UNIVERSITY OF CALIFORNIA SAN DIEGO

Dowling set partitions, and Positional marked patterns

A dissertation submitted in partial satisfaction of the requirements for the degree Doctor of Philosophy

in

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by

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The dissertation of Sittipong Thamrongpairoj is approved, and it is acceptable in quality and form for publication on microfilm and electronically:

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DEDICATION

To much beloved professor Jeffrey B. Remmel.

EPIGRAPH

Because I typeset this document myself, all errors can be blamed on my computer.

—Bruce Sagan

TABLE OF CONTENTS

LIST OF NOTATIONS

$A_m(n,k)$	the first generalized Eulerian polynomial
$B_m(n,k)$	the second generalized Eulerian polynomial
$E_{n,k}$	Eulerian number
$F_{m,n,1}(x)$	the first generating function for the Whitney number of the second kind
$F_{m,n,2}(x)$	the first generating function for the Whitney number of the first kind
$G(\sigma)$	the graph of σ
$MMP(a,b,c,d)$	marked mesh pattern
$\mathbb{N}$	the set of natural numbers $\{0,1,2,\ldots\}$
$[n]$	the set of integers $\{1,2,\ldots,n\}$
$OW^{(k)}_{m,p}(t)$	the generating function for ordered Dowling set partitions avoiding p
Π_n	the set of partitions of n
$\Pi_n(\tau)$	the set of partitions of n avoiding pattern τ
pmp_τ	positional marked pattern τ
$P(x)\big\vert_{x^n}$	the coefficient in front of x^n in $P(x)$
$P_{n,\tau}(x)$	the generating function for positional marked pattern τ
$red(w)$	reduction of w
S_n	the set of symmetric group of order n
S^*_n	the set of positional marked patterns of length n
$s(n,k)$	Stirling number of the first kind
$S(n,k)$	Stirling number of the second kind
$w_m(n,k)$	Whitney number of the first kind
$W_m(n,k)$	Whitney number of the second kind

LIST OF FIGURES

LIST OF TABLES

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ABSTRACT OF THE DISSERTATION

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and
Positional marked patterns

by

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This dissertation consists of two parts. First, we define and study positional marked patterns. A *positional marked pattern* is a permutation τ where one of elements in τ is underlined. Given a permutation σ, we say that σ is τ-*match* at position i if τ occurs in σ in such a way that σ_i plays the role of the underlined element in the occurrence. We let $pmp_\tau(\sigma)$ denote the number of position i which σ is τ-match. This defines a new class of statistics on permutations, where we study such statistics and prove a number of results. In particular, we prove that two positional marked patterns $\underline{1}23$ and $\underline{1}32$ give

rise to two statistics that have the same distribution. The equidistibution phenomenon also occurs in other several pairs of patterns like $1\underline{2}3$ and $1\underline{3}2$, which we prove in this dissertation. The second part of the dissertation focuses on the Whitney numbers of Dowling lattices. In the papers [2, 3], Benoumhani defined two polynomials $F_{m,n,1}(x)$ and $F_{m,n,2}(x)$. Then, he defined $A_m(n,k)$ and $B_m(n,k)$ to be the polynomials satisfying $F_{m,n,1}(x) = \sum_{k=0}^{n} A_m(n,k)x^{n-k}(x+1)^k$ and $F_{m,n,2}(x) = \sum_{k=0}^{n} B_m(n,k)x^{n-k}(x+1)^k$. In this dissertation, we give a combinatorial interpretation of coeffcients of $A_{m+1}(n,k)$ and prove a symmetry of the coefficients, namely $A_{m+1}(n,k)|_{m^s} = A_{m+1}(n,n-k)|_{m^{n-s}}$. We also give a combinatorial interpretation of $B_{m+1}(n,k)$, prove that $B_{m+1}(n,n-1)$ is a polynomial in m with non-negative integer coefficients, and prove that $B_{m+1}(n,n-2)$ is a polynomial in m with non-negative integer coefficients except for the coefficient of m^{n-1} which is $-(n-1)$ for $n \geq 6$.

Chapter 1

Introduction

This dissertation studies two main topics: positional marked patterns, and Dowling set partitions.

1.1 Brief history of Permutation Patterns

Let S_n be the set of symmetric group of order n. In this dissertation, we use one-line notation to represent permutations. We say that a permutation σ *contains* a pattern τ if σ has a subsequence that is order-isomorphic to τ, and σ *avoids* τ if σ does not contain τ. Let $S_n(\tau)$ denotes the set of symmetric group of permutations avoiding τ of order n. We say that two patterns τ_1 and τ_2 are *Wilf-equivalent* if $|S_n(\tau_1)| = |S_n(\tau_2)|$ for all positive integers n. The study of permutation patterns is a central part of enumerative combinatorics.

As one of the early results [22], MacMahon studied the permutations which can be divided into two decreasing subsequences. Such permutations are exactly the permutations that avoid the pattern 123. He showed that the number of such permutations is enumerated by the Catalan numbers. A permutation is *stack-sortable* if it can be sorted using only single stack data structure as internal storage. Knuth [23] showed that a permutation π is stack-sortable if the permutation avoids 231. He proved that the number of stack-sortable permutations is also enumerated by the Catalan numbers, and so are permutations avoiding 231. Thus, MacMahon and Knuth together proved that the number of permutations avoiding 123 and the number of permutations avoiding 231 are the same.

Simion and Schmidt [33] systematically studied enumeration of pattern avoiding permutations. They gave a bijective proof that the number of permutations avoidng 123 is the same as the number of permutations avoiding 231. Since then, the study of permutation patterns has been growing and become one of the main research interests in enumerative combinatorics as it has rich connections to other fields like algebraic combinatorics, computer science and physics.

1.2 Marked Mesh Patterns and Positional Marked Patterns

Several generalizations of patterns has been defined and studied. In particular, we are interested is a notion of *quadrant marked mesh patterns* defined by Kitaev and Remmel [13].

Throughout this dissertation, we let $\mathbb{N} = \{0, 1, 2, \ldots\}$ denote the set of natural numbers and $[n]$ denote the set of integers $\{1, 2, \ldots, n\}$. Let S_n denote the symmetric group of permutations of $[n]$, then we consider the *graph* of $\sigma, G(\sigma)$, to be the set of points (i, σ_i), we were interested in the points that lie in the four quadrants I, II, III, IV of the Cartesian coordinate system as pictured in Figure 1.1.

Definition 1.1. *For any $a, b, c, d \in \mathbb{N}$ and any $\sigma = \sigma_1 \ldots \sigma_n \in S_n$, we say that σ_i matches* the simple marked mesh pattern $MMP(a, b, c, d)$ *if in $G(\sigma)$ relative to the coordinate system which has the point (i, σ_i) as its origin, there are at least a points in quadrant I, at least b points in quadrant II, at least c points in quadrant III, and at least d points in quadrant IV. We let $mmp^{(a,b,c,d)}(\sigma)$ denote the number of elements i such that σ_i matches*

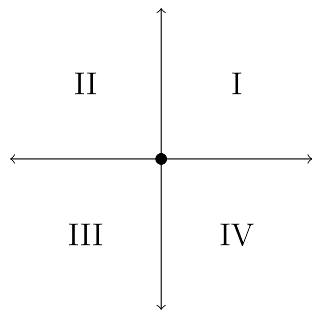


Figure 1.1: Diagram for marked mesh pattern

the marked mesh pattern $MMP(a,b,c,d)$ *in* σ*.*

As an example, let $\sigma = 53168724$ and consider marked mesh pattern $MMP(1,1,0,0)$. The graph of σ and the illustration of $MMP(1,1,0,0)$ can be seen in Figure 1.2. In this case, we see that 3, 1 and 2 matches $MMP(1,1,0,0)$ in σ, so $mmp^{(1,1,0,0)}(\sigma) = 3$.

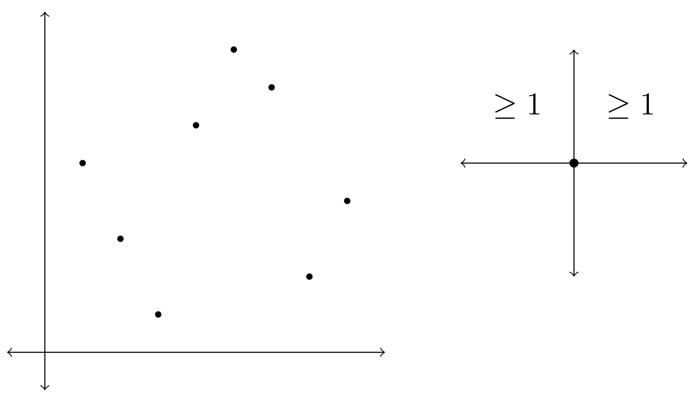


Figure 1.2: Example for marked mesh pattern

Remmel and Kitaev [13] provided explicit formulas for the generating functions of distributions of quadrant marked mesh patterns in several general cases, and developed recursions in other cases. The notion was studied further by Kitaev and Remmel [14, 15], Davis [8], Kitaev, Remmel and Tiefenbruck [16, 17, 18], and Qiu and Remmel [30].

Having quadrant marked mesh patterns as a motivation, we define the notion of *positional marked pattern*. A positional marked pattern of length k is a permutation of

$[k]$ with one of the elements underlined. Given any positional marked pattern τ, let $\pi(\tau)$ denote the element in S_k obtained from τ by removing the underline, and let $u(\tau)$ be the position of the underlined element in τ.

Definition 1.2. *Given $\sigma = \sigma_1\sigma_2\cdots\sigma_n \in S_n$ and positional marked pattern τ, we say that σ is a τ-*match *at position l if σ contains a pattern $\pi(\tau)$ in such a way that l-th element in σ plays the role of the underlined elements in τ. Let $pmp_\tau(\sigma)$ denote the number of position l such that σ is a τ-match at position l.*

As an example, let $\sigma = 145236$. Consider positional marked patterns $\underline{1}23, 1\underline{2}3$, and $12\underline{3}$. Note that there are 8 subsequences of length 3 in σ that are order-isomorphic to the pattern 123 , namely

$$145, 146, 156, 123, 126, 136, 456, 236.$$

One can see that 1, 4 and 2 play the role of 1 in at least one of the occurrences, so each will contribute 1 to $pmp_\tau(\sigma)$. In this case, we see that $pmp_\tau(\sigma) = 3$. Similarly, $pmp_{1\underline{2}3}(\sigma) = 4$ and $pmp_{12\underline{3}}(\sigma) = 3$. Our main goal is that, for each positional marked pattern τ, we study polynomials of the form

$$P_{n,\tau}(x) = \sum_{\sigma \in S_n} x^{pmp_\tau(\sigma)}.$$

Given two positional marked patterns τ_1, τ_2, we say that τ_1 *and* τ_2 *are positional Wilf-equivalent* if, for all $n \geq 1$,

$$P_{n,\tau_1}(x) = P_{n,\tau_2}(x).$$

We shall see that in Chapter 2 positional marked patterns are refinement of quadrant marked mesh patterns. For the main result, we show that there are only two classes of positional marked patterns of length 3. In particular, we prove the following new theorem in Chapter 2:

Theorem 1.1. *Two positional marked patterns* $1\underline{2}3$ *and* $2\underline{3}1$ *are positional Wilf-equivalent.*

This result is significant as we generalize one of the most fundamental theorems in classical permutation patterns.. As we mentioned that one of the first results that motivates the study of permutation patterns is the fact that two patterns 123 and 231 are equinumerous. Theorem 1.1 simply says that not only 123-avoiders and 231-avoiders are equinumerous, but also the distributions of the number of midpoints of patterns 123 and 231 are the same. In Chapter 2 We also prove the following equivalence of patterns of length 4 and arbitrary length.

Theorem 1.2. *Four positional marked patterns* $1\underline{2}34, \underline{2}134, 1\underline{2}43$ *and* $\underline{2}143$ *are positional Wilf-equivalent.*

Theorem 1.3. *Given two pmp* $P_1 = 1\underline{2}p_1 \ldots p_{l-2}$ *and* $P_2 = \underline{2}1p_1 \ldots p_{l-2}$*. Then,* P_1 *and* P_2 *are pmp-Wilf equivalent.*

1.3 Dowling set partitions and Whitney numbers

Chapter 3 of the dissertation mainly focuses on Dowling set partitions and Whitney numbers, which can be realized as generalizations of set partitions and the Stirling numbers of the second kind, $S(n, k)$.

Remmel and Wachs [31] defined a class of combinatorial objects called *Dowling set partition* as follows: Given a set partition $\pi = (A_1, \ldots, A_k, A_{k+1})$ of $[n+1]$, we shall assume that it is written in a standard form which means that

1. $\min(A_1) < \cdots < \min(A_k)$ and

2. $n+1 \in A_{k+1}$.

In such a situation, we let $Min(\pi) = \{\min(A_1), \ldots, \min(A_k)\}$ and $Last(\pi) = A_{k+1}$.

Definition 1.3. a Dowling set partition of $[n]$ with k parts and m colors *is a pair* (π, c) *where* $\pi = (A_1, \ldots, A_k, A_{k+1})$ *is a set partition of* $[n+1]$ *into* $k+1$ *parts and* c *is a map from* $[n+1] - (Min(\pi) \cup Last(\pi))$ *into* $[m]$. *We shall view c as coloring of the non-minimal elements in each of parts* $A_1, \ldots, A_k$ *with one of* m *colors from* $\{1, \ldots, m\}$. *We let* $\mathcal{W}_m(n,k)$ *denote the set of Dowling set partitions of* $[n]$ *with* k *parts and* m *colors.*

In what follows, we shall represent (π, c) in the form $A_1/A_2/\ldots/A_k/A_{k+1}$ where we put superscripts on the elements to indicate the color. We will also distinguish the largest element $n+1$ by making it boldface. For example,

$$1/2\ 6^1/4\ 5^1\ 7^3\ 8^2/3\ \mathbf{9}$$

is a Dowling set partition of 8 with 3 parts and 3 colors. Dowling set partitions of $[n]$ with k parts and m colors are enumerated by *the Whitney numbers of the second kind*, $W_m(n,k)$.When $m = 1$, the Whitney numbers of the second kind is the Stirling numbers of the second kind, so the Whitney numbers of the second kind can be seen as generalizations of the Stirling numbers of the second kind.

The Stirling numbers of the second kind, denoted by $S(n,k)$, are one of the most well-known combinatorial numbers. $S(n,k)$ counts the number partitions of $[n]$ into k non-empty subsets. The Stirling numbers of the second kind have been studied extensively and many results are known. For example, $S(n,k)$ have the following generating functions:

$$\sum_{n=1}^{\infty} S(n,k)x^n = \frac{x^k}{(1-x)(1-2x)\dots(1-kx)}$$
$$\sum_{n=1}^{\infty} S(n,k)\frac{x^n}{n!} = \frac{1}{k!}(e^x-1)^k$$

We are mainly interested in a result that relates the Stirling numbers of the second kind to the Eulerian numbers. Given a permutation $\sigma = \sigma_1\sigma_2\dots\sigma_n$, let the *descent number of* σ, denoted by $des(\sigma)$, be the number of i such that $\sigma_i > \sigma_{i+1}$. Denoted by $E_{n,k}$, the Eulerian numbers is the number of $\sigma \in S_n$ such that $des(\sigma) = k$. A well-known identity relating the Stirling numbers of the second kind to the Eulerian numbers is that

Proposition 1.4.
$$\sum_{k=1}^{n} k!S(n,k)x^k = \sum_{k=0}^{n-1} E_{n,k}x^k(x+1)^{n-k-1}$$

Proposition 1.4 is easily derived (See [7]). Here, we give a combinatorial proof to the proposition. Let $\mathcal{OSP}_n$ denote the set of all ordered set partitions of $[n]$ and for any $\pi \in \mathcal{OSP}_n$, let $\ell(\pi)$ denote the number of parts of π. Let $\mathcal{E}_{n,k}$ denote the set of permutations in S_n with k descents. Then the left-hand side of (3.1) is the generating function $x^{\ell(\pi)}$ over all elements $\pi \in \mathcal{OSP}_n$. Given an ordered set partition $\pi = A_1/A_2/\dots/A_j$ where the sets A_i are written in increasing order, let σ_π denote the permutation that arises by removing the slashes. For example, if $\pi = 3\ 5\ 6/1\ 7\ 8/2\ 4/9$, then $\sigma_\pi = 356178249$. Then given any $\sigma \in \mathcal{E}_{n,k}$, it is easy to see that

$$\sum_{\pi \in \mathcal{OSP}_n, \sigma_\pi = \sigma} x^{\ell(\pi)}$$

is $x^k(x+1)^{n-k-1}$. That is we must put a slash after every descent of σ and after every rise of σ we can either put a slash or not. For example, $\sigma = 142563$ gives rise to the following ordered set partitions π such that $\sigma_\pi = \sigma$:

$$1\ 4/2\ 5\ 6/3 \quad 1/4/2\ 5\ 6/3 \quad 1\ 4/2/5\ 6/3 \quad 1/4/2/5\ 6/3$$

$$1\ 4/2\ 5/6/3 \quad 1/4/2\ 5/6/3 \quad 1\ 4/2/5/6/3 \quad 1/4/2/5/6/3.$$

In [3], Benoumhani asked for the generalization the equation (1.4) for the Whitney numbers of the second kind. In specific, he defined two polynomials based on the left-hand side of the equation (1.4):

$$F_{m,n,1}(x) = \sum_{k=0}^{n} k! m^k W_m(n,k) x^k$$

and

$$F_{m,n,2}(x) = \sum_{k=0}^{n} k! W_m(n,k) x^k.$$

He then asked for a combinatorial interpretation of the analogue of the right-hand side of (1.4). That is, he asked for combinatorial interpretations of $A_m(n,k)$ and $B_m(n,k)$ where

$$F_{m,n,1}(x) = \sum_{k=0}^{n} A_m(n,k) x^{n-k} (x+1)^k$$

and

$$F_{m,n,2}(x) = \sum_{k=0}^{n} B_m(n,k) x^{n-k} (x+1)^k.$$

We note that Benoumhani [3] was able to derive generating functions for the polynomials $F_{m,n,1}(x)$ and $F_{m,n,2}(x)$. That is, he proved that

$$\begin{aligned}\sum_{n\geq 0} F_{m,n,1}(x)\frac{z^n}{n!} &= \frac{e^z}{1-x(e^{mz}-1)} \text{ and}\\ \sum_{n\geq 0} F_{m,n,2}(x)\frac{z^n}{n!} &= \frac{e^z}{1-(x/m)(e^{mz}-1)}.\end{aligned}$$

Benoumhani [3] also proved that the polynomials $F_{m,n,1}(x)$ and $F_{m,n,2}(x)$ have only real zeros from which it follows that sequence $\{k!W_m(n,k)\}_{0\leq k\leq n}$ are log-concave.

Mezö [27] found another analogue of the relationship between the Stirling numbers and the Eulerian numbers for the Whitney numbers of Dowling lattices. His point of departure was that fact that

$$\begin{aligned}\sum_{n=0}^{k} k!S_{n,k} &= \sum_{k=0}^{n-1} A_{n,k}2^{n-k} \text{ and}\\ k!S_{n,k} &= \sum_{i=0}^{n-1} E_{n,k}\binom{n-i}{k-1}.\end{aligned}$$

Mezö [27] found an analogue $E_m(n,k)$ of the Eulerian numbers in Dowling lattices such that

$$\sum_{n=0}^{k} k!m^k W_m(n,k) = \sum_{k=0}^{n} E_m(n+1,k+1)2^{n-k}$$

as well as a refinement $E_{m,d}(n,k)$ of these numbers such that

$$k!m^k W_m(n,k) = \sum_{d=1}^{n+1-k}\sum_{i=d}^{k+d} E_{m,d}(n+1,i+1)\binom{n-i}{k-k+d}.$$

For our main results in Chapter 3, we give complete combinatorial interpretation of coefficients of $A_{m+1}(n,k)$. Given a polynomial $p(x)$, let $p(x)|_{x^n}$ denotes the coeffient in front of x^n in $p(x)$. We also prove a symmetry of the coefficient: $A_{m+1}(n,k)|_{m^s} = A_{m+1}(n,n-k)|_{m^{n-s}}$. We also give a combinatorial interpretation $B_{m+1}(n,k)$ and prove that coefficients of $B_{m+1}(n,k)$ is positive in some cases.

We shall see in Chapter 3 that the Whitney numbers was defined from more general structures called Dowling lattices [9]. Several authors further studied Dowling lattices and Whitney numbers [2, 3, 6, 12, 27]. However, for our interests, we will mainly use Dowling set partitions as the way to define the Whitney numbers of the second kind.

1.4 Patterns in set partitions

The notion of patterns was generalized to other combinatorial objects such as words and posets. We are interested in study of patterns in set partitions and signed permutations. Many authors defined and studied patterns in set partitions [19, 20, 21, 32, 11] and colored permutations [24, 25]. In particular, Sagan [32] defined patterns in set partitions as follows.

A partition π of $[n]$ is a set of nonempty subset $B_1, B_2, \ldots, B_k$ of $[n]$ such that the disjoint union of B_i is $[n]$. For our convenience, we shall omit braces and comma, and denote a set partition π writing π in the form $B_1/B_2/\ldots/B_k$. For example, $\pi = 135/2/46$ is a set partition of [6]. We let Π_n denotes the set of partitions of $[n]$

Given a partition τ of $[k]$ and a partition π of $[n]$, we say that π contains τ if we can obtain τ from π by eliminating some of the elements of $[n]$ from π then reducing the result set partition by replacing the i-th largest element by i, throwing away any empty parts. For example, if $\pi = 135/2/46$ and $\tau = 1/23$, then π contains τ as we can eliminate every elements except 1,4 and 6 and reduce. Notice that keeping 2,3 and 5 would show the containment too as the order of the blocks does not matter. We say that π avoids τ if π does not contain τ. We let $\Pi_n(\tau)$ denotes the set of partitions of $[n]$ avoiding τ.

Sagan [32] proved the carnality of $\Pi_n(\tau)$ for various τ including all partitions τ of

[3]. For example, he proved that

$$\begin{aligned}\Pi_n(13/2) &= 2^{n-1}\\ \Pi_n(12/3/\ldots/m) &= 1+\sum_{k=1}^{n-1}\sum_{j=1}^{m-2} a^{\mathbf{P}}_{n-k,j}\sum_{i=1}^{j}\binom{j-1}{i-1}(k)_i.\end{aligned}$$

We are interested in generalizing the notion of patterns to Dowling set partitions the same way Sagan defined for set partition. Recall from Definition 1.3, a Dowling set partition is a partition whose non-minimal and non-last elements are colored with one of m colors. Given a Dowling set partition π and a colored set partition τ, we say that π contains τ if we can obtain τ from π by eliminating some of the elements of $[n]$ from π then reducing the result colored set partition by replacing the i-th largest element by i *and replacing i-th largest color by* i. For example, if $\pi = 13_14_3/26_4/4\mathbf{7}$ and $\tau = 12_1/3_2/\mathbf{4}$, then π contains τ as we can eliminate every elements except $1, 3_1, 6_4$ and $\mathbf{7}$. We say that π avoids τ if π does not contain τ. We let $\mathcal{W}_{m,\tau}(n,k)$ denotes the set of Dowling set partitions of $[n]$ with k parts and m colors which avoids τ.

In Chapter 4, we enumerate $|\mathcal{W}_{m,\tau}(n,k)|$ for all patterns τ of length 3. In each case, we either give explicit formula, recursive formula or recursive formula for their refinements. In particular, the table shows enumeration results for each τ.

Table 1.1: Results for the $|\mathcal{W}_{m,\tau}(n,k)|$ for every τ

τ	Results
$1/2/\mathbf{3}$	Generating function: $\frac{1}{1-t}$
$1/2_1/\mathbf{3}$	Generating function: $\frac{t^k}{(1-(1+m)t)^k(1-t)}$
$1_1/2/\mathbf{3}$	Generating function: $\frac{t^k}{(1-(1+mk)t)(1-t)^k}$
$2/1_1/\mathbf{3}$	Every Dowling set partition avoids τ
$2_1/1/\mathbf{3}$	Generating function: $\frac{t^k}{(1-(1+m)t)^k(1-t)}$
$1_1/2_1/\mathbf{3}$, $2_1/1_1/\mathbf{3}$	Recursive formula
$1_1/2_2/\mathbf{3}$	Some Generating function for its refinement
$1_12_1/\mathbf{3}$	Recursive formula
$1_12_2\mathbf{3}$	Recursive formula
$1/2\mathbf{3}$	Explicit formula: $\sum_{s=k}^{n} S(s,k)m^{s-k}$
$1_1/2\mathbf{3}$	Recursive formula and explicit formula
$2_1/1\mathbf{3}$	GF: $\frac{t^k}{(1-t)^{k+1}} + \sum_{l=1}^{k} \left(\frac{t}{1-t}\right)^{k+1-l} \frac{mlt^l}{\prod_{i=1}^{l}(1-imt)}$
$12\mathbf{3}$	Explicit formula: $S(n,k)m^{n-k} + nS(n-1,k)m^{n-1-k}$

Chapter 2

Positional marked patterns

In this chapter, we prove some results on positional marked patterns, including *pmp*-Wilf equivalences of patterns of length 3, patterns of length 4 and pattern of arbitrary length n.

2.1 Introduction

The notion of mesh patterns was first introduced by Brändén and Claesson [4]. Many authors have further studied this notion. In particular, the notion of marked mesh pattern was introduced by Úlfarsson [35], and the study of the distributions of quadrant marked mesh patterns in permutations was initiated by Kitaev and Remmel in [13].

Here, we recall the definition of quadrant marked mesh pattern. Let $\mathbb{N} = \{0, 1, 2, \ldots\}$ denote the set of natural numbers and S_n denote the symmetric group of permutations of $1, \ldots, n$, then we consider the graph of $\sigma, G(\sigma)$, to be the set of points (i, σ_i), we are interested in the points that lie in the four quadrants I, II, III, IV of that coordinate system. For any $a, b, c, d \in \mathbb{N}$ and any $\sigma = \sigma_1 \ldots \sigma_n \in S_n$, we say that σ_i matches *the simple marked mesh pattern* $MMP(a, b, c, d)$ if in $G(\sigma)$ relative to the coordinate system which has the point (i, σ_i) as its origin, there are at least a points in quadrant I, at least b points in

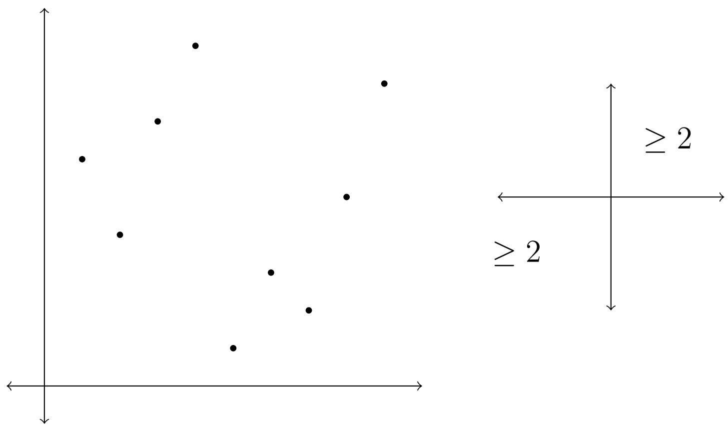


Figure 2.1: Example for marked mesh pattern

quadrant II, at least c points in quadrant III, and at least d points in quadrant IV. We let $mmp^{(a,b,c,d)}(\sigma)$ denote the number of i such that σ_i matches the marked mesh pattern $MMP(a,b,c,d)$ in σ.

For example, let $\sigma = 647913258$ and consider simple marked mesh pattern $MMP(2,0,2,0)$. We look for points (i,σ_i) in the graph of σ such that there are at least 2 points to the top right of (i,σ_i) and at least 2 points to the bottom left of (i,σ_i). The graph of σ and an illustration of $MMP(1,0,1,0)$ are shown in Figure 2.1. As we see in Figure 2.2, at point $(3,7)$, there are two points to the top right and 2 points to the bottom left of point $(3,7)$, so 7 matches $MMP(2,0,2,0)$ in σ. At point $(6,3)$, on the other hand, there are 2 points to the top right but only 1 point to the bottom left of point $(6,3)$. Therefore, 3 does not match $MMP(2,0,2,0)$ in σ. In this case, only 7 matches $MMP(2,0,2,0)$ in σ, so $mmp^{(2,0,2,0)}(\sigma) = 1$.

We studied a generalization of quadrant marked mesh patterns. As a motivation, consider a statistic $mmp(1,0,1,0)$. An equivalent way to think about $mmp(1,0,1,0)$ for any permutation σ is to count the number of elements in σ that can be the midpoint of the pattern 123 in σ. One can ask similar questions but for different patterns and positions. For example, one may consider the number of elements in σ that can be the starting point

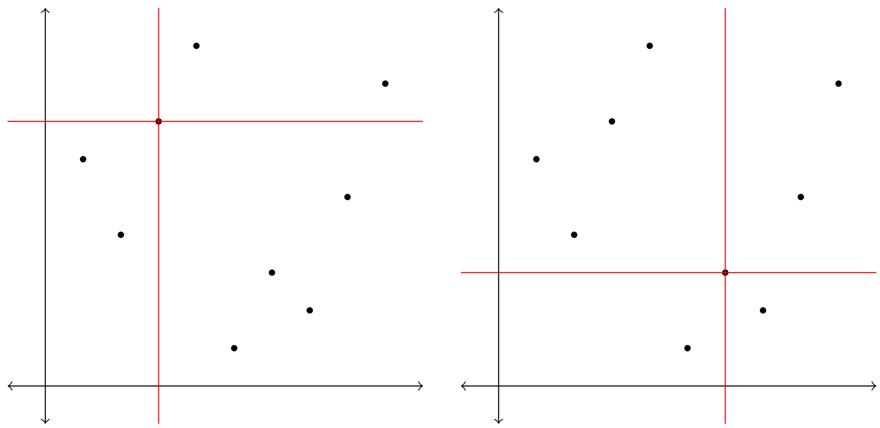


Figure 2.2: 7 matches $MMP(2,0,2,0)$ in σ but 3 does not

of pattern 2314.

Here, we briefly define the notion of *positional marked patterns (pmp)* which enumerate the described statistics. A positional marked pattern of length k is a permutation of $[k]$ with one of the elements underlined. Given any positional marked pattern τ, let $\pi(\tau)$ denote the element in S_k obtained from τ by removing the underline, and let $u(\tau)$ be the position of the underlined element in τ. Given $\sigma = \sigma_1\sigma_2\ldots\sigma_n \in S_n$ and positional marked pattern τ, we say that σ is τ-match at position l if σ contains a pattern $\pi(\tau)$ in such a way that l-th element in σ plays the role of the underlined elements in τ. Let $pmp_\tau(\sigma)$ denote the number of position l such that σ is τ-match at position l. We will carefully define positional marked patterns in Section 2.2.

Recall that, in classical patterns, two patterns τ_1 and τ_2 are Wilf-equivalent if the number of σ in S_n avoiding τ_1 is the same as that of τ_2 for all $n \in \mathbb{N}$. Here, we adopt the vocabulary from the classical definition to our definition. Given two positional marked patterns τ_1 and τ_2, we say that τ_1 and τ_2 are *pmp-Wilf equivalent* if pmp_{τ_1} and pmp_{τ_2} has the same distribution over S_n for all n. Our main goal is to classify *pmp*-Wilf equivalent classes for positional marked patterns.

The chapter outlines as follows: In section 2.2, we give a precise definition and some preliminary results on positional marked patterns. In section 2.3, we study positional marked patterns of length 3. We prove that there are only 2 *pmp*-Wilf equivalent classes for positional marked patterns of length 3. The result follows from the following theorem. We will prove in Section 2.3 that there are only 2 *pmp*-Wilf equivalent classes for positional marked patterns of length 3. The result follows from the following theorem.

Theorem 2.1. *Two positional marked patterns* $\underline{1}23$ *and* $\underline{1}32$ *are pmp-Wilf equivalent, and two positional marked patterns* $1\underline{2}3$ *and* $1\underline{3}2$ *are pmp-Wilf equivalent.*

In Section 2.9, we prove some non-trivial equivalences of pairs of positional marked patterns of length 4 as well as providing numerical results for every patterns of length 4. In Section 2.5, we prove the following theorem, which gives an equivalence of a pair of positional marked patterns of arbitrary length.

Theorem 2.2. *Given two positional marked patterns* $P_1 = 1\underline{2}p_1 \dots p_{l-2}$ *and* $P_2 = \underline{2}1p_1 \dots p_{l-2}$. *Then,* P_1 *and* P_2 *are pmp-Wilf equivalent.*

In Section 2.6, we discuss further research possibilities, as well as precise connection between positional marked patterns and quadrant marked mesh patterns.

2.2 Definitions

In this section, we define and give some preliminary results on positional marked patterns.

Definition 2.1. *Let* S_k^* *denote a set of permutations of* $[k]$ *with one of the elements underlined. Given any* $\tau \in S_k^*$, *let* $\pi(\tau)$ *denote a element in* S_k *obtained from* τ *by removing the underline, and let* $u(\tau)$ *be the position of the underlined element in* τ. *We shall call an element in* S_k^* *a positional marked pattern (pmp).*

For example, an element in S_4^* looks like $\tau = 1\underline{4}32$. In this case, $\pi(\tau) = 1432$ and $u(\tau) = 2$.

Definition 2.2. *Given a word w where alphabets are taken from $\mathbb{Z}_{\geq 0}$, a reduction of w, denoted by $red(w)$, is the word obtained from w by replacing the i-th smallest alphabet by i. In particular, if w is a word with distinct alphabets of length k, then $red(w) \in S_k$.*

For example, if $w = 25725$, then $red(w) = 12312$. Here, we are ready to define a statistics on S_n that we mainly focus on in this chapter.

Definition 2.3. *Given $\sigma = \sigma_1\sigma_2 \dots \sigma_n \in S_n$ and $\tau \in S_k^*$, we say that σ is τ-match at position l if there is a subsequence $\sigma_{i_1}\sigma_{i_2}\dots\sigma_{i_k}$ in σ such that*

1. *$i_l = u(\tau)$*
2. *$red(\sigma_{i_1}\sigma_{i_2}\dots\sigma_{i_k}) = \pi(\tau)$*

In other words, σ contains a pattern $\pi(\tau)$ in such a way that l-th element in σ plays the role of the underlined elements in τ. Let $pmp_\tau(\sigma)$ denote the number of position l such that σ is τ-match at position l.

For example, if $\sigma = 26481573$ and $\tau = 1\underline{4}32$. Then σ is τ-match at positions 2 and 4, as we find subsequences 2653 and 2853 respectively. Thus, $pmp_\tau(\sigma) = 2$. We are interested in the generating function $P_{n,\tau}(x) = \sum_{\sigma \in S_n} x^{pmp_\tau(\sigma)}$. We say that two positional marked patterns τ_1 and τ_2 are *pmp*-Wilf equivalent if $P_{n,\tau_1}(x) = P_{n,\tau_2}(x)$ for all n. Note that, by looking at the constant terms of generating functions, if τ_1 and τ_2 are *pmp*-Wilf equivalent, then $\pi(\tau_1)$ and $\pi(\tau_2)$ are Wilf-equivalent. Thus, one might think of *pmp*-Wilf equivalent as a stronger version of Wilf-equivalent. In this chapter, we classify equivalence classes of S_3^* and S_4^*.

We associate a positional marked pattern with a permutation matrix-like diagram. Given any $\tau \in S_k^*$, the diagram associated to τ is a k by k array with the following filling:

For the cell at i-th row and j-th column (i) the cell is filled with $\circ$ if $\tau_i = j$ and $u(\tau) = i$, (ii) the cell is filled with $\times$ if $\tau_i = j$ and $u(\tau) \neq i$, or (iii) the cell is empty otherwise. By convention, we count rows and columns of an array from left to right and from bottom to top. For example, if $\tau = 1\underline{4}32$, then the corresponding diagram is

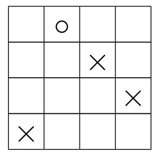


Figure 2.3: Diagram for $1\underline{4}32$

Similarly, we associate $\sigma \in S_n$ with an n by n diagram with the cell at i-th row and j-column is (i) filled with $\times$ if $\sigma_i = j$ and (ii) empty otherwise. For example, the diagram corresponding to $\pi = 26481573$ is

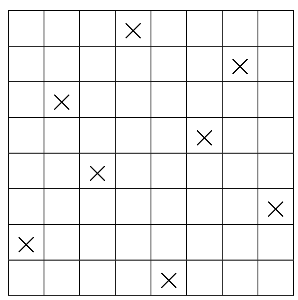


Figure 2.4: Diagram for 26481573

We can visualize *pmp* matching using diagrams. Given a permutation σ and a *pmp* τ, σ is τ-match at position i if when replacing $\times$ in the i-th column of the diagram of σ by $\circ$, then it contains a subdiagram τ. For example, $\sigma = 26481573$ is τ-match at position 2 and 4 by looking at subdiagrams marked in Figure 2.5.

With diagrams, we prove equivalences of *pmp* by symmetry. Given two *pmp* τ_1, τ_2 such that the diagram of τ_2 can be obtained from by applying a series of rotations and reflections to the diagram of τ_1, then we can construct a map $\theta : S_n \to S_n$ by applying the

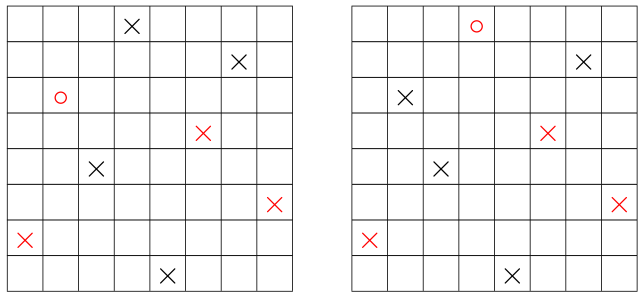


Figure 2.5: Matching of $1\underline{4}32$ in 26481573 through diagrams

same series of rotations and reflections to elements in S_n. The map will obviously have the property that $pmp_{\tau_1}(\sigma) = pmp_{\tau_2}(\theta(\sigma))$ for all $\sigma \in S_n$, and so τ_1 and τ_2 are *pmp*-Wilf equivalent. We proved the following lemma.

Lemma 2.1. *Given $\tau_1, \tau_2 \in S_k^*$, such that τ_2 can be obtained by applying series of reflections and rotation to τ_1. Then τ_1 and τ_2 are pmp-Wilf equivalent.*

Lemma 2.1 reduces the problem tremendously. There are $3! \cdot 3 = 18$ *pmp* of length 3. However, with Lemma 2.1, there are at most 4 equivalence classes, which are represented by $\underline{1}23, 1\underline{2}3, \underline{1}32, 1\underline{3}2$. Their diagrams are shown below:

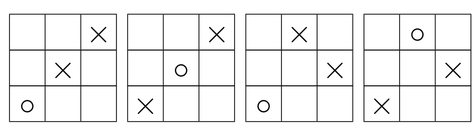


Figure 2.6: Diagrams for $\underline{1}23$, $1\underline{2}3$, $\underline{1}32$, $1\underline{3}2$

2.3 Equivalence classes of S_3^*

In this section, we prove that $\underline{1}23$ and $\underline{1}32$ are *pmp*-Wilf equivalent, and $1\underline{2}3$ and $1\underline{3}2$ are *pmp*-Wilf equivalent.

2.3.1 Equivalence of $\underline{1}23$ and $\underline{1}32$

In this section, we will prove the following theorem:

Theorem 2.3. *Two pmp $\underline{1}23$ and $\underline{1}32$ are pmp-Wilf equivalent.*

For our convenience, we let $\tau_1 = \underline{1}23$ and $\tau_2 = \underline{1}32$. We will first prove by showing that two generating functions satisfy a same recursive formula. We later will construct a bijection from S_n to itself that maps pmp_{τ_1} to pmp_{τ_2}, that is, a map θ such that $pmp_{\tau_1}(\sigma) = pmp_{\tau_2}(\theta(\sigma))$.

Given any permutation $\sigma = \sigma_1\sigma_2 \dots \sigma_n \in S_n$. We will look at the position where the last ascent occurs. In particular, σ is either has no ascent or there is k such that $\sigma_k < \sigma_{k+1} > \sigma_{k+2} > \dots > \sigma_n$. In that case, we say that the last ascent of σ is at position k. Let $P_{n,\tau_1,k}(x) = \sum x^{pmp_{\tau_1}(\sigma)}$ where the sum is over all permutation in S_n with the last ascent is at position k. Then, we have $P_{n,\tau_1}(x) = 1 + \sum_{k=1}^{n-1} P_{n,\tau_1,k}(x)$.

Given any permutation $\sigma = \sigma_1\sigma_2 \dots \sigma_n \in S_n$. Similar to τ_1, for τ_2, we will look at the position where the last descent occurs. σ is either $1\ 2\ \dots (n-1)\ n$, or there is k such that $\sigma_k > \sigma_{k+1} < \sigma_{k+2} < \dots < \sigma_n$. We say that the last descent of σ is at position k. Let $P_{n,\tau_2,k}(x) = \sum x^{pmp_{\tau_2}(\sigma)}$ where ths sum is over all permutation in S_n with the last descent is at position k. Then, we have $P_{n,\tau_2}(x) = 1 + \sum_{k=1}^{n-1} P_{n,\tau_2,k}$

First, we derive recursive formula for $P_{n,\tau_1,k}(x)$.

Lemma 2.2. *Let $\tau_1 = \underline{1}23$ and let $P_{n,\tau_1,k}(x) = \sum x^{pmp_{\tau_1}(\sigma)}$ where the sum is over all $\sigma \in S_n$ with the last ascent of σ is at position k. Then, $P_{n,\tau_1,k}(x)$ satisfies the following recursive formula:*

$$P_{n,\tau_1,k}(x) = (k-1)xP_{n-1,\tau_1,k-1}(x) + 1 + \sum_{l=1}^{k} P_{n-1,\tau_1,l}(x).$$

where $P_{n,\tau_1,0}(x) = P_{n,\tau_1,n}(x) = 0$ by convention.

Proof. We derive a recursive formula of $P_{n,\tau_1,k}(x)$ by looking at position of 1 in σ where the last ascent is at position k. So, σ has a following form:

$$\sigma = \sigma_1\ \sigma_2 \dots \sigma_{k-1}\ \sigma_k < \sigma_{k+1} > \sigma_{k+2} > \dots > \sigma_n.$$

Let t be the position of 1. We have 3 cases:

1. $1 \leq t \leq k-1$. In this case, $1, \sigma_k, \sigma_{k+1}$ form a pattern τ_1. Thus, σ is τ_1-match at postion t. Moreover, 1 does not inflence *pmp*-match at other positions. Thus, we can remove 1 from σ and reduce other elements. The last ascent will be at position $k-1$. Therefore, this case contributes $x(k-1)P_{n-1,\tau_1,k-1}(x)$.

2. $t = k$. In this case, σ is not τ_1-match at position k. We can remove 1 from σ and reduce other elements. The remaining permutation will either has no ascent, or the last ascent appears at some position between 1 and $k-1$. Thus, this case contributes $1 + \sum_{l=1}^{k-1} P_{n-1,\tau_1,l}(x)$

3. $t = n$. In this case, σ is not τ_1-match at position n. We can remove 1 from σ and reduce other elements. The remaining permutation will have the last ascent at position k. Thus, this case contributes $P_{n-1,\tau_1,k}(x)$

In total, we have

$$\begin{aligned} P_{n,\tau_1,k}(x) &= (k-1)xP_{n-1,\tau_1,k-1}(x) + 1 + \left(\sum_{l=1}^{k-1} P_{n-1,\tau_1,l}(x)\right) + P_{n-1,\tau_1,k}(x) \\ &= (k-1)xP_{n-1,\tau_1,k-1}(x) + 1 + \sum_{l=1}^{k} P_{n-1,\tau_1,l}(x). \end{aligned}$$

Note that, the first case does not exist when $k = 1$, but the formula is correct since $P_{n-1,\tau_1,0}(x) = 0$. Also, the third case does not exist when $k = n-1$. However, the formula is also correct since $P_{n,\tau_1,n}(x) = 0$.

$\square$

Here, we prove a similar result for τ_2.

Lemma 2.3. *Let* $\tau_2 = \underline{1}32$ *and let* $P_{n,\tau_2,k}(x) = \sum x^{pmp_{\tau_2}(\sigma)}$ *where the sum is over all* $\sigma \in S_n$ *with the last descent of* σ *is at position* k*. Then,* $P_{n,\tau_2,k}(x)$ *satisfies the following recursive formula:*

$$P_{n,\tau_2,k}(x) = (k-1)xP_{n-1,\tau_2,k-1}(x) + 1 + \sum_{l=1}^{k} P_{n-1,\tau_2,l}(x).$$

where $P_{n,\tau_2,0}(x) = P_{n,\tau_2,n}(x) = 0$ *by convention.*

Proof. We use the same strategy as in the case for τ_1. We derive a recursive formula for $P_{n,\tau_2,k}(x)$ by looking at the position of 1 in σ where the last descent is at position k. So, σ has a following form:

$$\sigma = \sigma_1\ \sigma_2 \ldots \sigma_{k-1}\ \sigma_k > \sigma_{k+1} < \sigma_{k+2} < \ldots < \sigma_n.$$

Let t be the position of 1. We have 2 cases:

1. $1 \leq t \leq k-1$. In this case, $1, \sigma_k, \sigma_{k+1}$ form a pattern τ_2. Thus, σ is τ_2-match at postion t. Moreover, 1 does not inflence τ_2-match at other positions. Thus, we can remove 1 from σ and reduce other elements. The last descent will be at position $k-1$. Therefore, this case contributes $x(k-1)P_{n-1,\tau_2,k-1}(x)$.

2. $t = k+1$. In this case, σ is not τ-match at position $k+1$. We can remove 1 from σ and reduce other elements. The remaining permutation will either has no descent, or the last ascent appears at some position between 1 and k. Thus, this case contributes $1 + \sum_{l=1}^{k} P_{n-1,\tau_1,l}(x)$

In total, we have

$$P_{n,\tau_2,k}(x) = (k-1)xP_{n-1,\tau_2,k-1}(x) + 1 + \sum_{l=1}^{k} P_{n-1,\tau_2,l}(x).$$

Note that, the first case does not exist when $k = 1$, but the formula is correct since $P_{n-1,\tau_2,0}(x) = 0$. Also, for the second case, if $k = n-1$, the last descent cannot be at position $k = n-1$. However, the formula is still correct since $P_{n-1,\tau_2,n}(x) = 0$. □

One can check that $P_{2,\tau_1,1}(x) = 1 = P_{2,\tau_2,1}(x)$. Since $P_{n,\tau_1,k}(x)$ and $P_{n,\tau_2,k}(x)$ satisfy the same recursive formula and have the same initial values, we proved that $P_{n,\tau_1,k}(x) = P_{n,\tau_2,k}(x)$ for all n, k such that $1 \le k \le n-1$, and hence $P_{n,\tau_1}(x) = P_{n,\tau_2}(x)$ for all $n \ge 1$. Therefore, we prove Theorem 2.3.

To conclude this section, we give a bijection $\theta : S_n \to S_n$ such that $pmp_{\tau_1}(\sigma) = pmp_{\tau_2}(\theta(\sigma))$. The bijection is constructed based on the recursive formula. We start with $\theta(1) = 1$, $\theta(12) = 21$ and $\theta(21) = 12$. In general, θ will have following properties

1. σ has no ascent if and only if $\theta(\sigma)$ has no descent.

2. The last ascent of σ is at the same position as the last descent of $\theta(\sigma)$.

3. $pmp_{\tau_1}(\sigma) = pmp_{\tau_2}(\theta(\sigma))$

By observation, θ satisfies the properties for S_1 and S_2. For S_{n+1} where $n \ge 2$, we define the map recursively. Given $\sigma \in S_n$, we have $\sigma' = \theta(\sigma) \in S_n$, where the last ascent of σ is at the same position as the last descent of σ'. Let k be the position. We decompose σ and σ' at k.

$$\sigma = \sigma_1\sigma_2\ldots\sigma_k \qquad \sigma_{k+1}\ldots\sigma_n$$

$$\sigma' = \sigma'_1\sigma'_2\ldots\sigma'_k \qquad \sigma'_{k+1}\ldots\sigma'_n$$

If σ has no ascent, and so σ' has no descent, we decompose σ and σ' by having the "first part" empty, or, equivalently, set $k = 0$.

Here, we obtain an element $\hat{\sigma} \in S_{n+1}$ by increase every element in σ by 1 and insert 1 at some position, then $\theta(\hat{\sigma})$ will be obtained from increasing elements in σ' by 1 and insert 1 at some position based on the position of 1 in $\hat{\sigma}$. Suppose we insert 1 at position r from the left in σ, then the position r' of 1 inserted in σ' is

$$r' = \begin{cases} r & r \leq k \\ r+1 & k+1 \leq r \leq n \\ k+1 & r = n+1. \end{cases}$$

The corresponding positions of 1 can be viewed from the diagram below:

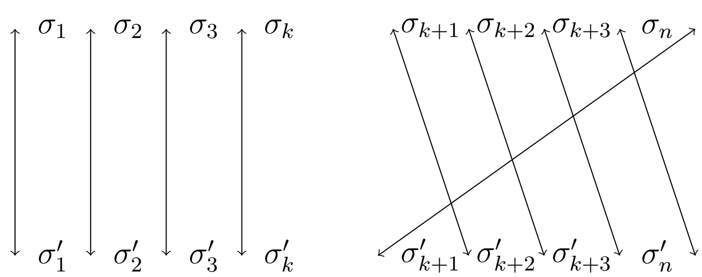
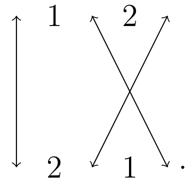


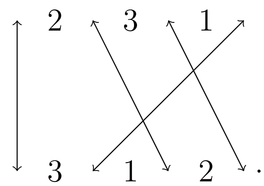
Figure 2.7: Bijection for equivalence of $\underline{1}23$ and $\underline{1}32$

By observation, the postion of the last ascent of $\hat{\sigma}$ is the same as the position of the last descent of $\hat{\sigma}$. Also, pmp_{τ_1} will increase by 1 if and only if 1 is inserted at the first k positions, which is the same condition for pmp_{τ_2} to increase by 1. Thus, θ satisfies the conditions.

As an example, we start with $\theta(12) = 21$. Here, the diagram for inserting 1 looks like



Suppose we insert 1 to 12 at the last position, so we get 231. According to the diagram, we should insert 1 to 21 at the second position, so we get 312. Thus, $\theta(231) = 312$. Now the diagram looks like



Here, we insert 1 to 231 at the first position, so we get 1342. We should also insert 1 to 312 at the first position, so we get 1423. Thus, $\theta(1342) = 1423$.

Below are enumerations of $P_{n,\underline{1}23}(x)$ for the first few n:

$$
\begin{aligned}
P_{1,\underline{1}23}(x) &= 1 \\
P_{2,\underline{1}23}(x) &= 2 \\
P_{3,\underline{1}23}(x) &= 5 + x \\
P_{4,\underline{1}23}(x) &= 14 + 8x + 2x^2 \\
P_{5,\underline{1}23}(x) &= 42 + 47x + 25x^2 + 6x^3 \\
P_{6,\underline{1}23}(x) &= 132 + 244x + 216x^2 + 104x^3 + 24x^4 \\
P_{7,\underline{1}23}(x) &= 429 + 1186x + 1568x^2 + 1199x^3 + 538x^4 + 120x^5 \\
P_{8,\underline{1}23}(x) &= 1430 + 5536x + 10232x^2 + 11264x^3 + 7814x^4 + 3324x^5 + 720x^6.
\end{aligned}
$$

2.3.2 Special values for coefficients

Even though we do not know a formula for $P_{n,\underline{1}23}(x)$ in general, we can still explain some of the coeffients. For example, $P_{n,\underline{1}23}(x)|_{x^0} = C_n$, the n-th Catalan number. This is obvious since a permutation is 123 avoiding if and only if it is not 123-match at any position. There are other coefficients that have a nice formula.

Theorem 2.4. *For $n \geq 3$, the degree of $P_{n,\underline{1}23}(x)$ is $n-2$, and $P_{n,\underline{1}23}(x)|_{x^{n-2}} = (n-2)!$*

Proof. For any $\sigma \in S_n$, it is clear that σ is not $\underline{1}23$-match at position n or $n-1$. Thus, $pmp_{\underline{1}23}(\sigma) \leq n-2$. For any $\sigma \in S_n$, we claim that $pmp_{\underline{1}23}(\sigma) = n-2$ if and only if $\sigma_{n-1} = n-1$ and $\sigma_n = n$. The coverse obviously true since σ would be $\underline{1}23$-match at position i for $1 \leq i \leq n-2$. Suppose that $pmp_{\underline{1}23}(\sigma) = n-2$. Let p_i be the position of i in σ. That is $\sigma_{p_i} = i$. Note that $n-1$ and n cannot be a starting point of the pattern 123, so σ is not $\underline{1}23$-match at positions p_{n-1} and p_n. However, σ is $\underline{1}23$-match at every position except $n-1$ and n. Thus, $\{p_{n-1}, p_n\} = \{n-1, n\}$. Consider position p_{n-2}. σ must be $\underline{1}23$-match at position p_{n-2}. That is $n-2$ is a starting point of a pattern 123 in σ. However, the only way for $n-2$ to be a starting point of 123 is that $n-2, n-1, n$ form a pattern 123. That is $n-1$ is to the left of n. So, $p_{n-1} = n-1$ and $p_n = n$, which means $\sigma_{n-1} = n-1$ and $\sigma_n = n$.

Thus, we know that all σ such that $pmp_{\underline{1}23}(\sigma) = n-2$ are precisely all σ of the form:

$$\sigma = \sigma_1\ \sigma_2\ \ldots\ \sigma_{n-3}\ \sigma_{n-2}\ n-1\ n.$$

There are $(n-2)!$ such σ's, so $P_{n,\underline{1}23}(x)|_{x^{n-2}} = (n-2)!$ □

This theorem can generalized to any positional marked patterns.

Theorem 2.5. *Given any positive integers n, k such that $n \geq k$, and any $\tau \in S_k^*$, the* *degree of* $P_{n,\tau}(x)$ *is* $n-k+1$*, and* $P_{n,\tau}(x)|_{x^{n-k+1}} = (n-k+1)!$.

Proof. Given $k \leq n$ and $\tau \in S_k^*$. Suppose τ has the form

$$\tau = \tau_1 \ \tau_2 \ \ldots \underline{\tau_l} \ldots \ \tau_{k-1} \ \tau_k.$$

That is $u(\tau) = l$ and $\pi(\tau) = \tau_1\tau_2 \ldots \tau_k$. Note that there are $l-1$ numbers to the left of the underlined number in τ and there are $k-l$ numbers to the right of the underlined number in τ. Given $\sigma \in S_n$. Suppose σ is τ-match at position i. Then, there must be at least $l-1$ numbers to the left of position i in σ and there must be at least $k-l$ to the right of position i in σ. Thus, $l \leq i \leq n-k+l$. Therefore, $pmp_\tau(\sigma) \leq n-k+1$. Thus, the degree of $P_{n,\tau}(x)$ is at most $n-k+1$.

Here, we count the number of $\sigma \in S_n$ such that $pmp_\tau(x) = n-k+1$. We shall prove the following claim:

Claim 2.6. *Given* $\sigma = \sigma_1 \ldots \sigma_n \in S_n$*,* $pmp_\tau(\sigma) = n-k+1$ *if and only if all of the followings hold:*

(1) $\{\sigma_1, \sigma_2, \ldots, \sigma_{l-1}, \sigma_{n-k+l+1}, \ldots, \sigma_n\} = \{1, 2, \ldots, \tau_l - 1, n-k+\tau_l+1, \ldots, n\}$

(2) $red(\sigma_1\sigma_2 \ldots \sigma_{l-1}\sigma_{n-k+l+1} \ldots \sigma_n) = red(\tau_1\tau_2 \ldots \tau_{l-1}\tau_{l+1} \ldots \tau_k)$

That is, the first $l-1$ *numbers and last* $k-l$ *numbers in* σ *is a rearrangement of* $\{1, 2, \ldots, \tau_l - 1, n-k+\tau_l+1, \ldots, n\}$ *in the way that they have the same relative order as* $\tau_1\tau_2 \ldots \tau_{l-1}\tau_{l+1} \ldots \tau_k$.

Proof. (of the claim)

We first prove the converse. Suppose σ satisfies (1) and (2). We want to show that σ is τ-match at any position i when $l \leq i \leq n-k+l$. Given any such position i. Consider the following subsequence

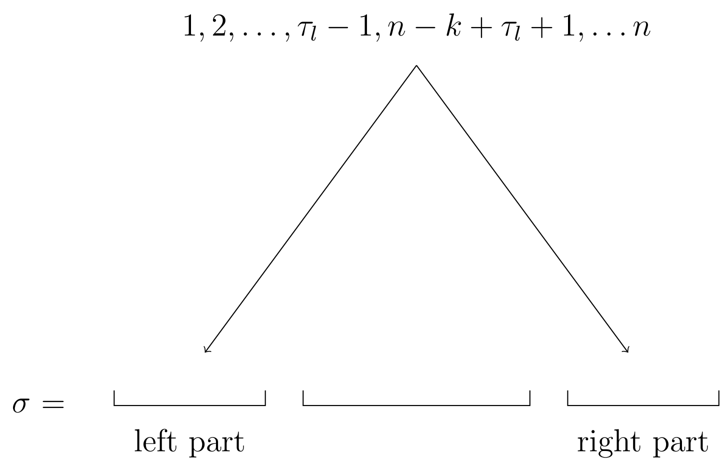
$$\sigma_1\sigma_2\ldots\sigma_{l-1}\ \sigma_i\ \sigma_{n-k+l+1}\ldots\sigma_n.$$

Note that σ_i is the τ_l-th smallest number in the subsequence, and τ_l is the τ_l-th smallest number in subsequence $\tau_1\tau_2\ldots\tau_k$. Thus, we can insert σ_i to $\sigma_1\ldots\sigma_{l-1}\sigma_{l+1}\ldots\sigma_n$ and τ_l to $\tau_1\ldots\tau_{l-1}\tau_{l+1}\tau_k$. So, $\sigma_1\sigma_2\ldots\sigma_{l-1}\ \sigma_i\ \sigma_{n-k+l+1}\ldots\sigma_n$ have the same relative order as $\tau_1\ldots\tau_k$. However, $\tau_1\ldots\tau_k = \pi(\tau)$ is a permutation, so $red(\sigma_1\sigma_2\ldots\sigma_{l-1}\ \sigma_i\ \sigma_{n-k+l+1}\ldots\sigma_n) = \pi(\tau)$. So, σ is τ-match at position i.

Here, we prove the forward direction. Suppose $pmp_\tau(\sigma) = n-k+1$, then σ is τ-match at all positions i when $l \leq i \leq n-k+l$, and σ is not τ-match at all positions $j \in \{1, 2, \ldots, l-1, n-k+l+1, \ldots, n\}$. Note that, in order for σ to match at postion i, there must be at least $\tau_l - 1$ numbers less than σ_i and there must be at least $k-\tau_l$ numbers greater than σ_i. Therefore $\tau_l \leq \sigma_i \leq n-k+\tau_l$. In other words, σ is not τ-match at positions of $1, 2, \ldots, \tau_l - 1$ nor positions of $n-k+\tau_l, n-k+\tau_l+1, \ldots, n$. There are $k-1$ such positions, therefore $\{p_1, p_2, \ldots, p_{\tau_l-1}, p_{n-k+\tau_l+1}, \ldots, p_n\} = \{1, 2, \ldots, l-1, n-k+l+1, \ldots, n\}$, or in other words, $\{\sigma_1, \sigma_2, \ldots, \sigma_{l-1}, \sigma_{n-k+l+1}, \ldots, \sigma_n\} = \{1, 2, \ldots, \tau_l - 1, n-k+\tau_l+1, \ldots, n\}$. Thus, we prove (1).

To prove (2), we only need to show that there is at most one rearrangement of $\{1, 2, \ldots, \tau_l - 1, n-k+\tau_l+1, \ldots, n\}$ such that $pmp_\tau(\sigma) = n-k+1$. Then, by converse direction, we know that the rearrangement in (2) make $pmp_\tau(\sigma) = n-k+1$. In that case, we conclude that the unique rearrangement that make $pmp_\tau(\sigma) = n-k+1$ exists and must be (2).

Let's call numbers appearing at positions $1, 2, \ldots, l-1$ the *left part of* σ, while call the numbers appearing at positions $n-k+l+1, n-k+l+2, \ldots, n$ the *right part of* σ. Then, σ has a following structure:



We shall prove that ways to rearrange $1, 2, \ldots, \tau_l - 1, n - k + \tau_l + 1, \ldots n$ in left and right part of σ so that $pmp_\tau(\sigma) = n - k + 1$ is unique if exist.

Consider the number τ_l in σ. We know that σ is τ-match at position p_{τ_l}. Then, numbers $1, 2, \ldots, \tau_l - 1$ in σ must involve in τ-match at position p_{τ_l} since we need $\tau_l - 1$ numbers smaller than τ_l. Therefore, for each number i less than τ_l, we can determine whether i is in the left part or right part of σ based on the relative postion of i and τ_l in τ. Also, consider the number $n - k + \tau_l$. σ must be τ-match at position $p_{n-k+\tau_l}$. By the same reasoning, for each number $n - k + \tau_l + i$, we determine whether $n - k + \tau_l + i$ is in the left part or right part of σ based on the relative position of $\tau_l + i$ and τ_l in τ. Thus, we determine both left and right part as sets.

Now, consider position l in σ. σ must be τ-match at position l. Thus, numbers in first $l-1$ positions in σ must involve in τ-match at postion l, and so, the first $l-1$ numbers in σ must have the same relative order as first $l-1$ elements in τ. By the same reasoning, by considering position $n - k + l$, the last $k - l$ numbers in σ have the same relative order as the last $k - l$ numbers in τ. Therefore, we completely determine both left and right part of σ. Thus, there is at most one way to rearrange $\{1, 2, \ldots, \tau_l - 1, n - k + \tau_l + 1, \ldots, n\}$.

Since the rearrangement in (2) makes $pmp(\sigma) = n - k + 1$, σ, then σ must satisfies

(2).

∎

As an example, if $\tau = 164\underline{3}52$, any $\sigma \in S_9$ with $pmp_\tau(\sigma) = 9 - 6 + 1 = 4$ must have a following form:

$$\sigma = 1\ 9\ 7\ \ \sigma_4\ \sigma_5\ \sigma_6\ \sigma_7\ 8\ 2.$$

There are $(n-k+1)!$ ways to rearrange the "middle" part of σ. Thus, $P_{n,\tau}(x)|x^{n-k+1} = (n-k+1)!$

∎

Another coefficient we can describe is $P_{n,\underline{1}32}(x)|_x$. The sequence are A029760 and A139262 on OEIS. The sequence A139262 counts the sum of all inversion of all elements in $S_n(132)$, the set of 132-avoiders in S_n.

Theorem 2.7. $P_{n,\underline{1}32}(x)|_x = \sum_{\sigma \in S_n(132)} inv(\sigma)$

Proof. To prove the theorem, we construct sets whose cardinality are $\sum_{\sigma \in S_n(132)} inv(\sigma)$.

Definition 2.4. *Let $IMS_n(132)$ be the set of 132-avoiding σ that a pair of elements causing an inversion are marked with $*$.*

As an example, $IMS_3(132)$ contains 8 elements, which are $\overset{*}{2}\overset{*}{1}3$, $\overset{*}{2}3\overset{*}{1}$, $2\overset{*}{3}\overset{*}{1}$, $\overset{*}{3}\overset{*}{1}2$, $\overset{*}{3}1\overset{*}{2}$, $\overset{*}{3}\overset{*}{2}1$, $\overset{*}{3}2\overset{*}{1}$, and $3\overset{*}{2}\overset{*}{1}$. It is easy to see that $|IMS_n(132)| = \sum_{\sigma \in S_n(132)} inv(\sigma)$

Definition 2.5. *Given $\sigma \in S_n$, $i, j \in \mathbb{Z}$, $1 \le i \le n$, $1 \le j \le n+1$. Let $\theta(\sigma, i, j)$ be an element in S_{n+1} obtained from σ by*

1. *increases every number greater than or equal to i by 1*

2. *insert i at the j-th position from the left. (The first position is in front of the first element, and the $n+1$-th position is behind the last element)*

For example, to find $\theta(2143, 3, 2)$, first, we increase 3,4 by 1 so we have 2154. Then, we insert 3 at the second position, so we have 23154. Thus, $\theta(2143, 3, 2) = 23154$.

Here, we are ready to define a map $\Phi : IMS_n(132) \to \{\sigma \in S_{n+1} | pmp_{\underline{1}32}(\sigma) = 1\}$

Definition 2.6. *Given $\bar{\sigma} \in IMS_n(132)$. Let σ be the underlying permutation of $\bar{\sigma}$. Let j be the position of the first * in σ, and let i be the number underneath the second * in σ. Define $\Phi(\bar{\sigma}) = \theta(\sigma, i, j)$*

Lemma 2.4. *Given $\bar{\sigma} \in IMS_n(132)$ and $\hat{\sigma} = \Phi(\bar{\sigma})$, then $mmp_{\underline{1}32}(\hat{\sigma}) = 1$. Thus, Φ is a map from $IMS_n(132)$ to $\{\sigma \in S_{n+1} | pmp_{\underline{1}32}(\sigma) = 1\}$*

Proof. Let $\bar{\sigma} \in IMS_n(132)$ has a form

$$\bar{\sigma} = \sigma_1 \dots \sigma_{k-1} \overset{*}{\sigma}_k \sigma_{k+1} \dots \sigma_{l-1} \overset{*}{\sigma}_l \sigma_{l+1} \dots \sigma_n$$

and σ is the underlying permutation. Then, $\Phi(\bar{\sigma}) = \theta(\sigma, \sigma_l, k)$ looks like

$$\Phi(\bar{\sigma}) = \sigma'_1 \dots \sigma'_{k-1} \sigma_l \sigma'_k \sigma'_{k+1} \dots \sigma'_{l-1} \sigma'_l \sigma'_{l+1} \dots \sigma'_n$$

where

$$\sigma'_i = \begin{cases} \sigma_i + 1 & \text{for } \sigma_i \geq \sigma_l \\ \sigma_i & \text{for } \sigma_i < \sigma_l. \end{cases}$$

Note that $\hat{\sigma} = \sigma'_l = \sigma_l + 1$. Also, since σ_k and σ_l cause an inversion, so $\sigma_k > \sigma_l$, and thus $\sigma'_k = \sigma_k + 1$. Therefore, $\sigma_l < \sigma'_l < \sigma'_k$. Equivialently, $\sigma_l, \sigma'_k, \sigma'_l$ form a pattern 132 in $\hat{\sigma}$. Thus, $mmp_{\underline{1}32}(\hat{\sigma}) \geq 1$. To prove that $pmp_{\underline{1}32}(\hat{\sigma}) = 1$, we need to show that $\hat{\sigma}$ does not contain a pattern 132 that starts at and element other than σ_l.

Suppose otherwise. That is, $\hat{\sigma}$ contains a pattern 132 where the starting element is σ'_t for some t. We have several cases:

1. If the pattern does not involve σ_l, then there must be u, v such that $\sigma'_t, \sigma'_u, \sigma'_v$ form a pattern 132. It is easy to see that $\sigma_t, \sigma_u, \sigma_v$ also form a pattern 132 in σ, which is a contradiction since σ is a 132-avoider.

2. If the pattern involves σ_l, then $t < k$ and $\sigma'_t < \sigma_l$. In this case, it is easy to see that $\sigma'_t, \sigma'_k, \sigma'_l$ also form a pattern 132. Thus, $\sigma_t, \sigma_k, \sigma_l$ form a pattern 132 in σ, which is again a contradiction.

Thus, $\hat{\sigma}$ does not contain the pattern 132 that starts at any elements except σ_l. So, $pmp_{\underline{1}32}(\hat{\sigma}) = 1$. □

The map Φ is clearly an injection. Let $\sigma = \sigma_1 \ldots \sigma_n$ be in the image of Φ, and let σ_i be the starting point of the pattern 132. Then, i determines the first starred element, and σ_i determines the second starred element. To prove that the map Φ is a surjective, we need another lemma:

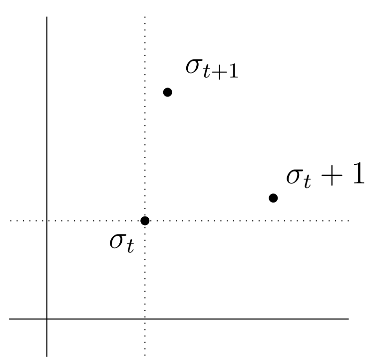
Lemma 2.5. *Given $\sigma = \sigma_1 \ldots \sigma_n \in S_n$ such that $pmp_{\underline{1}32}(\sigma) = 1$. Suppose the pattern 132 in σ starts at σ_t, then the following must be true:*

(1) $\sigma_{t+1} \geq \sigma_t$

(2) $\sigma_t + 1$ *is on the right of* σ_t

(3) $\sigma_{t+1} \neq \sigma_t + 1$

That is, the graph of σ should look like



Proof. Suppose (1) is not true. There must be u, v such that $\sigma_t, \sigma_u, \sigma_v$ form a pattern 132. Then, $\sigma_{t+1}, \sigma_u, \sigma_v$ also form a pattern 132. So, $pmp_{\underline{1}32}(\sigma) \geq 2$.

Suppose (2) is not true. There must be u, v such that $\sigma_t, \sigma_u, \sigma_v$ form a pattern 132. Then, $\sigma_t + 1, \sigma_u, \sigma_v$ also form a pattern 132. So, $pmp_{\underline{1}32}(\sigma) \geq 2$.

Suppose (3) is not true. There must be u, v such that $\sigma_t, \sigma_u, \sigma_v$ form a pattern 132. Then, $\sigma_{t+1}, \sigma_u, \sigma_v$ also form a pattern 132. So, $pmp_{\underline{1}32}(\sigma) \geq 2$.a

In all case, we have a contradiction. Therefore, (1), (2) and (3) are true. □

Lemma 2.6. *The map* $\Phi : IMS_n(132) \rightarrow \{\sigma \in S_{n+1} | \underline{1}32(\sigma) = 1\}$ *is surjective.*

Proof. Given any $\sigma \in \{\sigma \in S_{n+1} | \underline{1}32(\sigma) = 1\}$. Let σ_t be the starting poing of the pattern 132 in σ. By lemma 2.5, we know that $\sigma_t, \sigma_{t+1}, \sigma_t + 1$ form a pattern 132. Let $\tilde{\sigma}$ be a permutation obtained from σ by removing σ_t and reducing. Let $\tilde{\sigma}^*$ be $\tilde{\sigma}$ with * marked on elements corresponding to σ_{t+1} and $\sigma_t + 1$ before reducing ($\sigma_{t+1} - 1, \sigma_t$ after reducing). $\tilde{\sigma}^*$ is an element is $IMS_n(132)$ since $\sigma_{t+1}, \sigma_t + 1$ form an inversion. Clearly, $\Phi(\tilde{\sigma}^*) = \sigma$ □

For example, consider $\sigma = 785269314 \in S_9$ with $pmp_{132}(\sigma) = 1$, with only possible starting point of the pattern 132 is 2. So, in this example, $\sigma_t, \sigma_{t+1}, \sigma_t + 1$ are 2,6,3. Then, removing 2 and reducing give $\tilde{\sigma} = 67458213$. Then, put * at elements corresponding to 6,3 before reducing, which are 5,2 after reducing. So, $\tilde{\sigma}^* = 674\overset{*}{5}8\overset{*}{2}13$.

Therefore Φ is a bijection, and so we prove Theorem 2.7

□

2.3.3 Equivalence of $1\underline{2}3$ and $1\underline{3}2$

By observing diagrams of $1\underline{3}2$ and $13\underline{2}$, we see that those two patterns are equivalent by lemma 2.1. Thus, we will instead prove that $1\underline{2}3$ and $13\underline{2}$ are equivalent.

In this section, we prove the following theorem

Theorem 2.8. *Two generalized pmp* $1\underline{2}3$ *and* $13\underline{2}$ *are pmp-Wilf equivalent.*

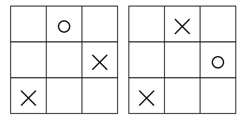


Figure 2.8: Diagrams of $1\underline{3}2$ and $13\underline{2}$

For our convenience, we let $\tau_1 = 1\underline{2}3$ and $\tau_2 = 13\underline{2}$. We will first prove by showing that two generating functions satisfy the same recursive formula. We later will construct a bijection from S_n to itself that maps an pmp_{τ_1} to pmp_{τ_2}, that is, a map θ such that $pmp_{\tau_1}(\sigma) = pmp_{\tau_2}(\theta(\sigma))$.

Given any permutation $\sigma \in S_n$. We will look at the position of 1 in σ. Let $P_{n,\tau_1,k}(x) = \sum x^{pmp_{\tau_1}(\sigma)}$, where the sum is over all permutation in S_n with 1 at position k. Then, we have $P_{n,\tau_1}(x) = \sum_{k=1}^{n} P_{n,\tau_1,k}(x)$.

Here, we derive a recursive formula for $P_{n,\tau_1,k}(x)$

Lemma 2.7. *Let $\tau_1 = 1\underline{2}3$ and let $P_{n,\tau_1,k} = x^{pmp_{\tau_1}(\sigma)}$ where the sum is over all $\sigma \in S_n$ with 1 at position k. Then, $P_{n,\tau_1,k}(x)$ satisfies the following recursive formula:*

$$P_{n,\tau_1,k}(x) = \left(\sum_{l=1}^{k-1} P_{n-1,\tau_1,l}(x)\right) + (x(n-k-1)+1)P_{n-1,\tau_1,k}(x)$$

where $P_{n,\tau_1,N}(x) = 0$ for $N > n$ by convention.

Proof. We derive a recursive formula for $P_{n,\tau_1,k}(x)$ by looking at the position of 2 in σ where the position of 1 is k. σ has a following form

$$\sigma = \sigma_1\ \sigma_2\ \ldots\ \sigma_{k-1}\ 1\ \sigma_{k+1}\ \ldots\ \sigma_n.$$

Let l be position of 2. We have 3 cases:

1. $1 \le l \le k-1$. In this case, we can remove 1 from σ and reduce other elements without effecting τ_1-match at other position. To see this, first note that if σ is not

τ_1-match at a particular position, removing 1 and reducing will not change it. Also, if σ is τ_1-match at some position before 1, then removing 1 will not change τ_1-match at the position, since 1 can only serve as 1 in $1\underline{2}3$. However, 1 appears after the position that τ_1-match occur. Thus, σ is τ_1-match at the position without considering 1. Lastly, if σ is τ_1-match at some position t after 1. Note that 2 appears before 1 in σ. Thus, 2 can serve as 1 in $1\underline{2}3$. Therefore, σ is still τ_1-match at position t when not considering 1. So, we can remove 1 and reduce without effecting τ_1-match.

After removing 1 and reducing, the position of 1 will be at l. Thus, this case contribute $\sum_{l=1}^{k-1} P_{n-1,\tau_1,k}(x)$.

2. $k+1 \leq l \leq n-1$. In this case, σ is τ_1-match at position l since 1 2 σ_n form a pattern τ_1. Morever, by the same reason as in the first case, removing 2 and reducing will not effect τ_1-match at other position.

   After removing 2 and reducing, the position of 1 is still at l. Thus, this case contribute $x(n-k-1)P_{n-1,\tau_1,k}(x)$

3. $l = n$. In this case, σ is not τ_1-match at position l, and removing 2 will not effect τ_1-match at other position. Thus, this case contribute $P_{n-1,\tau_1,k}(x)$

In total, we have

$$
\begin{aligned}
P_{n,\tau_1,k}(x) &= \left(\sum_{l=1}^{k-1} P_{n-1,\tau_1,l}(x)\right) + x(n-k-1)P_{n-1,\tau_1,k}(x) + P_{n-1,\tau_1,k}(x) \\
&= \left(\sum_{l=1}^{k-1} P_{n-1,\tau_1,l}(x)\right) + (x(n-k-1)+1)P_{n-1,\tau_1,k}(x).
\end{aligned}
$$

Note that, the first case vanishes when $k=1$, but the formula is correct since it would contribute an empty summation. The second case vanishes when $n-1 \leq k \leq n$, but the formula is still correct since $n-k-1=0$ for $k=n-1$ and $P_{n-1,\tau_1,n}(x)=0$ for $k=n$. The last case also vanishes when $k=n$, but the formula is still correct as $P_{n-1,\tau_1,k}(x)=0$.

$\square$

Here, we prove a similar result of $\tau_2 = 13\underline{2}$.

Lemma 2.8. *Let $\tau_2 = 13\underline{2}$ and let $P_{n,\tau_2,k}(x) = x^{pmp_{\tau_2}(\sigma)}$ where the sum is over all $\sigma \in S_n$ with 1 at position k. Then, $P_{n,\tau_2,k}(x)$ satisfies the following recursive formula:*

$$P_{n,\tau_2,k}(x) = \left(\sum_{l=1}^{k-1} P_{n-1,\tau_2,l}(x)\right) + x(n-k-1)P_{n-1,\tau_2,k}(x).$$

Proof. We derive a recursive formula for $P_{n,\tau_2,k}(x)$ by looking at the position of 2 in σ where the position of 1 is k. σ has a following form

$$\sigma = \sigma_1\ \sigma_2\ \ldots\ \sigma_{k-1}\ 1\ \sigma_{k+1}\ \ldots\ \sigma_n.$$

Let l be position of 2. We have 3 cases:

1. $1 \le l \le k-1$. In this case, by the same reason as the first case in the proof of Lemma 2.7, we can remove 1 from σ and reduce other elements without effecting τ_1-match at other positions.

   After removing 1 and reducing, the position of 1 will be at l. Thus, this case contribute $\sum_{l=1}^{k-1} P_{n-1,\tau_2,k}(x)$.

2. $l = k+1$. In this case, σ is not τ_2-match at position l, and removing 2 will not effect τ_2-match at other position by the same reason as above. Thus, this case contribute $P_{n-1,\tau_2,k}(x)$

3. $k+2 \le l \le n$. In this case, σ is τ_2-match at position l since $1\ \sigma_{k+1}\ 2$ form a pattern τ_2. Moreover, by the same reason as above, removing 2 and reducing will not effect τ_2-match at other position.

After removing 2 and reducing, the position of 1 is still at l. Thus, this case contribute $x(n-k-1)P_{n-1,\tau_2,k}(x)$

In total, we have

$$\begin{aligned} P_{n,\tau_2,k}(x) &= \left(\sum_{l=1}^{k-1} P_{n-1,\tau_2,l}(x)\right) + x(n-k-1)P_{n-1,\tau_2,k}(x) + P_{n-1,\tau_2,k}(x) \\ &= \left(\sum_{l=1}^{k-1} P_{n-1,\tau_2,l}(x)\right) + (x(n-k-1)+1)P_{n-1,\tau_2,k}(x). \end{aligned}$$

Note that the first case vanishes when $k = 1$, but the formula is still correct as it contributes an empty summation. The second case vanishes when $k = n$. The formula is still correct as $P_{n-1,\tau_2,n}(x) = 0$. The last case vanishes when $k = n-1, n$. When $k = n-1$, the formula is correct as $n-k-1 = 0$. When $k = n$, the formula is correct as $P_{n-1,\tau_2,n}(x) = 0$.

□

It is easy to see that $P_{1,\tau_1,1}(x) = 1 = P_{1,\tau_2,1}(x)$. Since, $P_{n,\tau_1,k}(x)$ and $P_{n,\tau_2,k}(x)$ satisfy the same recursive formula and have the same initial values, we prove that $P_{n,\tau_1,k}(x) = P_{n,\tau_2,k}(x)$ for all n, k such that $1 \leq k \leq n$, and thus $P_{n,\tau_1}(x) = P_{n,\tau_2}(x)$ for all $n \geq 1$. Hence, we prove Theorem 2.8.

We also construct a bijection $\theta : S_n \to S_n$ such that $pmp_{\tau_1}(\sigma) = pmp_{\tau_2}(\theta(\sigma))$ based on the recursive formula. We define $\theta(1) = 1, \theta(12) = 12$ and $\theta(21) = 21$. In general, θ will satisfies the following properties

1. The position of 1 in σ is the same as the position of 1 in $\theta(\sigma)$.

2. $pmp_{\tau_1}(\sigma) = pmp_{\tau_2}(\theta(\sigma))$

θ satisfies the properties for S_1 and S_2. For S_{n+1} where $n \geq 2$, we define the map recursively. Given $\sigma \in S_n$, we have $\sigma' = \theta(\sigma)$. We know that the position of 1 in σ is the same as the position of 1 in σ'. Let k be the position.

Here, we obtain an element $\hat{\sigma} \in S_{n+1}$ by applying one of the following:

1. Increase every element by 1, and insert 1 at or after position $k+1$.

2. Increase every element except 1 by 1, and insert 2 at or after position $k+1$.

Then, we obtain $\theta(\hat{\sigma})$ by applying similar action to σ' based on an action applied to σ:

1. If 1 was inserted to σ, then increase every element in σ' by 1 and insert 1 at the same position as inserted in σ.

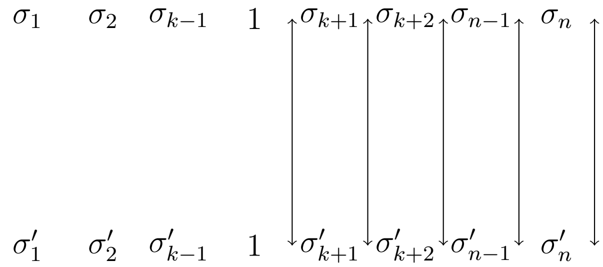
2. If 2 was inserted to σ at position r, then increase every element in σ' except 1 by 1 and insert 2 at position r' where

$$r' = \begin{cases} r+1 & k+1 \leq r \leq n \\ k+1 & r = n+1 \end{cases}$$

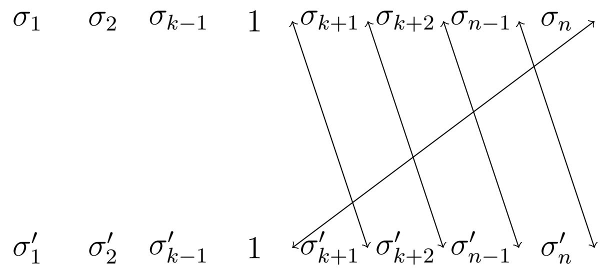
Corresponding positions when inserting 1 or 2 can be viewed from the diagram below:

The position of 1 in $\hat{\sigma}$ and $\theta(\hat{\sigma})$ will be the same. Also, $pmp_{\tau_1}(\sigma)$ would increase by 1 if and only if 2 was inserted at a non-last position, while $pmp_{\tau_2}(\sigma')$ will increase by 1 if and only if 2 was inserted at any position but $k+1$. Thus, $pmp_{\tau_1}(\hat{\sigma}) = pmp_{\tau_2}(\theta(\hat{\sigma}))$. Also, it is not hard to see that every $\sigma \in S_n$ can be obtained by inserting 1 or 2 repeatedly in a unique way. Therefore, θ is a bijection.

As an example, we start with $\theta(12) = 12$. Say, we would like to insert 2, then the diagram looks like

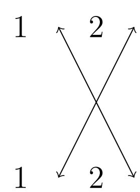


Corresponding positions when inserting 1

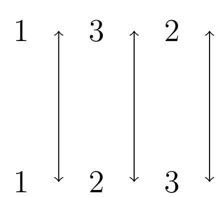


Corresponding positions when inserting 2

Figure 2.9: Bijection for equivalence of 1$\underline{3}$2 and 13$\underline{2}$

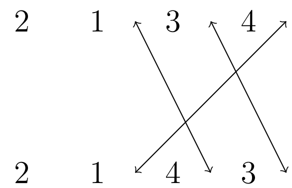


Suppose we insert 2 to the preimage 12 at the last position, so we get 132. According to the diagram, we should insert 2 to the image 12 at the first position, so we get 123. Thus, $\theta(132) = 123$. Here, suppose we would like to insert 1, then the diagram looks like:



Suppose we insert 1 to 132 at the first position, so we get 2143. According to the diagram, we insert 1 to 123 at the same position, so we get 2134. Thus, $\theta(2143) = 2134$.

Now, suppose we want to insert 2. The diagram looks like:



Suppose we insert 2 to 2134 at the second position, so we get 31245. According to the diagram, we insert 2 to 2143 at the last position, so we get 31542. Thus, $\theta(31245) = 31542$.

Below are enumerations of $P_{n,1\underline{2}3}(x)$ for the first few n.

$$
\begin{aligned}
P_{1,1\underline{2}3}(x) &= 1 \\
P_{2,1\underline{2}3}(x) &= 2 \\
P_{3,1\underline{2}3}(x) &= 5 + x \\
P_{4,1\underline{2}3}(x) &= 14 + 8x + 2x^2 \\
P_{5,1\underline{2}3}(x) &= 42 + 46x + 26x^2 + 6x^3 \\
P_{6,1\underline{2}3}(x) &= 132 + 232x + 220x^2 + 112x^3 + 24x^4 \\
P_{7,1\underline{2}3}(x) &= 429 + 1093x + 1527x^2 + 1275x^3 + 596x^4 + 120x^5 \\
P_{8,1\underline{2}3}(x) &= 1430 + 4944x + 9436x^2 + 11384x^3 + 8638x^4 + 3768x^5 + 720x^6.
\end{aligned}
$$

Remark In fact, the equivalence of $1\underline{2}3$ and $\underline{2}13$ follows from a general result from Theorem 2.10.

2.4 Equivalence classes of S_4^*

There are $4! \cdot 4 = 96$ *pmp* in S_4^*. However, there are at most 16 equivalent classes by Lemma 2.1. All the representatives are listed in Figure 2.10.

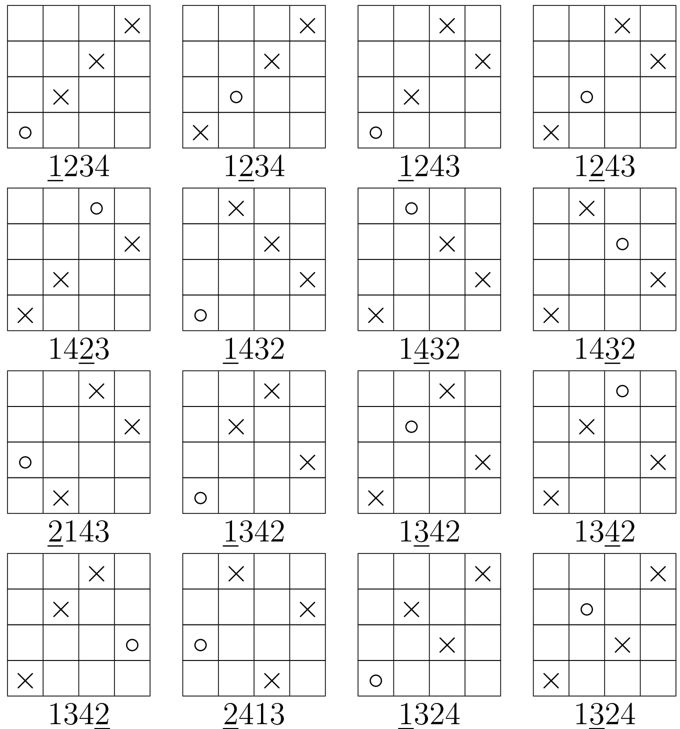


Figure 2.10: Geometrically non-equivalence positional marked patterns of length 4

In this section, we present numerical results suggesting that there 10 equivalent classes in S_4^*. We then will show 4 following patterns are equivalent: $1\underline{2}34, 1\underline{2}43, 2\underline{1}34, 2\underline{1}43$.

2.4.1 Numerical data

Here, for each pattern $\tau \in S_4^*$, we enumerate polynomials $P_{n,\tau}(x)$ for small n. We also group patterns together if they seem to provide the same polynomials.

1. $\tau = \underline{1}234, \underline{1}243, \underline{1}432$

$$
\begin{aligned}
P_{1,\tau}(x) &= 1 \\
P_{2,\tau}(x) &= 2 \\
P_{3,\tau}(x) &= 6 \\
P_{4,\tau}(x) &= 23 + x \\
P_{5,\tau}(x) &= 103 + 15x + 2x^2 \\
P_{6,\tau}(x) &= 513 + 158x + 43x^2 + 6x^3 \\
P_{7,\tau}(x) &= 2761 + 1466x + 619x^2 + 170x^3 + 24x^4 \\
P_{8,\tau}(x) &= 15767 + 12864x + 7598x^2 + 3121x^3 + 850x^4 + 120x^5
\end{aligned}
$$

2. $\tau = 1\underline{2}34, 1\underline{2}43, 12\underline{4}3, \underline{2}143$

$$
\begin{aligned}
P_{1,\tau}(x) &= 1 \\
P_{2,\tau}(x) &= 2 \\
P_{3,\tau}(x) &= 6 \\
P_{4,\tau}(x) &= 23 + x \\
P_{5,\tau}(x) &= 103 + 15x + 2x^2 \\
P_{6,\tau}(x) &= 513 + 157x + 44x^2 + 6x^3 \\
P_{7,\tau}(x) &= 2761 + 1439x + 638x^2 + 178x^3 + 24x^4 \\
P_{8,\tau}(x) &= 15767 + 12420x + 7764x^2 + 3341x^3 + 908x^4 + 120x^5
\end{aligned}
$$

3. $\tau = 1\underline{4}32$

$$\begin{aligned}
P_{1,\tau}(x) &= 1\\
P_{2,\tau}(x) &= 2\\
P_{3,\tau}(x) &= 6\\
P_{4,\tau}(x) &= 23 + x\\
P_{5,\tau}(x) &= 103 + 15x + 2x^2\\
P_{6,\tau}(x) &= 513 + 157x + 44x^2 + 6x^3\\
P_{7,\tau}(x) &= 2761 + 1438x + 640x^2 + 177x^3 + 24x^4\\
P_{8,\tau}(x) &= 15767 + 12393x + 7809x^2 + 3332x^3 + 899x^4 + 120x^5
\end{aligned}$$

4. $\tau = 14\underline{3}2$

$$\begin{aligned}
P_{1,\tau}(x) &= 1\\
P_{2,\tau}(x) &= 2\\
P_{3,\tau}(x) &= 6\\
P_{4,\tau}(x) &= 23 + x\\
P_{5,\tau}(x) &= 103 + 15x + 2x^2\\
P_{6,\tau}(x) &= 513 + 156x + 45x^2 + 6x^3\\
P_{7,\tau}(x) &= 2761 + 1415x + 655x^2 + 185x^3 + 24x^4\\
P_{8,\tau}(x) &= 15767 + 12058x + 7895x^2 + 3524x^3 + 956x^4 + 120x^5
\end{aligned}$$

5. $\tau = \underline{1}342$

$$
\begin{aligned}
P_{1,\tau}(x) &= 1 \\
P_{2,\tau}(x) &= 2 \\
P_{3,\tau}(x) &= 6 \\
P_{4,\tau}(x) &= 23 + x \\
P_{5,\tau}(x) &= 103 + 15x + 2x^2 \\
P_{6,\tau}(x) &= 512 + 160x + 42x^2 + 6x^3 \\
P_{7,\tau}(x) &= 2740 + 1500x + 614x^2 + 162x^3 + 24x^4 \\
P_{8,\tau}(x) &= 15485 + 13207x + 7700x^2 + 3016x^3 + 792x^4 + 120x^5
\end{aligned}
$$

6. $\tau = 1\underline{3}42$

$$
\begin{aligned}
P_{1,\tau}(x) &= 1 \\
P_{2,\tau}(x) &= 2 \\
P_{3,\tau}(x) &= 6 \\
P_{4,\tau}(x) &= 23 + x \\
P_{5,\tau}(x) &= 103 + 15x + 2x^2 \\
P_{6,\tau}(x) &= 512 + 158x + 44x^2 + 6x^3 \\
P_{7,\tau}(x) &= 2740 + 1451x + 646x^2 + 179x^3 + 24x^4 \\
P_{8,\tau}(x) &= 15485 + 12455x + 7912x^2 + 3427x^3 + 921x^4 + 120x^5
\end{aligned}
$$

7. $\tau = 13\underline{4}2, \underline{2}413$

$$
\begin{aligned}
P_{1,\tau}(x) &= 1\\
P_{2,\tau}(x) &= 2\\
P_{3,\tau}(x) &= 6\\
P_{4,\tau}(x) &= 23 + x\\
P_{5,\tau}(x) &= 103 + 15x + 2x^2\\
P_{6,\tau}(x) &= 512 + 158x + 44x^2 + 6x^3\\
P_{7,\tau}(x) &= 2740 + 1454x + 644x^2 + 178x^3 + 24x^4\\
P_{8,\tau}(x) &= 15485 + 12533x + 7897x^2 + 3377x^3 + 908x^4 + 120x^5
\end{aligned}
$$

8. $\tau = 134\underline{2}$

$$
\begin{aligned}
P_{1,\tau}(x) &= 1\\
P_{2,\tau}(x) &= 2\\
P_{3,\tau}(x) &= 6\\
P_{4,\tau}(x) &= 23 + x\\
P_{5,\tau}(x) &= 103 + 15x + 2x^2\\
P_{6,\tau}(x) &= 512 + 159x + 43x^2 + 6x^3\\
P_{7,\tau}(x) &= 2740 + 1475x + 629x^2 + 172x^3 + 24x^4\\
P_{8,\tau}(x) &= 15485 + 12817x + 7781x^2 + 3244x^3 + 873x^4 + 120x^5
\end{aligned}
$$

9. $\tau = \underline{1}324$

$$\begin{aligned}
P_{1,\tau}(x) &= 1\\
P_{2,\tau}(x) &= 2\\
P_{3,\tau}(x) &= 6\\
P_{4,\tau}(x) &= 23 + x\\
P_{5,\tau}(x) &= 103 + 15x + 2x^2\\
P_{6,\tau}(x) &= 513 + 158x + 43x^2 + 6x^3\\
P_{7,\tau}(x) &= 2762 + 1464x + 620x^2 + 170x^3 + 24x^4\\
P_{8,\tau}(x) &= 15793 + 12820x + 7608x^2 + 3129x^3 + 850x^4 + 120x^5
\end{aligned}$$

10. $\tau = 1\underline{3}24$

$$\begin{aligned}
P_{1,\tau}(x) &= 1\\
P_{2,\tau}(x) &= 2\\
P_{3,\tau}(x) &= 6\\
P_{4,\tau}(x) &= 23 + x\\
P_{5,\tau}(x) &= 103 + 15x + 2x^2\\
P_{6,\tau}(x) &= 513 + 156x + 45x^2 + 6x^3\\
P_{7,\tau}(x) &= 2762 + 1414x + 654x^2 + 186x^3 + 24x^4\\
P_{8,\tau}(x) &= 15793 + 12041x + 7861x^2 + 3539x^3 + 966x^4 + 120x^5
\end{aligned}$$

2.4.2 Equivalence of $1\underline{2}34$, $1\underline{2}43$, $\underline{2}134$ and $\underline{2}143$

Here, we prove an equivalence of four patterns: $1\underline{2}34, 1\underline{2}43, \underline{2}134$ and $\underline{2}143$. Note that, they are equivalent to four patterns appearing in section 2, since $2\underline{1}34$ and $12\underline{4}3$ are equivalent.

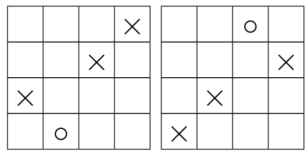


Figure 2.11: Diagrams of $2\underline{1}34$ and $12\underline{4}3$

The equivalence of $1\underline{2}34$ and $\underline{2}134$ as well as the equivalence of $\underline{2}134$ and $\underline{2}143$ follow from a more general Theorem 2.10. Thus, in this section, we only prove the equivalence of $1\underline{2}34$ and $1\underline{2}43$.

Theorem 2.9. *Two pmp* $1\underline{2}34$ *and* $1\underline{2}43$ *are pmp-Wilf equivalent.*

For our convenient, let $\tau_1 = 1\underline{2}34$. Let $P_{n,\tau_1,s}(x) = \sum_\sigma x^{pmp_{\tau_1}(\sigma)}$, where the sum is over all $\sigma \in S_n$ with the last ascent of σ is at position s. Let $P_{n,\tau_1,s,t}(x) = \sum_\sigma x^{pmp_{\tau_1}(\sigma)}$, where the sum is over all $\sigma \in S_n$ with the last ascent of σ is at position s and the 1 is at position t. All feasible values of s are $1, 2, \ldots, n-1$. All feasible values of t are $1, 2, \ldots, s-2, n$ for $s \neq n-1$, and $1, 2, \ldots, n-1$ for $s = n-1$. Thus,

$$P_{n,\tau_1}(x) = 1 + \sum_{s=1}^{n-1} P_{n,\tau_1,s}(x).$$

Also,

$$P_{n,\tau_1,s}(x) = \left(\sum_{t=1}^{s-1} P_{n,\tau_1,s,t}(x) \right) + P_{n,\tau_1,s,n}(x)$$

for $s \neq n-1$, and

$$P_{n,\tau_1,n-1}(x) = \sum_{t=1}^{n-1} P_{n,\tau_1,n-1,t}(x).$$

Here, we derive a recursive formula for $P_{n,\tau_1,s,t}(x)$.

Lemma 2.9. *$P_{n,\tau_1,s,t}(x)$ satisfies the following recursive formulas:*

$$P_{n,\tau_1,s,t}(x) = \sum_{i=1}^{t-1} P_{n-1,\tau_1,s-1,i-1}(x) + (s-1-t)xP_{n-1,\tau_1,s-1,t}(x) + \sum_{j=t}^{s-1} P_{n-1,\tau_1,j,t}(x) + P_{n-1,\tau_1,s,t}(x)$$

for $1 \leq t < s \leq n-1$,

$$P_{n,\tau_1,s,s}(x) = 1 + \sum_{i=1}^{s-1} P_{n-1,\tau_1,i}(x)$$

for $1 \leq s \leq n-1$, and

$$P_{n,\tau_1,s,n}(x) = P_{n-1,\tau_1,s}(x)$$

for $1 \leq s < n-1$.

By convention, Let $P_{n,\tau_1,s,t}(x) = 0$ for infeasible value of s, t.

Proof. First, consider the formula for $P_{n,\tau_1,s,t}(x)$ when $1 \leq t < s \leq n-1$. Let l be the position of 2 in σ, where $\sigma \in S_n$ with the last ascent of σ is at position s and 1 is at position t. We have 4 cases:

1. $1 \leq l \leq t-1$. In this case, we can remove 1 from σ without effecting τ_1-match at other positions. That is because σ is not τ_1-match at the position of 2, and if σ is τ_1-match at other positions and matching involves 1, it can still be τ_1-match by using 2 instead of 1.

   After removing and reducing, the position of 1 is l, and the position of the last ascent is $s-1$. Thus, this case contributes $\sum_{i=1}^{t-1} P_{n-1,\tau_1,s-1,i}(x)$.

2. $t+1 \le l \le s-1$. In this case, σ is τ_1-match at position l since 1 2 σ_s σ_{s+1} form the pattern τ_1. By the same reasoning as the first case, we can remove 2 without effecting τ_1-match at any other positions.

   After removing and reducing, the position of 1 stays at t and the position of the last ascent is $s-1$. Thus, this case contribute $x(s-1-t)P_{n-1,\tau_1,s-1,t}(x)$.

3. $l = s$. In this case, we can remove 2 without effecting τ_1-match at other positions.

   After removing and reducing, the position of 1 stays at t and the last ascent appears at some position between t and $s-1$. Thus, this case contributes $\sum_{j=t}^{s-1} P_{n-1,\tau_1,j,t}(x)$.

4. $l = n$. In this case, we can remove 2 without effecting τ_1-match at other positions. After removing and reducing, the position of 1 stays at t and the position of the last ascent stays at s. Thus, this case contributes $P_{n-1,\tau_1,s,t}(x)$.

In total, we have

$$P_{n,\tau_1,s,t}(x) = \sum_{i=1}^{t-1} P_{n-1,\tau_1,s-1,i-1}(x) + (s-1-t)xP_{n-1,\tau_1,s-1,t}(x) + \sum_{j=t}^{s-1} P_{n-1,\tau_1,j,t}(x) + P_{n-1,\tau_1,s,t}(x).$$

Note that, the first case vanishes if $t = 1$, but the formula is consistent as it contributes an empty summation. The second case vanishes when $t = s-1$, but the formula is still correct as $P_{n-1,\tau_1,s-1,s}(x) = 0$. The last case vanishes when $s = n-1$, but the formula is correct as $P_{n-1,\tau_1,n-1,t}(x) = 0$. Thus, we prove the first formula.

Here, we derive a recursive formula for $P_{n,\tau_1,s,s}(x)$ when $1 \le s \le n-1$. Given any $\sigma \in S_n$ such that the position of 1 and the position of the last ascent is s. Removing 1 does not effect τ-matching at other positions. After removing and reducing, the permutation either has no ascent, or the last ascent is at some position between 1 and $s-1$. Thus, we have

$$P_{n,\tau_1,s,s}(x) = 1 + \sum_{i=1}^{s-1} P_{n-1,\tau_1,i}(x).$$

Lastly, we derive a formula for $P_{n,\tau_1,s,n}(x)$ when $1 \leq s < n-1$. Given any σ such that the position of last secent is s and 1 is at the last position. Removing 1 will not effect τ_1-match at other positions. The position of last ascent is still at s. Thus, we have

$$P_{n,\tau_1,s,n}(x) = P_{n-1,\tau_1,s}(x).$$

□

We shall prove similar result for $1\underline{2}43$. Let $\tau_2 = 1\underline{2}43$. Let $P_{n,\tau_2,s}(x) = \sum_\sigma x^{pmp_{\tau_2}(\sigma)}$, where the sum is over all $\sigma \in S_n$ with the *last descent* of σ is at position s. Let $P_{n,\tau_2,s,t}(x) = \sum_\sigma x^{pmp_{\tau_2}(\sigma)}$, where the sum is over all $\sigma \in S_n$ with the last ascent of σ is at position s and the 1 is at position t. All feasible values of s are $t = 1, 2, \ldots, s-1$ and $s+1$. Note that, when $s = 1$, the only possible value for t is 2. Thus, we have

$$P_{n,\tau_2}(x) = 1 + \sum_{s=1}^{n-1} P_{n,\tau_2,s}(x).$$

Also,

$$P_{n,\tau_2,s}(x) = \left(\sum_{t=1}^{s-1} P_{n,\tau_1,s,t}(x) \right) + P_{n,\tau_1,s+1,n}(x).$$

Here, we derive a recursive formula for $P_{n,\tau_2,s}(x)$.

Lemma 2.10. *$P_{n,\tau_2,s,t}(x)$ satisfies the following recursive formulas:*

$$\begin{aligned} P_{n,\tau_2,s,t}(x) &= \sum_{i=1}^{t-1} P_{n-1,\tau_2,s-1,i-1}(x) + (s-1-t)xP_{n-1,\tau_2,s-1,t}(x) \\ &\quad + \sum_{j=t}^{s-1} P_{n-1,\tau_2,j,t}(x) + P_{n-1,\tau_2,s,t}(x) \end{aligned}$$

for $1 \leq t < s \leq n-1$,

$$P_{n,\tau_2,s,s+1}(x) = 1 + \sum_{i=1}^{s} P_{n-1,\tau_2,i}(x)$$

for $1 \leq s \leq n-1$.

By convention, Let $P_{n,\tau_2,s,t}(x) = 0$ *for infeasible value of* s, t.

Proof. First, consider the formula for $P_{n,\tau_2,s,t}(x)$ when $1 \leq t < s \leq n-1$. Let l be the position of 2 in σ, where $\sigma \in S_n$ with the last descent of σ is at position s and 1 is at position t. we have 4 cases:

1. $1 \leq t \leq l-1$. In this case, σ is not τ_2-match at position l. We can remove 1 from σ without effecting τ_2-match at other positions. That is because if σ is τ_2-match at other positions and the matching involves 1, it is still τ_2-match by using 2 instead of 1.

   After removing and reducing, the position of 1 is l and the position of the last descent is $s-1$. Thus, this case contributes $\sum_{i=1}^{t-1} P_{n-1,\tau_2,s-1,i}(x)$.

2. $t+1 \leq l \leq s-1$. In this case, σ is τ_2-match at position l since 1 2 σ_s σ_{s+1} form the pattern τ_2. By the same reasoning as the first case, we can remove 2 without effecting τ_2-match at any other positions.

   After removing and reducing, the position of 1 stays at t and the position of the last ascent is $s-1$. Thus, this case contribute $x(s-1-t)P_{n-1,\tau_1,s-1,t}(x)$.

3. $l = s$. In this case, we can remove 2 without effecting τ_2-match at other positions.

   After removing and reducing, the position of 1 stays at t and the last descent appears at some position between $t+1$ and s. Thus, this case contributes $\sum_{j=t}^{s-1} P_{n-1,\tau_2,j,t}(x)$.

4. $l = n$. In this case, we can remove 2 without effecting τ_2-match at other positions. After removing and reducing, the position of 1 stays at t and the position of the last ascent stays at s. Thus, this case contributes $P_{n-1,\tau_2,s,t}(x)$.

In total, we have

$$\begin{aligned} P_{n,\tau_2,s,t}(x) &= \sum_{i=1}^{t-1} P_{n-1,\tau_2,s-1,i-1}(x) + (s-1-t)xP_{n-1,\tau_2,s-1,t}(x) \\ &\quad + \sum_{j=t+1}^{s} P_{n-1,\tau_2,j,t}(x) + P_{n-1,\tau_2,t-1,t}(x). \end{aligned}$$

Note that, the first case vanishes if $t = 1$, but the formula is consistent as it contributes an empty summation. The second case vanishes when $t = s - 1$, but the formula is still correct as $P_{n-1,\tau_2,s-1,s}(x) = 0$. The last case vanishes when $s = n - 1$, but the formula is correct as $P_{n-1,\tau_2,n-1,t}(x) = 0$. Thus, we prove the first formula.

Next, we derive the recursive formula for $P_{n,\tau_2,s,s+1}(x)$ for $1 \leq s \leq n-1$. Given any $\sigma \in S_n$ such that the position of the last descent is s and the position of 1 is $s+1$. Removing 1 will not effect τ_2-matching at other positions. Afer removing and reducing, the remaining permutation either has no descent, or the last desent is at some position between 1 and s. Thus, we have

$$P_{n,\tau_2,s,s+1}(x) = 1 + \sum_{i=1}^{s} P_{n-1,\tau_2,i}(x).$$

Thus, we prove the second formula.

□

Here, we state another lemma proving equality $P_{n,\tau_1,s,t}(x)$ and $P_{n,\tau_2,s,t}(x)$.

Lemma 2.11.

$$P_{n,\tau_1,s,t}(x) = P_{n,\tau_2,s,t}(x)$$

for $1 \leq t < s \leq n-1$*, and*

$$P_{n,\tau_1,s,s}(x) + P_{n,\tau_1,s,n}(x) = P_{n,\tau_2,s,s+1}(x)$$

for $1 \leq s \leq n-1$*.*

And so, as a consequence, $P_{n,\tau_1,s}(x) = P_{n,\tau_2,s}(x)$ *for all* $1 \leq s \leq n-1$, $P_{n,\tau_1}(x) = P_{n,\tau_2}(x)$*, and so* τ_1 *and* τ_2 *are pmp-Wilf equivalent.*

Proof. One can check that both equations are satisfied for small n. Then, we proceed by induction. For the first equation, consider $P_{n,\tau_1,s,t}(x) - P_{n,\tau_2,s,t}(x)$, for $1 \leq t < s \leq n-1$. Using recursive formula for $P_{n,\tau_1,s,t}(x)$ and $P_{n,\tau_2,s,t}(x)$ as well as inductive hypothesis, we have

$$P_{n,\tau_1,s,t}(x) - P_{n,\tau_2,s,t}(x) = P_{n-1,\tau_1,t,t}(x) - P_{n-1,\tau_2,t-1,t}(x).$$

Using recursive formula and inductive hypothesis, $P_{n-1,\tau_1,t,t}(x) = P_{n-1,\tau_2,t-1,t}(x)$, thus we prove the first equation. For the second equation:

$$\begin{aligned} P_{n,\tau_1,s,s}(x) + P_{n,\tau_1,s,n}(x) &= 1 + \sum_{i=1}^{s} P_{n-1,\tau_1,i}(x) \\ &= 1 + \sum_{i=1}^{s} P_{n-1,\tau_1,i}(x) \\ &= P_{n,\tau_1,s,s+1}(x). \end{aligned}$$

Thus, we prove the second equation. As a consequence, $P_{n,\tau_1,s}(x) = P_{n,\tau_2,s}(x)$ for all $1 \leq s \leq n-1$, and $P_{n,\tau_1}(x) = P_{n,\tau_2}(x)$ and so, τ_1 and τ_2 are *pmp*-Wilf equivalent.

□

As a consequence, $P_{n,\tau_1,s}(x) = P_{n,\tau_2,s}(x)$ for all $1 \leq s \leq n-1$, $P_{n,\tau_1}(x) = P_{n,\tau_2}(x)$, and so τ_1 and τ_2 are *pmp*-Wilf equivalent. Hence, we prove Theorem 2.9.

2.5 Patterns of arbitrary length

In this section, we present equivalence of two positional marked pattern $1\underline{2}p_1 \dots p_n$ and $\underline{2}1p_1 \dots p_n$.

Theorem 2.10. *Given two pmp $P_1 = 1\underline{2}p_1 \dots p_{l-2}$ and $P_2 = \underline{2}1p_1 \dots p_{l-2}$. Then, P_1 and P_2 are pmp-Wilf equivalent.*

We apply the technique introduced in [1] to prove Theorem 2.10. To prove the theorem, we need to give several definitions. Let $\tau = red(p_1 p_2 \dots p_{l-2})$.

Definition 2.7. *Given $\pi \in S_n$. Consider the diagram of π. For each cell (i, j) in the diagram of π, the cell is* dominant *if there is an occurrence of τ in the diagram of π when only considering row $i+1, i+2, \dots, n$ and columns $j+1, j+2, \dots, n$. A cell is* non-dominant *if it is not dominant.*

It is clear that given any dominant cell, every cell to the left and below the dominant cell is also dominant. Thus, the collection of dominant cells form a Ferrers board.

Let $ND(\pi) = \{(i, j) \in [n] \times [n] \mid \text{cell } (i, j) \text{ is non-dominant and contains } \times\}$. Note that, if a cell is dominant, then one could find an occurrence of τ above and to the right of the cell such that all $\times$ involved are in non-dominant cell. If not, then every occurence of τ above and to the right of the cell (i, j) contains $\times$ in a dominant cell. Pick a copy T_1 of τ in which a dominant cell (i', j') containing $\times$ is the rightmost among all dominant cells containing $\times$ in all copies of τ above and to the right of (i, j). Since cell (i', j') is also dominant, one can find a copy T_2 of τ above and to the right of (i', j'), which is also above and to the right of cell (i, j). Thus, this copy will also contain a dominant cell containing $\times$, contradicting to the fact that the T_1 contains the rightmost dominant cell.

Thus, if $ND(\pi)$ is known, one can recover the set of dominant cells in the diagram of π completely. Given any $Q \subseteq [n] \times [n]$, Let $S_n^Q = \{\pi \in S_n | ND(\pi) = Q\}$. We shall prove that

$$\sum_{\pi \in S_n^Q} x^{pmp_{P_1}(\pi)} = \sum_{\pi \in S_n^Q} x^{pmp_{P_2}(\pi)}.$$

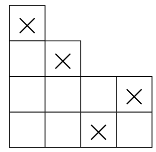
Here, we analyse the set S_n^Q. First, we only need to consider those Q such that $S_n^Q \neq \emptyset$. Given any such Q, we can obtain an element in S_n^Q by filling $\times$ in the dominant part of the diagram until every row and column contains precisely one $\times$. This gives all elements in S_n^Q since filling $\times$ in the dominant part does not alter whether a cell is dominant or non-dominant.

To fill $\times$ in dominant cells, we start will all dominant cells and eliminate all rows and columns that already contain $\times$ from the set Q. The remaining cells form a Ferrers board. Let $\lambda(Q)$ denote the shape of the Ferrers board obtained from the process above.

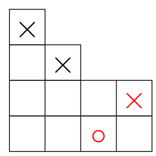
Definition 2.8. *Given any Ferrers board of shape $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k)$ with $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_k$,* a filling of λ *is an assignment of $\times$ in the Ferrers board of shape λ such that every row and columns contain precisely one $\times$.*

In order for λ to have a filling, the number of rows of λ must be the same as the number of columns in λ. More specifically, if $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k)$, λ has a filling if and only if $(k - i + 1) \leq \lambda_i \leq k$ for all k. Let S_λ denote the set of fillings of λ.

Given any $\pi \in S_\lambda$, and $p \in S_k$, we say that π contains p if π contains a subdiagram of p. Note that, in order for π to contain p, the *entire* diagram of p has to present as a subdiagram of π including cells not containing $\times$. For example, the filling below contains $p = 12$ but does not contain $q = 21$.



Here, we define positional marked pattern on S_λ similar to S_n. Given any $\pi \in S_\lambda$ and $p \in S_k^*$, we say that π is p-match at position t if when replacing $\times$ in the t-th column of π of by $\circ$, then it contains a subdiagram p. For example, the diagram above is $\underline{1}2$-match at position 3 as shown below



Let $pmp_p(\pi)$ denote the number of position t such that π is p-match at position t. Here, we state a lemma which will be our main tool to prove Theorem 2.10.

Lemma 2.12. *Given any $Q \subseteq [n] \times [n]$ such that $S_n^Q \neq \emptyset$. Then*

$$\sum_{\pi \in S_n^Q} x^{pmp_{P_1}(\pi)} = \sum_{\pi \in S_\lambda(Q)} x^{pmp_{1\underline{2}}(\pi)}$$

and

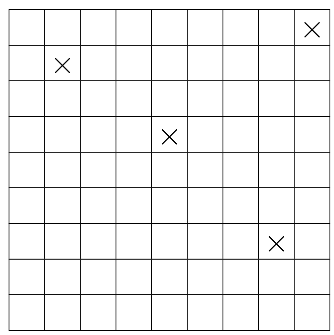
$$\sum_{\pi \in S_n^Q} x^{pmp_{P_2}(\pi)} = \sum_{\pi \in S_\lambda(Q)} x^{pmp_{\underline{2}1}(\pi)}.$$

Proof. Note that there is a natural bijection $\phi : S_{\lambda(Q)} \to S_n^Q$ which maps any filling $\pi \in \lambda(Q)$ to a permutation with non-dominant part corresponding to Q and dominant part having the same filling as π. We will show that π is $1\underline{2}$-match at a position if and only if $\phi(\pi)$ is P_1-match at the corresponding position according to the map ϕ

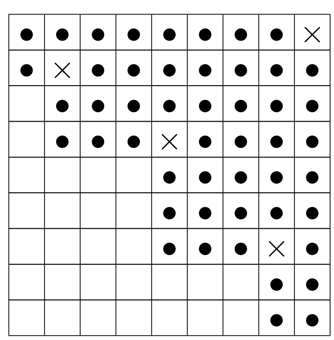
The converse is obvious. For the forward direction, suppose $\pi \in S_{\lambda(Q)}$ is $1\underline{2}$-match at a certain position. Thus, the corresponding position in $\phi(\pi)$ is $1\underline{2}$-match with every cells involved are dominant. Thus, we can find a copy of τ above and to the right of every cells involved in $1\underline{2}$-match. Therefore, the $1\underline{2}$-match together with τ makes $\phi(\pi)$ P_1-match at the position corresponding to the $1\underline{2}$-match in π.

Therefore, ϕ is a bijection between $S_{\lambda(Q)}$ and S_n^Q such that $pmp_{1\underline{2}}(\pi) = pmp_{P_1}(\phi(\pi))$. Thus, it proves the first equality of the lemma. The second equation can be proved with the exact same reasoning. □

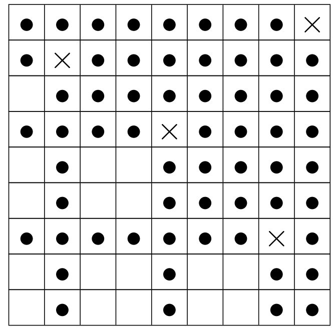
As an example, let $n = 9$, $\tau = 12$, (so $P_1 = 1\underline{2}34$ and $P_2 = \underline{2}134$), and Q is as below



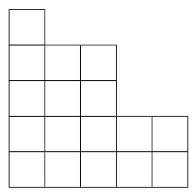
With Q, we recover dominant and non-dominant cells. We fill $\bullet$ in non-dominant cells.



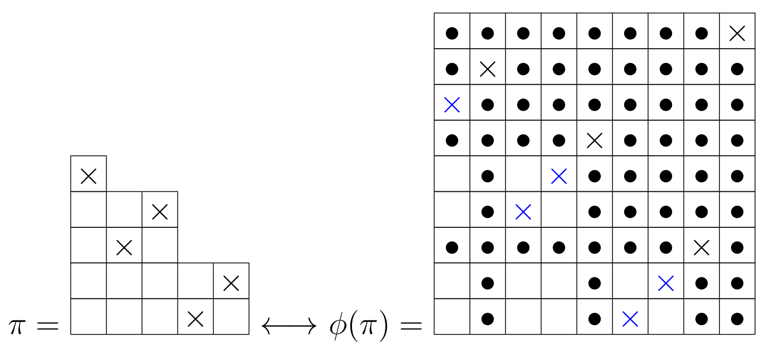
We want to fill $\times$ into the diagram so that every row and column contains precisely one $\times$, thus we eliminate all empty cells that are in same rows or columns with cell containing $\times$. We fill $\bullet$ in such cells.



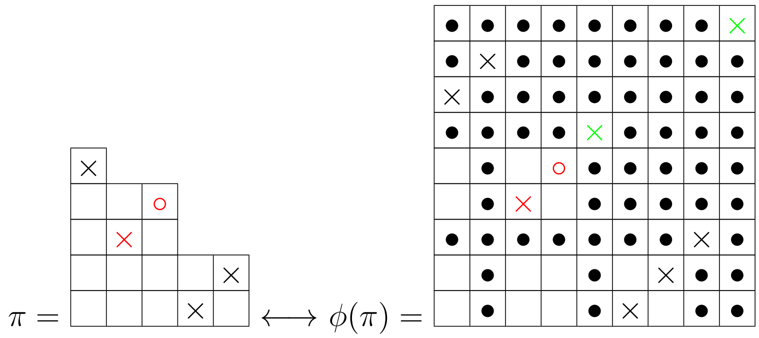
Thus, the remaining cells form a Ferrers board $\lambda(Q) = (5, 5, 3, 3, 1)$, as shown below:



Then, to define the one-to-one corresponding between S_9^Q and $S_{\lambda(Q)}$, we fill cells in $\lambda(Q)$ the same way we fill available cells in a diagram Q. For example, below is an example of the correspondence:



With this map, $1\underline{2}$-matching at position 3 of π corresponds to $1\underline{2}34$-matching at position 4 of $\phi(\pi)$.



Here, in order to prove theorem 2.10, we only need to prove that, given any Ferrers board λ, $\sum_{\pi\in S_\lambda} x^{pmp_{1\underline{2}}(\pi)} = \sum_{\pi\in S_\lambda} x^{pmp_{\underline{2}1}(\pi)}$. In fact, we find a formula for both polynomials.

Lemma 2.13. *Let $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k)$ be a Ferrers board such that $(k - i + 1) \le \lambda_i \le k$ for all $1 \le i \le k$. Then,*

$$\sum_{\pi\in S_\lambda} x^{pmp_{1\underline{2}}(\pi)} = \sum_{\pi\in S_\lambda} x^{pmp_{\underline{2}1}(\pi)} = \prod_{i=1}^{k}(1 + (\lambda_i - (k - i + 1))x).$$

Proof. Given any λ satisfying $(k - i + 1) \le \lambda_i \le k$ for all $1 \le i \le k$, let $\bar{\lambda}$ be a Ferrers board obtained from λ by removing the top most row and left most column. That is $\bar{\lambda} = (\lambda_1 - 1, \lambda_2 - 1, \ldots, \lambda_{k-1} - 1)$. Then, to prove the formula, we only need to prove that

$$\sum_{\pi\in S_\lambda} x^{pmp_{1\underline{2}}(\pi)} = (1 + (\lambda_k - 1)x) \sum_{\pi\in S_{\bar{\lambda}}} x^{pmp_{1\underline{2}}(\pi)}.$$

To prove the equation above, we consider the topmost row in λ. The filling is $1\underline{2}$-match at the position of $\times$ in the topmost row if and only if $\times$ is not in the rightmost possible cell. So, there are $\lambda_k - 1$ positions to fill $\times$ so that it is $1\underline{2}$-match, and 1 position otherwise. Once the top row is filled, we consider filling the rest of λ by remove the top row and the column containing $\times$. The remaining cells form a shape $\bar{\lambda}$. Thus, the equation above is proved.

For the pattern $\underline{2}1$, a similar reasoning can also be applied. The filling of λ is $\underline{2}1$-match at the position of $\times$ in the topmost row if and only if $\times$ is not in the *leftmost* possible cell. Hence, we have

$$\sum_{\pi \in S_\lambda} x^{pmp_{\underline{2}1}(\pi)} = (1 + (\lambda_k - 1)x) \sum_{\pi \in S_{\bar{\lambda}}} x^{pmp_{\underline{2}1}(\pi)}.$$

Therefore, we proved the lemma.

□

Here, we prove the main theorem:

Proof. (of Theorem 2.10) Note that S_n is a disjoint union of S_n^Q for all Q such that $S_n^Q \neq \emptyset$. So, we have

$$\begin{aligned}
P_{n,P_1}(x) &= \sum_{\sigma \in S_n} x^{pmp_{P_1}(\sigma)} \\
&= \sum_{Q} \sum_{\sigma \in S_n^Q} x^{pmp_{P_1}(\sigma)} \\
&= \sum_{Q} \sum_{\sigma \in S_{\lambda(Q)}} x^{pmp_{1\underline{2}}(\sigma)} && \text{By Lemma 2.12} \\
&= \sum_{Q} \sum_{\sigma \in S_{\lambda(Q)}} x^{pmp_{\underline{2}1}(\sigma)} && \text{By Lemma 2.13} \\
&= \sum_{Q} \sum_{\sigma \in S_n^Q} x^{pmp_{P_2}(\sigma)} && \text{By Lemma 2.12} \\
&= \sum_{\sigma \in S_n} x^{pmp_{P_2}(\sigma)} = P_{n,P_2}(x).
\end{aligned}$$

Thus, P_1 and P_2 are *pmp*-Wilf equivalent. □

Remark Let $P_3 = 2\underline{1}p_1p_2 \dots p_{l-2}$. By applying the same reasoning, we can prove that P_3 and P_2 are *pmp*-Wilf equivalent by considering the rightmost column of any Ferrers board λ. Thus, we prove that P_1, P_2 and P_3 are *pmp*-Wilf equivalent.

2.6 Future research

One way to generalize positional marked patterns is to consider multiple patterns. Given a collection of positional marked patterns Γ and $\sigma \in S_n$, we say that σ is Γ-match at position l if σ is τ-match at position l for some $\tau \in \Gamma$. Given $\sigma \in S_n$, let $pmp_\Gamma(\sigma)$ denote the number of position l such that σ is Γ-match at position l. Let $P_{n,\Gamma}(x) = \sum_{\sigma \in S_n} x^{pmp_\Gamma(\sigma)}$. Given two collections of positional marked patterns Γ_1 and Γ_2, they are Wilf-equivalent if $P_{n,\Gamma_1}(x) = P_{n,\Gamma_2}(x)$. This is a nice generalization to positional marked patterns since the constant term of $P_{n,\Gamma}(x)$ enumerates the number of $\sigma \in S_n$ avoiding every patterns in Γ, which replicates what $P_{n,\tau}(x)$ provides in single pattern cases. Here, we stated, without proof, a result on collection of positional marked patterns similar to Theorem 2.10. The proof also follows the same logic as in the proof of Theorem 2.10.

Theorem 2.11. *Let $\tau_1, \tau_2, \ldots, \tau_l$ be rearrangements of $\{3, 4, \ldots, k\}$, so that $1\underline{2}\tau_i$ and $\underline{2}1\tau_i$ are elements of S_k^*. Let $\Gamma_1 = \{1\underline{2}\tau_i | 1 \leq i \leq l\}$ and $\Gamma_2 = \{\underline{2}1\tau_i | 1 \leq i \leq l\}$. Then Γ_1 and Γ_2 are Wilf-equivalent.*

Moreover, define Γ-matching would let us realize positional marked pattern as a refinement of marked mesh pattern defined by Remmel and Kitaev [13]. Given $a, b, c, d \in \mathbb{Z}_{\geq 0}$, and given $\sigma \in S_n$, we say σ matchs $MMP(a, b, c, d)$ at position l if in the diagram of σ relative to the coordinate system which has $\times$ in the l-th column as its origin there are at least a $\times$'s in quadrant I, at least b $\times$'s in quadrant II, at least c $\times$'s in quadrant III, and at least d $\times$'s in quadrant IV. As an example, $\sigma = 25417683$ matches $MMP(3, 0, 1, 1)$ at position 3:

It is easy to see that, for any a, b, c, d, $MMP(a, b, c, d)$ is equivalent to Γ-matching for some collection of positional marked patterns Γ. For example, to match $MMP(2, 0, 0, 0)$ at some position, it is the same as to match $\Gamma = \{\underline{1}23, \underline{1}32\}$ at the same position. Therefore, as we introduce multiple positional marked patterns, one can realize posional marked

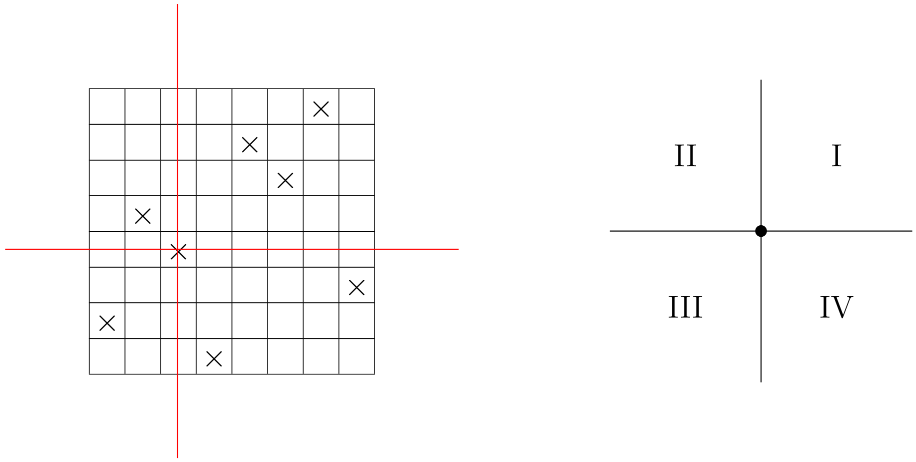


Figure 2.12: Marked mesh pattern on $\sigma = 25417683$

patterns as a refinement of marked mesh pattern.

Chapter 3

Whitney numbers and generalized Eulerian polynomials

In this chapter, we investigate the combinatorial properties of two polynomials introduced by Benoumhani [2, 3]. The polynomials involve the Whitney numbers of the second kind $W_m(n, k)$ which enumerate Dowling set partitions.

3.1 Introduction

Remmel and Wachs [31] defined a class of combinatorial objects called *Dowling set partition* as follows: Given a set partition $\pi = (A_1, \ldots, A_k, A_{k+1})$ of $[n+1]$, we shall assume that it is written in a standard form which means that

1. $\min(A_1) < \cdots < \min(A_k)$ and

2. $n + 1 \in A_{k+1}$.

In such a situation, we let $Min(\pi) = \{\min(A_1), \ldots, \min(A_k)\}$ and $Last(\pi) = A_{k+1}$.

Definition 3.1. a Dowling set partition of $[n]$ with k parts and m colors *is a pair* (π, c) *where* $\pi = (A_1, \ldots, A_k, A_{k+1})$ *is a set partition of* $[n + 1]$ *into* $k + 1$ *parts and* c *is a map* *from* $[n+1]-(Min(\pi)\cup Last(\pi))$ *into* $[m]$. *We shall view* c *as coloring of the non-minimal elements in each of parts* $A_1,\ldots,A_k$ *with one of* m *colors from* $\{1,\ldots,m\}$.

Dowling set partitions of $[n]$ with k parts and m colors is enumerated by *the Whitney numbers of the second kind*, $W_m(n,k)$.When $m=1$, the Whitney numbers of the second kind is the Stirling numbers of the second kind, that is:

$$W_1(n,k)=S_{n+1,k+1}.$$

Our point of interest is the relationship between the Stirling numbers of the second kind and the Eulerian numbers $E_{n,k}$. A well known identity relating the Stirling numbers of the second kind to the Eulerian numbers is that

$$\sum_{k=1}^{n} k!S_{n,k}x^k=\sum_{k=0}^{n-1} E_{n,k}x^k(x+1)^{n-k-1}.$$

Benoumhani [2] asked if the equation above can be generalized to the Whitney numbers of the second kind. In particular, he defined two polynomials $A_m(n,k)$ and $B_m(n,k)$ to be polynomials satisfying the equations

$$\sum_{k=0}^{n} k!m^kW_m(n,k)x^k=\sum_{k=0}^{n} A_m(n,k)x^{n-k}(x+1)^k$$

and

$$\sum_{k=0}^{n} k!W_m(n,k)x^k=\sum_{k=0}^{n} B_m(n,k)x^{n-k}(x+1)^k$$

and asked for combinatorial interpretations of $A_m(n,k)$ and $B_m(n,k)$.

In this chapter, we gave complete combinatorial interpretations of polynomials $A_m(n,k)$ and $B_m(n,k)$. The outline of this chapter is as follows: In section 3.2, we give precise definitions of Dowling lattices and Whitney numbers. In section 3.3, we give our combinatorial interpretation of the coefficients $A_{m+1}(n,k)$. In particular, one can compute that

$$\begin{array}{rcrrrrr}
A_m(4,4) & = & & & & & 1 \\
A_m(4,3) & = & m^4 & +4m^3 & +6m^2 & +4m & -4 \\
A_m(4,2) & = & 11m^4 & +12m^3 & -6m^2 & -12m & +6 \\
A_m(4,1) & = & 11m^4 & -12m^3 & -6m^2 & +12m & -4 \\
A_m(4,0) & = & m^4 & -4m^3 & +6m^2 & -4m & +1.
\end{array}$$

However, when we expand the polynomials $A_m(n,k)$ in terms of powers of $(m-1)$ or, equivalently, when we expand $A_{m+1}(n,k)$ in terms of powers of m, we see a much nicer structure.

$$\begin{array}{rcrrrrr}
A_{m+1}(4,4) & = & & & & & 1 \\
A_{m+1}(4,3) & = & m^4 & +8m^3 & +24m^2 & +32m & +11 \\
A_{m+1}(4,2) & = & 11m^4 & +56m^3 & +96m^2 & +56m & +11 \\
A_{m+1}(4,1) & = & 11m^4 & +32m^3 & +24m^2 & +8m & +1 \\
A_{m+1}(4,0) & = & m^4 & & & & .
\end{array}$$

In this case, we see that all the coefficients are non-negative integers and there is a natural symmetry in the table. For any polynomial $P(x) = \sum_{k\geq 0} a_k x^k$, we let $P(x)|_{x^k} = a_k$ denote the coefficient of x^k in $P(x)$. As the table above suggest, we will prove the following theorem.

Theorem 3.1. $A_{m+1}(n,k)|_{m^s} \geq 0$ *for all* n,k,s *with* $0 \leq k,s \leq n$. *Moreover,* $A_{m+1}(n,k)|_{m^s} = A_{m+1}(n,n-k)|_{m^{n-s}}$ *for all* n,k,s *with* $0 \leq k,s \leq n$.

In section 3.4, we will give our combinatorial interpretation of the coefficients $B_{m+1}(n,k)$ and prove some properties of $B_{m+1}(n,k)$.

3.2 Definitions

Throughout the dissertation, we mainly use Dowling set partitions as the way to work with the Whitney numbers. However, the Whitney numbers were defined originally based on more general structures called Dowling lattices.

The Dowling lattice $Q_n(G)$ can be defined as follows. Let G be a multiplicative group of order n. A **G-labeled set** (T, α) is a set T together with a function $\alpha : T \to G$. We say that two G-labeled sets (T, α) and (T, β) are equivalent if there is a $g \in G$ such that $\beta = g\alpha$. That is, for all $t \in T$, $\beta(t) = g\alpha(t)$. The equivalence class of (T, α) will be denoted by $[T, \alpha]$. A **partial G-partition** of $[n] = \{1, \ldots, n\}$ is a set $P = \{[A_1, \alpha_1], \ldots [A_k, \alpha_k]\}$ of equivalence classes of G-labeled sets such that $(A_1, \ldots, A_k)$ is a set partition of some subset of $[n]$. One can define a partial order on the set of all partial G-partitions of $[n]$ as follows. If $P_1 = \{[A_1, \alpha_1], \ldots [A_k, \alpha_k]\}$ and $P_2 = \{[B_1, \beta_1], \ldots, [B_\ell, \beta_\ell]\}$, then we say that $P_1 \leq P_2$ if and only if

1. Each B_i is a union of the parts of the set partition $(A_1, \ldots, A_k)$ and
2. for all i and j, if $A_i \subseteq B_j$, then (A_i, α_i) is equivalent to $(A_i, \beta_j \restriction A_i)$ where $\beta_j \restriction A_i$ is the restriction of β_j to A_i.

The Dowling lattice $Q_n(G)$ is the set of all partial G-partitions of $[n]$ under $\leq$.

Dowling [9] proved that $Q_n(G)$ is a lattice and that it only depends on the cardinality m of G up to isomorphism. Thus $Q_n(G)$ has a rank function **rank** and a co-rank function **co-rank**. We let $C_k(Q_n(G))$ denote the set of all partial G-partitions of $[n]$ whose co-rank is k. Then the Whitney number of the first kind of $Q_n(G)$ is defined

$$w_G(n, k) = \sum_{P \in C_k(Q_n(G))} \mu(0, P).$$

where 0 is the zero of the lattice $Q_n(G)$ and μ is the Möbius function of $Q_n(G)$. The

Whitney number of the second kind of $Q_n(G)$ is defined

$$W_G(n,k) = |C_k(Q_n(G))|.$$

Since $Q_n(G)$ depends only on the cardinality of G, we let $w_m(n,k) = w_G(n,k)$ and $W_m(n,k) = W_G(n,k)$ where $|G| = m$.

Whitney numbers and Dowling lattices have been studied further in [6, 12]. Wachs and the Remmel [31] gave a combinatorial interpretation of $W_m(n,k)$ as follows. Given a set partition $\pi = (A_1,\ldots,A_k,A_{k+1})$ of $[n+1]$, we shall assume that it is written in a standard form which means that

1. $\min(A_1) < \cdots < \min(A_k)$ and
2. $n+1 \in A_{k+1}$.

In such a situation, we let $Min(\pi) = \{\min(A_1),\ldots,\min(A_k)\}$ and $Last(\pi) = A_{k+1}$. For example, if $n = 7$ and $\pi = (\{1,4,6\},\{3,5\},\{2,7,8\})$, then $Min(\pi) = \{1,3\}$ and $Last(\pi) = \{2,7,8\}$.

Definition 3.2. a Dowling set partition of $[n]$ with k parts *is a pair* (π,c) *where* $\pi = (A_1,\ldots,A_k,A_{k+1})$ *is a set partition of* $[n+1]$ *into* $k+1$ *parts and* c *is a map from* $[n+1]-(Min(\pi)\cup Last(\pi))$ *into* $[m]$. *We shall view* c *as coloring of the non-minimal elements in each of parts* $A_1,\ldots,A_k$ *with one of* m *colors from* $\{1,\ldots,m\}$.

In what follows, we shall represent (π,c) in the form $A_1/A_2/\ldots/A_k/A_{k+1}$ where we put subsscripts on the elements to indicate the color. We will also distinguish the largest element $n+1$ by making it boldface. For example,

$$1/2\ 6_1/4\ 5_1\ 7_3\ 8_2/3\ \mathbf{9}$$

is a Dowling set partition of 8 with 3 parts for $m = 3$.

Note that it follows that $W_1(n,k) = S_{n+1,k+1}$ where $S_{n,k}$ is the Stirling number of the second kind which counts the number of set partitions of $[n]$ into k-parts.

Let S_n denote the set symmetric group. Given a permutation $\sigma = \sigma_1 \ldots \sigma_n \in S_n$, we let

$$\begin{aligned} Des(\sigma) &= \{i : \sigma_i > \sigma_{i+1}\},\ \mathrm{des}(\sigma) = |Des(\sigma)|, \\ Ris(\sigma) &= \{i : \sigma_i < \sigma_{i+1}\},\ \text{and } \mathrm{ris}(\sigma) = |Ris(\sigma)|. \end{aligned}$$

The Eulerian number $E_{n,k}$ is the number of $\sigma \in S_n$ such that $\mathrm{des}(\sigma) = k$. Then a well known identity relating the Stirling numbers $S_{n,k}$ to the Eulerian numbers $E_{n,k}$ is that

$$\sum_{k=1}^{n} k! S_{n,k} x^k = \sum_{n=0}^{n-1} E_{n,k} x^k (x+1)^{n-k-1}. \tag{3.1}$$

Now Benoumhani [3] defined the following analogues of the left-hand side of (3.1).

$$F_{m,n,1}(x) = \sum_{k=0}^{n} k! m^k W_m(n,k) x^k \tag{3.2}$$

and

$$F_{m,n,2}(x) = \sum_{k=0}^{n} k! W_m(n,k) x^k. \tag{3.3}$$

He then asked for a combinatorial interpretation of the analogue of the right-hand side of (3.1). That is, he asked for combinatorial interpretations of $A_m(n,k)$ and $B_m(n,k)$ where

$$F_{m,n,1}(x) = \sum_{k=0}^{n} A_m(n,k) x^{n-k} (x+1)^k \tag{3.4}$$

and

$$F_{m,n,2}(x) = \sum_{k=0}^{n} B_m(n,k) x^{n-k} (x+1)^k. \tag{3.5}$$

3.3 A combinatorial interpretation of $A_{m+1}(n,k)$.

In this section, we shall give a combinatorial interpretation of the polynomials $A_{m+1}(n,k)$. Our point of departure is to write (3.4) in matrix notation. Let W and A be the column vectors

$$W = \begin{bmatrix} m^n n! W_m(n,n) \\ m^{n-1}(n-1)! W_m(n,n-1) \\ \vdots \\ m W_m(n,1) \\ W_m(n,0) \end{bmatrix} \quad \text{and } A = \begin{bmatrix} A_m(n,0) \\ A_m(n,1) \\ \vdots \\ A_m(n,n) \end{bmatrix}.$$

Let X and $\overline{X}$ be the row vectors

$$\begin{aligned} X &= \left[x^n, x^{n-1}, \ldots, x, 1\right] = \left[x^{n-k}\right]_{k=0,\ldots,n} \text{ and} \\ \overline{X} &= \left[x^n, x^{n-1}(x+1), x^{n-2}(x+1)^2, \ldots, x(x+1)^{n-1}, (x+1)^n\right] \\ &= \left[x^{n-k}(x+1)^k\right]_{k=0,\ldots,n}. \end{aligned}$$

Then we can rewrite (3.4) as $XW = \overline{X}A$.

Next we write $\overline{X} = XM$ where $M = [m_{i,j}]_{i,j=0,\ldots,n}$. It follows that

$$m_{i,j} = [x^{n-i}]x^{n-j}(x+1)^j = [x^{j-i}](x+1)^j = \binom{j}{j-i} = \binom{j}{i}$$

for $i \leq j$, and $m_{i,j} = 0$ otherwise. Hence, the matrix M looks like

$$M = \begin{bmatrix} \binom{0}{0} & \binom{1}{0} & \binom{2}{0} & \dots & \binom{n}{0} \\ 0 & \binom{1}{1} & \binom{2}{1} & \dots & \binom{n}{1} \\ 0 & 0 & \binom{2}{2} & \dots & \binom{n}{2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & \binom{n}{n} \end{bmatrix}.$$

It follows that $XW = XMA$ so $W = MA$. Hence $A = M^{-1}W$.

It is well known that $M^{-1} = [n_{i,j}]_{i,j=0,\dots,n}$ where $n_{i,j} = (-1)^{j-i}\binom{j}{i}$ for $j \geq i$, and $n_{i,j} = 0$ otherwise, see Stanley [34]. It follows that

$$A_m(n,k) = \sum_{j=k}^{n} (-1)^{j-k} \binom{j}{k} m^{n-j}(n-j)!W_m(n, n-j)$$

and, hence,

$$A_{m+1}(n,k) = \sum_{j=k}^{n} (-1)^{j-k} \binom{j}{k} (m+1)^{n-j}(n-j)!W_{m+1}(n, n-j). \tag{3.6}$$

Next we want to give a combinatorial interpretation of the right-hand side of (3.6). First we can interpret the term $(m+1)^{n-j}(n-j)!W_{m+1}(n, n-j)$ as starting with a Dowling set partition of n with $n-j$ parts

$$\pi = A_1/A_2/\dots/A_{n-j}/A_{n-j+1}$$

and interpreting the factor $(m+1)^{n-j}$ as all the ways to color each element of $Min(\pi) = \{\min(A_1), \dots, \min(A_{n-j})\}$ with one of $m+1$ colors. This will ensure that each element of $[n] - Last(\pi) = [n] - A_{n-j+1}$ is colored with one of $m+1$ colors. Then we interpret the factor $(n-j)!$ as the number of ways to order $A_1/A_2/\ldots/A_{n-j}$. We let $ODSP_{n,n-j}$ denote the set of all ordered set partitions $\pi = B_1/B_2/\ldots/B_{n-j}/B_{n-j+1}$ of $[n+1]$ into $n-j+1$ parts such that $n+1 \in B_{n-j+1} = Last(\pi)$ and all the elements in $[n] - Last(\pi)$ are colored with one of $m+1$ colors. It follows that if we let $\text{last}(\pi) = |Last(\pi) - \{n+1\}|$ and $\text{min}(\pi) = |Min(\pi)|$, then

$$\begin{aligned} A_{m+1}(n,k) &= \sum_{j=k}^{n} (-1)^{j-k} \binom{j}{k} \sum_{\pi \in ODSP_{n,n-j}} (m+1)^{n-\text{last}(\pi)} \qquad (3.7) \\ &= \sum_{j=k}^{n} (-1)^{j-k} \binom{j}{k} \sum_{\pi \in ODSP_{n,n-j}} \sum_{s=0}^{n-\text{last}(\pi)} \binom{n-\text{last}(\pi)}{s} m^s. \end{aligned}$$

On the right-hand side of (3.7), we see that for fixed j, we sum over ordered Dowling set partitions of $\pi = B_1/B_2/\ldots/B_{n-j}/B_{n-j+1}$ so $j = n - |\min(\pi)| = |[n] - Min(\pi)|$. Thus we interpret the binomial coefficient $\binom{j}{k}$ as the number of ways of picking k elements of $[n] - Min(\pi)$ and we can interpret the binomial coefficient $\binom{n-\text{last}(\pi)}{s}$ as the number of ways of pick s elements from $[n] - Last(\pi)$. Thus we let $MODSP_{n,n-j,s,k}$ denote the set of ordered Dowling set partitions $\pi = B_1/B_2/\ldots/B_{n-j}/B_{n-j+1}$ in $ODSP_{n,n-j}$ such that

1. we have underlined s of elements in $[n] - Last(\pi)$ and

2. we have starred k elements in $[n] - Min(\pi)$.

For example, $\pi = \underline{2}\ \underline{5}^*\ 6/7\ 8^*\ \underline{9}/1\ \underline{10}^*/3\ 4^*\ \mathbf{11}$ is an element of $MODSP_{10,3,4,4}$.

We let

$$\begin{aligned} MODSP_{n,n-j} &= \bigcup_{s\geq 0, k \geq 0} MODSP_{n,n-j,s,k} \text{ and} \\ MODSP_n &= \bigcup_{0 \leq j \leq n} MODSP_{n,n-j}. \end{aligned}$$

We shall call an element $\pi \in MODSP_n$ a marked ordered Dowling set partition of

n. Thus given some marked ordered Dowling set partition $\pi = B_1/\dots/B_r/B_{r+1}$ of n, we let

1. $\ell(\pi) = r$ denote the number of parts of π minus 1,
2. $Min(\pi) = \{\min(B_1),\dots,\min(B_r)\}$ equal the set of minimal elements in $B_1,\dots,B_r$ and $\min(\pi) = |Min(\pi)| = \ell(\pi)$,
3. $Last(\pi) = B_{r+1}$ equal the last part of π and $\mathrm{last}(\pi) = |Last(\pi) - \{n+1\}|$ equal the number of elements of $[n]$ in $Last(\pi)$,
4. $ST(\pi)$ denote the set of elements of $[n]$ which are starred in π and $\mathrm{st}(\pi) = |ST(\pi)|$,
5. $UN(\pi)$ denote the set of elements of $[n]$ which are underlined in π and $\mathrm{un}(\pi) = |UN(\pi)|$, and
6. $FREE(\pi)$ denote the set of elements in $[n] - Min(\pi)$ which are not starred and $\mathrm{free}(\pi) = |FREE(\pi)|$.

For example, if $\pi = \underline{2}\ \underline{5}^*\ 6/7\ 8^*\ \underline{9}/1\ \underline{10}^*/3\ 4^*\ \mathbf{11}$, then $\ell(\pi) = 3$, $Min(\pi) = \{1,2,7\}$, $\min(\pi) = 3$, $Last(\pi) = \{3,4,11\}$, $\mathrm{last}(\pi) = 2$, $STAR(\pi) = \{4,5,8,10\}$, $\mathrm{st}(\pi) = 4$, $UN(\pi) = \{2,5,9,10\}$, $\mathrm{un}(\pi) = 4$, $FREE(\pi) = \{3,6,9\}$ and $\mathrm{free}(\pi) = 3$.

It follows that we have the following combinatorial interpretation of $[m^s]A_{m+1}(n,k)$.

Lemma 3.1.

$$\begin{aligned}[m^s]A_{m+1}(n,k) &= \sum_{j=k}^{n} \sum_{\substack{\pi\in MODSP_{n,n-j}\\ \mathrm{st}(\pi)=k\\ \mathrm{un}(\pi)=s}} (-1)^{j-k} \\ &= \sum_{j=k}^{n} \sum_{\pi\in MODSP_{n,n-j,s,k}} (-1)^{\mathrm{free}(\pi)}. \end{aligned} \tag{3.8}$$

Our next goal is to define a pair of involutions $I = I(n,k,s)$ and $J = J(n,k,s)$ which will cancel all the elements with negative signs on the right-hand side of (3.8). Let $\Gamma(n,k,s)$ denote the set of all $\pi \in MODSP_n$ such that $\mathrm{st}(\pi) = k$ and $\mathrm{un}(\pi) = s$.

First Involution $I : \Gamma(n,k,s) \to \Gamma(n,k,s)$**.**

We define an involution $I : \Gamma(n,k,s) \to \Gamma(n,k,s)$ as follows. Given $\pi = A_1/A_2/\dots/A_\ell/A_{\ell+1}$ in $\Gamma(n,k,s)$, we say that an element $b \in [n] - Last(\pi)$ is *first movable* if either

1. b is free in π, i.e. $b \in FREE(\pi)$, or
2. b is the only element in A_i where $i < \ell$ and $b > \min(A_{i+1})$.

For example, if $\pi = \underline{2}/1\ 5^*/3\ 4\ \underline{7}^*/6^*\ 8\ \mathbf{9}$, then $\underline{2}$ and 4 are first movable. Note that 8 is free, but it is not first movable because it is in the last part.

Then for any given $\pi = A_1/A_2/\dots/A_\ell/A_{\ell+1} \in \Gamma(n,k,s)$, we let $I(\pi) = \pi$ if π has no first movable elements. Otherwise let b be the largest first movable element of π. Then we have two cases.

Case 1. $b \in A_i$ and b is free.

Then we let

$$I(\pi) = A_1/\dots/A_{i-1}/\{b\}/A_i - \{b\}/A_{i+1}/\dots/A_\ell/A_{\ell+1}.$$

Here if b is underlined in π, then it remains underlined in $I(\pi)$.

Case 2. $A_i = \{b\}$ for some $i < \ell$. Then we set

$$I(\pi) = A_1/\dots/A_{i-1}/A_{i+1} \cup \{b\}/\dots/A_\ell/A_{\ell+1}.$$

Here if b is underlined in π, then it remains underlined in $I(\pi)$.

For example, if $\pi = 3/\underline{4}/1\ \underline{5}\ \underline{7}^*/2^*\ 6\ \mathbf{8}$, then 5 is the largest first movable element and we are in Case 1 so

$$I(3/\underline{4}/1\ \underline{5}\ \underline{7}^*/2^*\ 6\ \mathbf{8}) = 3/\underline{4}/\underline{5}/1\ \underline{7}^*/2^*\ 6\ \mathbf{8}.$$

But then 5 is the largest first movable element of $I(\pi)$ so $I(I(\pi)) = \pi$.

It is easy to see that if π is in Case 1, then b is a first movable element in $I(\pi)$. We claim that b is in fact the largest first movable element of $I(\pi)$. Since b was free in π, it must be larger than all the other free elements in π. Moving b to a singleton in $I(\pi)$ makes b not free in $I(\pi)$. However it is easy to see that for all $x \in [n] - \{b\}$, x is free in π if and only if x is free in $I(\pi)$. The only possible new first movable element in $I(\pi)$ would be if $A_{i-1} = \{c\}$ and $c > b$. But then $c > b > \min(A_i)$. So c would also be first movable in π which would violate our choice of b. Hence b is the largest first movable element in $I(\pi)$ so $I(\pi)$ is in Case 2 and $I(I(\pi)) = \pi$.

It is easy to see that if π is in Case 2, then b is a first movable element in $I(\pi)$ because it is free in $I(\pi)$. We claim that b is in fact the largest first movable element of $I(\pi)$. b must be larger than all the free elements in π. Moving b into in A_{i+1} makes b free in $I(\pi)$ but all other free elements in π remain free elements in $I(\pi)$. The only possible new first movable element in $I(\pi)$ would be if $A_{i-1} = \{c\}$ and $c > \min(A_{i+1} \cup \{b\})$. But it is impossible that $c > b$ since then c would be first movable in π. Hence $c < b$ so b is still the largest first movable element in $I(\pi)$. Thus $I(\pi)$ is in Case 1 so $I(I(\pi)) = \pi$.

Hence, I is an involution. Clearly if $I(\pi) \neq \pi$, then the number of free elements changes by ± 1 in going from π to $I(\pi)$ so $(-1)^{\text{free}(\pi)} + (-1)^{\text{free}(I(\pi))} = 0$. Let $Fix(I)$ denote the set of fixed points in $\Gamma(n,k,s)$. Thus I shows the following lemma.

Lemma 3.2. *For any positive integer n and integers k and s between 0 and n,*

$$[m^s]A_{m+1}(n,k) = \sum_{\pi \in Fix(I)} (-1)^{\text{free}(\pi)}.$$

Thus we must examine the fixed points of I. It is easy to see that $\pi = A_1/A_2/\ldots/A_\ell/A_{\ell+1}$ is an element of $Fix(I)$ if and only if

1. every element in $[n] - (Min(\pi) \cup Last(\pi))$ is starred and
2. if $A_i = \{b\}$ or $A_i = \{\underline{b}\}$ where $i < \ell$, then $b < \min(A_{i+1})$.

Second Involution $J : Fix(I) \to Fix(I)$

Given $\pi = A_1/A_2/\ldots/A_\ell/A_{\ell+1} \in Fix(I)$, we say that $b \in [n]$ is *second movable* if either

1. b is free which means that b must be in $Last(\pi)$ or
2. b is not underlined, there is an $i \leq l$ such that $A_i = \{b\}$ and for all $i < j \leq \ell$, $A_i = \{\underline{c}^{(j)}\}$ for some $c^{(j)}$ and, hence, $b < c^{(i+1)} < \ldots < c^{(\ell)}$. Thus π is of the form

$$A_1/\ldots/A_{i-1}/\{b\}/\{\underline{c}^{(i+1)}\}/\ldots/\{\underline{c}^{(\ell)}\}/A_{\ell+1}.$$

Then given $\pi = A_1/A_2/\ldots/A_\ell/A_{\ell+1} \in Fix(I)$, we define $J(\pi) = \pi$ if π has no second movable elements. Otherwise, let b be the largest second movable element in π. Then we have two cases.

Case 1. b is free and hence $b \in A_{\ell+1}$.

Then write π in the form

$$A_1/\ldots/A_{i-1}/\{\underline{c}^{(i)}\}/\ldots/\{\underline{c}^{(\ell)}\}/A_{\ell+1}$$

where $b < c^{(i)} < c^{(i+1)} < \cdots < c^{(\ell)}$ and either (i) $|A_{i-1}| \geq 2$, (ii) $A_{i-1} = \{c^{(i-1)}\}$ or $A_{i-1} = \{\underline{c}^{(i-1)}\}$ and $c^{(i-1)} < b$. Here we allow $i-1=\ell$ so the sequence $c^{(i)}, c^{(i+1)}, \ldots, c^{(\ell)}$ might be empty. Then we let

$$J(\pi) = A_1/\ldots/A_{i-1}/\{b\}/\{\underline{c}^{(i)}\}/\ldots/\{\underline{c}^{(\ell)}\}/A_{\ell+1} - \{b\}.$$

Case 2. π is of the form $A_1/\ldots/A_{i-1}/\{b\}/\{\underline{c}^{(i+1)}\}/\ldots/\{\underline{c}^{(\ell)}\}/A_{\ell+1}$.
In such a case our conditions on the fixed points of I ensure that
$b < c^{(i+1)} < \cdots < c^{(\ell)}$. Then

$$J(\pi) = A_1/\ldots/A_{i-1}/\{\underline{c}^{(i+1)}\}/\ldots/\{\underline{c}^{(\ell)}\}/A_{\ell+1} \cup \{b\}.$$

For example if $\pi = 1/\underline{3}/6\ 7^*/\underline{2}/\underline{8}/4\ 5\ 9^*\ \mathbf{10}$, then the largest second movable element is 5 and we are in Case 1 so

$$J(\pi) = 1/\underline{3}/6\ 7^*/\underline{2}/5/\underline{8}/4\ 9^*\ \mathbf{10}.$$

Then 5 is still the largest second movable element in $J(\pi)$ so $J(\pi)$ is as in Case 2 and $J(J(\pi)) = \pi$.

Suppose that π is as described in Case 1. Then we know that b is second movable in $J(\pi)$. We also know that b is bigger than all the free elements in $A_{\ell+1}$. Now consider the effect on the second movable elements π by moving b to become a singleton in $J(\pi)$.

If $|A_{i-1}| \geq 2$, then none of the elements in $A_1, \ldots, A_{i-1}$ are second movable in π or $J(\pi)$. Thus in this case, the set of second movable elements in π and $J(\pi)$ are the same so b is also the largest second movable element in $J(\pi)$. If $A_{i-1} = \{c^{(i-1)}\}$ where $c^{(i-1)} < b$, then none of the elements in $A_1, \ldots, A_{i-2}$ are second movable in π or $J(\pi)$ and $c^{(i-1)}$ is second movable in π but not second movable in $J(\pi)$. However, in this case, it still is the case that b is the largest second movable element in $J(\pi)$. If $A_{i-1} = \{\underline{c}^{(i-1)}\}$ where $c^{(i-1)} < b$, then it could be that there is a second movable element to the left of A_{i-1}. That is, there could be some $s < i-1$ such that π is of the form

$$A_1/\ldots/A_{j-1}/\{c^{(s)}\}/\{\underline{c}^{(s+1)}\}/\ldots/\{\underline{c}^{(\ell)}\}/A_{\ell+1},$$

but in this case $c^{(s)} < b$. Thus $c^{(s)}$ is second movable in π but not second movable in $J(\pi)$. However, in this case, b would still be the largest second movable element in $J(\pi)$. Thus in any case, b is the largest second movable element in $J(\pi)$ so $J(\pi)$ is in Case 2 and $J(J(\pi)) = \pi$.

Next suppose that π is as described in Case 2. Then we know that b is second movable in $J(\pi)$. We also know that b is bigger than all the free elements in $A_{\ell+1}$. Now consider the effect on the second movable elements π by moving b into $A_{\ell+1}$. If $|A_{i-1}| \geq 2$, then none of the elements in $A_1, \ldots, A_{i-1}$ are second movable in π or $J(\pi)$. Thus in this case, the set of second movable elements in π and $J(\pi)$ are the same so b is also the largest second movable element in $J(\pi)$. If $A_{i-1} = \{c^{(i-1)}\}$, then our conditions on the fixed points of I ensure that $c^{(i-1)} < b$. Then none of the elements in $A_1, \ldots, A_{i-2}$ are second movable in π or $J(\pi)$ and $c^{(i-1)}$ is second movable in $J(\pi)$ but not second movable in π. However, in this case, it is still the case that b is the largest second movable element in $J(\pi)$. If $A_{i-1} = \{\underline{c}^{(i-1)}\}$ where $c^{(i-1)} < c^{(i+1)}$, then none of the elements in $A_1, \ldots A_{i-1}$ are second movable in either π or $J(\pi)$. Thus in this case, b is still the largest second movable element in $J(\pi)$. If $A_{i-1} = \{\underline{c}^{(i-1)}\}$ where $c^{(i-1)} < c^{(i+1)}$, then by our arguments in Case 1, it could be that there is an element d to the left of A_{i-1} which is not second movable in π but becomes second movable in $J(\pi)$. But in such a case, it would follow that $d < c^{(i-1)} < b$ so again b is the largest second movable element in $J(\pi)$. Thus in any case, b is the largest second movable element in $J(\pi)$ so $J(\pi)$ is in Case 1 and $J(J(\pi)) = \pi$.

Thus J is an involution. Clearly if $J(\pi) \neq \pi$, then the number of free elements changes by ± 1 in going from π to $J(\pi)$ so $(-1)^{\text{free}(\pi)} + (-1)^{\text{free}(J(\pi))} = 0$. Let $Fix(J)$ denote the set of fixed points in $\Gamma(n, k, s)$. Then J proves the following lemma.

Lemma 3.3. *For any positive integer n and integers k and s between 0 and n,*

$$[m^s]A_{m+1}(n,k) = \sum_{\pi \in Fix(J)} (-1)^{\text{free}(\pi)}.$$

Thus we must examine the fixed points of J. Now suppose that $\pi = A_1/A_2/\dots/A_\ell/A_{\ell+1} \in Fix(J)$. Every element in $\bigcup_{i=1}^{\ell} A_i - Min(\pi)$ is starred since π is a fixed point of I and every element in $Last(\pi) - \{n+1\}$ is starred since it is a fixed point of J. Thus π has no free elements and, hence, $(-1)^{\text{free}(\pi)} = 1$. This shows that $[m^s]A_{m+1}(n,k) = |Fix(J)|$ so we have shown that $[m^s]A_{m+1}(n,k)$ is always a non-negative integer.

It follows that an ordered Dowling set partition $\pi = A_1/\dots/A_\ell/A_{\ell+1}$ is in $Fix(J)$ if and only if the following conditions are satisfied.

1. π has k starred elements and no free elements. This implies that every element of $[n] - Min(\pi)$ is starred so $\ell(\pi) = \min(\pi) = n - k$.

2. if $A_i = \{b\}$ or $A_i = \{\underline{b}\}$ is a singleton and $i < \ell$, then $b < \min(A_{i+1})$.

3. s elements of $[n] - Last(\pi)$ are underlined.

4. π cannot end in a sequence of the form

$$A_i/A_{i+1}/A_{i+2}/\dots/A_\ell/A_{\ell+1}$$

where

$$A_i = \{c^{(i)}\} \text{ and } A_j = \{\underline{c}^{(j)}\} \text{ for } j = i+1, i+2, \dots, \ell$$

where $c^{(i)} < c^{(i+1)} < \dots < c^{(\ell)}$. Thus if π ends in a series of singletons, then all the elements in these singletons must be underlined.

Hence we have proved the following theorem.

Theorem 3.2. *For all* $0 \le k \le n$, $[m^s]A_{m+1}(n,k)$ *is equal to the number of marked ordered Dowling set partitions* $\pi = A_1/\dots/A_{n-k}/A_{n-k+1}$ *such that*

1. π *has* k *stars so all elements of* $[n] - Min(\pi)$ *are starred,*
2. s *elements of* $[n] - Last(\pi)$ *are underlined,*
3. *if* $A_i = \{b\}$ *or* $A_i = \{\underline{b}\}$ *is a singleton and* $i < \ell$, *then* $b < \min(A_{i+1})$, *and*
4. *if* π *ends in a series of singletons, then all the elements in these singletons must be underlined.*

For example, to compute $A_{m+1}(3,2)$, we list all the marked ordered Dowling set partitions π which satisfy all criteria above:

$$\begin{array}{llll}
\underline{1}/2^*\ 3^*\ \mathbf{4} & \underline{2}/1^*\ 3^*\ \mathbf{4} & \underline{3}/1^*\ 2^*\ \mathbf{4} & \\
1\ 2^*/3^*\ \mathbf{4} & \underline{1}\ 2^*/3^*\mathbf{4} & 1\ \underline{2}^*/3^*\ \mathbf{4} & \underline{1}\ \underline{2}^*/3^*\ \mathbf{4} \\
1\ 3^*/2^*\ \mathbf{4} & \underline{1}\ 3^*/2^*\ \mathbf{4} & 1\ \underline{3}^*/2^*\ \mathbf{4} & \underline{1}\ \underline{3}^*/2^*\ \mathbf{4} \\
2\ 3^*/1^*\ \mathbf{4} & \underline{2}\ 3^*/1^*\ \mathbf{4} & 2\ \underline{3}^*/1^*\ \mathbf{4} & \underline{2}\ \underline{3}^*/1^*\mathbf{4} \\
1\ 2^*\ 3^*/\mathbf{4} & 1\ 2^*\ \underline{3}^*/\mathbf{4} & 1\ \underline{2}^*\ 3^*/\mathbf{4} & 1\ \underline{2}^*\ \underline{3}^*/\mathbf{4} \\
\underline{1}\ 2^*\ 3^*/\mathbf{4} & \underline{1}\ 2^*\ \underline{3}^*/\mathbf{4} & \underline{1}\ \underline{2}^*\ 3^*/\mathbf{4} & \underline{1}\ \underline{2}^*\ \underline{3}^*/\mathbf{4}
\end{array} .$$

Then the coefficient of m^s in $A_{m+1}(3,2)$ is the number of π above with s underlines so $A_{m+1}(3,2) = m^3 + 6m^2 + 12m + 4$.

There are several coefficients $[m^s]A_{m+1}(n,k)$ that are easy to compute from our combinatorial interpretation of $A_{m+1}(n,k)$.

Theorem 3.3.

1. *$A_{m+1}(n,n) = 1$.*
2. *$A_{m+1}(n,0) = m^n$.*
3. *$[m^0]A_{m+1}(n,n-1) = 2^n - n - 1$.*
4. *$[m^s]A_{m+1}(n,n-1) = \binom{n}{s}2^{n-s}$ for $s \geq 1$.*
5. *$[m^n]A_{m+1}(n,1) = 2^n - n - 1$.*
6. *$[m^s]A_{m+1}(n,1) = \binom{n}{n-s}2^s$ for $s \leq n-1$.*

Proof. For $A_{m+1}(n,n)$, we must consider the fixed points π of J which have 0 parts and n starred elements. There is only one such marked ordered Dowling set partition, namely, $\pi = /1^* \ 2^* \ \ldots \ n^* \ \mathbf{n+1}$. That is, all the elements of $[n]$ are in the last part and all of them are starred. There can be no underlined elements since we cannot underline elements in the last part. Since the weight of π is $m^{\mathrm{un}(\pi)}$, we see that $A_{m+1}(n,n) = 1$.

For $A_{m+1}(n,0)$, we must consider the fixed points π of J which have n parts and 0 starred elements. There is only one such marked ordered Dowling set partition, namely, $\pi = \underline{1}/\underline{2}/\ldots/\underline{n}/\mathbf{n+1}$. Since the weight of π is $m^{\mathrm{un}(\pi)}$, we see that $A_{m+1}(n,n) = m^n$.

For $[m^0]A_{m+1}(n,n-1)$, we must consider the fixed points π of J with 1 part, $n-1$ starred elements, and no underlined elements. That is, we must count the number of marked ordered Dowling set partitions of n of the form $\pi = A_1/A_2$ where all the elements of $[n]$ except $\min(A_1)$ are starred. Of the possible choices of subsets that we can choose for

A_1, we cannot choose $A_1 = \emptyset$ since π must have one part and we cannot choose $A_1 = \{j\}$ for any $j \in [n]$ since to be a fixed point of J, the elements of any co-final sequence of singletons in π must be underlined. Thus there are $2^n - n - 1$ choices for A_1.

For $[m^s]A_{m+1}(n, n-1)$ where $s \geq 1$, we must consider the fixed points π of J with 1 part, $n-1$ starred elements, and s underlined elements. That is, we must count the number of marked ordered Dowling set partitions of n of the form $\pi = A_1/A_2$ where all the elements of $[n]$ except $\min(A_1)$ are starred and s of the elements of A_1 are underlined. First we can choose the set B of the s underlined in elements of π in $\binom{n}{s}$ ways. All the elements of B must be put into A_1 since no underlined elements can appear in the last part. Once we have chosen B, then we have 2 choices for each element x of $[n] - B$, namely, we can put x into A_1 or put x into A_2. Hence $[m^s]A_{m+1}(n, n-1) = \binom{n}{s}2^{n-s}$.

For $[m^n]A_{m+1}(n, 1)$, we must consider the fixed points π of J that have $n-1$ parts, one starred element, and n underlined elements. This means the last part of π is just $Last(\pi) = \{\mathbf{n+1}\}$ since no underlined elements can go in the last part. Hence π must have $n-2$ parts of size 1 and one part of size 2. Moreover, the non-minimal element in the part of size 2 must be starred. Thus π must be of the form

$$\pi = \underline{a_1}/\ldots/\underline{a_k}/\underline{a_{k+1}}\ \underline{b}^*/\underline{b_1}/\ldots/\underline{b_{n-k-2}}/\mathbf{n+1}$$

where $a_1 < a_2 < \cdots < a_{k+1} < b$ and $b_1 < \cdots < b_{n-k-2}$. It follows that we can specify π by picking any subset $S = \{a_1 < a_2 < \cdots < a_{k+1} < b\}$ of size greater than or equal to 2. Since there are $2^n - n - 1$ such sets, it follows that $[m^n]A_{m+1}(n, 1) = 2^n - n - 1$.

For $[m^s]A_{m+1}(n, 1)$ where $0 \leq s \leq n-1$, we must consider two cases.

Case 1. $s = n-1$.

In this case, we must consider the fixed points π of J that have $n-1$ parts, one starred element, and $n-1$ underlined elements. In this situation, there are two possibilities. That is, it could be that the starred element is in the last part of π so π is of the form

$$\pi = \underline{a_1}/\dots/\underline{a_{n-1}}/j^* \ \mathbf{n+1}$$

where $a_1 < \dots < a_{n-1}$. Clearly, there are n elements of this type. The other possibility is that

$$\pi = a_1/\dots/a_k/a_{k+1} \ b^*/\underline{b_1}\dots\underline{b_{n-k-2}}/\mathbf{n+1}$$

where $a_1 < a_2 < \dots < a_{k+1} < b$ and $b_1 < \dots < b_{n-k-2}$ and where exactly one of $a_1, \dots, a_{k+1}, b$ is not underlined. Since the set $S = \{a_1 < a_2 < \dots < a_{k+1} < b\}$ must be size greater than or equal to 2, there are

$$\sum_{s=2}^{n} s\binom{n}{s} = n\sum_{s=2}^{n}\binom{n-1}{s-1} = n(2^{n-1}-1) = n2^{n-1} - n$$

ways to pick such a Dowling set partitiont π in this case. Thus $[m^{n-1}]A_{m+1}(n,1) = n2^{n-1} - n + n = n2^{n-1}$.

Case 2. $1 \le s \le n-2$.

In this case, we must consider the fixed points π of the J that have $n-1$ parts, one starred element, and s underlined elements. In this situation, we cannot have the starred element in the last part of π since otherwise π is forced to have the form

$$\pi = \underline{a_1}/\dots/\underline{a_{n-1}}/j^* \ \mathbf{n+1}$$

where $a_1 < \dots < a_{n-1}$ which would mean that π has $n-1$ underlined elements. Thus π

must be of the form

$$\pi = a_1/\dots/a_k/a_{k+1}\ b^*/\underline{b_1}/\underline{b_2}/\dots/\underline{b_{n-k-2}}/\mathbf{n+1}$$

where $a_1 < a_2 < \cdots < a_{k+1} < b$ and $b_1 < \cdots < b_{n-k-2}$. Again we can specify π by the set $S = \{a_1 < a_2 < \cdots < a_{k+1} < b\}$. However, in this case the $n-s$ elements of $[n]$ which are not underlined in π must all be in S. Thus we have $\binom{n}{n-s}$ ways of picking the set T of $n-s$ elements which are not underlined in π. Since $s \le n-2$, it follows that $n-s \ge 2$ so T must have at least 2 elements. For the rest of the elements $x \in [n]-T$, we can chose to put x in S or not. Thus we have $\binom{n}{n-s}2^s$ such π. Hence, $[m^s]A_{m+1}(n,1) = \binom{n}{n-s}2^s$ for $0 \le s \le n-2$. □

Next we prove the remarkable symmetry that holds between the polynomials $A_{m+1}(n,k)$ and $A_{m+1}(n,n-k)$.

Theorem 3.4. *If n is a positive integer, and k and s are both integers between 0 and n, then*

$$[m^s]A_{m+1}(n,k) = [m^{n-s}]A_{m+1}(n,n-k).$$

Proof. By Theorem 3.2, we must show that

$$|Fix(J(n,k,s))| = |Fix(J(n,n-k,n-s))|.$$

We shall define a bijection $F : Fix(J(n,k,s)) \to Fix(J(n,n-k,n-s))$.

Given any $\pi \in Fix(J(n,k,s))$, we define the standard factorization of π to be the sequence

$$\pi = P_1P_2\dots P_lAB$$

where each P_i is of the form $P_i = s_1/s_2/\dots/s_{a_i}/Q_i$ where $|Q_i| \ge 2$, $A = \underline{x}_1/\underline{x}_2/\dots/\underline{x}_t$, and B is the last part. That is, reading from left to right, we partition the parts of π into groups of singletons followed by a part which has cardinality greater than or equal to 2, A is a (possibly empty) sequence of singletons, and $B = Last(\pi)$. For example, if

$$\begin{aligned}\pi^{(1)} &= \underline{2}/4/\underline{5}/6\ \underline{9}^*/1/\underline{7}\ 10^*\ 11^*/\underline{3}/8^*\ 12^*\ \mathbf{13} \text{ and}\\ \pi^{(2)} &= 2/7^*/\underline{1}/\underline{4}/\underline{5}\ 6^*/3^*\ 8^*\ 9^*\ \mathbf{10},\end{aligned}$$

then we would write $\pi^{(1)}$ and $\pi^{(2)}$ as follows:

$$\pi^{(1)} = \overbrace{\underline{2}/4/\underline{5}/6\ \underline{9}^*}^{P_1}/\overbrace{1/\underline{7}\ 10^*\ 11^*}^{P_2}/\overbrace{\ \underline{3}\ }^{A}/\overbrace{8^*\ 12^*\ \mathbf{13}}^{B}$$

$$\pi^{(2)} = \overbrace{2\ 7^*}^{P_1}/\overbrace{\underline{1}/\underline{4}/\underline{5}\ 6^*}^{P_2}/\overbrace{3^*\ 8^*\ 9^*\ \mathbf{10}}^{B}.$$

Given $\pi = P_1P_2\dots P_lAB \in Fix(J(n,k,s))$, we let

$$F(\pi) = T(P_1)\dots T(P_l)T(AB)$$

where T is the transformation described according to the following two cases.

Case 1. Suppose $i \leq l$ and $P_i = s_1/s_2/\dots/s_{a_i}/Q_i$ where $Q_i = \{s_{a_i+1}, s^*_{a_i+2}, \dots, s^*_{a_i+b_i}\}$ and $b_i \geq 2$. Our conditions to be a fixed point of $J(n,k,s)$ ensure that $s_1 < \dots < s_{a_i} < s_{a_i+1} < \dots < s_{a_i+b_i}$, each of the elements s_j for $j > a_i + 1$ must be starred, and any of the elements s_i may or may not be underlined. In this case,

$$T(P_i) = s_1/s_2/\dots/s_{b_i-2}/\overline{Q}_i$$

where $\overline{Q}_i = \{s_{b_i-1}, s^*_{b_i}, \dots, s^*_{a_i+b_i}\}$, each of the elements s_r for $r \geq b_i$ are starred, and for all

$1 \leq j \leq a_i + b_i$, s_j is underlined in $T(P_i)$ if and only if s_j is not underlined in P_i.

In P_i, there are $a_i + 1$ minimal elements and $b_i - 1$ non-minimal elements while in $T(P_i)$ there are $b_i - 1$ minimal elements and $a_i + 1$ non-minimal elements. It easily follows that $T(T(P_i)) = P_i$.

Case 2 Suppose $A = \underline{u}_1/\ldots/\underline{u_s}$ and $B = \{v_1^*, \ldots, v_t^*, \mathbf{n+1}\}$. (It may be the case that either s or t equals 0.) Our conditions for π to be a fixed point of $J(n, k, s)$ ensure that all the v_is must be starred and all the u_js must be underlined. Moreover, if $s > 1$, then $u_1 < \cdots < u_s$ and if $t > 1$, then $v_1 < \cdots < v_t$. Then we let

$$T(AB) = \underline{v_1}/\ldots/\underline{v_t}/u_1^* \ \ldots \ u_s^* \ \mathbf{n+1}.$$

In AB there are s minimal elements and t non-minimal elements while in $T(AB)$ there are t minimal elements and s non-mimimal elements. Thus $T(T(AB)) = AB$. Note that for all $x \in [n]$ that lie in AB, x is underlined in AB if and only if x is not underlined in $T(AB)$.

For example, suppose that $\pi = \underline{2}/4/\underline{5}/6\ \underline{9}^*/1/\underline{7}\ 10^*\ 11^*/\underline{3}/8^*\ 12^*\ \mathbf{13}$. Then

1. $P_1 = \underline{2}/4/\underline{5}/6\ \underline{9}^*$ so $T(P_1) = \overline{Q}_1 = \{2, \underline{4}^*, 5^*, \underline{6}^*, 9^*\}$,

2. $P_2 = 1/\underline{7}\ 10^*\ 11^*$ so $T(P_2) = \underline{1}/7\ \underline{10}^*\ \underline{11}^*$, and

3. $A = \underline{3}$ and $B = \{8^*, 12^*, \mathbf{13}\}$ so $T(A) = \underline{8}/\underline{12}$ and $T(B) = \{3^*, \mathbf{13}\}$.

Hence $F(\pi) = 2\ \underline{4}^*\ 5^*\ \underline{6}^*\ 9^*/\underline{1}/7\ \underline{10}^*\ \underline{11}^*/\underline{8}/\underline{12}/3^*\mathbf{13}$.

It is easy to check the following facts.

1. If $\pi \in Fix(J(n, k, s))$, $F(\pi)$ meets the conditions 1-4 of Theorem 3.2.

2. If the standard factorization of π is $P_1 \dots P_l AB$, then the standard factorization of $F(\pi)$ is $T(P_1)\dots T(P_l)T(AB)$. Hence it follows that $F(F(\pi)) = \pi$.

3. An element x in $[n]$ is underlined in π if and only if x is not underlined in $F(\pi)$. Thus if π has s underlined elements, then $F(\pi)$ has $n-s$ underlined elements.

4. Our observations about the number of minimal elements in each P_i as compared to the number of minimal elements in $T(P_i)$ and the number of minimal elements in AB as compared to the number of minimal elements in $T(AB)$ ensure that if $\min(\pi) = k$, then $\min(F(\pi)) = n-k$.

It follows that for all $\pi \in Fix(J(n,k,s))$, $F(\pi) \in Fix(J(n,n-k,n-s))$. Thus $|Fix(J(n,k,s))| = |Fix(J(n,n-k,n-s))|$ which is what we wanted to prove. □

3.4 The polynomials $B_{m+1}(n,k)$

In this section, we shall study the polynomials $B_m(n,k)$ defined by

$$F_{m,n,2}(x) = \sum_{k=0}^{n} k! W_m(n,k) x^n = \sum_{k=0}^{n} B_m(n,k) x^{n-k}(x+1)^n. \tag{3.9}$$

Let $\overline{W}$ and B be the column vectors

$$\overline{W} = \begin{bmatrix} n!W_m(n,n) \\ (n-1)!W_m(n,n-1) \\ \vdots \\ W_m(n,1) \\ W_m(n,0) \end{bmatrix} \quad \text{and } B = \begin{bmatrix} B_m(n,0) \\ B_m(n,1) \\ \vdots \\ B_m(n,n) \end{bmatrix}.$$

Recall that X and $\overline{X}$ are the row vectors $X = [x^{n-k}]_{k=0,\dots,n}$ and $\overline{X} = [x^{n-k}(x+1)^k]_{k=0,\dots,n}$. Then we can rewrite (3.9) as $X\overline{W} = \overline{X}B = XMB$ where M is the matrix defined in Section 3.3 such that $\overline{X} = MX$. It follows that $B = M^{-1}\overline{W}$ and, hence,

$$B_{m+1}(n,k) = \sum_{j=k}^{n} (-1)^{j-k} \binom{j}{k} (n-j)! W_{m+1}(n, n-j). \tag{3.10}$$

Next we want to give a combinatorial interpretation of the right-hand side of (3.10). In this case, we can interpret the term $(n-j)!W_{m+1}(n, n-j)$ as starting with a Dowling set partition of n with $n-j$ parts

$$\pi = A_1/A_2/\ldots/A_{n-j}/A_{n-j+1}$$

and allowing us to permute the parts $A_1/A_2/\ldots/A_{n-j}$ in all possible ways. We let $\overline{ODSP}_{n,n-j}$ denote the set of all ordered set partitions

$$\pi = B_1/B_2/\ldots/B_{n-j}/B_{n-j+1}$$

of $[n+1]$ into $n-j+1$ parts such that $n+1 \in B_{n-j+1} = Last(\pi)$ and all the elements in $[n] - (Last(\pi) \cup Min(\pi))$ are colored with one of $m+1$ colors. It follows that if we let $\text{last}(\pi) = |Last(\pi) - \{n+1\}|$ and $\min(\pi) = |Min(\pi)|$, then

$$\begin{aligned} &B_{m+1}(n,k) = \\ &\sum_{j=k}^{n} (-1)^{j-k} \binom{j}{k} \sum_{\pi \in \overline{ODSP}_{n,n-j}} (m+1)^{n-\text{last}(\pi)-\min(\pi)} = \\ &\sum_{j=k}^{n} (-1)^{j-k} \binom{j}{k} \sum_{\pi \in \overline{ODSP}_{n,n-j}} \sum_{s=0}^{n-\text{last}(\pi)-\min(\pi)} \binom{n-\text{last}(\pi)-\min(\pi)}{s} m^s. \end{aligned} \tag{3.11}$$

Then we can interpret the binomial coefficient $\binom{j}{k}$ as the number of ways of picking k elements of $[n] - Min(\pi)$ and we can interpret the binomial coefficient $\binom{n-\text{last}(\pi)-\min(\pi)}{s}$ as the number of ways of picking s elements from $[n] - (Last(\pi) \cup Min(\pi))$. Thus we let $\overline{MODSP}_{n,n-j,s,k}$ denote the set of ordered Dowling set partitions $\pi = B_1/B_2/\ldots/B_{n-j}/B_{n-j+1}$ in $\overline{ODSP}_{n,n-j}$ such that

1. we have underlined s of elements in $[n] - (Last(\pi) \cup Min(\pi))$ and
2. we have starred k elements in $[n] - Min(\pi)$.

For example, $\pi = 2\ \underline{5}^*\ 6/7\ 8^*\ \underline{9}/1\ \underline{10}^*/3\ 4^*\mathbf{11}$ is in $\overline{MODPS}_{10,3,3,4}$. We let

$$\begin{aligned}\overline{MODSP}_{n,n-j} &= \bigcup_{s\geq 0,k\geq 0} \overline{MODSP}_{n,n-j,s,k} \text{ and}\\ \overline{MODSP}_{n} &= \bigcup_{0\leq j\leq n} \overline{MODSP}_{n,n-j}.\end{aligned}$$

We shall call an element $\pi \in \overline{MODSP}_n$, a non-min marked ordered Dowling set partition of n. Given some non-min marked ordered Dowling set partition $\pi = B_1/\ldots/B_r/B_{r+1}$ of n, we let

1. $\ell(\pi) = r$ denote the number of parts of π minus 1,
2. $Min(\pi) = \{\min(B_1), \ldots, \min(B_r)\}$ equal the set of minimal elements in $B_1, \ldots, B_r$ and $\min(\pi) = |Min(\pi)| = \ell(\pi)$,
3. $Last(\pi) = B_{r+1}$ equal the last part of π and $\mathrm{last}(\pi) = |Last(\pi) - \{n+1\}|$ equal the number of elements of $[n]$ in $Last(\pi)$,
4. $ST(\pi)$ denote the set of elements of $[n]$ which are starred in π and $\mathrm{st}(\pi) = |ST(\pi)|$,
5. $UN(\pi)$ denote the set of elements of $[n]$ which are underlined in π and $\mathrm{un}(\pi) = |UN(\pi)|$,
6. $FREE(\pi)$ denote the set of elements in $[n] - Min(\pi)$ which are not starred and $\mathrm{free}(\pi) = |FREE(\pi)|$, and

7. $TFREE(\pi)$ denotes the set of elements in $[n] - Min(\pi)$ which are not starred and not underlined and $\text{tfree}(\pi) = |TFREE(\pi)|$. We shall call the elements in $TFREE$ the totally free elements of π.

For example, if $\pi = 2\ \underline{5}^*\ 6/7\ 8^*\ \underline{9}/1\ \underline{10}^*/3\ 4^*\ \mathbf{11}$, then $\ell(\pi) = 3$, $Min(\pi) = \{1,2,7\}$, $\min(\pi) = 3$, $Last(\pi) = \{3,4,11\}$, $\text{last}(\pi) = 2$, $STAR(\pi) = \{4,5,8,10\}$, $\text{st}(\pi) = 4$, $UN(\pi) = \{5,9,10\}$, $\text{un}(\pi) = 3$, $FREE(\pi) = \{3,6,9\}$, $\text{free}(\pi) = 3$, $TFREE(\pi) = \{3,6\}$ and $\text{tfree}(\pi) = 2$.

It follows that we have the following combinatorial interpretation of $[m^s]B_{m+1}(n,k)$.

Lemma 3.4. *For positive integer n and integers k and s between 0 and n,*

$$\begin{aligned}
[m^s]B_{m+1}(n,k) &= \sum_{j=k}^{n} \sum_{\substack{\pi \in \overline{MODSP}_{n,n-j} \\ \text{st}(\pi)=k \\ \text{un}(\pi)=s}} (-1)^{j-k} \\
&= \sum_{j=k}^{n} \sum_{\pi \in \overline{MODSP}_{n,n-j,s,k}} (-1)^{\text{free}(\pi)}. \qquad (3.12)
\end{aligned}$$

This is almost the same interpretation that we had for $[m^s]A_{m+1}(n,k)$. The difference here is that for $[m^s]A_{m+1}(n,k)$, we allow elements of $Min(\pi)$ to be underlined where for $[m^s]B_{m+1}(n,k)$, we do not allow elements of $Min(\pi)$ to be underlined. We shall see that this difference forces a significant difference in our final combinatorial interpretation for $[m^s]B_{m+1}(n,k)$.

Our next goal is to define a pair of involutions $K = K(n,k,s)$ and $L = L(n,k,s)$ which will cancel many of the elements with negative signs on the right-hand side of (3.12). Let $\overline{\Gamma}(n,k,s)$ denote the set all $\pi \in \overline{MODSP}_n$ such that $\text{st}(\pi) = k$ and $\text{un}(\pi) = s$.

First Involution $K : \overline{\Gamma}(n,k,s) \to \overline{\Gamma}(n,k,s)$**.**

We define an involution $K : \overline{\Gamma}(n,k,s) \to \overline{\Gamma}(n,k,s)$ as follows. Given $\pi = A_1/A_2/\ldots/A_\ell/A_{\ell+1}$ in $\overline{\Gamma}(n,k,s)$, we say that an element $b \in [n] - Last(\pi)$ is *first movable* if either

1. b is totally free in π, i.e. $b \in TFREE(\pi)$, or

2. b is the only element in A_i where $i < \ell$ and $b > \min(A_{i+1})$.

For example, if $\pi = 2/1\ 5^*/3\ 4\ \underline{7}^*/6^*\ 8\ \mathbf{9}$, then 2 and 4 are first movable. Note that 8 is totally free, but it is not first movable because it is in the last part.

Then for any given $\pi = A_1/A_2/\ldots/A_\ell/A_{\ell+1} \in \overline{\Gamma}(n,k,s)$, we let $K(\pi) = \pi$ if π has no first movable elements. Otherwise let b be the largest first movable element of π. Then we have two cases.

Case 1. $b \in A_i$ and b is totally free.

Then we let

$$K(\pi) = A_1/\ldots/A_{i-1}/\{b\}/A_i - \{b\}/A_{i+1}/\ldots/A_\ell/A_{\ell+1}.$$

Case 2. $A_i = \{b\}$ for some $i < \ell$. Then we let

$$K(\pi) = A_1/\ldots/A_{i-1}/A_{i+1} \cup \{b\}/\ldots/A_\ell/A_{\ell+1}.$$

For example, if $\pi = 3/\underline{4}/1\ 5\ \underline{7}^*/2^*\ 6\ \mathbf{8}$, then 5 is the largest first movable element and we are in Case 1 so

$$K(3/\underline{4}/1\ \underline{5}\ \underline{7}^*/2^*\ 6\ \mathbf{8}) = 3/\underline{4}/5/1\ \underline{7}^*/2^*\ 6\ \mathbf{8}.$$

But then 5 is the largest first movable element of $K(\pi)$ so $K(K(\pi)) = \pi$.

By a similar argument given to the one given for I in Section 3.3, it is easy to see that K is an involution, and if π is not a fixed point of K, then $\text{free}(\pi) = \text{free}(K(\pi)) \pm 1$.

So $(-1)^{\text{free}(\pi)} + (-1)^{\text{free}(K(\pi))} = 0$. Let $Fix(K)$ denote the set of fixed points in $\overline{\Gamma}(n,k,s)$. Then K shows the following lemma.

Lemma 3.5. *For positive integer n and integers k and s between 0 and n,*

$$[m^s]B_{m+1}(n,k) = \sum_{\pi \in Fix(K)} (-1)^{\text{free}(\pi)}.$$

Thus we must examine the fixed points of K. It is easy to see that $\pi = A_1/A_2/\ldots/A_\ell/A_{\ell+1}$ is in $Fix(K)$ if and only if

1. Every element in $[n] - (Min(\pi) \cup Last(\pi))$ must be starred and/or underlined and
2. if $A_i = \{b\}$ where $i < \ell$, then $b < \min(A_{i+1})$.

Second Involution $L : Fix(K) \to Fix(K)$

Given $\pi = A_1/A_2/\ldots/A_\ell/A_{\ell+1} \in Fix(K)$, we say that $b \in [n]$ is *second movable* if either

1. b is totally free which means that b must be in $Last(\pi)$ or
2. $A_\ell = \{b\}$.

Then given $\pi = A_1/A_2/\ldots/A_\ell/A_{\ell+1} \in Fix(K)$, we define $L(\pi) = \pi$ if π has no second movable elements. Otherwise, let b be the largest second movable element in π. Then we have two cases.

Case 1. b is totally free and hence $b \in A_{\ell+1}$.
Then

$$L(\pi) = A_1/\ldots/A_\ell/\{b\}/A_{\ell+1} - \{b\}.$$

In particular, if $A_\ell = \{c\}$, then by assumption $c < b$ so $L(\pi)$ will be a fixed point of K.

Case 2. π is of the form $A_1/\ldots/A_{\ell-1}/\{b\}/A_{\ell+1}$.

Then

$$L(\pi) = A_1/\ldots/A_{\ell-1}/A_{\ell+1} \cup \{b\}.$$

For example if $\pi = 1/3/6\ 7^*/2/5/4\ 8\ 9^*\ \mathbf{10}$, then the largest second movable element is 8 and we are in Case 1 so

$$L(\pi) = 1/3/6\ 7^*/2/5/8/4\ 9^*\ \mathbf{10}.$$

Then 8 is still the largest second movable element in $L(\pi)$ so $L(\pi)$ is in Case 2 and $L(L(\pi)) = \pi$.

By a similar argument to the one given for J in Section 3.3, it is easy to see that L is an involution, and if π is not a fixed point of L, then $\text{free}(\pi) = \text{free}(L(\pi)) \pm 1$. So $(-1)^{\text{free}(\pi)} + (-1)^{\text{free}(L(\pi))} = 0$. Let $Fix(L)$ denote the set of fixed points in $\overline{\Gamma}(n,k,s)$. Then L shows that

Lemma 3.6. *For positive integer n and integers k and s between 0 and n,*

$$[m^s]B_{m+1}(n,k) = \sum_{\pi \in Fix(L)} (-1)^{\text{free}(\pi)}.$$

Thus we must examine the fixed points of L. Now suppose that $\pi = A_1/A_2/\ldots/A_\ell/A_{\ell+1} \in Fix(L)$. It is possible that there are no totally free elements in π. In that case, every element $x \in (\bigcup_{i=1}^{k} A_i) - Min(\pi)$ must be either starred, underlined, or both starred and underlined. Similarly, every element of $[n]$ in $A_{\ell+1}$ must be starred. It follows that a non-min marked ordered Dowling set partition $\pi = A_1/\ldots/A_\ell/A_{\ell+1}$ is in $Fix(L)$ if and only if the following conditions are satisfied.

1. π has k starred elements in $[n] - Min(\pi)$.

2. π has no totally free elements. Thus every element of $[n]$ in $Last(\pi)$ must be starred and every elements of $[n] - (Last(\pi) \cup Min(\pi))$ must be either starred, underlined, or both starred and underlined.

3. A_ℓ cannot be a singleton.

4. If $A_i = \{b\}$ is a singleton, then $i < \ell$ and $b < \min(A_{i+1})$.

5. s elements of $[n] - (Last(\pi) \cup Min(\pi))$ are underlined.

Thus we have proved the following theorem.

Theorem 3.5. *For all* $0 \leq k \leq n$,

$$[m^s]B_{m+1}(n,k) = \sum_{j=k}^{n} \sum_{\substack{\pi \in \overline{MODSP}_{n,n-j} \cap Fix(L) \\ \mathrm{st}(\pi)=k \\ \mathrm{un}(\pi)=s}} (-1)^{\mathrm{free}(\pi)} \tag{3.13}$$

where $\pi = A_1/\ldots/A_\ell/A_{\ell+1}$ *is in* $Fix(L)$ *if and only if*

1. *π has no totally free elements,*

2. *A_ℓ cannot be a singleton, and*

3. *If $A_i = \{b\}$ is a singleton, then $i < \ell$ and $b < \min(A_{i+1})$.*

To illustrate Theorem 3.5, let us go through the computation of the polynomials $B_{m+1}(3,k)$ for $0 \leq k \leq 3$.

$\mathbf{B_{m+1}(3,3)}$. In this case, we must consider the non-min marked ordered set Dowling set partitions of 3 in $Fix(L)$ which have 3 starred elements. Since we cannot put stars on minimal elements, this means there can be no minimal elements. Moreover, to be a fixed point of L all the elements in the last part must be starred. Hence the only fixed point of L with 3 stars is $\pi = \{1^*, 2^*, 3^*, \mathbf{4}\}$. Since π has no free elements and no underlined elements, it has weight 1 so $B_{m+1}(3,3) = 1$. This argument works in general for $B_{m+1}(n,n)$. That is, the only non-min marked ordered Dowling set partition of n which has n stars which is a fixed point of L is $\pi = \{1^*, 2^*, \ldots, n^*, \mathbf{n+1}\}$. Thus we have the following theorem.

Theorem 3.6. *For all $n \geq 1$, $B_{m+1}(n,n) = 1$.*

$\mathbf{B_{m+1}(3,2)}$. In this case, we must consider the non-min marked ordered Dowling set partitions of 3 with either 0 or 1 parts and 2 starred elements. From our argument above, the only non-min marked ordered Dowling set partition of 3 with 0 parts which is a fixed point of L has 3 stars so it cannot contribute to $B_{m+1}(3,2)$. This leaves us with four possibilities, namely, $1\ 2^*/3^*\ \mathbf{4}$, $1\ 3^*/2^*\ \mathbf{4}$, $2\ 3^*/1^*\ \mathbf{4}$, and $1\ 2^*\ 3^*/\mathbf{4}$. In each case, there are no free elements so the sign of such non-min marked ordered Dowling set partitions will be 1. In the first three cases, we can either underline or not underline the non-minimal elements in the first part which means that each element gives a contribution of $1 + m$ to $B_{m+1}(3,2)$. In the fourth case, we can underline or not underline the 2 and 3 so this element gives a contribution of $(1+m)^2$ to $B_{m+1}(3,2)$. It follows that $B_{m+1}(3,2) = 1+5m+m^2$.

$\mathbf{B_{m+1}(3,1)}$. In this case, we must consider the non-min marked ordered Dowling set partitions of 3 with either 0, 1, or 2 parts and 1 starred element which are fixed points of L. For the non-min marked ordered Dowling set partitions with 1 part, there are six possibilities:

$$1\ \underline{2}/3^*\ \mathbf{4},\ 1\ \underline{3}/2^*\ \mathbf{4},\ 2\ \underline{3}/1^*\ \mathbf{4},\ 1\ 2^*\ \underline{3}/\mathbf{4},\ 1\ \underline{2}\ 3^*/\mathbf{4},\ 1\ \underline{2}^*\ \underline{3}/\mathbf{4},\ \text{and } 1\ \underline{2}\ \underline{3}^*/\mathbf{4}.$$

The first four of these have weight $-m$ and the last two have weight $-m^2$.

Finally there are only two fixed points of L in our case with 2 parts, namely, $1/2\ 3^*/\mathbf{4}$ and $1/2\ \underline{3}^*/\mathbf{4}$, which have weights 1 and m, respectively. That is, if $\pi = A_1/A_2/A_3$ is a fixed point of L in this case, we must have $|A_1| = 1$, $|A_2| = 2$, and $\min(A_1) < \min(A_2)$ so the only possibility is that $A_1 = \{1\}$ and $A_2 = \{2, 3\}$. It follows that $B_{m+1}(3, 1) = 1-4m-2m^2$.

$\mathbf{B_{m+1}(3, 0)}$. In this case, we must consider the non-min marked ordered Dowling set partitions π of 3 with either 0, 1, 2, or 3 parts and no starred elements which are fixed points of L. Since there are no starred elements, $Last(\pi) = \{\mathbf{4}\}$. Since the part preceding $Last(\pi)$ must be of size 2, it is easy to see that there are two possibilities: $1\ \underline{2}\ \underline{3}/\mathbf{4}$ which has weight m^2 and $1/2\ \underline{3}/\mathbf{4}$ which has weight $-m$. Thus $B_{m+1}(3, 0) = -m + m^2$.

Thus in general, the polynomial $B_{m+1}(n, k)$ might have negative coefficients. Nevertheless, our combinatorial interpretation does allow us to give a lot of information about the coefficients $[m^s]B_{m+1}(n, k)$. We have already seen that $B_{m+1}(n, n) = 1$. Thus we shall only consider cases where $k < n$.

Theorem 3.7. *For all $1 \leq k < n$,*

$$[m^n]B_{m+1}(n,k) = 0, \tag{3.14}$$

$$\left[m^{n-1}\right] B_{m+1}(n,k) = (-1)^{n-1-k}\binom{n-1}{k}, \text{ and} \tag{3.15}$$

$$\left[m^{n-2}\right] B_{m+1}(n,k) = (-1)^{n-2-k}(2^n - n - 1)\binom{n-2}{k} + (-1)^{n-1-k}(2n-1)\binom{n-2}{k-1}. \tag{3.16}$$

Proof. For (3.14), since we can only underline elements in $[n] - (Last(\pi) \cup Min(\pi))$, we cannot have a non-min marked ordered Dowling set partition with n underlined elements. Thus $[m^n]B_{m+1}(n,k) = 0$ for all n and k.

For (3.15), the only non-min marked ordered Dowling set partition with $n-1$ underlined elements is $\pi = 1\ \underline{2}\ \ldots\ \underline{n}/\mathbf{n+1}$. Thus for $[m^{n-1}]B_{m+1}(n,k)$, we need only choose which of the elements $2,\ldots,n$ should be starred. Once we have chosen the starred elements, the number of elements of $[n] - Min(\pi)$ which are not starred is $n-1-k$. It follows that $[m^{n-1}]B_{m+1}(n,k) = (-1)^{n-1-k}\binom{n-1}{k}$.

For (3.16), we claim that we have only three possibilities for a non-min marked ordered Dowling set partition with $n-2$ underlined elements, namely, (i) $\pi = A_1/\mathbf{n+1}$, (ii) $\pi = A_1/i^*\ \mathbf{n+1}$, or (iii) $\pi = A_1/A_2/\mathbf{n+1}$. That is, since only elements in $[n]-(Last(\pi)\cup Min(\pi))$ can be marked, we must have $\text{last}(\pi) + \min(\pi) \leq 2$. If $\min(\pi) = 1$, then our only choices are (i) and (ii) depending on whether $last(\pi)$ equals 0 or 1. If $\min(\pi) = 2$, then $\text{last}(\pi)$ must be 0 so we get elements of type (iii).

For the elements of type (i), the elements $2,\ldots,n$ must have one starred element without an underline and $n-2$ underlined elements since we can have no totally free elements. Hence we have $n-1$ choices for the starred elements without an underline and then we must choose $k-1$ of the underlined elements to have stars. Hence the elements of type (i) contribute $(-1)^{n-1-k}(n-1)\binom{n-2}{k-1}$ to $[m^{n-2}]B_{m+1}(n,k)$. For the elements of type (ii), we have n choices for the i^* in the last part. This means all the elements in $A_1 - \{\min(A_1)\}$ must be underlined. Then we have $\binom{n-2}{k-1}$ ways to pick the remaining starred elements from $A_1 - \{\min(A_1)\}$. Hence the elements of type (ii) contribute $(-1)^{n-1-k}n\binom{n-2}{k-1}$ to $[m^{n-2}]B_{m+1}(n,k)$. Finally, for the elements of type (iii), all the elements in $[n] - Min(\pi)$ must be underlined and we have $\binom{n-2}{k}$ ways to place the stars on these elements. Thus we need only consider the choices for A_1 since $[n] - A_1 = A_2$. Of the 2^n possible ways to pick the set A_1, we can not have $A_1 = \emptyset$ or $A_1 = [n]$. We can also not have $|A_1| = n-1$ since then $|A_2| = 1$ and this is impossible for a fixed point of L. This rules out another n choices for A_1. Finally, if $A_1 = 1$, then to be a fixed point of L, we must have $min(A_1) < min(A_2)$ so A_1 must be $\{1\}$. This rules out $n-1$ more choices for A_1, namely, $A_1 = \{j\}$ for $j = 2, \ldots, n$. Hence the type (iii) elements contribute $(-1)^{n-2-k}(2^n - 2n - 1)\binom{n-2}{k}$ to $[m^{n-2}]B_{m+1}(n,k)$. It follows that

$$[m^{n-2}]B_{m+1}(n,k) = (-1)^{n-2-k}(2^n - 2n - 1)\binom{n-2}{k} + (-1)^{n-1-k}(2n-1)\binom{n-2}{k-1}.$$

□

The polynomial $B_{m+1}(n, n-1)$ is a polynomial in m with non-negative coefficients as we will prove in our next theorem.

Theorem 3.8. *For all $n \geq 2$ and $0 \leq s < n$,*

$$[m^s]B_{m+1}(n, n-1) = \sum_{k=\max(s+1,2)}^{n} \binom{k-1}{s}\binom{n}{k}. \tag{3.17}$$

Proof. By Theorem 3.5, to compute $[m^s]B_{m+1}(n, n-1)$, we must consider the non-min marked ordered Dowling set partitions in $Fix(L)$ with either 0 or 1 parts and with $n-1$

elements starred. We have already seen that a non-min marked ordered Dowling set partitions in $Fix(L)$ with 0 parts cannot contribute to $[m^s]B_{m+1}(n, n-1)$ so we need only consider fixed points of L of the form $\pi = A_1/A_2$. Suppose $\pi = A_1/A_2$ is a fixed point of L, $|A_1| \geq 2$ and have s underlined elements. Thus $|A_1| \geq s+1$ since only elements of $[n] - (Last(\pi) \cup Min(\pi))$ can be underlined. Hence the size k of A_1 can range from $\max(2, s+1)$ to n. We then have $\binom{n}{k}$ ways of choosing the elements in A_1. Once we have chosen the elements of A_1 we must star all elements of $[n]$ except $\min(A_1)$ since we must have $n-1$ starred elements. Finally we have $\binom{k-1}{s}$ ways to pick which elements of $A_1 - \{\min(A_1)\}$ are underlined. □

The polynomial $B_{m+1}(n, n-2)$ is not a polynomial in m with non-negative coefficients since by (3.9), $[m^{n-1}]B_{m+1}(n, n-2) = -(n-1)$. However, for $n \geq 6$, it is the only negative coefficient in $B_{m+1}(n, n-2)$. For example, we have computed that

$$
\begin{aligned}
B_{m+1}(2,0) &= -m, \\
B_{m+1}(3,1) &= 1 - 4m - 2m^2, \\
B_{m+1}(4,2) &= 11 + m - 7m^2 - 3m^3, \\
B_{m+1}(5,3) &= 66 + 94m + 36m^2 - 6m^3 - 4m^4, \\
B_{m+1}(6,4) &= 302 + 683m + 567m^2 + 183m^3 + 7m^4 - 5m^5, \\
B_{m+1}(7,5) &= 1191 + 3520m + 4138m^2 + 2373m^3 + 684m^4 + 48m^5 - 6m^6, \\
B_{m+1}(8,6) &= 4293 + 15461m + 23165m^2 + 18401m^3 + 8129m^4 + 1853m^5 \\
&\quad +149m^6 - 7m^7.
\end{aligned}
$$

We can give an exact formula for $[m^s]B_{m+1}(n, n-2)$.

Theorem 3.9. *For $n \geq 2$ and $0 \leq s \leq n-1$, $[m^s]B_{m+1}(n, n-2)$ equals*

$$\sum_{k=3}^{n} \binom{n}{k}\binom{k-2}{s} + \sum_{k=4}^{n} \binom{n}{k}\binom{k-2}{s}(2^k - 2k - 2) -$$
$$\sum_{k=2}^{n} \binom{n}{k}(k-1)\binom{k-2}{s-1}. \tag{3.18}$$

Proof. By Theorem 3.5, to compute $[m^s]B_{m+1}(n, n-2)$, we must consider the non-min marked ordered Dowling set partitions in $Fix(L)$ with either 0, 1, or 2 parts and with $n-2$ starred elements and s underlined elements. We have already seen that a non-min marked ordered Dowling set partitions in $Fix(L)$ with 0 parts cannot contribute to $[m^s]B_{m+1}(n, n-2)$ so we need only consider fixed points of L of the form $\pi = A_1/A_2$ or $\pi = A_1/A_2/A_3$.

For π of the form A_1/A_2, we must have that $|A_1| \geq 2$ by our conditions for π to be a fixed point of L. If $|A_1| = k$ where $2 \leq k \leq n$, then we have $\binom{n}{k}$ ways to pick the elements of A_1. Once we have picked the elements in A_1, we have to star all the elements in $A_2 - \{n+1\}$ because to be a fixed point of L, π must have no totally free elements. This leaves us $|A_1] - 2$ stars to use on the elements of A_1. Since we can have no totally free elements, this means there is an $x \in A_1 - \{\min(A_1)\}$ which is underlined but not starred. We have $k-1$ ways to choose x. All the elements of $A_1 - \{\min(A_1), x\}$ must be starred so we have $\binom{k-2}{s-1}$ to pick the underlined elements of $A_1 - \{\min(A_1), x\}$. In this case, x will be the only free element of π. Thus the Dowling set partition π of the form A_1/A_2 will contribute $-\sum_{k=2}^{n}\binom{n}{k}(k-1)\binom{k-2}{s-1}$ to $[m^s]B_{m+1}(n, n-2)$

For π of the form $A_1/A_2/A_3$, we have two cases. Namely, we know that $|A_2| \geq 2$, but for A_1, it could be the case that $|A_1| = 1$ or $|A_1| \geq 2$.

Case 1. $|A_1| = 1$.

In this case, to be a fixed point of L, we must have $\min(A_1) < \min(A_2)$. Since $|A_2| \geq 2$, it must be the case that $|A_1 \cup A_2| \geq 3$. Thus for any $3 \leq k \leq n$, we have $\binom{n}{k}$ ways to pick $A_1 \cup A_2$. Once we have picked $A_1 \cup A_2$, the smallest element of $A_1 \cup A_2$ must be put in A_1 and the remaining elements of $A_1 \cup A_2$ must go into A_2. Since we have $n-2$ starred elements, all the elements of $[n] - \{\min(A_1), \min(A_2)\}$ must be starred. Then we have $\binom{k-2}{s}$ ways to pick the elements of $A_2 - \{\min(A_2)\}$ which are underlined. It follows that π in Case 1 will contribute $\sum_{k=3}^{n} \binom{n}{k}\binom{k-2}{s}$ to $[m^s]B_{m+1}(n, n-2)$.

Case 2. $|A_1| \geq 2$.

In this case, since $|A_2| \geq 2$, it must be the case that $|A_1 \cup A_2| \geq 4$. Thus for any $4 \leq k \leq n$, we have $\binom{n}{k}$ ways to pick $A_1 \cup A_2$. Once we have picked $A_1 \cup A_2$, we have $2^k - 2k - 2$ ways to pick A_1. That is A_1 can be any subset of $A_1 \cup A_2$ except that the choices where $|A_1| = 0$, $|A_1| = 1$, $|A_1| = k$, and $|A_1| = k-1$ are not allowed. Since we have $n-2$ starred elements, all the elements of $[n] - \{\min(A_1), \min(A_2)\}$ must be starred. Then we have $\binom{k-2}{s}$ ways to pick the elements of $A_1 \cup A_2 - Min(\pi)$ which are underlined. It follows that π in Case 2 will contribute $\sum_{k=4}^{n} \binom{n}{k}\binom{k-2}{s}(2^k - 2k - 2)$ to $[m^s]B_{m+1}(n, n-2)$.

Hence (3.18) follows.

□

Our next goal is to show that for $n \geq 6$, $[m^s]B_{m+1}(n, n-2) \geq 0$ for $0 \leq s \leq n-2$. Thus we must analyze the behavior of $[m^s]B_{m+1}(n, n-2)$.

First we consider $[m^0]B_{m+1}(n, n-2)$. Our proof of Theorem 3.9 shows that the term $-\sum_{k=2}^{n}\binom{n}{k}(k-1)\binom{k-2}{s-1}$ arises only when $s \geq 1$. Thus

$$\begin{aligned}[m^0]B_{m+1}(n,n-2) &= \sum_{k=3}^{n}\binom{n}{k} + \sum_{k=4}^{n}\binom{n}{k}(2^k-2k-2) \\ &= \binom{n}{3} + \sum_{k=4}^{n}\binom{n}{k}(2^k-2k-1). \end{aligned} \tag{3.19}$$

The sequence $([m^0]B_{m+1}(n,n-2))_{n\geq 2}$ starts out

$$0, 1, 11, 66, 302, 1191, 4293, 14608, 47840, \ldots.$$

This is sequence A000460 in the OEIS which counts the number of permutations of the symmetric group S_n with 2 descents.

Next we consider $[m]B_{m+1}(n,n-2)$.

$$\begin{aligned}[m]B_{m+1}(n,n-2) &= \sum_{k=4}^{n}\binom{n}{k}(k-2)(2^k-2k-2) + \sum_{k=3}^{n}\binom{n}{k}(k-2) \\ &\quad -\sum_{k=2}^{n}\binom{n}{k}(k-1) \\ &= \sum_{k=4}^{n}\binom{n}{k}(k-2)(2^k-2k-2) - \sum_{k=2}^{n}\binom{n}{k} \\ &= \left(\sum_{k=4}^{n}\binom{n}{k}(k-2)(2^k-2k-2)\right) - (2^n-n-1). \end{aligned}$$

For $n \geq 4$, the $k = n$ term in the first sum is $(n-2)(2^n-2n-2)$. But it is easy to check that for $n \geq 4$,

$$(n-2)(2^n-2n-2) - (2^n-n-1) \geq 0.$$

Thus $[m]B_{m+1}(n,n-2) \geq 0$ for $m \geq 4$.

Next suppose that $s \geq 2$. Then the $k = 3$ term in $\sum_{k=3}^{n}\binom{n}{k}\binom{k-2}{s}$ is equal to 0. This implies the sum

$$\sum_{k=3}^{n}\binom{n}{k}\binom{k-2}{s}+\sum_{k=4}^{n}\binom{n}{k}\binom{k-2}{s}(2^k-2k-2)$$

can be rewritten as

$$\sum_{k=s+2}^{n}\binom{n}{k}\binom{k-2}{s}(2^k-2k-1).$$

Similarly, the term $\sum_{k=2}^{n}\binom{n}{k}(k-1)\binom{k-2}{s-1}$ can be rewritten as

$$\begin{aligned}&\sum_{k=s+1}^{n}\binom{n}{k}(k-1)\binom{k-2}{s-1}=\\&s\binom{n}{s+1}+\sum_{k=s+2}^{n}\binom{n}{k}(k-1)\binom{k-2}{s-1}=\\&s\binom{n}{s+1}+\sum_{k=s+2}^{n}\binom{n}{k}(k-1)\binom{k-2}{s}\frac{s}{k-s-1}.\end{aligned}$$

Thus if $s\geq 2$, then

$$\begin{aligned}&[m^s]B_{m+1}(n,n-2)=\\&\qquad -s\binom{n}{s+1}+\sum_{k=s+2}^{n}\binom{n}{k}\binom{k-2}{s}(2^k-2k-1-\frac{(k-1)s}{k-s-1}).\end{aligned}\tag{3.20}$$

Now consider the first term plus the $k=s+2$ term in the sum in (3.20) which is equal to

$$\begin{aligned}&\binom{n}{s+2}(2^{s+2}-2(s+2)-1-s(s+1))-s\binom{n}{s+1}=\\&\binom{n}{s+2}(2^{s+2}-2(s+2)-1-s(s+1))-s\binom{n}{s+2}\frac{s+2}{n-s-1}=\\&\binom{n}{s+2}(2^{s+2}-2(s+2)-1-s(s+1)-\frac{s(s+2)}{n-s-1}).\end{aligned}$$

Since we are only considering coefficients $[m^s]B_{m+1}(n,n-2)$ where $n\geq s+2$, it follows that $n-s-1 \geq 1$. Hence

$$\binom{n}{s+2}(2^{s+2} - 2(s+2) - 1 - s(s+1) - \frac{s(s+2)}{n-s-1}) \geq$$
$$\binom{n}{s+2}(2^{s+2} - 2(s+2) - 1 - s(s+1) - s(s+2)).$$

But one can easily check that $(2^{s+2} - 2(s+2) - 1 - s(s+1) - s(s+2)) \geq 0$ for $s \geq 4$. It follows that for $n-2 \geq s \geq 4$, $[m^s]B_{m+1}(n, n-2) \geq 0$. Thus to complete our proof, we need only consider the cases $s = 2$ and $s = 3$.

When $s = 2$, the term $\sum_{k=2}^{n} \binom{n}{k}(k-1)\binom{k-2}{s-1}$ simplifies as in the following lemma.

Lemma 3.7. *For any integer $n \geq 3$,*

$$\sum_{k=3}^{n} \binom{n}{k}(k-1)(k-2) = 2^{n-2}(n^2 - 5n + 8) - 2.$$

We have used Mathematica to verify Lemma 3.7. In fact, it is not hard to prove the formula by hand. One can use the facts that $\sum_k k^2\binom{n}{k} = n(n+1)2^{n-2}$, $\sum_k k\binom{n}{k} = n2^{n-1}$ and $\sum_k \binom{n}{k} = 2^n$ to prove the lemma above. Now the $k = n$ term from $\sum_{k=4}^{n} \binom{n}{k}\binom{k-2}{s}(2^k - 2k - 1)$ is $\binom{n-2}{2}(2^n - 2n - 1)$. But is easy to check that

$$\binom{n-2}{2}(2^n - 2n - 1) - 2^{n-2}(n^2 - 5n - 8) + 2 \geq 0$$

for $n \geq 4$. Thus it follows that $[m^2]B_{m+1}(n, n-2) \geq 0$ for $n \geq 4$.

When $s = 3$, the term $\sum_{k=2}^{n} \binom{n}{k}(k-1)\binom{k-2}{s-1}$ simplifies as in the following lemma

Lemma 3.8. *For any integer $n \geq 4$,*

$$\sum_{k=4}^{n} \binom{n}{k}(k-1)\binom{k-2}{2} = 2^{n-4}(n^3 - 9n^2 - 32n - 48) + 3.$$

We have used Mathematica to verify this identity. Again, Lemma 3.8 can be proved by hand using facts mentioned above together with $\sum_k k^3\binom{n}{k} = 2^{n-3}(n^3+3n^2)$. Now the $k=n$ term from $\sum_{k=5}^{n}\binom{n}{k}\binom{k-2}{s}(2^k-2k-1)$ is $\binom{n-2}{3}(2^n-2n-1)$. But is easy to check that

$$\binom{n-2}{3}(2^n-2n-1) - 2^{n-4}(n^3-9n^2-32n-48) - 3 \geq 0$$

for $n \geq 6$. Thus it follows that $[m^3]B_{m+1}(n,n-2) \geq 0$ for $n \geq 6$.

Thus we have proved the following theorem.

Theorem 3.10. *For all $n \geq 2$, $[m^{n-1}]B_{m+1}(n,n-2) = -(n-1)$ and for all $n \geq 6$ and $0 \leq s \leq n-2$, $[m^s]B_{m+1}(n,n-2) \geq 0$.*

3.5 Future research

3.5.1 Whitney numbers of the first kind

In the paper [31], Remmel and Wachs also defined combinatorial objects which describe the Whitney numbers of the first kind. We let $S_m(n,k)$ denote the set of colored permutations π such that

1. π is composed of at least k disjoint cycles.
2. In each cycles, every elements except the smallest elements are colored with one of the colors in $\{0,1,2,\ldots,m\}$.
3. Smallest elements in each cycles are colored with one of the colors in $\{0,1\}$. Precisely, k such elements are colored 0.

For example, an element of $S_4(7,2)$ looks like $(1_0 3_0 5_2)(2_1)(4_0 6)_4(7_1)$. Remmel and Wachs proved that the signless Whitney numbers of the first kind $|w_m(n,k)|$ counts the number of elements in $S_m(n,k)$. In particular, we have

$$w_m(n,k) = (-1)^{n-k}|S_m(n,k)|$$

Similar to the case of the Whitney numbers of the second kind, the Whitney numbers of the first kind $w_m(n,k)$ is the Stirling numbers of the first kind $s(n,k)$ when $m = 1$. Thus, one can think of the Whitney numbers of the first kind as a generalization of the Stirling numbers of the first kind, and one might be able to generalized results related to the Stirling numbers of the first kind to the Whitney numbers of the first kind. However, we have a combinatorial interpretation of the Whitney numbers of the first kind. It would be interesting to see how would the combinatorial interpretation help us generalize results for the Stirling numbers of the first kind to the Whitney number of the first kind.

3.5.2 r-Whitney numbers

In the paper [26], Mezö defined a generalization of the Whitney numbers. The n-th falling factorial of x, denoted by $(x)_n$, is defined to be the polynomial $x(x-1)\dots(x-n+1)$. It was proved in [2, 9] that the Whitney numbers of the first and second kind satisfy the following equations:

$$m^n(x)_n = \sum_{k=0}^{n} w_m(n,k)(mx+1)^k,$$

$$(mx+1)^n = \sum_{k=0}^{n} W_m(n,k)x^k(x)_k.$$

Mezö then defined r-Whitney numbers of first and second kind by generalizing equations above. In particular, he defined r-Whitney numbers of the first and second kind, denoted by $w_{m,r}(n,k)$ and $W_{m,r}(n,k)$ respectively, as numbers satisfying following equations.

$$m^n(x)_n = \sum_{k=0}^{n} w_{m,r}(n,k)(mx+r)^k,$$

$$(mx+r)^n = \sum_{k=0}^{n} W_{m,r}(n,k)x^k(x)_k.$$

Mezö proved a number of results related r-Whitney numbers including their generating functions and relations to Bernoulli numbers. Several authors have further studied the topic [36, 10, 28, 5, 29]. In this dissertation, we study a kind of generalization of Eulerian numbers by generalize the left had side of the following equation to the Whitney numbers.

$$\sum_{k=1}^{n} k!S(n,k)x^k = \sum_{k=0}^{n-1} E_{n,k}x^k(x+1)^{n-k-1}.$$

It would be interesting to see what results we get out of the equation above if we generalize the left hand side to r-Whitney numbers.

Chapter 3 contains material from J.B. Remmel, S. Thamrongpairoj. "The combinatorial properties of the Benoumhani polynomials for the Whitney numbers of Dowling lattices" *Discrete Mathematics* v. 342, Issue 11 (2019) 2966-2983, 2018. The dissertation author is a primary author on this paper.

Chapter 4

Patterns on Dowling set partition

In this chapter, we define and study patterns on Dowling set partitions. We prove a number of results about pattern avoidance in Dowling set partitions and ordered Dowling set partitions including computing a generating function, explicit formula, or recursive for formula for computation for each patterns.

4.1 Introduction

Remmel and Wachs [31] gave a combinatorial interpretation to the Whitney numbers $W_m(n,k)$ of the second kind for any $m \geq 1$. We recall the definition of Dowling set partitions. Let $\mathcal{W}(n,k)$ denote the set of partitions $\pi = (A_1, \ldots, A_{k+1})$ of $[n+1] = \{1, 2, \ldots, n+1\}$ into $k+1$ parts, $\min(A_1) < \min(A_2) < \ldots < \min(A_k)$ and $n+1 \in A_{k+1}$. In such a situation, we let $\min(\pi)$ denote the set $\{\min(A_1), \ldots, \min(A_k)\}$ and $\mathrm{last}(\pi) = A_{k+1}$. For $m \geq 2$, we let $\mathcal{W}_m(n,k)$ be the set of all pairs (π, c) where $\pi \in \mathcal{W}(n,k)$ and $c : [n+1] - (\min(\pi) \cup \mathrm{last}(\pi)) \to \{1, \ldots, m\}$ and $\mathrm{W}_m(n,k) = |\mathcal{W}_m(n,k)|$. We call the elements of $\mathcal{W}_m(n,k)$ Dowling set partitions of size n with k parts of order m. We let $\mathcal{OW}_m(n,k)$ denote the set of ordered set partitions that arise from elements of $\mathcal{W}_m(n,k)$ by permuting the first k parts and we let $\mathrm{OW}_m(n,k) = |\mathcal{OW}_m(n,k)|$.

The set of patterns that we consider arise in a similar manner from the way patterns arise in the study of patterns in set partitions studied by Sagan [32] and in ordered set partitions by Godbole et. all [11]. That is, the set of patterns for Dowling set partitions arise by starting with a Dowling set partition $\pi = A_1/A_2/\ldots,/A_{k+1}$ and eliminating some of the elements of $\{1,\ldots,n\}$ and then reducing the resulting colored set partition by replacing the i-th largest element by i, throwing away any empty parts, and reducing the colors by replacing j-th largest color that remain by j.

We call the set of patterns that arise from Dowling set partitions in this way *Dowling patterns*. Then we say that a Dowling set partition $\pi \in \mathcal{W}_m(n,k)$ *contains a Dowling pattern* p if p can be obtained from π by eliminating some elements of $1,\ldots,n$ from π and reducing. We say that a Dowling set partition $\pi \in \mathcal{W}_m(n,k)$ *avoids a Dowling pattern* p if p is not contained in π. Thus Dowling patterns are a much richer class of colored set partitions in that we distinguish elements in the last part, distinguish possible minimal elements in the remaining parts, and keep track of colored elements. Given a Dowling pattern p, we let $\mathcal{W}_{m,p}(n,k)$ denote the set of all $\pi \in \mathcal{W}_m(n,k)$ such that π avoids p and we let $\mathrm{W}_{m,p}(n,k) = |\mathcal{W}_{m,p}(n,k)|$.

Patterns arising from ordered Dowling set partitions arise in the same fashion. We call the set of patterns that arise from Dowling set partitions in this way *ordered Dowling patterns* . We defined containment and avoidance for ordered Dowling set partitions the same way we defined in the case of Dowling set partitions. We say that an ordered Dowling set partition $\pi \in \mathcal{OW}_m(n,k)$ *avoids a ordered Dowling pattern* p if p is not contained in π. Given an ordered Dowling pattern p, we let $\mathcal{OW}_{m,p}(n,k)$ denote the set of all $\pi \in \mathcal{OW}_m(n,k)$ such that π avoids p and we let $\mathrm{OW}_{m,p}(n,k) = |\mathcal{OW}_{m,p}(n,k)|$.

This chapter outlines as follow. In Section 4.2, we recall the definitions of Dowling set partitions and define ordered Dowling set partitions and their patterns. In Section 4.3, we prove a number of results for Dowling patterns of length 3. In Section 4.4, we prove similar results for ordred Dowling patterns of length 4. Our main goal is to compute following generating functions for Dowling patterns p of size 3.

$$W_{m,p}^{(k)}(t) = \sum_{n \geq k} \mathrm{W}_{m,p}(n,k)t^n.$$

Similarly, we study the following generating function for ordered Dowling patterns p of size 3.

$$OW_{m,p}^{(k)}(t) = \sum_{n \geq k} \mathrm{OW}_{m,p}(n,k)t^n.$$

4.2 Definitions

In this section, we carefully define Dowling set partitions and Dowling patterns. We recall the definition of Dowling set partitions given by Wachs and the Remmel [31]. Given a set partition $\pi = (A_1, \ldots, A_k, A_{k+1})$ of $[n+1]$, we shall assume that it is written in a standard form which means that

1. $\min(A_1) < \cdots < \min(A_k)$ and

2. $n + 1 \in A_{k+1}$.

In such a situation, we let $Min(\pi) = \{\min(A_1), \ldots, \min(A_k)\}$ and $Last(\pi) = A_{k+1}$. For example, if $n = 7$ and $\pi = (\{1,4,6\},\{3,5\},\{2,7,8\})$, then $Min(\pi) = \{1,3\}$ and $Last(\pi) = \{2,7,8\}$.

Definition 4.1. a Dowling set partition of $[n]$ with k parts *is a pair* (π, c) *where* $\pi = (A_1, \ldots, A_k, A_{k+1})$ *is a set partition of* $[n+1]$ *into* $k+1$ *parts and* c *is a map from* $[n+1] - (Min(\pi) \cup Last(\pi))$ *into* $[m]$*. We shall view c as coloring of the non-minimal elements in each of parts* $A_1, \ldots, A_k$ *with one of* m *colors from* $\{1, \ldots, m\}$*.*

We now define ordered Dowling set partitions based on Dowling set partitions.

Definition 4.2. An ordered Dowling set partition of $[n]$ with k parts *is a Dowling set partition* (π, c) *of* $[n]$ *with* k *parts where we permute the first* k *parts of* π*.*

We shall represent (π, c) in the form $A_1/A_2/\ldots/A_l/A_{l+1}$ where we put subscript on the elements to indicate the color. We will also distinguish the largest element $n+1$ by making it boldface.

For example,

$$\pi = 1\ 4_2\ 5_1/3\ 7_1\ 8_2/6\ 11_3/2\ 9\ 10\ \mathbf{12}$$

is a Dowling set partition of size 11 with 3 parts of order m for any $m \geq 3$ Similarly,

$$\overline{\pi} = 3\ 7_1\ 8_2/1\ 4_2\ 5_1/6\ 11_3/2\ 9\ 10\ \mathbf{12}$$

is an ordered Dowling set partition of size 11 with 3 parts of order m for any $m \geq 3$.

The set of patterns that we can obtain from colored set partitions. *Dowling patterns* are colored ordered set partitions such that

(i) the largest element is in the last part

(ii) all the elements in the last part are not colored

(iii) there is at most one element in the remaining parts which is not colored

(iv) the non-colored elements which are not in the last part are increasing, reading from left to right

(v) if a color $i > 1$ occurs, then the color $i-1$ must occur.

For example, for Dowling patterns of p whose underlying set partition is $1/2/\mathbf{3}$, there are 6 Dowling patterns, $p = 1/2/\mathbf{3}$, $p = 1/2_1/\mathbf{3}$, $p = 1_1/2/\mathbf{3}$, $p = 1_1/2_1/\mathbf{3}$, $p = 1_1/2_2/\mathbf{3}$, and $p = 1_2/2_1/\mathbf{3}$.

Ordered Dowling patterns are colored ordered set partitions such that

(i) the largest element is in the last part

(ii) all the elements in the last part are not colored

(iii) there is at most one element in the remaining parts which is not colored

(iv) if a color $i > 1$ occurs, then the color $i-1$ must occur.

We define occurrence and avoidance of pattens on Dowling set partitions and ordered Dowling set partitions the same way as in the study of patterns in set partitions studied by Sagan [32] and in ordered set partitions by Godbole et. all [11]. That is, we say that Dowling pattern τ is contained in Dowling set partition π if we can obtain τ from π by eliminating some of the elements of $\{1,\ldots,n\}$ and then reducing the resulting colored set partition by replacing the i-th largest element by i, throwing away any empty parts, and reducing the colors by replacing j-th largest color that remain by j.

For example, starting with the Dowling set partition,

$$\pi = 1\ 4_3\ 5_1/3\ 7_1\ 8_1/6\ 11_3/2\ 9\ 10\ \mathbf{12}$$

and eliminating $1, 6, 9, 10$, and 11, we would obtain

$$\pi = 4_3\ 5_1/3\ 7_1\ 8_1/2\ \mathbf{12}$$

and reducing we would obtain

$$\pi = 3_2\ 4_1/2\ 5_1\ 6_1/1\ \mathbf{7}.$$

Then we say that a Dowling set partition π *avoids a Dowling pattern* τ if τ is not contained in π. Thus Dowling patterns are a much richer class of colored set partitions in that we distinguish elements in the last part, distinguish possible minimal elements in the remaining parts, and keep track of colored elements. Given a Dowling pattern p, we let $\mathcal{W}_{m,p}(n,k)$ denote the set of all $\pi \in \mathcal{W}_m(n,k)$ such that π avoids p and we let $\mathrm{W}_{m,p}(n,k) = |\mathcal{W}_{m,p}(n,k)|$.

We say that two Dowling patterns p_1, p_2 are *Wilf-equivalent* if $W_{m,p_1}(n,k) = W_{m,p_2}(n,k)$ for all n, k.

We define occurrence and avoidance of ordered Dowling patterns in ordered Dowling set partitions in the same fashion. We say that an ordered Dowling set partition $\pi \in \mathcal{OW}_m(n,k)$ *contains an ordered Dowling pattern* p if p can be obtained from π by eliminating some elements of $1, \ldots, n$ from π and reducing. We say that an ordered Dowling set partition $\pi \in \mathcal{OW}_m(n,k)$ *avoids a ordered Dowling pattern* p if p is not contained in π. Given an ordered Dowling pattern p, we let $\mathcal{OW}_{m,p}(n,k)$ denote the set of all $\pi \in \mathcal{OW}_m(n,k)$ such that π avoids p and we let $\mathrm{OW}_{m,p}(n,k) = |\mathcal{OW}_{m,p}(n,k)|$.

4.3 Unordered case

This section concerns the case where the set partition is ordered by the minimal elements. We explored all the cases, and computed the generating function, recursive formula, or enumerate the first few terms.

For any pattern τ, let $W^{(k)}_{m,\tau}(t) = \sum_{n \geq k} W_{m,\tau}(n,k)t^n$.

Case $\tau = 1/2/\mathbf{3}$

Theorem 4.1. *Let $\tau = 1/2/\mathbf{3}$, then*

$$W^{(0)}_{m,\tau}(t) = \frac{1}{1-t},$$

$$W^{(1)}_{m,\tau}(t) = \frac{t}{(1-(1+m)t)(1-t)},$$

$$W^{(k)}_{m,\tau}(t) = 0$$

for $k \geq 2$.

Proof. Since parts are ordered by the minimal element, then, in order for π to avoid τ, π can only have 0 or 1 parts. This proves that $W^{(k)}_{m,\tau}(t) = 0$ for $k \geq 2$. For $k = 0$ part, π contain every element in the last part, so $\mathrm{W}_{m,\tau}(n, 0) = 1$, and

$$W^{(0)}_{m,\tau}(t) = \frac{1}{1-t}.$$

For $k = 1$, then for each $\pi \in \mathcal{W}_{m,p}(n, k)$, $\pi = A_1/A_2$, where $A_1 \neq \emptyset$. So, $W_{m,\tau}(n, 1)$ can be computed by counting the number of way we can choose A_1, which is $\frac{(1+m)^n-1}{m}$.

$$\begin{aligned} W^{(1)}_{m,\tau}(t) &= \sum_{n\geq 1} \frac{(1+m)^n - 1}{m} t^n \\ &= \frac{t}{(1-(1+m)t)(1-t)}. \end{aligned}$$

□

Case $\tau = 1/2_1/\mathbf{3}$

Theorem 4.2. *Let* $\tau = 1/2_1/\mathbf{3}$*, then*

$$W^{(k)}_{m,\tau}(t) = \frac{t^k}{(1-(1+m)t)^k(1-t)}$$

.

Proof. We will prove the theorem in two ways: using recursive formula, and direct counting.

We first derive a recursive formula for $W_{m,\tau}(n, k)$ by considering the position of n, removing n and replacing $\mathbf{n+1}$ by $\mathbf{n}$

In this case, n is either in the last part, a part by itself, or a non-minimal element in a non-last part. If n is in the last part, n does not influence the pattern τ, so this case will contribute $W_{m,\tau}(n-1,k)$. If $\{n\}$ is a part, n also does not influence the pattern τ, so this case will contribute $W_{m,\tau}(n,k-1)$. Finally, if n is a non-minimal element in a non-last part, we know that n has to be in the first part in order to avoid τ, and we have m choices for the color of n, which yields m copies of $W_{m,\tau}(n-1,k)$. Therefore, we have

$$W_{m,\tau}(n,k) = (1+m)W_{m,\tau}(n-1,k) + W_{m,\tau}(n-1,k-1)$$

which give the generating function

$$W_{m,\tau}^{(k)}(t) = \frac{t^k}{(1-(1+m)t)^k(1-t)}.$$

Another way to understand $1/2_1/\mathbf{3}$-avoiding Dowling set partition is to observe that if $\pi = A_1/A_2/\ldots/A_k/A_{k+1} \in W_{m,\tau}(n,k)$, then $|A_2| = |A_3| = \ldots = |A_k| = 1$. That is because if $|A_i| > 1$ for some $i > 1$, let b be a non-minimal element in A_i, then we have $min(A_1) < min(A_i) < b$. Thus, $min(A_1), b, \mathbf{n+1}$ will produce the pattern $1/2_1/\mathbf{3}$. As a consequence, any $\pi \in \mathcal{W}_{m,\tau}(n,k)$ will be in the form

$$\pi = A_1/c_2/c_3/\ldots/c_k/A_{k+1}.$$

with $c_2 < c_3 < \ldots < c_k$. We shall count the number of way to contruct such π, which can be described as follow:

1. Choose $a \in \{1,2,\ldots,n-k+1\}$ to be the minimal element in A_1.

2. Choose $\{c_2,\ldots,c_k\} \subset \{n-a+1,\ldots,n\}$ to be minimal elements in $A_2,\ldots,A_k$.

3. For $1,2,\ldots,a-1$, they must be in the last part.

4. For $\{a, a+1, \ldots, n\} - \{a, c_2, \ldots, c_k\}$, there are $1+m$ choices for each elements, either put it in the last part or in A_1 with one of m colors.

So, the number of such π is $\sum_{a=1}^{n-k+1} \binom{n-a}{k-1}(m+1)^{n-a-k+1}$, which can be rewritten as $\sum_{l=0}^{n-k} \binom{k+l-1}{k-1}(1+m)^l = \frac{t^k}{(1-(1+m)t)^k(1-t)}\Big|_{t^n}$. □

Case $\tau = 1_1/2/\mathbf{3}$

Theorem 4.3. *Let* $\tau = 1_1/2/\mathbf{3}$*, then*

$$W_{m,\tau}^{(k)}(t) = \frac{t^k}{(1-(1+mk)t)(1-t)^k}$$

.

Proof. We first derive the recursive formula by considering the position of n. If n is in the last part, n cannot influence the pattern τ, so this case will contruibute $W_{m,\tau}(n-1,k)$. If n has a non-minimal in a non-last part, n cannot influece the pattern τ. We start with an element in $\mathcal{W}_{m,\tau}(n-1,k)$, then there are k choices for a part containing n, and m choices for the color of n, so we have $mkW_{m,\tau}(n-1,k)$ in this case.

If $\{n\}$ is a part, then $\{n\}$ must be right before the last part due to ordering of minimal elements. To avoid τ, every part before $\{n\}$ cannot have colors. Then, if $\pi \in \mathcal{W}_{m,\tau}(n,k)$ and $\{n\}$ is a part in the π, then π must be in the form

$$\pi = c_1/c_2/\ldots/c_{k-1}/n/A_{k+1}$$

with $c_1 < c_2 < \ldots < c_{k-1}$. So, the number of such π is the number of choices of $\{c_1, \ldots, c_k\} \subset [n-1]$, which is $\binom{n-1}{k-1}$. Combine all cases together, we can write the generating function as

$$W_{m,\tau}(n,k) = (1+mk)W_{m,\tau}(n-1,k) + \binom{n-1}{k-1}$$

. which gives the generating function

$$W^{(k)}_{m,\tau}(t) = \frac{t^k}{(1-(1+mk)t)(1-t)^k}.$$

□

Case $\tau = 2/1_1/\mathbf{3}$

This case is trivial, due to the structure of elements in $W_{m,\tau}(n,k)$. For any $\pi = A_1/\ldots/A_k/A_{k+1} \in \mathcal{W}_m(n,k)$, we claim π does not contain τ. Let $a \in A_i, b \in A_j, i < j$, so that a is the minimal element in A_i, and b has a color in A_j. Then, there must be a minimal element $c \in A_j$. Since A_i appears before A_j, then $a < c$. Since both b and c are in A_j and c is the minimal elements, we have $c < b$. Therefore, $a < b$, which proves that picking a, b does not produce the pattern τ in π, which proves that π always avoid τ

Case $\tau = 2_1/1/\mathbf{3}$

Theorem 4.4. *Let $\tau = 2_1/1/\mathbf{3}$, then*

$$W^{(k)}_{m,\tau}(t) = \frac{t^k}{(1-(1+mk)t)(1-t)^k}$$

.

That is, pattern $2_1/1/\mathbf{3}$ and pattern $1/2_1/\mathbf{3}$ are Wilf-equivalent.

Proof. We will prove this theorem in three ways: Using recursive formula, direct counting, and constructing a bijection to $1/2_1/\mathbf{3}$-avoiding Dowling set partitions.

We first derive a recursive formula for $W_{m,\tau}(n,k)$. When n is in the last part, n cannot influence the pattern τ, so this case contributes $W_{m,\tau}(n-1,k)$. If $\{n\}$ is a part, then n also cannot influence the pattern τ, so this case contributes $W_{m,\tau}(n-1,k-1)$. If n has a color, then n must be in the part before the last part in order for π to avoid τ. So, there will be m choices for the color of n, and there will be $W_{m,\tau}(n-1,k)$ to start with. Therefore, we have

$$W_{m,\tau}(n,k) = (1+m)W_{m,\tau}(n-1,k) + W_{m,\tau}(n-1,k-1).$$

Note the formula is exactly the same as the recursive formula in the case $\tau = 1/2_1/\mathbf{3}$, and both cases have the same initial conditions: $W_{m,\tau}(n,0) = W_{m,\tau}(n,n) = 1$. Therefore, they have the same number of pattern avoiding Dowling set partitions.

Like the case $\tau = 1/2_1/\mathbf{3}$, we can count the number of ways we can construct $\pi \in \mathcal{W}_{m,\tau}(n,k)$. Note that if $\pi = A_1/A_2/\dots/A_k/A_{k+1} \in W_{m,\tau}(n,k)$, then we must have $max(A_i) < min(A_{i+1})$. To see this, consider 2 cases. If $|A_i| = 1$, then $max(A_i) = min(A_i) < min(A_{i+1}$ by minimal ordering condition. Otherwise, A_i has at least one colored element. In particular, $max(A_i)$ is colored. Therefore, $max(A_i), min(A_{i+1}), \mathbf{n+1}$ will produce $2_1/1/\mathbf{3}$. Now, we decribe how to construct π:

1. Choose $a \in \{1,2,\dots,n-k+1\}$ to be the minimal element in A_1.
2. Choose $c_2,\dots,c_k \in \{n-a+1,\dots,n\}$ to be minimal elements in $A_2,\dots,A_k$.
3. For $1,2,\dots,a-1$, they must be in the last part.
4. For $k \in \{a,a+1,\dots,n\} - \{a,c_2,\dots,c_k\}$, there are $1+m$ choices for each elements, either put it in the last part or in A_i such that $c_i < k < c_{i+1}$.

So the number of way to construct such π is $\sum_{a=1}^{n-k+1}\binom{n-a}{k-1}(m+1)^{n-a-k+1} = \left.\frac{t^k}{(1-(1+m)t)^k(1-t)}\right|_{t^n}$.

With this construction, we can easily construction a bijection $I : \mathcal{W}_{m,2_1/1/\mathbf{3}}(n,k) \to \mathcal{W}_{m,1/2_1/\mathbf{3}}$ by moving all elements with color to the first part. For example, if $\pi = 2\ 3_1/5\ 6_2/7\ 9_1/1\ 4\ 8\ \mathbf{10}$, then $I(\pi) = 2\ 3_1\ 6_2\ 9_1/5/7/1\ 4\ 8\ \mathbf{10}$.

□

Case $\tau = 1\ 2_1/\mathbf{3}$

In order for π to avoid τ, π cannot contain any colored elements. Thus, π must be in the form $c_1/c_2/\ldots/c_k/A_{k+1}$, with $c_1 < c_2 < \ldots < c_k$. Thus, the number of such π is $\binom{n}{k}$.

Case $\tau = 1_1/2_1/\mathbf{3}$

This case is the first case that a recursive formula could not be written easily. We need some refinement in order to write down a recursive formula in this case.

Let $W_{m,\tau,\alpha_1,\ldots,\alpha_m}(n,k)$ be the number of Dowling set partitions avoiding $\tau = 1_1/2_1/\mathbf{3}$ such that the first set that the color i appears is the α_i-th set. If no color i appears, then $\alpha_i = 0$. Then, we derive a recursive formula for $W_{m,\tau,\alpha_1,\ldots,\alpha_m}(n,k)$ by considering the position of n. If n is in the last part, then n cannot influece the pattern τ, so this case will contribute $W_{m,\tau,(\alpha_1,\ldots,\alpha_m)}(n-1,k)$. $\{n\}$ can be a part, given that $\alpha_i \neq k$ for all i. In this case, n cannot influence the pattern τ, so this case contributes $W_{m,\tau,(\alpha_1,\ldots,\alpha_{k-1})}(n-1,k-1)\chi(\alpha_i \neq k \text{ for all } i)$. If n has color i, then n must be in the first set with color i, which is the α_i-th set. Removing n would make the first set with color i become $\alpha_i, \alpha_i+1, \ldots, k$ or contain no color i at all. This case will contribute

$$W_{m,\tau,(\alpha_1,\ldots,\alpha_{i-1},0,\alpha_{i+1},\alpha_m)}(n,k) + \sum_{j=\alpha_i}^{k} W_{m,\tau,\alpha_1,\ldots,\alpha_{i-1},j,\alpha_{i+1},\alpha_m}(n-1,k).$$

In total, we derive a recursive formula as

$$\begin{aligned} W_{m,\tau,\alpha_1,\ldots,\alpha_m}(n,k) &= W_{m,\tau,\alpha_1,\ldots,\alpha_m}(n-1,k) \\ &+W_{m,\tau,\alpha_1,\ldots,\alpha_m}(n-1,k-1)\chi(\alpha_i \neq k \text{ for all } i) \\ &+\sum_{i=1}^{m} W_{m,\tau,\alpha_1,\ldots,\alpha_{i-1},0,\alpha_{i+1},\alpha_m}(n-1,k) \\ &+\sum_{i=1}^{m}\sum_{j=\alpha_i}^{k} W_{m,\tau,\alpha_1,\ldots,\alpha_{i-1},j,\alpha_{i+1},\alpha_m}(n-1,k). \end{aligned}$$

Below are series expansion of $W_{2,\tau}^{(k)}(t)$ for the first few k's:

$$\begin{aligned} W_{2,\tau}^{(0)}(t) &= 1+t+t^2+t^3+t^4+t^5+t^6+\ldots \\ W_{2,\tau}^{(1)}(t) &= t+4t^2+13t^3+40t^4+121t^5+364t^6+1093t^7+\ldots \\ W_{2,\tau}^{(2)}(t) &= t^2+9t^3+54t^4+268t^5+1189t^6+4901t^7+19190t^8+\ldots \\ W_{2,\tau}^{(3)}(t) &= t^3+16t^4+150t^5+1060t^6+6283t^7+33078t^8+160036t^9+\ldots \\ W_{2,\tau}^{(4)}(t) &= t^4+25t^5+335t^6+3157t^7+23806t^8+154200t^9+895216t^{10}+\ldots \\ W_{2,\tau}^{(5)}(t) &= t^5+36t^6+651t^7+7840t^8+72822t^9+565934t^{10}+3863944t^{11}+\ldots \end{aligned}$$

Case $\tau = 2_1/1_1/\mathbf{3}$

In this case, we define a refinement of $W_{m,\tau}(n,k)$ in order to derive a recursive formula.

Let $W_{m,\tau,(\alpha_1,\ldots,\alpha_m)}(n,k)$ be the number of Dowling set partition avoding $\tau = 2_1/1_1/\mathbf{3}$ such that the last set that the color i appears is in the α_i-th set. If no color i appears, then $\alpha_i = 0$. Then, we derive a recursive formula for $W_{m,\tau,(\alpha_1,\ldots,\alpha_m)}(n,k)$ by considering the position of n.If n is in the last part, then n could not influece the pattern τ, so this case will contribute $W_{m,\tau,(\alpha_1,\ldots,\alpha_m)}(n-1,k)$. $\{n\}$ can be a part, given that $\alpha_i \neq k$ for all i. In this case, n cannot influence the pattern τ, so this case contributes $W_{m,\tau,(\alpha_1,\ldots,\alpha_{k-1})}(n-1,k-1)\chi(\alpha_i \neq k$ for all $i)$. If n has color i, then n must be in the last set with color i, which is the α_i^{th} set. Removing n would make the last set with color i become $1,2,\ldots,\alpha_i$ or contain no color i at all. This case will contribute

$$\sum_{j=0}^{\alpha_i} W_{m,\tau,\alpha_1,\ldots,\alpha_{i-1},j,\alpha_{i+1},\alpha_m}(n-1,k)$$

.

Therefore, we have

$$\begin{aligned} W_{m,\tau,\alpha_1,\ldots,\alpha_m}(n,k) &= W_{m,\tau,\alpha_1,\ldots,\alpha_m}(n-1,k) \\ &\quad + W_{m,\tau,\alpha_1,\ldots,\alpha_m}(n-1,k-1)\chi(\alpha_i \neq k \text{ for all } i) \\ &\quad + \sum_{i=1}^{m}\sum_{j=0}^{\alpha_i} W_{m,\tau,\alpha_1,\ldots,\alpha_{i-1},j,\alpha_{i+1},\alpha_m}(n-1,k). \end{aligned}$$

Let $W^{(k)}_{m,\tau,s_1,\ldots,s_m}(t) = \sum_{n\geq k} W_{m,\tau,s_1,\ldots,s_m}(n,k)t^n$. The recursive formula above gives us a recursive formula for $W^{(k)}_{m,\tau,s_1,\ldots,s_m}(t)$ as follows:

$$\begin{aligned} W^{(k)}_{m,\tau,s_1,\ldots,s_m}(t) &= \frac{t}{1-(1+m)t}\Bigg(W^{(k-1)}_{m,\tau,s_1,\ldots,s_m}(t)\chi(k > s_i \text{ for all } i) \\ &\quad + \sum_{i=1}^{m}\sum_{j=0}^{s_i-1} W^{(k)}_{m,\tau,s_1,\ldots,s_{i-1},j,s_{i+1},\ldots,s_m}\Bigg) \end{aligned}$$

Here we consider the special case when $m = 2$. We let $W^{(k)}_{2,\tau,\alpha_1,\alpha_2}(t) = \sum_{n\geq k} W_{2,\tau,\alpha_1,\alpha_2}(n,k)t^n$. We derive an explicit formula for $W^{(k)}_{2,\tau,2,2}(t)$.

Theorem 4.5. *Let* $\tau = 2_1/1_1/\mathbf{3}$ *then*

$$W^{(k)}_{2,\tau,2,2}(t) = \frac{t^{k+2}k(k-1)}{(1-t)(1-3t)^{k+2}}.$$

Proof. For any π avoiding $2_1/1_1/\mathbf{3}$ with 2 colors, so that the last part that each color appear is the second part, π must have the following form

$$\pi = a**\dots/bc_1**d_2**\dots/x^{(1)}/\dots/x^{(k-2)}/**\dots.$$

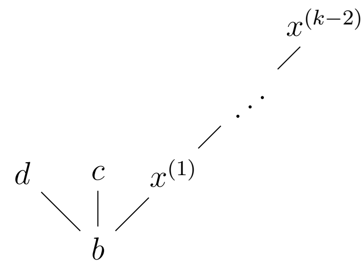
where a is the minimal element in the first part, b is the minimal element in the second part, c is the minimal element with color 1 in the second part, d is the minimal element with color 2 in the second part, and $x^{(i)}$'s are minimal elements in the later parts. Note that, the form above is misleading, since c is not necessary less than d.

We construct such π as follows:

1. Pick a. This will corresponds the the generating function $\frac{t}{1-t}$, since every elements smaller than a has to go to the last part.

2. Pick $k+1$ elements, to be rearranged later as $b, c, d, x^{(1)}, \dots, x^{(k-2)}$. Then, every other number in between has three choices, either having color 1, 2 or being in the last part. This corresponds to the generating function $\frac{t^{k+1}}{(1-3t)^{k+2}}$.

3. Pick a feasible choices for $b, c, d, x^{(1)}, \dots, x^{(k-2)}$, selected from the set from part (2). Restrictions are $b < c, b < d$, and $b < x^{(1)} < \dots < x^{(k-2)}$. So, it is equivalent to filling numbers from $[k+1]$ in to the following poset:



It is not hard to show that the number of way to fill numbers from $[k+1]$ to the poset is $k(k-1)$.

Therefore, in total, we have $G_{k,2,2} = \frac{t}{1-t} \cdot \frac{t^{k+1}}{(1-3t)^{k+2}} \cdot k(k-1) = \frac{t^{k+2}k(k-1)}{(1-t)(1-3t)^{k+2}}$ □

In fact, the proof above works for $m \geq 3$. Thus, we have the following result:

Theorem 4.6.

$$W^{(k)}_{m,\tau,\underbrace{2,2,\ldots,2}_{m}}(t) = \frac{t^{k+m}(m+k-2)(m+k-3)\ldots(k-1)}{(1-t)(1-t)^{k+m}}$$

.

Below are series expansion of $W^{(k)}_{2,\tau}(t)$ for the first few k's:

$$\begin{aligned}
W^{(0)}_{2,\tau}(t) &= 1 + t + t^2 + t^3 + t^4 + t^5 + t^6 + \ldots \\
W^{(1)}_{2,\tau}(t) &= t + 4t^2 + 13t^3 + 40t^4 + 121t^5 + 364t^6 + 1093t^7 + \ldots \\
W^{(2)}_{2,\tau}(t) &= t^2 + 9t^3 + 56t^4 + 296t^5 + 1421t^6 + 6389t^7 + 27368t^8 + \ldots \\
W^{(3)}_{2,\tau}(t) &= t^3 + 16t^4 + 160t^5 + 1266t^6 + 8635t^7 + 53008t^8 + 300472t^9 + \ldots \\
W^{(4)}_{2,\tau}(t) &= t^4 + 25t^5 + 365t^6 + 4011t^7 + 36586t^8 + 291636t^9 + 2096562t^{10} + \ldots \\
W^{(5)}_{2,\tau}(t) &= t^5 + 36t^6 + 721t^7 + 10472t^8 + 122658t^9 + 1227036t^{10} + 10865080t^{11} + \ldots
\end{aligned}$$

Case $\tau = 1_1/2_2/\mathbf{3}$ and $1_2/2_1/\mathbf{3}$

First, it is easy to see that the pattern $1_1/2_2/\mathbf{3}$ is equivalent to the pattern $1_2/2_1/\mathbf{3}$. Replacing the color i by $m+1-i$ is a bijection between $1_1/2_2/\mathbf{3}$-avoiders to $1_2/2_1/\mathbf{3}$-avoiders. We will only focus on the pattern $\tau = 1_1/2_2/\mathbf{3}$.

In this case, we do not know how the write a recursive formula for general m. Assume that $m = 2$. Let $W_{2,\tau,s}(n,k)$ denote the number set of Dowling set partition with 2 colors avoiding τ such that the the first set which color 1 appear is the s-th set , and $W_{2,\tau,0}(n,k)$ is for the case when there is no color 1. Then, we can write a recursive formula for $W_{2,\tau,0}(n,k)$ as

$$W_{2,\tau,0}(n,k) = W_{2,\tau,0}(n-1,k) + kW_{2,\tau,0}(n,k) + W_{2,\tau,0}(n,k-1)$$

by consindering positions of n.

For $W_{2,\tau,s}$ where $s \geq 1$, if n is in the last part, we have $W_{2,\tau,s}(n-1,k)$. If $\{n\}$ is a part, we have $W_{2,\tau,s}(n-1,k-1)$ provided that $s < k$.

If n has color 1, then there are two subcases. n could be the only element with color 1, which contributes $W_{2,\tau,0}(n-1,k)$. n could be the only element with color 1 in the s^{th} set, but there is other element with color 1 after n. This case gives $\sum_{t=s+1}^{k} W_{2,\tau,t}(n-1,k)$. If n has color 1 and there is other element in the s^{th} set with color 1, we have $(k-(s-1))W_{2,\tau,s}(n-1,k)$.

If n has color 2, then we have $sW_{2,\tau,s}(n-1,k)$. In total, we have a recursive formula for $W_{2,\tau,s}(n,k)$:

$$\begin{aligned}
W_{2,\tau,s}(n,k) &= W_{2,\tau,s}(n-1,k) + W_{2,\tau,s}(n-1,k-1)\chi(s<k) \\
&\quad + W_{2,\tau,s}(n-1,0) + \sum_{t=s+1}^{k} W_{2,\tau,t}(n-1,k) \\
&\quad + (k-(s-1))W_{2,\tau,s}(n-1,k) + sW_{2,\tau,s}(n-1,k) \\
&= (k+2)W_{2,\tau,s}(n-1,k) + W_{2,\tau,s}(n-1,k-1)\chi(s<k) \\
&\quad + \sum_{t=s+1}^{k} W_{2,\tau,s}(n-1,k) + W_{2,\tau,0}(n-1,k).
\end{aligned}$$

Let $W^{(k)}_{2,\tau,s}(t) = \sum_{n\geq k} W_{2,\tau,s}(n,k)t^n$. Then, with recursive formula above, we could calculate the following:

$$\begin{aligned}
W^{(0)}_{2,\tau,0}(t) &= \frac{1}{1-t} \\
W^{(k)}_{2,\tau,0}(t) &= \frac{1}{1-t}\prod_{j=2}^{k+1}\frac{t}{1-jt} \\
W^{(k)}_{2,\tau,k}(t) &= \frac{1}{1-t}\prod_{j=2}^{k+2}\frac{t}{1-jt}.
\end{aligned}$$

We can apply the same strategy to the case when $m=3$. However, the recursive formula become much more complicated. Let $W^{(k)}_{3,\tau,s_1,s_2}(n,k)$ denote the number of set of Dowling set partition with 3 colors avoiding τ such that the first set which color 1 apears is the s_1-th set, and the first set wich color 2 appear is the s_2-set. We can derive a recursive formula for $W^{(k)}_{3,\tau,s_1,s_2}(n,k)$, but the formula will depend on relative positions of s_1 and s_2. For example, if $0 < s_1 < s_2$, then n cannot has color 2, which makes a recursive formula looks like

$$
\begin{aligned}
W_{3,\tau,s_1,s_2}(n,k) &= (k+2)W_{3,\tau,s_1,s_2}(n-1,k) \\
&\quad + W_{3,\tau,s_1,s_2}(n-1,k-1)\chi(s_1 \neq k \text{ and } s_2 \neq k) \\
&\quad + \sum_{l=s_1+1}^{k} W_{3,\tau,l,s_2}(n-1,k) \\
&\quad + W_{3,\tau,0,s_2}(n-1,k).
\end{aligned}
$$

However, if $s_1 \geq s_2 > 0$, then n could have color 2, which makes a recursive formula looks slightly different:

$$
\begin{aligned}
W_{3,\tau,s_1,s_2}(n,k) &= (k+3)W_{3,\tau,s_1,s_2}(n-1,k) \\
&\quad + W_{3,\tau,s_1,s_2}(n-1,k-1)\chi(s_1 \neq k \text{ and } s_2 \neq k) \\
&\quad + \sum_{l=s_1+1}^{k} W_{3,\tau,l,s_2}(n-1,k) \\
&\quad + \sum_{l=s_2+1}^{k} W_{3,\tau,s_1,l}(n-1,k) \\
&\quad + W_{3,\tau,0,s_2}(n-1,k) + W_{3,\tau,s_1,0}(n-1,k).
\end{aligned}
$$

The recursive formulas are complicated and depend on relative order of s_1, s_2, which make computations even more complicated. Thus, we decided not to move forward in this case using this method.

Case $\tau = 1_1\ 2_1/\mathbf{3}$

In this case, we need some refinement in order to write a recursive formula.

Let $\mathcal{W}_{m,\tau,c_1,\ldots,c_k}(n,k)$ be the set of Dowling set partition avoiding $1_1 2_1/\mathbf{3}$ such that the i-th part has c_i elements. If n is in the last part, then n cannot influence the pattern τ, so this case contributes $W_{m,\tau,c_1,\dots,c_k}(n-1,k)$. If $\{n\}$ is a part, then n cannot influence the pattern τ, so this case contributes $W_{m,\tau,c_1,\dots,c_{k-1}}(n-1,k-1)$ given that $c_k = 1$. If n has a color in the i^{th} part, then there are $W_{m,\tau,c_1,\dots,c_{i-1},c_i-1,c_{i+1},\dots,c_k}(n-1,k)$ set partitions to start with and there are $m-(c_i-2)$ colors for n. So, we have a recursive formula as follows:

$$\begin{aligned} W_{m,\tau,c_1,\dots,c_k}(n,k) &= W_{m,\tau,c_1,\dots,c_k}(n-1,k) \\ &\quad + W_{m,\tau,c_1,\dots,c_{k-1}}(n-1,k-1)\chi(c_k=1) \\ &\quad + \sum_{i=1}^{k}(m-(c_i-2))W_{m,\tau,c_1,\dots,c_{i-1},c_i-1,c_{i+1},\dots,c_k}(n-1,k)\chi(c_i \geq 2). \end{aligned}$$

Below are tables of $W_{m,\tau}(n,k)$ for small n,k.

Case $\tau = 1_1\ 2_2/3$ and $1_2\ 2_1/3$

In this case, we need a refinement in order to write a recursive formula.

Let $\mathcal{W}_{m,\tau,c_1,\dots,c_k}(n,k)$ be the set of Dowling set partition avoiding $1_1 2_2/\mathbf{3}$ such that the i-th part has c_i elements.. If n is in the last part, then n cannot influence the pattern τ, so this case contributes $W_{m,\tau,c_1,\dots,c_k}(n-1,k)$. If $\{n\}$ is a part, then n cannot influence the pattern τ, so this case contributes $W_{m,\tau,c_1,\dots,c_{k-1}}(n-1,k-1)$ provided that $c_k = 1$.

The last case is when n has a color. If A is a set with l elements, then there are $\binom{l+m-2}{m-1}$ ways to color non-minimal elements in A such that if $i<j$ are non-minimal elements in A, then $c(i) > c(j)$. We requrie such coloring in order to avoid τ. Suppose n is in the i-th part, then there are $W_{m,\tau,c_1,\dots,c_{i-1},c_i-1,c_{i+1},\dots,c_k}(n,k)$ Dowling set partition to start with. Then we remove colors in the i-th part, so there are $\frac{1}{\binom{c_i+m-3}{m-1}} W_{m,\tau,c_1,\dots,c_{i-1},c_i-1,c_{i+1},\dots,c_k}(n-1,k)$ Dowling set partition with no color in the i^{th} part. Then we add n and color the i^{th} part with one of $\binom{c_i+m-2}{m-1}$ ways. Therefore, in total, we have the following recursive formula:

Table 4.1: Results for the $|\mathcal{W}_{m,\tau}(n,k)|$ when $\tau = 1_1\ 2_1/3$

$W_{m,\tau}(2,0)$	1
$W_{m,\tau}(2,1)$	$m+2$
$W_{m,\tau}(2,2)$	1
$W_{m,\tau}(3,0)$	1
$W_{m,\tau}(3,1)$	m^2+2m+3
$W_{m,\tau}(3,2)$	$3m+3$
$W_{m,\tau}(3,3)$	1
$W_{m,\tau}(4,0)$	1
$W_{m,\tau}(4,1)$	m^3+m^2+4m+4
$W_{m,\tau}(4,2)$	$7m^2+8m+6$
$W_{m,\tau}(4,3)$	$6m+4$
$W_{m,\tau}(4,4)$	1
$W_{m,\tau}(5,0)$	1
$W_{m,\tau}(5,1)$	$m^4-m^3+6m^2+4m+5$
$W_{m,\tau}(5,2)$	$15m^3+10m^2+20m+10$
$W_{m,\tau}(5,3)$	$25m^2+20m+10$
$W_{m,\tau}(5,4)$	$10m+5$
$W_{m,\tau}(5,5)$	1
$W_{m,\tau}(6,0)$	1
$W_{m,\tau}(6,1)$	$m^5-4m^4+14m^3-9m^2+13m+6$
$W_{m,\tau}(6,2)$	$31m^4-11m^3+61m^2+24m+15$
$W_{m,\tau}(6,3)$	$90m^3+45m^2+60m+20$
$W_{m,\tau}(6,4)$	$65m^2+40m+15$
$W_{m,\tau}(6,5)$	$15m+6$
$W_{m,\tau}(6,6)$	1

$$\begin{aligned} W_{m,\tau,c_1,\ldots,c_k}(n,k) &= W_{m,\tau,c_1,\ldots,c_k}(n-1,k) \\ &\quad + W_{m,\tau,c_1,\ldots,c_{k-1}}(n-1,k-1)\chi(c_k=1) \\ &\quad + \sum_{i=1}^{k} \frac{\binom{c_i+m-2}{m-1}}{\binom{c_i+m-3}{m-1}} W_{m,\tau,c_1,\ldots,c_{i-1},c_i-1,c_{i+1},\ldots,c_k}(n-1,k)\chi(c_i \geq 2). \end{aligned}$$

Below is the table for $W_{m,\tau}(n,k)$ for small n,k.

Table 4.2: Results for the $|\mathcal{W}_{m,\tau}(n,k)|$ when $\tau = 1_1\ 2_2/3$ and $1_2\ 2_1/3$

$W_{m,\tau}(2,0)$	1
$W_{m,\tau}(2,1)$	$m+2$
$W_{m,\tau}(2,2)$	1
$W_{m,\tau}(3,0)$	1
$W_{m,\tau}(3,1)$	$1/2m^2+7/2m+3$
$W_{m,\tau}(3,2)$	$3m+3$
$W_{m,\tau}(3,3)$	1
$W_{m,\tau}(4,0)$	1
$W_{m,\tau}(4,1)$	$1/6m^3+5/2m^2+25/3m+4$
$W_{m,\tau}(4,2)$	$5m^2+14m+6$
$W_{m,\tau}(4,3)$	$6m+4$
$W_{m,\tau}(4,4)$	1
$W_{m,\tau}(5,0)$	1
$W_{m,\tau}(5,1)$	$1/24m^4+13/12m^3+191/24m^2+203/12m+5$
$W_{m,\tau}(5,2)$	$35/6m^3+65/2m^2+125/3m+10$
$W_{m,\tau}(5,3)$	$20m^2+35m+10$
$W_{m,\tau}(5,4)$	$10m+5$
$W_{m,\tau}(5,5)$	1
$W_{m,\tau}(6,0)$	1
$W_{m,\tau}(6,1)$	$1/120m^5+1/3m^4+103/24m^3+62/3m^2+317/10m+6$
$W_{m,\tau}(6,2)$	$21/4m^4+49m^3+521/4m^2+203/2m+15$
$W_{m,\tau}(6,3)$	$95/2m^3+315/2m^2+125m+20$
$W_{m,\tau}(6,4)$	$55m^2+70m+15$
$W_{m,\tau}(6,5)$	$15m+6$
$W_{m,\tau}(6,6)$	1

Case $\tau = 1/2\ \mathbf{3}$ and $2/1\ \mathbf{3}$

Suppose $\tau = 1/2\ \mathbf{3}$, then there must be s such that $A_{k+1} = \{1, 2, \ldots, s, \mathbf{n+1}\}$. Otherwise, τ will appear in the set partition. Once s is fixed, then the number of avoiding Dowling set partition is $S(n-s,k)m^{n-s-k}$. So, the total number is

$$W_{m,\tau}(n,k) = \sum_{s=0}^{n-k} S(n-s,k)m^{n-s-k}$$

.

For $\tau = 2/1\ \mathbf{3}$, it is the opposite of $1/2\ \mathbf{3}$. Smaller elements are in non-last parts. So, the number of pattern avoiding Dowling set partitions is

$$W_{m,\tau}(n,k) = \sum_{s=k}^{n} S(s,k)m^{s-k}$$

.

By substituting s by $n-s$, we have $W_{m,1/2\ \mathbf{3}}(n,k) = W_{m,2/1\ \mathbf{3}}(n,k)$

Case $1_1/2\ \mathbf{3}$

Theorem 4.7. *Let $\tau = 1_1/2\ \mathbf{3}$, then*

$$W_{m,\tau}^{(k)}(t) = t^{k+1} \sum_{l=1}^{k} \left(\frac{1}{1-t}\right)^{l+1} \prod_{j=l}^{k} \frac{1}{1-mjt} + \frac{t^k}{1-t} \prod_{j=1}^{k} \frac{1}{1-mjt}.$$

Proof. We first derive a recursive formula by considering the position of n. If n is in the last part, then there could not be a color element. So, π must be in the form $c_1/c_2/\ldots/c_k/A_{k+1}$. So, the number of such π is the number of choices of $c_1, \ldots, c_k$, which is $\binom{n-1}{k}$. If $\{n\}$ is a part, then n cannot influence the pattern τ, so this case contributes $W_{m,\tau}(n-1,k-1)$. If n has a color, then n cannot influence the pattern τ. There are m colors for n and k parts to add n, so this case contributes $mkW_{m,\tau}(n-1,k)$.

Therefore, we have

$$W_{m,\tau}(n,k) = \binom{n-1}{k} + W_{m,\tau}(n-1,k-1) + mkW_{m,\tau}(n-1,k)$$

which gives a following recursive formula for $W_{m,\tau}^{(k)}(t)$:

$$W_{m,\tau}^{(k)}(t) = \frac{1}{1-mkt}\left(\frac{t}{1-t}\right)^{k+1} + \frac{t}{1-mkt}W_{m,\tau}^{(k-1)}(t)$$

which gives us a closed formula for $W_{m,\tau}^{(k)}(t)$:

$$W_{m,\tau}^{(k)}(t) = t^{k+1}\sum_{l=1}^{k}\left(\frac{1}{1-t}\right)^{l+1}\prod_{j=l}^{k}\frac{1}{1-mjt} + \frac{t^k}{1-t}\prod_{j=1}^{k}\frac{1}{1-mjt}$$

□

Below are the first few terms of $W_{m,\tau}^{(k)}(t)$ for small k:

$$\begin{aligned}
W_{m,\tau}^{(0)}(t) &= 1 + t + t^2 + t^3 + \dots \\
W_{m,\tau}^{(1)}(t) &= t + (2+m)t^2 + (3+2m+m^2)t^3 + (4+3m+2m^2+m^3)t^4 + \dots \\
W_{m,\tau}^{(2)}(t) &= t^2 + (3+3m)t^3 + (6+8m+7m^2)t^4 + (10+15m+18m^2+15m^3)t^5 \dots \\
W_{m,\tau}^{(3)}(t) &= t^3 + (4+6m)t^4 + (10+20m+25m^2)t^5 + (20+45m+78m^2+90m^3)t^6 + \dots
\end{aligned}$$

Case $\tau = 2_1/1\ \mathbf{3}$

Theorem 4.8. *Let* $\tau = 2_1/1\ \mathbf{3}$*, and let* $S_k(t) = \frac{mkt^k}{\prod_{l=1}^{k}(1-lmt)}$ *then*

$$W_{m,\tau}^{(k)}(t) = \frac{t^k}{(1-t)^{k+1}} + \sum_{l=0}^{k}\left(\frac{t}{1-t}\right)^{k+1-l} S_l(t)$$

.

Proof. We first derive a recursive formula by considering the position of n. If n is in the last part, then n cannot influence the pattern τ, so this case contributes $W_{m,\tau}(n-1,k)$. If n is a part by itself, then n cannot influence the pattern τ, so this case contributes $W_{m,\tau}(n-1,k-1)$. If n has a color, there cannot be any elements in the last part except $\mathbf{n+1}$. So, there are $S(n-1,k)$ ways to get a partition of $n-1$ into k parts, k ways to pick a set to put n in, and m^{n-k} ways to color every non-minimal elements. Therefore, we have the following recursive formula

$$W_{\tau,m}(n,k) = W_{\tau,m}(n-1,k) + W_{\tau,m}(n-1,k-1) + km^{n-k}S(n-1,k).$$

which give a recursive formula for $W_{m,\tau}^{(k)}(t)$

$$W_{m,\tau}^{(k)}(t) = \frac{t}{1-t}\left(W_{m,\tau}^{(k-1)}(t) + \frac{mkt^k}{(1-mt)(1-2mt)\dots(1-kmt)}\right).$$

If we let $S_k(t) = \frac{mkt^k}{\prod_{l=1}^{k}(1-lmt)}$, then we have a formula for $W_{m,\tau}^{(k)}(t)$ as follows:

$$W_{m,\tau}^{(k)}(t) = \frac{t^k}{(1-t)^{k+1}} + \sum_{l=0}^{k}\left(\frac{t}{1-t}\right)^{k+1-l} S_l(t).$$

□

Below are the first few terms of $W_{m,\tau}^{(k)}(t)$ for small k:

$$\begin{aligned}
W_{m,\tau}^{(0)}(t) &= 1+t+t^2+t^3+\dots \\
W_{m,\tau}^{(1)}(t) &= t+(2+m)t^2+(3+m+m^2)t^3+(4+m+m^2+m^3)t^4+\dots \\
W_{m,\tau}^{(2)}(t) &= t^2+(3+3m)t^3+(6+4m+7m^2)t^4+(10+5m+8m^2+15m^3)t^5+\dots \\
W_{m,\tau}^{(3)}(t) &= t^3+(4+6m)t^4+(10+10m+25m^2)t^5+(20+15m+33m^2+90m^3)t^6+\dots
\end{aligned}$$

Case $\tau = 1\ 2\ \mathbf{3}$

We will derive an explicit formula for $W_{m,\tau}(n,k)$. In order to avoid $1\ 2\ \mathbf{3}$, the last part can only have 0 or 1 element. If the last part has no element, then there are $S(n,k)$ way to partition n elements to k sets, then there are m^{n-k} ways to color each non-minimal elements. If the last part has 1 element, then there are n ways to pick the element in the last part, $S(n-1,k)$ to partition the remaining $n-1$ elements into k parts, and m^{n-1-k} ways to color non-minimal elements. Therefore, we have a formula for $W_{m,\tau}(n,k)$

$$W_{m,\tau}(n,k) = S(n,k)m^{n-k} + nS(n-1,k)m^{n-1-k}.$$

4.4 Ordered case

This section concerns the case where the set partition *may not* be ordered by the minimal elements. We explored all the cases, and computed the generating function, recursive formula, or enumerate the first few terms.

Like the unordered case, for any pattern τ, let $OW_{m,\tau}^{(k)}(t) = \sum_{n\geq k} OW_{m,\tau}(n,k)t^n$.

We simplify the computation by introducing trivial Wilf-equivalence in the ordered case. For any pattern $\tau = \tau_1/\dots/\tau_k/\tau_{k+1}$, let $\tau^r = \tau_k/\dots/\tau_1/\tau_{k+1}$ be a reverse of τ, and τ^c be a pattern obtained from τ by replacing a color c by a color $m+1-c$. Let $A = A_1/\dots/A_k/A_{k+1}$ be . Let $A^r = A_k/\dots/A_1/A_{k+1}$ and A^c be a ordered Dowling set partition obtained from A by replacing a color c by $m+1-c$. It is easy to see that the map $A \mapsto A^r$ is a bijection between $\mathcal{OW}_{m,\tau}(n,k)$ and $\mathcal{OW}_{m,\tau^r}(n,k)$, and the map $A \mapsto A^c$ is a bijection between $\mathcal{OW}_{m,\tau}(n,k)$ and $\mathcal{OW}_{m,\tau^c}(n,k)$. We proved the following lemma:

Lemma 4.1.

$$OW_{m,\tau}(n,k) = OW_{m,\tau^r}(n,k) = OW_{m,\tau^c}(n,k)$$

.

Case $\tau = 1/2/\mathbf{3}$ and $2/1/\mathbf{3}$

The pattern $1/2/\mathbf{3}$ and $2/1/\mathbf{3}$ are equivalent by lemma 4.1. We will only focus on $\tau = 2/1/\mathbf{3}$.

In order for π to avoid $\tau = 2/1/\mathbf{3}$, non-last parts must be ordered by their minimal elements, from smallest to largest, which is precisely the description of the unordered Dowling set partition. So $OW_{m,\tau}(n,k) = W_m(n,k)$.

Case $\tau = 1/2_1/\mathbf{3}$ and $2_1/1/\mathbf{3}$

Theorem 4.9. *Let $\tau = 1/2_1/\mathbf{3}$ or $2_1/1/\mathbf{3}$, then*

$$OW^{(k)}_{m,\tau}(t) = \frac{k!t^k}{(1-t)(1-(1+m)t)^k}.$$

First, the pattern $1/2_1/\mathbf{3}$ and the pattern $2_1/1/\mathbf{3}$ are equivalent by lemma 4.1. We will only forcus on the pattern $\tau = 1/2_1/\mathbf{3}$.

Consider $\tau = 1/2_1/\mathbf{3}$. We will prove the the formula for the generating function in two ways: using recursive formula, and direct counting. We derive a recursive formula for $OW_{m,\tau}(n,k)$ by considering the position of n. If n is in the last part, then n cannot influence the pattern τ, so this case contributes $OW_{m,\tau}(n-1,k)$. If $\{n\}$ is a part, then n cannot influence the pattern τ. There are $OW_{m,\tau}(n-1,k-1)$ ordered Dowling set partitions to start with, k choices for the position of $\{n\}$. Thus, this case contributes $kOW_{m,\tau}(n-1,k-1)$. If n has a color, then n has to be in the first part, and there are m choices for the color of n, so this case contributes $mW_{m,\tau}(n-1,k)$.

Thus, we have

$$OW_{m,\tau}(n,k) = (1+m)OW_{m,\tau}(n-1,k) + kOW_{m,\tau}(n-1,k-1)$$

which gives us a recursive formula for generating functions:

$$OW^{(k)}_{m,\tau}(t) = \frac{kt}{(1-(1+m)t)} OW^{(k-1)}_{m,\tau}(t)$$

which gives the formula for $OW^{(k)}_{m,\tau}(t)$:

$$OW^{(k)}_{m,\tau}(t) = \frac{1}{1-t} \cdot \frac{k!t^k}{(1-(1+m)t)^k}.$$

We can also count $OW_{m,\tau}(n,k)$ directly by counting the number of ways we can construct such Dowling set partition. Suppose $\pi = A_1/A_2/\dots/A_k/A_{k+1} \in \mathcal{OW}_{m,\tau}(n,k)$. Let a_i be the minimal element in A_i, then for any $b \in [n] - \{a_1,\dots,a_k\}$, b is either in the last part, or $b \in A_i$ and $b > a_j$ for any $j < i$. That is, scanning from left to right, b must be in the first set with a minimal element smaller than b. Otherwise, if $b \in A_i$, and $b > a_j$ for some $j < i$, then $a_j, b, \mathbf{n}+1$ will form $1/2_1/\mathbf{3}$. Therefore, we can construct $\pi \in \mathcal{OW}_{m,\tau}(n,k)$ as follows:

1. Pick $a \in \{1,2,\dots,n-k+1\}$ to be the smallest minimal elements.

2. Pick $a_2, a_3, \ldots, a_k \in \{a+1, \ldots, n\}$ to be the rest of the minimal elements.

3. Arrange $a, a_2, \ldots, a_k$.

4. For each $b \in \{1, \ldots, a-1\}$, b must be in the last part.

5. For each $b \in \{a, \ldots, n\} - \{a, a_2, \ldots\}$, there are $1+m$ choices, either be in the last part, or have one of m colors and located at the first set with the minimal element smaller than b.

So, the number of way to construct π is $\sum_{a=1}^{n-k+1} k!\binom{n-a}{k-1}(1+m)^{n-a+1-k} = \frac{k!t^k}{(1-t)(1-mt)^k}\Big|_{t^n}$

Case $\tau = 1_1/2/\mathbf{3}$ and $2/1_1/\mathbf{3}$

The pattern $1_1/2/\mathbf{3}$ and the pattern $2/1_1/\mathbf{3}$ are equivalent by lemma 4.1. We will only forcus on the pattern $\tau = 1_1/2/\mathbf{3}$.

We derive a recursive formula for $OW_{m,\tau}(n,k)$. If n is in the last part, then n cannot influence the pattern τ, so this case contributes $OW_{m,\tau}(n-1,k)$. if $\{n\}$ is a part, then any part before n cannot have a color element, otherwise it will produce $1_1/2/\mathbf{3}$. So, if $\pi \in \mathcal{O}W_{m,\tau}(n,k)$ with $\{n\}$ is a part, then π must be in the form

$$\pi = a_1/a_2/\ldots/a_l/n/A_{l+1}/\ldots/A_k/A_{k+1}$$

so it contributes $\sum_{l=0}^{k-1}\binom{n-1}{l} l! OW_{m,\tau}(n-1-l, k-1-l)$. If n has a color, there are $OW_{m,\tau}(n-1,k)$ ordered Dowling set partitions to start with, k choices for the position of n, and m choices for the color of n. Therefore, we have a recursive formula for $OW_{m,\tau}(n,k)$:

$$OW_{m,\tau}(n,k) = (1+mk)OW_{m,\tau}(n-1,k) + \sum_{l=0}^{k-1}\binom{n-1}{l} l! OW_{m,\tau}(n-1-l, k-1-l).$$

Let $EOW^{(k)}_{m,\tau}(t) = \sum_{n\geq k} OW_{m,\tau}(n,k)\frac{t^k}{k!}$ denote the exponential generating function for $OW_{m,tau}(n,k)$. Then a recursive formula for exponential generating functions can be derived as follow:

$$\frac{\partial}{\partial t}EOW^{(k)}_{m,\tau}(t) = (1+mk)EOW^{(k)}_{m,\tau}(t) + \sum_{l=1}^{k-1} t^l EOW^{(k-1-l)}_{m,\tau}(t).$$

We let $H_\tau = \sum_{k\geq 0} EOW^{(k)}_{m,\tau}(t)z^k$, then we have the following differential equation of H_τ:

$$\frac{\partial}{\partial t}H_\tau = H_\tau + mz\frac{\partial}{\partial z}H_\tau + \frac{z}{1-tz}H_\tau.$$

Case $\tau = 1_1/2_1/\mathbf{3}$ and $2_1/1_1/\mathbf{3}$

First, two patterns are equivalent by lemma 4.1. We will focus on the case when $\tau = 1_1/2_1/\mathbf{3}$.

In this case, we need a refinement in order to derive a recursive formula.

Let $OW_{m,\tau,\alpha_1,\ldots,\alpha_m}(n,k)$ be the number of ordered Dowling set partition avoiding τ so that the first set that color i appears is the α_i-th set. If there is no element with color i, then $\alpha_i = 0$. If n is in the last part, n cannot influence the pattern τ, so this case contribues $OW_{m,\tau,\alpha_1,\ldots,\alpha_m}(n-1,k)$. If $\{n\}$ is the j-th part, then n cannot influence the pattern τ. We define $\alpha_{i,j}$ as

$$\alpha_{i,j} = \begin{cases} \alpha_i & \text{for } j > \alpha_i \\ \alpha_i - 1 & \text{for } j < \alpha_i. \end{cases}$$

Then, this case contributes $OW_{m,\tau,\alpha_{1,j},\ldots,\alpha_{m,j}}(n-1,k-1)$ given that non of α_i is equal to j for all i. If n has a color i, then n has to be in the first set containing color i, which is the α_i-th set. Removing n would make the first set with color i become $\alpha_i, \alpha_i+1, \ldots, k$ or contain no color i at all. This case will contribute

$$OW_{m,\tau,\alpha_1,\ldots,\alpha_{i-1},0,\alpha_{i+1},\alpha_m}(n-1,k) + \sum_{j=\alpha_i}^{k} OW_{m,\tau,\alpha_1,\ldots,\alpha_{i-1},j,\alpha_{i+1},\alpha_m}(n-1,k).$$

We derive a recursive formula for $OW_{m,\tau,\alpha_1,\ldots,\alpha_m}(n,k)$ as follows:

$$\begin{aligned} OW_{m,\tau,\alpha_1,\ldots,\alpha_m}(n,k) &= OW_{m,\tau,\alpha_1,\ldots,\alpha_m}(n-1,k) \\ &+ \sum_{j=1}^{k} OW_{m,\tau,\alpha_{1,j},\ldots,\alpha_{m,j}}(n-1,k-1)\chi(\alpha_i \neq j \text{ for all } i) \\ &+ \sum_{i=0}^{m} OW_{m,\tau,\alpha_1,\ldots,\alpha_{i-1},0,\alpha_{i+1},\alpha_m}(n-1,k) \\ &+ \sum_{i=0}^{m} \sum_{j=\alpha_i}^{k} OW_{m,\tau,\alpha_1,\ldots,\alpha_{i-1},j,\alpha_{i+1},\alpha_m}(n-1,k). \end{aligned}$$

Case $\tau = 1_1/2_2/\mathbf{3}$, $1_2/2_1/\mathbf{3}$, $2_1/1_2/\mathbf{3}$ and $2_2/1_1/\mathbf{3}$

First, four patterns mentioned are equivalent by lemma 4.1. We will only focus on the pattern $1_1/2_2/\mathbf{3}$.

In this case, we do not know how to derive the generating function in general. Consider the case when $m = 2$. Let $OW_{2,\tau,s}(n,k)$ be the number of ordered Dowling set partitions with 2 colors avoiding τ such that the first set that color 1 appear is the s-th set, and $OW_{2,\tau,0}(n,k)$ is for the case when no elements have color 1. By considering the position of n, we can write a recursive formula for $OW_{2,\tau,0}(n,k)$ as

$$OW_{m,\tau,0}(n,k) = (1+k)OW_{m,\tau,0}(n-1,k) + kOW_{m,\tau,0}(n-1,k-1).$$

For $OW_{2,\tau,s}(n,k)$ where $s \geq 1$, if n is in the last part, then n cannot influence the pattern τ, so this case contributes $OW_{2,\tau,s}(n-1,k)$. If $\{n\}$ is the r-th part, then n cannot influence the pattern τ. The contribution in this case would depend on r. If $r < s$, we have $OW_{2,\tau,s-1}(n-1,k-1)$. For $r > s$, we have $OW_{2,\tau,s}(n-1,k-1)$

If n has color 1, then n cannot influence the pattern τ. Removing n will potentially change the value of the variable s. If n is the only element with color 1, it will contribute $OW_{2,\tau,0}(n-1,k)$. If n is the only element with color 1 in the s-th set and there is other element with color 1 after n, it will contribute $\sum_{t=s+1}^{k} OW_{2,\tau,t}(n-1,k)$. If n has color 1 and there is other element in the s-th set with color 1, we have $(k-(s-1))OW_{2,\tau,s}(n-1,k)$.

If n has color 2, then n cannot appear after any part with color 1, which means that n has to be in the first s sets. So, this case contributes $sOW_{2,\tau,s}(n-1,k)$. In total, we have a recursive formula for $OW_{2,\tau,s}(n,k)$:

$$
\begin{aligned}
OW_{m,\tau,s}(n,k) &= (k+2)OW_{m,\tau,s}(n-1,k) \\
&= (s-1)OW_{m,\tau,s-1}(n-1,k-1) + (k-s)OW_{m,\tau,s}(n-1,k-1) \\
&\quad + \left(\sum_{t=s+1}^{k} OW_{m,\tau,t}(n-1,k)\right) + OW_{m,\tau,0}(n-1,k).
\end{aligned}
$$

Case $\tau = 1_1\ 2_1/\mathbf{3}$

In this case, we need some a refinement in order to write a recursive formula.

Let $OW_{m,\tau,c_1,\ldots,c_k}(n,k)$ be the number of ordered Dowling set partition avoiding $1_1 2_1/\mathbf{3}$ such that the i-th part has c_i elements. If n is in the last part, then n cannot influence the pattern τ, so this case contributes $OW_{m,\tau,c_1,\ldots,c_k}(n-1,k)$. If $\{n\}$ is a part, then n cannot influence the pattern τ. $\{n\}$ could be any part l as long as $c_l = 1$, so this case contributes $\sum_{l=1}^{k} OW_{m,\tau,c_1,\ldots,\hat{c}_l,\ldots,c_k}(n-1,k-1)\chi(c_l = 1)$. If n has a color in the i-th part, then there are $W_{m,\tau,c_1,\ldots,c_{i-1},c_i-1,c_{i+1},\ldots,c_k}(n-1,k)$ set partitions to begin with and there are $m-(c_i-2)$ colors for n. So, we have a recursive formula as follows:

$$\begin{aligned} OW_{m,\tau,c_1,\ldots,c_k}(n,k) &= OW_{m,\tau,c_1,\ldots,c_k}(n-1,k) \\ &+ \sum_{l=1}^{k} OW_{m,\tau,c_1,\ldots,\hat{c}_l,\ldots,c_k}(n-1,k-1)\chi(c_l=1) \\ &+ \sum_{i=1}^{k}(m-(c_i-2))OW_{m,\tau,c_1,\ldots,c_{i-1},c_i-1,c_{i+1},\ldots,c_k}(n-1,k)\chi(c_i \geq 2). \end{aligned}$$

Case $1_1\ 2_2/\mathbf{3}$ and $1_2\ 2_1/\mathbf{3}$

First, pattern $1_1\ 2_2/\mathbf{3}$ and pattern $1_2\ 2_1/\mathbf{3}$ are equivalent by lemma 4.1.

In this case, we need a refinement in order to write a recursive formula.

Let $\mathcal{O}W_{m,\tau,(c_1,\ldots,c_k)}(n,k)$ be the number of ordered Dowling set partition avoiding $1_1 2_2/\mathbf{3}$ such that the i^{th} part has c_i elements. If n is in the last part, then n cannot influence the pattern τ, so this case contributes $OW_{m,\tau,c_1,\ldots,c_k}(n-1,k)$. If $\{n\}$ is a part, then n cannot influence the pattern τ. $\{n\}$ could be any part l as long as $c_l=1$, so this case contributes $\sum_{l=1}^{k} OW_{m,\tau,(c_1,\ldots,\hat{c}_l,\ldots,c_k)}(n-1,k-1)\chi(c_l=1)$.

The last case is when n has a color. If A is a set with l elements, then there are $\binom{l+m-2}{m-1}$ ways to color non-minimal elements in A such that if $i<j$ are non-minimal elements in A, then $c(i)>c(j)$. We requrie such coloring in order to avoid τ. Suppose n is in the i^{th} part, then there are $OW_{m,\tau,c_1,\ldots,c_{i-1},c_i-1,c_{i+1},\ldots,c_k}$ ordered Dowling set partition to start with. Then we remove color in the i^{th} part, so there are $\frac{1}{\binom{c_i+m-3}{m-1}}OW_{m,\tau,c_1,\ldots,c_{i-1},c_i-1,c_{i+1},\ldots,c_k}(n-1,k)$ Dowling set partition with no color in the i^{th} part. Then we add n and color the i^{th} part with one of $\binom{c_i+m-2}{m-1}$ ways. Therefore, in total, we have the following recursive formula:

$$\begin{aligned} OW_{m,\tau,(c_1,\ldots,c_k)}(n,k) &= OW_{m,\tau,(c_1,\ldots,c_k)}(n-1,k) \\ &+ \sum_{l=1}^{k} OW_{m,\tau,(c_1,\ldots,\hat{c}_l,\ldots,c_k)}(n-1,k-1)\chi(c_l=1) \\ &+ \sum_{i=1}^{k} \frac{\binom{c_i+m-2}{m-1}}{\binom{c_i+m-3}{m-1}} OW_{m,\tau,(c_1,\ldots,c_{i-1},c_i-1,c_{i+1},\ldots,c_k)}(n-1,k)\chi(c_i \geq 2). \end{aligned}$$

Case $\tau = 1/2\ \mathbf{3}$ and $\tau = 2/1\ \mathbf{3}$

For $\tau = 1/2\ \mathbf{3}$, there must be an element s such that the last part is $\{1, 2, \ldots, s, \mathbf{n+1}\}$. We have the following formula for $OW_{m,\tau}(n,k)$:

$$OW_{m,\tau}(n,k) = \sum_{s=0}^{n-k} S(n-s,k)k!m^{n-s-k},$$

and for $\tau = 2/1\ \mathbf{3}$, we must have larger elements be in the last part. We have to following formula for $OW_{m,\tau}(n,k)$

$$OW_{m,\tau}(n,k) = \sum_{s=k}^{n} S(s,k)k!m^{s-k},$$

and $OW_{m,1/2\ \mathbf{3}}(n,k) = OW_{m,2/1\ \mathbf{3}}(n,k)$.

Case $\tau = 1_1/2\ \mathbf{3}$

Theorem 4.10. *Let $\tau = 1_2/2\ \mathbf{3}$, then*

$$OW^{(k)}_{m,\tau}(t) = k!t^{k+1} \sum_{l=1}^{k} \left(\frac{1}{1-t}\right)^{l+1} \prod_{j=l}^{k} \frac{1}{1-mjt} + k!\frac{t^k}{1-t} \prod_{j=1}^{k} \frac{1}{1-mjt}.$$

Proof. We first derive a recursive formula by considering the position of n. If n is in the last part, then there could not be a color element. So, π must be in the form $c_1/c_2/\ldots/c_k/A_{k+1}$. So, the number of such π is the number of choices of $c_1,\ldots,c_k$, which is $k!\binom{n-1}{k}$. If $\{n\}$ is a part, then n cannot influence the pattern τ, so this case contributes $kOW_{m,\tau}(n-1,k-1)$. If n has a color, then n cannot influence the pattern τ. There are m colors for n and k sets to add n, so this case contributes $mkOW_{m,\tau}(n-1,k)$.

Thus, we have

$$OW_{m,\tau}(n,k) = k!\binom{n-1}{k} + kOW_{m,\tau}(n-1,k-1) + mkOW_{m,\tau}(n-1,k)$$

which gives a following recursive formula for $OW^{(k)}_{m,\tau}(t)$:

$$OW^{(k)}_{m,\tau}(t) = \frac{k!}{1-mkt}\left(\frac{t}{1-t}\right)^{k+1} + \frac{kt}{1-mkt}OW^{(k-1)}_{m,\tau}(t)$$

which give us a closed formula for $W^{(k)}_{m,\tau}(t)$:

$$OW^{(k)}_{m,\tau}(t) = k!t^{k+1}\sum_{l=1}^{k}\left(\frac{1}{1-t}\right)^{l+1}\prod_{j=l}^{k}\frac{1}{1-mjt} + k!\frac{t^k}{1-t}\prod_{j=1}^{k}\frac{1}{1-mjt}.$$

□

Case $\tau = 2_1/1\ \mathbf{3}$

Theorem 4.11. *Let* $\tau = 2_1/1\ \mathbf{3}$, *and let* $S_k(t) = \frac{mkt^k}{\prod_{l=1}^{k}(1-lmt)}$, *then*

$$OW^{(k)}_{m,\tau}(t) = k!\left(\frac{t^k}{(1-t)^{k+1}} + \sum_{l=1}^{k}\left(\frac{t}{1-t}\right)^{k+1-l}S_l(t)\right).$$

Proof. We first derive a recursive formula for $OW_{m,\tau}(n,k)$. If n is in the last part, then n cannot influence the pattern τ, so this case contributes $OW_{m,\tau}(n-1,k)$. If $\{n\}$ is a part, then n cannot influence the pattern τ, so this case contributes $kOW_{m,\tau}(n-1,k-1)$. If n has a color, then there could be no elements in the last part except $\mathbf{n+1}$. There are k positions for n, and $k!m^{n-k}S(n-1,k)$ to rearrange the rest $n-1$ numbers and color them. In total we have

$$OW_{m,\tau}(n,k) = OW_{m,\tau}(n-1,k) + kOW_{m,\tau}(n-1,k-1) + S(n-1,k)kk!m^{n-k}$$

which give a recursive formula for $OW^{(k)}_{m,\tau}(t)$

$$OW^{(k)}_{m,\tau}(t) = \frac{t}{1-t}\left(kOW^{(k-1)}_{m,\tau}(t) + \frac{k!mt^k}{(1-mt)(1-2mt)\dots(1-kmt)}\right).$$

If we let $S_k(t) = \frac{mkt^k}{\prod_{l=1}^{k}(1-lmt)}$, then we have a formula for $OW^{(k)}_{m,\tau}(t)$ as follows:

$$OW^{(k)}_{m,\tau}(t) = k!\left(\frac{t^k}{(1-t)^{k+1}} + \sum_{l=1}^{k}\left(\frac{t}{1-t}\right)^{k+1-l} S_l(t)\right).$$

□

Case $\tau = 1\ 2\ \mathbf{3}$

We will derive an explicit formula for $OW_{m,\tau}(n,k)$. In order to avoid $1\ 2\ \mathbf{3}$, the last part can only have 0 or 1 element. If the last part has no element, then there are $S(n,k)$ ways to partition n elements to k sets, $k!$ ways to arrange k parts, and m^{n-k} ways to color each non-minimal elements. If the last part has 1 element, then there are n ways to pick the element in the last part, $S(n-1,k)$ to partition the remaining $n-1$ elements into k parts, $k!$ ways to arrange k parts, and m^{n-1-k} ways to color non-minimal elements.

Therefore, we have a formula for $OW_{m,\tau}(n,k)$

$$OW_{m,\tau}(n,k) = S(n,k)k!m^{n-k} + nS(n-1,k)k!m^{n-1-k}.$$

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