Title
A Vector Field Method for Non-Trapping Spacetimes

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Publication Date
2013

Peer reviewed|Thesis/dissertation

UNIVERSITY OF CALIFORNIA, SAN DIEGO

A Vector Field Method for Non-Trapping Spacetimes

A dissertation submitted in partial satisfaction of the
requirements for the degree
Doctor of Philosophy

in

Mathematics

by

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2013

The dissertation of Jesús Oliver is approved, and it is acceptable in quality and form for publication on microfilm and electronically:

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Chair

University of California, San Diego

2013

DEDICATION

To the memory of my father Luis Oliver and my grandfather Rafael Bartolozzi.

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ACKNOWLEDGEMENTS

Many thanks go to my advisor Jacob Sterbenz who suggested the problem and who has patiently and generously provided me with many valuable insights thoroughout the years. Your knowledge and your enthusiasm are an inspiration to me.

I thank my mom, Olivia, for having the courage to bring me and my brother into this country in order to give us a chance at a better future. Your example, your love and your faith in me are the foundation upon which my success is built.

I thank my family – Venezuelan, Mexican and American – for your unconditional love and support. I am very lucky to have you in my life and I will always be grateful for all you've done for me.

I thank my wife, Taylor who has been my biggest supporter, my best friend and my greatest source of inspiration ever since we've been together. Thank you for your patience, your kindness and your help; I couldn't have done this without you. I love you always and forever.

VITA

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ABSTRACT OF THE DISSERTATION

A Vector Field Method for Non-Trapping Spacetimes

by

Jesús Oliver

Doctor of Philosophy in Mathematics

University of California, San Diego, 2013

Professor Jacob Sterbenz, Chair

Assuming a local smoothing estimate we prove that sufficiently regular solutions to the wave equation $\Box_g \phi = F(t,x)$ on radiating, non-trapping, time-dependent, 4-dimensional space-times satisfy a conformal energy estimate and higher order conformal energy estimate with vector fields. We also establish a global pointwise decay estimate of the form $|\phi(t,x)| \lesssim \frac{1}{\langle t+r \rangle \langle u \rangle^{\frac{1}{2}}}$ for sufficiently regular solutions to the free wave equation.

Chapter 1

Introduction

In this work we initiate the study of the global pointwise decay properties of solutions to the inhomogeneous linear wave equation in time-dependent, radiating spacetimes. Let $(\mathcal{M}, g_{\alpha\beta})$ be a 4-dimensional, smooth, asymptotically flat Lorentzian manifold with non-trapping metric g. Let η denote the Minkowski metric $diag[-1,1,1,1]$. By *asymptotically flat* we mean an a priori assignment of decay rates for the difference $(g-\eta)$ and its derivatives of all orders as $|x| \to \infty$. By non-trapping we mean that there do not exist null geodesics that are confined to compact regions of $\mathcal{M}$ for all time. We also assume that this spacetime is of the form $\mathbb{R}\times\mathbb{R}^3$ and that there exists a globally defined set of coordinates (t, x^i) such that the level sets of t are space-like. Let $\Box_g$ be the *Laplace-Beltrami* operator associated to this metric, given in local coordinates by $\Box_g = \frac{1}{\sqrt{|g|}}\partial_\alpha(g^{\alpha\beta}\sqrt{|g|}\partial_\beta)$. We study solutions to the wave equation:

$$\Box_g\phi = F(t,x) , \qquad \phi(0,x) = \phi_0(x) , \partial_t\phi(0,x) = \phi_1(x) , \tag{1.1}$$

with $(\phi_0, \phi_1) \in C_c^\infty(\mathbb{R}^3)$.

For this initial value problem the case with $F = \mathcal{N}(\phi, \nabla\phi)$ and $\mathcal{N}$ a non-linear function is of great mathematical and physical interest. In large part this is because systems of such non-linear equations encompass essentially all relativistic field theories on isolated backgrounds: Maxwell-Klein-Gordon, Yang-Mills, Maxwell-Dirac, and Wave Maps. For a number of these applications, one of the main open problems is to show a small-data global existence result. This amounts to proving that for sufficiently small, smooth, compactly supported initial data a smooth global solution for the non-linear problem exists for all time. In order to prove this type of result it is crucial in a number of cases to first show that solutions to the linear wave equation in a fixed background, evolving from smooth, compactly supported initial data, decay at rates similar to those in flat space. Once we have established this robust pointwise decay result for the linear wave equation, we may then apply the linear decay to control solutions to some these non-linear problems in fixed backgrounds with small data. We point out that the method of proof for the linear decay has to be robust enough to handle not just stationary metrics but perturbations as well. This is essential in order for this strategy to work for the type of systems of nonlinear wave equations one would see in most interesting applications.

For these reasons, the problem of pointwise decay for the linear wave equation on asymptotically flat backgrounds has seen a tremendous surge in activity and understanding lately. In the last few years, the main focus has been on the black hole metrics – Schwarzschild and Kerr – due to interest in proving the non-linear stability of the Kerr solution in General Relativity. The state-of-the-art for the linear problem in the black hole spacetimes is the work of Metcalfe-Tataru-Tohaneanu [MTT12] and Dafermos-Rodnianski [DR10a]. In the first work, a global pointwise decay estimate is proved for solutions to the linear homogeneous wave equation for a general class of time-dependent perturbations of the slowly rotating Kerr metric $|a| \ll M$ and as well as fast-decaying, time-dependent non-trapping metrics. In [DR10a] a global pointwise decay estimate is proved for solutions to the linear homogeneous wave equation for a general class of stationary perturbations of the slowly rotating Kerr metric $|a| \ll M$ axisymmetric solutions ϕ in the case $|a| < M$. Both of these results build on much earlier work by [BS05], [BS06], [DR09]. In particular, the work on Dafermos-Rodnianski, makes use of the vector field method which was first established by Klainerman in [Kla85]. This robust method has been applied to prove small data global existence results for non-linear problems in a variety of backgrounds: Minkowski [Kla86], Minkowski with obstacles [MS06], and Kerr [Luk10b]. This method also played a significant role in the quintessential work of Christodoulou- Klainerman [CK93] showing the global non-linear stability of the Minkowski space within the framework of the initial value problem for General Relativity. In some sense this result showed, for the first time, that the vector field method is particularly well-suited for attacking problems with a smallness condition on the Ricci curvature – in this case $R_{\alpha\beta} = 0$ due to the Einstein vacuum equations.

The black hole case notwithstanding, in this work we focus on a different but related problem: pointwise linear decay for solutions to (1.1) on radiating, time-dependent, non-trapping spacetimes. The motivation to look at this particular problem is both physical and mathematical. From a physical point of view these space-times are interesting since they correspond to the "far exterior" portion of a dynamic perturbation of a Black Hole space-time. In particular, it is of interest to study the properties of the wave flow with the outgoing conditions on the metric being consistent with the work of Bondi-Sachs (1962) and Christodoulou-Klainerman (1993) on the the radiation of gravitational waves. From a mathematical point of view, we are also interested in how far can we push the decay assumptions on the metric before the pointwise decay rates proved in Minkowski space start to break down for solutions to the linear problem. In particular, as a first step in this direction we would like for our decay assumptions to, at the very least, connect to the weak asymptotic assumptions in recent work of Bieri [BZ09].

Assuming an local integrated energy estimate, in this work we are able to produce a vector field method yielding a pointwise decay result for linear waves analogous to what is available on Minkowski (via such methods) for a general class of radiating, time-dependent, non-trapping space-times satisfying both the radiating conditions and the weak asymptotic assumptions discussed above. We will now give a precise description of the space-times we work with.

1.1 Decay Assumptions on the Metric

There are mainly two regions that need to be considered separately: a sufficiently large compact set and its exterior. In general, what happens outside of a compact set only needs a detailed description in the "wave-zone" $r \sim t$. For our metric the first condition is the following:

Assumption 1.1.1 (The wave-zone is well defined)**.** *In the regions* $\mathcal{I} := \{\frac{1}{10}t < r < \frac{2}{3}t\}$ *and* $\mathcal{F} := \{r > \frac{3}{2}t\}$ *there exists* $\delta > 0$ *such that:*

$$|\partial^\alpha_{t,x}(g - \eta)| \ \lesssim \ \langle t + r\rangle^{-\delta - |\alpha|} \ . \tag{1.2}$$

Assumption 1.1.2 (Decay For the Metric in The Interior Region)**.** *In the interior region* $\mathcal{I}$ *there exists* $\gamma > 0$ *such that the metric* g *satisfies:*

$$|\partial^k_t \partial^J_x (g - \eta)| \ \lesssim \ \langle r\rangle^{-\delta - |J| - k} \langle r/t\rangle^{\gamma k} \ . \tag{1.3}$$

This leaves us the task of describing the metric in the wave zone $\mathcal{W} := \{\frac{2}{3}t < r < \frac{3}{2}t\}$.

Assumption 1.1.3 (Existence of normalized coordinates)**.** *There exists a function* $u(t, x) \in C^\infty(\mathcal{M})$ *satisfying the following properties:*

i) Inside $\mathcal{W}$*,* u *is an optical function – that is,* u *solves the* Eikonal equation*:*

$$g^{\alpha\beta}\partial_\alpha u \partial_\beta u \ = \ 0 \ .$$

ii) $u = t - \langle r\rangle$ *on the set* $\{r \leqslant \frac{t}{2}\} \cup \{2t \leqslant r\}$*.*

iii) In the region $\{\frac{t}{2} \leqslant r \leqslant 2t\}$*, the function* u *is approximately equal to* $t - r$ *in the following sense:*

$$|\partial^J_{t,x}\big(u - (t - r)\big)| \ \lesssim \ \langle r\rangle^{-\delta}\langle u\rangle^{1-|J|} \ , \qquad |J| \geqslant 1 \ .$$

iv) In the wave zone we define the following Lie algebra of vector-fields:

$$\mathbb{L} \ = \ \big\{\widetilde{T} = \frac{1}{u_t}\partial_t \, , \, \widetilde{S} = \frac{u - ru_r}{u_t}\partial_t + r\partial_r \, , \, \widetilde{\Omega_{ij}} = \Omega_{ij} - \frac{\Omega_{ij}u}{u_t}\partial_t\big\} \ , \tag{1.4}$$

then we have the L^∞ *bounds,*

$$|\mathcal{L}^k_T \hat{\mathcal{L}}^l_{\widetilde{S}} \mathcal{L}^J_{\widetilde{\Omega}_{ij}} g| \lesssim \ \langle r\rangle^{-\delta}\langle u\rangle^{-k} \ , \qquad k + l + |J| \geq 1 \ , \tag{1.5}$$

where $\hat{\mathcal{L}}^l_{\widetilde{S}} = \mathcal{L}^l_{\widetilde{S}} - 2Id$ is the trace-free part of the modified scaling field. We also assume the corresponding bound for the metric coefficients:

$$|g_{\alpha\beta} - \eta_{\alpha\beta}| \ \lesssim \ \langle r\rangle^{-\delta} \ , \tag{1.6}$$

where $\eta = diag[-1, 1, 1, 1]$.

v) (Radiation Condition) In an open neighborhood of the wave zone there exists a system of normalized coordinates $y^i = y^i(u, x^j)$ such that the inverse metric has the following symbol properties:

$$|\partial_u^\alpha \widetilde{\partial}_l^\beta (\sqrt{|g|} g^{ui} + \omega^i)| \ \lesssim \ \langle r\rangle^{-\delta-|\beta|} \langle u\rangle^{-|\alpha|} \left(\frac{\langle u\rangle}{\langle r\rangle} \right)^{\frac{1}{2}} \ , \tag{1.7a}$$

$$|\partial_u^\alpha \widetilde{\partial}_l^\beta (g^{ui} - \omega^i g^{ur})| \ \lesssim \ \langle r\rangle^{-\delta-|\beta|} \langle u\rangle^{-|\alpha|} \left(\frac{\langle u\rangle}{\langle r\rangle} \right) \ , \tag{1.7b}$$

with $\widetilde{\partial}_l$ the coordinate derivatives of y^l, $g^{ur} \ = \ g^{ui}\omega_i$.

Remark 1.1.4. *In the sequel we will often refer to the set of normalized coordinates for the metric g by the name"Bondi" coordinates.*

Remark 1.1.5. *These decay rates allow for g to be a large perturbation of η for all time inside $\{r \leqslant \frac{1}{10}t\}$. Inside the wave zone this hierarchy of decay for different (u, i) components of the metric along outgoing null directions is consistent with the radiation of gravitational waves as in [BvdBM62], [Sac62] and [CK93]. These decay rates are also consistent with the types of metrics constructed in the stability of Minkowski space in wave coordinates [LR10].*

Remark 1.1.6. *The far exterior portion of the Schwarzschild and Kerr space-times are both examples of Lorentzian metrics having such normalized coordinates. Roughly, these correspond to using Regge-Wheeler coordinates near spatial infinity. As we mentioned before, the issue of when such a normalized coordinate system exists is an interesting problem in its own right. We will discuss this a bit more in the next chapter.*

1.2 Energies and Norms

We define the vector field:

$$L \;=\; -\nabla u \;=\; -g^{\alpha\beta}\partial_\alpha u \partial_\beta \ ,$$

which is an outgoing null generator in the wave zone. We let $C(u_0)\cap\mathcal{W}$ be the null hypersurfaces defined by $u = u_0$ in the wave zone and define $dV_{C(u_0)}$ to be the Euclidean volume element restricted to $C(u_0)$. We also define the vector:

$$\nabla\phi = (\partial_0\phi, ..., \partial_3\phi) \ ,$$

where $\partial_0, ..., \partial_3$ denotes any basis which can be written as a bounded linear combination of (t, x) coordinate derivatives. In the sequel we always use the convention that $\nabla u, \nabla t$ denote the geometric gradients of the functions u, t respectively, whereas $\nabla\phi$ denotes the vector defined above. We also define the vector:

$$\widetilde{\nabla\!\!\!/}\phi \;=\; (e_3\phi, e_4\phi) \ ,$$

with $\{e_a\}_{a=3,4}$ a basis for the tangent space to the $\{r = const\}\cap\{u = const\}$ hypersurfaces (see section 2.1.2 for precise definitions). Using this we define the *Energy* and *Characteristic Energy*:

$$\begin{aligned} E[\phi(t)] \ &:= \ \frac{1}{2}\int_{\mathbb{R}^3} |\nabla\phi(t,x)|^2 \ \sqrt{|g|}dx \ , \\ E_{ch}[\phi(u_0)] \ &:= \ \frac{1}{2}\int_{C(u_0)\cap\mathcal{W}} \Big((L\phi)^2 + |\widetilde{\nabla\!\!\!/}\phi|^2 \Big) \ dV_{C(u_0)} \ , \end{aligned}$$

with $|\nabla\phi|^2$ and $|\widetilde{\nabla\!\!\!/}\phi|^2$ denoting the squared ℓ^2 norm of the vectors $\nabla\phi$, $\widetilde{\nabla\!\!\!/}\phi$ respectively. We introduce now the *Conformal Energy* which is the main energy in this work:

$$\begin{aligned} CE[\phi(t)] \ :=& \ \frac{1}{2}\int_{\mathbb{R}^3} \langle u\rangle^2 \Big(|\nabla\phi(t)|^2 + |\frac{\phi(t)}{r}|^2 \Big) \ \sqrt{|g|}dx \\ &+ \ \frac{1}{2}\int_{\mathcal{W}} \Big(\langle t+r\rangle^2 (|L\phi(t)|^2 + |\widetilde{\nabla\!\!\!/}\phi(t)|^2 + |\frac{\phi(t)}{r}|^2) \Big) \ \sqrt{|g|}dx. \end{aligned}$$

We will also use two conjugated versions of the energy above. Let the *Conjugated Conformal Energies of the First and Second Kind* be given by:

$$
\begin{aligned}
&{}^{I}CE[\phi(t)] \\
&\qquad := \ \frac{1}{2}\int_{\mathbb{R}^3} \langle t+r\rangle^2\Big(\Big|\frac{L(\langle r\rangle\phi)}{\langle r\rangle}\Big|^2 + |\,\widetilde{\nabla}\phi|^2\Big) + \langle u\rangle^2 \Big|\frac{\partial_t(\langle r\rangle\phi)}{\langle r\rangle}\Big|^2 \ \sqrt{|g|}\ dx \ , \quad (1.8)\\
&{}^{II}CE[\phi(t)] \\
&\qquad := \ \frac{1}{2}\int_{\mathbb{R}^3} \langle t+r\rangle^2\Big(\Big|\frac{L(\langle u\rangle\langle \underline{u}\rangle\phi)}{\langle u\rangle\langle \underline{u}\rangle}\Big|^2 + |\,\widetilde{\nabla}\phi|^2\Big) + \langle u\rangle^2 \Big|\frac{\partial_t(\langle u\rangle\langle \underline{u}\rangle\phi)}{\langle u\rangle\langle \underline{u}\rangle}\Big|^2 \ \sqrt{|g|}\ dx \ , \quad (1.9)
\end{aligned}
$$

where $\underline{u} = u + 2r$. The *Conformal Characteristic Energies* are similarly defined:

$$
\begin{aligned}
{}^{I}CE_{ch}[\phi(u_0)] \ &:= \ \frac{1}{2}\int_{C(u_0)\cap\mathcal{W}} \left(\langle t+r\rangle^2 \Big|\frac{L(\langle r\rangle\phi)}{\langle r\rangle}\Big|^2 + \langle u\rangle^2 |\,\widetilde{\nabla}\phi|^2\right) dV_{C(u_0)} \ ,\\
{}^{II}CE_{ch}[\phi(u_0)] \ &:= \ \frac{1}{2}\int_{C(u_0)\cap\mathcal{W}} \left(\langle t+r\rangle^2 \Big|\frac{L(\langle u\rangle\langle \underline{u}\rangle\phi)}{\langle u\rangle\langle \underline{u}\rangle}\Big|^2 + \langle u\rangle^2 |\,\widetilde{\nabla}\phi|^2\right) dV_{C(u_0)} \ .
\end{aligned}
$$

Define ϕ_m to be the vector:

$$
\phi_m \ := \ (\partial^J_{t,x}\widetilde{S}\phi, \partial^J_{t,x}\widetilde{\Omega}\phi, \partial^K_{t,x}\phi) \ , \qquad\qquad |J| = m-1, \ |K| = m \ ,
$$

and we take $\phi_0 := \phi$. In the sequel, when dealing with vectors we will use the standard convention for differentiating and taking norms:

$$
\begin{aligned}
\partial_{t,x}\phi_m \ &:= \ \Big(\partial_{t,x}(\partial^J_{t,x}\widetilde{S}\phi), \partial_{t,x}(\partial^J_{t,x}\widetilde{\Omega}\phi), \partial_{t,x}(\partial^K_{t,x}\phi)\Big) \ ,\\
\|\,\phi_m\,\| \ &:= \ \|\,\partial^J_{t,x}\widetilde{S}\phi\,\| + \|\,\partial^J_{t,x}\widetilde{\Omega}\phi\,\| + \|\,\partial^K_{t,x}\phi\,\|, \qquad |J| = m-1, \ |K| = m \ .
\end{aligned}
$$

For any $k \in \mathbb{Z}^+$ we also define the vector $\nabla^k\phi$ in the obvious way. We also define the higher order energies to be:

$$
\begin{aligned}
E_k[\phi(t)] \ &:= \ \sum_{m\leqslant k} E[\phi_m(t)] \ , \qquad & CE_k[\phi(t)] \ &:= \ \sum_{m\leqslant k} CE[\phi_m(t)] \ ,\\
\mathring{E}_k[\phi(t)] \ &:= \ \sum_{|J|\leqslant k} E[\partial^J\phi(t)] \ , \qquad & \mathring{CE}_k[\phi(t)] \ &:= \ \sum_{|J|\leqslant k} CE[\partial^J\phi(t)] \ ,
\end{aligned}
$$

with analogous definitions for $E_{ch,k}, \ CE_{ch,k}, \ \mathring{E}_{ch,k}, \ \mathring{CE}_{ch,k}$.

1.3 Local Smoothing

Let $\epsilon > 0$. We introduce the Local Smoothing (LS) norm:

$$\| \phi \|_{LS[t_0,t_1]} \ := \ \| \langle x \rangle^{-\frac{1}{2}-\epsilon} \phi \|_{L^2([t_0,t_1]\times\mathbb{R}^3)} \ .$$

As well as the weighted norms:

$$\| \phi \|_{LS^{a,b}[t_0,t_1]} \ := \ \| \langle \underline{u} \rangle^{a-b} \langle u \rangle^{b} \phi \|_{LS[t_0,t_1]} \ , \qquad \| \phi \|_{LS^{a}[t_0,t_1]} \ := \ \| \phi \|_{LS^{a,0}[t_0,t_1]} \ .$$

Using this we define the *Modified Local Smoothing* norms to be:

$$\begin{aligned} \| \phi \|_{LSM[t_0,t_1]} \ &:= \ \| \nabla \phi \|_{LS[t_0,t_1]} \ + \ \| \langle x \rangle^{-1} \phi \|_{LS[t_0,t_1]} \ , \\ \| \phi \|_{LSM_{int}[t_0,t_1]} \ &:= \ \| \chi_{r\leqslant \epsilon t} \nabla \phi \|_{LS[t_0,t_1]} \ + \ \| \langle x \rangle^{-1} \chi_{r\leqslant \epsilon t} \phi \|_{LS[t_0,t_1]} \ , \\ \| \phi \|_{LSM_{r\leqslant \frac{1}{10}t}[t_0,t_1]} \ &:= \ \| \chi_{r\leqslant \frac{1}{10}t} \nabla \phi \|_{LS[t_0,t_1]} \ + \ \| \langle x \rangle^{-1} \chi_{r\leqslant \frac{1}{10}t} \phi \|_{LS[t_0,t_1]} \ , \end{aligned}$$

with $\chi_{r\leqslant \epsilon t}$, $\chi_{r\leqslant \frac{1}{10}t}$ smooth cutoff functions supported on the sets $\{r \leqslant \epsilon t\}$, $\{r \leqslant \frac{1}{10}t\}$ respectively. We also define the dual norm:

$$\| F \|_{LS^*[t_0,t_1]} \ := \ \| \langle x \rangle^{\frac{1}{2}+\epsilon} F \|_{L^2([t_0,t_1]\times\mathbb{R}^3)} \ .$$

We also define the weighted norms $\| \phi \|_{LSM^{a,b}[t_0,t_1]}$, $\| \phi \|_{LSM^{a,b}_{int}[t_0,t_1]}$, $\| \phi \|_{LSM^{a,b}_{r\leqslant \frac{1}{10}t}[t_0,t_1]}$ and $\| F \|_{LS^{*,a,b}[t_0,t_1]}$ in the obvious way. For all non-negative integers k we let the higher order norms to be:

$$\begin{aligned} \| \phi \|_{\mathring{LS}_k[t_0,t_1]} \ &:= \ \sum_{|J|\leqslant k} \| \nabla^J \phi \|_{LS[t_0,t_1]} \ , \\ \| \phi \|_{\mathring{LSM}_{int,k}[t_0,t_1]} \ &:= \ \sum_{|J|\leqslant k} \| \nabla^J \phi \|_{LSM_{int}[t_0,t_1]} \ , \\ \| \phi \|_{LS_k[t_0,t_1]} \ &:= \ \sum_{m\leqslant k} \| \phi_m \|_{LS[t_0,t_1]} \ , \\ \| \phi \|_{LSM_{int,k}[t_0,t_1]} \ &:= \ \sum_{m\leqslant k} \| \phi_m \|_{LSM_{int}[t_0,t_1]} \ , \end{aligned}$$

with analogous definitions for the weighted norms and the dual norms.

We now proceed to make the non-trapping assumption precise:

Assumption 1.3.1 (Quantitative Non-Trapping for Null Geodesics). *For any fixed* $r_0 > 0$ *there exists* $C = C(r_0, g)$ *such that if* γ *is any forward null geodesic starting inside the set* $r \leqslant r_0$ *with affine parameter* s *satisfying* $\dot{\gamma}t\big|_{s=0} = 1$ *with* $\dot{\gamma}s \equiv 1$, $\gamma(0) \in \{|x| \leqslant r_0\}$, *then* $\gamma(s) \notin \{|x| \leqslant r_0\}$ *for all* $s \geq C$.

Using the LS norms we may now state the final decay assumption on the metric $g_{\alpha\beta}$:

Assumption 1.3.2 (Local Smoothing Estimate). *For sufficiently small* $0 < \epsilon$ *and times* $0 \leqslant t_0 \leqslant t_1$ *the evolution satisfies the estimate:*

$$\| \phi \|_{LSM[t_0,t_1]} \ \lesssim \ E^{\frac{1}{2}}[\phi(t_0)] \ + \ \| F \|_{LS^*[t_0,t_1]} \ . \qquad (1.10)$$

The assumption of non-trapping is a necessary condition in for (1.10) to hold in this form. If there's trapped null rays, the work of Ralston [Ral69] shows that the estimate must necessarily lose derivatives in a neighborhood of the set where trapping occurs. In the case of non-trapping spacetimes, estimates such as (1.10) go all the way back to work of Morawetz [Mor68] and are known to hold in a variety of settings. In the simplest case of Minkowski space, Keel-Smith-Sogge proved a limiting version of this estimate and used it to prove small-data, almost global existence for general nonlinearities [KSS02] (see also [Ste05]). For uniformly small, time-dependent perturbations of Minkowski, Alinhac [Ali10] and Metcalfe-Tataru [MT12] both established this result. The work of Bony-Hafner [BH10] extended the validity of this estimate to the case of large, stationary, non-trapping perturbations of Minkowski space. Recently, Sterbenz-Tataru [ST13] proved the LS estimate for the Maxwell field on spherically symmetric Black Hole space-times. We point out that both the Schwarzschild and Kerr solutions have trapped null geodesics and therefore our work will not apply to perturbations of such spacetimes.

The importance of the LS estimate for the vector field method took some time to understand. By now, however, it is clear that a pointwise decay estimate using the vector field method should be built out of the following three more basic decay/boundedness statements:

a) An LS estimate

b) An estimate for $E[\phi(t)]$ in terms of initial data (energy boundedness)

c) An estimate for $CE[\phi(t)]$ in terms of initial data (conformal energy estimate) or a suitable replacement (see [DR10b])

All instances in which the vector field method has been used to prove a pointwise decay estimate like (1.25) all three of these ingredients are available. We point out that in some applications Stricharz estimates are proved instead of *CE* boundedness or pointwise decay. we will not discuss such applications (see Lindblad-Sogge-Tohaneanu-Wang in [LMS+13]). The LS estimate, in particular, plays a special role in the proof of pointwise decay. This is because its existence already implies an integrated local energy result. An alternative way to think of this is to make the observation that since (1.10) is an $L^2_T(L^2_x)$ estimate, the large deformation tensor errors in a compact set arising from *b*) above can be put into this space by a simple application of the Cauchy-Schwarz inequality, the LS estimate is naturally suited to handle the error terms in a compact set arising in the proof of boundedness of energy. Therefore, at least locally, boundedness of $E[\phi(t)]$ is a direct consequence of the LS estimate. Therefore, in some ways, the LS estimate should be thought of as the starting point for any proof of decay for solutions to the wave equation using vector field methods. Once this estimate is in place, the vector field method bridges the gap between the linear and non- linear dynamics – at least for small data – by providing a method for proving pointwise decay. Certain types of non-linear problems then become tractable by using the pointwise decay estimate. There are numerous examples in the literature to illustrate this point. In [KSS02], for instance, the authors establish a different version of the LS estimate in Minkowski space and use it to prove almost global existence for problem (1.1) with $F = \mathcal{N}(\phi, \nabla\phi)$ a general quadratic nonlinearity. A generalization of this result is in [SW10] where the authors use the LS estimates shown in [BH10] to prove almost global existence as well as Stricharz estimates and the Strauss conjecture for semi-linear problems with g now a large, stationary, nontrapping perturbation of η.

Finally, we note that it is generally expected that the LS estimate (1.10) will hold for a large class of metrics – in particular, it should hold for generic, time-dependent, non-trapping, asymptotically flat metrics, and it should also hold, with loss of derivatives, for the Kerr metric with $|a| < M$ and for small time-dependent perturbations as well. In view of this, in this work we just assume its existence and focus on the problem of developing a vector field method robust enough to yield a pointwise decay result analogous to the Minkowski case. This constitutes the necessary first step in proving a small-data global existence result for some non-linear problems with special structure.

1.4 Statement of the Main Theorem

Theorem 1.4.1 (Main Theorem)**.** *Let $(\mathcal{M}, g_{\alpha\beta})$ be 4-dimensional, smooth, asymptotically flat, Lorentzian manifold of the form $\mathbb{R}\times\mathbb{R}^3$ with a globally defined set of coordinates (t, x^i) such that the level sets of t are space-like. Assume the metric g satisfies assumptions 1.1.1–1.3.2 with $0 < \gamma < \delta$. Let ϕ be a solution to the wave equation* (1.1) *with smooth initial data such that $CE^{\frac{1}{2}}[\phi(t_0)]$ and $\| F \|_{LS^{*,1+\gamma',\frac{1}{2}}[t_0,t_1]}$ are finite for some $0 < \gamma' < \gamma$. Then there exists $\epsilon > 0$ sufficiently small with:*

$$\epsilon < \gamma' \ , \qquad \epsilon + \gamma < \frac{\delta}{2} \ , \tag{1.11}$$

such that for all $t_0, t_1 \in [0, \infty)$ the function ϕ satisfies:

Conformal Energy Estimate

$$\sup_{t_0\leqslant t\leqslant t_1} CE^{\frac{1}{2}}[\phi(t)] + \sup_{\mathcal{W}\cap[t_0,t_1]} \left(CE^{\frac{1}{2}}_{ch}[\phi(u)]\right) + \| \phi \|_{LSM^{1-\gamma'}_{r\leqslant\frac{1}{10}t}[t_0,t_1]} \tag{1.12}$$
$$\lesssim \ CE^{\frac{1}{2}}[\phi(t_0)] + \| F \|_{LS^{*,1+\gamma',\frac{1}{2}}[t_0,t_1]} \ .$$

Additionally, if the initial data are such that $CE^{\frac{1}{2}}_k[\phi(t_0)]$, $\| F \|_{LS_k^{*,1+\gamma',\frac{1}{2}}[t_0,t_1]}$, and $\| F \|_{LS_{k-1}^{2-\gamma',\frac{1}{2}}[t_0,t_1]}$ finite for some fixed $k \in \mathbb{Z}^+$, $0 < \gamma' < \frac{\gamma}{10}$, then there exists $\epsilon > 0$ sufficiently small meeting the conditions of (1.11) *such that for all $t_0, t_1 \in [0, \infty)$ the function ϕ satisfies:*

Higher Order Conformal Energy Estimate

$$\sup_{t_0\leqslant t\leqslant t_1} CE_k^{\frac{1}{2}}[\phi(t)] \ + \sup_{\mathcal{W}\cap[t_0,t_1]} \big(CE_{ch,k}^{\frac{1}{2}}[\phi(u)]\big) + \|\,\phi\,\|_{LSM^{1-\gamma'}_{int,k}[t_0,t_1]} \lesssim \ CE_k^{\frac{1}{2}}[\phi(t_0)] + \|\,F\,\|_{LS_k^{*,1+\gamma',\frac{1}{2}}[t_0,t_1]} + \|\,F\,\|_{LS_{k-1}^{2-\gamma',\frac{1}{2}}[t_0,t_1]} \ . \tag{1.13}$$

Next, we state the L^∞ estimate below as a standard application:

Theorem 1.4.2 (Global Pointwise Decay)**.** *Let $(\mathcal{M}, g_{\alpha\beta})$ satisfy all the assumptions of the main theorem. Let ϕ be a solution to the free wave equation:*

$$\Box_g\phi = 0 \ , \qquad \phi(0,x) \ = \ \phi_0(x) \ , \partial_t\phi(0,x) \ = \ \phi_1(x) \ ,$$

with smooth initial data such that $CE_2^{\frac{1}{2}}[\phi(t_0)]$ is finite. Then, for all $(t,x) \in [0,\infty)\times\mathbb{R}^3$, the function ϕ satisfies the pointwise decay estimate:

$$|\phi(t,x)| \ \lesssim \ \frac{1}{\langle t+r\rangle\langle u\rangle^{\frac{1}{2}}}\Big(CE_2^{\frac{1}{2}}[\phi(t_0)]\Big) \ . \tag{1.14}$$

Remark 1.4.3. *The parameter ϵ only depends on $\|\,g\,\|_{L^\infty}, \gamma, \delta$ and all the constants involved in assumptions 1.3.1–1.3.2. In principle, an explicit choice of ϵ involving such constants can be made. However, as we will only use this parameter to close a finite number of estimates it is not necessary – or particularly illuminating – to keep track of its size. From this point on we simply fix ϵ sufficiently small with* (1.11) *satisfied.*

Remark 1.4.4. *For our results below, we will work with data (ϕ_0, ϕ_1) which are smooth and compactly supported. This condition can be relaxed so that the only regularity requirements for the data are that they belong to the weighted Sobolev spaces discussed in the statement of the main theorem. This relaxation can be achieved by standard approximation arguments which we will omit.*

Remark 1.4.5. *These results in effect reduce the problem of establishing the optimal pointwise decay rates consistent with the vector field method for solutions to the free wave equation for a large class of time-dependent, radiating metrics to just proving that the LS estimate* (1.10) *holds in such spacetimes.*

To the best of our knowledge, for outgoing metrics there's no previous results in the field other than the stability of Minkowski space [CK93] and [BZ09]. Additionally, no previous work has been done for metrics with the type of 'mixed' conditions we deal where both the slow decay for $\partial_t g$ in a compact set and the slow decay for $g-\eta$ in $\mathcal{W}$ must be dealt with simultaneously. As an application of this result, we hope to soon be able to prove a small data global existence result for some simple non-linear problems with special structure. We leave it to future work to treat the separate problem of reducing these coordinate assumptions to more basic, geometrically invariant assumptions.

The state-of-the-art for linear decay on time-dependent, asymptotically flat metrics boils down to two very recent results. The first one is the work of S. Yang [Yan10] which establishes a pointwise decay rate of $\langle t+r\rangle^{-1}$ for the linear problem using a vector field method for compactly supported metrics. As an application, the author then uses this estimate to establish a small data global existence result for nonlinearities satisfying the null condition. Just as we do in our work, the estimate (1.10) is assumed for large perturbations. However, in contrast to our result, the author only assumes $|\partial_t^k g| \leqslant \epsilon^k$ for $\epsilon > 0$ sufficiently small inside a compact set which allows for a wider class of perturbations there. On the other hand, this is paired with $(g-\eta) \equiv 0$ outside a compact set so there's no new analysis or results there. The second pointwise decay result for non-trapping, time- dependent metrics is the work of [MTT12] where the authors prove a decay rate of $\langle t+r\rangle^{-1}\langle t-r\rangle^{-2}$ (Price Law) for non-trapping, time-dependent metrics. The authors assume the LS estimate together with the interior decay rates $\partial_t^k g \sim \langle t\rangle^{-k}\langle r\rangle^{-1}$ and wave zone decay rates $\partial^k g \sim \langle r\rangle^{-1-k}$ which are much more restrictive than ours. Despite the fact that a lot of decay is achieved for the linear problem, we point out that the norms for F involved in getting such decay are very costly from the point of view of working with non-linear problems. For this reason no applications to non-linear problems using this result have been shown up to date. We also mention that this work also deals with black-hole spacetimes: the authors also establish the t^{-3} interior decay rate for the linear problem for a class of small, time dependent perturbations of the Kerr metric with $|a| \ll M$ by assuming the corresponding trapping version of the LS estimate.

As we mentioned before, our result differs from the previous two mainly in the fact that we are able to deal with a radiating metric in the wave zone with while keeping the slow decay for $\partial_t g$ in a compact set. In the wave zone, the radiation condition for g requires that we modify our vector fields so that they have better commutators – at least relative to the standard Minkowski vector fields. In the wave zone, the process of controlling the errors coming from vector field multipliers and commutators is very delicate and relies heavily on the Bondi coordinates – plus a bootstrap procedure. This is in stark contrast with the previous two results since the conditions imposed on the metric in both of those cases allow for control of the standard Minkowski vector fields more or less trivially in the wave zone. In the interior, since $t\partial_t(g) \sim t^{1-\gamma}$ is not bounded, commuting with even a single scaling vector field in this region becomes non-trivial and requires us to prove a weighted L^2 elliptic estimate as well as a higher order T-weighted LS estimate with vector fields.

1.5 Overview of The Vector Field Method

In this section we'll recall the geometric identities used in the vector field method. We also review how these identities are used in the case of Minkowski to prove a pointwise decay result and discuss how that proof differs from our case.

1.5.1 Energy Formalism

Define the *Energy-Momentum Tensor* associated to g, ϕ by:

$$T_{\alpha\beta}[\phi] \ = \ \partial_\alpha\phi\partial_\beta\phi - \frac{1}{2}g_{\alpha\beta}\partial^\lambda\phi\partial_\lambda\phi \ .$$

$T_{\alpha\beta}[\phi]$ is related to $\Box_g\phi$ by the following formula:

$$D^\alpha T_{\alpha\beta}[\phi] \ = \ F\cdot\partial_\beta\phi \ ,$$

with D denoting the Levi-Civita connection. Given a smooth vector field X, we define the 1-form:

$$^{(X)}P_\alpha[\phi] \ := \ T_{\alpha\beta}[\phi]X^\beta \ .$$

Taking the divergence of this 1-form we arrive at the well-known formula:

$$D^{\alpha(X)}P_\alpha[\phi] \ = \ F\phi \cdot X\phi + \frac{1}{2}{}^{(X)}\pi^{\alpha\beta}T_{\alpha\beta}[\phi] \ , \tag{1.15}$$

where,

$$\mathcal{L}_X g \ = \ {}^{(X)}\pi(Y,Z) \ = \ \langle D_Y X, Z\rangle + \langle D_Z X, Y\rangle \ ,$$

is the *Deformation Tensor* of g with respect to X. This symmetric 2-tensor measures the change of g under the flow generated by X. Integrating (1.15) over the time slab $\{(t,x)|\ t_0 \leq t \leq t_1\}$ and using Stokes' theorem:

$$\begin{aligned}\int_{\{t=t_0\}\cap\mathbb{R}^3} N^\alpha({}^{(X)}P_\alpha[\phi])\sqrt{|g|}dx \ &= \ \int_{\{t=t_1\}\cap\mathbb{R}^3} N^\alpha({}^{(X)}P_\alpha[\phi])\sqrt{|g|}dx \\ &+ \int_{t_0}^{t_1}\int_{\mathbb{R}^3}\left(F\cdot X\phi + \frac{1}{2}{}^{(X)}\pi^{\alpha\beta}T_{\alpha\beta}[\phi]\right)\sqrt{|g|}dxdt\end{aligned} \tag{1.16}$$

with $N = -\nabla t$. Since this identity comes from multiplying the vector field $X\phi$ times the equation and performing an integration by parts, this is often referred to as the *Multiplier Method* and we say that the vector field X is used as a multiplier. We refer to the integrand on the left hand side as the *energy density* of X associated to the foliation by spacelike hypersurfaces $t = const.$

In order to obtain a useful estimate, in all the classical applications of the multiplier method one looks for X such that the LHS above is non-negative and such that the error term on the RHS ${}^{(X)}\pi^{\alpha\beta}T_{\alpha\beta}[\phi]$ is as small as possible. – this is where the choice of the vector field X and how it interacts with g comes in. To deal with the positivity of the LHS, we recall that for any Loretzian metric g the energy momentum tensor $T_{\alpha\beta}[\phi]$ satisfies the *Dominant Energy Condition*: For any two timelike, future-directed vector fields Y, Z:

$$\sum_\alpha(\partial_\alpha\phi)^2 \ \lesssim \ T(Y,Z) \ = \ Y^\alpha T_{\alpha\beta}[\phi]Z^\beta \ . \tag{1.17}$$

1.5.2 Killing and Conformal Killing Vector Fields

For the error term on the RHS of equation (1.16) there's no general way to deal with an arbitrary vector field X, so we must insist that X interact well with g. The simplest case is when ${}^{(X)}\pi \equiv 0$, that is, X is a *Killing* field – or equivalently, the flow of X generates an isometry for g. In this case, we see that equation (1.16) immediately leads to conservation of the energy density associated to X. This result is a manifestation of Noether's theorem relating symmetries of the space-time to conserved quantities for the flow. As it is to be expected, most spacetimes do have have any symmetries, so in some sense this case is an idealized model. The next simplest case is when the flow of X is a conformal transformation – that is:

$$ {}^{(X)}\pi \; = \; \Omega g \; , \tag{1.18}$$

for some scalar function $\Omega = e^{\lambda}$. In this case we say that the vector field X is *Conformal Killing* with respect to the metric g. The classical way of proving an energy identity for such a field is to use it directly as a multiplier via (1.16). To take advantage of (1.18) we use the following identity which follows from (1.16) by an integration by parts:

$$\begin{aligned}
&\int_{\{t=t_0\}\cap\mathbb{R}^3} \left(N^\alpha({}^{(X)}P_\alpha[\phi]) - \frac{1}{4}\phi^2\langle N, \nabla\Omega\rangle + \frac{1}{2}\Omega\phi\langle N, \nabla\phi\rangle \right) \sqrt{|g|}dx \\
&= \int_{\{t=t_1\}\cap\mathbb{R}^3} \left(N^\alpha({}^{(X)}P_\alpha[\phi]) - \frac{1}{4}\phi^2\langle N, \nabla\Omega\rangle + \frac{1}{2}\Omega\phi\langle N, \nabla\phi\rangle \right) \sqrt{|g|}dx \\
&+ \int_{t_0}^{t_1}\int_{\mathbb{R}^3} F\left(X\phi + \frac{\Omega}{2}\phi \right) + \frac{1}{2}T_{\alpha\beta}[\phi]({}^{(X)}\pi^{\alpha\beta} - \Omega g^{\alpha\beta}) - \frac{1}{4}\phi^2\Box_g(\Omega)\sqrt{|g|}dxdt \; .
\end{aligned} \tag{1.19}$$

The deformation tensor term vanishes if X is conformal Killing. This cancellation effectively eliminates the most dangerous term on the RHS. The price for doing this is that even if X is timelike and future-directed, the energy density could now have a negative sign due to the two extra terms we have added.

1.5.3 Conformal Energy Estimate in Minkowski Space

Let us recall how this setup helps us prove energy estimates and pointwise decay in Minkowski space. Let $g = \eta$ and $X = \partial_t$ – the infinitesimal generator for time translations. In this setting, the vector field ∂_t is timelike and future-directed everywhere. Since the $t = t_0$ hypersurfaces are spacelike everywhere, the vector field $N = -\nabla t$ is also timelike and future directed everywhere, therefore equation (1.17) together with (1.16) imply that we have fixed-time estimates for the positive energy density associated to ∂_t through the foliation by $t = t_0$ hypersurfaces as long as we can control the error term $T_{\alpha\beta}[\phi]^{(\partial_t)}\pi^{\alpha\beta}$. In the case of Minkowski, ∂_t is *Killing* so ${}^{(\partial_t)}\pi \;\equiv\; 0$ – the symmetry in this case corresponding to time-translation invariance. If ϕ a solution to the Minkowski wave equation $\Box\phi = F$, an application of (1.16) yields the energy estimate:

$$\begin{aligned}\int_{\{t=t_1\}\cap\mathbb{R}^3} (\partial_t\phi)^2 + \sum_i (\partial_i\phi)^2 \; dx \;\lesssim\; & \int_{\{t=t_0\}\cap\mathbb{R}^3} (\partial_t\phi)^2 + \sum_i (\partial_i\phi)^2 \; dx \\ & + \int_{t_0}^{t_1}\int_{\mathbb{R}^3} |F\cdot\partial_t\phi| \; dxdt \; . \end{aligned} \tag{1.20}$$

Next we define the Morawetz vector field by:

$$K_0 \;=\; (t^2 + r^2)\partial_t + 2tr\partial_r \; . \tag{1.21}$$

This is a timelike vector field except on the set $t = r$ where it becomes null. The Morawetz vector field is connected to the conformal structure of Minkowski space since it is the generator of the inverted time translations – a conformal isometry for η. One can readily compute:

$${}^{(K_0)}\pi \;=\; 4t\eta \; , \qquad \Box(4t) \;=\; 0 \; .$$

Therefore applying (1.19), using some Hardy estimates and adding the energy estimate (1.20) one can, after some computation, show the *Conformal Energy Estimate*:

$$\begin{aligned} & \int_{\{t=t_1\}\cap\mathbb{R}^3} \langle \underline{u}\rangle^2 (L\phi)^2 + \langle u\rangle^2 (\underline{L}\phi)^2 + (\langle u\rangle^2 + \langle \underline{u}\rangle^2)|\nabla\!\!\!/\,\phi|^2 + \frac{(t\phi)^2}{r^2}\, dx \\ \lesssim\; & \int_{\{t=t_0\}\cap\mathbb{R}^3} \langle \underline{u}\rangle^2 (L\phi)^2 + \langle u\rangle^2 (\underline{L}\phi)^2 + (\langle u\rangle^2 + \langle \underline{u}\rangle^2)|\nabla\!\!\!/\,\phi|^2 + \frac{(t\phi)^2}{r^2}\, dx \\ & \quad + \int_{t_0}^{t_1}\int_{\mathbb{R}^3} \left| F\left(\overline{K_0}\phi - 2t\phi\right)\right| dxdt \; , \end{aligned} \tag{1.22}$$

with $\overline{K_0} = K_0 + \partial_t$, optical functions $u = t - r$, $\underline{u} = t + r$ and $L = -\nabla u = \partial_t + \partial_r$ and $L = -\nabla\underline{u} = \partial_t - \partial_r$ the outgoing (resp. incoming) null generators.

1.5.4 Commutators and Pointwise Decay in Minkowski Space

With the result (1.22) now in place, the commutator method then allows us to upgrade this estimate to higher order L^2 estimates by commuting the equation $\Box\phi = F$ with vector fields. The main idea is the following: let Γ be a smooth vector field. The function $\Gamma\phi$ then satisfies the Wave equation:

$$\Box\Gamma\phi \ = \ \Gamma F + [\Box, \Gamma]\phi \ . \tag{1.23}$$

Applying the conformal energy estimate (1.22) to $\Gamma\phi$ then yields:

$$\begin{aligned} \sup_{t_0 \leqslant t \leqslant t_1} CE^{\frac{1}{2}}[\Gamma\phi(t)] \ \lesssim \ & CE^{\frac{1}{2}}[\Gamma\phi(t_0)] + \int_{t_0}^{t_1} \| \langle t+r \rangle \Gamma F \|_{L^2_x} \, dt \\ & + \int_{t_0}^{t_1} \| \langle t+r \rangle [\Gamma, \Box]\phi \|_{L^2_x} \, dt \ . \end{aligned} \tag{1.24}$$

If one has good control of the commutators above, a conformal energy estimate for $\Gamma\phi$ immediately follows. This pairing of (1.23) with an energy estimate to produce an energy bound for $\Gamma\phi$ is referred to as using Γ as a *commutator*. In Minkowski space we then commute the equation with $\Gamma \in \{\partial_\alpha \ , \Omega_{\alpha\beta} \ , S\}$ where:

$$\Omega_{0i} \ = \ x^i\partial_t + t\partial_i \ , \quad \Omega_{ij} \ = \ x^i\partial_j - x^j\partial_i \ , \qquad S \ = \ t\partial_t + r\partial_r \ .$$

the *Lorentz boosts*, *rotations*, and *scaling* vector fields, respectively. The commutators are:

$$[\Box, \partial] \ = \ 0 \ , \qquad [\Box, \Omega] \ = \ 0 \ , \qquad [\Box, S] \ = \ 2\Box \ .$$

Commuting with all Γ and using the *CE* estimate (1.22) leads to an estimate for $CE[\Gamma\phi]$ as well. Once this is in pace, the pointwise decay then follows via the Klainerman-Sobolev estimates:

$$\langle t+r \rangle \langle t-r \rangle^{\frac{1}{2}} |\phi(t,x)| \ \lesssim \ \sum_{|\alpha| \leqslant 2} \| \Gamma^\alpha \phi \|_{L^2_x} \ . \tag{1.25}$$

It is important to note here that since this is the maximum pointwise decay rate that the vector field method achieves at (and near) flat space, where the best results are to be expected, it therefore represents the optimal decay rate one would also expect to get out of a vector field approach in a curved, asymptotically flat 4-dimensional space-time which is not close to flat space – such as ours. However, we point out that other methods can obtain more decay – see in particular [MTT12]. However, it is not yet clear if any other alternative approaches can also work for small data non-linear problems. By contrast, the vector field method works particularly well for small data-global existence problems for non-linear wave equations, where $F = \mathcal{N}(\phi, \nabla\phi)$ has special algebraic structure – the so-called null condition. This was first established for Minkowski space by S. Klainerman in [Kla86]. Such a condition on $\mathcal{N}$ is necessary in 4 dimensions since global existence for arbitrary $\mathcal{N}$ fails even in Minkowski space – see[Joh79].

The original proof of estimate (1.22) was done by Morawetz in [Mor62] and was used to prove the pointwise estimate (1.25) in the work of Klainerman [Kla85]. In the Schwarzschild case, the conformal energy estimate was first proved by Blue-Sterbenz [BS06] and later on by Dafermos-Rodnianski [DR09]. The case of slowly rotating Kerr, the conformal energy estimate was proved in [DR11] and then used by Luk in [Luk12] to prove a pointwise decay result for the free wave equation of similar form to (1.25).

1.5.5 Conformal Energy Estimate in the General Case

In the case we're interested in we will be able to use the fact that g is a non-trapping metric satisfying all of our decay assumptions to show that the vector field ∂_t is timelike *everywhere*. Since $-\nabla t$ is assumed timelike and future directed, we may again apply (1.17) to prove that the energy density associated to ∂_t is a positive quantity that controls $\sum_\alpha (\partial_\alpha \phi)^2$. However, since g in general could be time-dependent, this vector field will no longer be Killing in such cases. Nevertheless, our decay assumptions on $g_{\alpha\beta}$ will imply that ∂_t is *asymptotically* Killing outside of $r \leqslant \epsilon t$. Thus the deformation tensor of X will satisfy some decay estimates outside this region which will allow us to control these error terms. Therefore, modulo control of the deformation tensor and the proof that ∂_t is timelike, we may once again apply the strategy discussed above for Minkowski space and produce an energy estimate for the vector field ∂_t.

For the conformal energy estimate, we define an analog of K_0 for our curved spacetime $(\mathcal{M}, g_{\alpha\beta})$ as follows:

Definition 1.5.1. *Define the* modified Morawetz *vector field by:*

$$\widetilde{K_0} = \frac{u^2 - 2(u+r)r\partial_r u}{\partial_t u}\partial_t + 2(u+r)r\partial_r \ , \qquad (t,x) \ \textit{coordinates} \ , \quad (1.26)$$

$$\widetilde{K_0} = u^2\partial_u + 2(u+r)r\widetilde{\partial}_r \ , \qquad (u,x) \ \textit{coordinates} \ . \quad (1.27)$$

One option for us at this point would be to follow the classical proof of the conformal energy estimate and simply emulate what we did in the case of Minkowski. More specifically, the strategy be to apply (1.19) directly with $\widetilde{K}_0$ and deal with the proof of positivity of the LHS as it stands. However, in a curved, time-dependent background satisfying our assumptions, there's several problems with this strategy. The first difficulty is that since we want to use Bondi coordinates in the wave zone order to take advantage of the decay rates (1.7a)–(2.8), the proof of positivity requires some extra cancellations and integrations by parts and wouldn't follow in any simple, natural way from asymptotic flatness. A second problem with this approach is also its lack of characteristic energy estimates. It is well-known that outside the interior region, null energy estimates are optimal with respect to decay and peeling requirements for the metric – thus to get a result that is close to critical in the decay of g we need to make use of energies on null cones in the wave zone. In order to add energies associated with null cones $u = const$ in the wave zone, the setup of identity (1.19) would require another round of computations and integrations by parts. This process would generate some additional error terms, would not be very natural or illuminating, and would not necessarily make use of the Bondi coordinates in the most optimal way. However, there's an alternative approach which addresses all of these problems all at once and seems robust enough to handle the proof of the conformal energy estimate in this and many other Lorentzian backgrounds.

1.6 The Method of Lindblad-Sterbenz.

The work of Lindblad-Sterbenz [LS06] provides an alternative method of proof for the conformal energy estimate (1.22) in Minkowski space. Let's review the basic elements of this method, compare it with the standard method of proof and highlight some of the features which are advantageous in a curved background.

Let be a (smooth) X be a vector-field and $\widetilde{g} = \Omega^{-2}g$ for some weight function $\Omega = e^{\lambda}$. We have the identity:

$$\mathcal{L}_X\widetilde{g} \ = \ \Omega^{-2}\mathcal{L}_X g - 2\Omega^{-3}X(\Omega)g \ .$$

Recall that in the case of Minkowski we have $\mathcal{L}_{K_0}\eta = 4t\eta$ and it is then natural to look for conformal factors Ω such that:

$$4t\Omega \ = \ 2K_0(\Omega) \ .$$

This leads to two choices: ${}^{I}\Omega = r$ and ${}^{II}\Omega = t^2 - r^2 = u\underline{u}$. The crucial observation now is that the Morawetz field K_0 is now a *Killing field* with respect to the conformal metrics:

$${}^{I}\widetilde{\eta} = \frac{1}{r^2}\eta \ , \qquad {}^{II}\widetilde{\eta} = \frac{1}{(u\underline{u})^2}\eta \ ,$$

thus $\mathcal{L}_{K_0}\widetilde{g} = 0$. Next we recall that the wave equation has the following invariance with respect to conformal transformations:

$$\Box\psi = \Omega^3\Box\phi - V\phi \ , \tag{1.28}$$

with ψ the rescaled solution $\psi = \Omega\phi$ and V the scalar curvature $V = \Omega^3\Box(\Omega^{-1})$. In Minwkoski space, our choices $\Omega = r, u\underline{u}$ make $V \equiv 0$. Therefore, we can now apply Integrating with respect to the volume elements

$$dV_I \ = \ \frac{1}{r^4}dV_\eta \ , \qquad dV_{II} \ = \ \frac{1}{(u\underline{u})^4}dV_\eta \ .$$

Using K_0 as a multiplier for the metrics ${}^{I}\widetilde{\eta}$, ${}^{II}\widetilde{\eta}$ via identity (1.16) and computing with the energy momentum tensors corresponding to each of these metrics immediately leads to the *First and Second Morawetz* estimates:

$$\int_{\{t=t_1\}\cap\mathbb{R}^3} \underline{u}^2 \left|\frac{L(\Omega\phi)}{\Omega}\right|^2 + u^2 \left|\frac{\underline{L}(\Omega\phi)}{\Omega}\right|^2 + (u^2+\underline{u}^2)|\nabla\!\!\!/\,\phi|^2 + \frac{(t\phi)^2}{r^2}\,dx$$
$$\lesssim \int_{\{t=t_0\}\cap\mathbb{R}^3} (t_0^2+r^2)\left|\frac{\nabla(\Omega\phi)}{\Omega}\right|^2 dx + \int_{t_0}^{t_1}\int_{\mathbb{R}^3} F\cdot\frac{K_0(\Omega\phi)}{\Omega}\,dxdt\ , \qquad (1.29)$$

for $\Omega = r, u\underline{u}$. Notice the similarity between this form of the conformal energy and the conjugated energies (1.8) and (1.9). It is then pretty simple to remove the conjugation and derive the usual conformal energy estimate (1.22) via the pointwise estimates:

$$\underline{u}^2|L\phi|^2 + u^2|\underline{L}\phi|^2 \leqslant 2\left(\underline{u}^2\left|\frac{L(r\phi)}{r}\right|^2 + u^2\left|\frac{\underline{L}(r\phi)}{r}\right|^2\right) + 2(\underline{u}^2+u^2)|\frac{\phi}{r}|^2\ ,$$
$$(\underline{u}^2+u^2)|\frac{\phi}{r}|^2 \leqslant \sum_{\Omega=r,u\underline{u}} \underline{u}^2\left|\frac{L(\Omega\phi)}{\Omega}\right|^2 + u^2\left|\frac{\underline{L}(\Omega\phi)}{\Omega}\right|^2\ .$$

One of the key novelties in our work is that we develop a generalization of this method and use it to prove the conformal energy estimate for a metric with our weak decay assumptions. It will turn out that the method of Lindblad-Sterbenz will have some features which make it much better suited for dealing with curved spacetimes– and we discuss, in general terms some of these features now. Firstly, we note that using this approach we immediately get an estimate for a positive energy density via the dominant energy condition as long as $\widetilde{K}_0$ is timelike, future-directed – which will be the case for the interior region. Therefore we have eliminated the need to do any computations precisely in the region where g could stay far from the Minkowski metric. A second advantage is that, in contrast with the classical method, in the wave zone the Bondi coordinates will actually help us prove positivity for this energy density just by subtracting off the Minkowski energy density and treating the difference as a small bootstrap error term. Once positivity is proved it is then pretty straightforward to remove the conjugation and get the conformal energy estimate using some Hardy estimates. A third reason why this method is effective is that by using $\widetilde{K}_0$ as a multiplier for the conformal metrics we immediately can add the null energies in conjugated form – without any extra error terms or integrations by parts. Finally, another good reason to use this method we in a curved spacetime is that the identity (1.28) continues to hold verbatim – that is, the wave equation has the same invariance under conformal transformations in our setting. Consequently, the energy formalism for conformally deformed metrics has a natural form suitable for proving energy estimates regardless of the fact that g now has nontrivial curvature.

1.7 Notation

The following is a list of the notation that will be used in throughout the work.

As usual we denote $A \lesssim B$ (resp. "$A \ll B$"; "$A \approx B$") if $A \leqslant CB$" for some fixed $C > 0$ which may change from line to line (resp. $A \leqslant \epsilon B$ for a small ϵ; both $A \lesssim B$ and $B \lesssim A$).

$\widetilde{\partial}_i$ (resp. ∂_i) will denote coordinate derivatives in the Bondi coordinates (resp. coordinate derivatives is the (t, x) coordinates). In general, quantities with tilde are associated to Bondi coordinates.

$\widetilde{\Omega}$ will denote the collection of modified rotations. On the other hand Ω will denote the collection of weights ${}^{I}\Omega = \langle r \rangle$, ${}^{II}\Omega = \langle u \rangle \langle \underline{u} \rangle$.

Chapter 2

Geometry Setup

2.1 Normalized Coordinates

2.1.1 Properties

One of the main properties of the normalized coordinates which makes them natural to problem (1.1) is the fact that they provide a natural basis in which one can write the vector fields $X \in \mathbb{L}$. Furthermore, in these coordinates, the vector fields have a very simple form compared to their definition in (t, x) coordinates and it is in this simple form that we can more easily see that the fields commute well with each other.

Lemma 2.1.1. *In* (u, x) *coordinates the vector fields defined in* (1.4) *are given by:*

$$\mathbb{L} \ = \ \{\widetilde{T} = \partial_u \, , \, \widetilde{S} = u\partial_u + r\widetilde{\partial}_r \, , \, \widetilde{\Omega_{ij}} = \ x^i\widetilde{\partial}_j - x^j\widetilde{\partial}_i\} \ , \tag{2.1}$$

and $\mathbb{L}$ *forms a Lie algebra on* $\mathcal{M}$*.*

Proof. Using the chain rule, we get the following formula relating coordinate derivates in Bondi coordinates to coordinate derivatives in (t, x) coordinates:

$$\partial_u \ = \ (u_t)^{-1}\partial_t \ , \qquad \widetilde{\partial}_{x^i} \ = \ \partial_{x^i} - \frac{u_i}{u_t}\partial_t \ , \qquad \widetilde{\partial}_r \ = \ \partial_r - \frac{u_r}{u_t}\partial_t \ . \tag{2.2}$$

Using these identities and applying the chain rule to each vector field we get (2.1). A simple computation then shows that the commutators are equal to:

$$[\partial_u, \widetilde{S}] = \partial_u \ , \qquad [\partial_u, \widetilde{\Omega}] = 0 \ , \qquad [\widetilde{S}, \widetilde{\Omega}] = 0 \ , \qquad [\widetilde{\Omega}_{ij}, \widetilde{\Omega}_{kl}] = -\delta_{(ik}\widetilde{\Omega}_{jl)} \ . \tag{2.3}$$

The Lie algebra property follows directly.

□

Now we will derive some useful decay conditions for all the metric coefficients in the wave zone without the extra factor $\sqrt{|g|}$:

Lemma 2.1.2. *In the wave zone we have:*

$$|\partial_u^J \widetilde{\partial}_l^K (g^{\alpha\beta} - \widetilde{\eta}^{\alpha\beta})| \ \lesssim \ \langle r \rangle^{-\delta - |K|} \langle u \rangle^{-|J|} \ , \tag{2.4}$$

Proof. This is done by induction on the number of derivatives. The case $|J| + |K| = 0$ follows at once from (1.6). For the remaining cases we recall that in local coordinates we have the well-known formula:

$$(\mathcal{L}_X g)^{\alpha\beta} \ = \ {}^{(X)}\pi^{\alpha\beta} \ = \ -X(g^{\alpha\beta}) + g^{\alpha\gamma}\partial_\gamma(X^\beta) + g^{\beta\gamma}\partial_\gamma(X^\alpha) \ . \tag{2.5}$$

From this identity the result follows directly for any number of ∂_u derivatives by computing in Bondi coordinates. For a single $\widetilde{\partial}_r$ derivative:

$$\begin{aligned}(\widehat{\mathcal{L}_{\widetilde{S}} g})^{\alpha\beta} \ &= \ -\widetilde{S}(g^{\alpha\beta}) + g^{\alpha\gamma}\partial_\gamma(x^\beta) + g^{\beta\gamma}\partial_\gamma(x^\alpha) - 2g^{\alpha\beta} \ . \\ &= \ -u\partial_u(g^{\alpha\beta}) - r\widetilde{\partial}_r(g^{\alpha\beta}) \ .\end{aligned}$$

Therefore, since we already have the result for ∂_u we conclude:

$$|\widetilde{S}(g^{\alpha\beta} - \widetilde{\eta}^{\alpha\beta})| \lesssim \langle r \rangle^{-\delta} \ .$$

For a rotation $\widetilde{\Omega}_{jk} = x^j \widetilde{\partial}_k - x^k \widetilde{\partial}_j$ we recall:

$$\begin{aligned}
g^{\alpha\gamma}\partial_\gamma(\widetilde{\Omega}_{jk}^\beta) + g^{\beta\gamma}\partial_\gamma(\widetilde{\Omega}_{jk}^\alpha) \ &= \ g^{jj} - g^{kk} \ , && \alpha = j \ , \beta = k \ , \\
g^{\alpha\gamma}\partial_\gamma(\widetilde{\Omega}_{jk}^\beta) + g^{\beta\gamma}\partial_\gamma(\widetilde{\Omega}_{jk}^\alpha) \ &= \ -g^{k\beta} \ , && \alpha = j \ , \beta \neq k, j \ , \\
g^{\alpha\gamma}\partial_\gamma(\widetilde{\Omega}_{jk}^\beta) + g^{\beta\gamma}\partial_\gamma(\widetilde{\Omega}_{jk}^\alpha) \ &= \ 2g^{jk} \ , && \alpha = \beta = j \ , \\
g^{\alpha\gamma}\partial_\gamma(\widetilde{\Omega}_{jk}^\beta) + g^{\beta\gamma}\partial_\gamma(\widetilde{\Omega}_{jk}^\alpha) \ &= \ 0 \ , && \alpha \neq k, j \ , \beta \neq k, j \ .
\end{aligned}$$

With the exception of the $\alpha = j, \beta = u$ and $\alpha = k, \beta = u$ components, these terms all have the right form to apply (1.5) and get the decay. For the (j, u) case we observe:

$$\begin{aligned}(\mathcal{L}_{\widetilde{\Omega}_{jk}} g)^{ju} &= -\widetilde{\Omega}_{jk}(g^{ju}) + g^{j\gamma}\partial_\gamma(\widetilde{\Omega}^u_{jk}) + g^{u\gamma}\partial_\gamma(\widetilde{\Omega}^j_{jk}) \\ &= -\widetilde{\Omega}_{jk}(g^{ju} + \omega^j) + \widetilde{\Omega}_{jk}(\omega^j) - g^{ku} \ . \\ &= -\widetilde{\Omega}_{jk}(g^{ju} + \omega^j) - \omega^k - g^{ku} \ . \end{aligned} \tag{2.6}$$

The case (k, u) follows similarly. Therefore by (2.5) we conclude for all rotations:

$$|\widetilde{\Omega}(g^{\alpha\beta} - \widetilde{\eta}^{\alpha\beta})| \lesssim \langle r \rangle^{-\delta}$$

Since $\widetilde{\partial}_i = \frac{x^i}{r}\widetilde{\partial}_r + \frac{x^j}{r^2}\widetilde{\Omega}_{ji}$ the result follows immediately for $\widetilde{\partial}_i$ as well. A simple induction argument proves the remaining cases.

□

Since we will be using the volume forms $\sqrt{|g|}dx$ to integrate the relevant geometric quantities, it is important to have some decay estimates for the determinant in Bondi coordinates as well. The result above implies:

Corollary 2.1.3. *The determinant in Bondi coordinates $\sqrt{|g|}$ satisfies:*

$$|\partial_u^J \widetilde{\partial}_l^K(\sqrt{|g|} - 1)| \lesssim \langle u \rangle^{-|J|}\langle r \rangle^{-|K|-\delta} \ . \tag{2.7}$$

Proof. By continuous dependence of the determinant on the components of the matrix $(g)_{\alpha\beta}$ we may take the supremum over all components. The decay rates (2.4) then give us the result.

□

By combining (2.4) and (2.7) get the decay rates for $\sqrt{|g|}g^{ij}$ as well:

Corollary 2.1.4 (Decay rates for the (i, j) components with $\sqrt{|g|}$.)**.** *In the wave zone:*

$$|\partial_u^J \widetilde{\partial}_l^K(\sqrt{|g|}g^{ij} - \widetilde{\eta}^{ij})| \ \lesssim \ \langle r \rangle^{-\delta-|K|}\langle u \rangle^{-|J|} \ , \tag{2.8}$$

It is also important to point out that although the Bondi coordinates are originally defined only on an open set of $\mathcal{W}$, we may extend them in a natural way to the region of the space-time in which where we want to prove estimates for our solution ϕ.

Lemma 2.1.5. *In the region* $[t_0, \infty) \times \mathbb{R}^3$ *for* $t_0 > 0$ *sufficiently large, the function* $u_t = \partial_t u$ *does not vanish and the Bondi coordinates can be extended to this region with decay rates given by* (1.7a)*,* (1.7a)*, and*(2.8)*.*

Proof. Equation (2.1) gives such an extension provided that $\partial_t u$ in the (t, x) coordinates does not vanish. By assumption 1.1.3iii, in the neighborhood of the wave zone:

$$\partial_t u \ = \ 1 + O(\langle r \rangle^{-\delta}) \ .$$

Therefore for r sufficiently large these functions do not vanish. However in this region $r \approx t$ holds so this is satisfied by requiring t_0 to be sufficiently large. In the rest of the space-time we have $u = t - \langle r \rangle$, therefore $\partial_t u = 1$. The decay rates are simply a restatement of the decay rates for the metric $g_{\alpha\beta}$ plus the fact that the weights relative to each other satisfy:

i) In the interior $r \leqslant \epsilon t$ and $u \approx t$.

ii) In the wave zone $|u| \leqslant r$, $t \approx r$.

iii) In the region $\{\epsilon t \leqslant r \leqslant \frac{2}{3}t\} \cup \{\frac{3}{2}t \leqslant r \leqslant 2t\}$, therefore $t \approx r$ and $u \approx \underline{u}$

iv) In the far exterior $2t \leqslant r$.

□

Lastly, we mention that in normalized coordinates the light cones for the metric $g_{\alpha\beta}$ are within $O(1)$ of the Minkowski light cones – in particular, we do not have any logarithmic divergences. This is simply due to the fact that near spatial infinity these conditions allow us to perform a transformation $r \to r^*$ analogous to using Regge-Wheeler coordinates in Schwarzschild or Kerr. We will use this result throughout the rest of the paper.

2.1.2 The Radial/Angular Decomposition

In order to better capture the decay and peeling properties of the metric it is necessary to make a distinction between 'radial' and 'angular' Bondi derivatives. In particular, this becomes essential whenever we encounter a g^{ui} term of the metric since our decay rates (1.7a)-(1.7b) will distinguish between radial and angular components. This fundamental fact not only lies at the heart of our proof of boundedness of conformal energy, but also is very useful in making calculations more manageable since, in the wave zone, the conformal and characteristic energies are defined using L, e_3, e_4 derivatives and not $\widetilde{\partial}_i$ derivatives. We now proceed to make this decomposition for the Bondi coordinate derivatives and to record how the decay rates (1.7a)-(1.7b) will change in this new basis.

Let $\{e_a\}_{a=3,4}$ be a local basis on $\{r = const\} \cap \{u = const\}$. We locally split the coordinate derivatives $\widetilde{\partial}_i$ in terms of $\widetilde{\partial}_r$ and this basis by:

$$\widetilde{\partial}_i \ = \ \omega_i \widetilde{\partial}_r + \omega_i^a e_a \ ,$$

where,

$$\omega_a^i \ = \ e_a(x^i) \ , \qquad \omega_i \ = \ \omega^i \ = \ x^i/r \ , \qquad \partial_r \omega_i^a \ = \ 0 \ , \qquad [\widetilde{\partial}_r, e_a] \ = \ r^{-1} e_a \ , \tag{2.9}$$

with the functions ω^i, ω_i^a being homogeneous of degree zero in the variable r. Note that these conditions imply that the e_a cannot be coordinate derivatives. Since the $\{\omega_i^a\}$ form the components of a rotation matrix, we also have the following basic formulae:

$$\omega^i \omega_i^a \ = \ 0 \ , \qquad \omega_b^i \omega_i^a \ = \ \delta_b^a \ , \qquad \omega_i^a \omega_a^j \ = \ \delta_i^j - \omega_i \omega^j \ . \tag{2.10}$$

We define the *Bondi Frame* to be:

$$\mathcal{B} \ := \ \{\partial_u, \ \widetilde{\partial}_r, \ e_3, \ e_4\} \ . \tag{2.11}$$

By lemma 2.1.5 this frame is well-defined everywhere on $[t_0, \infty) \times \mathbb{R}^3$ for $t_0 > 0$ sufficiently large. Using this basis our first observation is the following:

Lemma 2.1.6. *For the components* g^{uj} *in the basis* $\mathcal{B}$*, conditions* (1.7a)- (1.7b) *imply:*

$$|\partial_u^\alpha \widetilde{\partial}_l^\beta (\sqrt{|g|} g^{ur} + 1)| \lesssim \langle r \rangle^{-\delta - |\beta|} \langle u \rangle^{-\alpha} \left(\frac{\langle u \rangle}{\langle r \rangle} \right)^{\frac{1}{2}} , \tag{2.12}$$

$$|\partial_u^\alpha \widetilde{\partial}_l^\beta (g^{ua})| \lesssim \langle r \rangle^{-\delta - |\beta|} \langle u \rangle^{-\alpha} \left(\frac{\langle u \rangle}{\langle r \rangle} \right) . \tag{2.13}$$

Proof. We induct on the number of derivatives. For (2.12), we start with the identity:

$$(\sqrt{|g|} g^{ur} + 1) = \sum_i \omega^i (\sqrt{|g|} g^{ui} + \omega^i) . \tag{2.14}$$

Case $\alpha = \beta = 0$: since $\omega^i = O(1)$, $\sqrt{|g|} - 1 = O(r^{-\delta})$, using (1.7a) and (1.7b) we get:

$$|(\sqrt{|g|} g^{ur} + 1)| \lesssim \sum_i |(\sqrt{|g|} g^{ui} + \omega^i)| \lesssim \langle r \rangle^{-\delta} \left(\frac{\langle u \rangle}{\langle r \rangle} \right)^{\frac{1}{2}} . \tag{2.15}$$

Case $\alpha + \beta = k$: by induction we may assume the result holds for $\alpha + \beta \leqslant k - 1$. Applying $\partial_u^\alpha \widetilde{\partial}_l^\beta$ to (2.14):

$$\partial_u^\alpha \widetilde{\partial}_l^\beta (\sqrt{|g|} g^{ur} + 1) = \sum_i \sum_{\nu + \theta = \beta} (\widetilde{\partial}_l^\nu \omega^i)(\partial_u^\alpha \widetilde{\partial}_l^\theta (\sqrt{|g|} g^{ui} + \omega^i)) .$$

Combining $|\widetilde{\partial}_l^\nu \omega^i| \lesssim r^{-|\nu|}$ and (2.7) for the determinant together with estimates (1.7a) and (1.7b) gives the result. Next, for (2.13) we begin with:

$$g^{ui} = \omega^i g^{ur} + \omega_a^i g^{au} .$$

By $\omega_i^a \omega_b^i = \delta_b^a$:

$$g^{au} = \omega_i^a (g^{ui} - \omega^i g^{ur}) .$$

Differentiating, using $|\widetilde{\partial}_l^\nu \omega_i^a| \lesssim r^{-|\nu|}$ and applying (1.7b):

$$\sum_{a,b} |\partial_u^\alpha \widetilde{\partial}_l^\beta (g^{au})| \lesssim \sum_{a,b} \sum_{\gamma + \lambda = \beta} |(\widetilde{\partial}_l^\gamma \omega_i^a) \partial_u^\alpha \widetilde{\partial}_l^\lambda (g^{ui} - \omega^i g^{ur})|$$
$$\lesssim r^{-\delta - |\beta|} \langle u \rangle^{-\alpha} \left(\frac{\langle u \rangle}{\langle r \rangle} \right) .$$

$\square$

An important consequence of these decay rates is the following:

Corollary 2.1.7. *Let $L = -\nabla u$ and ϕ be a test function. Inside the wave zone $\mathcal{W}$ the following estimates hold:*

$$|L\phi - \widetilde{\partial}_r\phi| \lesssim \langle r\rangle^{-\delta}|\widetilde{\partial}_r\phi| + \Big(\frac{\langle u\rangle}{\langle r\rangle}\Big)\langle r\rangle^{-\delta}\sum_a |e_a\phi| \ , \tag{2.16}$$

$$|g(L - \widetilde{\partial}_r, L - \widetilde{\partial}_r)| \lesssim \langle r\rangle^{-2\delta} \ . \tag{2.17}$$

Thus $\widetilde{\partial}_r$ is asymptotically null in this region. In the region $\{\epsilon t \leqslant \frac{2}{3}r\} \cup \{\frac{3}{2}t \leqslant r\}$:

$$|L\phi - \widetilde{\partial}_r\phi| \lesssim \langle r\rangle^{-\delta}(|\partial_u\phi| + |\widetilde{\partial}_r\phi| + |e_3\phi| + |e_4\phi|) \ . \tag{2.18}$$

Proof. Computing using the basis $\mathcal{B}$ inside $\mathcal{W}$, we have $L^u = 0$ trivially since u is an optical function. Using this together with the decay rates in (2.12) and (2.13):

$$|L^r - 1| = |-(g^{ur}+1)| \lesssim \langle r\rangle^{-\delta} \ ,$$
$$|L^a| \lesssim |g^{ua}| \lesssim \Big(\frac{\langle u\rangle}{\langle r\rangle}\Big)\langle r\rangle^{-\delta} \ .$$

Therefore (2.16) holds. Similarly:

$$\begin{aligned} g(L - \widetilde{\partial}_r, L - \widetilde{\partial}_r) &\lesssim |g^{ur}+1|^2|g_{rr}| + 2|g^{ur}+1||g^{ua}||g_{ra}| + \sum_b |g^{ua}|^2|g_{ab}| \\ &\lesssim \langle r\rangle^{-2\delta} + \Big(\frac{\langle u\rangle}{\langle r\rangle}\Big)\langle r\rangle^{-2\delta} + \Big(\frac{\langle u\rangle}{\langle r\rangle}\Big)^2\langle r\rangle^{-2\delta} \\ &\lesssim \langle r\rangle^{-2\delta} \to 0 \ , \end{aligned}$$

as $r \to \infty$. For the region $\{\epsilon t \leqslant \frac{2}{3}t\} \cup \{\frac{3}{2}t \leqslant r\}$, $L^u \neq 0$. However, using assumption 1.1.3iii and the decay rates (2.12), (2.13), the rest of the result follows by a similar calculation.

□

2.1.3 Relationship to Bondi-Sachs Metrics and Examples

Recall that a *Bondi-Sachs* metric is one for which given a world tube T and a set of coordinates $x^\alpha = (u, r, \theta, \phi)$, with u null, r radial, and θ, ϕ angular, the smooth curves on which (u,θ,ϕ) are constant are null geodesics. This implies that the components of metric tensor in this coordinate basis satisfy:

$$g_{11} \ = \ g_{12} \ = \ g_{13} \ = 0 \ ,$$

or equivalently:

$$g^{00} \ = \ g^{02} \ = \ g^{03} \ = 0 \ ,$$

An additional requirement – which has nothing to do with the null geodesic structure – is that the determinant of the angular part of the metric is that of a unit sphere metric. This definition was originally introduced in [BvdBM62] in order to study the relationship between gravitational radiation at infinity and mass loss. This type of metric has also provided a framework for computations in numerical relativity – see [BV06]. Our normalized coordinates can thought of as a generalized Bondi coordinate system whose derivatives are in Cartesian form. The link between a Bondi-Sachs metric and our normalized coordinates is provided by the frame $\mathcal{B}$. The main observation in this regard is the following: if a metric $g^{\alpha\beta}$ has normalized coordinates, then by the remarks above it also has a $\mathcal{B}$ basis on $[t_0,\infty)\times\mathbb{R}^3$ for $t_0>0$ sufficiently large. The metric components in this basis will then converge pointwise to a *Bondi-Sachs* metric as $r\to\infty$ in the wave zone, with explicit quantitative decay rates given by (2.12), (2.13) and (2.8).

In particular, we note that any metric which is exactly in Bondi-Sachs form is an example of a metric which has normalized coordinates. This is because, at least locally, we can always go from radial/angular derivatives to Cartesian coordinates since all the transformations involved are local diffeomeorphisms. With this in mind, we proceed to give some examples of metrics which have the normalized coordinates:

1) Minkowski: In (t,x^i) coordinates the optical function is given by $u=t-r$. The Bondi coordinates are $\{\partial_t,\ \partial_i-\omega^i\partial_t\}$. The inverse Minkowski metric in this basis is equal to the matrix $\widetilde{\eta}^{\alpha\beta}$ in definition (1.1.3). We can compute $\det\widetilde{\eta}=-1$ and thus it is immediately clear that this satisfies all the conditions in assumption

1.1.3. Transforming this into a Bondi frame via $\partial_u = \partial_t$, $\widetilde{\partial}_r = \partial_t - \partial_r$, e_a, e_b we find:

$$\eta^{\alpha\beta}(u,r,a,b) = \begin{pmatrix} 0 & -1 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & 0 & \multicolumn{2}{c}{\multirow{2}{*}{δ^{ab}}} \\ 0 & 0 & & \end{pmatrix}$$

where δ is the restriction of η to the spheres $r = \mathit{const}$ and $\det|\eta^{-1}| = r^{-2}(\sin\theta)^{-2}$.

2) Schwarzshild: Define the *Regge-Wheeler tortoise coordinate* to be $r^* = r + 2M\ln(\frac{r}{2M} - 1)$. Choosing $u = t - r^*$ and computing the metric in outgoing Eddington-Finkelstein coordinates (u, r, θ, ϕ):

$$g_{\mathbf{S}} = -\left(1 - \frac{2M}{r}\right)du^2 - 2dudr + r^2 d\sigma^2 \ .$$

Therefore the inverse metric is given by:

$$g_{\mathbf{S}}^{-1}(u,r,a,b) = \begin{pmatrix} 0 & -1 & 0 & 0 \\ -1 & (1-\frac{2M}{r}) & 0 & 0 \\ 0 & 0 & \multicolumn{2}{c}{\multirow{2}{*}{δ^{ab}}} \\ 0 & 0 & & \end{pmatrix} .$$

with, $\sqrt{|g_S|} = r^2\sin^2\theta$. We see that in the Wave zone assumption 1.1.3 holds with the Bondi frame being equivalent to (u, r, θ, ϕ).

3) Kerr: The authors in [BV06] compute the Kerr metric in Bondi-Sachs form up to r^{-1} errors to be:

$$\begin{aligned} g^{01} &= -1, & g^{11} &= 1 - \frac{2M}{r}, & g^{12} &= O(r^{-3}), \\ g^{13} &= O(r^{-3}), & g^{22} &= \frac{1}{r^2}, & g^{23} &= O(r^{-3}), \\ g^{33} &= \frac{1}{r^2\sin^2\theta} \ , \end{aligned}$$

with $\sqrt{|g_S|} = r^2\sin^2\theta + O(r^{-1})$.

More generally, the existence of the function u and the set of normalized coordinates should follow for any asymptotically flat metric with $|\partial_{t,x}^k g| \lesssim r^{-\delta-k}$ (in original coords) and with the mild curvature conditions $|R_{\alpha\beta\gamma\delta}| \lesssim r^{-2-\frac{1}{4}-\epsilon}$ and $|R_{\alpha\beta}| \lesssim r^{-2-\frac{1}{2}-\epsilon}$. We leave the proof of this assertion to future work.

2.1.4 Deformation Tensor Identities

We define the *conformal deformation tensor* to be:

$$ {}^{(X)}\widehat{\pi}_{\alpha\beta} \ = \ {}^{(X)}\pi_{\alpha\beta} - \frac{1}{2} g_{\alpha\beta}\, tr[{}^{(X)}\pi] \ . \tag{2.19} $$

A simple calculation then shows:

$$ T_{\alpha\beta}[\phi]^{(X)}\pi^{\alpha\beta} \ = \ \partial_\alpha\phi\partial_\beta\phi^{(X)}\pi^{\alpha\beta} - \frac{1}{2}g^{\alpha\beta}\partial_\alpha\phi\partial_\beta\phi\, tr[{}^{(X)}\pi] \ = \ \partial_\alpha\phi\partial_\beta\phi^{(X)}\widehat{\pi}^{\alpha\beta} \ . $$

For simplicity, we also use $\widehat{{}^{(X)}\pi}$ to define a *Conformal Killing* vector field X with respect to the metric g, with conformal factor Ω if:

$$ \widehat{{}^{(X)}\pi} + \Omega g \ = \ 0 \ , $$

holds. This is equivalent to the standard definition of conformal Killing since:

$$ \widehat{{}^{(X)}\pi} \ = \ \Omega g - \frac{1}{2} g(tr\pi) = \Omega g - 2\Omega g = -\Omega g \ . $$

To compute $\widehat{{}^{(X)}\pi}$ in local coordinates, we rely on the following:

Lemma 2.1.8. *For any smooth vector field X we have, in local coordinates:*

$$ \sqrt{|g|}(\widehat{{}^{(X)}\pi})^{\alpha\beta} \ = \ - X(\sqrt{|g|}g^{\alpha\beta}) + \sqrt{|g|}\left(g^{\alpha\gamma}\partial_\gamma(X^\beta) + g^{\beta\gamma}\partial_\gamma(X^\alpha) - g^{\alpha\beta}(\partial_\gamma X^\gamma)\right) \ . \tag{2.20} $$

Proof. Plugging in $\frac{1}{2}tr[{}^{(X)}\pi] = \nabla_\alpha X^\alpha$ into the identity (2.5) and simplifying:

$$ \begin{aligned} (\widehat{{}^{(X)}\pi})^{\alpha\beta} \ &= \ {}^{(X)}\pi^{\alpha\beta} - \frac{1}{2}g^{\alpha\beta} tr[{}^{(X)}\pi] \\ &= \ -X(g^{\alpha\beta}) + \partial^\alpha X^\beta + \partial^\beta X^\alpha - g^{\alpha\beta}(\frac{1}{\sqrt{|g|}}\partial_\gamma(\sqrt{|g|}X^\gamma)) \\ &= \ -X(g^{\alpha\beta}) - g^{\alpha\beta}\frac{1}{\sqrt{|g|}}X(\sqrt{|g|}) + \partial^\alpha X^\beta + \partial^\beta X^\alpha - g^{\alpha\beta}\partial_\gamma X^\gamma \\ &= \ -\frac{1}{\sqrt{|g|}}X(\sqrt{|g|}g^{\alpha\beta}) + \partial^\alpha X^\beta + \partial^\beta X^\alpha - g^{\alpha\beta}\partial_\gamma X^\gamma. \end{aligned} $$

□

The formula above works only in local coordinates. However, in the wave zone the conformal energies are defined in terms of the basis $\mathcal{B}$. To remedy this we could, of course, derive an identity equivalent to (2.20) using a frame but that would make our calculations below unnecessarily long and tedious. Instead we note that since (1.7a)–(2.8) have the worst decay rates for g^{ui} terms, it suffices for us to use the basis $\mathcal{B}$ to compute only the components$(\widehat{{}^{(X)}\pi})^{ui}$, $i = r, a, b$. To achieve this, we start with (2.20) in Bondi coordinates and commute with ω^i, ω^i_a to get the identity corresponding identity with all the metric coefficients computed with respect to the frame $\mathcal{B}$. For our applications, it will suffice for us to get formulas for:

$$\sqrt{|g|}(\widehat{{}^{(\widetilde{X})}\pi})^{ui}(\widetilde{\partial}_i\phi\partial_u\phi) \ , \qquad \sqrt{|g|}(\widehat{{}^{(\widetilde{X})}\pi})^{ui}\widetilde{\partial}_i\partial_u(\phi) \ , \tag{2.21}$$

with $\widetilde{X} = \widetilde{K}_0, \widetilde{S}, \widetilde{\Omega}$ in the $\mathcal{B}$ frame. All occurrences of $(\widehat{{}^{(\widetilde{X})}\pi})^{ui}$ in the rest of this work will be of this form.

Definition 2.1.9. *Let X a smooth vector field and let the quantities $\sqrt{|g|}$, $\partial_\gamma X^\gamma$ be written in Bondi coordinates. We define:*

$$\sqrt{|g|}(\widehat{{}^{(X)}\pi})^{ui}_{\mathcal{B}} \ = \ -X(\sqrt{|g|}g^{ui}) + \sqrt{|g|}\left(g^{u\gamma}\partial_\gamma(X^i) + g^{i\gamma}\partial_\gamma(X^u) - g^{ui}(\partial_\gamma X^\gamma)\right) \ . \tag{2.22}$$

with $i = r, a, b$.

Lemma 2.1.10 (Deformation Tensor Identities for $(\widehat{{}^{(X)}\pi})^{ui}$ with $i = r, a, b$)**.** *For* (2.21) *computed in Bondi coordinates, the following identities hold:*

$$\sqrt{|g|}(\widehat{{}^{(\widetilde{K}_0)}\pi})^{ui}(\widetilde{\partial}_i\phi\partial_u\phi) \ = \ \sqrt{|g|}(\widehat{{}^{(\widetilde{K}_0)}\pi})^{ui}_{\mathcal{B}}(e_i\phi\partial_u\phi) + 2(u+r)\sqrt{|g|}g^{ua}(e_a\phi\partial_u\phi) \tag{2.23}$$

$$\sqrt{|g|}(\widehat{{}^{(\widetilde{S})}\pi})^{ui}(\widetilde{\partial}_i\phi\partial_u\phi) \ = \ \sqrt{|g|}(\widehat{{}^{(\widetilde{S})}\pi})^{ui}_{\mathcal{B}}(e_i\phi\partial_u\phi) \tag{2.24}$$

$$\sqrt{|g|}(\widehat{{}^{(\widetilde{\Omega})}\pi})^{ui}(\widetilde{\partial}_i\phi\partial_u\phi) \ = \ \sqrt{|g|}(\widehat{{}^{(\widetilde{\Omega})}\pi})^{ui}_{\mathcal{B}}(e_i\phi\partial_u\phi) + {}^{(\widetilde{\Omega})}\Theta^i\sqrt{|g|}g^{ui}(e_i\phi\partial_u\phi) \tag{2.25}$$

$$\sqrt{|g|}(\widehat{{}^{(\widetilde{K}_0)}\pi})^{ui}\widetilde{\partial}_i\partial_u(\phi) \ = \ \sqrt{|g|}(\widehat{{}^{(\widetilde{K}_0)}\pi})^{ui}_{\mathcal{B}}(e_i\partial_u\phi) + 2(u+r)\sqrt{|g|}g^{ui}(e_i\partial_u\phi) \tag{2.26}$$

$$\sqrt{|g|}(\widehat{{}^{(\widetilde{S})}\pi})^{ui}\widetilde{\partial}_i\partial_u(\phi) \ = \ \sqrt{|g|}(\widehat{{}^{(\widetilde{S})}\pi})^{ui}_{\mathcal{B}}(e_i\partial_u\phi) \tag{2.27}$$

$$\sqrt{|g|}(\widehat{{}^{(\widetilde{\Omega})}\pi})^{ui}\widetilde{\partial}_i\partial_u(\phi) \ = \ \sqrt{|g|}(\widehat{{}^{(\widetilde{\Omega})}\pi})^{ui}_{\mathcal{B}}(e_i\partial_u\phi) + {}^{(\widetilde{\Omega})}\Theta^i\sqrt{|g|}g^{ui}(e_i\partial_u\phi) \ , \tag{2.28}$$

with:

$$\partial_u({}^{(\widetilde{\Omega})}\Theta^r) \;=\; 0\ , \qquad |\widetilde{\partial}_i^k({}^{(\widetilde{\Omega})}\Theta^r)| \;\lesssim\; \langle r\rangle^{-k}\ , \tag{2.29}$$

$$\partial_u({}^{(\widetilde{\Omega})}\Theta^a) \;=\; 0\ , \qquad |\widetilde{\partial}_i^k({}^{(\widetilde{\Omega})}\Theta^a)| \;\lesssim\; \langle r\rangle^{-k}\ . \tag{2.30}$$

Proof. Let's start with the cases $X = \widetilde{K}_0, \widetilde{S}$. Using (2.20) in Bondi coordinates and contracting with w_i to get the (u,r) terms:

$$\begin{aligned}
&- X(\sqrt{|g|}g^{ui}) + \sqrt{|g|}\left(g^{u\gamma}\partial_\gamma(X^i) + g^{i\gamma}\partial_\gamma(X^u) - g^{ui}(\partial_\gamma X^\gamma)\right)(\omega_i\widetilde{\partial}_r\phi\partial_u\phi)\\
&= -X(\sqrt{|g|}g^{ur}) + \sqrt{|g|}\left(\omega_i g^{u\gamma}\partial_\gamma(X^i) + \omega_i g^{i\gamma}\partial_\gamma(X^u) - g^{ur}(\partial_\gamma X^\gamma)\right)(\widetilde{\partial}_r\phi\partial_u\phi)\\
&\qquad + [X,\omega_i](\sqrt{|g|}g^{ui})
\end{aligned}$$

When $X = \widetilde{K}_0, \widetilde{S}$, the term $[X,\omega_i] = 0$. For the two terms missing a g^{ur} above we compute:

$$\begin{aligned}
\omega_i g^{u\gamma}\partial_\gamma(\widetilde{K}_0^i) &= \omega_i g^{uj}2((u+r)\delta_j^i + x^i\omega_j) \;=\; g^{ur}(2u+4r) \;=\; g^{ur}\widetilde{\partial}_r\widetilde{K}_0^r\\
\omega_i g^{i\gamma}\partial_\gamma(\widetilde{K}_0^u) &= g^{ur}\partial_u(\widetilde{K}_0^u)\\
\omega_i g^{u\gamma}\partial_\gamma(\widetilde{S}^i) &= g^{ur}\widetilde{\partial}_r\widetilde{S}^r\\
\omega_i g^{i\gamma}\partial_\gamma(\widetilde{S}^u) &= g^{ur}\partial_u(\widetilde{S}^u)\ .
\end{aligned}$$

To get the (u,a) terms we use (2.20) in Bondi coordinates and contract with w_i^a:

$$\begin{aligned}
&- X(\sqrt{|g|}g^{ui}) + \sqrt{|g|}\left(g^{u\gamma}\partial_\gamma(X^i) + g^{i\gamma}\partial_\gamma(X^u) - g^{ui}(\partial_\gamma X^\gamma)\right)(\omega_i^a e_a\phi\partial_u\phi)\\
&= -X(\sqrt{|g|}g^{ua}) + \sqrt{|g|}\left(\omega_i^a g^{u\gamma}\partial_\gamma(X^i) + \omega_i^a g^{i\gamma}\partial_\gamma(X^u) - g^{ua}(\partial_\gamma X^\gamma)\right)(\widetilde{\partial}_r\phi\partial_u\phi)\\
&\qquad + [X,\omega_i^a](\sqrt{|g|}g^{ui})\ ,
\end{aligned}$$

with $[X,\omega_i^a] = 0$ again. We compute the two terms missing g^{ua} above to be:

$$\begin{aligned}
\omega_i^a g^{u\gamma}\partial_\gamma(\widetilde{K}_0^i) &= \omega_i^a g^{uj}2((u+r)\delta_j^i + x^i\omega_j) \;=\; g^{ua}2(u+r)\\
\omega_i^a g^{i\gamma}\partial_\gamma(\widetilde{K}_0^u) &= 2u\omega_i^a g^{iu} \;=\; g^{ua}\partial_u(\widetilde{K}_0^u)\\
\omega_i^a g^{u\gamma}\partial_\gamma(\widetilde{S}^i) &= \omega_i^a g^{uj}\delta_j^i \;=\; g^{ua}\\
\omega_i^a g^{i\gamma}\partial_\gamma(\widetilde{S}^u) &= \omega_i^a g^{iu} \;=\; g^{ua}\partial_u(\widetilde{S}^u)\ .
\end{aligned}$$

Next we deal with the rotations $X = \widetilde{\Omega}_{kl} = x^k\widetilde{\partial}_l - x^l\widetilde{\partial}_k$. Since the commutators $[X,\omega_i^a]$ and $[X,\omega_i]$ no longer vanish, the proof will be a little different in this case. Start by noting that $\partial_\gamma \widetilde{\Omega}^\gamma = 0$ and $\widetilde{\Omega}^u = 0$ so these terms drop out in equation (2.20) (in Bondi coordinates). Adding and subtracting $w^i g^{ur}$ we get:

$$\begin{aligned} & -\widetilde{\Omega}(\sqrt{|g|}g^{ui}) + \sqrt{|g|}\left(g^{u\gamma}\partial_\gamma(\widetilde{\Omega}^i) + g^{i\gamma}\partial_\gamma(\widetilde{\Omega}^u) - g^{ui}(\partial_\gamma\widetilde{\Omega}^\gamma)\right)(\widetilde{\partial}_i\phi\partial_u\phi) \\ &= -\widetilde{\Omega}(\sqrt{|g|}(g^{ui} - \omega^i g^{ur})) + \sqrt{|g|}\left((g^{u\gamma} - \omega^\gamma g^{ur})\partial_\gamma(\widetilde{\Omega}^i)\right)(\widetilde{\partial}_i\phi\partial_u\phi) \\ &\quad - \widetilde{\Omega}(\sqrt{|g|}\omega^i g^{ur}) + \sqrt{|g|}\left(\omega^\gamma g^{ur}\partial_\gamma(\widetilde{\Omega}^i)\right)(\widetilde{\partial}_i\phi\partial_u\phi) \end{aligned} \tag{2.31}$$

The second line above corresponds to g^{ua} terms. To prove this it suffices to show that the radial part of this line is zero. This, in turn, reduces to showing:

$$\omega_i[-\widetilde{\Omega}(\sqrt{|g|}(g^{ui} - \omega^i g^{ur})) + \sqrt{|g|}\left((g^{u\gamma} - \omega^\gamma g^{ur})\partial_\gamma(\widetilde{\Omega}^i)\right)] = 0 \; . \tag{2.32}$$

Since $\omega_i(g^{ui} - \omega^i g^{ur}) = 0$, commuting with w^i further reduces this to establishing:

$$\widetilde{\Omega}(\omega_i)(g^{ui} - \omega^i g^{ur}) + \left(\omega_i(g^{u\gamma} - \omega^\gamma g^{u\gamma})\partial_\gamma(\widetilde{\Omega}^i)\right) = 0 \; .$$

Computing each of these terms shows:

$$\begin{aligned} \widetilde{\Omega}(\omega_i)(g^{ui} - \omega^i g^{ur}) &= \frac{x^k\delta_i^l - x^l\delta_i^k}{r}(g^{ui} - \omega^i g^{ur}) \\ &= \omega^k(g^{ul} - \omega^l g^{ur}) - \omega^l(g^{uk} - \omega^k g^{ur}) \; , \\ \omega_i(g^{u\gamma} - \omega^\gamma g^{ur})\partial_\gamma(\widetilde{\Omega}^i) &= \omega^l(g^{uk} - \omega^k g^{ur}) - \omega^k(g^{ul} - \omega^l g^{ur}) \; , \end{aligned}$$

which is enough to prove our claim. In a similar vein, the last line of (2.31) corresponds to the (u, r) terms. Simplifying, commuting with w^i and using the identity $\widetilde{\Omega}(\omega^i) - \left(\omega^\gamma\partial_\gamma(\widetilde{\Omega}^i)\right) = 0$ yields:

$$\begin{aligned} & -\widetilde{\Omega}(\sqrt{|g|}\omega^i g^{ur}) + \sqrt{|g|}\left(\omega^\gamma g^{ur}\partial_\gamma(\widetilde{\Omega}^i)\right) \\ &= -\widetilde{\Omega}(\sqrt{|g|}g^{ur})\omega^i - \sqrt{|g|}g^{ur}\left[\widetilde{\Omega}(\omega^i) - \left(\omega^\gamma\partial_\gamma(\widetilde{\Omega}^i)\right)\right] \\ &= -\widetilde{\Omega}(\sqrt{|g|}g^{ur})\omega^i \; . \end{aligned} \tag{2.33}$$

Applying (2.32) and (2.33) to identity (2.31) gives us:

$$\begin{aligned} &= \left[-\widetilde{\Omega}(\sqrt{|g|}(g^{ua})) + \sqrt{|g|}\left((g^{ub}e_b(\widetilde{\Omega}^a) + (g^{ub}e_b(\omega_i^a)\widetilde{\Omega}^i\right)\right](e_a\phi\partial_u\phi) \\ &+ \left[-\widetilde{\Omega}(\sqrt{|g|}g^{ur}) - \widetilde{\Omega}(\omega^i)\sqrt{|g|}g^{ur}\right](\widetilde{\partial}_r\phi\partial_u\phi) + \left(-\widetilde{\Omega}(\omega_a^i)\sqrt{|g|}g^{ua}\right)(\widetilde{\partial}_i\phi\partial_u\phi) \; . \end{aligned}$$

□

2.2 No Superradiance

The fundamental fact which allows us to achieve control of our solution ϕ inside the region $r \leqslant \epsilon t$ is the fact that ∂_t is timelike everywhere. In order to prove this fact we will need the following:

Lemma 2.2.1 (Estimate for the coefficients)**.** *Let* $(\mathcal{M}, g_{\alpha\beta})$ *satisfy all the assumptions in the Main Theorem* (1.4.1)*. Let* ξ *be a forward null geodesic given in* (t, x^i) *coordinates by* ξ^α *with affine parameter* s *satisfying* $\dot{\gamma}t\big|_{s=0} = 1$ *with* $\dot{\gamma}s \equiv 1$, $\gamma(0) \in \{|x| \leqslant r_0\}$*. Then* $\exists$ *constant* $A(r_0) > 0$ *such that:*

$$\sup_\alpha |\xi_0^\alpha|_\infty A^{-1} \ \leqslant \ \sup_\alpha |\xi^\alpha|_\infty \ \leqslant \ \sup_\alpha |\xi_0^\alpha|_\infty A$$

Proof of Lemma. By our non-trapping assumption $\exists$ $C(\lambda) > 0$ such that for all $s \geq C$, $\gamma(s) \in \{|g - \eta| \ \leqslant \ \lambda\}$ with $\lambda > 0$. By choosing λ sufficiently small, it is clear that it suffices to prove the result only for the range $s \leq C$. By continuity of solutions to ODE and compactness we can choose $A > 0$ with:

$$\sup_{x_0 \in \{r \leqslant r_0\}} \{\sup_\alpha \dot{\gamma}^\alpha(C)\} \ \leqslant \ A \ ,$$
$$\inf_{x_0 \in \{r \leqslant r_0\}} \{\sup_\alpha \dot{\gamma}^\alpha(C)\} \ \geqslant \ A^{-1} \ .$$

□

Proposition 2.2.2 (∂_t is timelike everywhere)**.** *Let* $(\mathcal{M}, g_{\alpha\beta})$ *be a 4-dimensional, smooth, asymptotically flat Lorentzian manifold with metric* g *satisfying assumption 1.3.1. Assume that the spacetime is of the form* $\mathbb{R} \times \mathbb{R}^3$*, with the level sets of* t *being space-like and with* $|\mathcal{L}_{\partial_t} g| \leqslant \epsilon$*. Let* A *be as in the previous lemma, then there exists* $\epsilon^*(A) > 0$ *sufficiently small such that for all* $\epsilon < \epsilon^*(A)$, ∂_t *is timelike everywhere on* $\mathcal{M}$*.*

Proof. Assume for a contradiction that there exists a $p \in \mathcal{M}$ such that ∂_t is not timelike. By continuity we may in fact take $\langle \partial_t, \partial_t \rangle\big|_p = 0$. Let $\gamma(s)$ be the unique, affinely parametrized forward null geodesic with $\gamma(s_0) \in T_p\mathcal{M}$ with affine parameter s such that $\gamma(s_0) = \partial_t$ at p. For sufficiently small $r \in (-r_0, r_0)$ we may then define a smooth 1-parameter family of curves $\gamma_r(s)$ with $\gamma_0(s) = \gamma(s)$. Let $\frac{d}{ds} = \xi$. Then, along the geodesic γ we have:

$$\begin{aligned} \xi\langle\partial_t, \xi\rangle &= \langle\nabla_\xi\partial_t, \xi\rangle + \langle\partial_t, \nabla_\xi\xi\rangle = -\langle[\partial_t, \xi], \xi\rangle + \langle\nabla_{\partial_t}\xi, \xi\rangle \\ &= -\langle\mathcal{L}_{\partial_t}\xi, \xi\rangle + \frac{1}{2}\partial_t\langle\xi, \xi\rangle \\ &= -\mathcal{L}_{\partial_t}g(\xi, \xi) - \frac{1}{2}\partial_t\langle\xi, \xi\rangle + \frac{1}{2}\partial_t\langle\xi, \xi\rangle \ . \end{aligned}$$

Integrating this identity along γ:

$$\langle\partial_t, \xi\rangle\Big|_{s_0}^{s_1} = \int_{s_0}^{s_1} \mathcal{L}_{\partial_t}g(\xi, \xi)\ ds$$

Combining assumption 1.3.1 together with $\partial_t = \xi$ at $s = s_0$ and taking absolute value gives us:

$$\left|\langle\partial_t, \xi\rangle\Big|_{s=s_1}\right| \lesssim A\epsilon|s_1 - s_0| \ . \tag{2.34}$$

Let $\delta_1 > 0$, and let $s^*_{\delta_1}$ be as above. The claim is that if the following holds:

$$|g - \eta| \leqslant \delta_1 \qquad \langle\xi, \xi\rangle = 0 \ , \qquad \delta_2 = \sup_\alpha |\xi^\alpha|_\infty \ ,$$

then:

$$|\langle\partial_t, \xi\rangle| > C\delta_2 \ .$$

Since δ_1 is small, we have:

$$\xi = [(\partial_t + \omega\cdot\nabla_x) + O(\delta_1)]\, O(\delta_2)$$

Thus:

$$\langle\partial_t, \xi\rangle \approx \delta_2 \approx A^{-1}$$

Choosing:

$$\epsilon < C_0\frac{A^{-1}}{|s^*_{\delta_1} - s_0|}$$

gives a contradiction. □

2.3 Preliminary Reductions and Setup for the Proof of the Main Theorem

2.3.1 Reduction to the Case $t_1 > t_0 \geq t^* >> 1$

Without loss of generality in all that follows we may reduce to the case where $t_0 \gg 1$. This follows from the fact that for any fixed $t^* \geq 1$, estimates (1.12) and (1.13) will hold for $t_0, t_1 < t^*$ with constants that depend on the parameter t^*. This, in turn, is a direct consequence of the (local) energy estimate:

$$\mathring{E}_k^{\frac{1}{2}}[\phi(t)] \ \leqslant \ C_{k,t^*} \left(\mathring{E}_k^{\frac{1}{2}}[\phi(t_0)] + \int_{t_0}^{t^*} \| \, F(\tau, \cdot) \, \|_{H^k} \ d\tau \right) \tag{2.35}$$

which will hold for ϕ a solution to the wave equation (1.1), $k \in \mathbb{Z}^+$, $t_0 \leqslant t \leqslant t_1 < t^*$. Multiplying this estimate by t-weights:

$$\mathring{E}_k^{\frac{1}{2}}[t\phi(t)] \ \leqslant t^* \mathring{E}_k^{\frac{1}{2}}[\phi(t)] \ \leqslant \ \widetilde{C}_{k,t^*} \left(\mathring{E}_k^{\frac{1}{2}}[\phi(t_0)] + \int_{t_0}^{t^*} \| \, F(\tau, \cdot) \, \|_{H^k} \ d\tau \right) \tag{2.36}$$

with $\widetilde{C}_{k,t^*} = t^* C_{k,t^*}$. Estimates (1.12) and (1.13) follow in the range $t_0 \leqslant t \leqslant t_1 < t^*$. Thus, the non-trivial problem will be to prove (1.12) and (1.13) with $t_1 > t_0 \geq t^*$ with t^* sufficiently large and with constants that do not depend on t. In order to simplify the arguments that follow we will now make an explicit choice of t^*. Start with the following:

Definition 2.3.1. *Let β be given by:*

$$\beta = \min\{\frac{1}{2}(\gamma' - \epsilon) \ , \gamma - \gamma' \ , \frac{\delta}{2} \ , \gamma'\} \ . \tag{2.37}$$

For our purposes, since ϵ is fixed and small enough to beat all the constants in the finite number of estimates in use, we take t^* sufficiently large so that:

$$\langle t_0 \rangle^{-\beta} \ \leqslant \ \langle t^* \rangle^{-\beta} \ < \ \epsilon^2 \ . \tag{2.38}$$

2.3.2 Bootstrap Notation and Conventions

Here we set up some notations to help us deal with bootstrap arguments. First, every time we encounter an estimate of the form:

$$LHS \ \leqslant \ C_0(RHS + \epsilon LHS)$$

with absolute constant $C_0 > 0$ and we say that we will "bootstrap the term ϵLHS" we simply mean that the parameter ϵ has already been chosen small enough so that $C_0\epsilon < 1$ holds and the last term above can be absorbed on the LHS of the estimate. Second, since (2.37) and (2.38) are both satisfied, whenever we encounter estimates of the form:

$$LHS_{[t_0,t_1]} \leqslant C_0(RHS_{[t_0,t_1]} + \langle t_0\rangle^{-\beta} LHS_{[t_0,t_1]}) \ , \tag{2.39}$$

$$LHS_{[t_0,t_1]} \leqslant C_0(RHS_{[t_0,t_1]} + \epsilon^{-1}\langle t_0\rangle^{-\beta} LHS_{[t_0,t_1]}) \ , \tag{2.40}$$

when we say that we will "bootstrap the term $\langle t_0\rangle^{-\beta} LHS_{[t_0,t_1]}$" or "bootstrap the term $\epsilon^{-1}\langle t_0\rangle^{-\beta} LHS_{[t_0,t_1]}$" we simply mean that the since parameter t^* has already been chosen large enough so that (2.38) holds and since $C_0\epsilon < 1$, we may absorb these terms on the LHS of the estimate to get:

$$LHS_{[t_0,t_1]} \leqslant C_1(RHS_{[t_0,t_1]}) \ .$$

Chapter 3

Conformal Energy Estimate

Theorem 3.0.2 (Conformal Energy Estimate)**.** *Let ϕ be a test function and all the hypotheses from the Main Theorem 1.4.1 be satisfied. Then, for all $0 \leq t_0 \leq t \leq t_1$ the conformal energy estimate* (1.12) *holds:*

$$\sup_{t_0 \leqslant t \leqslant t_1} CE^{\frac{1}{2}}[\phi(t)] + \sup_{\mathcal{W} \cap [t_0,t_1]} \big(CE^{\frac{1}{2}}_{ch}[\phi(u)]\big) + \| \, \phi \, \|_{LSM^{1-\gamma'}_{r \leqslant \frac{1}{10}t}[t_0,t_1]}$$
$$\lesssim \; CE^{\frac{1}{2}}[\phi(t_0)] + \| \, F \, \|_{LS^{*,1+\gamma',\frac{1}{2}}[t_0,t_1]} \; .$$

Definition 3.0.3. *We define the hybrid vector field $\overline{K_0}$ in (t,x) coordinates by:*

$$\overline{K_0} \;=\; \widetilde{K_0} + (u_t)^{-1}\partial_t \;=\; \left(\frac{1 + u^2 - 2(u+r)r\partial_r u}{\partial_t u}\right)\partial_t + 2(u+r)r\partial_r \; ,$$

with $\widetilde{K}_0$ given by (1.26)*. Equivalently, in (u,x) coordinates this has the form:*

$$\overline{K_0} \;=\; \widetilde{K_0} + \partial_u \;=\; (1+u^2)\partial_u + 2(u+r)r\widetilde{\partial}_r \; .$$

with $\widetilde{K}_0$ in this coordinate system given by (1.27)*.*

The proof of the conformal energy estimate is a direct consequence of the following two statements:

Proposition 3.0.4 (Output From Using $\overline{K_0}$ as a Multiplier)**.** *Let ϕ be a test function and assume that all the hypotheses from the Main Theorem 1.4.1 are* *satisfied. Then, for all $t_0, t_1 \in [t^*, \infty)$ with t^* satisfying* (2.38) *we have:*

$$\sup_{t_0 \leqslant t \leqslant t_1} CE^{\frac{1}{2}}[\phi(t)] + \sup_{\mathcal{W}\cap[t_0,t_1]} \big(CE_{ch}^{\frac{1}{2}}[\phi(u)]\big)$$
$$\lesssim \ CE^{\frac{1}{2}}[\phi(t_0)] + \epsilon^{-1}\| F \|_{LS^{*,1+\gamma',\frac{1}{2}}[t_0,t_1]} + \epsilon\| \phi \|_{LSM^{1-\gamma'}_{r\leqslant\frac{1}{10}t}[t_0,t_1]} \ . \tag{3.1}$$

Proposition 3.0.5 (T-Weighted LS Estimate)**.** *Let ϕ be a test function and assume that all the hypotheses from the Main Theorem 1.4.1 are satisfied. Then, for all $t_0, t_1 \in [t^*, \infty)$ with t^* satisfying* (2.38) *we have:*

$$\| \phi \|_{LSM^{1-\gamma'}_{r\leqslant\frac{1}{10}t}[t_0,t_1]} \ \lesssim \ CE[\phi(t_0)]^{\frac{1}{2}} + \| F \|_{LS^{*,1-\gamma',\frac{1}{2}}[t_0,t_1]} + \epsilon \sup_{t_0\leqslant t\leqslant t_1} CE[\phi(t)]^{\frac{1}{2}} \ . \tag{3.2}$$

Proof of Theorem 3.0.2. By the reduction in section 2.3.1 it suffices to prove the result for $t_1, t_0 > t^*$. Adding estimates (3.1) and (3.2):

$$\sup_{t_0 \leqslant t \leqslant t_1} CE^{\frac{1}{2}}[\phi(t)] + \sup_{\mathcal{W}\cap[t_0,t_1]} \big(CE_{ch}^{\frac{1}{2}}[\phi(u)]\big) + \| \phi \|_{LSM^{1-\gamma'}_{r\leqslant\frac{1}{10}t}[t_0,t_1]}$$
$$\lesssim \ CE^{\frac{1}{2}}[\phi(t_0)] + \epsilon^{-1}\| F \|_{LS^{*,1+\gamma',\frac{1}{2}}[t_0,t_1]} + \epsilon\| \phi \|_{LSM^{1-\gamma'}_{r\leqslant\frac{1}{10}t}[t_0,t_1]}$$
$$+ \epsilon \sup_{t_0\leqslant t\leqslant t_1} CE[\phi(t)]^{\frac{1}{2}} \ .$$

Bootstrapping the last two terms finishes the proof. □

3.1 Proof of the T-Weighted LS Estimate

The T-weighted LS estimate is the main tool used to control all the error terms generated in the region where $\{r \leqslant \frac{1}{10}t\}$ when using vector fields as multipliers and where $\{r \leqslant \epsilon t\}$ when using vector fields as commutators. The existence of this estimate is a direct consequence the basic LS estimate (1.10).

Proof of Proposition 3.0.5. Apply LS estimate to the function $t^{1-\gamma'}\chi_{r\leqslant\frac{1}{10}t}\phi$:

$$\begin{aligned}
&\| \phi \|_{LSM^{1-\gamma'}_{r\leqslant\frac{1}{10}t}[t_0,t_1]} \\
&\lesssim E^{\frac{1}{2}}[t^{1-\gamma'}\chi_{r\leqslant\frac{1}{10}t}\phi](t_0) + \| \langle x\rangle^{\frac{1}{2}+\epsilon}\Box_g(t^{1-\gamma'}\chi_{r\leqslant\frac{1}{10}t}\phi) \|_{L^2_{t,x}[t_0,t_1]} \\
&+ \| \frac{t^{1-\gamma'}}{r^{\frac{3}{2}+\epsilon}}\chi_{r\sim\frac{1}{10}t}\phi \|_{L^2_{t,x}[t_0,t_1]} \\
&\lesssim E^{\frac{1}{2}}[t^{1-\gamma'}\chi_{r\leqslant\frac{1}{10}t}\phi](t_0) + \| F \|_{LS^{*,1-\gamma',\frac{1}{2}}[t_0,t_1]} \\
&+ \| \langle x\rangle^{\frac{1}{2}+\epsilon}[\Box_g, t^{1-\gamma'}\chi_{r\leqslant\frac{1}{10}t}]\phi \|_{L^2_{t,x}[t_0,t_1]} + \| \frac{t^{1-\gamma'}}{r^{\frac{3}{2}+\epsilon}}\chi_{r\sim\frac{1}{10}t}\phi \|_{L^2_{t,x}[t_0,t_1]} .
\end{aligned}$$

For the last term above we use the fact that $(\chi_{r\leqslant\frac{1}{10}t})' \sim r^{-1}\chi_{r\sim\frac{1}{10}t}$ is supported where $r \sim \frac{1}{10}t$. Taking $\sup_{t_0\leqslant t\leqslant t_1}$ and integrating:

$$\| \frac{t^{1-\gamma'}}{r^{\frac{3}{2}+\epsilon}}\chi_{r\sim\frac{1}{10}t}\phi \|_{L^2_{t,x}[t_0,t_1]} \lesssim \langle t_0\rangle^{-\beta} \sup_{t_0\leqslant t\leqslant t_1} CE[\phi(t)]^{\frac{1}{2}} .$$

Expanding the commutator:

$$\begin{aligned}
[\Box_g, t^{1-\gamma'}\chi_{r\leqslant\frac{1}{10}t}]\phi &= \Box_g(t^{1-\gamma'})\chi_{r\leqslant\frac{1}{10}t}\phi + \Box_g(\chi_{r\leqslant\frac{1}{10}t})t^{1-\gamma'}\phi + 2\partial^\alpha(t^{1-\gamma'})\partial_\alpha(\chi_{r\leqslant\frac{1}{10}t})\phi \\
&+ 2t^{1-\gamma'}\partial_\alpha(\chi_{r\leqslant\frac{1}{10}t})\partial^\alpha\phi + 2\partial_\alpha(t^{1-\gamma'})\chi_{r\leqslant\frac{1}{10}t}\partial^\alpha\phi \\
&:= I + II + III + IV + V.
\end{aligned}$$

Since we are in the region $\{r \leqslant \frac{1}{10}t\}$ we compute these terms using ∂_t, ∂_x derivatives. For I we have:

$$\begin{aligned}
&\| \langle x\rangle^{\frac{1}{2}+\epsilon}\Box_g(t^{1-\gamma'})\chi_{r\leqslant\frac{1}{10}t}\phi \|_{L^2_{t,x}[t_0,t_1]} \\
&\lesssim \| \frac{\chi_{r\leqslant\frac{1}{10}t}\phi}{t^{\frac{1}{2}+\gamma'-\epsilon}} \|_{L^2_{t,x}[t_0,t_1]} + \| \frac{t^{-\gamma'}\chi_{r\leqslant\frac{1}{10}t}\phi}{r^{\frac{1}{2}+\delta-\epsilon}} \|_{L^2_{t,x}[t_0,t_1]} \\
&\lesssim \| \frac{\chi_{r\leqslant\frac{1}{10}t}\phi}{t^{\frac{1}{2}+\gamma'-\epsilon}} \|_{L^2_{t,x}[t_0,t_1]} + \| \frac{\chi_{r\leqslant\frac{1}{10}t}t\phi}{t^{\frac{1}{2}+\gamma'}r} \|_{L^2_{t,x}[t_0,t_1]} \\
&\lesssim \langle t_0\rangle^{-\beta} \sup_{t_0\leqslant t\leqslant t_1} CE[\phi(t)]^{\frac{1}{2}} .
\end{aligned}$$

Since II is supported where $r \sim \frac{1}{10}t$ we get:

$$\begin{aligned}
\| \langle x\rangle^{\frac{1}{2}+\epsilon}\Box_g(\chi_{r\leqslant\frac{1}{10}t})t^{1-\gamma'}\phi \|_{L^2_{t,x}[t_0,t_1]} &\lesssim \| \frac{t^{1-\gamma'}}{r^{\frac{3}{2}}}\chi_{r\sim\frac{1}{10}t}\phi \|_{L^2_{t,x}[t_0,t_1]} \\
&\lesssim \langle t_0\rangle^{-\beta} \sup_{t_0\leqslant t\leqslant t_1} CE[\phi(t)]^{\frac{1}{2}} .
\end{aligned}$$

The term *III* is controlled by the same type of proof. *IV* is controlled since $r \leqslant \frac{1}{10}t$ implies $u = t - \langle r \rangle$ and $(1-\frac{1}{10})t \leqslant |t - \langle r \rangle| \leqslant t$:

$$\begin{aligned} \| \langle x \rangle^{\frac{1}{2}+\epsilon} t^{1-\gamma'} \partial_\alpha (\chi_{r \leqslant \frac{1}{10}t}) \partial^\alpha \phi \|_{L^2_{t,x}[t_0,t_1]} &\lesssim \| t^{-\frac{1}{2}-\gamma'+\epsilon} \chi_{r\sim \frac{1}{10}t} t \nabla \phi \|_{L^2_{t,x}[t_0,t_1]} \\ &\lesssim \Big(\int_{t_0}^{t_1} \int_{\mathbb{R}^3} \frac{(\chi_{r\sim\frac{1}{10}t})^2 (\langle u \rangle \nabla \phi)^2}{t^{1+2\gamma'-2\epsilon}} dxdt \Big)^{\frac{1}{2}} \\ &\lesssim \langle t_0 \rangle^{-\beta} \sup_{t_0 \leqslant t \leqslant t_1} CE[\phi(t)]^{\frac{1}{2}} \ . \end{aligned}$$

For V we use the same reasoning:

$$\begin{aligned} \| \langle x \rangle^{\frac{1}{2}+\epsilon} \chi_{r \leqslant \frac{1}{10}t} \partial_\alpha (t^{1-\gamma'}) \partial^\alpha \phi \|_{L^2_{t,x}[t_0,t_1]} &\lesssim \| t^{-\frac{1}{2}-\gamma'+\epsilon} \chi_{r\leqslant \frac{1}{10}t} t \nabla \phi \|_{L^2_{t,x}[t_0,t_1]} \\ &\lesssim \Big(\int_{t_0}^{t_1} \int_{\mathbb{R}^3} \frac{(\chi_{r\leqslant\frac{1}{10}t} \langle u \rangle \partial_\alpha \phi)^2}{t^{1+2\gamma'-2\epsilon}} dxdt \Big)^{\frac{1}{2}} \\ &\lesssim \langle t_0 \rangle^{-\beta} \sup_{t_0 \leqslant t \leqslant t_1} CE[\phi(t)]^{\frac{1}{2}} \ . \end{aligned}$$

The energy term can be handled similarly:

$$\begin{aligned} E^{\frac{1}{2}}[t^{1-\gamma'} \chi_{r\leqslant \frac{1}{10}t} \phi](t_0) &\lesssim \Big(\int_{\{t=t_0\}\cap \mathbb{R}^3} t^{-2\gamma'} \Big| \chi_{r\leqslant\frac{1}{10}t} (t - \langle r \rangle)(\nabla \phi, r^{-2}\phi) \Big|^2 dx \Big)^{\frac{1}{2}} \\ &\quad + \Big(\int_{\{t=t_0\}\cap\mathbb{R}^3} \Big| \frac{t^{2-\gamma'} \chi_{r\sim\frac{1}{10}t} \phi}{r^2} \Big|^2 dx \Big)^{\frac{1}{2}} \\ &\lesssim CE[\phi(t_0)]^{\frac{1}{2}}. \end{aligned}$$

□

3.2 Conformal Changes for Vector-Fields: Multipliers

In this section we develop all the necessary geometric machinery for the method of Lindblad-Sterbenz in section 1.6. In simple terms, the main goal is to record how all the formulas associated with the multiplier method (see section 1.5.1) change under conformal deformations of the metric $g_{\alpha\beta}$.

Let $ds^2 = g_{\alpha\beta} dx^\alpha dx^\beta$ be a Lorentzian metric on an 4 dimensional spacetime. We consider a conformally equivalent metric $\widetilde{ds^2} = \widetilde{g}_{\alpha\beta} dx^\alpha dx^\beta$ where $\Omega^2 \widetilde{g}_{\alpha\beta} = g_{\alpha\beta}$

for some weight function $\Omega = e^{\lambda}$. Let $\Box_g = D^{\alpha} D_{\alpha}$ be the wave equation for g, and $\Box_{\widetilde{g}} = \widetilde{D}^{\alpha}\widetilde{D}_{\alpha}$ the corresponding one for $\widetilde{g}$. Since $\widetilde{g} = \frac{1}{\Omega^2} g$ we get $\sqrt{|\widetilde{g}|} = \Omega^{-4}\sqrt{|g|}$. The following basic result shows how the wave equation transforms under such a conformal change:

Lemma 3.2.1. *Let* $\psi \ = \ \Omega\phi$ *then,*

$$\Box_{\widetilde{g}}\psi + V\psi \ = \ G \ = \ \Omega^3 F \ , \tag{3.3}$$

where:

$$V \ = \ \Omega^3 \Box_g(\Omega^{-1}) \ . \tag{3.4}$$

Proof. A quick calculation shows:

$$\Box_{\widetilde{g}} \ = \ \Omega^2(\Box_g - 2g^{\alpha\beta}\partial_\alpha \ln(\Omega)\partial_\beta) \ .$$

Rescaling $\Box_g\psi = \Box_g(\Omega\phi)$ and computing:

$$\begin{aligned} \Box_g(\Omega\phi) \ &= \Omega F + 2g^{\alpha\beta}\partial_\alpha(\ln\Omega)\partial_\beta\psi - [-\Box_g(\Omega)\phi + 2\Omega g^{\alpha\beta}\partial_\alpha(\ln\Omega)\partial_\beta(\ln\Omega)\phi] \\ &= \Omega F + 2g^{\alpha\beta}\partial_\alpha(\ln\Omega)\partial_\beta\psi - (W\phi) \ , \end{aligned} \tag{3.5}$$

with:

$$W \ = \ \Omega^2\Box_g(\Omega^{-1}) \ .$$

Therefore it suffices to show:

$$-\Box_g(\Omega) \ = \ \Omega^2\Box_g(\Omega^{-1}) - 2\partial_\alpha(\ln\Omega)\partial^\alpha(\ln\Omega)\Omega, \tag{3.6}$$

But this follows directly by:

$$\begin{aligned} \Box_g(\Omega) \ &= \Omega^2\Box_g(\Omega^{-1}) + \Box_g(\Omega^2)\Omega^{-1} + 2\partial_\alpha(\Omega^2)\partial^\alpha(\Omega^{-1}) \\ &= \ \Omega^2\Box_g(\Omega^{-1}) + 2\Box_g(\Omega) - 2\partial_\alpha(\ln\Omega)\partial^\alpha(\ln\Omega)\Omega \ . \end{aligned}$$

Plugging (3.6) into (3.5) gives us the result.

□

Let ${}^{I}\Omega = \langle r \rangle$, ${}^{II}\Omega = \langle u \rangle \langle \underline{u} \rangle$ and ${}^{I}\psi = {}^{I}\Omega\phi$, ${}^{II}\psi = {}^{II}\Omega\phi$. Since ${}^{J}\psi$ satisfies the wave equation with potential (3.3) for each respective weight, we define the *Energy Momentum Tensors of the First and Second Kind* to be:

$$
\begin{aligned}
{}^{I}\widetilde{T}_{\alpha\beta}[\phi] \ &= \partial_\alpha({}^{I}\Omega\phi)\partial_\beta({}^{I}\Omega\phi) \qquad (3.7)\\
&\quad - \ \frac{1}{2}\widetilde{g}_{\alpha\beta}\left(\widetilde{g}^{\gamma\omega}\partial_\gamma({}^{I}\Omega\phi)\partial_\omega({}^{I}\Omega\phi) - {}^{I}V({}^{I}\Omega\phi)^2\right) \ , \\
{}^{II}\widetilde{T}_{\alpha\beta}[\phi] \ &= \partial_\alpha({}^{II}\Omega\phi)\partial_\beta({}^{II}\Omega\phi) \qquad (3.8)\\
&\quad - \ \frac{1}{2}\widetilde{g}_{\alpha\beta}\left(\widetilde{g}^{\gamma\omega}\partial_\gamma({}^{II}\Omega\phi)\partial_\omega({}^{II}\Omega\phi) - {}^{II}V({}^{II}\Omega\phi)^2\right) \ ,
\end{aligned}
$$

Using this we define the *Conformal Energy Densities Corresponding to* $(X, {}^{I}\Omega)$ *and* $(X, {}^{II}\Omega)$ by:

$$
{}^{(X)I}\widetilde{P}_\alpha[\phi] \ = \ {}^{I}\widetilde{T}_{\alpha\beta}[\phi]X^\beta \ , \qquad {}^{(X)II}\widetilde{P}_\alpha[\phi] \ = \ {}^{II}\widetilde{T}_{\alpha\beta}[\phi]X^\beta \ .
$$

Lemma 3.2.2. *Let X be a smooth vector field and $J = I, II$. Let ${}^{I}\widetilde{D}^\alpha$, ${}^{II}\widetilde{D}^\alpha$ denote the Levi-Civita connections of ${}^{I}\widetilde{g}$ and ${}^{II}\widetilde{g}$ respectively. The following identities hold:*

$$
\begin{aligned}
&\int_{\{t=t_0\}\cap\mathbb{R}^3} \frac{N^\alpha({}^{(X)J}\widetilde{P}_\alpha[\phi])}{({}^{J}\Omega)^2} \ \sqrt{|g|}dx \\
&= \int_{\{t=t_1\}\cap\mathbb{R}^3} \frac{N^\alpha({}^{(X)J}\widetilde{P}_\alpha[\phi])}{({}^{J}\Omega)^2} \ \sqrt{|g|}dx + \int_{t_0}^{t_1}\int_{\mathbb{R}^3} \frac{{}^{J}\widetilde{D}^\alpha({}^{(X)J}\widetilde{P}_\alpha[\phi])}{({}^{J}\Omega)^4} \ \sqrt{|g|}dxdt \ , \qquad (3.9)
\end{aligned}
$$

$$
\begin{aligned}
&\int_{\{t=t_0\}\cap\mathbb{R}^3\cap\mathcal{D}} \frac{N^\alpha({}^{(X)J}\widetilde{P}_\alpha[\phi])}{({}^{J}\Omega)^2} \ \sqrt{|g|}dx \\
&= \int_{\{t=t_1\}\cap\mathbb{R}^3\cap\mathcal{D}} \frac{N^\alpha({}^{(X)J}\widetilde{P}_\alpha[\phi])}{({}^{J}\Omega)^2} \ \sqrt{|g|}dx + \int_{C(u_0)\cap\mathcal{D}} \frac{L^\alpha({}^{(X)J}\widetilde{P}_\alpha[\phi])}{({}^{J}\Omega)^2} \ dV_{C(u_0)} \\
&\quad + \iint_{\mathcal{D}} \frac{{}^{J}\widetilde{D}^\alpha({}^{(X)J}\widetilde{P}_\alpha[\phi])}{({}^{J}\Omega)^4} \ \sqrt{|g|}dxdt \ , \qquad (3.10)
\end{aligned}
$$

where $N = -\nabla t$ and the domain $\mathcal{D}$ consists of the region bounded by the null hypersurface $\{u = u_0\} \cap \mathcal{W}$ and the spacelike hypersurfaces $\{t = t_1\}$ and $\{t = t_0\}$.

Proof. Identity (3.9) follows immediately by integrating ${}^{J}\widetilde{D}^\alpha({}^{(X)J}\widetilde{P}_\alpha[\phi])$ with respect to the volume form $\widetilde{dV} = \sqrt{|\widetilde{g}|}dx = \Omega^{-4}\sqrt{|g|}dx$ over time slabs of the form $t_0 \leqslant t \leqslant t_1$ and applying Stokes' theorem. Identity (3.10) similarly follows by integrating ${}^{J}\widetilde{D}^\alpha({}^{(X)J}\widetilde{P}_\alpha[\phi])$ with respect to the volume form $\widetilde{dV}_{{}^{J}\widetilde{g}}$ on $\mathcal{D}$. Since the normal to the $u = u_0$ hypersurface is $L = -\nabla u$, the identity follows by an application of Stokes' theorem.

□

In general, computing the divergence on the RHS will lead to error terms arising from the deformation tensor. We compute these in the following lemma.

Lemma 3.2.3. *For $J = I, II$, the 1-forms ${}^{(X)J}\widetilde{P}_\alpha[\phi]$ satisfy the following divergence laws with respect to ${}^J\widetilde{g}$:*

$$\begin{aligned}
&{}^J\widetilde{D}^\alpha({}^{(X)J}\widetilde{P}_\alpha[\phi]) \\
&\quad= {}^J\Omega^3 F \cdot X({}^J\Omega\phi) + \frac{1}{2}{}^J\Omega^2\big(\widehat{\mathcal{L}_X g} + 2X\ln({}^J\Omega)g\big)^{\alpha\beta}\partial_\alpha({}^J\Omega\phi)\partial_\beta({}^J\Omega\phi) \\
&\quad+ \frac{1}{4}{}^J\Omega^2\big(g^{\alpha\beta}(\mathcal{L}_X g)_{\alpha\beta}{}^J V - 8X\ln({}^J\Omega){}^J V + 2X({}^J V)\big)\phi^2 \ . \qquad (3.11)
\end{aligned}$$

Proof. Rewrite the conformal deformation tensor defined in equation (2.19) as:

$$\widehat{\mathcal{L}_X\widetilde{g}} \ := \ \mathcal{L}_X\widetilde{g} - \frac{1}{2}\widetilde{g}^{\alpha\beta}(\mathcal{L}_X\widetilde{g})_{\alpha\beta}\cdot\widetilde{g} \ . \qquad (3.12)$$

Computing the RHS of this identity in terms of g:

$$\begin{aligned}
&\mathcal{L}_X\widetilde{g} - \frac{1}{2}\widetilde{g}^{\alpha\beta}(\mathcal{L}_X\widetilde{g})_{\alpha\beta}\cdot\widetilde{g} \\
&= \Omega^{-2}\mathcal{L}_X g - 2\Omega^{-3}X(\Omega)g - \frac{1}{2}\Omega^2 g^{\alpha\beta}(\Omega^{-2}(\mathcal{L}_X g)_{\alpha\beta} - 2\Omega^{-3}X(\Omega)g_{\alpha\beta})g\cdot\Omega^{-2} \\
&= \Omega^{-2}\mathcal{L}_X g - 2\Omega^{-2}X(\ln\Omega)g - \frac{1}{2}(g^{\alpha\beta}(\mathcal{L}_X g)_{\alpha\beta})\Omega^{-2}g + X(\ln\Omega)4g\Omega^{-2} \\
&= \Omega^{-2}\big(\widehat{\mathcal{L}_X g} + 2X\ln(\Omega)g\big). \qquad (3.13)
\end{aligned}$$

Applying this together with equation (3.13) gives:

$$\begin{aligned}
&{}^J\widetilde{D}^\alpha({}^{(X)J}\widetilde{P}_\alpha[\phi]) \\
&= {}^J\Omega^3 F\cdot X({}^J\Omega\phi) + \frac{1}{2}(\widehat{\mathcal{L}_X\widetilde{g}})^{\alpha\beta}\partial_\alpha({}^J\Omega\phi)\partial_\beta({}^J\Omega\phi) \\
&\quad+ \frac{1}{4}\big(\widetilde{g}^{\alpha\beta}(\mathcal{L}_X\widetilde{g})_{\alpha\beta}{}^J V + 2X({}^J V)\big)({}^J\Omega\phi)^2 \ , \\
&= {}^J\Omega^3 F\cdot X({}^J\Omega\phi) + \frac{1}{2}{}^J\Omega^2\big(\widehat{\mathcal{L}_X g} + 2X\ln({}^J\Omega)g\big)^{\alpha\beta}\partial_\alpha({}^J\Omega\phi)\partial_\beta({}^J\Omega\phi) \\
&\quad+ \frac{1}{4}{}^J\Omega^2\big(g^{\alpha\beta}(\mathcal{L}_X g)_{\alpha\beta}{}^J V - 8X\ln({}^J\Omega){}^J V + 2X({}^J V)\big)\phi^2 \ .
\end{aligned}$$

□

We will modify the lower order terms by adding a smooth cutoff χ. With this addition the energy momentum tensors become:

$$
\begin{aligned}
{}^{I}\widetilde{T}^{\chi}_{\alpha\beta}[\phi] &= \partial_\alpha({}^{I}\Omega\phi)\partial_\beta({}^{I}\Omega\phi) && (3.14)\\
&- \frac{1}{2}\widetilde{g}_{\alpha\beta}\left(\widetilde{g}^{\gamma\omega}\partial_\gamma({}^{I}\Omega\phi)\partial_\omega({}^{I}\Omega\phi) - {}^{I}V\chi({}^{I}\Omega\phi)^2\right) \ , \\
{}^{II}\widetilde{T}^{\chi}_{\alpha\beta}[\phi] &= \partial_\alpha({}^{II}\Omega\phi)\partial_\beta({}^{II}\Omega\phi) && (3.15)\\
&- \frac{1}{2}\widetilde{g}_{\alpha\beta}\left(\widetilde{g}^{\gamma\omega}\partial_\gamma({}^{II}\Omega\phi)\partial_\omega({}^{II}\Omega\phi) - {}^{II}V\chi({}^{II}\Omega\phi)^2\right) \ ,
\end{aligned}
$$

Defining ${}^{(X)I}\widetilde{P}^{\chi}_{\alpha}[\phi]$ and ${}^{(X)II}\widetilde{P}^{\chi}_{\alpha}[\phi]$ in the same manner as before and using the same proof as lemma 3.2.2 we get:

Corollary 3.2.4. *Let X be a smooth vector field and $J = I, II$. Then:*

$$
\begin{aligned}
&\int_{\{t=t_0\}\cap\mathbb{R}^3} \frac{N^\alpha({}^{(X)J}\widetilde{P}^{\chi}_{\alpha}[\phi])}{({}^{J}\Omega)^2}\sqrt{|g|}dx \\
&= \int_{\{t=t_1\}\cap\mathbb{R}^3} \frac{N^\alpha({}^{(X)J}\widetilde{P}^{\chi}_{\alpha}[\phi])}{({}^{J}\Omega)^2}\sqrt{|g|}dx + \int_{t_0}^{t_1}\int_{\mathbb{R}^3} \frac{{}^{J}\widetilde{D}^\alpha({}^{(X)J}\widetilde{P}_{\alpha}[\phi])}{({}^{J}\Omega)^4}\sqrt{|g|}dxdt \ . \qquad (3.16)\\
&\int_{\{t=t_0\}\cap\mathbb{R}^3\cap\mathcal{D}} \frac{N^\alpha({}^{(X)J}\widetilde{P}_{\alpha}[\phi])}{({}^{J}\Omega)^2}\sqrt{|g|}dx \\
&= \int_{\{t=t_1\}\cap\mathbb{R}^3\cap\mathcal{D}} \frac{N^\alpha({}^{(X)J}\widetilde{P}^{\chi}_{\alpha}[\phi])}{({}^{J}\Omega)^2}\sqrt{|g|}dx + \int_{C(u_0)\cap\mathcal{D}} \frac{L^\alpha({}^{(X)J}\widetilde{P}^{\chi}_{\alpha}[\phi])}{({}^{J}\Omega)^2}\, dV_{C(u_0)} \\
&+ \iint_{\mathcal{D}} \frac{{}^{J}\widetilde{D}^\alpha({}^{(X)J}\widetilde{P}^{\chi}_{\alpha}[\phi])}{({}^{J}\Omega)^4}\sqrt{|g|}dxdt \ . \qquad (3.17)
\end{aligned}
$$

Likewise, by following the proof of lemma 3.2.3 we get:

Corollary 3.2.5. *For $J = I, II$, the 1-forms ${}^{(X)J}\widetilde{P}^{\chi}_{\alpha}[\phi]$ satisfy the following divergence laws with respect to ${}^{J}\widetilde{g}$:*

$$
\begin{aligned}
&{}^{J}\widetilde{D}^\alpha({}^{(X)J}\widetilde{P}^{\chi}_{\alpha}[\phi]) \qquad (3.18)\\
&= {}^{J}\Omega^3 F\cdot X({}^{J}\Omega\phi) + \frac{1}{2}{}^{J}\Omega^2\left(\widehat{{}^{(X)}\pi} + 2X\ln({}^{J}\Omega)g\right)^{\alpha\beta}\partial_\alpha({}^{J}\Omega\phi)\partial_\beta({}^{J}\Omega\phi) \\
&+ {}^{J}\Omega(\chi-1){}^{J}V\phi X({}^{J}\Omega\phi) + \frac{1}{4}{}^{J}\Omega^2\big(g^{\alpha\beta}(\mathcal{L}_X g)_{\alpha\beta}({}^{J}V)\chi - 8X\ln({}^{J}\Omega)({}^{J}V)\chi \\
&+ 2X({}^{J}V\chi)\big)\phi^2 \ .
\end{aligned}
$$

Remark 3.2.6. *Choosing $\Omega = 1$ and $\chi \equiv 0$ above we recover the standard multiplier method discussed in Section 1.5.*

3.3 Proof of Proposition 3.0.4

In order to prove proposition 3.0.4 we will need the results in the following lemma. The proof of this lemma will be done in the next section.

Lemma 3.3.1. *For all $t_1, t_0 > t^*$ the following estimates hold:*

(First and Second Morawetz Estimates)

$$\sup_{t_0 \leqslant t \leqslant t_1} {}^{I}CE[\phi(t)] + \sup_{\mathcal{W}\cap[t_0,t_1]} \big({}^{I}CE_{ch}[\phi(u)] \big)$$

$$\lesssim \ RHS^2(3.1) + \epsilon \Big(\sup_{t_0 \leqslant t \leqslant t_1} CE[\phi(t)] + \sup_{\mathcal{W}\cap[t_0,t_1]} \sum_{J=I,II} {}^{J}CE_{ch}[\phi(u)] \Big) \ , \qquad (3.19)$$

$$\sup_{t_0 \leqslant t \leqslant t_1} {}^{II}CE[\phi(t)] + \sup_{\mathcal{W}\cap[t_0,t_1]} \big({}^{II}CE_{ch}[\phi(u)] \big)$$

$$\lesssim \ RHS^2(3.1) + \epsilon \Big(\sup_{t_0 \leqslant t \leqslant t_1} CE[\phi(t)] + \sup_{\mathcal{W}\cap[t_0,t_1]} \big({}^{II}CE_{ch}[\phi(u)] \big) \Big) \ . \qquad (3.20)$$

(Energy Boundedness)

$$\sup_{t_0 \leqslant t \leqslant t_1} E^{\frac{1}{2}}[\phi(t')] + \sup_{\mathcal{W}\cap[t_0,t_1]} \big(E^{\frac{1}{2}}_{ch}[\phi(u)] \big) \ \lesssim \ E^{\frac{1}{2}}[\phi(t_0)] \ + \ \| F \|_{LS^*[t_0,t_1]} \ . \qquad (3.21)$$

(Conjugation Removal) *For any fixed $t' \in [t_0, t_1]$:*

$$CE[\phi(t')] \ \lesssim \ \sum_{J=I,II} {}^{J}CE[\phi(t')] + E[\phi(t')] \ \lesssim \ CE[\phi(t')] \ . \qquad (3.22)$$

Proof of Proposition 3.0.4. Adding the estimates (3.19) and (3.20) and bootstrapping the small characteristic energy error terms:

$$\sum_{J=I,II} {}^{J}CE[\phi(t)] + \sup_{\mathcal{W}\cap[t_0,t_1]} \sum_{J=I,II} {}^{J}CE_{ch}[\phi(u)]$$

$$\lesssim \ RHS^2(3.1) + \epsilon \sup_{t_0 \leqslant t \leqslant t_1} CE[\phi(t)] \ .$$

Since we now have the sum of the two conjugated energies we add estimate (3.21) for fixed t' and get:

$$\sum_{J=I,II} {}^{J}CE[\phi(t)] + E[\phi(t')] + \sup_{\mathcal{W}\cap[t_0,t_1]} \sum_{J=I,II} {}^{J}CE_{ch}[\phi(u)]$$

$$\lesssim \ RHS^2(3.1) + \epsilon \sup_{t_0 \leqslant t \leqslant t_1} CE[\phi(t)]$$

Using (3.22) we can trade the first two terms on the LHS for conformal energy. Taking $\sup_{t_0 \leqslant t \leqslant t_1}$, bootstrapping the last term on the RHS and taking square roots finishes the proof.

□

3.4 Proof of Lemma 3.3.1

Next we list all the error bounds we need in order to prove lemma 3.3.1. The proof of the estimates below will be done the next section.

Lemma 3.4.1 (Estimates for Error Terms). *Let $t' \in [t_0, t_1]$, $N = -\nabla t$ with ${}^{I}\Omega = \langle r \rangle$, ${}^{II}\Omega = \langle \underline{u} \rangle \langle u \rangle$ and $J = I, II$ respectively. Take ${}^{I}V = ({}^{I}\Omega)^3 \Box_g(({}^{I}\Omega)^{-1})$ and ${}^{II}V = ({}^{II}\Omega)^3 \Box_g(({}^{II}\Omega)^{-1})$ as defined in section 3.2. The following bounds hold for the error terms in the conformal energy estimate:*

Space-time Error Estimates: *In the wave zone $\mathcal{W}$:*

$$\left| \frac{{}^{I}V}{({}^{I}\Omega)^2} \right| \lesssim \frac{1}{\langle r \rangle^{\frac{3}{2}+\delta} \langle u \rangle^{\frac{1}{2}}} \ , \tag{3.23}$$

$$\left| \frac{{}^{II}V}{({}^{II}\Omega)^2} \right| \lesssim \frac{1}{\langle r \rangle^{\frac{3}{2}+\delta} \langle u \rangle^{\frac{1}{2}}} \ . \tag{3.24}$$

Outside the wave zone:

$$\left| \frac{{}^{I}V}{({}^{I}\Omega)^2} \right| \lesssim \frac{1}{\langle r \rangle^{2+\delta}} \ , \tag{3.25}$$

$$\left| \frac{{}^{II}V}{({}^{II}\Omega)^2} \right| \lesssim \frac{1}{\langle u \rangle \langle r \rangle^{1+\delta}} \ , \tag{3.26}$$

$$\left| \frac{\widetilde{K_0}({}^{I}V \chi_{r \leq \frac{1}{2}\epsilon t})}{({}^{I}\Omega)^2} \right| \lesssim \epsilon \chi_{r \leqslant \frac{1}{10} t} \frac{t^{2-\gamma'}}{\langle r \rangle^{3+\epsilon}} \ , \tag{3.27}$$

$$\left| \frac{\widetilde{K_0}({}^{II}V \chi_{r \leq \frac{1}{2}\epsilon t})}{({}^{II}\Omega)^2} \right| \lesssim \epsilon \chi_{r \leqslant \frac{1}{10} t} \frac{t^{2-\gamma'}}{\langle u \rangle \langle r \rangle^{2+\epsilon}} \ . \tag{3.28}$$

For the Lie derivative space-time error terms:

$$\int_{t_0}^{t_1}\int_{\mathbb{R}^3}\Big|({}^J\Omega)^{-2}\big(\widehat{\mathcal{L}_{\widetilde{K_0}}g}+2\widetilde{K_0}\ln({}^J\Omega)g\big)^{\alpha\beta}\partial_\alpha({}^J\Omega\phi)\partial_\beta({}^J\Omega\phi)\Big| \ \sqrt{|g|}dxdt$$
$$\lesssim \ \epsilon\left(\|\,\phi\,\|^2_{LSM^{1-\gamma'}_{r\leqslant\frac{1}{10}t}[t_0,t_1]}+\sup_{t_0\leqslant t\leqslant t_1}CE[\phi(t)]+\sup_{\mathcal{W}\cap[t_0,t_1]}\big({}^JCE_{ch}[\phi(u)]\big)\right) \ , \quad (3.29)$$

$$\int_{t_0}^{t_1}\int_{\mathbb{R}^3}\left|\big(g^{\alpha\beta}(\mathcal{L}_{\widetilde{K_0}}g)_{\alpha\beta}-8\widetilde{K_0}\ln({}^J\Omega)\big)\frac{({}^JV)\chi_{r\leq\frac{1}{2}\epsilon t}}{({}^J\Omega)^2}-\frac{2\widetilde{K_0}({}^JV\chi_{r\leq\frac{1}{2}\epsilon t})}{({}^J\Omega)^2}\right|\phi^2 \ \sqrt{|g|}dxdt$$
$$\lesssim \ \epsilon\|\,\phi\,\|^2_{LSM^{1-\gamma'}_{r\leqslant\frac{1}{10}t}[t_0,t_1]} \ . \quad (3.30)$$

Conjugated Energy Comparison Estimates. *For any fixed $t'\in[t_0,t_1]$:*

$$\begin{aligned}
&{}^JCE[\phi(t')]\\
&\lesssim \int_{\{t=t'\}\cap\mathbb{R}^3}\Big|\frac{\langle u\rangle\nabla({}^J\Omega\phi)}{({}^J\Omega)^2}\Big|^2 \ \sqrt{|g|}dx\\
&\ + \int_{\{t=t'\}\cap\mathcal{W}\cap\mathbb{R}^3}\left(\langle t+r\rangle^2\big(\Big|\frac{\widetilde{\partial}_r({}^J\Omega\phi)}{({}^J\Omega)^2}\Big|^2+|\,\widetilde{\nabla\!\!\!/}\phi|^2\big)+\langle u\rangle^2\Big|\frac{\partial_t({}^J\Omega\phi)}{({}^J\Omega)^2}\Big|^2\right) \ \sqrt{|g|}dx\\
&\lesssim \ \int_{\{t=t'\}\cap\mathbb{R}^3}\frac{N^\alpha\big({}^{(\overline{K_0})J}\widetilde{P}_\alpha[\phi]\big)}{({}^J\Omega)^2} \ \sqrt{|g|}dx\\
&\lesssim \ {}^JCE[\phi(t')] \ ,
\end{aligned} \quad (3.31)$$

where $N=-\nabla t$.

Conjugated Null Energy Comparison Estimates. *In the wave zone, for any null hypersurface $u=u_0$:*

$$\begin{aligned}
{}^JCE_{ch}[\phi(u_0)] \ &\lesssim \ \int_{C(u_0)\cap\mathcal{W}\cap[t_0,t_1]}\left(\langle t+r\rangle^2\Big|\frac{\widetilde{\partial}_r({}^J\Omega\phi)}{({}^J\Omega)^2}\Big|^2+\langle u_0\rangle^2|\,\widetilde{\nabla\!\!\!/}\phi|^2\right) \ dV_{C(u_0)}\\
&\lesssim \ \int_{C(u_0)\cap\mathcal{W}\cap[t_0,t_1]}\frac{L^\alpha\big({}^{(\overline{K_0})J}\widetilde{P}_\alpha[\phi]\big)}{({}^J\Omega)^2} \ dV_{C(u_0)}\\
&\lesssim \ {}^JCE_{ch}[\phi(u_0)] \ .
\end{aligned} \quad (3.32)$$

Fixed-Time Error Estimates. *For any* $t' \in [t_0, t_1]$*:*

$$\sum_{\beta=t,r} \int_{\{t=t'\}\cap\mathbb{R}^3} \left| \frac{\overline{K_0}^{\beta}({}^{I}V)\chi_{r\leq\frac{1}{2}\epsilon t}\phi^2}{({}^{I}\Omega)^2} \right| \sqrt{|g|}dx$$

$$\lesssim \int_{\{t=t'\}\cap\mathbb{R}^3} \left| \frac{u\nabla({}^{II}\Omega\phi)}{{}^{II}\Omega} \right|^2 \sqrt{|g|}dx + \epsilon \cdot \sup_{t_0\leqslant t\leqslant t_1} CE[\phi(t)] \ , \tag{3.33}$$

$$\sum_{\beta=t,r} \int_{\{t=t'\}\cap\mathbb{R}^3} \left| \frac{\overline{K_0}^{\beta}({}^{II}V)\chi_{r\leq\frac{1}{2}\epsilon t}\phi^2}{({}^{II}\Omega)^2} \right| \sqrt{|g|}dx \ \lesssim \ \epsilon \cdot \sup_{t_0\leqslant t\leqslant t_1} CE[\phi(t)] \ . \tag{3.34}$$

Hardy Estimates. *For any fixed* $t' \in [t_0, t_1]$ *and* $\chi_{r\leqslant 2}$*,* $\chi_{|u|\leqslant 2}$ *smooth cutoff functions supported on the sets* $\{r \leqslant 2\}$*,* $\{|u| \leqslant 2\}$*:*

$$\int_{\{t=t'\}\cap\mathbb{R}^3} \left|\chi_{r\leqslant 2}\frac{t\phi}{r}\right|^2 dx \ \lesssim \int_{\{t=t'\}\cap\mathbb{R}^3} u^2 \left(\left|\frac{\widetilde{\partial}_r({}^{II}\Omega\phi)}{{}^{II}\Omega}\right|^2 + \left|\frac{\partial_u({}^{II}\Omega\phi)}{{}^{II}\Omega}\right|^2 \right) dx$$

$$+ \int_{\{t=t'\}\cap\mathbb{R}^3} \left|\chi_{r\sim 2}\frac{t\phi}{r}\right|^2 dx + E[\phi(t')] \ . \tag{3.35}$$

$$\int_{\{t=t'\}\cap\mathbb{R}^3} |\chi_{|u|\leqslant 2}\phi|^2 \ dx \ \lesssim \ \int_{\{t=t'\}\cap\mathbb{R}^3} u^2 \left(\left|\frac{\widetilde{\partial}_r({}^{I}\Omega\phi)}{{}^{I}\Omega}\right|^2 + \left|\frac{\partial_u({}^{I}\Omega\phi)}{{}^{I}\Omega}\right|^2 \right) dx$$

$$+ \int_{\{t=t'\}\cap\mathbb{R}^3} \left|\chi_{|u|\sim 2}\phi\right|^2 + \left|\chi_{|u|\leqslant 2}\frac{u\phi}{r}\right|^2 dx \ . \tag{3.36}$$

Remark 3.4.2. *Note, in particular, that estimate* (3.31) *shows* $\widetilde{\partial}_r$ *can be traded for* L *in the conjugate conformal energies in the wave zone and this exchange yields a norm equivalent to* ${}^{J}CE[\phi]$*. Similarly,* (3.32) *shows* $\widetilde{\partial}_r$ *can be traded for* L *in the null energies and this exchange yields a norm equivalent to* ${}^{J}CE_{ch}[\phi]$*.*

3.4.1 Proof of Lemma 3.3.1

Proof of (3.19) *and* (3.20)*.* Plug in $\overline{K_0}$ into (3.17) with $J = I, II$, and $N = -\nabla t$. Choose χ to be supported in the set $\{r \leqslant \frac{1}{2}\epsilon t\}$ and apply the identity:

$$\frac{N^{\alpha}({}^{J}\widetilde{T}^{\chi}_{\alpha\beta}[\phi])(\overline{K_0}^{\beta})}{({}^{J}\Omega)^2} - \sum_{\beta} \frac{(\overline{K}_0^{\beta})^{J}V\chi_{r\leq\frac{1}{2}\epsilon t}\phi^2}{({}^{J}\Omega)^2} \ = \ \frac{N^{\alpha}({}^{J}\widetilde{T}_{\alpha\beta}[\phi])(\overline{K_0}^{\beta})}{({}^{J}\Omega)^2} \ ,$$

to the $t = t_1$ term. Re-arranging the resulting identity we get:

$$\int_{C(u_0)\cap\mathcal{D}} \frac{L^\alpha(^{(\overline{K_0})\,J}\widetilde{P}^\chi_\alpha[\phi])}{(^J\Omega)^2}\, dV_{C(u_0)} + \int_{\{t=t_1\}\cap\mathbb{R}^3\cap\mathcal{D}} \frac{N^\alpha(^{(\overline{K_0})\,J}\widetilde{P}_\alpha[\phi])}{(^J\Omega)^2}\, \sqrt{|g|}dx$$
$$= \int_{\{t=t_0\}\cap\mathbb{R}^3\cap\mathcal{D}} \left(\frac{N^\alpha(^{(\overline{K_0})\,J}\widetilde{P}^\chi_\alpha[\phi])}{(^J\Omega)^2}\right) \sqrt{|g|}dx - \iint_{\mathcal{D}} \frac{^J\widetilde{D}^\alpha(^{(\overline{K_0})\,J}\widetilde{P}^\chi_\alpha[\phi])}{(^J\Omega)^4}\, \sqrt{|g|}dxdt$$
$$- \sum_\beta \int_{\{t=t_1\}\cap\mathbb{R}^3\cap\mathcal{D}} \frac{(\overline{K_0}^\beta)\,^J V\chi_{r\le\frac{1}{2}\epsilon t}\phi^2}{(^J\Omega)^2}\, \sqrt{|g|}dx$$

Using (3.32) and the positivity given by (3.31), taking absolute value and then $\sup_{u_0\in\mathcal{W}\cap[t_0,t_1]}$:

$$\sup_{u_0\in\mathcal{W}\cap[t_0,t_1]} {}^JCE_{ch}[\phi(u_0)] + \int_{\{t=t_1\}\cap\mathbb{R}^3} \frac{N^\alpha(^{(\overline{K_0})\,J}\widetilde{P}_\alpha[\phi])}{(^J\Omega)^2}\, \sqrt{|g|}dx$$
$$\lesssim \int_{\{t=t_0\}\cap\mathbb{R}^3} \left|\frac{N^\alpha(^{(\overline{K_0})\,J}\widetilde{P}^\chi_\alpha[\phi])}{(^J\Omega)^2}\right| \sqrt{|g|}dx \ + \int_{t_0}^{t_1}\int_{\mathbb{R}^3} \left|\frac{^J\widetilde{D}^\alpha(^{(\overline{K_0})\,J}\widetilde{P}^\chi_\alpha[\phi])}{(^J\Omega)^4}\right| \sqrt{|g|}dxdt$$
$$+\sum_\beta \int_{\{t=t_1\}\cap\mathbb{R}^3} \left|\frac{(\overline{K_0}^\beta)\,^J V\chi_{r\le\frac{1}{2}\epsilon t}\phi^2}{(^J\Omega)^2}\right| \sqrt{|g|}dx\ .$$

Applying (3.25) and (3.26), together with (3.31) to the $t = t_0$ term:

$$\sup_{u_0\in\mathcal{W}\cap[t_0,t_1]} {}^JCE_{ch}[\phi(u_0)] + \int_{\{t=t_1\}\cap\mathbb{R}^3} \frac{N^\alpha(^{(\overline{K_0})\,J}\widetilde{P}_\alpha[\phi])}{(^J\Omega)^2}\, \sqrt{|g|}dx$$
$$\lesssim {}^JCE[\phi(t_0)] + \int_{t_0}^{t_1}\int_{\mathbb{R}^3} \left|\frac{^J\widetilde{D}^\alpha(^{(\overline{K_0})\,J}\widetilde{P}^\chi_\alpha[\phi])}{(^J\Omega)^4}\right| \sqrt{|g|}dxdt$$
$$+ \sum_\beta \int_{\{t=t_1\}\cap\mathbb{R}^3} \left|\frac{(\overline{K_0}^\beta)\,^J V\chi_{r\le\frac{1}{2}\epsilon t}\phi^2}{(^J\Omega)^2}\right| \sqrt{|g|}dx\ .$$

We start by controlling the spacetime error terms on the RHS. The errors corresponding to ∂_u are just simpler cases of what we prove below for $\widetilde{K}_0$, therefore we omit those estimates. For the spacetime errors on the RHS corresponding to $\widetilde{K}_0$ we expand using (3.18) followed by (3.29) applied to the terms that are quadratic in the derivatives. For the terms quadratic in ϕ we use (3.30). Combining these results we get:

$$
\begin{aligned}
&\sup_{u_0 \in \mathcal{W}\cap[t_0,t_1]} {}^J CE_{ch}[\phi(u_0)] + \int_{\{t=t_1\}\cap\mathbb{R}^3} \frac{N^\alpha ({}^{(\overline{K_0})J}\widetilde{P}_\alpha[\phi])}{({}^J\Omega)^2}\ \sqrt{|g|}dx \\
&\lesssim\ {}^J CE[\phi(t_0)] + \epsilon \| \, \phi \, \|^2_{LSM^{1-\gamma'}_{r\leqslant \frac{1}{10}t}[t_0,t_1]} + \epsilon \left(\sup_{t_0\leqslant t\leqslant t_1} CE[\phi(t)] + \sup_{\mathcal{W}\cap[t_0,t_1]} \left({}^J CE_{ch}[\phi(u)] \right) \right) \\
&\ +\ \int_{t_0}^{t_1}\int_{\mathbb{R}^3} \left| F\cdot {}^J\Omega^{-1}\widetilde{K_0}({}^J\Omega\phi)\right| + \left| (\chi_{r\leqslant \frac{1}{2}\epsilon t} - 1) {}^J V\phi \cdot ({}^J\Omega)^{-3}\widetilde{K_0}({}^J\Omega\phi)\right|\ \sqrt{|g|}dtdx\ . \\
&\ +\ \sum_\beta \int_{\{t=t_1\}\cap\mathbb{R}^3} \left| \frac{(\overline{K_0}^\beta)^J V\chi_{r\leq\frac{1}{2}\epsilon t}\phi^2}{({}^J\Omega)^2} \right|\ \sqrt{|g|}dx\ .
\end{aligned}
\tag{3.37}
$$

For the term with F we use Young's inequality. In the wave zone:

$$
\begin{aligned}
&\iint_{\mathcal{W}\cap[t_0,t_1]} \left| F\cdot {}^J\Omega^{-1}\widetilde{K_0}({}^J\Omega\phi)\right|\ \sqrt{|g|}dtdx \\
&\qquad \lesssim\ \| \, F \, \|^2_{LS^{*,1+\gamma',\frac{1}{2}}[t_0,t_1]} + \epsilon \left(\sup_{t_0\leqslant t\leqslant t_1} CE[\phi(t)] + \sup_{\mathcal{W}\cap[t_0,t_1]} \left({}^J CE_{ch}[\phi(u)] \right) \right)\ .
\end{aligned}
$$

For the interior we use $|\frac{\partial(\Omega\phi)}{(\Omega)}| \lesssim |\partial\phi| + |\frac{\phi}{r}|$ and apply Young's inequality again to get:

$$
\begin{aligned}
&\int_{t_0}^{t_1}\int_{\mathbb{R}^3} \left| \chi_{r\leqslant\frac{1}{2}\epsilon t} F\cdot {}^J\Omega^{-1}\widetilde{K_0}({}^J\Omega\phi)\right|\ \sqrt{|g|}dtdx \\
&\qquad \lesssim\ \epsilon^{-1}\| \, F \, \|^2_{LS^{*,1+\gamma',\frac{1}{2}}[t_0,t_1]} + \epsilon \| \, \phi \, \|^2_{LSM^{1-\gamma'}_{r\leqslant\frac{1}{10}t}[t_0,t_1]}\ .
\end{aligned}
$$

For the last spacetime term on the RHS of (3.37) we bound ${}^J V({}^J\Omega)^{-2}$ by using (3.23), (3.24) in the wave zone and (3.25), (3.26) outside. This yields:

$$
\begin{aligned}
&\int_{t_0}^{t_1}\int_{\mathbb{R}^3} \left| (\chi_{r\leqslant\frac{1}{2}\epsilon t} - 1) {}^J V\phi\cdot ({}^J\Omega)^{-3}\widetilde{K_0}({}^J\Omega\phi)\right|\ \sqrt{|g|}dtdx \\
&\qquad \lesssim\ \epsilon \left(\sup_{t_0\leqslant t\leqslant t_1} CE[\phi(t)] + \sup_{\mathcal{W}\cap[t_0,t_1]} \left({}^J CE_{ch}[\phi(u)] \right) \right)\ .
\end{aligned}
$$

Applying these last three estimates to equation (3.37):

$$\sup_{u_0 \in \mathcal{W}\cap[t_0,t_1]} {}^{J}CE_{ch}[\phi(u_0)] + \int_{\{t=t_1\}\cap\mathbb{R}^3} \frac{N^\alpha\big({}^{(\overline{K_0})\,J}\widetilde{P}_\alpha[\phi]\big)}{({}^{J}\Omega)^2}\ \sqrt{|g|}dx$$
$$\lesssim\ {}^{J}CE[\phi(t_0)] + \| F \|^2_{LS^{*,1+\gamma',\frac{1}{2}}[t_0,t_1]}$$
$$+\ \epsilon\left(\sup_{t_0\leqslant t\leqslant t_1} CE[\phi(t)] + \sup_{\mathcal{W}\cap[t_0,t_1]}\big({}^{J}CE_{ch}[\phi(u)]\big)\right) +\ \epsilon\| \phi \|^2_{LSM^{1-\gamma'}_{r\leqslant\frac{1}{10}t}[t_0,t_1]}$$
$$+\ \sum_\beta \int_{\{t=t_1\}\cap\mathbb{R}^3} \left| \frac{(\overline{K_0}^\beta)^{J}V\chi_{r\leq\frac{1}{2}\epsilon t}\phi^2}{({}^{J}\Omega)^2} \right|\ \sqrt{|g|}dx\ . \tag{3.38}$$

At this point we must treat the estimates for $J = I$ and $J = II$ separately. This is because we will need to use the estimate for ${}^{II}\Omega$ in the proof of the estimate for ${}^{I}\Omega$. This, in turn, is a result of the fact that ${}^{II}V$ has better decay estimates in the interior region and gives us a small fixed-time error which can be bootstrapped – whereas that does not happen in the other case. Choosing $J = II$ in (3.38) then applying (3.34) to the last term:

$$\sup_{u \in \mathcal{W}\cap[t_0,t_1]} {}^{II}CE_{ch}[\phi(u_0)] + \int_{\{t=t_1\}\cap\mathbb{R}^3} \frac{N^\alpha({}^{II}\widetilde{T}_{\alpha\beta}[\phi])(\overline{K_0}^\beta)}{({}^{II}\Omega)^2}\ \sqrt{|g|}dx$$
$$\lesssim\ {}^{J}CE[\phi(t_0)] + \epsilon^{-1}\| F \|^2_{LS^{*,1+\gamma',\frac{1}{2}}[t_0,t_1]}$$
$$+\ \epsilon\left(\sup_{t_0\leqslant t\leqslant t_1} CE[\phi(t)] + \sup_{\mathcal{W}\cap[t_0,t_1]}\big({}^{II}CE_{ch}[\phi(u)]\big) + \| \phi \|^2_{LSM^{1-\gamma'}_{r\leqslant\frac{1}{10}t}[t_0,t_1]}\right)\ .$$

Using estimate (3.31) rearranging terms and taking $\sup_{t_0\leqslant t\leqslant t_1}$:

$$\sup_{u \in \mathcal{W}\cap[t_0,t_1]} {}^{II}CE_{ch}[\phi(u_0)] + \sup_{t_0\leqslant t\leqslant t_1} {}^{II}CE[\phi(t)]$$
$$\lesssim\ RHS^2(3.1) + \epsilon\left(\sup_{t_0\leqslant t\leqslant t_1} CE[\phi(t)] + \sup_{\mathcal{W}\cap[t_0,t_1]}\big({}^{II}CE_{ch}[\phi(u)]\big)\right),$$

which is exactly (3.20). Next we work on the ${}^{I}\Omega$ weight. Choosing $J = II$ in (3.38) and applying (3.33) to the last term:

$$\sup_{u\in\mathcal{W}\cap[t_0,t_1]} {}^{I}CE_{ch}[\phi(u_0)] + \int_{\{t=t_1\}\cap\mathbb{R}^3} \frac{N^\alpha({}^{I}\widetilde{T}_{\alpha\beta}[\phi])(\overline{K_0}^\beta)}{({}^{I}\Omega)^2} \sqrt{|g|}dx$$
$$\lesssim {}^{J}CE[\phi(t_0)] + \epsilon^{-1}\| F \|^2_{LS^{*,1+\gamma',\frac{1}{2}}[t_0,t_1]}$$
$$+ \epsilon \left(\sup_{t_0\leqslant t\leqslant t_1} CE[\phi(t)] + \sup_{\mathcal{W}\cap[t_0,t_1]} \left({}^{I}CE_{ch}[\phi(u)]\right) \right) + \epsilon\| \phi \|^2_{LSM^{1-\gamma'}_{r\leqslant\frac{1}{10}t}[t_0,t_1]}$$
$$+ \int_{\{t=t_1\}\cap\mathbb{R}^3} u^2 \left| \frac{\nabla({}^{II}\Omega)}{{}^{II}\Omega} \right|^2 dx \ .$$

In contrast with the estimate for ${}^{II}\Omega$, the fixed-time error term on the RHS cannot be bootstrapped. Instead, we apply estimate (3.20) to this term and get:

$$\sup_{u\in\mathcal{W}\cap[t_0,t_1]} {}^{I}CE_{ch}[\phi(u_0)] + \int_{\{t=t_1\}\cap\mathbb{R}^3} \frac{N^\alpha({}^{I}\widetilde{T}_{\alpha\beta}[\phi])(\overline{K_0}^\beta)}{({}^{I}\Omega)^2} \sqrt{|g|}dx$$
$$\lesssim RHS^2(3.1) + \epsilon \left(\sup_{t_0\leqslant t\leqslant t_1} CE[\phi(t)] + \sup_{\mathcal{W}\cap[t_0,t_1]} \sum_{J=I,II} {}^{J}CE_{ch}[\phi(u)] \right) \ .$$

Using estimate (3.31) rearranging terms and taking $\sup_{t_0\leqslant t\leqslant t_1}$:

$$\sup_{u\in\mathcal{W}\cap[t_0,t_1]} {}^{I}CE_{ch}[\phi(u_0)] + \sup_{t_0\leqslant t\leqslant t_1} {}^{I}CE[\phi(t)]$$
$$\lesssim RHS^2(3.1) + \epsilon \left(\sup_{t_0\leqslant t\leqslant t_1} CE[\phi(t)] + \sup_{\mathcal{W}\cap[t_0,t_1]} \sum_{J=I,II} {}^{J}CE_{ch}[\phi(u)] \right) \ .$$

this proves (3.19).

□

Remark 3.4.3. *Suppose that $\mathcal{W} \subset supp\chi$ and let's consider the spacetime error term that would arise for ϕ^2 in this case. A quick computation shows in the wave zone:*

$$\left| g^{\alpha\beta}(\mathcal{L}_{\widetilde{K_0}}g)_{\alpha\beta} - 8\widetilde{K_0}\ln({}^{J}\Omega) \right| \lesssim \langle r\rangle^{1-\delta} \ , \tag{3.39}$$

Multiplying this by the decay rates in (3.23)-(3.24) *we are left with:*

$$\frac{\langle \underline{u}\rangle}{\langle r\rangle^{\frac{3}{2}+\delta}\langle u\rangle^{\frac{1}{2}}}\phi^2 \ ,$$

which is not integrable in the wave zone. This is the main reason we set the cutoff χ to be supported in the set $\{r \leqslant \frac{1}{2}\epsilon t\}$.

Proof of (3.21). This follows as a special case of the proof of (3.19) and (3.20) with $\widetilde{K}_0 = 0$, ${}^{I}\Omega = {}^{II}\Omega = 1$. The main point here is that we use the LS estimate (1.10) to control the large deformation tensor errors in the interior. The estimate in the exterior closes by a simple bootstrap argument. □

Proof of (3.22). It suffices to prove the bound:

$$CE[\phi(t')] \ \lesssim \ \sum_{J=I,II} {}^{J}CE[\phi(t')] + E[\phi(t')] \ .$$

By (3.31) this reduces to showing:

$$\int_{\{t=t'\}\cap\mathbb{R}^3} \langle \underline{u}\rangle^2 (\widetilde{\partial}_r\phi)^2 + \langle u\rangle^2(\partial_u\phi)^2 \ dx$$
$$\lesssim \ \int_{\{t=t'\}\cap\mathbb{R}^3} \langle \underline{u}\rangle^2 \left|\frac{\widetilde{\partial}_r(\langle r\rangle\phi)}{\langle r\rangle}\right|^2 + \langle u\rangle^2 \left|\frac{\partial_u(\langle r\rangle\phi)}{\langle r\rangle}\right|^2 + \frac{(t\phi)^2}{r^2} \ dx + E[\phi(t')] \ , \quad (3.40)$$

$$\int_{\{t=t'\}\cap\mathbb{R}^3} \frac{(t\phi)^2}{r^2} \ dx$$
$$\lesssim \ \sum_{J=I,II} \int_{\{t=t'\}\cap\mathbb{R}^3} \underline{u}^2 \left|\frac{\widetilde{\partial}_r({}^{J}\Omega\phi)}{{}^{J}\Omega}\right|^2 + u^2 \left|\frac{\partial_u({}^{J}\Omega\phi)}{{}^{J}\Omega}\right|^2 \ dx + E[\phi(t')] \ . \quad (3.41)$$

For (3.40) we start with the identities:

$$\underline{u}\left(\frac{\widetilde{\partial}_r(\langle r\rangle\phi)}{\langle r\rangle}\right) \ = \ \underline{u}\left(\widetilde{\partial}_r\phi + \frac{r\phi}{\langle r\rangle^2}\right) \ , \quad (3.42)$$

$$\underline{u}\left(\frac{\widetilde{\partial}_r(\langle u\rangle\langle \underline{u}\rangle\phi)}{\langle u\rangle\langle \underline{u}\rangle}\right) \ = \ \underline{u}\left(\widetilde{\partial}_r\phi + \frac{2\underline{u}\phi}{\langle \underline{u}\rangle^2}\right) \ . \quad (3.43)$$

Using the first identity:

$$\langle \underline{u}\rangle^2(\widetilde{\partial}_r\phi)^2 + \langle u\rangle^2(\partial_u\phi)^2$$
$$= \ \langle \underline{u}\rangle^2\left((\widetilde{\partial}_r\phi)^2 + 2\widetilde{\partial}_r\phi\cdot\frac{r\phi}{\langle r\rangle^2} + \frac{(r\phi)^2}{\langle r\rangle^4}\right) - \langle \underline{u}\rangle^2\left(2\widetilde{\partial}_r\phi\cdot\frac{r\phi}{\langle r\rangle^2} + \frac{(r\phi)^2}{\langle r\rangle^4}\right)$$
$$+ \ \langle u\rangle^2\left|\frac{\partial_u(\langle r\rangle\phi)}{\langle r\rangle}\right|^2$$
$$\leq \langle \underline{u}\rangle^2\left|\frac{\widetilde{\partial}_r(\langle r\rangle\phi)}{\langle r\rangle}\right|^2 + \langle \underline{u}\rangle^2\left(\frac{1}{2}(\widetilde{\partial}_r\phi)^2 + 2\frac{(r\phi)^2}{\langle r\rangle^4}\right) + \langle u\rangle^2\left|\frac{\partial_u(\langle r\rangle\phi)}{\langle r\rangle}\right|^2 - \langle \underline{u}\rangle^2\frac{(r\phi)^2}{\langle r\rangle^4},$$

which implies:

$$\langle \underline{u}\rangle^2(\widetilde{\partial}_r\phi)^2 + \langle u\rangle^2(\partial_u\phi)^2 \leq 2\left(\langle \underline{u}\rangle^2\left|\frac{\widetilde{\partial}_r(\langle r\rangle\phi)}{\langle r\rangle}\right|^2 + \langle u\rangle^2\left|\frac{\partial_u(\langle r\rangle\phi)}{\langle r\rangle}\right|^2 + \langle \underline{u}\rangle^2|\frac{\phi}{r}|^2\right).$$

Estimate (3.40) follows immediately. For (3.41) we note that the Hardy estimates (3.36) and (3.35) imply that we already control the desired quantity on the sets $r \leqslant 1$ and $u \leqslant 1$, respectively. Therefore, it suffices to control $(\frac{u}{r}\phi)^2$, ϕ^2, and $(\frac{\underline{u}}{r}\phi)^2$ everywhere else . To bound $(\frac{u}{r}\phi)^2$ we subtract (3.42) from (3.43)

$$\begin{aligned}
\frac{\underline{u}\widetilde{\partial}_r({}^{II}\Omega)}{{}^{II}\Omega} - \frac{\underline{u}\widetilde{\partial}_r({}^{I}\Omega)}{{}^{I}\Omega} &= 2\phi - \frac{\underline{u}}{r}\phi - \frac{2}{\langle \underline{u}\rangle^2}\phi + \underline{u}\left(\frac{1}{\langle r\rangle^2 r}\right)\phi \\
&= -\frac{u}{r}\phi - \frac{2}{\langle \underline{u}\rangle^2}\phi + \underline{u}\left(\frac{1}{\langle r\rangle^2 r}\right)\phi \\
&= -\left(1 - \frac{1}{\langle r\rangle^2}\right)\frac{u}{r}\phi - \frac{2}{\langle \underline{u}\rangle^2}\phi + \frac{2}{\langle r\rangle^2}\phi \, . \qquad (3.44)
\end{aligned}$$

Rearranging, squaring and using $r \leqslant |\underline{u}|$:

$$\left(1 - \frac{1}{\langle r\rangle^2}\right)^2\left(\frac{u}{r}\phi\right)^2 \leqslant 8\left(|\frac{\underline{u}\widetilde{\partial}_r({}^{I}\Omega)}{{}^{I}\Omega}|^2 + |\frac{\underline{u}\widetilde{\partial}_r({}^{II}\Omega)}{{}^{II}\Omega}|^2 + \frac{\phi^2}{r^2}\right) \, . \qquad (3.45)$$

Therefore on the set $r \geq 1$ we get:

$$\left(\frac{u}{r}\phi\right)^2 \lesssim |\frac{\underline{u}\widetilde{\partial}_r({}^{I}\Omega)}{{}^{I}\Omega}|^2 + |\frac{\underline{u}\widetilde{\partial}_r({}^{II}\Omega)}{{}^{II}\Omega}|^2 + \frac{\phi^2}{r^2} \, . \qquad (3.46)$$

Integrating this, using a Hardy estimate for the undifferentiated term and combining with (3.35) we get:

$$\int_{\{t=t'\}\cap\mathbb{R}^3}\left(\frac{u}{r}\phi\right)^2 \, dx \lesssim RHS(3.41) \qquad (3.47)$$

To control ϕ^2 we use the linear combination $\bar{\partial} = 2\partial_u - \widetilde{\partial}_r$. This combination has the properties:

$$\bar{\partial}(\underline{u}) = 0 \, , \qquad \bar{\partial}(u) = 2, \qquad \bar{\partial}(r) = -1 \, . \qquad (3.48)$$

Collectively these imply:

$$u\left(\frac{\bar{\partial}({}^{II}\Omega\phi)}{{}^{II}\Omega}\right) = u\bar{\partial}\phi + \frac{2u^2}{\langle u\rangle^2}\phi, \qquad u\left(\frac{\bar{\partial}({}^{I}\Omega\phi)}{{}^{I}\Omega}\right) = u\left(\bar{\partial}\phi - \frac{r}{\langle r\rangle^2}\phi\right) \, .$$

$$\frac{u\bar{\partial}({}^{II}\Omega\phi)}{{}^{II}\Omega} - \frac{u\bar{\partial}({}^{I}\Omega\phi)}{{}^{I}\Omega} = 2\phi + \frac{u}{r}\phi - \frac{2}{\langle u\rangle^2}\phi + ur\left(\frac{1}{\langle r\rangle^2} - \frac{1}{r^2}\right)\phi$$
$$= \frac{\underline{u}}{r}\phi - \frac{2}{\langle u\rangle^2}\phi - u\left(\frac{1}{r\langle r\rangle^2}\right)\phi$$
$$= 2\left(1 - \frac{1}{\langle u\rangle^2}\right)\phi + \left(1 - \frac{1}{\langle r\rangle^2}\right)\frac{u}{r}\phi \tag{3.49}$$

Rearranging and squaring:

$$4\left(1 - \frac{1}{\langle u\rangle^2}\right)^2\phi^2 \leqslant 2\left(|\frac{u\bar{\partial}({}^{II}\Omega\phi)}{{}^{II}\Omega}|^2 + |\frac{u\bar{\partial}({}^{I}\Omega\phi)}{{}^{I}\Omega}|^2 + \left(1 - \frac{1}{\langle r\rangle^2}\right)^2(\frac{u}{r}\phi)^2\right) \tag{3.50}$$

Applying (3.45) to the last term, integrating, and using a Hardy estimate together with (3.36) gives us:

$$\int_{\{t=t'\}\cap\mathbb{R}^3} \phi^2 \, dx \lesssim RHS(3.41) \ , \tag{3.51}$$

The result for $\int \frac{\underline{u}^2}{r^2}\phi$ follows by rearranging (3.49), integrating and applying (3.47) and (3.51).

□

3.5 Estimates for The Error Terms

The following lemmas are necessary for the proof of Lemma 3.4.1.

Lemma 3.5.1. *Let $\widetilde{\eta}$ denote the Minkowski metric in Bondi coordinates. Then:*

$${}^{I}\Omega\Box_{\widetilde{\eta}}(\frac{1}{{}^{I}\Omega}) = \frac{-3}{\langle r\rangle^4} \tag{3.52}$$
$${}^{II}\Omega\Box_{\widetilde{\eta}}(\frac{1}{{}^{II}\Omega}) = \frac{-4}{\langle u\rangle^3\langle \underline{u}\rangle^3} \ . \tag{3.53}$$

Lemma 3.5.2 (Estimates For $\widehat{({}^{(\widetilde{K}_0)}\pi)}$ and $\widehat{({}^{(\partial_t)}\pi)}$)**.** *Inside the wave zone:*

$$
\begin{aligned}
\widehat{({}^{(\widetilde{K}_0)}\pi)}^{uu} + 4(u+r)g^{uu} \ &= \ 0 \ , \\
|\widehat{({}^{(\widetilde{K}_0)}\pi} + 4(u+r)g)^{ur}_{\mathcal{B}}| \ &\lesssim \ \langle r\rangle^{\frac{1}{2}-\delta}\langle u\rangle^{\frac{1}{2}} \ , \\
|\widehat{({}^{(\widetilde{K}_0)}\pi} + 4(u+r)g)^{ua}_{\mathcal{B}}| \ &\lesssim \ \langle u\rangle\langle r\rangle^{-\delta} \ , \\
|\widehat{({}^{(\widetilde{K}_0)}\pi)}^{ij} + 4(u+r)g^{ij}| \ &\lesssim \ \langle r\rangle^{1-\delta} \ , \\
\widehat{({}^{(\partial_t)}\pi)}^{uu} \ &= \ 0 \ , \\
|\partial_u^\alpha \widetilde{\partial}_l^\beta \widehat{({}^{(\partial_t)}\pi)}^{ur}| \ &\lesssim \ \langle r\rangle^{-\delta-\frac{1}{2}-|\beta|}\langle u\rangle^{-|\alpha|-\frac{1}{2}} \ , \\
|\partial_u^\alpha \widetilde{\partial}_l^\beta \widehat{({}^{(\partial_t)}\pi)}^{ua}| \ &\lesssim \ \langle r\rangle^{-\delta-1-|\beta|}\langle u\rangle^{-|\alpha|} \ , \\
|\partial_u^\alpha \widetilde{\partial}_l^\beta \widehat{({}^{(\partial_t)}\pi)}^{ij}| \ &\lesssim \ \langle r\rangle^{-\delta-|\beta|}\langle u\rangle^{-|\alpha|-1} \ .
\end{aligned}
$$

Outside the wave zone:

$$
\begin{aligned}
|\widehat{({}^{(\widetilde{K}_0)}\pi)}^{\alpha\beta} + 4tg^{\alpha\beta}| \ &\lesssim \ \left(\frac{\langle t\rangle^{2-\gamma}}{\langle r\rangle^{1+\delta-\gamma}}\right) \ , \\
|\partial_t^k \partial_x^J \widehat{({}^{(\partial_t)}\pi)}^{\alpha\beta}| \ &\lesssim \ \langle r\rangle^{-\delta-|J|-k-1}\langle r/t\rangle^{\gamma(k+1)} \ .
\end{aligned}
$$

Lemma 3.5.3 (Approximate Normal Lemma)**.** $\widetilde{N} = -\nabla(u+r)$ *is an approximate normal vector to the* $t = const$ *hypersurfaces such that for any* $t' \in [t_0, t_1]$*:*

$$
\begin{aligned}
&\left| N^\alpha({}^{(\overline{K}_0)}P_\alpha[\phi]) - \widetilde{N}^\alpha({}^{(\overline{K}_0)}P_\alpha[\phi]) \right| \\
&\qquad \lesssim \ \langle r\rangle^{-\beta}\left[\langle \underline{u}\rangle^2\left(\left|\widetilde{\partial}_r\phi\right|^2 + |\widetilde{\nabla\!\!\!/}\phi|^2\right) + \langle u\rangle^2 \left|\partial_t\phi\right|^2\right] \ . \qquad (3.54)
\end{aligned}
$$

3.5.1 Proof of Lemma 3.4.1

Proof of (3.23)– (3.28). For estimate (3.23) we compute in Bondi coordinates inside the wave zone:

$$
\begin{aligned}
&({}^I\Omega)^{-2}({}^I V) \\
=\ & {}^I\Omega \Box_g \left(\frac{1}{{}^I\Omega}\right) \\
=\ & \sqrt{|g|}\big({}^I\Omega(\sqrt{|g|}\Box_g - \Box_{\widetilde{\eta}})\left(\frac{1}{{}^I\Omega}\right) + {}^I\Omega\Box_{\widetilde{\eta}}\left(\frac{1}{{}^I\Omega}\right)\big) \\
=\ & \sqrt{|g|}\big(- \frac{(\sqrt{|g|}g^{\alpha\beta} - \widetilde{\eta}^{\alpha\beta})\widetilde{\partial}^2_{\alpha\beta}\langle r\rangle}{\langle r\rangle} + 2\frac{(\sqrt{|g|}g^{\alpha\beta} - \widetilde{\eta}^{\alpha\beta})\widetilde{\partial}_\alpha\langle r\rangle\widetilde{\partial}_\beta\langle r\rangle}{\langle r\rangle^2} \\
& \qquad - \frac{\widetilde{\partial}_\alpha(g^{\alpha\beta}\sqrt{|g|} - \widetilde{\eta}^{\alpha\beta})\widetilde{\partial}_\beta\langle r\rangle}{\langle r\rangle} + \langle r\rangle(\widetilde{\partial}^2_r(\frac{1}{\langle r\rangle}) + \frac{2}{r}\widetilde{\partial}_r(\frac{1}{\langle r\rangle}))\big) \\
=\ & \sqrt{|g|}\big(\frac{(\sqrt{|g|}g^{ij} - \widetilde{\eta}^{ij})\widetilde{\partial}^2_{ij}\langle r\rangle}{\langle r\rangle} + 2\frac{(g^{ij}\sqrt{|g|} - \widetilde{\eta}^{ij})\widetilde{\partial}_i\langle r\rangle\widetilde{\partial}_j\langle r\rangle}{\langle r\rangle^2} \\
& \qquad - \frac{\widetilde{\partial}_\alpha(g^{\alpha j}\sqrt{|g|} - \widetilde{\eta}^{\alpha j})\widetilde{\partial}_j\langle r\rangle}{\langle r\rangle} - \frac{3}{\langle r\rangle^4}\big)\ ,
\end{aligned}
$$

where we used equation (3.52) on the last line. Inside the wave zone, by (1.7a) – (2.8) the worst decay occurs in the contraction on (u, i) indices. Therefore:

$$
\big|({}^I\Omega)^{-2}({}^I V)\big| \ \lesssim\ \frac{1}{\langle r\rangle^{\frac{3}{2}+\delta}\langle u\rangle^{\frac{1}{2}}} + O\left(\frac{1}{\langle r\rangle^{2+\delta}}\right). \tag{3.55}
$$

Outside the wave zone the only change to the computations above is that terms containing g^{uu} no longer vanish. By our extension lemma we can swap $\partial_u, \widetilde{\partial}_i$ derivatives for ∂_t, ∂_x derivatives in this equation. The worst term outside the wave zone occurs when a spatial derivative ∂_x lands on the metric. Using (2.7) to take care of the determinant, and the decay rates for the metric:

$$
\big|({}^I\Omega)^{-2}({}^I V)\big| \ \lesssim\ \frac{1}{\langle r\rangle^{2+\delta}} + \frac{1}{\langle r\rangle^4}. \tag{3.56}
$$

The inequalities (3.55) and (3.56) together prove the estimates (3.23) and (3.25). For the other weight, computing in Bondi coordinates in the wave zone:

$$\begin{aligned}
&({}^{II}\Omega)^{-2}({}^{II}V)\\
&= {}^{II}\Box_g\left(\frac{1}{{}^{II}\Omega}\right)\\
&= \sqrt{|g|}({}^{II}\Omega(\sqrt{|g|}\Box_g - \Box_{\widetilde{\eta}})\left(\frac{1}{{}^{II}\Omega}\right) - {}^{II}\Omega\Box_{\widetilde{\eta}}\left(\frac{1}{{}^{II}\Omega}\right))\\
&= \sqrt{|g|}(-\frac{(\sqrt{|g|}g^{\alpha\beta} - \widetilde{\eta}^{\alpha\beta})\widetilde{\partial}^2_{\alpha\beta}(\langle u\rangle\langle \underline{u}\rangle)}{\langle u\rangle\langle \underline{u}\rangle}\\
&+ 2\frac{(\sqrt{|g|}g^{\alpha\beta} - \widetilde{\eta}^{\alpha\beta})\widetilde{\partial}_\alpha(\langle u\rangle\langle \underline{u}\rangle)\widetilde{\partial}_\beta(\langle u\rangle\langle \underline{u}\rangle)}{(\langle u\rangle^2\langle \underline{u}\rangle^2)} - \frac{\widetilde{\partial}_\alpha(g^{\alpha\beta}\sqrt{|g|} - \widetilde{\eta}^{\alpha\beta})\widetilde{\partial}_\beta(\langle u\rangle\langle \underline{u}\rangle)}{\langle u\rangle\langle \underline{u}\rangle}\\
&- \frac{4}{\langle u\rangle^3\langle \underline{u}\rangle^3}) \ ,
\end{aligned}$$

where we used equation (3.53) on the last line. Inside the wave zone, by (1.7a) – (2.8), once again the worst decay occurs in the contraction on (u, i) indices. Using this together with the estimate (2.7) for the determinant:

$$|({}^{II}\Omega)^{-2}({}^{II}V)| \lesssim \frac{1}{\langle r\rangle^{\frac{3}{2}+\delta}\langle u\rangle^{\frac{1}{2}}} + O\left(\frac{1}{\langle r\rangle^\delta\langle u\rangle^2\langle \underline{u}\rangle^2}\right) \ . \tag{3.57}$$

Outside the wave zone the only change to the identity above is that terms containing g^{uu} no longer vanish. By our extension lemma we can swap $\partial_u, \widetilde{\partial}_i$ derivatives for ∂_t, ∂_x derivatives in the computations. The worst term outside the wave zone occurs when a spatial derivative ∂_x lands on the metric. Using (2.7) to take care of the determinant, we get outside the wave zone:

$$|({}^{II}\Omega)^{-2}({}^{II}V)| \lesssim \frac{1}{\langle r\rangle^{1+\delta}\langle u\rangle} + O\left(\frac{1}{\langle r\rangle^\delta\langle u\rangle\langle \underline{u}\rangle}\right) \ . \tag{3.58}$$

The inequalities (3.57) and (3.58) together prove the estimates (3.24) and (3.26).

For $({}^{I}\Omega)^{-2}\widetilde{K_0}({}^{I}V\chi_{r\leq\frac{1}{2}\epsilon t})$ in the set $\{r \leqslant \frac{1}{2}\epsilon t\}$, the worst case happens when the operator $t^2\partial_t$ lands. However, this operator commutes with the weight, so the only cases of consequence are when it lands on the metric coefficients or the cutoff.

Applying (3.25) followed by (1.11) for the exponents:

$$\begin{aligned}\left|\frac{\widetilde{K_0}({}^{I}V\chi_{r\leq\frac{1}{2}\epsilon t})}{({}^{I}\Omega)^2}\right| &\lesssim \chi_{r\leqslant\frac{1}{2}\epsilon t}\frac{t^{2-\gamma}}{\langle r\rangle^{3-\gamma+\delta}}+\chi_{r\sim\frac{1}{2}\epsilon t}\frac{t^2}{\langle r\rangle^{3+\delta}}\\ &\lesssim \langle t_0\rangle^{-\beta}(1+\epsilon^{-1})\chi_{r\leqslant\epsilon t}\frac{t^{2-\gamma'}}{\langle r\rangle^{3+\epsilon}}.\end{aligned}$$

For $({}^{II}\Omega)^{-2}\widetilde{K_0}({}^{II}V\chi_{r\leq\frac{1}{2}\epsilon t})$, the operator $t^2\partial_t$ can now land on the weight. Let's consider this case: since $u=t-\langle r\rangle$ in this region, it follows that $\langle u\rangle\sim\langle t\rangle$ and $\langle \underline{u}\rangle\sim\langle t\rangle$. Therefore, bounding all weights from above by t, counting weights and using (3.26) and (1.11) again to fix the exponents gives the result. For all other cases, the proof is the same as the ${}^{I}\Omega$ case.

□

Proof of (3.29). Inside $r\leqslant\frac{3}{4}t$ we have in (t,x) coordinates:

$$\begin{aligned}\big(\widehat{\mathcal{L}_{\widetilde{K}_0}g}+2\widetilde{K}_0\ln({}^{I}\Omega)g\big)^{\alpha\beta} &= (\widehat{{}^{(\widetilde{K}_0)}\pi})^{\alpha\beta}+4tg^{\alpha\beta}+O(\frac{t}{\langle r\rangle^2})\\ \big(\widehat{\mathcal{L}_{\widetilde{K}_0}g}+2\widetilde{K}_0\ln({}^{II}\Omega)g\big)^{\alpha\beta} &= (\widehat{{}^{(\widetilde{K}_0)}\pi})^{\alpha\beta}+4tg^{\alpha\beta}+O(\frac{1}{\langle t\rangle})\end{aligned}$$

We drop the faster-decaying terms, use lemma 3.5.2 and apply $|\frac{\partial(\Omega\phi)}{(\Omega)}|\lesssim|\partial\phi|+|\frac{\phi}{\langle r\rangle}|$ to get:

$$\begin{aligned}&\int_{t_0}^{t_1}\int_{\mathbb{R}^3}\chi^2_{r\leqslant\frac{3}{4}t}\left|{}^{J}\Omega^{-2}\big(\widehat{\mathcal{L}_{\widetilde{K}_0}g}+2\widetilde{K}_0\ln({}^{J}\Omega)g\big)^{\alpha\beta}\partial_\alpha({}^{J}\Omega\phi)\partial_\beta({}^{J}\Omega\phi)\right|\sqrt{|g|}dxdt\\ &\lesssim \int_{t_0}^{t_1}\int_{\mathbb{R}^3}\left(\frac{t^{2-\gamma}(\chi_{r\leqslant\frac{1}{10}t}\nabla\phi)^2}{\langle r\rangle^{1-\gamma+\delta}}+\frac{t^{2-\gamma}(\chi_{r\leqslant\frac{1}{10}t}\phi)^2}{\langle r\rangle^{3-\gamma+\delta}}\right)\sqrt{|g|}dtdx\\ &+\int_{t_0}^{t_1}\int_{\mathbb{R}^3}\left(\frac{t^{2-\gamma}(\chi_{\frac{t}{20}\leqslant r\leqslant\frac{3t}{4}}\nabla\phi)^2}{\langle r\rangle^{1-\gamma+\delta}}+\frac{t^{2-\gamma}(\chi_{\frac{t}{20}\leqslant r\leqslant\frac{3t}{4}}\phi)^2}{\langle r\rangle^{3-\gamma+\delta}}\right)\sqrt{|g|}dtdx\\ &\lesssim \langle t_0\rangle^{-\beta}\left(\|\phi\|^2_{LSM^{1-\gamma'}_{r\leqslant\frac{1}{10}t}[t_0,t_1]}+\sup_{t_0\leqslant t\leqslant t_1}CE[\phi(t)]\right).\end{aligned}$$

For the term supported inside $\frac{t}{20}\leqslant r\leqslant\frac{3t}{4}$ in the last line, we traded r decay for t decay and integrated. In the wave zone:

$$\begin{aligned}\big(\widehat{\mathcal{L}_{\widetilde{K}_0}g}+2\widetilde{K}_0\ln({}^{I}\Omega)g\big)^{\alpha\beta} &= (\widehat{{}^{(\widetilde{K}_0)}\pi})^{\alpha\beta}+4(u+r)g^{\alpha\beta}+O(\frac{1}{\langle r\rangle})\\ \big(\widehat{\mathcal{L}_{\widetilde{K}_0}g}+2\widetilde{K}_0\ln({}^{II}\Omega)g\big)^{\alpha\beta} &= (\widehat{{}^{(\widetilde{K}_0)}\pi})^{\alpha\beta}+4(u+r)g^{\alpha\beta}+O(\frac{1}{\langle u\rangle})+O(\frac{1}{\langle\underline{u}\rangle})\end{aligned}$$

We drop the faster-decaying terms and use the rates in Lemma 3.5.2 to bound all components. When $\alpha = u, \beta \neq u$, we apply Lemma 2.1.10 and use (2.22) to compute in the frame $\mathcal{B}$. In the case $\beta = r$, this gives us:

$$\begin{aligned}
&\int_{t_0}^{t_1}\int_{\mathbb{R}^3} \chi w \left| {}^J\Omega^{-2}\big(\widehat{\mathcal{L}_{\widetilde{K}_0} g + 2\widetilde{K}_0 \ln({}^J\Omega) g}\big)^{ur} \partial_u({}^J\Omega\phi)\widetilde{\partial}_r({}^J\Omega\phi)\right| \, dV_g \\
&\quad\lesssim \int_{t_0}^{t_1}\int_{\mathbb{R}^3} \chi w \left| {}^J\Omega^{-2}\left(\widehat{{}^{(\widetilde{K}_0)}\pi + 4(u+r) g}\right)^{ur}_{\mathcal{B}} \partial_u({}^J\Omega\phi)\widetilde{\partial}_r({}^J\Omega\phi)\right| \, dV_g \\
&\quad\lesssim \int_{t_0}^{t_1}\int_{\mathbb{R}^3} \chi w \frac{|\langle u\rangle \partial_u({}^J\Omega\phi)|}{\langle r\rangle^{\frac{1}{2}+\delta/2}\, {}^J\Omega} \cdot \frac{|\langle \underline{u}\rangle \widetilde{\partial}_r({}^J\Omega\phi)|}{\langle u\rangle^{\frac{1}{2}+\delta/2}\, {}^J\Omega} \, dV_g \\
&\quad\lesssim \langle t_0\rangle^{-\beta}\left(\sup_{t_0\leqslant t\leqslant t_1} CE[\phi(t)] + \sup_{\mathcal{W}\cap[t_0,t_1]}\left({}^J CE_{ch}[\phi(u)]\right)\right) ,
\end{aligned}$$

where in the last line we used Young's inequality followed by (3.32) to trade $\widetilde{\partial}_r$ for L in the null energy. Similarly, when $\beta = a$:

$$\begin{aligned}
&\int_{t_0}^{t_1}\int_{\mathbb{R}^3} \chi w \left| {}^J\Omega^{-2}\big(\widehat{\mathcal{L}_{\widetilde{K}_0} g + 2\widetilde{K}_0 \ln({}^J\Omega) g}\big)^{ua} \partial_u({}^J\Omega\phi) e_a({}^J\Omega\phi)\right| \, dV_g \\
&\quad= \int_{t_0}^{t_1}\int_{\mathbb{R}^3} \chi w \left| {}^J\Omega^{-2}\left(\widehat{{}^{(\widetilde{K}_0)}\pi + 4(u+r) g}\right)^{ua}_{\mathcal{B}} \partial_u({}^J\Omega\phi) e_a({}^J\Omega\phi)\right| \, dV_g \\
&\quad\lesssim \int_{t_0}^{t_1}\int_{\mathbb{R}^3} \chi w \frac{|\langle u\rangle \partial_u({}^J\Omega\phi)|}{\langle r\rangle^{\frac{1}{2}+\delta/2}\, {}^J\Omega} \cdot \frac{|\langle \underline{u}\rangle e_a({}^J\Omega\phi)|}{\langle r\rangle^{\frac{1}{2}+\delta/2}\, {}^J\Omega} \, dV_g \\
&\quad\lesssim \langle t_0\rangle^{-\beta} \sup_{t_0\leqslant t\leqslant t_1} CE[\phi(t)] .
\end{aligned}$$

For the components without a u coordinate:

$$\begin{aligned}
&\int_{t_0}^{t_1}\int_{\mathbb{R}^3} \chi w \left| {}^J\Omega^{-2}\big(\widehat{\mathcal{L}_{\widetilde{K}_0} g + 2\widetilde{K}_0 \ln({}^J\Omega) g}\big)^{ij} \widetilde{\partial}_i({}^J\Omega\phi)\widetilde{\partial}_j({}^J\Omega\phi)\right| \, dV_g \\
&\quad\lesssim \sum_{i,j}\int_{t_0}^{t_1}\int_{\mathbb{R}^3} \chi w \frac{|\langle r\rangle \widetilde{\partial}_i({}^J\Omega\phi)|}{\langle r\rangle^{\frac{1}{2}+\delta/2}\, {}^J\Omega} \cdot \frac{|\langle \underline{u}\rangle \widetilde{\partial}_j({}^J\Omega\phi)|}{\langle r\rangle^{\frac{1}{2}+\delta/2}\, {}^J\Omega} \, dV_g \\
&\quad\lesssim \langle t_0\rangle^{-\beta} \sup_{t_0\leqslant t\leqslant t_1} CE[\phi(t)] .
\end{aligned}$$

For the far exterior the result will follow by a similar argument.

□

Proof of (3.30). For both $\Omega = {}^I\Omega,\ {}^{II}\Omega$:

$$g^{\alpha\beta}(\mathcal{L}_{\widetilde{K}_0} g)_{\alpha\beta} - 8\widetilde{K}_0 \ln(\Omega) \;=\; O\Big(\frac{t^{2-\gamma}}{\langle r\rangle^{1-\gamma+\delta}}\Big) .$$

Combining this with estimates (3.25)–(3.28):

$$\int_{t_0}^{t_1}\int_{\mathbb{R}^3} \big|\big(g^{\alpha\beta}(\mathcal{L}_{\widetilde{K}_0} g)_{\alpha\beta} - 8\widetilde{K_0}\ln({}^J\Omega)\big)\frac{({}^J V)\chi_{r\leq\frac{1}{2}\epsilon t}}{({}^J\Omega)^2} - \frac{2\widetilde{K_0}({}^J V\chi_{r\leq\frac{1}{2}\epsilon t})}{({}^J\Omega)^2}\big|\phi^2\ \sqrt{|g|}dxdt$$

$$\lesssim\ \epsilon\int_{t_0}^{t_1}\int_{\mathbb{R}^3}\big(\frac{t^{2-\gamma'}\chi_{r\leqslant\frac{1}{10}t}\phi^2}{\langle r\rangle^{3+\epsilon}}\big)\ \sqrt{|g|}dtdx$$

$$\lesssim\ \epsilon\|\,\phi\,\|^2_{LSM^{1-\gamma'}_{r\leqslant\frac{1}{10}t}[t_0,t_1]}\ .$$

□

Next we will do the proof of (3.31) – (3.32). The strategy of the proof will be to compute the energy densities and observe that by the decay rates for the optical function u, $\widetilde{K}_0^\alpha \to K_0^\alpha$ with K_0 the Minkowski Morawetz field. For the normal vectors we have $N \to -\partial_t$ and $L \to \partial_t + \partial_r$ by the decay rates for $g_{\alpha\beta}$. This will allow us to show that these energy densities are upper bounds to the conjugated conformal energies modulo bootstrap error terms. This specially needs some care inside $\mathcal{W}\cap[t_0,t_1]$ since we do not want to run into bad combinations of weights and derivatives in the error terms.

Proof of (3.31). Let $\chi_{r\leqslant 2\epsilon t}$, $\chi_{\epsilon t\leqslant r}$ be a smooth partition of unity. For the interior region we aim to show:

$$\begin{aligned}
\chi_{r\leqslant 2\epsilon t}{}^J CE[\phi(t')] &\lesssim \int_{\{t=t'\}\cap\mathbb{R}^3}\chi_{r\leqslant 2\epsilon t}\frac{\langle u\rangle^2|\nabla_{t,x}({}^J\Omega\phi)|^2}{({}^J\Omega)^2}\ \sqrt{|g|}dx\\
&\lesssim \int_{\{t=t'\}\cap\mathbb{R}^3}\chi_{r\leqslant 2\epsilon t}\frac{N^\alpha({}^{(\widetilde{K_0})J}\widetilde{P}_\alpha[\phi])}{({}^J\Omega)^2}\ \sqrt{|g|}dx\\
&\lesssim \chi_{r\leqslant 2\epsilon t}{}^J CE[\phi(t')]\ ,
\end{aligned}\tag{3.59}$$

for $J = I, II$. We start by noting that $N = -\nabla t$ is timelike everywhere by assumption and ∂_t is timelike everywhere by lemma (2.2.2). Since $t \gg r$ we can apply the dominant energy condition and Young's inequality to produce constants

$C_i > 0$, $i = 1, 2, 3$, such that for $J = I, II$:

$$\begin{aligned} &N^\alpha({}^{(\overline{K_0})J}\widetilde{P}_\alpha[\phi]) \\ &= \left(t^2 - 2t\langle r\rangle + \langle r\rangle^2 + \frac{2(t+r-\langle r\rangle)r^2}{\langle r\rangle}\right) {}^J\widetilde{T}(\partial_t, N) - \frac{2(t+r-\langle r\rangle)r^2}{\langle r\rangle} {}^J\widetilde{T}(\partial_r, N) \\ &\quad + {}^J\widetilde{T}(\partial_t, N) \\ &\geqslant C_1 t^2 |\nabla_{t,x}({}^J\Omega\phi)|^2 - C_2 O(t\langle r\rangle)|\nabla_{t,x}({}^J\Omega\phi)|^2 + C_1|\nabla_{t,x}({}^J\Omega)\phi)|^2 \\ &\geqslant (C_1 - \epsilon C_3)\langle u\rangle^2 |\nabla_{t,x}({}^J\Omega\phi)|^2 , \end{aligned} \tag{3.60}$$

and on the last line we used the fact that $u = t - \langle r\rangle$ in this set. Choosing ϵ sufficiently small ensures that this last quantity is nonnegative. For the upper bound of $N^\alpha({}^{(\overline{K_0})J}\widetilde{P}_\alpha[\phi])$ we apply Young's inequality:

$$N^\alpha({}^{(\overline{K_0})J}\widetilde{P}_\alpha[\phi]) \lesssim \langle u\rangle^2 |\nabla_{t,x}({}^J\Omega\phi)|^2 . \tag{3.61}$$

Combining the pointwise bounds (3.60) and (3.61), multiplying by the cutoff, dividing by $({}^J\Omega)^2$ and integrating gives us (3.59).

For the region where $\{\epsilon t < r\}$ we have asymptotic flatness. Therefore we define the Minkowski energy density associated to $K_0 + \partial_t$ to be:

$${}^J\mathcal{M}[\phi] = \langle u\rangle^2(\partial_u {}^J\psi)^2 + \frac{1}{2}(\langle u\rangle^2 + 2(u+r)r)\sum_i(\widetilde{\partial}_i {}^J\psi)^2 - \langle u\rangle^2 \partial_u {}^J\psi \widetilde{\partial}_r {}^J\psi . \tag{3.62}$$

We note that (3.62) is simply what would come out if we computed $N^\alpha({}^{(\overline{K_0})J}\widetilde{P}_\alpha[\phi])$ using $u = t - r$ and $g = \widetilde{\eta}$. We claim:

$$\begin{aligned} \langle t+r\rangle^2 \left(\left|\widetilde{\partial}_r({}^J\Omega\phi)\right|^2 + |{}^J\Omega\,\widetilde{\nabla}\phi|^2\right) + \langle u\rangle^2 \left|\partial_t({}^J\Omega\phi)\right|^2 &\lesssim {}^J\mathcal{M}[\phi] \\ &\lesssim N^\alpha({}^{(\overline{K_0})J}\widetilde{P}_\alpha[\phi]) . \end{aligned} \tag{3.63}$$

To show the claim we expand $\widetilde{N}^{\alpha(\overline{K_0})J}P_\alpha[\phi]$ with approximate normal vector $\widetilde{N} = -\nabla(u + r)$ computed in the frame $\mathcal{B}$. As usual, in the computations below for the (u, i) terms are done using the $\mathcal{B}$ frame and the (i, j) components are done using the Bondi coordinates. In the wave zone we use $g^{uu} = 0$ and cancel the mixed terms $\widetilde{K}_0^r g^{ur}\partial_u{}^J\psi\partial_r{}^J\psi$ below to get:

$$\begin{aligned}
&\widetilde{N}^{\alpha(\overline{K_0})\,J}P_\alpha[\phi] \\
&= -\widetilde{K}_0^r g^{ur}\partial_u{}^J\psi\partial_r{}^J\psi + \widetilde{K}_0^u g^{ur}(\partial_u{}^J\psi)^2 - (g^{ur}+g^{rr})\partial_r{}^J\psi\widetilde{K}^\beta\partial_\beta{}^J\psi \\
&\quad + \frac{1}{2}(K_0^u + \widetilde{K}_0^r)\big[2(g^{ur}\partial_u{}^J\psi\partial_r{}^J\psi + g^{au}\partial_u{}^J\psi e_a{}^J\psi) + g^{ij}\widetilde{\partial}_i{}^J\psi\widetilde{\partial}_j{}^J\psi\big] \\
&\quad + (g^{ua}+g^{ra})e_a{}^J\psi K^\beta\partial_\beta{}^J\psi - g^{ur}(\partial_u{}^J\psi)^2 - (g^{ur}+g^{rr})\partial_r{}^J\psi\partial_u{}^J\psi \\
&\quad + \frac{1}{2}\Big[2(g^{ur}\partial_u{}^J\psi\partial_r{}^J\psi + g^{au}\partial_u{}^J\psi e_a{}^J\psi) + g^{ij}\widetilde{\partial}_i{}^J\psi\widetilde{\partial}_j{}^J\psi\Big] + (g^{ua}+g^{ra})e_a{}^J\psi\partial_u{}^J\psi \\
&= -(\widetilde{K}_0^u+1)g^{ur}(\partial_u{}^J\psi)^2 - (g^{ur}+g^{rr})\partial_r{}^J\psi(\widetilde{K}^\beta\partial_\beta{}^J\psi + \partial_u{}^J\psi) \\
&\quad + (\widetilde{K}_0^u+1)g^{ur}\partial_u{}^J\psi\partial_r{}^J\psi + (\widetilde{K}_0^r + \widetilde{K}_0^u + 1)g^{ua}\partial_u{}^J\psi e_a{}^J\psi \\
&\quad + \frac{1}{2}(K_0^u + \widetilde{K}_0^r + 1)g^{ij}\widetilde{\partial}_i{}^J\psi\widetilde{\partial}_j{}^J\psi + (g^{ua}+g^{ra})e_a{}^J\psi(K^\beta\partial_\beta{}^J\psi + \partial_u{}^J\psi)\ . \qquad (3.64)
\end{aligned}$$

Subtracting ${}^J\mathcal{M}[\phi]$ from this:

$$\begin{aligned}
&{}^J\mathcal{M}[\phi] \\
&= \widetilde{N}^{\alpha(\overline{K_0},\Omega)}P_\alpha[{}^J\psi] + ({}^{(\Omega)}\mathcal{M}[{}^J\psi] - \widetilde{N}^{\alpha(\widetilde{K}_0+\partial_t,\Omega)}P_\alpha[{}^J\psi]) \\
&= \widetilde{N}^{\alpha(\overline{K_0},\Omega)}P_\alpha[{}^J\psi] + \langle u\rangle^2(g^{ur}+1)(\partial_u{}^J\psi)^2 - (g^{ur}+g^{rr})\partial_r{}^J\psi(\widetilde{K}^\beta\partial_\beta{}^J\psi + \partial_u{}^J\psi) \\
&\quad + \langle u\rangle^2(g^{ur}+1)\partial_u{}^J\psi\partial_r{}^J\psi + (\langle u\rangle^2 + 2(u+r)r)g^{ua}\partial_u{}^J\psi e_a{}^J\psi \\
&\quad + \frac{1}{2}(\langle u\rangle^2 + 2(u+r)r)(g^{ij}-\widetilde{\eta}^{ij})\widetilde{\partial}_i{}^J\psi\widetilde{\partial}_j{}^J\psi \\
&\quad + (g^{ua}+g^{ra})e_a{}^J\psi(K^\beta\partial_\beta{}^J\psi + \partial_u{}^J\psi).
\end{aligned}$$

By the decay rates in (2.4) we get, inside the wave zone:

$$\begin{aligned}
\big|\langle u\rangle^2(g^{ur}+1)\partial_u{}^J\psi\partial_r{}^J\psi\big| &\lesssim \frac{{}^J\mathcal{M}[\phi]}{\langle r\rangle^\delta} \\
\big|(\langle u\rangle^2 + 2(u+r)r)(g^{ij}-\delta^{ij})e_i{}^J\psi e_j{}^J\psi\big| &\lesssim \frac{{}^J\mathcal{M}[\phi]}{\langle r\rangle^\delta} \\
\big|(g^{ur}+1)\langle u\rangle^2(\partial_u{}^J\psi)^2\big| &\lesssim \frac{{}^J\mathcal{M}[\phi]}{\langle r\rangle^\delta} \\
\big|(g^{ua}+g^{ra})e_a{}^J\psi K^\beta\partial_\beta{}^J\psi\big| &\lesssim \frac{\langle u\rangle r^2}{\langle r\rangle^{1+\delta}}|\partial_u{}^J\psi e_a{}^J\psi| \lesssim \frac{{}^J\mathcal{M}[\phi]}{\langle r\rangle^\delta}
\end{aligned}$$

By the special decay rate (2.13) we also have:

$$\left|(\langle u\rangle^2 + 2(u+r)r)g^{ua}\partial_u {}^J\psi e_a {}^J\psi\right| \lesssim \frac{\langle u\rangle r^2}{\langle r\rangle^{1+\delta}}|\partial_u {}^J\psi e_a {}^J\psi| \lesssim \frac{{}^J\mathcal{M}[\phi]}{\langle r\rangle^\delta} \tag{3.65}$$

In the complement to the wave zone within the region where $\epsilon t < r$ we can get the same type of estimates due to the fact that we can trade u weights for $\underline{u}$ freely. Combining these results we have, everywhere inside the region $\epsilon t < r$:

$$^J\mathcal{M}[\phi] \lesssim \widetilde{N}^\alpha({}^{(\overline{K_0})J}\widetilde{P}_\alpha[\phi]) + \frac{{}^J\mathcal{M}[\phi]}{\langle r\rangle^\delta} .$$

Bootstrapping the terms with $\langle r\rangle^{-\delta}$ gives bounds for ${}^I\mathcal{M}[\phi]$, ${}^{II}\mathcal{M}[\phi]$ in terms of $\widetilde{N}^\alpha({}^{(\overline{K_0})I}\widetilde{P}_\alpha[\phi])$ and $\widetilde{N}^\alpha({}^{(\overline{K_0})II}\widetilde{P}_\alpha[\phi])$ respectively. To swap $\widetilde{N}$ for $N = -\nabla t$ we note by (3.54)

$$\begin{aligned}
&{}^J\mathcal{M}[\phi] \\
&\lesssim N^{\alpha(\overline{K_0})J}P_\alpha[\phi] + (\widetilde{N}^\alpha - N^\alpha)^{(\overline{K_0})J}P_\alpha[\phi] \\
&\lesssim N^{\alpha(\overline{K_0})J}P_\alpha[\phi] + \langle r\rangle^{-\beta}\left(\langle\underline{u}\rangle^2\left(\left|\widetilde{\partial}_r({}^J\psi)\right|^2 + |\widetilde{\nabla\!\!\!/}({}^J\psi)|^2\right) + \langle u\rangle^2\left|\partial_t({}^J\psi)\right|^2\right) ,
\end{aligned}$$

the last term can be bootstrapped onto ${}^J\mathcal{M}[\phi]$. This proves the upper bounds for ${}^J\mathcal{M}[\phi]$ in (3.63). To finish the proof of the claim it suffices to show:

$$\langle\underline{u}\rangle^2\left(\left|\widetilde{\partial}_r({}^J\Omega\phi)\right|^2 + |{}^J\Omega\,\widetilde{\nabla\!\!\!/}\phi|^2\right) + \langle u\rangle^2\left|\partial_t({}^J\Omega\phi)\right|^2 \lesssim {}^J\mathcal{M}[\phi] \tag{3.66}$$

In the wave zone this is a simple consequence of (3.62), Young's inequality and the estimate $|u| \leqslant r$ together with the identity:

$$K^r + \frac{K^u}{2} = 2(u+r)r + \frac{u}{2} = \frac{u + 4ur + 4r^2}{2} = \frac{(u+2r)^2}{2} = \frac{\underline{u}^2}{2}. \tag{3.67}$$

In the region $\{\epsilon t \leqslant r \leqslant \frac{2}{3}t\} \cup \{\frac{3}{2}t \leqslant r \leqslant 2t\}$, we have $u \approx \underline{u}$, therefore Young's inequality again proves the claim. In the far exterior $2t \leqslant r$, this holds by switching the roles of t and r and repeating the proof of the case $t \gg r$. Since $\langle t+r\rangle \approx \langle\underline{u}\rangle$ this finishes the proof of the claim everywhere.

Next we observe that the following two estimates hold:

$$\begin{aligned}
&\langle t+r\rangle^2\left(\left|L({}^J\Omega\phi)\right|^2 + |{}^J\Omega\,\widetilde{\nabla\!\!\!/}\phi|^2\right) + \langle u\rangle^2\left|\partial_t({}^J\Omega\phi)\right|^2 \\
&\qquad \lesssim \langle t+r\rangle^2\left(\left|\widetilde{\partial}_r({}^J\Omega\phi)\right|^2 + |{}^J\Omega\,\widetilde{\nabla\!\!\!/}\phi|^2\right) + \langle u\rangle^2\left|\partial_t({}^J\Omega\phi)\right|^2 , \\
&\widetilde{N}^\alpha({}^{(\overline{K_0})J}\widetilde{P}_\alpha[\phi]) \lesssim \langle t+r\rangle^2\left(\left|L({}^J\Omega\phi)\right|^2 + |{}^J\Omega\,\widetilde{\nabla\!\!\!/}\phi|^2\right) + \langle u\rangle^2\left|\partial_t({}^J\Omega\phi)\right|^2 .
\end{aligned}$$

The first one follows directly by Young's inequality while the second one follows by using the basis $\mathcal{C} = \{\partial_t, L, e_3, e_4\}$ to expand $\widetilde{N}^\alpha({}^{(\overline{K_0})J}\widetilde{P}_\alpha[\phi])$ followed by Young's inequality again. Combining these results with our estimate (3.63) dividing by $({}^J\Omega)^2$ and integrating gives us, in the region where $\epsilon t \leqslant r$:

$$\begin{aligned}
&\chi_{\epsilon t\leqslant r}{}^J CE[\phi(t')] \\
\lesssim\ & \int_{\{t=t'\}\cap\mathbb{R}^3} \chi_{\epsilon t\leqslant r}\left(\langle t+r\rangle^2\big(\Big|\frac{\widetilde{\partial}_r({}^J\Omega\phi)}{({}^J\Omega)^2}\Big|^2 + |\,\widetilde{\nabla}\!\!\!/\,\phi|^2\big) + \langle u\rangle^2\left|\frac{\partial_t({}^J\Omega\phi)}{({}^J\Omega)^2}\right|^2\right)\sqrt{|g|}dx \\
\lesssim\ & \int_{\{t=t'\}\cap\mathbb{R}^3} \chi_{\epsilon t\leqslant r}\frac{{}^J\mathcal{M}[\phi]}{({}^J\Omega)^2}\sqrt{|g|}dx \\
\lesssim\ & \int_{\{t=t'\}\cap\mathbb{R}^3} \chi_{\epsilon t\leqslant r}\frac{N^\alpha({}^{(\overline{K_0})J}\widetilde{P}_\alpha[\phi])}{({}^J\Omega)^2}\sqrt{|g|}dx \\
\lesssim\ & \chi_{\epsilon t\leqslant r}{}^J CE[\phi(t')]\ ,
\end{aligned} \tag{3.68}$$

for $J = I, II$. Adding the two estimates (3.59) and (3.68) finishes the proof of the lemma.

□

Proof of Lemma (3.32). Inside the wave zone $\mathcal{W}$ we define the Minkowski characteristic energy densities associated to $K_0 + \partial_t$ to be:

$${}^J\mathcal{M}_{ch}[\phi] = (\frac{3}{2} + K^r + \frac{K^u}{2})(\widetilde{\partial}_r{}^J\psi)^2 + \frac{1}{2}(1+K^u)\sum(e_a{}^J\psi)^2\ , \tag{3.69}$$

with ${}^J\psi = {}^J\Omega\phi$. We note that (3.69) is simply what would come out if we computed $L^\alpha({}^{(\overline{K_0})J}\widetilde{P}_\alpha[\phi])$ using $u = t - r$ and $g = \widetilde{\eta}$. We claim the following pointwise estimates hold inside $\mathcal{W}$:

$$\langle t+r\rangle^2\left|\widetilde{\partial}_r({}^J\Omega\phi)\right|^2 + \langle u\rangle^2|{}^J\Omega\,\widetilde{\nabla}\!\!\!/\,\phi|^2 \ \lesssim\ {}^J\mathcal{M}_{ch}[\phi]\ \lesssim\ L^\alpha({}^{(\overline{K_0})J}\widetilde{P}_\alpha[\phi])\ , \tag{3.70}$$

with $J = I, II$. To prove this we begin by computing the outer unit normal to the null hypersurfaces $u = const$ using the Bondi frame and applying (2.16):

$$L\ =\ -\nabla u\ =\ -g^{u\alpha}\partial_\alpha - g^{ur}\widetilde{\partial}_r - g^{ua}e_a\ =\ \widetilde{\partial}_r + O(r^{-\delta})\widetilde{\partial}_r + O(\frac{\langle u\rangle}{\langle \underline{u}\rangle}r^{-\delta})\sum_a e_a\ . \tag{3.71}$$

Since we are in the wave zone we use this plus $g^{uu} = 0$ and cancel the mixed terms $\widetilde{K}_0^r g^{ur}\partial_u{}^J\psi\partial_r{}^J\psi$ below to get:

$$
\begin{aligned}
&L^\alpha({}^{(\overline{K_0})J}\widetilde{P}_\alpha[\phi]) \\
&= -g^{u\alpha}\partial_\alpha{}^J\psi\partial_r{}^J\psi K^r + \frac{1}{2}g^{rr}(\widetilde{\partial}_r{}^J\psi)^2K^u + K^u g^{ra}\widetilde{\partial}_r{}^J\psi e_a{}^J\psi + \frac{1}{2}K^u g^{ab}e_a{}^J\psi e_b{}^J\psi \\
&\qquad - \ g^{ur}\partial_r{}^J\widetilde{\psi}\partial_u{}^J\psi - g^{ua}e_a{}^J\psi\partial_u{}^J\psi + \frac{1}{2}g^{rr}(\widetilde{\partial}_r{}^J\psi)^2 + \frac{1}{2}g^{ra}\widetilde{\partial}_r{}^J\psi e_a{}^J\psi \\
&\ + \ \frac{1}{2}g^{ab}e_a{}^J\psi e_b{}^J\psi \\
&= -g^{ur}(\partial_r{}^J\psi)^2(1+K^r) + (\frac{1+K^u}{2})g^{rr}(\widetilde{\partial}_r{}^J\psi)^2 + \frac{1}{2}(1+K^u)g^{ab}e_a{}^J\psi e_b{}^J\psi \\
&\qquad + \ g^{ua}e_a{}^J\psi\widetilde{\partial}_r{}^J\psi(1+K^r) \qquad (3.72)\\
&= \ \frac{1}{2}(1+\underline{u}^2)(\widetilde{\partial}_r{}^J\psi)^2 + \frac{1}{2}(1+u^2)|\,\widetilde{\nabla}{}^J\psi|_g^2 + (1+2(u+r)r)g^{ua}e_a{}^J\psi\widetilde{\partial}_r{}^J\psi\ .
\end{aligned}
$$

Subtracting ${}^J\mathcal{M}_{ch}[\phi]$ from the above we get:

$$
\begin{aligned}
&{}^J\mathcal{M}_{ch}[\phi] \\
&= \ L^\alpha({}^{(\overline{K_0})J}\widetilde{P}_\alpha[\phi]) + ({}^J\mathcal{M}_{ch}[\phi] - L^{\alpha(\overline{K_0})J}\widetilde{P}_\alpha[\phi]) \\
&= \ L^\alpha({}^{(\overline{K_0})J}\widetilde{P}_\alpha[\phi]) + (g^{ur}+1)(\partial_r{}^J\psi)^2(1+K^r) - (\frac{1+K^u}{2})(g^{rr}+1)(\widetilde{\partial}_r{}^J\psi)^2 \\
&\qquad - \frac{1}{2}(1+K^u)(g^{ab}-\eta^{ab})e_a{}^J\psi e_b{}^J\psi - g^{ua}e_a{}^J\psi\widetilde{\partial}_r{}^J\psi(1+K^r)\ .
\end{aligned}
$$

By the decay rates in (2.4) in the wave zone we get:

$$
\begin{aligned}
\left|(g^{ur}+1)(\partial_r{}^J\psi)^2(1+K^r)\right| \ &\lesssim\ \frac{{}^J\mathcal{M}_{ch}[\phi]}{\langle r\rangle^\delta}\ , \\
\left|(\frac{1+K^u}{2})(g^{rr}+1)(\widetilde{\partial}_r{}^J\psi)^2\right| \ &\lesssim\ \frac{{}^J\mathcal{M}_{ch}[\phi]}{\langle r\rangle^\delta}\ , \\
\left|\frac{1}{2}(1+K^u)(g^{ab}-\eta^{ab})e_a{}^J\psi e_b{}^J\psi\right| \ &\lesssim\ \frac{{}^J\mathcal{M}_{ch}[\phi]}{\langle r\rangle^\delta}\ .
\end{aligned}
$$

By the special decay rate (2.13) we also have:

$$
\left|g^{ua}e_a{}^J\psi\widetilde{\partial}_r{}^J\psi(1+K^r)\right| \ \lesssim\ \frac{\langle u\rangle r^2}{\langle r\rangle^{1+\delta}}|\widetilde{\partial}_r{}^J\psi e_a{}^J\psi| \ \lesssim\ \frac{{}^J\mathcal{M}_{ch}[\phi]}{\langle r\rangle^\delta}\ . \qquad (3.73)
$$

Combining these results we have inside the region $\mathcal{W}$:

$$
{}^J\mathcal{M}_{ch}[\phi] \ \lesssim\ L^\alpha({}^{(\overline{K_0})J}P_\alpha[\phi]) + \frac{{}^J\mathcal{M}_{ch}[\phi]}{\langle r\rangle^\delta}
$$

Bootstrapping leads to bounds ${}^{I}\mathcal{M}_{ch}[\phi]$, ${}^{II}\mathcal{M}_{ch}[\phi]$ in terms of $L^{\alpha}({}^{(\overline{K_0})I}\widetilde{P}_{\alpha}[\phi])$ and $L^{\alpha}({}^{(\overline{K_0})II}\widetilde{P}_{\alpha}[\phi])$ respectively. To finish the proof of (3.70) it suffices to show:

$$\langle t+r\rangle^2 \left|\widetilde{\partial}_r({}^{J}\Omega\phi)\right|^2 + \langle u\rangle^2 |{}^{J}\Omega\,\widetilde{\nabla\!\!\!/}\phi|^2 \ \lesssim\ {}^{J}\mathcal{M}_{ch}[\phi]$$

this follows by applying (3.67) to $(\frac{3}{2}+K^r+\frac{K^u}{2})$ and observing that $\underline{u}\sim t+r$. Therefore we have shown (3.70).

Next we observe that squaring identity (3.71) and using Young's inequality gives us:

$$\begin{aligned}
&\langle t+r\rangle^2 \left|L({}^{J}\Omega\phi)\right|^2 + \langle u\rangle^2 |{}^{J}\Omega\,\widetilde{\nabla\!\!\!/}\phi|^2 \\
&\quad\lesssim\ \langle t+r\rangle^2 |\widetilde{\partial}_r({}^{J}\Omega\phi)|^2 + \langle t+r\rangle^2 \left|\frac{\langle u\rangle}{\langle \underline{u}\rangle}\langle r\rangle^{-\delta}\right|^2 \sum_a |{}^{J}\Omega e_a({}^{J}\Omega\phi)|^2 \\
&\quad\lesssim\ \langle t+r\rangle^2 \left|\widetilde{\partial}_r({}^{J}\Omega\phi)\right|^2 + \langle u\rangle^2 |{}^{J}\Omega\,\widetilde{\nabla\!\!\!/}\phi|^2\ ,
\end{aligned} \tag{3.74}$$

as well as:

$$L^{\alpha}({}^{(\overline{K_0})J}\widetilde{P}_{\alpha}[\phi])\ \lesssim \langle t+r\rangle^2 \left|L({}^{J}\Omega\phi)\right|^2 + \langle u\rangle^2 |{}^{J}\Omega\,\widetilde{\nabla\!\!\!/}\phi|^2\ . \tag{3.75}$$

Combining (3.70), (3.74), and (3.75), dividing by $({}^{J}\Omega)^2$ and integrating finishes the proof of the lemma.

□

Remark 3.5.4. *Notice that in estimates* (3.65) *and* (3.73) *we needed the full decay for g^{ua} given by* (2.13) *in order to finish the proof of* (3.31) *and* (3.32).

Proof of (3.33)-(3.34). Since these two estimates contain cutoffs $\chi_{r\leqslant\frac{1}{2}\epsilon t}$ we will be careful when computing the constants in order to keep track of all instances of ϵ^{-1} in the proof. For (3.33) we use (3.23) to get:

$$\sum_{\beta=t,r}\int_{\{t=t'\}\cap\mathbb{R}^3}\left|\frac{(\overline{K}_0^{\beta})^{I}V\chi_{r\leq\frac{1}{2}\epsilon t}\phi}{({}^{I}\Omega)^2}\right|^2\sqrt{|g|}dx\ \lesssim\ \int_{\{t=t'\}\cap\mathbb{R}^3}\frac{(t\phi\chi_{r\leqslant\frac{1}{2}\epsilon t})^2}{r^{2+\delta}}\ dx.$$

Applying a Hardy estimate with ∂_r in (t,x) coordinates to the RHS:

$$\int_{\{t=t'\}\cap\mathbb{R}^3} \frac{(t\phi\chi_{r\leqslant\frac{1}{2}\epsilon t})^2}{r^{2+\delta}}\, dx$$
$$\leqslant\ C_\delta \left(\int_{\{t=t'\}\cap\mathbb{R}^3} \frac{(\chi_{r\leqslant\frac{1}{2}\epsilon t} t\partial_r\phi)^2}{r^\delta}\, dx + \int_{\{t=t'\}\cap\mathbb{R}^3} \frac{\left(\chi_{r\sim\frac{1}{2}\epsilon t} t\phi\right)^2}{r^{2+\delta}}\, dx \right)\ , \quad (3.76)$$

where we compute the (non-sharp) constant to be $C_\delta = \frac{4}{1-2\delta}$. For the last term of the inequality above, since $r \sim \frac{1}{2}\epsilon t$:

$$\int_{r\sim\frac{1}{2}\epsilon t} \frac{(\chi_{r\sim\frac{1}{2}\epsilon t}\phi)^2}{r^{2+\delta}}\, dx \ \leqslant\ \epsilon^{-\delta}\cdot t^{-\delta} \int_{\{t=t'\}\cap\mathbb{R}^3} \frac{t^2\phi^2}{r^2}\, dx$$
$$\leqslant\ \epsilon^{-1}\cdot\langle t_0\rangle^{-\beta} \left(\sup_{t_0\leqslant t\leqslant t_1} CE[\phi(t)] \right) \quad (3.77)$$

For the remaining term:

$$t^2(\partial_r\phi)^2 \ =\ t^2 \left(\left| \frac{\partial_r({}^{II}\Omega\phi)}{{}^{II}\Omega} \right|^2 - 2\frac{\partial_r({}^{II}\Omega)\partial_r\phi\cdot\phi}{{}^{II}\Omega} + \left| \frac{\partial_r({}^{II}\Omega)\phi}{{}^{II}\Omega} \right|^2 \right)\ .$$

An application of Young's inequality yields:

$$t^2(\partial_r\phi)^2 \ \leqslant\ t^2 \left(\left| \frac{\partial_r({}^{II}\Omega\phi)}{{}^{II}\Omega} \right|^2 + \frac{1}{2}(\partial_r\phi)^2 + 3\left| \frac{\partial_r({}^{II}\Omega)\phi}{{}^{II}\Omega} \right|^2 \right)\ .$$

Since $u = t - \langle r\rangle$ in this region:

$$\left| \frac{\partial_r(\langle \underline{u}\rangle\langle u\rangle)}{\langle \underline{u}\rangle\langle u\rangle} \right| \ \leqslant\ \frac{2}{\langle t\rangle}\ , \qquad \forall\ (t,r)\ \ni\ \{r\leqslant \frac{1}{2}\epsilon t\}\ .$$

Moving the term $\frac{1}{2}(\partial_r\phi)^2$ to the LHS and applying the above inequality:

$$t^2(\partial_r\phi)^2 \ \leqslant\ 24\cdot t^2 \left(\left| \frac{\partial_r({}^{II}\Omega\phi)}{{}^{II}\Omega} \right|^2 + \frac{\phi^2}{\langle t\rangle^2} \right)\ . \quad (3.78)$$

Applying this to the first term of inequality (3.76):

$$\int_{\{t=t'\}\cap\mathbb{R}^3} \frac{(t\chi_{r\leqslant\frac{1}{2}\epsilon t}\partial_r\phi)^2}{r^\delta}\, dx$$
$$\leqslant\ 24\left(\int_{\{t=t'\}\cap\mathbb{R}^3} \left| \frac{\chi^2_{r\leqslant\epsilon t} t\partial_r({}^{II}\Omega\phi)}{{}^{II}\Omega} \right|^2\, dx + \epsilon\cdot \int_{\{t=t'\}\cap\mathbb{R}^3} \frac{(\chi_{r\leqslant\frac{1}{2}\epsilon t} t\phi)^2}{r^{2+\delta}}\, dx \right)\ ,$$

we then bootstrap the last term onto the LHS of (3.76) and use the fact that ∂_r is a bounded linear combination of $\widetilde{\partial}_r$ and ∂_u. For (3.34) using (3.24) and the fact that $u = t - \langle r\rangle$:

$$\sum_{\beta=t,r}\int_{\{t=t'\}\cap\mathbb{R}^3}\left|\frac{(\overline{K}_0^{\beta})^{II}V\chi_{r\leq\frac{1}{2}\epsilon t}\phi}{({}^{II}\Omega)^2}\right|^2\sqrt{|g|}dx \lesssim \int_{\{t=t'\}\cap\mathbb{R}^3}\frac{\chi_{r\leq\frac{1}{2}\epsilon t}t^2\phi^2}{\langle u\rangle r^{1+\delta}}\sqrt{|g|}dx$$
$$\lesssim \frac{1}{2}\epsilon\int_{\{t=t'\}\cap\mathbb{R}^3}\frac{\chi_{r\leq\frac{1}{2}\epsilon t}t^2\phi^2}{r^2}\sqrt{|g|}dx$$
$$\lesssim \epsilon\cdot\sup_{t_0\leqslant t\leqslant t_1} CE[\phi(t)]\ .$$

□

Proof of (3.36). Since we have already reduced to the case $t_0 \gg 1$ and we are working inside the set $|u| \leqslant 1$ we have $t \approx r$ there, and without loss of generality $r \gg 1$ as well. With $dx = r^2 dr d\omega$ we can compute, with ∂_r in (t, x) coordinates:

$$\begin{aligned} 0 &= \int_{\{t=t'\}\cap\mathbb{R}^3}\partial_r(u\langle r\rangle^2\phi^2\chi_{|u|\leqslant 2})drd\omega \\ &= \int_{\{t=t'\}\cap\mathbb{R}^3}(\partial_r u)\phi^2\chi_{|u|\leqslant 2}\langle r\rangle^2 drd\omega + \int_{\{t=t'\}\cap\mathbb{R}^3}2u\partial_r\phi\phi\chi_{|u|\leqslant 2}\langle r\rangle^2 drd\omega \\ &\quad + \int_{\{t=t'\}\cap\mathbb{R}^3}u2\langle r\rangle\frac{r}{\langle r\rangle}\phi^2\chi_{|u|\leqslant 2}\langle r\rangle^2 drd\omega + \int_{\{t=t'\}\cap\mathbb{R}^3}\phi^2\chi_{u\sim 2}\langle r\rangle^2 drd\omega \\ &= I + II + III + IV\ . \end{aligned}$$

Since $\partial_r(u) = -1 + O(r^{-\delta})$ in this region:

$$I = (-1 + O(r^{-\delta}))\int_{\{t=t'\}\cap\mathbb{R}^3}\phi^2\chi_{|u|\leqslant 2}\langle r\rangle^2\ drd\omega\ ,$$

Therefore we may move this term to the LHS above and absorb the $O(r^{-\delta})$ term. For II, by Young's inequality and $\partial_r = \widetilde{\partial}_r + u_r\partial_u$:

$$\begin{aligned} |II| &\lesssim \lambda^{-1}\int_{\{t=t'\}\cap\mathbb{R}^3}\chi^2_{|u|\leqslant 2}u^2(\partial_r\phi)^2 dx + \lambda\int_{\{t=t'\}\cap\mathbb{R}^3}\chi^2_{|u|\leqslant 2}\phi^2\ dx \\ &\lesssim \lambda^{-1}\int_{\{t=t'\}\cap\mathbb{R}^3}\chi^2_{|u|\leqslant 2}\frac{(u\partial_u(\langle r\rangle\phi))^2}{\langle r\rangle^2}dx + \lambda^{-1}\int_{\{t=t'\}\cap\mathbb{R}^3}\chi^2_{|u|\leqslant 2}\frac{(u\widetilde{\partial}_r(\langle r\rangle\phi))^2}{\langle r\rangle^2}dx \\ &\quad + \lambda^{-1}\int_{\{t=t'\}\cap\mathbb{R}^3}\chi^2_{|u|\leqslant 2}\frac{(u\phi)^2}{\langle r\rangle^2}dx + \lambda\int_{\{t=t'\}\cap\mathbb{R}^3}\chi^2_{|u|\leqslant 2}\phi^2\ dx\ , \end{aligned}$$

and we may bootstrap the last term above by choosing λ sufficiently small in this estimate. Similarly:

$$|III| \lesssim \lambda^{-1}\int_{\{t=t'\}\cap\mathbb{R}^3}\chi_{|u|\leqslant 2}^2\frac{(u\phi)^2}{\langle r\rangle^2}dx + \lambda\int_{\{|u|\leq 2\}\cap\{r_0\leqslant r\}}\chi_{|u|\leqslant 2}^2\phi^2\ dx\ ,$$

bootstrapping the small error term finishes the proof.

□

Proof of (3.35). Applying the Hardy estimate (3.76) with ∂_r in (t,x) coordinates with $\delta = 0$:

$$\begin{aligned}
&\int_{\{t=t'\}\cap\mathbb{R}^3}\frac{(t\phi\chi_{r\leqslant 2})^2}{r^2}\ dx \\
&\lesssim \int_{\{t=t'\}\cap\mathbb{R}^3}(\chi_{r\leqslant 2}u\partial_r\phi)^2\ dx + \int_{\{t=t'\}\cap\mathbb{R}^3}(\chi_{r\leqslant 2}\langle r\rangle\partial_r\phi)^2\ dx \\
&\quad + \int_{\{t=t'\}\cap\mathbb{R}^3}\frac{(\chi_{r\sim 2}t\phi)^2}{r^2}\ dx \\
&\lesssim \int_{\{t=t'\}\cap\mathbb{R}^3}(\chi_{r\leqslant 2}u\partial_r\phi)^2\ dx + E[\phi(t')] + \int_{\{t=t'\}\cap\mathbb{R}^3}\frac{(\chi_{r\sim 2}t\phi)^2}{r^2}\ dx\ .
\end{aligned}$$

Applying (3.78) to the first term:

$$\int_{\{t=t'\}\cap\mathbb{R}^3}(\chi_{r\leqslant 2}u\partial_r\phi)^2\ dx \lesssim \int_{\{t=t'\}\cap\mathbb{R}^3}\chi_{r\leqslant 2}u\left|\frac{\partial_r({}^{II}\Omega\phi)}{{}^{II}\Omega}\right|^2 + \chi_{r\leqslant 2}\phi^2\ dx\ .$$

A Hardy estimate for the l.o.t. finishes the proof.

□

3.6 Proof of the Remaining Lemmas

3.6.1 Proof of Lemma 3.5.1

The Laplace-Beltrami operator for the Minkowski metric $\widetilde{\eta}$ in Bondi coordinates takes the form:

$$\Box_{\widetilde{\eta}} = -2\widetilde{\partial}_r\partial_u + \widetilde{\partial}_r^2 + \frac{2}{r}(\widetilde{\partial}_r - \partial_u) + \widetilde{\nabla\!\!\!/}^2\ . \tag{3.79}$$

Proof of (3.52). Computing using (3.79):

$$\langle r\rangle\Box_{\widetilde{\eta}}(\frac{1}{\langle r\rangle}) \ = \ 3\langle r\rangle\left(\frac{r^2-\langle r\rangle^2}{\langle r\rangle^5}\right) \ = \ \frac{-3}{\langle r\rangle^4}$$

□

Proof of (3.53). Once again we use the linear combination $\overline{\partial} = 2\partial_u - \widetilde{\partial}_r$. Using this, the wave equation (3.79) is then equivalent to:

$$\Box_{\widetilde{\eta}} \ = \ -\overline{\partial}\widetilde{\partial}_r + \frac{2}{r}(\widetilde{\partial}_r - \partial_u) + \widetilde{\nabla\!\!\!/}^2 \ . \tag{3.80}$$

Using properties (3.48):

$$-\overline{\partial}\widetilde{\partial}_r\left(\frac{1}{\langle u\rangle\langle \underline{u}\rangle}\right) \ = \ \frac{2\underline{u}}{\langle \underline{u}\rangle^3}\overline{\partial}\left(\frac{1}{\langle \underline{u}\rangle}\right) \ = \ \frac{-4u\underline{u}}{\langle u\rangle^3\langle \underline{u}\rangle^3} \ = \ \frac{-8ur-4u^2}{\langle u\rangle^3\langle \underline{u}\rangle^3} \ . \tag{3.81}$$

As well as:

$$\widetilde{\partial}_r\left(\frac{1}{\langle u\rangle\langle \underline{u}\rangle}\right) \ = \ \frac{-2\underline{u}}{\langle u\rangle\langle \underline{u}\rangle^3} \ , \qquad \partial_u\left(\frac{1}{\langle u\rangle\langle \underline{u}\rangle}\right) \ = \ -\frac{u\langle \underline{u}\rangle^2+\underline{u}\langle u\rangle^2}{\langle u\rangle^3\langle \underline{u}\rangle^3} \ . \tag{3.82}$$

Combining the last two identities above:

$$\begin{aligned}
\frac{2}{r}(\partial_r - \partial_u)\left(\frac{1}{\langle u\rangle\langle \underline{u}\rangle}\right) \ &= \ \frac{2}{r}\left(\frac{u\langle \underline{u}\rangle^2-\underline{u}\langle u\rangle^2}{\langle u\rangle^3\langle \underline{u}\rangle^3}\right) \\
&= \ \frac{2}{r}\left(\frac{u(u+2r)^2-(u+2r)u^2}{\langle u\rangle^3\langle \underline{u}\rangle^3} + \frac{u-\underline{u}}{\langle u\rangle^3\langle \underline{u}\rangle^3}\right) \\
&= \ \frac{2}{r}\left(\frac{2ru^2+4r^2u}{\langle u\rangle^3\langle \underline{u}\rangle^3} + \frac{-2r}{\langle u\rangle^3\langle \underline{u}\rangle^3}\right) \\
&= \ \frac{8ur+4u^2}{\langle u\rangle^3\langle \underline{u}\rangle^3} + \frac{-4}{\langle u\rangle^3\langle \underline{u}\rangle^3} \ .
\end{aligned} \tag{3.83}$$

Combining this last identity with (3.81) gives us the result.

□

3.6.2 Proof of The Estimates For $(\widehat{^{(\widetilde{K}_0)}\pi})$ and $(\widehat{^{(\partial_t)}\pi})$

Proof of Lemma 3.5.2. We first prove the bounds inside the wave zone. For the modified Morawetz vector field we have, in Bondi coordinates, $\partial_\gamma \widetilde{K}_0^\gamma = 8(u+r)$.

Computing:

$$
\begin{aligned}
&\big({}^{(\widetilde{K}_0)}\widehat{\pi}^{uu} + 4(u+r)g^{uu}\big)\\
&\quad = \ (\sqrt{|g|})^{-1}\sqrt{|g|}\big({}^{(\widetilde{K}_0)}\widehat{\pi}^{uu} + 4(u+r)\sqrt{|g|}g^{uu}\big)\\
&\quad = \ (\sqrt{|g|})^{-1}\Big(-\widetilde{K}_0(g^{uu}\sqrt{|g|}) + 2\sqrt{|g|}g^{u\gamma}\partial_\gamma(u^2) - 4(u+r)\sqrt{|g|}g^{uu}\Big) \ = \ 0 \ .
\end{aligned}
$$

For the (u,i) components we recall that Lemma 2.1.10 lets us use identity (2.22) in the frame $\mathcal{B}$ – modulo an error term of the form $2(u+r)g^{ua}$ which is a faster decaying term thanks to (2.13). This observation together with (2.12) and (2.13):

$$
\begin{aligned}
&|\big({}^{(\widetilde{K}_0)}\widehat{\pi} + 4(u+r)g\big)^{ur}_{\mathcal{B}}|\\
&\quad = \ |(\sqrt{|g|})^{-1}\sqrt{|g|}\big({}^{(\widetilde{K}_0)}\widehat{\pi} + 4(u+r)g\big)^{ur}_{\mathcal{B}}|\\
&\quad = \ |(\sqrt{|g|})^{-1}\big(-\widetilde{K}_0(g^{ur}\sqrt{|g|}) + \sqrt{|g|}g^{u\gamma}\partial_\gamma(2(u+r)r) + \sqrt{|g|}g^{r\gamma}\partial_\gamma(u^2)\\
&\qquad\quad - 4(u+r)\sqrt{|g|}g^{ur}\big)|\\
&\quad = \ |(\sqrt{|g|})^{-1}\big(-\widetilde{K}_0(g^{ur}\sqrt{|g|}+1)\big)|\\
&\quad \lesssim \ \Big(\langle r\rangle^{\frac{1}{2}-\delta}\langle u\rangle^{\frac{1}{2}}\Big) \ ,\\
&|\big({}^{(\widetilde{K}_0)}\widehat{\pi} + 4(u+r)g\big)^{ua}_{\mathcal{B}}|\\
&\quad = \ |-(\sqrt{|g|})^{-1}\widetilde{K}_0(g^{ua}\sqrt{|g|}) + 2ug^{au} - 2(u+r)g^{ua}|\\
&\quad \lesssim \ \Big(\langle u\rangle\langle r\rangle^{-\delta}\Big) \ .
\end{aligned}
$$

For the (i,j) components we use the Bondi coordinate derivatives and (2.8) to get:

$$
\begin{aligned}
&|\big({}^{(\widetilde{K}_0)}\widehat{\pi}^{ij} + 4(u+r)g^{ij}\big)|\\
&\quad = \ |(\sqrt{|g|})^{-1}\sqrt{|g|}\big({}^{(\widetilde{K}_0)}\widehat{\pi}^{ij} + 4(u+r)g^{ij}\big)|\\
&\quad = \ |(\sqrt{|g|})^{-1}\big(-\widetilde{K}_0(g^{ij}\sqrt{|g|} + \delta^{ij}) + \sqrt{|g|}g^{i\gamma}\partial_\gamma\widetilde{K}_0^j + \sqrt{|g|}g^{j\gamma}\partial_\gamma\widetilde{K}_0^i\\
&\qquad - \ 4(u+r)\sqrt{|g|}g^{ij}\big)|\\
&\quad = \ |(\sqrt{|g|})^{-1}\big(-\widetilde{K}_0(g^{ij}\sqrt{|g|} + \delta^{ij}) + 2\sqrt{|g|}\big[g^{iu}x^j + ug^{il}\delta^j_l + g^{il}\omega_l x^j + rg^{il}\delta^j_l\\
&\qquad + g^{ju}x^i + ug^{jl}\delta^i_l + g^{jl}\omega_l x^i + rg^{jl}\delta^i_l\big] - 4(u+r)\sqrt{|g|}g^{ij}\big)|\\
&\quad = \ |(\sqrt{|g|})^{-1}\big(-\widetilde{K}_0(g^{ij}\sqrt{|g|} + \delta^{ij}) + 2\sqrt{|g|}\big[x^j(g^{iu} + g^{il}\omega_l) + x^i(g^{ju} + g^{jl}\omega_l)\big]\big)|\\
&\quad = \ |(\sqrt{|g|})^{-1}\big(-\widetilde{K}_0(g^{ij}\sqrt{|g|} + \delta^{ij}) + 2x^j\big[(\sqrt{|g|}g^{iu} + \omega^i) + (\sqrt{|g|}g^{il}\omega_l - \omega^i)\big]\\
&\qquad + 2x^i\big[(\sqrt{|g|}g^{ju} + \omega^j) + (\sqrt{|g|}g^{jl}\omega_l - \omega^j)\big]\big)|\\
&\quad \lesssim \ \langle r\rangle^{1-\delta} \ .
\end{aligned}
$$

Outside the wave zone the main change is that the (u, u) component no longer vanishes since u is not an optical function. However, by a similar computation, we may use the decay rates (1.3) in the interior and get the result.

The decay rates for $(\widehat{^{(\partial_t)}\pi})$ and derivatives applied to this quantity are immediate consequences of (2.12), (2.13) and (2.8) in the wave zone, and (1.3) outside the wave zone.

□

3.6.3 Proof of The Approximate Normal Lemma

For computations in Bondi coordinates it is convenient to replace the timelike vector field $-\nabla t = N$ by its approximation $-\nabla(u+r) = \widetilde{N}$. However, Stokes' theorem demands that the vector in the energy densities be the normal to the hypersurfaces $t = t_0$. Looking at the difference of the two and using the decay dates for u we expect for the geometric gradient ∇:

$$|\nabla(t - (u+r))| \lesssim \langle r\rangle^{-\delta}\partial \ ,$$

therefore, a priori we cannot rule out the appearance of a bad combination of weights and derivatives when replacing N with $\widetilde{N}$. Lemma (3.5.3) shows, however, that such bad combinations do not occur when we do this replacement in the energy density associated with $\overline{K_0}$.

Proof of Lemma 3.5.3. Since $u_r = -1 + O(r^{-\delta})$, equation (2.2) gives:

$$\begin{aligned} \nabla t &= g^{\alpha\beta}\partial_\alpha t \partial_\beta \\ &= (u_t)^{-1} g^{u\beta}\partial_\beta - \frac{u_r}{u_t} g^{r\beta}\partial_\beta - \sum_a \frac{u_a}{u_t} g^{a\beta}\partial_\beta \\ &= (u_t)^{-1}(g^{u\beta}\partial_\beta + g^{r\beta}\partial_\beta) - \frac{(u_r+1)}{u_t} g^{r\beta}\partial_\beta \\ &= (u_t)^{-1}\nabla(u+r) - \frac{O(r^{-\delta})}{u_t} g^{r\beta}\partial_\beta \\ &= (1 + O(r^{-\delta}))\nabla(u+r) - \frac{O(r^{-\delta})}{u_t}(Error) \ . \end{aligned} \tag{3.84}$$

Contracting the error term with $T_{\alpha\beta}[\phi](\widetilde{K}_0^\beta + \partial_t)$:

$$
\begin{aligned}
& g^{r\beta}T_{\alpha\beta}[\phi](\overline{K}_0^\beta) \\
&= g^{r\alpha}\partial_\alpha\phi\widetilde{K}_0^\beta\partial_\beta\phi - \frac{1}{2}\widetilde{K}_0^r|\nabla\phi|_g^2 + g^{r\alpha}\partial_\alpha\phi\partial_u\phi \\
&= g^{ru}\partial_u\phi\widetilde{K}_0^r\partial_r\phi + g^{ru}(\partial_u\phi)^2\widetilde{K}_0^u + \widetilde{K}_0^u g^{rr}\partial_u\phi\partial_r\phi + \frac{1}{2}\widetilde{K}_0^r g^{rr}(\widetilde{\partial}_r\phi)^2 \\
&\quad - \frac{1}{2}\widetilde{K}_0^r[2(g^{au}\partial_u\phi e_a\phi + g^{ur}\partial_u\phi\widetilde{\partial}_r\phi + g^{ra}\widetilde{\partial}_r\phi e_a\phi) + g^{ab}e_a\phi e_b\phi] + g^{r\alpha}\partial_\alpha\phi\partial_u\phi \ .
\end{aligned}
$$

Inside the wave zone $\mathcal{W}$, we cancel the $\widetilde{K}_0^r\partial_u\phi\widetilde{\partial}_r\phi$ terms, use $g^{uu} = 0$ as well as the decay rates for g^{ua}, and apply Young's inequality to get:

$$
|g^{r\beta}T_{\alpha\beta}[\phi](\overline{K}_0^\beta)| \ \lesssim \ \langle \underline{u}\rangle^2\left(\left|\widetilde{\partial}_r\phi\right|^2 + |\widetilde{\nabla}\phi|^2\right) + \langle u\rangle^2\left|\partial_t\phi\right|^2 \ .
$$

Outside the wave zone it is clear by Young's inequality that the error satisfies the same estimate. Using Young's inequality and identity (3.64) we also get:

$$
|\widetilde{N}^\alpha({}^{(\overline{K}_0)}P_\alpha[\phi])| \ \lesssim \ \langle \underline{u}\rangle^2\left(\left|\widetilde{\partial}_r\phi\right|^2 + |\widetilde{\nabla}\phi|^2\right) + \langle u\rangle^2\left|\partial_t\phi\right|^2
$$

Combining all this with (3.84):

$$
\begin{aligned}
|N^\alpha({}^{(\overline{K}_0)}P_\alpha[\phi]) - \widetilde{N}^\alpha({}^{(\overline{K}_0)}P_\alpha[\phi])| \ &\lesssim \ \langle r\rangle^{-\delta}\widetilde{N}^\alpha({}^{(\overline{K}_0)}P_\alpha[\phi]) + |g^{r\beta}T_{\alpha\beta}[\phi](\overline{K}_0^\beta)| \\
&\lesssim \ \langle r\rangle^{-\delta}\left[\langle \underline{u}\rangle^2\left(\left|\widetilde{\partial}_r\phi\right|^2 + |\widetilde{\nabla}\phi|^2\right) + \langle u\rangle^2\left|\partial_t\phi\right|^2\right] \ .
\end{aligned}
$$

$\square$

Chapter 4

Commutators

The main goal in this section is to establish:

Theorem 4.0.1 (Higher Order Conformal Energy Estimate)**.** *Assume the conformal energy estimate* (1.12) *and the hypotheses of the Main Theorem. The function ϕ then satisfies the conformal energy estimate with vector fields* (1.13)*:*

$$\sup_{t_0\leqslant t\leqslant t_1} CE_k^{\frac{1}{2}}[\phi(t)] \ + \ \sup_{\mathcal{W}\cap[t_0,t_1]} \big(CE_{ch,k}^{\frac{1}{2}}[\phi(u)]\big) + \|\,\phi\,\|_{LSM_{int,k}^{1-\gamma'}[t_0,t_1]}$$
$$\lesssim \ CE_k^{\frac{1}{2}}[\phi(t_0)] + \|\,F\,\|_{LS_k^{*,1+\gamma',\frac{1}{2}}[t_0,t_1]} + \|\,F\,\|_{LS_{k-1}^{2-\gamma',\frac{1}{2}}[t_0,t_1]} \ .$$

4.1 Proof of The Higher Order Conformal Energy Estimate

In order to prove (1.13) we must start by showing the simplest case of $\Gamma = \partial_t$ and improve the estimates one step at a time by adding more vector fields and derivatives. This is because each successive estimate has error terms that must be controlled by the previous round of estimates. In order to simplify the proof below we set up the following notation:

Definition 4.1.1. *Define the following auxiliary norms:*

$$\mathring{CEH}_k^{\frac{1}{2}}[\phi]_{[t_0,t_1]} \ := \ \sup_{t_0\leqslant t\leqslant t_1} \mathring{CE}_k^{\frac{1}{2}}[\phi(t)] \ + \ \sup_{\mathcal{W}} \big(\mathring{CE}_{ch,k}^{\frac{1}{2}}[\phi(u)]\big) + \|\,\phi\,\|_{\mathring{LSM}_{int,k}^{1-\gamma'}[t_0,t_1]}$$
$$CEH_k^{\frac{1}{2}}[\phi]_{[t_0,t_1]} \ := \ \sup_{t_0\leqslant t\leqslant t_1} CE_k^{\frac{1}{2}}[\phi(t)] \ + \ \sup_{\mathcal{W}} \big(CE_{ch,k}^{\frac{1}{2}}[\phi(u)]\big) + \|\,\phi\,\|_{LSM_{int,k}^{1-\gamma'}[t_0,t_1]}$$

Theorem 4.0.1 is a consequence of the following:

Lemma 4.1.2 (Commutator Estimates)**.** *For all* $t_0, t_1 \in [t^*, \infty)$ *and* $k \in \mathbb{Z}^+$ *the following estimates hold:*

$$\begin{aligned}
&\| [\Box_g, \partial_t^k]\phi \|_{LS^{*,1+\gamma',\frac{1}{2}}[t_0,t_1]} \\
&\quad \lesssim \ \epsilon \cdot \sum_{j\leqslant k} \| \partial_t^j \phi \|_{LSM^{1-\gamma'}_{int}[t_0,t_1]} + \mathring{CE}_k^{\frac{1}{2}}[\phi(t_0)] + \| F \|_{\mathring{LS}_k^{*,1+\gamma',\frac{1}{2}}[t_0,t_1]} \\
&\quad\quad + \ \epsilon \cdot \mathring{CEH}_k^{\frac{1}{2}}[\phi]_{[t_0,t_1]} \ , \qquad (4.1) \\
&\| [\Box_g, \partial_{t,x}^k]\phi \|_{LS^{*,1+\gamma',\frac{1}{2}}[t_0,t_1]} \\
&\quad \lesssim \ \sum_{j\leqslant k} \| \partial_t^j \phi \|_{LSM^{1-\gamma'}_{int}[t_0,t_1]} + \mathring{CE}_k^{\frac{1}{2}}[\phi(t_0)] + \| F \|_{\mathring{LS}_k^{*,1+\gamma',\frac{1}{2}}[t_0,t_1]} \\
&\quad\quad + \ \epsilon \cdot \mathring{CEH}_k^{\frac{1}{2}}[\phi]_{[t_0,t_1]} \ , \qquad (4.2) \\
&\| [\Box_g, \widetilde{S}]\phi \|_{LS^{*,1+\gamma',\frac{1}{2}}[t_0,t_1]} + \| [\Box_g, \widetilde{\Omega}]\phi \|_{LS^{*,1+\gamma',\frac{1}{2}}[t_0,t_1]} \\
&\quad \lesssim \ \epsilon \cdot \| \phi \|_{LSM^{1-\gamma'}_{int,1}[t_0,t_1]} + CE_1^{\frac{1}{2}}[\phi(t_0)] + \| F \|_{LS_1^{*,1+\gamma',\frac{1}{2}}[t_0,t_1]} + \| F \|_{LS^{2-\gamma',\frac{1}{2}}[t_0,t_1]} \\
&\quad\quad + \epsilon \cdot CEH_1^{\frac{1}{2}}[\phi]_{[t_0,t_1]} \ , \qquad (4.3) \\
&\| [\Box_g, \partial_{t,x}^{k-1}\widetilde{S}]\phi, \|_{LS^{*,1+\gamma',\frac{1}{2}}[t_0,t_1]} + \| \Box_g, \partial_{t,x}^{k-1}\widetilde{\Omega}]\phi \|_{LS^{*,1+\gamma',\frac{1}{2}}[t_0,t_1]} \\
&\quad \lesssim \ \epsilon \cdot \| \phi \|_{LSM^{1-\gamma'}_{int,k}[t_0,t_1]} + CE_k^{\frac{1}{2}}[\phi(t_0)] + \| F \|_{LS_k^{*,1+\gamma',\frac{1}{2}}[t_0,t_1]} + \| F \|_{LS_{k-1}^{2-\gamma',\frac{1}{2}}[t_0,t_1]} \\
&\quad\quad + \ \epsilon \cdot CEH_k^{\frac{1}{2}}[\phi]_{[t_0,t_1]} \ . \qquad (4.4)
\end{aligned}$$

Proof of Theorem 4.0.1. By the reduction in section 2.3.1 it suffices to prove the result for $t_1, t_0 > t^*$. We start by commuting the equation with ∂_t. Applying the conformal energy estimate (1.12) to $\partial_t\phi$ and using (4.1) for the commutator:

$$\begin{aligned}
&\sup_{t_0\leqslant t\leqslant t_1} CE^{\frac{1}{2}}[\partial_t\phi(t)] \ + \ \sup_{\mathcal{W}} \big(CE_{ch}^{\frac{1}{2}}[\partial_t\phi(u)]\big) + \| \partial_t\phi \|_{LSM^{1-\gamma'}_{int}[t_0,t_1]} \\
&\quad \lesssim \ \epsilon \cdot \sum_{j\leqslant 1} \| \partial_t^j \phi \|_{LSM^{1-\gamma'}_{int}[t_0,t_1]} + \mathring{CE}_1^{\frac{1}{2}}[\phi(t_0)] + \| F \|_{\mathring{LS}_1^{*,1+\gamma',\frac{1}{2}}[t_0,t_1]} \\
&\quad\quad + \ \epsilon \cdot \mathring{CEH}_1^{\frac{1}{2}}[\phi]_{[t_0,t_1]} \ .
\end{aligned}$$

Adding the conformal energy estimate (1.12) to this gives us a LHS that can absorb the first term above. Thus:

$$\sum_{j\leqslant 1}\Big(\sup_{t_0\leqslant t\leqslant t_1} CE^{\frac{1}{2}}[\partial_t^j\phi(t)] \ + \ \sup_{\mathcal{W}}\big(CE^{\frac{1}{2}}_{ch}[\partial_t^j\phi(u)]\big) + \|\,\partial_t^j\phi\,\|_{LSM^{1-\gamma'}_{int}[t_0,t_1]}\Big)$$
$$\lesssim \ \mathring{CE}^{\frac{1}{2}}_1[\phi(t_0)] + \|\,F\,\|_{\mathring{LS}^{*,1+\gamma',\frac{1}{2}}_1[t_0,t_1]} + \epsilon\cdot \mathring{CEH}^{\frac{1}{2}}_1[\phi]_{[t_0,t_1]} \ . \tag{4.5}$$

Now we repeat the proof for spatial derivatives. Applying the conformal energy estimate (1.12) to $\partial_x\phi$, followed by estimates (4.2) and (4.5) in succession:

$$\sup_{t_0\leqslant t\leqslant t_1} CE^{\frac{1}{2}}[\nabla_x\phi(t)] \ + \ \sup_{\mathcal{W}}\big(CE^{\frac{1}{2}}_{ch}[\nabla_x\phi(u)]\big) + \|\,\nabla_x\phi\,\|_{LSM^{1-\gamma'}_{int}[t_0,t_1]}$$
$$\lesssim \ \sum_{j\leqslant k}\|\,\partial_t^j\phi\,\|_{LSM^{1-\gamma'}_{int}[t_0,t_1]} + \mathring{CE}^{\frac{1}{2}}_1[\phi(t_0)] + \|\,F\,\|_{\mathring{LS}^{*,1+\gamma',\frac{1}{2}}_1[t_0,t_1]}$$
$$+ \ \epsilon\cdot \mathring{CEH}^{\frac{1}{2}}_k[\phi]_{[t_0,t_1]}$$
$$\lesssim \ \mathring{CE}^{\frac{1}{2}}_1[\phi(t_0)] + \|\,F\,\|_{\mathring{LS}^{*,1+\gamma',\frac{1}{2}}_1[t_0,t_1]} + \epsilon\cdot \mathring{CEH}^{\frac{1}{2}}_1[\phi]_{[t_0,t_1]} \tag{4.6}$$

Adding (4.5) and (4.6) and bootstrapping the last terms gives the estimate for one derivative:

$$\mathring{CEH}^{\frac{1}{2}}_1[\phi]_{[t_0,t_1]} \ \lesssim \ \mathring{CE}^{\frac{1}{2}}_1[\phi(t_0)] + \|\,F\,\|_{\mathring{LS}^{*,1+\gamma',\frac{1}{2}}_1[t_0,t_1]} \ . \tag{4.7}$$

To get the estimate for any number of time derivatives we set up an induction: assume the estimate holds for all $\partial_t^j\phi$, $1\leqslant j\leqslant k-1$. Applying the conformal energy estimate (1.12) to $\partial_t^k\phi$ followed by (4.1) to control the error:

$$\sup_{t_0\leqslant t\leqslant t_1} CE^{\frac{1}{2}}[\partial_t^k\phi(t)] \ + \ \sup_{\mathcal{W}}\big(CE^{\frac{1}{2}}_{ch}[\partial_t^k\phi(u)]\big) + \|\,\partial_t^k\phi\,\|_{LSM^{1-\gamma'}_{int}[t_0,t_1]}$$
$$\lesssim \ \epsilon\cdot\sum_{j\leqslant k}\|\,\partial_t^j\phi\,\|_{LSM^{1-\gamma'}_{int}[t_0,t_1]} + \epsilon\cdot \mathring{CEH}^{\frac{1}{2}}_k[\phi]_{[t_0,t_1]}$$
$$+ \ \mathring{CE}^{\frac{1}{2}}_k[\phi(t_0)] + \|\,F\,\|_{\mathring{LS}^{*,1+\gamma',\frac{1}{2}}_k[t_0,t_1]} \ .$$

Adding the estimates from the induction hypothesis and bootstrapping the first error term above we have established:

$$\sum_{j\leqslant k}\Big(\sup_{t_0\leqslant t\leqslant t_1} CE^{\frac{1}{2}}[\partial_t^j\phi(t)] \ + \ \sup_{\mathcal{W}}\big(CE^{\frac{1}{2}}_{ch}[\partial_t^j\phi(u)]\big) + \|\,\partial_t^j\phi\,\|_{LSM^{1-\gamma'}_{int}[t_0,t_1]}\Big)$$
$$\lesssim \ \epsilon\cdot \mathring{CEH}^{\frac{1}{2}}_k[\phi]_{[t_0,t_1]} + \mathring{CE}^{\frac{1}{2}}_k[\phi(t_0)] + \|\,F\,\|_{\mathring{LS}^{*,1+\gamma',\frac{1}{2}}_k[t_0,t_1]} \ . \tag{4.8}$$

Next we claim the estimate (4.8) also holds for $\nabla^k\phi$ with $\nabla^k = \partial_t^j \nabla_x^L$. To prove this we set up an induction on $|L|$. The case $|L| = 0$ is already done by (4.8). Next, using our induction hypothesis we assume the following estimates hold:

$$\sum_{j+|L|\leqslant k} \Big(\sup_{t_0\leqslant t\leqslant t_1} CE^{\frac{1}{2}}[\partial_t^j \nabla_x^L \phi(t)] \ + \ \sup_{\mathcal{W}} \big(CE_{ch}^{\frac{1}{2}}[\partial_t^j \nabla_x^L \phi(u)]\big) + \| \, \partial_t^j \nabla_x^L \phi \, \|_{LSM_{int}^{1-\gamma'}[t_0,t_1]} \Big)$$
$$\lesssim \ \epsilon \cdot \mathring{CEH}_k^{\frac{1}{2}}[\phi]_{[t_0,t_1]} + \mathring{CE}_k^{\frac{1}{2}}[\phi(t_0)] + \| \, F \, \|_{\mathring{LS}_k^{*,1+\gamma',\frac{1}{2}}[t_0,t_1]} \ , \tag{4.9}$$

for all $|L| \leqslant k-1$. Commute ∂_x^k with the equation and observe that estimate (4.2) lets us trade ∂_x for ∂_t derivatives in the $LSM_{int}^{1-\gamma'}[t_0,t_1]$ error terms:

$$\sup_{t_0\leqslant t\leqslant t_1} CE^{\frac{1}{2}}[\nabla^k \phi(t)] \ + \ \sup_{\mathcal{W}} \big(CE_{ch}^{\frac{1}{2}}[\nabla^k \phi(u)]\big) + \| \, \nabla^k \phi \, \|_{LSM_{int}^{1-\gamma'}[t_0,t_1]}$$
$$\lesssim \ \sum_{j\leqslant k} \| \, \partial_t^j \phi \, \|_{LSM_{int}^{1-\gamma'}[t_0,t_1]} + \epsilon \cdot \mathring{CEH}_k^{\frac{1}{2}}[\phi]_{[t_0,t_1]} + \mathring{CE}_k^{\frac{1}{2}}[\phi(t_0)] + \| \, F \, \|_{\mathring{LS}_k^{*,1+\gamma',\frac{1}{2}}[t_0,t_1]}$$
$$\lesssim \ \epsilon \cdot \mathring{CEH}_k^{\frac{1}{2}}[\phi]_{[t_0,t_1]} + \mathring{CE}_k^{\frac{1}{2}}[\phi(t_0)] + \| \, F \, \|_{\mathring{LS}_k^{*,1+\gamma',\frac{1}{2}}[t_0,t_1]} \ , \tag{4.10}$$

where we have used the estimate (4.8) on the last line. Adding the estimates (4.8)-(4.10) and bootstrapping:

$$\mathring{CEH}_k^{\frac{1}{2}}[\phi]_{[t_0,t_1]} \ \lesssim \ \mathring{CE}_k^{\frac{1}{2}}[\phi(t_0)] + \| \, F \, \|_{\mathring{LS}_k^{*,1+\gamma',\frac{1}{2}}[t_0,t_1]} \ . \tag{4.11}$$

Next we do the scaling and the rotations. Commuting with $\widetilde{S}$ and $\widetilde{\Omega}$ and using estimate (4.3):

$$\sup_{t_0\leqslant t\leqslant t_1} CE^{\frac{1}{2}}[(\widetilde{S}\phi(t), \widetilde{\Omega}\phi(t))] \ + \ \sup_{\mathcal{W}} \big(CE_{ch}^{\frac{1}{2}}[\widetilde{S}\phi(u), \widetilde{\Omega}\phi(u)]\big)$$
$$+ \ \| \, (\widetilde{S}\phi, \widetilde{\Omega}\phi) \, \|_{LSM_{int}^{1-\gamma'}[t_0,t_1]}$$
$$\lesssim \ \epsilon \cdot \| \, \phi \, \|_{LSM_{int,1}^{1-\gamma'}[t_0,t_1]} + \epsilon \cdot CEH_1^{\frac{1}{2}}[\phi]_{[t_0,t_1]}$$
$$+ \ CE_1^{\frac{1}{2}}[\phi(t_0)] + \| \, F \, \|_{LS_1^{*,1+\gamma',\frac{1}{2}}[t_0,t_1]} + \| \, F \, \|_{LS^{2-\gamma',\frac{1}{2}}[t_0,t_1]}$$

Adding estimate (4.7) and the conformal energy estimate (1.12) to this and bootstrapping:

$$CEH_1^{\frac{1}{2}}[\phi]_{[t_0,t_1]} \ \lesssim \ CE_1^{\frac{1}{2}}[\phi(t_0)] + \| \, F \, \|_{LS_1^{*,1+\gamma',\frac{1}{2}}[t_0,t_1]} + \| \, F \, \|_{LS^{2-\gamma',\frac{1}{2}}[t_0,t_1]} \ . \tag{4.12}$$

To get the estimate for $\nabla^{k-1}\Gamma\phi$ we induct on the number of vector fields. The $k = 1$ case is just estimate (4.12). Assume we have the estimates for all $\nabla^j\Gamma\phi$, with $1 \leqslant j \leqslant k-2$. Apply the conformal energy estimate (1.12) to $\nabla^{k-1}\Gamma\phi$ followed by (4.4) for the commutator:

$$\begin{aligned} \sup_{t_0\leqslant t\leqslant t_1} CE^{\frac{1}{2}}[\nabla^{k-1}\Gamma\phi(t)] \ + \ \sup_{\mathcal{W}} \big(CE^{\frac{1}{2}}_{ch}[\nabla^{k-1}\Gamma\phi(u)]\big) + \|\, \nabla^{k-1}\Gamma\phi \,\|_{LSM^{1-\gamma'}_{int}[t_0,t_1]} \\ \lesssim \ \epsilon \cdot \|\, \phi \,\|_{LSM^{1-\gamma'}_{int.k}[t_0,t_1]} + \epsilon \cdot CEH^{\frac{1}{2}}_k[\phi]_{[t_0,t_1]} \\ + \ CE^{\frac{1}{2}}_k[\phi(t_0)] + \|\, F \,\|_{LS^{*,1+\gamma',\frac{1}{2}}_k[t_0,t_1]} + \|\, F \,\|_{LS^{2-\gamma',\frac{1}{2}}_{k-1}[t_0,t_1]} \ . \end{aligned} \tag{4.13}$$

Adding the estimates we get from the induction hypothesis as well as (1.12) to this gives us:

$$\begin{aligned} CEH^{\frac{1}{2}}_k[\phi]_{[t_0,t_1]} \ \lesssim \ \epsilon \cdot CEH^{\frac{1}{2}}_k[\phi]_{[t_0,t_1]} + \ \epsilon \cdot \|\, \phi \,\|_{LSM^{1-\gamma'}_{int,k}[t_0,t_1]} \\ + \ CE^{\frac{1}{2}}_k[\phi(t_0)] + \|\, F \,\|_{LS^{*,1+\gamma',\frac{1}{2}}_k[t_0,t_1]} + \|\, F \,\|_{LS^{2-\gamma',\frac{1}{2}}_{k-1}[t_0,t_1]} \ . \end{aligned}$$

Now the LHS can absorb the bootstrap terms. Therefore:

$$CEH^{\frac{1}{2}}_k[\phi]_{[t_0,t_1]} \ \lesssim \ CE^{\frac{1}{2}}_k[\phi(t_0)] + \|\, F \,\|_{LS^{*,1+\gamma',\frac{1}{2}}_k[t_0,t_1]} + \|\, F \,\|_{LS^{2-\gamma',\frac{1}{2}}_{k-1}[t_0,t_1]} \ . \tag{4.14}$$

$\square$

4.2 Proof of the Commutator Estimates

The proof of the commutator estimates (4.1)–(4.4) relies on the following support lemmas. We will state them here and prove them in the next section below.

Lemma 4.2.1. *Let X be a smooth vector field and ${}^{(X)}\widehat{\pi}_{\alpha\beta}$ be its conformal deformation tensor. The following commutator formula holds in local coordinates:*

$$[\Box_g, X]\phi \ = \ \frac{1}{\sqrt{|g|}}\partial_\alpha(\sqrt{|g|}\,{}^{(X)}\widehat{\pi}^{\alpha\beta}\partial_\beta\phi) + \frac{1}{\sqrt{|g|}}\partial_\alpha(\sqrt{|g|}X^\alpha)F \ . \tag{4.15}$$

Lemma 4.2.2. *In the wave zone* $(\widehat{^{(\widetilde{S})}\pi})$ *and* $(\widehat{^{(\widetilde{\Omega})}\pi})$*satisfy:*

$$\begin{aligned}
|\partial_u^\alpha \widetilde{\partial}_l^\beta \big((\widehat{^{(\widetilde{S})}\pi})^{uu} + 2g^{uu}\big)| &= 0 \ , \\
|\partial_u^\alpha \widetilde{\partial}_l^\beta \big(\widehat{^{(\widetilde{S})}\pi} + 2g\big)^{ur}_{\mathcal{B}}| &\lesssim \langle r\rangle^{-\delta-|\beta|}\langle u\rangle^{-|\alpha|}\left(\frac{\langle u\rangle}{\langle r\rangle}\right)^{\frac{1}{2}} \ , \\
|\partial_u^\alpha \widetilde{\partial}_l^\beta \big(\widehat{^{(\widetilde{S})}\pi} + 2g\big)^{ua}_{\mathcal{B}}| &\lesssim \langle r\rangle^{-\delta-|\beta|}\langle u\rangle^{-|\alpha|}\left(\frac{\langle u\rangle}{\langle r\rangle}\right) \ , \\
|\partial_u^\alpha \widetilde{\partial}_l^\beta \big((\widehat{^{(S)}\pi})^{ij} + 2g^{ij}\big)| &\lesssim \langle r\rangle^{-\delta-|\beta|}\langle u\rangle^{-|\alpha|} \ , \\
|\partial_u^\alpha \widetilde{\partial}_l^\beta \big((\widehat{^{(\widetilde{\Omega})}\pi})^{uu}\big)| &= 0 \ , \\
|\partial_u^\alpha \widetilde{\partial}_l^\beta \big((\widehat{^{(\widetilde{\Omega})}\pi})^{ur}\big)| &\lesssim \langle r\rangle^{-\delta-|\beta|}\langle u\rangle^{-|\alpha|}\left(\frac{\langle u\rangle}{\langle r\rangle}\right)^{\frac{1}{2}} \ , \\
|\partial_u^\alpha \widetilde{\partial}_l^\beta \big((\widehat{^{(\widetilde{\Omega})}\pi})^{ua}\big)| &\lesssim \langle r\rangle^{-\delta-|\beta|}\langle u\rangle^{-|\alpha|}\left(\frac{\langle u\rangle}{\langle r\rangle}\right) \ , \\
|\partial_u^\alpha \widetilde{\partial}_l^\beta \big((\widehat{^{(\widetilde{\Omega})}\pi})^{ij}\big)| &\lesssim \langle r\rangle^{-\delta-|\beta|}\langle u\rangle^{-|\alpha|} \ .
\end{aligned}$$

Outside the wave zone:

$$\begin{aligned}
|\partial_t^k \partial_x^J \big((\widehat{^{(\widetilde{S})}\pi})^{\alpha\beta} + 2g^{\alpha\beta}\big)| &\lesssim t\langle r\rangle^{-\delta-|J|-k-1}\langle r/t\rangle^{\gamma k} \ , \\
|\partial_t^k \partial_x^J \big((\widehat{^{(\widetilde{\Omega})}\pi})^{\alpha\beta}\big)| &\lesssim t\langle r\rangle^{-\delta-|J|-k-1}\langle r/t\rangle^{\gamma k} \ .
\end{aligned}$$

Lemma 4.2.3 (Weighted Elliptic Estimates for Two Derivatives)**.**

$$\begin{aligned}
&\sum_{|\alpha|=2} \| \chi_{r\leqslant \epsilon t}\nabla^\alpha \phi \|_{LS^{2-\gamma'}[t_0,t_1]} \\
&\qquad \lesssim \ \| \phi \|_{LSM^{1-\gamma'}_{int,1}[t_0,t_1]} + \| F \|_{LS^{2-\gamma',\frac{1}{2}}[t_0,t_1]} + \epsilon \cdot CEH^{\frac{1}{2}}_1[\phi]_{[t_0,t_1]} \qquad (4.16) \\
&\sum_{|\alpha|=2} \| \chi_{r\leqslant \epsilon t}\nabla^\alpha \phi \|_{LS^{1-\gamma'}[t_0,t_1]} \\
&\qquad \lesssim \ \sum_{j\leqslant 1} \| \partial_t^j \phi \|_{LSM^{1-\gamma'}_{int}[t_0,t_1]} + \| F \|_{\mathring{LS}^{*,1+\gamma',\frac{1}{2}}_1[t_0,t_1]} + \epsilon \cdot C\mathring{E}H^{\frac{1}{2}}_1[\phi]_{[t_0,t_1]} \ . \qquad (4.17)
\end{aligned}$$

Lemma 4.2.4. *In the wave zone:*

$$\begin{aligned}
&\int_{\{t=t'\}\cap\mathbb{R}^3\cap\mathcal{W}} |u^2\partial_u^2\phi|^2 + |r^2\widetilde{\partial}_r^2\phi|^2 + |ur\partial_u\widetilde{\partial}_r\phi|^2 \ \sqrt{|g|}dx \\
&\qquad + \int_{C(u_0)\cap\mathcal{W}} |u^2\partial_u^2\phi|^2 + |r^2\widetilde{\partial}_r^2\phi|^2 + |ur\partial_u\widetilde{\partial}_r\phi|^2 \ dV_{C(u_0)} \\
&\lesssim \ CE_1[\phi(t')] + CE_{ch,1}[\phi(u_0)] + \int_{\{t=t'\}\cap\mathbb{R}^3\cap\mathcal{W}} |\langle u\rangle\langle \underline{u}\rangle F|^2 \ \sqrt{|g|}dx \\
&\qquad + \int_{C(u_0)\cap\mathcal{W}} |\langle u\rangle\langle \underline{u}\rangle F|^2 \ dV_{C(u_0)} \ .
\end{aligned} \tag{4.18}$$

In addition to the lemmas above, we will need the following notation in the proof of the commutator estimates:

Definition 4.2.5. *Take* $\chi_{int} + \chi_{wave} + \chi_{rest}$ *to be a smooth partition of unity with:*

$$\chi_{int} \ := \ \chi_{r\leqslant \epsilon t} \ , \qquad \chi_{wave} \ := \ \chi_{\mathcal{W}} \ , \qquad \chi_{rest} \ := \ \chi_{\{t\leqslant \frac{2r}{\epsilon}\}\cap\{\frac{5}{6}t<r<\frac{6}{5}t\}} \ . \tag{4.19}$$

4.2.1 Proof of Commutator Estimate (4.3)

Since the proof for the rotations $\widetilde{\Omega}$ follows from the exact same argument as the scaling we will only deal with $\widetilde{S}$. By (4.19), the result will follow from:

$$\|\,\chi_{int}[\Box_g,\widetilde{S}]\phi\,\|_{LS^{*,1+\gamma'}} \ \lesssim \ \|\,F\,\|_{LS^{2-\gamma',\frac{1}{2}}[t_0,t_1]} + \epsilon\cdot\|\,\phi\,\|_{LS_1^{1-\gamma'}[t_0,t_1]} + \epsilon\cdot CEH_1^{\frac{1}{2}}[\phi]_{[t_0,t_1]} \ , \tag{4.20}$$

$$\|\,\chi_{wave}[\Box_g,\widetilde{S}]\phi\,\|_{LS^{*,1+\gamma',\frac{1}{2}}} \ \lesssim \ \|\,F\,\|_{LS^{2-\gamma',\frac{1}{2}}[t_0,t_1]} + \epsilon\cdot CEH_1^{\frac{1}{2}}[\phi]_{[t_0,t_1]} \ , \tag{4.21}$$

$$\|\,\chi_{rest}[\Box_g,\widetilde{S}]\phi\,\|_{LS^{*,1+\gamma',\frac{1}{2}}} \ \lesssim \ \|\,F\,\|_{LS^{2-\gamma',\frac{1}{2}}[t_0,t_1]} + \epsilon\cdot CEH_1^{\frac{1}{2}}[\phi]_{[t_0,t_1]} \ . \tag{4.22}$$

Using (4.15) with $\Gamma = \widetilde{S}$ and $(\partial_\gamma\widetilde{S}^\gamma) = 4$:

$$\begin{aligned}
[\Box_g,\widetilde{S}]\phi \ &= \ \frac{1}{\sqrt{|g|}}\partial_\alpha(\sqrt{|g|}{}^{(\widetilde{S})}\widehat{\pi}^{\alpha\beta}\partial_\beta\phi) + \frac{1}{\sqrt{|g|}}\partial_\alpha(\sqrt{|g|}\widetilde{S}^\alpha)F \\
&= \frac{1}{\sqrt{|g|}}\partial_\alpha\left(\sqrt{|g|}({}^{(\widetilde{S})}\widehat{\pi}^{\alpha\beta} + 2g^{\alpha\beta})\partial_\beta\phi\right) + (\frac{1}{\sqrt{|g|}}\widetilde{S}(\sqrt{|g|}) + 2)F \ .
\end{aligned}$$

Proof of (4.20). Using the condition $r \leqslant \epsilon t$, the decay rates in Lemma 4.2.2 together with estimate (2.7) to bound ($\sqrt{|g|}$):

$$
\begin{aligned}
&|\langle x\rangle^{\frac{1}{2}+\epsilon}t^{1+\gamma'}\chi_{r\leqslant\epsilon t}({}^{(S)}\widehat{\pi}+2g)^{\alpha\beta}\partial^2_{\alpha\beta}\phi| \\
&\qquad\qquad\lesssim \langle t_0\rangle^{-\beta}\frac{\chi_{r\leqslant\epsilon t}t^{2-\gamma'}}{\langle x\rangle^{\frac{1}{2}-\gamma'+\delta-\epsilon}}(|\partial_t^2\phi|+|\nabla_x^2\phi|+|\partial_t\nabla_x\phi|)\ , \\
&\Big|\langle x\rangle^{\frac{1}{2}+\epsilon}t^{1+\gamma'}\chi_{r\leqslant\epsilon t}\partial_\alpha(\sqrt{|g|}({}^{(\widetilde{S})}\widehat{\pi}^{\alpha\beta}+2g^{\alpha\beta}))\partial_\beta\phi\Big| \\
&\qquad\qquad\lesssim \langle t_0\rangle^{-\beta}\frac{\chi_{r\leqslant\epsilon t}t^{2-2\gamma'}}{\langle x\rangle^{\frac{3}{2}-2\gamma'+\delta-\epsilon}}(|\partial_t\phi|+|\nabla_x\phi|)\ , \\
&\left|\langle x\rangle^{\frac{1}{2}+\epsilon}t^{1+\gamma'}\chi_{r\leqslant\epsilon t}\big(\frac{\widetilde{S}(\sqrt{|g|})}{\sqrt{|g|}}+2\big)\cdot F\right| \lesssim \frac{t^{2-\gamma'}\chi_{r\leqslant\epsilon t}|F|}{\langle x\rangle^{\frac{1}{2}+\epsilon}}+\langle x\rangle^{\frac{1}{2}+\epsilon}t^{1+\gamma'}\chi_{r\leqslant\epsilon t}|F|\ .
\end{aligned}
$$

Using (1.11) to simplify the exponents:

$$
\begin{aligned}
&|\langle x\rangle^{\frac{1}{2}+\epsilon}t^{1+\gamma'}\chi_{int}[\Box_g,\widetilde{S}]\phi| \\
&\lesssim \langle t_0\rangle^{-\beta}\frac{\chi_{r\leqslant\epsilon t}t^{2-\gamma'}}{\langle x\rangle^{\frac{1}{2}+\epsilon}}\big(|\partial_t^2\phi|+|\nabla_x^2\phi|+|\partial_t\nabla_x\phi|+t^{-\gamma'}\langle x\rangle^{-1}(|\partial_t\phi|+|\nabla_x\phi|)\big) \\
&+\ \frac{t^{2-\gamma'}\chi_{r\leqslant\epsilon t}|F|}{\langle x\rangle^{\frac{1}{2}+\epsilon}}+\langle x\rangle^{\frac{1}{2}+\epsilon}t^{1+\gamma'}\chi_{r\leqslant\epsilon t}|F|\ .
\end{aligned}
$$

After squaring and integrating the above estimate, we can use the Hardy estimate (A.2) to add a space derivative to the lower order terms. The leads to a loss of $t^{\gamma'}$ since the weight must be above the $-\frac{3}{2}$ threshold – but we can absorb this since $\partial_t(g)$ gives us one extra $t^{-\gamma'}$. Note also that when the derivatives land on the cutoff, the term is supported where $r\sim\epsilon t$ – therefore using (2.38) we account for the left over $\epsilon^{-1}\langle t_0\rangle^{-\beta}$ factors and get a term that is bounded above by $\epsilon\sup_{t_0\leqslant t\leqslant t_1}CE[\phi]$. Combining these results:

$$
\begin{aligned}
&\|\,\chi_{int}[\Box_g,\widetilde{S}]\phi\,\|_{LS^{*,1+\gamma'}} \\
&\lesssim \epsilon\cdot\|\,\chi_{r\leqslant\epsilon t}\frac{t^{2-\gamma'}}{\langle x\rangle^{\frac{1}{2}+\epsilon}}(|\partial_t^2\phi|+|\nabla_x^2\phi|+|\partial_t\nabla_x\phi|)\,\|_{L^2_{t,x}[t_0,t_1]}+\|\,F\,\|_{LS^{2-\gamma'}[t_0,t_1]} \\
&+\ \|\,\chi_{int}F\,\|_{LS^{*,1+\gamma'}[t_0,t_1]}+\epsilon\|\,\phi\,\|_{LS_1^{1-\gamma'}[t_0,t_1]}+\epsilon\sup_{t_0\leqslant t\leqslant t_1}CE[\phi]\ ,
\end{aligned}
$$

Applying the weighted elliptic estimate (4.16) to the first term finishes the proof.

□

Proof of (4.21). For the (u,i) terms, lemma 2.1.10 lets us use identity (2.22) to expand. To control each combination of components we then use the decay obtained in Lemma 4.2.2:

$$|\langle u\rangle^{\frac{1}{2}+\epsilon}\langle\underline{u}\rangle^{1+\epsilon}\chi_{wave}({}^{(\widetilde{S})}\widehat{\pi}+2g)^{ur}_{\mathcal{B}}\partial_u\widetilde{\partial}_r\phi| \lesssim \frac{\langle\underline{u}\rangle\langle u\rangle}{\langle r\rangle^{\frac{1}{2}+\frac{\delta}{2}}}|\chi_{wave}\partial_u\widetilde{\partial}_r\phi| \ ,$$
$$|\langle u\rangle^{\frac{1}{2}+\epsilon}\langle\underline{u}\rangle^{1+\epsilon}\chi_{wave}({}^{(\widetilde{S})}\widehat{\pi}+2g)^{ua}_{\mathcal{B}}\partial_u e_a\phi| \lesssim \frac{\langle\underline{u}\rangle\langle u\rangle}{\langle r\rangle^{\frac{1}{2}+\frac{\delta}{2}}}|\chi_{wave}\partial_u e_a\phi| \ ,$$
$$|\langle u\rangle^{\frac{1}{2}+\epsilon}\langle\underline{u}\rangle^{1+\epsilon}\chi_{wave}({}^{(\widetilde{S})}\widehat{\pi}^{ij}+2g^{ij})\widetilde{\partial}^2_{ij}\phi| \lesssim \frac{\langle\underline{u}\rangle\langle u\rangle}{\langle u\rangle^{\frac{1}{2}+\frac{\delta}{2}}}|\chi_{wave}\widetilde{\partial}^2_{ij}\phi| \ .$$

Using (4.18) squaring and integrating yields the result for these terms. For (u,u) terms we have ${}^{(\widetilde{S})}\widehat{\pi}^{uu}=0$. For the l.o.t. we again use the decay obtained in Lemma 4.2.2 together with (2.7) for the determinant. For the (j,u) l.o.t. terms, after using Lemma 2.1.10:

$$\left|\langle u\rangle^{\frac{1}{2}+\epsilon}\langle\underline{u}\rangle^{1+\epsilon}(\sqrt{|g|})^{-1}\chi_{wave}e_j\big(\sqrt{|g|}({}^{(\widetilde{S})}\widehat{\pi}+2g)^{ju}_{\mathcal{B}}\big)\partial_u\phi\right| \lesssim \frac{\langle u\rangle}{\langle r\rangle^{\frac{1}{2}+\frac{\delta}{2}}}|\chi_{wave}\partial_u\phi| \ .$$

For the other terms we compute:

$$\left|\langle u\rangle^{\frac{1}{2}+\epsilon}\langle\underline{u}\rangle^{1+\epsilon}\chi_{wave}(\sqrt{|g|})^{-1}\partial_\beta\big(\sqrt{|g|}({}^{(\widetilde{S})}\widehat{\pi}^{\beta j}+2g^{\beta j})\big)\widetilde{\partial}_j\phi\right|$$
$$\lesssim \left(\frac{\langle\underline{u}\rangle}{\langle r\rangle^{\frac{\delta}{2}}\langle u\rangle^{\frac{1}{2}+\epsilon}}+\frac{\langle\underline{u}\rangle^{\frac{1}{2}}\langle u\rangle^{\frac{1}{2}}}{\langle r\rangle^{\frac{1}{2}+\frac{\delta}{2}}}\right)\sum_j|\chi_{wave}\widetilde{\partial}_j\phi| \ .$$

Lastly:

$$\left|\langle u\rangle^{\frac{1}{2}+\epsilon}\langle\underline{u}\rangle^{1+\epsilon}\chi_{wave}\big(\frac{\widetilde{S}(\sqrt{|g|})}{\sqrt{|g|}}+2\big)\cdot F\right| \lesssim \langle u\rangle^{\frac{1}{2}+\epsilon}\langle\underline{u}\rangle^{1+\epsilon}\chi_{wave}|F| \ .$$

Squaring the the inequalities, integrating, and taking supremum gives us (4.21).

□

Proof of (4.22). For this region the only change is that the (u,u) components no longer vanish and that all weights are equivalent. The result follows by the same type of proof as (4.21) above once we apply the decay rates obtained in (1.2). We also use $\frac{t}{2\epsilon}\leq r$ to trade r decay for t and account for the left over $\epsilon^{-1}\langle t_0\rangle^{-\delta}$ factors by using (2.38).

□

Remark 4.2.6. *We are limited to commuting one scaling vector field into the conformal energy due to the lower order terms in the interior. A quick computation shows that for two scalings we would get:*

$$l.o.t.\ |\langle x\rangle^{\frac{1}{2}+\epsilon} t^{1-\gamma'} \chi_{r\leqslant\epsilon t}[S^2, \Box_g]\phi| \ \lesssim\ \epsilon\chi_{r\leqslant\epsilon t} t^{3-3\gamma'} \left[\langle x\rangle^{-\frac{5}{2}-2\delta+\epsilon}\nabla_x\phi\right]$$

Since the Hardy estimate then forces the power of the weight to be greater than $-\frac{3}{2}$, we gain a derivative at the cost of losing $r^{-1-3\delta+\epsilon}$. The best we could do in this case is:

$$\begin{aligned} &l.o.t.\ |\langle x\rangle^{\frac{1}{2}+\epsilon} t^{1-\gamma'} \chi_{r\leqslant\epsilon t}[S^2, \Box_g]\phi| \\ &\quad\lesssim\ \epsilon\chi_{r\leqslant\epsilon t} t^{3-3\gamma'} \left[\langle x\rangle^{-\frac{1}{2}-\epsilon}\nabla_x^2\phi\right] \ \sim\ \epsilon t^{1-\gamma'}\chi_{r\leqslant\epsilon t}\langle x\rangle^{-\frac{1}{2}-\epsilon} S^2\phi \end{aligned}$$

which is off from being a bootstrap term exactly by the power of r that we lost. This is a problem where $r \sim \epsilon t$; here we are completely off of anything we can control with a t-weighted LS estimate for $S^2\phi$. Conformal energy with two scalings wouldn't close using this method.

4.2.2 Proof of Commutator Estimate (4.1)

Using (4.19) the result will follow from:

$$\begin{aligned} \|\,\chi_{int}[\Box_g, \partial_t^k]\phi\,\|_{LS^{*,1+\gamma'}} \ &\lesssim\ \epsilon\cdot\sum_{j\leqslant k}\|\,\chi_{r\leqslant\epsilon t}\partial_t^j\phi\,\|_{LSM^{1-\gamma'}[t_0,t_1]} + \|\,F\,\|_{\mathring{LS}_k^{*,1+\gamma',\frac{1}{2}}[t_0,t_1]} \\ &\qquad + \epsilon\cdot C\mathring{E}H_k^{\frac{1}{2}}[\phi]_{[t_0,t_1]}\ . \end{aligned} \tag{4.23}$$

$$\|\,\chi_{wave}[\Box_g, \partial_t^k]\phi\,\|_{LS^{*,1+\gamma',\frac{1}{2}}} \ \lesssim\ \|\,F\,\|_{\mathring{LS}_k^{*,1+\gamma',\frac{1}{2}}[t_0,t_1]} + \epsilon\cdot C\mathring{E}H_k^{\frac{1}{2}}[\phi]_{[t_0,t_1]} \tag{4.24}$$

$$\|\,\chi_{rest}[\Box_g, \partial_t^k]\phi\,\|_{LS^{*,1+\gamma',\frac{1}{2}}} \ \lesssim\ \|\,F\,\|_{\mathring{LS}_k^{*,1+\gamma',\frac{1}{2}}[t_0,t_1]} + \epsilon\cdot C\mathring{E}H_k^{\frac{1}{2}}[\phi]_{[t_0,t_1]} \tag{4.25}$$

Applying (4.15) with $\Gamma^k = \partial_t^k$:

$$\begin{aligned} &[\Box_g, \partial_t^k]\phi \\ &= \sum_{m+j=k-1} (\sqrt{|g|})^{-1}\left[\partial_t^j(\sqrt{|g|}\,{}^{(\partial_t)}\widehat{\pi}^{\alpha\beta})\partial_t^m\partial_{\alpha\beta}^2\phi + \partial_t^j\partial_\alpha(\sqrt{|g|}\,{}^{(\partial_t)}\widehat{\pi}^{\alpha\beta})\partial_t^m\partial_\beta\phi\right] \\ &\quad +\ \partial_t^j\left(\frac{\partial_t(\sqrt{|g|})}{\sqrt{|g|}}\right)\partial_t^m F\ , \end{aligned} \tag{4.26}$$

Proof of (4.23). By the same type of proof as (4.20):

$$
\begin{aligned}
&\| \chi_{int}[\Box_g, \partial_t^k]\phi \|_{LS^{*,1+\gamma'}} \\
&\lesssim \langle t_0\rangle^{-\beta} \sum_{j\leqslant k-1} \Big(\| \chi_{int} \frac{t^{1-\gamma'}}{\langle x\rangle^{\frac{1}{2}+\epsilon}} (|\partial_t^{j+2}\phi| + |\nabla_x^2 \partial_t^j \phi| + |\nabla_x \partial_t^{j+1}\phi|) \|_{L^2_{t,x}[t_0,t_1]} \\
&+ \| \chi_{int}\partial_t^j F \|_{LS^{1+\gamma'}[t_0,t_1]} + \epsilon \sup_{t_0\leqslant t\leqslant t_1} CE[\partial_t^j \phi] \Big) \ .
\end{aligned}
$$

Applying estimate (4.17) with $\partial_t^j\phi$ and a Hardy estimate for the term with F finishes the proof. □

Proof of (4.24). It suffices to do the proof using ∂_u derivatives since $\partial_u = (u_t)^{-1}\partial_t$ and $u_t \approx 1$. We expand using the Bondi frame $\mathcal{B}$ and lemma 2.1.10. To control each combination of components we then use the decay in Lemma 4.2.2:

$$
\begin{aligned}
&\sum_{j+m=k-1} |\langle u\rangle^{\frac{1}{2}+\epsilon}\langle \underline{u}\rangle^{1+\epsilon}\chi_{wave}(\sqrt{|g|})^{-1}\partial_u^j(\sqrt{|g|}({}^{(\partial_u)}\widehat{\pi})^{ur}_{\mathcal{B}})\partial_u^{m+1}\widetilde{\partial}_r\phi| \\
&\qquad\qquad \lesssim \frac{\langle \underline{u}\rangle}{\langle r\rangle^{\frac{1}{2}+\frac{1}{2}\delta}} \sum_{m\leqslant k-1} |\chi_{wave}\partial_u^{m+1}\widetilde{\partial}_r\phi| \ , \\
&\sum_{j+m=k-1} |\langle u\rangle^{\frac{1}{2}+\epsilon}\langle \underline{u}\rangle^{1+\epsilon}\chi_{wave}(\sqrt{|g|})^{-1}\partial_u^j(\sqrt{|g|}({}^{(\partial_u)}\widehat{\pi})^{ua}_{\mathcal{B}})\partial_u^{m+1}e_a\phi| \\
&\qquad\qquad \lesssim \frac{\langle \underline{u}\rangle^{\frac{1}{2}}\langle u\rangle^{\frac{1}{2}}}{\langle r\rangle^{\frac{1}{2}+\frac{1}{2}\delta}} \sum_{m\leqslant k-1} |\chi_{wave}\partial_u^{m+1}e_a\phi| \ , \\
&\sum_{j+m=k-1} |\langle u\rangle^{\frac{1}{2}+\epsilon}\langle \underline{u}\rangle^{1+\epsilon}\chi_{wave}(\sqrt{|g|})^{-1}\partial_u^j(\sqrt{|g|}{}^{(\partial_u)}\widehat{\pi}^{ij})\partial_u^{m}\partial^2_{ij}\phi| \\
&\qquad\qquad \lesssim \frac{\langle \underline{u}\rangle}{\langle u\rangle^{\frac{1}{2}+\frac{\delta}{2}}} \sum_{m\leqslant k-1} |\chi_{wave}\partial_u^{m}\widetilde{\partial}^2_{ij}\phi| \ ,
\end{aligned}
$$

and ${}^{(\partial_u)}\widehat{\pi}^{uu} = 0$. For the l.o.t. with (j,u):

$$
\begin{aligned}
&\sum_{j+m=k-1} \left| \langle u\rangle^{\frac{1}{2}+\epsilon}\langle \underline{u}\rangle^{1+\epsilon}\chi_{wave}(\sqrt{|g|})^{-1}\partial_u^j\widetilde{\partial}_l\left(\sqrt{|g|}{}^{(\partial_u)}\widehat{\pi}^{lu}\right)\partial_u^{m+1}\phi\right| \\
&\qquad\qquad \lesssim \frac{\langle u\rangle}{\langle r\rangle^{\frac{1}{2}+\frac{\delta}{2}}} \sum_{m\leqslant k-1} |\chi_{wave}\partial_u^{m+1}\phi| \ .
\end{aligned}
$$

For the other components a similar result holds. Lastly:

$$\sum_{m+j=k} \left| \langle u \rangle^{\frac{1}{2}+\epsilon} \langle \underline{u} \rangle^{1+\epsilon} \chi_{wave} \partial_u^j \left(\frac{\partial_u(\sqrt{|g|})}{\sqrt{|g|}} \right) \partial_u^m F \right| \\ \lesssim \langle u \rangle^{\frac{1}{2}+\epsilon} \langle \underline{u} \rangle^{1+\epsilon} \langle r \rangle^{-\delta} \chi_{wave} \sum_{m \leqslant k-1} |\partial_u^m F| \ .$$

By (4.26), squaring the inequalities, integrating and taking supremum finishes the proof.

□

Proof of (4.25). Once again, for this region the only change is that the (u, u) components no longer vanish and that all weights are equivalent. The result follows by the same proof as (4.24) above once we use the decay rates in (1.2). We also use $\frac{t}{2\epsilon} \leq r$ to trade r decay for t and account for the left over $\epsilon^{-1}\langle t_0 \rangle^{-\delta}$ factors by using (2.38).

□

4.2.3 Proof of Commutator Estimate (4.2)

Using (4.19) the result will follow from:

$$\| \chi_{int}[\Box_g, \partial_{t,x}^k]\phi \|_{LS^{*,1+\gamma'}} \lesssim \sum_{j \leqslant k} \| \chi_{r \leqslant \epsilon t} \partial_t^j \phi \|_{LSM^{1-\gamma'}[t_0,t_1]} + \| F \|_{\mathring{LS}_k^{*,1+\gamma',\frac{1}{2}}[t_0,t_1]} \\ + \epsilon \cdot C\mathring{E}H_k^{\frac{1}{2}}[\phi]_{[t_0,t_1]} \ . \tag{4.27}$$

$$\| \chi_{wave}[\Box_g, \partial_{t,x}^k]\phi \|_{LS^{*,1+\gamma',\frac{1}{2}}} \lesssim \| F \|_{\mathring{LS}_k^{*,1+\gamma',\frac{1}{2}}[t_0,t_1]} + \epsilon \cdot C\mathring{E}H_k^{\frac{1}{2}}[\phi]_{[t_0,t_1]} \tag{4.28}$$

$$\| \chi_{rest}[\Box_g, \partial_{t,x}^k]\phi \|_{LS^{*,1+\gamma',\frac{1}{2}}} \lesssim \| F \|_{\mathring{LS}_k^{*,1+\gamma',\frac{1}{2}}[t_0,t_1]} + \epsilon \cdot C\mathring{E}H_k^{\frac{1}{2}}[\phi]_{[t_0,t_1]} \tag{4.29}$$

Since $\nabla^k\phi$ with $\nabla^k = \partial_t^j \nabla_x^L$ we set up an induction on $|L|$. The case $l = 0$ is already done by (4.1). Next we assume that estimates (4.27)-(4.29) hold for all $j + |L| \leqslant k - 1$. Therefore, it suffices to commute the equation with all ∂_x^k and prove the estimates (4.27)-(4.29) with spatial derivatives only. Applying (4.15):

$$\begin{aligned}
&[\Box_g, \partial_x^k]\phi \\
&= \sum_{m+j=k-1} (\sqrt{|g|})^{-1} \left(\partial_x^j ({}^{(\partial_t)}\widehat{\pi}^{\alpha\beta}) \partial_x^m \partial_{\alpha\beta}^2 \phi + \partial_x^j \partial_\alpha (\sqrt{|g|}\,{}^{(\partial_t)}\widehat{\pi}^{\alpha\beta}) \partial_x^m \partial_\beta \phi \right) \\
&+ \ \partial_x^j \left(\frac{\partial_x(\sqrt{|g|})}{\sqrt{|g|}} \right) \partial_x^m F \ ,
\end{aligned} \tag{4.30}$$

Proof of (4.27). By the same proof as in (4.23):

$$\begin{aligned}
&\| \chi_{int} [\Box_g, \partial_x^k]\phi \|_{LS^{*,1+\gamma'}} \\
&\lesssim \sum_{j\leqslant k-1} \| \frac{t^{1-\gamma'}\chi_{int}}{\langle x \rangle^{\frac{1}{2}+\epsilon}} (|\partial_t^2 \nabla_x^j \phi| + |\nabla_x^{j+2}\phi| + |\partial_t \nabla_x^{j+1}\phi|) \|_{L^2_{t,x}[t_0,t_1]} \\
&+ \ \| F \|_{\mathring{LS}_k^{*,1+\gamma',\frac{1}{2}}[t_0,t_1]} \ .
\end{aligned}$$

Since $[\partial_x, \partial_t] = 0$ we can apply the estimate (4.17) $k-1$ times and get the result. □

Proof of (4.28). It suffices to do the proof using $\partial_u, \widetilde{\partial}_i$ derivatives since by lemma 2.1.5 these are bounded linear combinations of ∂_t, ∂_x derivatives. By (4.24), it suffices to control $\| \chi_{wave} [\Box_g, \widetilde{\partial}_x^k]\phi \|_{LS^{*,1+\gamma',\frac{1}{2}}}$. But this just follows by the same type of proof as (4.24).

□

Proof of (4.29). This follows immediately by the decay rates in Lemma 3.5.2 plus a simple interpolation between (4.27) and (4.28). As before, $\frac{t}{2\epsilon} \leq r$ lets us r decay for t and we account for the left over $\epsilon^{-1}\langle t_0 \rangle^{-\delta}$ factors by using (2.38).

□

4.2.4 Proof of Commutator Estimate (4.4)

This will follow easily by induction on the number of vector fields. The case $k = 1$ follows by previous results. For $k > 1$ we assume that the result holds for all $1 \leqslant j \leqslant k-1$ and note that the commutator, schematically, looks like:

$$\begin{aligned}
[\partial^{k-1}\Gamma, \Box_g]\phi \; &= \partial^j(g)(\partial^m\partial^2\Gamma\phi + \langle x\rangle^{-1}\partial^m\partial\Gamma\phi) \\
&+ \; \partial^j(\Gamma g)(\partial^m\partial^2\phi + \langle x\rangle^{-1}\partial^m\partial\Gamma\phi) \\
&+ \; [\Gamma,\partial](\partial^{j-1}(g)(\partial^m\partial^2\phi + \langle x\rangle^{-1}\partial^m\partial\phi)) \\
&+ \; [\Gamma,\partial](\partial^j(g)(\partial^{m-1}\partial^2\phi + \langle x\rangle^{-1}\partial^{m-1}\partial\phi))
\end{aligned}$$

with $1 \leqslant j, \; m + j = k$. By (2.3), it then follows by induction hypothesis that we must only deal with the highest order terms. The estimate will follow by a combination of (4.1)-(4.3).

4.3 Proof of The Supporting Lemmas

4.3.1 Proof of Lemma 4.2.1

Proof. Let D denote the covariant derivative associated to g. Computing in local coordinates, we have the following identity proved in [Ali10]:

$$[\Box_g, X]\phi = {}^{(X)}\pi^{\alpha\beta}D^2_{\alpha\beta}\phi + (D_\alpha{}^{(X)}\pi^{\alpha\beta})\partial_\beta\phi - \frac{1}{2}\partial^\alpha(tr\pi)\partial_\alpha\phi \; .$$

By the definition of $\widehat{\pi}$:

$$\begin{aligned}
{}^{(X)}\pi^{\alpha\beta}D^2_{\alpha\beta}\phi \; &= \; {}^{(X)}\widehat{\pi}^{\alpha\beta}D^2_{\alpha\beta}\phi + \frac{1}{2}(tr^{(X)}\widehat{\pi})F \; , \\
(D_\alpha{}^{(X)}\pi^{\alpha\beta})\partial_\beta\phi \; &= \; (D_\alpha{}^{(X)}\widehat{\pi}^{\alpha\beta})\partial_\beta\phi + \frac{1}{2}D_\alpha(tr^{(X)}\widehat{\pi})g^{\alpha\beta}\partial_\beta\phi \; .
\end{aligned}$$

Therefore:

$$\begin{aligned}
[\Box_g, X]\phi \; &= {}^{(X)}\widehat{\pi}^{\alpha\beta}D^2_{\alpha\beta}\phi + \frac{1}{2}(tr^{(X)}\pi)F + (D_\alpha{}^{(X)}\widehat{\pi}^{\alpha\beta})\partial_\beta\phi \\
&\quad + \frac{1}{2}D_\alpha(tr^{(X)}\widehat{\pi})g^{\alpha\beta}\partial_\beta\phi - \frac{1}{2}\partial^\alpha(tr^{(X)}\pi)\partial_\alpha\phi \\
&= D_\alpha({}^{(X)}\widehat{\pi}^{\alpha\beta}\partial_\beta\phi) + \frac{1}{2}(tr^{(X)}\pi)F \\
&= \frac{1}{\sqrt{|g|}}\partial_\alpha(\sqrt{|g|}{}^{(X)}\widehat{\pi}^{\alpha\beta}\partial_\beta\phi) + \frac{1}{\sqrt{|g|}}\partial_\alpha(\sqrt{|g|}X^\alpha)F \; .
\end{aligned}$$

□

4.3.2 Proof of Lemma 4.2.2

The wave zone $\mathcal{W}$: For the scaling we have in Bondi coordinates $\partial_\gamma \widetilde{S}^\gamma = 4$. Computing and using estimate (2.7) to bound $(\sqrt{|g|})^{-1}$ we get:

$$\begin{aligned}
&\big({}^{(\widetilde{S})}\widehat{\pi}^{uu} + 2g^{uu}\big)\\
&\quad = \ (\sqrt{|g|})^{-1}\sqrt{|g|}\big({}^{(\widetilde{S})}\widehat{\pi}^{uu} + 2\sqrt{|g|}g^{uu}\big)\\
&\quad = \ (\sqrt{|g|})^{-1}\Big(-\widetilde{S}(g^{uu}\sqrt{|g|}) + 2\sqrt{|g|}g^{u\gamma}(\partial_\gamma u) - 2\sqrt{|g|}g^{uu}\Big) \ = \ 0 \ .
\end{aligned}$$

For the (u, i) components with we recall that Lemma 2.1.10 lets us use identity (2.20) in the frame $\mathcal{B}$. Therefore by (2.12) and (2.13):

$$\begin{aligned}
&|\big({}^{(\widetilde{S})}\widehat{\pi} + 2g\big)^{ur}_{\mathcal{B}}|\\
&\quad = \ |(\sqrt{|g|})^{-1}\sqrt{|g|}\big({}^{(\widetilde{S})}\widehat{\pi} + 2g\big)^{ur}_{\mathcal{B}}|\\
&\quad = \ |(\sqrt{|g|})^{-1}\big(- \widetilde{S}(g^{ur}\sqrt{|g|}) + \sqrt{|g|}g^{u\gamma}(\partial_\gamma r) + \sqrt{|g|}g^{r\gamma}(\partial_\gamma u) - 2\sqrt{|g|}g^{ur}\big)|\\
&\quad = \ |(\sqrt{|g|})^{-1}\big(- \widetilde{S}(g^{ur}\sqrt{|g|} + 1)\big)|\\
&\quad \lesssim \ \left(\frac{\langle u\rangle^{\frac{1}{2}}}{\langle r\rangle^{\frac{1}{2}+\delta}}\right) \ ,
\end{aligned}$$

$$|\big({}^{(\widetilde{S})}\widehat{\pi} + 2g\big)^{ua}_{\mathcal{B}}| \ = \ | - (\sqrt{|g|})^{-1}\widetilde{S}(g^{ua}\sqrt{|g|}) + g^{au} - 2g^{ua}| \ \lesssim \ \left(\frac{\langle u\rangle}{\langle r\rangle^{1+\delta}}\right) \ .$$

For the (i, j) components we use the Bondi coordinate derivatives and (2.8) to get:

$$\begin{aligned}
&|\big({}^{(\widetilde{S})}\widehat{\pi}^{ij} + 2g^{ij}\big)|\\
&\quad = \ |(\sqrt{|g|})^{-1}\sqrt{|g|}\big({}^{(\widetilde{S})}\widehat{\pi}^{ij} + 2g^{ij}\big)|\\
&\quad = \ |(\sqrt{|g|})^{-1}\big(- \widetilde{S}(\sqrt{|g|}g^{ij} - \delta^{ij}) + \sqrt{|g|}g^{i\gamma}\partial_\gamma x^j + \sqrt{|g|}g^{j\gamma}\partial_\gamma x^i - 2\sqrt{|g|}g^{ij}\big)|\\
&\quad = \ | - (\sqrt{|g|})^{-1}\widetilde{S}(\sqrt{|g|}g^{ij} - \delta^{ij})|\\
&\quad \lesssim \ \langle r\rangle^{-\delta} \ .
\end{aligned}$$

Similarly, outside the wave zone, in Bondi coordinates:

$$\begin{aligned}
&|\big({}^{(\widetilde{S})}\widehat{\pi}^{\alpha\beta} + 2g^{\alpha\beta}\big)| \\
&= \ |(\sqrt{|g|})^{-1}\sqrt{|g|}\big({}^{(\widetilde{S})}\widehat{\pi}^{\alpha\beta} + 2g^{\alpha\beta}\big)| \\
&= \ |(\sqrt{|g|})^{-1}\big(- \widetilde{S}(\sqrt{|g|}g^{\alpha\beta} - \delta^{\alpha\beta}) + \sqrt{|g|}g^{i\alpha\gamma}\partial_\gamma x^\beta + \sqrt{|g|}g^{\beta\gamma}\partial_\gamma x^\alpha - 2\sqrt{|g|}g^{\alpha\beta}\big)| \\
&= \ | - (\sqrt{|g|})^{-1}\widetilde{S}(\sqrt{|g|}g^{\alpha\beta} - \delta^{\alpha\beta})| \\
&\lesssim \ t\langle r\rangle^{-\delta-1-\gamma}\langle r/t\rangle^{\gamma} \ .
\end{aligned}$$

To get the result with derivatives we differentiate these identities and note that by Leibniz rule the derivatives either land on $(\sqrt{|g|})^{-1}$ – in which case we may again apply estimate (2.7) – or land on terms to which we can still apply (2.12)–(2.13) or (2.8) directly. For the rotations the proof is very similar so we omit it.

4.3.3 Proof of Lemma 4.2.3

We will need the following results in the proof of (4.16):

Lemma 4.3.1.

$$\| \frac{\chi_{r\leqslant\epsilon t}t^{2-\gamma'}\partial_t^2\phi}{\langle x\rangle^{\frac{1}{2}+\epsilon}} \|_{L^2_{t,x}[t_0,t_1]} \ \lesssim \| \phi \|_{LSM^{1-\gamma'}_{int,1}[t_0,t_1]} + \epsilon\| \frac{\chi_{r\leqslant\epsilon t}t^{2-\gamma'}\partial_t\nabla_x\phi}{\langle x\rangle^{\frac{1}{2}+\epsilon}} \|_{L^2_{t,x}[t_0,t_1]} + \ \epsilon\cdot CEH_1^{\frac{1}{2}}[\phi]_{[t_0,t_1]} \tag{4.31}$$

$$\| \frac{\chi_{r\leqslant\epsilon t}t^{2-\gamma'}\nabla_x^2\phi}{\langle x\rangle^{\frac{1}{2}+\epsilon}} \|_{L^2_{t,x}[t_0,t_1]} \ \lesssim \| \phi \|_{LSM^{1-\gamma'}_{int,1}[t_0,t_1]} + \| \frac{\chi_{r\leqslant\epsilon t}t^{2-\gamma'}}{\langle x\rangle^{\frac{1}{2}+\epsilon}}\partial_t^2\phi \|_{L^2_{t,x}[t_0,t_1]} + \ \epsilon\cdot CEH_1^{\frac{1}{2}}[\phi]_{[t_0,t_1]} + \| F \|_{LS^{2-\gamma'}[t_0,t_1]} \tag{4.32}$$

Let's first use these results to finish up the proof of lemma 4.2.3. We will establish lemma 4.3.1 immediately after.

Proof of (4.16). In the region $r \leqslant \epsilon t$ we have $u = t - \langle r\rangle$ and therefore in (t,x) coordinates the scaling vector field is:

$$\widetilde{S} \ = \ (t - \frac{1}{\langle r\rangle})\partial_t + r\partial_r \ = \ S - \frac{1}{\langle r\rangle}\partial_t$$

Therefore, modulo lower order terms we may use $S = t\partial_t + r\partial_r$ as our scaling in this region. In the sequel, we will use this reduction and ignore the lower order terms below since they do not affect the results in any significant way. We start with the simple pointwise bound:

$$\begin{aligned}\Big|\chi_{r\leqslant\epsilon t}\frac{t^{2-\gamma'}\partial_t\nabla_x\phi}{\langle x\rangle^{\frac{1}{2}+\epsilon}}\Big| \ &= \ \chi_{r\leqslant\epsilon t}\Big|\frac{t^{1-\gamma'}\widetilde{S}\nabla_x\phi}{\langle x\rangle^{\frac{1}{2}+\epsilon}}\Big| + \Big|\frac{t^{1-\gamma'}r\partial_r\nabla_x\phi}{\langle x\rangle^{\frac{1}{2}+\epsilon}}\Big| \\ &\lesssim \ \chi_{r\leqslant\epsilon t}\Big|\frac{t^{1-\gamma'}\nabla_x\widetilde{S}\phi}{\langle x\rangle^{\frac{1}{2}+\epsilon}}\Big| + \epsilon\Big|\frac{t^{2-\gamma'}\nabla_x^2\phi}{\langle x\rangle^{\frac{1}{2}+\epsilon}}\Big| + \Big|\frac{t^{1-\gamma'}\nabla_x\phi}{\langle x\rangle^{\frac{1}{2}+\epsilon}}\Big| \ .\end{aligned}$$

Squaring and integrating this gives us:

$$\| \frac{\chi_{r\leqslant\epsilon t}t^{2-\gamma'}\partial_t\nabla_x\phi}{\langle x\rangle^{\frac{1}{2}+\epsilon}} \|_{L^2_{t,x}[t_0,t_1]} \ \lesssim \ \|\phi\|_{LS^{1-\gamma'}_{int,1}[t_0,t_1]} + \epsilon\| \frac{\chi_{r\leqslant\epsilon t}t^{2-\gamma'}}{\langle x\rangle^{\frac{1}{2}+\epsilon}}\nabla_x^2\phi \|_{L^2_{t,x}[t_0,t_1]} \ . \tag{4.33}$$

Next we combine this with our previous estimates to finish the rest of the proof. We apply (4.32) and (4.33) in succession to estimate the RHS of (4.31):

$$\begin{aligned}&\| \frac{\chi_{r\leqslant\epsilon t}t^{2-\gamma'}\partial_t^2\phi}{\langle x\rangle^{\frac{1}{2}+\epsilon}} \|_{L^2_{t,x}[t_0,t_1]} \\ &\lesssim \ \|\phi\|_{LS^{1-\gamma'}_{int,1}[t_0,t_1]} + \epsilon\| \frac{\chi_{r\leqslant\epsilon t}t^{2-\gamma'}\partial_t\nabla_x\phi}{\langle x\rangle^{\frac{1}{2}+\epsilon}} \|_{L^2_{t,x}[t_0,t_1]} \\ &\lesssim \ \|\phi\|_{LS^{1-\gamma'}_{int,1}[t_0,t_1]} + \epsilon^2\| \frac{\chi_{r\leqslant\epsilon t}t^{2-\gamma'}}{\langle x\rangle^{\frac{1}{2}+\epsilon}}\nabla_x^2\phi \|_{L^2_{t,x}[t_0,t_1]} \\ &\lesssim \ \|\phi\|_{LS^{1-\gamma'}_{int,1}[t_0,t_1]} + \epsilon^2\| \frac{\chi_{r\leqslant\epsilon t}t^{2-\gamma'}}{\langle x\rangle^{\frac{1}{2}+\epsilon}}\partial_t^2\phi \|_{L^2_{t,x}[t_0,t_1]} + \|F\|_{LS^{2-\gamma',\frac{1}{2}}[t_0,t_1]} \\ &+ \ \epsilon^3\cdot CEH_1^{\frac{1}{2}}[\phi]_{[t_0,t_1]}\end{aligned}$$

Bootstrapping the second term gives us the result for ∂_t^2:

$$\begin{aligned}\| \frac{\chi_{r\leqslant\epsilon t}t^{2-\gamma'}\partial_t^2\phi}{\langle x\rangle^{\frac{1}{2}+\epsilon}} \|_{L^2_{t,x}[t_0,t_1]} \ &\lesssim \ \|\phi\|_{LS^{1-\gamma'}_{int,1}[t_0,t_1]} + \|F\|_{LS^{2-\gamma',\frac{1}{2}}[t_0,t_1]} \\ &+ \ \epsilon^3\cdot CEH_1^{\frac{1}{2}}[\phi]_{[t_0,t_1]}\end{aligned}$$

Now that we have the estimate for ∂_t^2, we apply this result to the RHS of (4.32) and get the estimate for two space derivatives:

$$\begin{aligned}\| \frac{\chi_{r\leqslant\epsilon t}t^{2-\gamma'}\nabla_x^2\phi}{\langle x\rangle^{\frac{1}{2}+\epsilon}} \|_{L^2_{t,x}[t_0,t_1]} \ &\lesssim \ \|\phi\|_{LS^{1-\gamma'}_{int,1}[t_0,t_1]} + \|F\|_{LS^{2-\gamma',\frac{1}{2}}[t_0,t_1]} \\ &+ \ \epsilon\cdot CEH_1^{\frac{1}{2}}[\phi]_{[t_0,t_1]} \ .\end{aligned}$$

Applying this result to (4.33) closes the estimate for $\partial_t \nabla_x$:

$$\| \frac{\chi_{r\leqslant\epsilon t} t^{2-\gamma'} \partial_t \nabla_x \phi}{\langle x\rangle^{\frac{1}{2}+\epsilon}} \|_{L^2_{t,x}[t_0,t_1]} \lesssim \| \phi \|_{LS^{1-\gamma'}_{int,1}[t_0,t_1]} + \| F \|_{LS^{2-\gamma',\frac{1}{2}}[t_0,t_1]} + \epsilon \cdot CEH^{\frac{1}{2}}_1[\phi]_{[t_0,t_1]} \ .$$

Adding these last three estimates proves estimate (4.16).

□

Proof of (4.17). It suffices to do the proof for two spatial derivatives only. Since the hypersurfaces $t = t_0$ are spacelike everywhere, the metric $h = g\big|_{t=t_0}$ is Riemannian. Therefore, the Laplace-Beltrami operator for h is an elliptic operator which in (t, x) coordinates has the form:

$$\begin{aligned} \Delta_h &= \frac{1}{\sqrt{|g|}} \partial_i (g^{ij} \sqrt{|g|} \partial_j) \\ &= \Box_g - g^{00} \partial_t - \frac{1}{\sqrt{|g|}} \partial_t (g^{0j} \sqrt{|g|} \partial_j) - \frac{1}{\sqrt{|g|}} \partial_j (g^{0j} \sqrt{|g|} \partial_t) \ . \end{aligned}$$

Consequently we may apply estimate the weighted L^2 estimate (A.2) and expand to get:

$$\begin{aligned} &\| \frac{t^{1-\gamma'} \chi_{r\leqslant\epsilon t}}{\langle x\rangle^{\frac{1}{2}+\epsilon}} \nabla_x^2 \phi) \|_{L^2_{t,x}[t_0,t_1]} \\ &\lesssim \| \frac{t^{1-\gamma'} \chi_{r\leqslant\epsilon t}}{\langle x\rangle^{\frac{1}{2}+\epsilon}} \Delta_h \phi \|_{L^2_{t,x}[t_0,t_1]} \\ &\lesssim \| \frac{t^{1-\gamma'} \chi_{r\leqslant\epsilon t}}{\langle x\rangle^{\frac{1}{2}+\epsilon}} g^{0j} \partial_t \partial_j \phi) \|_{L^2_{t,x}[t_0,t_1]} + \| \frac{t^{1-\gamma'} \chi_{r\leqslant\epsilon t}}{\langle x\rangle^{\frac{1}{2}+\epsilon}} \frac{1}{\sqrt{|g|}} \partial_t (g^{0j} \sqrt{|g|}) \partial_j \phi \|_{L^2_{t,x}[t_0,t_1]} \\ &\quad + \| \frac{t^{1-\gamma'} \chi_{r\leqslant\epsilon t}}{\langle x\rangle^{\frac{1}{2}+\epsilon}} \frac{1}{\sqrt{|g|}} \partial_j (g^{0j} \sqrt{|g|}) \partial_t \phi \|_{L^2_{t,x}[t_0,t_1]} + \| \frac{t^{1-\gamma'} \chi_{r\leqslant\epsilon t}}{\langle x\rangle^{\frac{1}{2}+\epsilon}} g^{00} \partial_t^2 \phi \|_{L^2_{t,x}[t_0,t_1]} \\ &\quad + \| F \|_{LS^{1-\gamma',\frac{1}{2}}[t_0,t_1]} \\ &\lesssim \langle t_0\rangle^{-\beta} \| \frac{t^{1-2\gamma'} \chi_{r\leqslant\epsilon t}}{\langle x\rangle^{\frac{3}{2}+\epsilon}} \nabla_x \phi \|_{L^2_{t,x}[t_0,t_1]} + \sum_{j\leqslant 1} \| \partial_t^j \phi \|_{LSM^{1-\gamma'}_{int}[t_0,t_1]} + \| F \|_{LS^{1-\gamma',\frac{1}{2}}[t_0,t_1]} \ . \end{aligned}$$

Using a Hardy estimate on the first term and bootstrapping finishes the proof.

□

Proof of 4.2.4. It suffices to establish the pointwise estimates:

$$\begin{aligned} & u^2|\partial_u^2\phi| + r\underline{u}|\widetilde{\partial}_r^2\phi| \\ & \lesssim \sum_{|J|\leqslant 1} \big(\langle r\rangle|(\widetilde{\partial}_r\Gamma^J\phi,\, \not\nabla\Gamma^J\phi)| + \langle u\rangle|\partial_u\Gamma^J\phi|\big) + \langle u\rangle\langle\underline{u}\rangle|F| + O(\langle r\rangle^{-\delta})|ur||\partial_u\widetilde{\partial}_r\phi| \ , \end{aligned} \tag{4.34}$$

$$|ur\partial_u\widetilde{\partial}_r\phi| \ \lesssim \ \sum_{|J|\leqslant 1} \big(\langle r\rangle|(\widetilde{\partial}_r\Gamma^J\phi,\, \not\nabla\Gamma^J\phi)| + \langle u\rangle|\partial_u\Gamma^J\phi|\big) + \langle u\rangle\langle\underline{u}\rangle|F| \ , \tag{4.35}$$

with $\Gamma\in\mathbb{L}$. To prove (4.34) we start with the identities:

$$u\partial_u(\widetilde{S}\phi) \ = \ u^2\partial_u^2\phi + ur\widetilde{\partial}_r\partial_u\phi + u\partial_u\phi \tag{4.36}$$

$$r\widetilde{\partial}_r(\widetilde{S}\phi) \ = \ r^2\widetilde{\partial}_r^2\phi + ur\widetilde{\partial}_r\partial_u\phi + r\widetilde{\partial}_r\phi \tag{4.37}$$

Since the calculation for $u^2\partial_u^2$ is very similar we only do the case $r\underline{u}\widetilde{\partial}_r^2$. Adding and subtracting urF and $ur\Box_{\widetilde{\eta}}\phi$ to twice identity (4.37):

$$\begin{aligned} & 2r^2\widetilde{\partial}_r^2\phi \\ & = \ 2r\widetilde{\partial}_r(\widetilde{S}\phi) - 2ur\widetilde{\partial}_r\partial_u\phi - 2r\widetilde{\partial}_r\phi - ur\widetilde{\partial}_r^2\phi - ur\Box_{\widetilde{\eta}}\phi + ur\Box_{\widetilde{\eta}}\phi - urF + urF \\ & = \ 2r\widetilde{\partial}_r(\widetilde{S}\phi) - 2r\widetilde{\partial}_r\phi - ur\widetilde{\partial}_r^2\phi - 2u(\widetilde{\partial}_r - \partial_u)\phi - ur\,\widetilde{\not\nabla}^2\phi + ur(\Box_{\widetilde{\eta}} - \Box_g)\phi + urF \ , \end{aligned}$$

where we have used the identity (3.79) to expand $\Box_{\widetilde{\eta}}$. Rearraging and taking absolute value:

$$\begin{aligned} & |r\underline{u}\widetilde{\partial}_r^2\phi| \\ & \lesssim \ |r\widetilde{\partial}_r(\widetilde{S}\phi)| + |r\widetilde{\partial}_r\phi| + |u(\widetilde{\partial}_r - \partial_u)\phi| + |ur\,\widetilde{\not\nabla}^2\phi| + |ur(\Box_{\widetilde{\eta}} - \Box_g)\phi| + |urF| \end{aligned}$$

By a straightforward computation:

$$\begin{aligned} & |ur(\Box_{\widetilde{\eta}} - \Box_g)\phi| \\ & \lesssim \ O(\langle r\rangle^{-\delta})|ur|\Big(|\partial_u\widetilde{\partial}_r\phi| + |\widetilde{\partial}_r^2\phi| + |\,\widetilde{\not\nabla}^2\phi| + |\widetilde{\partial}_r\not\nabla\phi| + |\frac{1}{\langle r\rangle}\partial_u\phi| \\ & \quad + \ |\frac{1}{\langle u\rangle^{\frac12}\langle r\rangle^{\frac12}}(\widetilde{\partial}_r\phi,\, \not\nabla\phi)|\Big) \ . \end{aligned}$$

We bootstrap the term with $\widetilde{\partial}_r^2$ and obtain (4.34). Estimate (4.35) follows by adding (4.36) and (4.37), using (4.34) and bootstrapping the term $ur\widetilde{\partial}_r\partial_u\phi$.

□

Lastly, we now prove the two estimates in Lemma 4.3.1.

Proof of (4.31). A computation gives us:

$$\begin{aligned}
&\| S\phi \|_{LSM^{1-\gamma'}_{int}} \\
&= \| \chi_{r\leqslant \epsilon t}\partial_t S\phi) \|_{LS^{1-\gamma'}} + \| \chi_{r\leqslant \epsilon t}\nabla_x (S\phi) \|_{LS^{1-\gamma'}} + \| \langle x\rangle^{-1}\chi_{r\leqslant \epsilon t} S\phi \|_{LS^{1-\gamma'}} \\
&= \| \frac{\chi_{r\leqslant \epsilon t}(t^{2-\gamma'}\partial_t^2\phi + t^{1-\gamma'} r\partial_r\partial_t\phi)}{\langle x\rangle^{\frac{1}{2}+\epsilon}} \|_{L^2_{t,x}[t_0,t_1]} \\
&+ \| \frac{\chi_{r\leqslant \epsilon t}(t^{2-\gamma'}\partial_t\nabla_x\phi + t^{1-\gamma'} r\nabla_x\partial_r\phi)}{\langle x\rangle^{\frac{1}{2}+\epsilon}} \|_{L^2_{t,x}[t_0,t_1]} \\
&+ \| \langle x\rangle^{-\frac{3}{2}-\epsilon}\chi_{r\leqslant \epsilon t} S\phi \|_{L^2_{t,x}[t_0,t_1]} \ .
\end{aligned}$$

Therefore,

$$\begin{aligned}
&\| \chi_{r\leqslant \epsilon t} t^{2-\gamma'}\langle x\rangle^{-\frac{1}{2}-\epsilon}\partial_t^2\phi \|_{L^2_{t,x}[t_0,t_1]} \\
&\lesssim \| \langle x\rangle^{-\frac{1}{2}-\epsilon} t^{1-\gamma'}\chi_{r\leqslant \epsilon t}\partial_t(S\phi) \|_{L^2_{t,x}[t_0,t_1]} + \| \langle x\rangle^{-\frac{1}{2}-\epsilon}\Big[\chi_{r\leqslant \epsilon t}(t^{1-\gamma'} r\partial_r\partial_t\phi)\Big] \|_{L^2_{t,x}[t_0,t_1]} \\
&+ \| \langle x\rangle^{-\frac{3}{2}-\epsilon}\chi_{r\leqslant \epsilon t} S\phi \|_{L^2_{t,x}[t_0,t_1]}
\end{aligned}$$

The third term can be grouped with the first to make up $\| \phi \|_{LS^{1-\gamma'}_{int,1}[t_0,t_1]}$ on the RHS. Using $r \leqslant \epsilon t$ then gives:

$$\| \frac{\chi_{r\leqslant \epsilon t} t^{2-\gamma'}\partial_t^2\phi}{\langle x\rangle^{\frac{1}{2}+\epsilon}} \|_{L^2_{t,x}[t_0,t_1]} \lesssim \| \phi \|_{LS^{1-\gamma'}_{int,1}[t_0,t_1]} + \epsilon \| \frac{\chi_{r\leqslant \epsilon t} t^{2-\gamma'}\partial_t\nabla_x\phi}{\langle x\rangle^{\frac{1}{2}+\epsilon}} \|_{L^2_{t,x}[t_0,t_1]} \ .$$

□

Proof of (4.32). Trading t for u then applying the weighted L^2 estimate (A.2) once again:

$$
\begin{aligned}
&\| \frac{t^{2-\gamma'}\chi_{r\leqslant \epsilon t}}{\langle x\rangle^{\frac{1}{2}+\epsilon}}\nabla_x^2\phi) \|_{L^2_{t,x}[t_0,t_1]} \\
\lesssim\ & \| \frac{\langle u\rangle^{2-\gamma'}}{\langle x\rangle^{\frac{1}{2}+\epsilon}}\nabla_x^2\phi) \|_{L^2_{t,x}[t_0,t_1]} \\
\lesssim & \| \frac{\langle u\rangle^{2-\gamma'}\chi_{r\leqslant \epsilon t}}{\langle x\rangle^{\frac{1}{2}+\epsilon}}\Delta_h\phi \|_{L^2_{t,x}[t_0,t_1]} + \| \frac{\langle u\rangle^{2-\gamma'}\chi_{\frac{1}{2}\epsilon t\leqslant r}}{\langle x\rangle^{\frac{1}{2}+\epsilon}}\Delta_h\phi \|_{L^2_{t,x}[t_0,t_1]} \\
\lesssim\ & \| \frac{t^{2-\gamma'}\chi_{r\leqslant \epsilon t}}{\langle x\rangle^{\frac{1}{2}+\epsilon}}\Box_g\phi \|_{L^2_{t,x}[t_0,t_1]} + \| \frac{t^{2-\gamma'}\chi_{r\leqslant \epsilon t}}{\langle x\rangle^{\frac{1}{2}+\epsilon}}\frac{1}{\sqrt{|g|}}\partial_t(g^{0j}\sqrt{|g|}\partial_j\phi) \|_{L^2_{t,x}[t_0,t_1]} \\
&+ \| \frac{t^{2-\gamma'}\chi_{r\leqslant \epsilon t}}{\langle x\rangle^{\frac{1}{2}+\epsilon}}\frac{1}{\sqrt{|g|}}\partial_j(g^{0j}\sqrt{|g|}\partial_t\phi) \|_{L^2_{t,x}[t_0,t_1]} + \| \frac{t^{2-\gamma'}\chi_{r\leqslant \epsilon t}}{\langle x\rangle^{\frac{1}{2}+\epsilon}}g^{00}\partial_t^2\phi \|_{L^2_{t,x}[t_0,t_1]} \\
&+ \| \frac{\langle u\rangle^{2-\gamma'}\chi_{\frac{1}{2}\epsilon t\leqslant r}}{\langle x\rangle^{\frac{1}{2}+\epsilon}}(\nabla^2\phi, \langle x\rangle^{-1-\delta}\nabla\phi) \|_{L^2_{t,x}[t_0,t_1]} \\
\lesssim\ & \| \frac{t^{2-\gamma'}\chi_{r\leqslant \epsilon t}}{\langle x\rangle^{\frac{1}{2}+\epsilon}}g^{0j}\partial_t\partial_j\phi) \|_{L^2_{t,x}[t_0,t_1]} + \| \frac{t^{2-\gamma'}\chi_{r\leqslant \epsilon t}}{\langle x\rangle^{\frac{1}{2}+\epsilon}}\frac{1}{\sqrt{|g|}}\partial_t(g^{0j}\sqrt{|g|})\partial_j\phi \|_{L^2_{t,x}[t_0,t_1]} \\
&+ \| \frac{t^{2-\gamma'}\chi_{r\leqslant \epsilon t}}{\langle x\rangle^{\frac{1}{2}+\epsilon}}\frac{1}{\sqrt{|g|}}\partial_j(g^{0j}\sqrt{|g|})\partial_t\phi \|_{L^2_{t,x}[t_0,t_1]} + \| \frac{t^{2-\gamma'}\chi_{r\leqslant \epsilon t}}{\langle x\rangle^{\frac{1}{2}+\epsilon}}g^{00}\partial_t^2\phi \|_{L^2_{t,x}[t_0,t_1]} \\
&+\ \| F \|_{LS^{2-\gamma',\frac{1}{2}}[t_0,t_1]} + \epsilon^{-\frac{1}{2}}\langle t_0\rangle^{-\frac{\beta}{2}} \sup_{t_0\leqslant t\leqslant t_1} CE_1^{\frac{1}{2}}[\phi(t)] \\
:=\ & I + II + III + IV + V + (\epsilon \sup_{t_0\leqslant t\leqslant t_1} CE_1^{\frac{1}{2}}[\phi(t)]) \ .
\end{aligned}
$$

where we've used the estimate (4.18) to control the terms supported on the set where $\{\frac{1}{2}\epsilon t \leqslant r\}$. Next we discuss the terms $I - V$; the terms IV and V are both acceptable on the RHS. For the term I:

$$
\begin{aligned}
I\ \lesssim\ & \| \frac{t^{1-\gamma'}\chi_{r\leqslant \epsilon t}}{\langle x\rangle^{\frac{1}{2}+\epsilon}}g^{0j}\partial_j(S\phi) \|_{L^2_{t,x}[t_0,t_1]} + \| \frac{t^{1-\gamma'}\chi_{r\leqslant \epsilon t}}{\langle x\rangle^{\frac{1}{2}+\epsilon}}g^{0j}\partial_j(r\partial_r\phi) \|_{L^2_{t,x}[t_0,t_1]} \\
\lesssim\ & \| \frac{t^{1-\gamma'}\chi_{r\leqslant \epsilon t}}{\langle x\rangle^{\frac{1}{2}+\epsilon}}g^{0j}\partial_j(S\phi) \|_{L^2_{t,x}[t_0,t_1]} + \| \frac{t^{1-\gamma'}\chi_{r\leqslant \epsilon t}}{\langle x\rangle^{\frac{1}{2}+\epsilon}}g^{0j}(\partial_r\phi) \|_{L^2_{t,x}[t_0,t_1]} \\
&+\ \| \frac{rt^{1-\gamma'}\chi_{r\leqslant \epsilon t}}{\langle x\rangle^{\frac{1}{2}+\epsilon}}g^{0j}(\partial_j\partial_r\phi) \|_{L^2_{t,x}[t_0,t_1]} \\
\lesssim\ & \| \phi \|_{LS^{1-\gamma'}_{int,1}[t_0,t_1]} + \epsilon\| \frac{t^{2-\gamma'}\chi_{r\leqslant \epsilon t}}{\langle x\rangle^{\frac{1}{2}+\epsilon}}\nabla_x^2\phi) \|_{L^2_{t,x}[t_0,t_1]} + \epsilon \sup_{t_0\leqslant t\leqslant t_1} CE_1^{\frac{1}{2}}[\phi(t)] \ ,
\end{aligned}
$$

where we used (2.38) on the last line. We bootstrap the last term onto the LHS of the preceding estimate. For II we use the fact that $\partial_t(g)$ gives us an extra $t^{-\gamma'}$ decay and therefore we can apply a Hardy estimate and get:

$$\begin{aligned} II &\lesssim \epsilon \| \langle x \rangle^{-\frac{1}{2}-\epsilon} \chi_{r \leqslant \epsilon t} (t^{2-\gamma'} \nabla_x^2 \phi) \|_{L^2_{t,x}[t_0,t_1]} + \| \langle x \rangle^{-\frac{3}{2}-\epsilon} \chi_{r \sim \epsilon t} (t^{2-\gamma'} \nabla_x \phi) \|_{L^2_{t,x}[t_0,t_1]} \\ &\lesssim \epsilon \| \langle x \rangle^{-\frac{1}{2}-\epsilon} \chi_{r \leqslant \epsilon t} (t^{2-\gamma'} \nabla_x^2 \phi) \|_{L^2_{t,x}[t_0,t_1]} + \epsilon^{-\frac{1}{2}} \langle t_0 \rangle^{-\frac{\beta}{2}} \sup_{t_0 \leqslant t \leqslant t_1} CE_1^{\frac{1}{2}}[\phi(t)] \\ &\lesssim \epsilon \| \langle x \rangle^{-\frac{1}{2}-\epsilon} \chi_{r \leqslant \epsilon t} (t^{2-\gamma'} \nabla_x^2 \phi) \|_{L^2_{t,x}[t_0,t_1]} + \epsilon \sup_{t_0 \leqslant t \leqslant t_1} CE_1^{\frac{1}{2}}[\phi(t)] \end{aligned}$$

Next:

$$\begin{aligned} III &\lesssim \| \langle x \rangle^{-\frac{3}{2}-\epsilon} t^{1-\gamma'} \chi_{r \leqslant \epsilon t} S\phi \|_{L^2_{t,x}[t_0,t_1]} + \| \langle x \rangle^{-\frac{1}{2}-\epsilon} t^{1-\gamma'} \chi_{r \leqslant \epsilon t} \partial_r \phi \|_{L^2_{t,x}[t_0,t_1]} \\ &\lesssim \| \phi \|_{LS^{1-\gamma'}_{int,1}[t_0,t_1]} . \end{aligned}$$

Combining all these estimates finishes the proof of lemma 4.3.1.

□

Chapter 5

Pointwise Decay

The main goal in this section is to use the higher order conformal energy estimate (1.13) to prove:

Theorem 5.0.2 (Global Pointwise Decay). *Let $(\mathcal{M}, g_{\alpha\beta})$ satisfy all the assumptions of the main theorem. Let ϕ be a solution to the free wave equation:*

$$\Box_g \phi = 0 \ , \qquad \phi(0,x) \ = \ \phi_0(x) \ , \partial_t \phi(0,x) \ = \ \phi_1(x) \ ,$$

with smooth initial data such that $CE_2^{\frac{1}{2}}[\phi(t_0)]$ is finite. Then, for all $(t,x) \in [0,\infty) \times \mathbb{R}^3$ with $t \geqslant t_0 \geqslant 0$, the function ϕ satisfies the pointwise decay estimate:

$$|\phi(t,x)| \ \lesssim \ \frac{1}{\langle t+r \rangle \langle u \rangle^{\frac{1}{2}}} \Big(CE_2^{\frac{1}{2}}[\phi(t_0)] \Big) \ .$$

We will need the following interior estimate in the proof:

Lemma 5.0.3 (Elliptic Estimates for Two Derivatives).

$$\sum_{|\alpha|=2} \| \, \chi_{r \leqslant \epsilon t} \langle t \rangle^2 \nabla^\alpha \phi \, \|_{L^2_x} \ \lesssim \ CE_2^{\frac{1}{2}}[\phi(t)] \tag{5.1}$$

Proof. By the reduction in section 2.3.1 it suffices to prove the result for $t > t^*$. By Lemma (2.1.1) we may apply the results in [Hör85] and get the pointwise estimate:

$$|\phi(t,x)| \ \lesssim \ \frac{1}{\langle t+r \rangle \langle u \rangle^{\frac{1}{2}}} \left(\sum_{|\alpha| \leqslant 2} \| \, \Gamma^\alpha \phi(t) \, \|_{L^2_x} \right) \ ,$$

with $\Gamma \in \mathbb{L}$. We use the cutoffs in (4.19) to break up the L^2 terms into pieces supported in the interior region, the wave zone and the complement of these two, respectively. In the interior region where $\{r \leqslant \epsilon t\}$ we use estimate (5.1):

$$\begin{aligned} \sum_{|\alpha|\leqslant 2} \| \chi_{int}\Gamma^\alpha \phi(t) \|_{L^2_x} &\lesssim \sum_{|\alpha|=2} \| \chi_{int}\langle t\rangle^2 \nabla^\alpha \phi \|_{L^2_x} + CE_1^{\frac{1}{2}}[\phi(t)] \\ &\lesssim CE_2^{\frac{1}{2}}[\phi(t)] \end{aligned}$$

In the wave zone we use (4.34) and (4.35) on the terms with two derivatives and absorb the small error terms to get:

$$\sum_{|\alpha|\leqslant 2} \| \chi_{wave}\Gamma^\alpha \phi(t) \|_{L^2_x} \lesssim CE_2^{\frac{1}{2}}[\phi(t)] \ .$$

In the complement, since asymptotic flatness holds and all weights are equivalent, we get, by the same type of argument:

$$\sum_{|\alpha|\leqslant 2} \| \chi_{rest}\Gamma^\alpha \phi(t) \|_{L^2_x} \lesssim CE_2^{\frac{1}{2}}[\phi(t)] \ .$$

An application of the higher order conformal energy estimate (1.13) with $k = 2$ finishes the proof.

□

Proof of Lemma (5.0.3). For ∂_t^2 we have:

$$\begin{aligned} \| \chi_{r\leqslant \epsilon t} t^2 \partial_t^2 \phi \|_{L^2_x} &\lesssim \| t\chi_{r\leqslant \epsilon t}\partial_t(S\phi) \|_{L^2_x} + \| \chi_{r\leqslant \epsilon t} tr \partial_r \partial_t \phi \|_{L^2_x} \\ &\lesssim CE_2^{\frac{1}{2}}[\phi(t)] + \epsilon \| \chi_{r\leqslant \epsilon t} t^2 \partial_t \nabla_x \phi \|_{L^2_x} \ . \end{aligned} \qquad (5.2)$$

For the mixed derivatives:

$$\begin{aligned} \left|\chi_{r\leqslant \epsilon t} t^2 \partial_t \nabla_x \phi\right| &= \chi_{r\leqslant \epsilon t}\left(\left|t\widetilde{S}\nabla_x\phi\right| + |tr\partial_r \nabla_x \phi|\right) \\ &\lesssim \chi_{r\leqslant \epsilon t}\left(\left|t\nabla_x \widetilde{S}\phi\right| + \epsilon\left|t^2\nabla_x^2\phi\right| + |t\nabla_x\phi|\right) \ . \end{aligned}$$

Squaring and integrating this gives us:

$$\| \chi_{r\leqslant \epsilon t} t^2 \partial_t \nabla_x \phi \|_{L^2_x} \lesssim CE_2^{\frac{1}{2}}[\phi(t)] + \epsilon \| \chi_{r\leqslant \epsilon t} t^2 \nabla_x^2 \phi \|_{L^2_x} \ . \qquad (5.3)$$

Lastly, for ∂_x^2 we apply the weighted L^2 estimate (A.2) once again and follow the proof of (4.32) to get:

$$\begin{aligned} \| \chi_{r\leqslant \epsilon t} t^2 \nabla_x^2 \phi \|_{L^2_x} &\lesssim \| \langle u \rangle^2 \nabla_x^2 \phi \|_{L^2_x} \\ &\lesssim \| \chi_{r\leqslant \epsilon t} t^2 \Delta_h \phi \|_{L^2_x} + \| \chi_{\frac{1}{2}\epsilon t \leqslant r} \langle u \rangle^2 (\nabla^2 \phi, \langle x \rangle^{-1-\delta} \nabla \phi \|_{L^2_x} \\ &\lesssim \| \chi_{r\leqslant \epsilon t} t^2 \partial_t^2 \phi \|_{L^2_x} + C E_2^{\frac{1}{2}} [\phi(t)] \ . \end{aligned} \tag{5.4}$$

where we've used (4.34) and (4.35) on the terms supported where $\{\frac{1}{2}\epsilon t \leqslant r\}$. Applying (5.3) to the RHS of (5.4) followed by (5.4) and a bootstrap closes the estimate for ∂_t^2. Adding the resulting estimate to (5.3) and (5.4) and bootstrapping finishes the proof.

□

Appendix A

Weighted L^2 Estimates

Our main estimate here is the following:

Theorem A.0.4 (Global weighted elliptic estimates)**.** *Let h be a Riemannian metric such that:*

$$|\partial_x^\alpha (h^{ij} - \delta^{ij})| \ \lesssim \ \langle r \rangle^{-|\alpha|-\delta} \ . \tag{A.1}$$

Then one has:

$$\| \langle x \rangle^\mu (\nabla^2 \phi, \langle x \rangle^{-1} \nabla \phi) \|_{L^2} \ \lesssim \ \| \langle x \rangle^\mu \Delta_h \phi \|_{L^2} \ , \qquad -\frac{1}{2} < \mu < \frac{3}{2} \ . \tag{A.2}$$

Proof. Write $\Delta_h \phi = F$. We approximately solve for F in terms of a Neumann series:

$$\widetilde{\phi} \ = \ \Delta^{-1} \sum_{i=0}^{k} R^i F \ , \qquad R \ = \ I - \Delta_h \Delta^{-1} \ .$$

Then we have:

$$\Delta_h (\widetilde{\phi} - \phi) \ = \ R^{k+1} F \ .$$

Therefore, setting $L^{2,\mu}$ for the norm on line (A.2) it suffices to show:

$$\nabla^2 \Delta^{-1} : L^{2,\mu} \ \to \ L^{2,\mu} \ , \qquad \langle x \rangle^{-1} \nabla \Delta^{-1} : L^{2,\mu} \ \to \ L^{2,\mu} \ , \qquad R : L^{2,\mu} \ \to \ L^{2,\mu+\delta} \ , \tag{A.3}$$

for the range $\frac{1}{2} < \mu$ and $\mu + \delta < \frac{3}{2}$, followed by the non-perturbative estimate:

$$\| (\langle x \rangle \nabla^2 \phi, \nabla \phi) \|_{L^2} \ \lesssim \ \| \langle x \rangle \Delta_h \phi \|_{L^2} \ . \tag{A.4}$$

□

Proof of (A.3). We decompose into dyadic scales $|x| \sim 2^i$, $|y| \sim 2^j$ with $|x| \lesssim 1$ when $i = 0$ since the weights are non-singular. To establish that $\nabla^2\Delta^{-1} : L^{2,\mu} \to L^{2,\mu}$ is bounded, it suffices to show:

$$\sum_{i,j} \left| \iint_{|x|\sim 2^i, |y|\sim 2^j} F(x)K_1(x-y)G(y)\ dxdy \right| \leq C\| F \|_{L^{2,-\mu}} \| G \|_{L^{2,\mu}} \ . \qquad \text{(A.5)}$$

where K_1 is the kernel:

$$\begin{aligned} &\sum_{i,j} \iint_{|x|\sim 2^i, |y|\sim 2^j} F(x)(\nabla^2\Delta^{-1})G(y)\ dxdy \\ &= \sum_{i,j} \iint_{|x|\sim 2^i, |y|\sim 2^j} F(x)K_1(x-y)G(y)\ dxdy \end{aligned}$$

We break up the proof into cases:

Case 1: When $|i-j| = O(1)$ the operator defined above is a singular integral operator. In this case the weights $2^{-\mu j} \approx 1$ and $2^{-\mu i} \approx 1$ balance since they are both approximately of size one. Therefore by Cauchy-Schwarz:

$$\begin{aligned} &\sum_{i+j=O(1)} \left| \iint_{|x|\sim 2^i, |y|\sim 2^j} F(x)K_1(x-y)G(y)\ dxdy \right| \\ &\lesssim \sum_{i,j} \| \chi_i F \|_{L^2} \| \chi_j G \|_{L^2} \\ &\lesssim \sum_{i,j} \| \chi_i F \|_{L^{2,-\mu}} \| \chi_j G \|_{L^{2,\mu}} \ , \end{aligned}$$

with χ_i, χ_j smooth cutoff functions in supported where $|x| \sim 2^i$, $|y| \sim 2^j$, respectively.

Case 2: When $i > j + c$ we have $|K_1(x-y)| = O(\frac{1}{|x|^3})$ therefore, since convolution with an L^1 function is a bounded operator in any L^p space with $p \geq 1$:

$$\begin{aligned} &\sum_{i>j+c} \left| \iint_{|x|\sim 2^i, |y|\sim 2^j} F(x)K_1(x-y)G(y)\ dxdy \right| \\ &\lesssim \sum_{i>j+c} 2^{-\frac{3}{2}(i-j)} \cdot 2^{\mu(i-j)} \| \chi_i F \|_{L^{2,-\mu}} \| \chi_j G \|_{L^{2,\mu}} \ , \end{aligned}$$

and we get $-\frac{3}{2}+\mu<0$ to have the sum above converge.

Case 3: When $j > i+c$ by switching the roles of x and y and using the same argument as case 2:

$$\sum_{j>i+c}\left|\iint_{|x|\sim 2^i,|y|\sim 2^j} F(x)K_1(x-y)G(y)\ dxdy\right|$$
$$\lesssim \sum_{j>i+c} 2^{\frac{3}{2}(i-j)}\cdot 2^{\mu(i-j)}\|\,\chi_i F\,\|_{L^{2,-\mu}}\|\,\chi_j G\,\|_{L^{2,\mu}}\ ,$$

which is convergent as long as $-\frac{3}{2}-\mu<0$. This proves (A.5) and shows that $\nabla^2\Delta^{-1} : L^{2,\mu} \rightarrow L^{2,\mu}$ is bounded. Next, to show $\langle x\rangle^{-1}\nabla\Delta^{-1} : L^{2,\mu} \rightarrow L^{2,\mu}$ is bounded we again aim to show:

$$\sum_{i,j}\left|\iint_{|x|\sim 2^i,|y|\sim 2^j} F(x)K_2(x-y)G(y)\ dxdy\right| \lesssim \|\,F\,\|_{L^{2,-\mu}}\|\,G\,\|_{L^{2,\mu}}\ , \tag{A.6}$$

with $K_2(x,y)$ the kernel given by:

$$\sum_{i,j}\iint_{|x|\sim 2^i,|y|\sim 2^j} F(x)(\langle x\rangle^{-1}\nabla\Delta^{-1})G(y)\ dxdy$$
$$= \sum_{i,j}\iint_{|x|\sim 2^i,|y|\sim 2^j} F(x)K_2(x-y)G(y)\ dxdy\ .$$

Case 1: When $|i-j|=O(1)$ we have $K_2(x,y)=O(\frac{1}{\langle x\rangle|x-y|^2})$, therefore we use the Hardy-Littlewood-Sobolev inequality to get:

$$\sum_{i+j=O(1)}\left|\iint_{|x|\sim 2^i,|y|\sim 2^j} F(x)K_2(x-y)G(y)\ dxdy\right|$$
$$\lesssim \sum_{i+j=O(1)}\left|\iint_{|x|\sim 2^i,|y|\sim 2^j} \frac{F(x)G(y)}{\langle x\rangle|x-y|^2}\ dxdy\right|$$
$$\lesssim \sum_{i,j} 2^{-i}\|\,\chi_i F\,\|_{L^{\frac{3}{2}}}\|\,\chi_j G\,\|_{L^{\frac{3}{2}}}$$
$$\lesssim \sum_{i,j} 2^{-i}\cdot 2^{\frac{1}{2}i}\cdot 2^{\frac{1}{2}j}\|\,\chi_i F\,\|_{L^{2,}}\|\,\chi_j G\,\|_{L^{2}}$$
$$\lesssim \sum_{i,j}\|\,\chi_i F\,\|_{L^{2,-\mu}}\|\,\chi_j G\,\|_{L^{2,\mu}}\ ,$$

where we've used $2^{-i}\approx 2^{\frac{1}{2}i}\cdot 2^{\frac{1}{2}j}\approx 1$ on the last line.

Case 2: When $i > j + c$ we have $|K_2(x-y)| = O(\frac{1}{\langle x\rangle |x|^2})$ therefore:

$$\sum_{i>j+c} \Big| \iint_{|x|\sim 2^i, |y|\sim 2^j} F(x)K_2(x-y)G(y)\ dxdy \Big| \lesssim \sum_{i>j+c} 2^{-\frac{1}{2}i} \cdot 2^{-(i-j)} \cdot 2^{\mu(i-j)} \| \chi_i F \|_{L^{2,-\mu}} \| \chi_j G \|_{L^{2,\mu}} \ ,$$

and since $i > j + c$ the extra $2^{-\frac{1}{2}i}$ helps us get $-\frac{3}{2} + \mu < 0$ once again.

Case 3: When $j > i + c$ we have $|K_2(x-y)| = O(\frac{1}{\langle x\rangle |y|^2})$ thus:

$$\sum_{j>i+c} \Big| \iint_{|x|\sim 2^i, |y|\sim 2^j} F(x)K_2(x-y)G(y)\ dxdy \Big| \lesssim \sum_{j>i+c} 2^{\frac{1}{2}i} \cdot 2^{(i-j)} \cdot 2^{\mu(i-j)} \| \chi_i F \|_{L^{2,-\mu}} \| \chi_j G \|_{L^{2,\mu}} \ ,$$

and since $j > i + c$ the extra $2^{\frac{1}{2}i}$ gives us the restriction $-\frac{1}{2} - \mu < 0$.

Lastly, for $R : L^{2,\mu} \ \rightarrow \ L^{2,\mu+\delta}$ we use the expansion:

$$\Delta_h - \Delta \ = \ \widetilde{h}\nabla^2 + (\widetilde{h})'\nabla$$

and by (A.1), $\widetilde{h}$, $(\widetilde{h})'$ obey the decay bounds:

$$|\widetilde{h}| \ \lesssim \ \langle r\rangle^{-\delta}$$
$$|(\widetilde{h})'| \ \lesssim \ \langle r\rangle^{-1-\delta} \ .$$

This observation together with the results we have shown above give this result.

□

Proof of (A.4). Let D denote the Levi-Civita connection for h and let dV_h denote the volume form. We claim:

$$\int_{\mathbb{R}^3} D^i\phi D_i\phi\ dV_h \ \lesssim \ \int_{\mathbb{R}^3} \langle x\rangle^2 |\Delta_h \phi|^2\ dV_h \ . \qquad \text{(A.7)}$$

This follows from Green's identity:

$$-\int_{\mathbb{R}^3} D^i\phi D_i\phi\ dV_h \ = \ \int_{\mathbb{R}^3} \Delta_h\phi \cdot \phi\ dV_h \ ,$$

by taking absolute value, applying Young's inequality and using the Hardy estimate:

$$\int_{\mathbb{R}^3} \left|\frac{\phi}{r}\right|^2 \, dV_h \;\lesssim\; \int_{\mathbb{R}^3} |D\phi|^2 \, dV_h \; . \tag{A.8}$$

To prove the estimate for two derivatives we integrate by parts:

$$\begin{aligned}
&\int_{\mathbb{R}^3} \langle x\rangle^2 (D_i D_j \phi)(D^i D^j \phi) \, dV_h \\
=\;& -\int_{\mathbb{R}^3} \langle x\rangle^2 (D^i D_i D_j \phi)(D^j \phi) \, dV_h - \int_{\mathbb{R}^3} 2\langle x\rangle D^i(\langle x\rangle)(D_i D_j \phi)(D^j \phi) \, dV_h \\
=\;& -\int_{\mathbb{R}^3} \langle x\rangle^2 (D_j D^i D_i \phi)(D^j \phi) \, dV_h - \int_{\mathbb{R}^3} 2\langle x\rangle D^i(\langle x\rangle)(D_i D_j \phi)(D^j \phi) \, dV_h \\
& - \int_{\mathbb{R}^3} \langle x\rangle^2 (D^i (R_{ij}\phi)(D^j \phi) \, dV_h - \int_{\mathbb{R}^3} \langle x\rangle^2 (R^i_j D_i \phi)(D^j \phi) \, dV_h \\
=\;& \int_{\mathbb{R}^3} \langle x\rangle^2 (D^i D_i \phi)(D_j D^j \phi) \, dV_h + \int_{\mathbb{R}^3} 2\langle x\rangle D^i(\langle x\rangle)(D_i \phi)(D_j D^j \phi) \, dV_h \\
& - \int_{\mathbb{R}^3} \langle x\rangle^2 (D^i (R_{ij}\phi)(D^j \phi) \, dV_h - \int_{\mathbb{R}^3} \langle x\rangle^2 (R^i_j D_i \phi)(D^j \phi) \, dV_h \\
& + \int_{\mathbb{R}^3} 2\langle x\rangle D_j(\langle x\rangle)(D^i D_i \phi)(D^j \phi) \, dV_h - \int_{\mathbb{R}^3} 2\langle x\rangle D^i(\langle x\rangle)(R_{ij}\phi)(D^j \phi) \, dV_h \\
& + \int_{\mathbb{R}^3} D_j(2\langle x\rangle D^i(\langle x\rangle))(D_i \phi)(D^j \phi) \, dV_h
\end{aligned}$$

Taking absolute value, using (A.1) on the terms with curvature components, and using the Hardy estimate (A.8) together with (A.7) we get:

$$\begin{aligned}
&\int_{\mathbb{R}^3} \langle x\rangle^2 (D_i D_j \phi)(D^i D^j \phi) \, dV_h \\
\lesssim\;& \int_{\mathbb{R}^3} \langle x\rangle^2 |\Delta_h \phi|^2 \, dV_h + \int_{\mathbb{R}^3} O(\langle x\rangle)|\Delta_h \phi||D\phi| \, dV_h \\
& + \int_{\mathbb{R}^3} O(\langle x\rangle^{-1-\delta})|\phi||D\phi| \, dV_h + \int_{\mathbb{R}^3} D_i\phi D^i \phi \, dV_h \\
\lesssim\;& \int_{\mathbb{R}^3} \langle x\rangle^2 |\Delta_h \phi|^2 \, dV_h + \int_{\mathbb{R}^3} D_i\phi D^i \phi \, dV_h \\
\lesssim\;& \int_{\mathbb{R}^3} \langle x\rangle^2 |\Delta_h \phi|^2 \, dV_h \; .
\end{aligned}$$

□

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