h\left(\frac{\epsilon_1(M)}{\mathcal{L}_{min}} + \frac{\epsilon_2(M)}{\mathcal{L}_{min}}\right) \\ &\implies \mathcal{L}^* + h\left(\frac{\epsilon_1(M)}{\mathcal{L}_{\min}} + \frac{\epsilon_2(M)}{\mathcal{L}_{\min}}\right) \geq \mathcal{L}_{\boldsymbol{\xi}}^{n,M}. \end{aligned}$$

We conclude that

$$\mathcal{L}_{\boldsymbol{\xi}}^{n,M} \leq \mathcal{L}^* + h\left(\frac{\epsilon_1(M)}{\mathcal{L}_{\min}} + \frac{\epsilon_2(M)}{\mathcal{L}_{\min}}\right) \leq \mathcal{L}^* + \epsilon.$$

In case (ii), subtracting $\frac{\epsilon_1(M)}{\mathcal{L}_{min}} + \frac{\epsilon_2(M)}{\mathcal{L}_{min}}$ from both sides of $g(\mathcal{L}_{\boldsymbol{\xi}}^{n-1,M}) + \frac{\epsilon_1(M)}{\mathcal{L}_{min}} + \frac{\epsilon_2(M)}{\mathcal{L}_{min}} \geq 1 - \delta$, applying f to both sides, and using the fact that f is decreasing, we obtain

$$\mathcal{L}_{\boldsymbol{\xi}}^{n,M} < \mathcal{L}_{\boldsymbol{\xi}}^{n-1,M} \leq \mathcal{L}^* + h\left(\frac{\epsilon_1(M)}{\mathcal{L}_{min}} + \frac{\epsilon_2(M)}{\mathcal{L}_{min}} + \delta\right) \leq \mathcal{L}^* + \epsilon.$$

In case (iii),

$$\frac{\mathcal{L}_{\boldsymbol{\xi}}^{n,M}}{\mathcal{L}_{\boldsymbol{\xi}}^{n-1,M}} \leq g(\mathcal{L}_{\boldsymbol{\xi}}^{n-1,M}) + \frac{\epsilon_1(M)}{\mathcal{L}_{min}} + \frac{\epsilon_2(M)}{\mathcal{L}_{min}} < 1 - \delta,$$

and so $\mathcal{L}_{\boldsymbol{\xi}}^{n,M} \leq \mathcal{L}_{\boldsymbol{\xi}}^{n-1,M}(1-\delta)$. Now we combine the three cases. Let N be such that for $n \geq N$, $C_c(1-\delta)^n \leq \mathcal{L}^* + \epsilon$. Then for $n \geq N$, let i_n be the largest natural number i less than or equal to n such that $\mathcal{L}_{\boldsymbol{\xi}}^{i-1}, \mathcal{L}_{\boldsymbol{\xi}}^{i}$ fall into Case (i) or Case (ii) (with i in place of n there), taking $i_n = 0$ if every $\mathcal{L}_{\boldsymbol{\xi}}^{i-1}, \mathcal{L}_{\boldsymbol{\xi}}^{i}$ has fallen into Case (iii) for all $1 \leq i \leq n$. Then we see that if $i_n > 0$, then Case (iii) applies from $i_n + 1$ to n, and so $\mathcal{L}_{\boldsymbol{\xi}}^{n,M} \leq (\mathcal{L}^* + \epsilon)(1-\delta)^{n-i_n} \leq \mathcal{L}^* + \epsilon$ and if $i_n = 0$, $\mathcal{L}_{\boldsymbol{\xi}}^{n,M} \leq C_c(1-\delta)^n \leq \mathcal{L}^* + \epsilon$. Taking $T = NM$, we see that for each $\boldsymbol{\xi} \in \mathbf{K}_c \setminus \{\mathbf{0}\}$, $\sup_{t \geq T}\{\mathcal{L}_{\boldsymbol{\xi}}(t)\} = \sup_{n \geq N} \mathcal{L}_{\boldsymbol{\xi}}^{n,M} \leq \mathcal{L}^* + \epsilon$.

We now apply similar logic to the other side. On this side we will only need two subcases: (i) $\frac{\mathcal{L}_{n,\boldsymbol{\xi}}^M}{\min\{\mathcal{L}_{n-1,\boldsymbol{\xi}}^M, \mathcal{L}_{n,\boldsymbol{\xi}}^M\}} \leq 1 + \frac{\delta}{2}$ and (ii) $\frac{\mathcal{L}_{n,\boldsymbol{\xi}}^M}{\min\{\mathcal{L}_{n-1,\boldsymbol{\xi}}^M, \mathcal{L}_{n,\boldsymbol{\xi}}^M\}} > 1 + \frac{\delta}{2}$. In case (i), using (10.3) we have $g(\min\{\mathcal{L}_{n-1,\boldsymbol{\xi}}^M, \mathcal{L}_{n,\boldsymbol{\xi}}^M\}) \leq \frac{\mathcal{L}_{n,\boldsymbol{\xi}}^M}{\min\{\mathcal{L}_{n-1,\boldsymbol{\xi}}^M, \mathcal{L}_{n,\boldsymbol{\xi}}^M\}} + \frac{\epsilon_1(M)}{\mathcal{L}_{\min}} \leq 1 + \delta$. Applying f to both sides, we obtain $\min\{\mathcal{L}^M_{n-1,\boldsymbol{\xi}}, \mathcal{L}^M_{n,\boldsymbol{\xi}}\} \geq \mathcal{L}^* - \epsilon$. In case (ii), it follows that $\mathcal{L}^M_{n,\boldsymbol{\xi}} \geq \left(1+\frac{\delta}{2}\right)\mathcal{L}^M_{n-1,\boldsymbol{\xi}}$. Using the same argument as we used for $\{\mathcal{L}^{n,M}_{\boldsymbol{\xi}}\}_{n=1}^\infty$, we conclude that there exists a $T > 0$ such that $\inf_{\boldsymbol{\xi}\in\mathbf{K}^J_c\setminus\{\mathbf{0}\}} \inf_{t\geq T}\{\mathcal{L}_{\boldsymbol{\xi}}(t)\} \geq \mathcal{L}^* - \epsilon$. Combining all of the above, we obtain $T_\epsilon > 0$ such that $\sup_{\mathcal{L}(\cdot)\in\mathscr{L}^J_c} \sup_{t\geq T_\epsilon} |\mathcal{L}(t) - \mathcal{L}^*| \leq \epsilon$. □

Theorem 10.2.2. *Let $(\boldsymbol{\alpha}, \boldsymbol{\mu}, \boldsymbol{p}, \boldsymbol{\vartheta})$ be a set of fluid model parameters satisfying Definition 3.0.1 with $\varrho \leq 1$. For $c > 0$, let $\mathbf{K}^J_c$ be defined as in Theorem 5.3.1. Then*

$$\sup_{\boldsymbol{\xi}\in\mathbf{K}^J_c} \mathcal{L}_{\boldsymbol{\xi}}(t) \to 0 \tag{10.6}$$

as $t \to \infty$.

Proof. Fix $c > 0$. Let $\epsilon > 0$. We will show that there exists $T_\epsilon > 0$ such that $\sup_{t\geq T_\epsilon} \mathcal{L}_{\boldsymbol{\xi}}(t) \leq \epsilon$ for all $\boldsymbol{\xi} \in \mathbf{K}^J_c$. In the $\rho \leq 1$ case, we know from (4.10) from the proof of Lemma 4.0.2 that for a fluid model solution $\boldsymbol{\zeta}(\cdot)$ with initial condition in $\mathbf{K}^J_c$,

$$B(t) = \sum_{j=1}^J \frac{1}{\mu_j}\langle 1, \zeta_j(t)\rangle \leq K(\varrho - 1)t + B(0), \quad 0 \leq t < t^*. \tag{10.7}$$

Furthermore, because $\mathcal{L}$ and B are positive linear combinations of the same J non-negative functions, there exist $C_1, C_2 > 0$ such that $B(\cdot) \leq C_1\mathcal{L}(\cdot) \leq C_2 B(\cdot)$. Define $\tilde{\epsilon} = \frac{\epsilon}{C_2}$. We claim it suffices to take $T_\epsilon \geq \sup_{\boldsymbol{\xi}\in\mathbf{K}^c_j} t^*_{\tilde{\epsilon},\boldsymbol{\xi}}$, where $t^*_{\tilde{\epsilon},\boldsymbol{\xi}} := \inf_{t\geq 0}\{\mathcal{L}_{\boldsymbol{\xi}}(t) \leq \tilde{\epsilon}\}$, provided that supremum is finite. Indeed, for $\boldsymbol{\xi} \in \mathbf{K}^J_c$, by continuity of $\mathcal{L}_{\boldsymbol{\xi}}$, we see that either $t^*_{\tilde{\epsilon},\boldsymbol{\xi}} = 0$ and $\mathcal{L}_{\boldsymbol{\xi}}(0) \leq \tilde{\epsilon}$, or $\mathcal{L}_{\boldsymbol{\xi}}(t^*_{\tilde{\epsilon},\boldsymbol{\xi}}) = \tilde{\epsilon}$. In either case, it follows that $B(t^*_{\tilde{\epsilon},\boldsymbol{\xi}}) \leq \tilde{\epsilon}C_1$. By Lemma 4.0.2 (iii), $B(\cdot)$ is nonincreasing when $\varrho \leq 1$. Therefore, $B(s) \leq \tilde{\epsilon}C_1$ for each $s \geq t^*_{\tilde{\epsilon},\boldsymbol{\xi}}$. It follows that $\mathcal{L}_{\boldsymbol{\xi}}(s) \leq C_2\tilde{\epsilon} = \epsilon$ for $s \geq t^*_{\tilde{\epsilon},\boldsymbol{\xi}}$.

In the $\varrho < 1$ case, we apply (10.7) to see that $t^*_{\tilde{\epsilon},\boldsymbol{\xi}} \leq T = \left(\frac{\sum_{j=1}^J \frac{1}{\mu_j}c - C_1\tilde{\epsilon}/C_2}{K(1-\varrho)}\right)^+$ for each $\boldsymbol{\xi} \in \mathbf{K}^J_c$. In the $\varrho = 1$ case, noting that $t^*_{\tilde{\epsilon},\boldsymbol{\xi}} = 0$ for $\boldsymbol{\xi} = 0$, we fix $\boldsymbol{\xi} \neq \mathbf{0} \in \mathbf{K}^J_c$ and examine the interval $[0, t^*_{\tilde{\epsilon},\boldsymbol{\xi}})$. As in the proof of Theorem 10.2.1, we define f to be the inverse of g. Because $\varrho = 1$, $g(0) = 1$. Therefore, there exists $\delta > 0$ such that $x \in [1-\delta, 1]$ implies that $f(x) \leq \tilde{\epsilon}$. Choose M sufficiently large that $\frac{\epsilon_1(M)}{\tilde{\epsilon}} + \frac{\epsilon_2(M)}{\tilde{\epsilon}} < \frac{\delta}{2}$. Then, once again, we use cases and Lemma 10.2.1. For $1 \leq n < \frac{t_{\tilde{\epsilon},\boldsymbol{\xi}}}{M} - 1$, case (i) will be when $\frac{\mathcal{L}_{\boldsymbol{\xi}}^{n,M}}{\mathcal{L}_{\boldsymbol{\xi}}^{n-1,M}} > 1 - \frac{\delta}{2}$ and case (ii) will be when $\frac{\mathcal{L}_{\boldsymbol{\xi}}^{n,M}}{\mathcal{L}_{\boldsymbol{\xi}}^{n-1,M}} \leq 1 - \frac{\delta}{2}$. In case (i), (10.2) gives $1 - \delta < g(\max\{\mathcal{L}_{\boldsymbol{\xi}}^{n-1,M}, \mathcal{L}_{\boldsymbol{\xi}}^{n,M}\})$. Applying f to both sides, we obtain $\max\{\mathcal{L}_{\boldsymbol{\xi}}^{n-1,M}, \mathcal{L}_{\boldsymbol{\xi}}^{n,M}\} \leq \tilde{\epsilon}$. In case (ii) we have $\mathcal{L}_{\boldsymbol{\xi}}^{n,M} \leq \mathcal{L}_{\boldsymbol{\xi}}^{n-1,M}\left(1 - \frac{\delta}{2}\right)$. Combining the two cases as was done in the proof of Theorem 10.2.1, we will obtain a uniform bound on $t^*_{\tilde{\epsilon},\boldsymbol{\xi}}$. In particular, let $N \geq 1$ be such that for all $n \geq N$, $C_c\left(1 - \frac{\delta}{2}\right)^n \leq \tilde{\epsilon}$. Assuming for the sake of contradiction that $t^*_{\tilde{\epsilon},\boldsymbol{\xi}} > (N+1)M$, then either there is some $1 \leq n \leq N$ such that case (i) occurs and so $t^*_{\tilde{\epsilon},\boldsymbol{\xi}} \leq (N+1)M$, or case (ii) occurs for $1 \leq n \leq N$ implying that $\mathcal{L}_{\boldsymbol{\xi}}^{N,M} \leq C_c\left(1 - \frac{\delta}{2}\right)^N \leq \tilde{\epsilon}$ and so $t^*_{\tilde{\epsilon},\boldsymbol{\xi}} \leq (N+1)M$. It follows that $t^*_{\tilde{\epsilon},\boldsymbol{\xi}} \leq (N+1)M$ and hence $\mathcal{L}_{\boldsymbol{\xi}}(t) \leq \epsilon$ for $t \geq (N+1)M$, $\boldsymbol{\xi} \in \mathbf{K}_c^J$. □

10.3 Proof of Uniform Convergence to the Invariant State

We now prove the last part of Theorem 5.3.1.

Proof. Fix $c > 0$. First, we prove the $\varrho \leq 1$ case. Let $\boldsymbol{\xi} \in \mathbf{K}_c^J$. Let $\boldsymbol{\zeta}^{\boldsymbol{\xi}}(\cdot)$ be the unique fluid model solution for the initial condition $\boldsymbol{\xi}$. For the remainder of this proof, we indicate quantities associated with $\boldsymbol{\zeta}^{\boldsymbol{\xi}}(\cdot)$ by appending a $\boldsymbol{\xi}$. If $\boldsymbol{\xi} = \mathbf{0}$, then $\boldsymbol{\zeta}^{\boldsymbol{\xi}}(t) = \mathbf{0} = \boldsymbol{\nu}$ for each $t \geq 0$ by Lemma 4.0.2 (iii). Otherwise, note that because $\mathcal{L}_{\boldsymbol{\xi}}(\cdot)$ is a positive linear combination of the nonnegative $z_j^{\boldsymbol{\xi}}(\cdot)$'s, Theorem 10.2.2 implies that $d(\boldsymbol{\zeta}^{\boldsymbol{\xi}}(t), \boldsymbol{\nu}) \to 0$ as $t \to \infty$ uniformly for all $\boldsymbol{\xi} \in \mathbf{K}_c^J$.

Next we prove the $\varrho > 1$ case. Let $\delta > 0$. Define $C_c = \sum_{j=1}^J \frac{p_j}{\mu_j}\left(c + \alpha_j\langle\chi, \vartheta_j\rangle\right)$, where by Lemma 6.2.1, $\sup_{t\geq 0} \mathcal{L}(t) \leq C_c$ for each $\mathcal{L}(\cdot) \in \mathscr{L}_c^J$, where $\mathscr{L}_c^J$ is as defined in Theorem 10.2.1. Similar to the proof of Lemma 5.3.1, define $g_j(x,y) := \int_0^\infty \exp\left(\frac{-Kp_js}{y}\right)\alpha_j N_j^c(x+s)ds$ for each $x \geq 0, y > 0, j \in \mathcal{J}$. It is easy to check that g_j is uniformly continuous on $\mathbb{R}_+ \times [\mathcal{L}_{\min}, \tilde{C}]$. To give a short argument for this fact, let $\tilde{\epsilon} > 0$. Because $\int_0^\infty \alpha_j N_j^c(s)ds = \alpha_j\langle\chi, \vartheta_j\rangle < \infty$, $N_j^c(\cdot)$ is decreasing, and

$\lim_{s\to\infty} N_j^c(s) = 0$, we see that $\lim_{x\to\infty} \int_0^\infty \alpha_j N_j^c(x+s)ds = 0$ by the dominated convergence theorem. Therefore, there exists $M \geq 0$ such that $g_j(x,y) \leq \int_0^\infty \alpha_j N_j^c(x+s)ds < \tilde{\epsilon}$ for each $(x,y) \in [M,\infty) \times [\mathcal{L}_{min}, \tilde{C}]$. Therefore, for each $(x,y),(x',y') \in [M,\infty) \times [\mathcal{L}_{min}, \tilde{C}]$, we see that $0 \leq g_j(x,y) \vee g_j(x',y') \leq \tilde{\epsilon}$, and so $|g_j(x,y) - g_j(x',y')| \leq \tilde{\epsilon}$. On the compact set $[0,M] \times [\mathcal{L}_{min}, \tilde{C}]$ the continuous function g_j must be uniformly continuous, so there exists $\tilde{\delta}$ such that for $(x,y),(x',y') \in [0,M] \times [\mathcal{L}_{min}, \tilde{C}]$ satisfying $||(x,y) - (x',y')||_2 < \tilde{\delta}$, we have $|g_j(x,y) - g_j(x',y')| < \tilde{\epsilon}$, where $||\cdot||_2$ is the Euclidean norm on $\mathbb{R}^2$. Combining the two facts, we can deduce uniform continuity on the entire domain. Thus, there exists $\delta_2 \in (0, \mathcal{L}^* \wedge 1)$ such that for each $(x,y) \in \mathbb{R}_+ \times [\mathcal{L}_{\min}, C_c]$,

$$\sup_{j\in\mathcal{J}} \sup_{0<h\leq\delta_2} |g_j(x,y) - g_j(x,y+h)| < \frac{\delta}{4J}. \tag{10.8}$$

Applying Theorem 10.2.1, there exists $T_1 > 0$ such that

$$\sup_{\mathcal{L}(\cdot)\in\mathscr{L}_c^J} \sup_{t\geq T_1} |\mathcal{L}(t) - \mathcal{L}^*| \leq \delta_2. \tag{10.9}$$

By Lemma 4.0.2 (i), for $\boldsymbol{\xi} \in \mathbf{K}_c^J$, $\tilde{\boldsymbol{\zeta}}^{\boldsymbol{\xi}}(\cdot) := \boldsymbol{\zeta}^{\boldsymbol{\xi}}(\cdot + T_1)$ is also a fluid model solution. Denote all of the variables associated with such shifted fluid model solutions with a tilde. We assume without loss of generality that $T_1 > t_0$ as given in Lemma 4.0.3 so that for the shifted fluid model solutions, the associated $\tilde{\mathcal{L}}(\cdot)$ take values in $[\mathcal{L}_{\min}, C_c]$. Then by (10.9),

$$\sup_{\mathcal{L}(\cdot)\in\mathscr{L}_c^J} \sup_{t\geq 0} |\tilde{\mathcal{L}}(t) - \mathcal{L}^*| \leq \delta_2. \tag{10.10}$$

Combining (10.10) with (6.3) applied to $\tilde{\boldsymbol{\zeta}}^{\boldsymbol{\xi}}(\cdot)$, we see that for each $x \geq 0, t \geq 0, j \in \mathcal{J}, \boldsymbol{\zeta}(\cdot) \in \mathscr{M}_c^J$, the associated $\tilde{M}_j^c$ has the property that

$$\begin{aligned}
&\int_0^t \exp\left(-\frac{Kp_j r}{\mathcal{L}^* - \delta_2}\right) \alpha_j N_j^c(r+x)dr \\
&\leq \tilde{M}_j^c(t,x) \\
&\leq \int_0^t \exp\left(-\frac{Kp_j r}{\mathcal{L}^* + \delta_2}\right) \alpha_j N_j^c(r+x)dr + \exp\left(\frac{-Kp_j t}{\mathcal{L}^* + \delta_2}\right) \tilde{M}_j^c(0, t+x).
\end{aligned}$$

Define the error terms

$$\epsilon_{1,j}(t) := \int_t^\infty \exp\left(\frac{-Kp_j s}{\mathcal{L}^* - \delta_2}\right) \alpha_j N_j^c(s)ds \quad \text{and} \quad \epsilon_{2,j}(t) := \exp\left(\frac{-Kp_j t}{\mathcal{L}^* + \delta_2}\right)(c + \alpha_j \langle \chi, \vartheta_j \rangle).$$

Recalling that by Lemma 6.2.1, $c + \alpha_j \langle \chi, \vartheta_j \rangle$ is a bound on $z_j(\cdot)$, it follows that for each $j \in \mathcal{J}, x \geq 0, t \geq 0, \boldsymbol{\zeta}(\cdot) \in \mathscr{M}_c^J$,

$$g_j(x, \mathcal{L}^* - \delta_2) - \epsilon_{1,j}(t) \leq \tilde{M}_j^c(t,x) \leq g_j(x, \mathcal{L}^* + \delta_2) + \epsilon_{2,j}(t).$$

Because the error terms converge to zero as $t \to \infty$, independent of which $\boldsymbol{\zeta}(\cdot) \in \mathscr{M}_c^J$ we are considering, one can find $T_2 > 0$ such that for each $t > T_2, x \geq 0, j \in \mathcal{J}$, $\max\{\epsilon_{1,j}(t), \epsilon_{2,j}(t)\} \leq \frac{\delta}{4J}$. Combining this with (10.8), we see that for $t > T_1 + T_2, x \geq 0, \boldsymbol{\zeta}(\cdot) \in \mathscr{M}_c^J, j \in \mathcal{J}$,

$$g_j(x, \mathcal{L}^*) - \frac{\delta}{2J} \leq M_j^c(t,x) \leq g_j(x, \mathcal{L}^*) + \frac{\delta}{2J}.$$

In particular, on setting $x = 0$, we obtain $z_j^* - \frac{\delta}{2J} \leq z_j(t) \leq z_j^* + \frac{\delta}{2J}$, where $z_j^* = \int_0^\infty \alpha_j N_j^c(s) \exp\left(\frac{-Kp_j s}{\mathcal{L}^*}\right) ds$. Combining the above inequality with this last inequality, we obtain

$$z_j^* - g_j(x, \mathcal{L}^*) - \frac{\delta}{J} \leq M_j(t,x) \leq z_j^* - g_j(x, \mathcal{L}^*) + \frac{\delta}{J}.$$

Using (5.3), we see that $z_j^* - \int_0^\infty \alpha_j N_j^c(s+x) \exp\left(\frac{-Kp_j s}{\mathcal{L}^*}\right) ds = \langle 1_{[0,x]}, \nu_j \rangle$, $x \geq 0$, and so invoking the definition of the distance metric in (1.1)-(1.2), it follows that $d(\boldsymbol{\zeta}(t), \boldsymbol{\nu}) \leq \delta$ for all $t > T_1 + T_2, \boldsymbol{\zeta}(\cdot) \in \mathscr{M}_c^J$. $\square$

Chapter 11

Proofs for the Case of Exponential Patience Times

Proof of Theorem 5.4.1. In this proof, we will first reduce to the interval $[0, t^*)$. We will then show that, in the case of exponentially distributed patience times and initial conditions, the system of equations given by (5.5)–(5.6) has a unique solution, and that solution is the $\boldsymbol{z}(\cdot)$ associated with the fluid model solution $\boldsymbol{\zeta}(\cdot)$. Lastly, we will check that (5.4) satisfies Definition 4.0.1. Since the fluid model solution $\boldsymbol{\zeta}(\cdot)$ is unique by Theorem 5.1.1, it must be given by (5.4), where $\boldsymbol{x}(\cdot) = \boldsymbol{z}(\cdot)$ is the unique solution to (5.5)–(5.6) on $[0, t^*)$.

First, we note that by Definition 4.0.1 (iv), $t^* = +\infty$ when $\varrho > 1$. Therefore, solving for the fluid model solution on $[0, t^*)$ is the same as solving for the fluid model solution on $[0, \infty)$ in this case. When $\varrho \leq 1$, it follows from Lemma 4.0.2 (iii) that the fluid model solution is the zero measure on $[t^*, \infty)$. Therefore, it suffices to characterize the fluid model solution on $[0, t^*)$.

Next, we know from Lemma 6.2.2 that the system of equations given by (6.4), (6.2) and (4.3) has a unique solution, which is the $\boldsymbol{z}(\cdot)$ associated with the fluid model solution $\boldsymbol{\zeta}(\cdot)$ for the given parameters. Using an integrating factor argument, we see that (6.4), (6.2), (4.3), with the complementary cdf of the exponential distribution, $\exp(-\gamma_j \cdot)$, substituted in for $N_j^c(\cdot)$ and $M_j^c(0, t) = z_{0,j} \exp(-\gamma_j t)$, is equivalent to (5.5)–(5.6) (with $\boldsymbol{z}(\cdot)$ in place of $\boldsymbol{x}(\cdot)$) for each $j \in \mathcal{J}$. Therefore, by the uniqueness of solutions of the former, the system of equations given by (5.5)–(5.6) has a unique solution. Since the total mass function $\boldsymbol{z}(\cdot)$ also satisfies these equations, it follows that $\boldsymbol{x}(\cdot) = \boldsymbol{z}(\cdot)$ is the unique solution of (5.5)–(5.6).

Now that we have determined that the total mass function $\boldsymbol{z}(\cdot)$ of the fluid model solution is the solution to (5.5)–(5.6), it remains to check that (5.4) satisfies (i)–(iv) of Definition (4.0.1), where (iii) only needs to be checked on $[0, t^*)$. It is immediate from (5.4) that (i) and (ii) of Definition 4.0.1 hold. Furthermore, because we have already shown that the unique solution of (5.5)–(5.6) is the total mass function of the fluid model solution, (iv) must hold as well. So the last thing we must check is that (iii) holds. It suffices to do this on $[0, t^*)$. Substituting (5.4), where $\boldsymbol{x}(\cdot) = \boldsymbol{z}(\cdot)$ satisfies (5.5), into the right hand side of (4.5), we obtain for each $j \in \mathcal{J}, f \in \mathscr{C}, t \in [0, t^*)$,

$$\langle f, z_{0,j}\vartheta_j\rangle - \int_0^t z_j(s)\langle f', \vartheta_j\rangle ds \tag{11.1}$$

$$- \int_0^t \frac{Kp_j z_j(s)}{\mathcal{L}(s)}\langle f, \vartheta_j\rangle ds + \alpha_j t\langle f, \vartheta_j\rangle \tag{11.2}$$

$$= z_{0,j}\langle f, \vartheta_j\rangle - \int_0^t \gamma_j z_j(s)\langle f, \vartheta_j\rangle ds \tag{11.3}$$

$$- \int_0^t \frac{Kp_j z_j(s)}{\mathcal{L}(s)}\langle f, \vartheta_j\rangle ds + \alpha_j t\langle f, \vartheta_j\rangle \tag{11.4}$$

$$= \langle f, \vartheta_j\rangle \left(z_{0,j} - \int_0^t \gamma_j z_j(s) ds - \int_0^t \frac{Kp_j z_j(s)}{\mathcal{L}(s)} ds + \alpha_j t \right) \tag{11.5}$$

$$= \langle f, \vartheta_j\rangle z_j(t) \tag{11.6}$$

$$= \langle f, z_j(t)\vartheta_j\rangle, \tag{11.7}$$

where (11.3) comes from integration by parts using the fact that $f(0) = 0$ and $\gamma_j \exp(-\gamma_j)$ is the density for ϑ_j, and (11.6) comes from applying (5.5). Thus (iii) of Definition 4.0.1 holds. By the uniqueness of fluid model solutions, (5.4) must be the unique fluid model solution. □

Proof of Theorem 5.4.2. By Lemma 5.3.1, $\mathcal{L}^*$ is the unique fixed point of (5.1). We show in the proof of Lemma 5.3.1 that (5.1) is equivalent to (5.2). Substituting $\exp(-\gamma_j \cdot)$ in for $N_j^c(\cdot)$, (5.2) simplifies to (5.8). We conclude that $\mathcal{L} = \mathcal{L}^*$ is the unique solution of (5.8). By (5.3), the unique invariant state $\boldsymbol{\nu}$ satisfies

$$\begin{aligned}\langle 1_{(x,\infty)}, \nu_j \rangle &= \alpha_j \int_0^\infty \exp(-\gamma_j(s+x)) \exp\left(\frac{-Kp_j s}{\mathcal{L}^*}\right) ds \\ &= \frac{\alpha_j \mathcal{L}^*}{\gamma_j \mathcal{L}^* + Kp_j} \exp(-\gamma_j x)\end{aligned}$$

for each $x \geq 0$. Hence $\boldsymbol{\nu}$ is of the form (5.7). □

Proof of Theorem 5.4.3. Assume that for each $j \in \mathcal{J}$, there exists a probability measure σ^j on $[0,\infty)$ such that $\zeta_j(\cdot) = z_j(\cdot)\sigma^j$. Suppose $\boldsymbol{\zeta}(\cdot) \not\equiv \boldsymbol{\nu}$, the invariant state. Fix $j \in \mathcal{J}$. For $\beta > 0$, taking the Laplace transform of the fluid model solution, $L_\beta^j(t) = \int_0^\infty \exp(-\beta s) d\zeta_j(s), t \geq 0$, we see by setting $f(x) = \exp(-\beta x) - 1$ in (4.5), that for each $t \in [0, t^*)$, where $t^* = \inf\{s \geq 0 : \mathcal{L}(s) = 0\}$,

$$\begin{aligned}L_\beta^j(t) - z_j(t) = L_\beta^j(0) - z_j(0) + \beta \int_0^t L_\beta^j(s) ds - \int_0^t \frac{Kp_j L_\beta^j(s)}{\mathcal{L}(s)} ds \\ + \int_0^t \frac{Kp_j z_j(s)}{\mathcal{L}(s)} ds + \alpha_j t(\langle \exp(-\beta \cdot), \vartheta_j \rangle - 1).\end{aligned}$$

Now, define $\sigma_\beta^j := \langle \exp(-\beta \cdot), \sigma^j \rangle$. Then $L_\beta^j(\cdot) = z_j(\cdot)\sigma_\beta^j$. Since $\beta > 0$, $\sigma_\beta^j < 1$. Then, on multiplying the above equation by $\frac{1}{\sigma_\beta^j - 1}$ we obtain for $t \in [0, t^*)$,

$$z_j(t) = z_j(0) - \frac{\beta \sigma_\beta^j}{1 - \sigma_\beta^j} \int_0^t z_j(s) ds - \int_0^t \frac{Kp_j z_j(s)}{\mathcal{L}(s)} ds + \alpha_j t \left(\frac{1 - \langle \exp(-\beta \cdot), \vartheta_j \rangle}{1 - \sigma_\beta^j} \right). \quad (11.8)$$

From (11.8) we see that $z_j(\cdot)$ is absolutely continuous on $[0, t^*)$. Taking the derivative of

$z_j(\cdot)$, we obtain

$$z_j'(t) = \left(\frac{-\beta\sigma_\beta^j}{1-\sigma_\beta^j} - \frac{Kp_j}{\mathcal{L}(t)} \right) z_j(t) + \alpha_j \left(\frac{1 - \langle \exp(-\beta\cdot), \vartheta_j \rangle}{1-\sigma_\beta^j} \right), \quad t \in [0, t^*). \tag{11.9}$$

For this equation, we reduce to the case where $\boldsymbol{z}(\cdot)$ is not constant. Indeed, if $\boldsymbol{z}(\cdot)$ were constant, t^* would either be 0 or $+\infty$. If $t^* = 0$, then $\varrho \leq 1$ by our assumptions, and so by Lemma 4.0.2 (iii) the fluid model solution is $\boldsymbol{\zeta}(\cdot) \equiv \mathbf{0}$, the invariant solution for $\varrho \leq 1$. If $t^* = \infty$, then $\zeta_j(t) = z_j(0)\sigma^j$ for all $t \geq 0$, $j \in \mathcal{J}$, which would mean that $\boldsymbol{\zeta}(\cdot)$ is invariant. In both cases, this contradicts the assumption that $\boldsymbol{\zeta}(\cdot) \not\equiv \boldsymbol{\nu}$. If $\boldsymbol{z}(\cdot)$ is not constant, there exist $s, t \in [0, t^*)$ such that $z_j(s) \neq z_j(t)$. Subtracting (11.9) at time s from (11.9) at time t yields

$$\frac{-\beta\sigma_\beta^j}{1-\sigma_\beta^j} = \frac{(z_j'(t) - z_j'(s)) + \left(\frac{Kp_j z_j(t)}{\mathcal{L}(t)} - \frac{Kp_j z_j(s)}{\mathcal{L}(s)} \right)}{z_j(t) - z_j(s)}$$

for each $\beta > 0$, where the right hand side does not depend on β. If we define

$$\gamma_j := \frac{(z_j'(t) - z_j'(s)) + \left(\frac{Kp_j z_j(t)}{\mathcal{L}(t)} - \frac{Kp_j z_j(s)}{\mathcal{L}(s)} \right)}{z_j(s) - z_j(t)}, \tag{11.10}$$

it follows that

$$\gamma_j = \frac{\beta\sigma_\beta^j}{1-\sigma_\beta^j}, \qquad \beta > 0,$$

and so

$$\sigma_\beta^j = \frac{\gamma_j}{\gamma_j + \beta}, \qquad \beta > 0,$$

which uniquely characterizes σ^j as an exponential distribution with mean $\frac{1}{\gamma_j}$. Revisiting (11.9), if one fixes t and substitutes in γ_j for $\frac{\beta\sigma_\beta^j}{1-\sigma_\beta^j}$, it follows that there exists some constant C such that

$$C = \left(\frac{1 - \langle \exp(-\beta\cdot), \vartheta_j \rangle}{1-\sigma_\beta^j} \right),$$

for each $\beta > 0$. Taking β to infinity we see that $C = 1$. Therefore the Laplace transform of ϑ_j is equal to the Laplace transform of σ^j. It follows that σ^j and ϑ_j are both exponentially distributed with mean $\frac{1}{\gamma_j}$.

□

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