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A Thurston Compactification of Stability Manifolds for some Local Calabi-Yau Varieties

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To my family.

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Abstract

This dissertation presents a number of results on the partial compactification of the Bridgeland stability manifolds associated to a few well-behaved complex varieties by considering their embedding in a certain real projective space. The first main result demonstrates that encoding stability conditions by the masses of a large set of objects always yields an embedding for a significant number of geometric examples — in almost all cases, however, this yields an embedding into an infinite-dimensional space. The second main result shows that a large class of geometric examples proposed by [KKO22] fail to admit a Thurston compactification. The last result is a description of the boundary of the compactification for a certain non-compact Calabi-Yau surface; this can be seen as a geometric realization of the original Thurston compactification for the A_2-quiver introduced by Bapat, Deopurkar, and Licata [BDL20]. This leads to a similar approach in one dimension higher with a much more complicated boundary structure.

Contents

1 Introduction

In heterotic superstring theory, one of the primary objects of study are gauge field configurations on a Calabi-Yau threefold $\mathfrak{Y}$ preserving $\mathcal{N} = 1$ supersymmetry in four dimensions. Due to the work of Donaldson, Uhlenbeck, and Yau, Hermitian Yang-Mills connections preserving this supersymmetry correspond to holomorphic vector bundles on $\mathfrak{Y}$ which are μ_H-stable for some choice of Kähler class H. As described by Douglas, objects depending on a choice of Kähler class in string theory are often modified by certain 'world-sheet instanton corrections'; in light of this, the authors of [DFR05] generalized the notion of stability for vector bundles to a quasi-topological condition for D-branes, known as Π-stability. With the precedent of Π-stability, Bridgeland gave a mathematically rigorous definition of stability conditions on a triangulated category in a series of seminal papers [Bri06, Bri07, Bri08]. One of the most significant results was that the space of all possible stability conditions has the structure of a complex manifold, thus prescribing a geometric realization to the purely algebraic data of triangulated categories. Combined with Douglas' original work, this also suggested that there is a space lying over the moduli of $\mathcal{N} = 2$ "framed" superconformal field theories — known as *Teichmüller space* — that should be closely related to the stability manifold of a certain triangulated category up to an action of automorphisms.

A notable property of Bridgeland's stability manifolds is that they are effectively always open due to several technical conditions in their construction; thus, a current problem in mathematics is how to compactify these spaces in a meaningful way. There have been several proposals in recent years for (partial) compactifications: modifying stability conditions to include massless objects ([BPPW22]), metric completions ([Bol20]), quasi-convergent paths ([HLR25]), and embeddings in a higher dimensional space of functionals ([BDL20]). Each partial compactification comes with its own flavor and interpretation of boundary points; for example, the space of augmented stability conditions proposed by [HLR25] interprets boundary points as certain nodal genus-0 curves (along with additional data) which give rise to highly non-trivial semiorthogonal decompositions of the underlying category. On the other hand, boundary points in the compactification proposed by [BPPW22] correspond to strata obtained from stability conditions on certain quotient triangulated categories.

Due to the proposed connection between Teichmüller space and stability manifolds (modulo the action of autoequivalences), Bapat-Deopurkar-Licata ([BDL20]) defined a categorical analog of Thurston's compactification of Teichmüller using an embedding under a certain "mass" function. One of the immediate advantages of this categorical Thurston compactification is that when the mass function does yield an embedding, the resulting closure of the embedding is almost always compact for standard applications. The biggest drawback, however, is that there is a large class of examples relevant to the minimal model program (MMP) for which the proposed mass map fails to embed the stability manifold, as originally shown by Kikuta-Koseki-Ouchi ([KKO22]) in the case of $\mathbb{P}^1$ — this issue will be highlighted in Section 4.1. The upshot for these examples is that the augmented stability manifold proposed by [HLR25] instead provides a much more meaningful compactification.

The categorical Thurston compactification was originally demonstrated for two stability manifolds associated to the finite A_1 and A_2 quivers in the initial article [BDL20]. The first geometric examples were computed by the authors of [KKO22] in the case of complex projective curves. In this text, we extend the results of [BDL20] and [KKO22] to show that the stability manifolds associated to certain local Calabi-Yau varieties admit well-behaved partial compactifications, while the Thurston compactification of categories with full exceptional collections effectively always loses data.

1.1 Thurston's Compactification of Teichmüller Space

We first provide brief details of Thurston's original compactification of Teichmüller space, but refer to [Thu88], [FLP12] for an in-depth discussion. Though Teichmüller space is very loosely portrayed in the above discussion as a manifold sitting over the space of framed $\mathcal{N} = 2$ superconformal field theories, we will refer to the standard definition as the universal covering space of the complex moduli of Riemann surfaces. In a certain sense, Teichmüller space should instead be thought of as parameterizing complex structures on a Riemann surface of genus g.

Given a smooth oriented surface S_g of genus $g \geq 0$, recall that a *marking* of S_g is the data of a compact Riemann surface X, together with a diffeomorphism $\phi : S_g \to X$. Two markings $\phi : S_g \to X$ and $\psi : S_g \to Y$ are said to be equivalent if $\psi \circ \phi^{-1}$ is isotopic to a holomorphic diffeomorphism. The *Teichmüller Space* of S_g — which we will denote by $\mathrm{Teich}(S_g)$ — is the set of markings on S_g up to equivalence.

Let $\mathcal{S}$ denote the collection of homotopy classes of simple closed curves on S_g. Given any simple closed curve γ and marking $\phi : S_g \to X$, it is a standard fact that there exists a unique closed geodesic on X which is homotopic (possibly up to reparameterization) to $\phi_* \gamma$. If we let $\ell_\phi(\gamma)$ denote the length of this closed geodesic with respect to the Riemann metric on X, we obtain a map

$$\begin{aligned} \ell : \mathrm{Teich}(S_g) &\to \mathbb{R}^{\mathcal{S}}_{\geq 0} \backslash \{0\} \\ \phi &\mapsto (\ell_\phi(\gamma))_{\gamma \in \mathcal{S}} \end{aligned}$$

Letting $\mathbb{P}\ell : \mathrm{Teich}(S_g) \to \mathbb{RP}^{\mathcal{S}}_{\geq 0}$ denote the composition of ℓ with the quotient map to projective space, Thurston showed that the closure of the image of $\mathbb{P}\ell$ yields an effective compactification of $\mathrm{Teich}(S_g)$; moreover, the boundary points in the closure have a meaningful interpretation as projective measured foliations on S_g.

Theorem 1.1. *([FLP12]) Let S_g be a smooth oriented surface of genus $g \geq 0$, $\mathcal{S}$ the collection of homotopy classes of simply closed curves on S_g, and $\mathbb{P}\ell : \mathrm{Teich}(S_g) \to \mathbb{RP}^{\mathcal{S}}_{\geq 0}$ the map as defined above. Then*

1. *The map $\mathbb{P}\ell$ is injective.*
2. *Moreover, the map $\mathbb{P}\ell$ is a homeomorphism onto its image.*
3. *Letting $\overline{\mathrm{Teich}(S_g)}$ denote the closure of the image $\mathbb{P}\ell(\mathrm{Teich}(S_g))$, the space $\overline{\mathrm{Teich}(S_g)}$ is compact.*
4. *Letting $i(\gamma_1, \gamma_2)$ denote the topological intersection number of $\gamma_1, \gamma_2 \in \mathcal{S}$, the bound-*

ary of the compactification can be described by $i(-,-)$ *via*

$$\partial\overline{\mathrm{Teich}(S_g)} = \overline{i_*(\mathcal{S})}$$

The first three properties of the theorem above serve as a basis for generalizing Thurston's compactification, while the fourth gives a meaningful interpretation of the boundary points added in the compactification process. This theorem will serve as a high-level criterion for when certain holomorphic manifolds may be compactified similarly.

1.2 Notation

Throughout the paper we will assume our base field $\Bbbk$ is the complex numbers, and use $\mathcal{D}$ to denote a $\Bbbk$-linear triangulated category. We will use $\mathrm{Hom}^{\bullet}_{\mathcal{D}}(-,-)$ to refer to the $\mathbb{Z}$-graded Hom-space, where $\mathrm{Hom}^i_{\mathcal{D}}(E,F) := \mathrm{RHom}_{\mathcal{D}}(E,F[i]) = \mathrm{RHom}_{\mathcal{D}}(E[-i],F)$; when the index is omitted, $\mathrm{Hom}_{\mathcal{D}}(E,F)$ will refer to the index-zero component of the complex. The group of automorphisms of $\mathcal{D}$ will be denoted $\mathrm{Aut}(\mathcal{D})$. When $S \in \mathcal{D}$ is a spherical object (Definition 2.43), we will use Tw_S to denote the spherical-twist functor associated with S; the subgroup of $\mathrm{Aut}(\mathcal{D})$ generated by spherical twists will be denoted $\mathrm{Aut}_{\mathrm{sph}}(\mathcal{D})$.

The Grothendieck group of $\mathcal{D}$ will be denoted $K(\mathcal{D})$; given an object $E \in \mathcal{D}$, we will typically use brackets $[E]$ to denotes its equivalence class in $K(\mathcal{D})$. The group $K(\mathcal{D})$ comes equipped with a bilinear form

$$\chi(E,F) = \sum_i (-1)^i \dim \mathrm{Hom}^i_{\mathcal{D}}(E,F)$$

known as the Euler pairing. Letting $K(\mathcal{D})^{\perp}$ denote the right kernel of the Euler pairing, we use $\mathcal{N}(\mathcal{D}) := K(\mathcal{D})/K(\mathcal{D})^{\perp}$ to refer to the numerical Grothendieck group.

When X is a smooth projective variety, we write $D^\flat(X) := D^\flat(\mathrm{Coh}(X))$ for the bounded derived category of coherent sheaves, and $\mathrm{Stab}(X) := \mathrm{Stab}(D^\flat(X))$ for the corresponding stability manifold. In this case, $K(D^\flat(X))$ simply coincides with the topological K-group. Using the standard notation for the Chern character

$$\mathrm{ch}: K(X) \to H^{\mathrm{even}}(X, \mathbb{Q})$$

we will denote the graded components by ch_i for $0 \leq i \leq \dim(X)$ and the Chern classes as c_i. Throughout the text, we will use $\Bbbk(x)$ to denote the skyscraper sheaf associated to a closed point $x \in X$.

For the beginning sections of the paper, we will let Λ denote a finite-rank lattice equipped with a surjective group homomorphism $\nu : K(\mathcal{D}) \to \Lambda$ [1]. By an abuse of notation we will typically omit the homomorphism $\nu : K(\mathcal{D}) \to \Lambda$ from functions defined on Λ — for example, we will often write $Z(E)$ in place of $Z(\nu(E))$ when considering the central charge of an object E. Additionally, we will write $\Lambda_{\mathbb{R}}$ for $\Lambda \otimes_{\mathbb{Z}} \mathbb{R}$ and $\Lambda_{\mathbb{C}}$ for $\Lambda \otimes_{\mathbb{Z}} \mathbb{C}$.

The gothic letters $\mathfrak{E}, \mathfrak{F}$ will almost exclusively be used to refer to exceptional collections and the letters $\mathfrak{R}, \mathfrak{S}$ will typically refer to spherical collections. Moreover, we will use $\overline{\mathbb{H}}$ to refer to the half-closed upper half-plane

$$\overline{\mathbb{H}} := \{r \exp(i\pi\phi) \mid r > 0 \text{ and } 0 < \phi \leq 1\}$$

[1] In almost all geometric situations, Λ is derived from $K(\mathcal{D})$ — for example $\Lambda = \mathcal{N}(\mathcal{D})$, the numerical K-group, will be most relevant to us. However, one may also consider the topological K^0-group or the image of the algebraic K^0-group under the Chern character.

2 Preliminaries

Before delving into any details of the Thurston compactification for stability manifolds, it will be important to revisit the general theory surrounding the stability of vector bundles and more modern treatment for triangulated categories. We will assume the reader is well-acquainted with introductory topics in algebraic geometry (for example, the contents of [Har77]), has a cursory understanding of the construction of the derived category $D(\mathcal{A})$ associated to an abelian category $\mathcal{A}$, and is familiar with the definition and axioms of a triangulated category.

2.1 Slope Stability of Vector Bundles

A significant portion of the topics that will be introduced later in this text are motivated by the classical theory of slope stability, which was originally introduced by Mumford in [MF82] as a means of producing well-behaved moduli spaces of sheaves in geometric invariant theory. However, even in the simplest cases of vector bundles over the projective line $\mathbb{P}^1$, the moduli problem turns out to be non-trivial — for example, it is not difficult to show that if we attempt to construct a scheme S parameterizing rank two vector bundles of degree 0 over $\mathbb{P}^1$, S will never be a finite-type scheme (in particular, it will always be non-Noetherian). It turns out the issue is closely related to the fact that no rank two vector bundle over $\mathbb{P}^1$ is stable.

Before explicitly defining slope stability, we first recall a few definitions. Suppose that X is a complex projective scheme with a fixed choice of ample line bundle $\mathcal{O}(1)$, and let $\mathcal{E}$ be a coherent sheaf. Recall that the *Euler character* of $\mathcal{F}$ is defined as

$$\chi(\mathcal{E}) := \sum_i (-1)^i \dim H^i(X, \mathcal{E})$$

It is a standard fact that the Euler character is additive on short exact sequences. Moreover, by Hirzebruch-Riemann-Roch, when $\mathcal{E}$ is locally free the Euler character may be computed in terms of the Chern character

$$\chi(\mathcal{E}) = \int_X \mathrm{ch}(\mathcal{E})\,\mathrm{td}(X)$$

where $\mathrm{td}(X)$ denotes the Todd class of the holomorphic tangent bundle. Define the *Hilbert function* $h(\mathcal{E})$ as

$$h(\mathcal{E}) : d \mapsto \chi(\mathcal{E}(d))$$

By Serre vanishing, this is a polynomial for $d \gg 0$ and the corresponding polynomial $p(\mathcal{E})$ is known as the *Hilbert polynomial*.

Definition 2.1. A coherent sheaf $\mathcal{E}$ is said to be *Gieseker semi-stable* if

$$\frac{p(\mathcal{F})(d)}{\mathrm{rk}(\mathcal{F})} \leq \frac{p(\mathcal{E})(d)}{\mathrm{rk}(\mathcal{E})} \quad \text{for } d \gg 0$$

for every non-trivial subsheaf $\mathcal{F} \subset \mathcal{E}$, and *Gieseker stable* when the strict inequality holds.

While Gieseker stability will be useful in later sections when we define a "twisted" version of the Hilbert function motivated by Hirzebruch-Riemann-Roch, we first want also to define a slightly different notion of stability.

Definition 2.2. Let H be an ample divisor on X, and write $n = \dim(X)$. Define the *slope* of a torsion-free, coherent sheaf $\mathcal{E}$ (with respect to H) by

$$\mu_H(\mathcal{E}) := \frac{\deg(\mathcal{E})}{\mathrm{rk}(\mathcal{E})} = \frac{H^{n-1} \cdot \mathrm{ch}_1(\mathcal{E})}{H^n \cdot \mathrm{ch}_0\,(\mathcal{E})}$$

For any torsion sheaf of with $\mathrm{rk}(\mathcal{E}) = 0$, we set the slope to be $\mu_H(\mathcal{E}) = +\infty$. The sheaf $\mathcal{E}$ is said to be *slope semi-stable* if

$$\mu_H(\mathcal{F}) \leq \mu_H(\mathcal{E})$$

for all subsheaves $\mathcal{F} \subset \mathcal{E}$ of positive rank, and *slope stable* when the strict inequality holds.

For curves, the two notions of stability coincide since $\chi(\mathcal{E}(m)) = \mathrm{rk}(\mathcal{E}) \cdot m + \deg(\mathcal{E})$;

however, for higher dimensional varieties it is only true that slope stability implies Gieseker stability. When the choice of ample divisor H is clear from the context, we will omit the subscript and simply write $\mu(\mathcal{F})$ for the slope of $\mathcal{F}$.

Lemma 2.3. *A sheaf $\mathcal{F}$ is slope stable if and only if*

$$\mu_H(\mathcal{F}) < \mu_H(\mathcal{G})$$

for all non-trivial quotients $\mathcal{F} \twoheadrightarrow \mathcal{G}$.

Proof. Let

$$0 \to \mathcal{E} \to \mathcal{F} \to \mathcal{G} \to 0$$

be a short-exact sequence in $\mathrm{Coh}(X)$. It is easy to see that $\mu_H(\mathcal{E}) < \mu_H(\mathcal{F})$ is equivalent to

$$\deg(\mathcal{F})\,\mathrm{rk}(\mathcal{F}) - \deg(\mathcal{F})\,\mathrm{rk}(\mathcal{E}) < \deg(\mathcal{F})\,\mathrm{rk}(\mathcal{F}) - \deg(\mathcal{E})\,\mathrm{rk}(\mathcal{F})$$

By additivity of $\deg(-)$ and $\mathrm{rk}(-)$ on short exact sequences, it follows that this is equivalent to

$$\mu_H(\mathcal{F}) = \frac{\deg(\mathcal{F})}{\mathrm{rk}(\mathcal{F})} < \frac{\deg(\mathcal{F}) - \deg(\mathcal{E})}{\mathrm{rk}(\mathcal{F}) - \mathrm{rk}(\mathcal{E})} = \mu_H(\mathcal{G})$$

$\square$

Corollary 2.4. *Let $\mathcal{E}$ and $\mathcal{F}$ be semi-stable vector bundles on a smooth projective variety X.*

(i) If $\mu(\mathcal{E}) > \mu(\mathcal{F})$ then $\mathrm{Hom}(\mathcal{E}, \mathcal{F}) = 0$.

(ii) If $\mathcal{E}$ and $\mathcal{F}$ are stable of the same slope, then any morphism $f : \mathcal{E} \to \mathcal{F}$ is either trivial or an isomorphism

Proof. Let $f : \mathcal{E} \to \mathcal{F}$ be non-trivial and $\mathcal{I} = \widetilde{\mathrm{im}(f)}$ denote the coherent image sheaf of f. As a subsheaf of $\mathcal{F}$, we have by definition of stability that $\mu(\mathcal{I}) \le \mu(\mathcal{F})$. By the first isomorphism theorem, $\mathcal{I}$ is isomorphic to a quotient sheaf of $\mathcal{E}$ so that $\mu(\mathcal{E}) \le \mu(\mathcal{I})$ by the previous lemma — thus the first result follows by contrapositive.

For the second statement, assume to the contrary that f is not an isomorphism. Then $\mathcal{K} = \ker(f)$ is a nontrivial subsheaf of $\mathcal{E}$, so by assumption we must have $\mu(\mathcal{K}) < \mu(\mathcal{E})$. By the first isomorphism theorem, $\mathcal{I} \cong \mathcal{E}/\mathcal{K}$ is a subsheaf of $\mathcal{F}$, so by assumption, we have $\mu(\mathcal{F}) = \mu(\mathcal{E}) < \mu(\mathcal{I})$ contradicting the stability of $\mathcal{F}$.

□

Corollary 2.5. *Let X be a smooth projective variety and $\mathcal{E}$ a slope stable vector bundle on X. Then* $\dim \operatorname{Hom}_{\mathcal{O}_X}(\mathcal{E}, \mathcal{E}) = 1$.

Proof. By the previous lemma, any nonzero map $f : \mathcal{E} \to \mathcal{E}$ is an automorphism. Clearly $\Bbbk \cdot \mathrm{id}_{\mathcal{E}} \subset \operatorname{End}(\mathcal{E})$. For reverse inclusion, fix a point $x \in X$ and let λ_x denote the eigenvalue of $f|_{\mathcal{E}(x)}$ (where $\mathcal{E}(x)$ denotes the fibre of $\mathcal{E}$ at the point x). Then $f - \lambda_x \cdot \mathrm{id}_{\mathcal{E}}$ drops rank at the closed point x so that it must be zero everywhere by upper-semicontinuity. □

Definition 2.6. Let $\mathcal{F}$ be a coherent sheaf on a smooth projective variety X with Chern classes $c_i(\mathcal{F})$ and positive rank. Define the *discriminant* of $\mathcal{F}$ to be the characteristic class

$$\Delta(\mathcal{F}) = \frac{c_2(\mathcal{F})}{\operatorname{rk}(\mathcal{F})} - \frac{(\operatorname{rk}(\mathcal{F}) - 1)c_1^2(\mathcal{F})}{2\operatorname{rk}(\mathcal{F})^2}$$

It is worth noting that there are multiple versions of the discriminant class in literature, which typically vary by a multiple or power of r. For example, by instead taking the class

$$2\operatorname{rk} \cdot c_2 - (\operatorname{rk} - 1)c_1^2 = \operatorname{ch}_1^2 - 2\operatorname{ch}_0 \operatorname{ch}_2$$

we are able to extend the definition to general torsion sheaves. However, for the sake of consistency with classical constructions (e.g. the Drézet-Le Potier curve) in later chapters, we will almost exclusively use the first definition.

Theorem 2.7 (Bogomolov-Gieseker Inequality)**.** *Let X be a smooth projective surface and $\mathcal{F}$ be a torsion-free, slope-stable sheaf on X. Then*

$$\Delta(\mathcal{F}) \geq 0$$

More generally, when X is a smooth projective n-dimensional variety with ample divisor

H

$$\Delta(\mathcal{F}) \cdot H^{n-2} \geq 0$$

2.2 t-structures, Tiltings, and Slicings

Generalizing the notion of stability to the derived category of a smooth projective variety initially presents significant hurdles since the objects are no longer vector bundles but bounded complexes of sheaves. However, by viewing vector bundles as complexes whose cohomology vanishes in positive and negative degrees, one can generalize the abelian category on which a stability function is defined. We first recall the following concepts, as originally introduced by [BBD82].

Definition 2.8. A *t-structure* on $\mathcal{D}$ is a full subcategory $\mathcal{T} \subset \mathcal{D}$ [2] satisfying:

(1) $\mathcal{T}[1] \subset \mathcal{T}$

(2) Every object $E \in \mathcal{D}$ fits into a distinguished triangle

$$A \to E \to B \to A[1]$$

where $A \in \mathcal{T}$ and $B \in \mathcal{T}^{\perp} := \{X \in \mathcal{D} \mid \mathrm{Hom}^{\bullet}_{\mathcal{D}}(T, X) = 0 \text{ for all } T \in \mathcal{T}\}$.

Furthermore, the t-structure $\mathcal{T}$ is said to be *bounded* if for every $E \in \mathcal{D}$ there exists some $a, b \in \mathbb{Z}$ such that $E \in \mathcal{T}[a] \cap \mathcal{T}^{\perp}[b]$. In this case the category $\mathcal{A} := \mathcal{T} \cap \mathcal{T}^{\perp}[1]$ is said to be the *heart* of the bounded t-structure.

In some references, a bounded t-structure is instead represented as a pair of full subcategories $(\mathcal{D}^{\leq 0}, \mathcal{D}^{\geq 0})$. However, it is worth noting that the second subcategory $\mathcal{D}^{\geq 0}$ is uniquely determined by the first — one can simply take $\mathcal{T} = \mathcal{D}^{\leq 0}$ to obtain $\mathcal{D}^{\geq 0} := \mathcal{T}^{\perp}[1]$, in which case the heart is conveniently written as $\mathcal{D}^{\leq 0} \cap \mathcal{D}^{\geq 0}$.

Lemma 2.9. *Let $\mathcal{T} \subset \mathcal{D}$ be a bounded t-structure and $A \to B \to C \to A[1]$ a distinguished triangle in $\mathcal{D}$. If A and C are in $\mathcal{T}$, then so is B.*

[2] It should be noted that the full subcategory $\mathcal{T}$ is almost never triangulated, since it is not invariant under negative shifts.

Proof. Let $\tau^{\geq 1} : \mathcal{D} \to \mathcal{T}^{\perp}$ denote the left adjoint to the inclusion functor. By definition of $\mathcal{T}^{\perp}$, we have that $\mathrm{Hom}^{\bullet}(A, \tau^{\geq 1}B) = \mathrm{Hom}^{\bullet}(C, \tau^{\geq 1}B) = 0$ so by applying $\mathrm{Hom}^{\bullet}(-, \tau^{\geq 1}B)$ to the triangle $A \to B \to C \to A[1]$ we obtain $\mathrm{Hom}^{\bullet}(B, \tau^{\geq 1}B) = 0$ which implies $\tau^{\geq 1}B = 0$. $\square$

An identical argument to that above shows that if A and C are in $\mathcal{T}^{\perp}[1]$ then so is B. It immediately follows that $\mathcal{A} = \mathcal{T} \cap \mathcal{T}^{\perp}[1]$ is an additive category by considering the distinguished triangles $A \to A \oplus B \to B \to A[1]$ for any $A, B \in \mathcal{A}$. It is in fact well-known that $\mathcal{A}$ carries significantly more structure:

Lemma 2.10. *The heart of a bounded t-structure is an abelian category.*

For a detailed proof of the above fact, we refer the reader to either [KS90, Proposition 10.1.11] or [MS17, Lemma 5.2] — a key detail is that a morphism $f : A \to B$ between two objects in the heart $\mathcal{A}$ is an inclusion if $\mathrm{cone}(f) \in \mathcal{A}$, and a surjection if $\mathrm{cone}(f) \in \mathcal{A}[1]$.

Lemma 2.11 ([Bri07, Lemma 3.2])**.** *Let $\mathcal{A} \subset \mathcal{D}$ be a full additive subcategory. Then $\mathcal{A}$ is the heart of a bounded t-structure if and only if*

(1) If $k_1 > k_2$ then $\mathrm{Hom}^{\bullet}_{\mathcal{D}}(A[k_1], B[k_2]) = 0$ for any $A, B \in \mathcal{A}$.

(2) For every $0 \neq E \in \mathcal{D}$ there exists a finite decreasing sequence of integers

$$k_1 > k_2 > \cdots > k_n$$

and collection of triangles

$$\begin{tikzcd}
0 = E_0 \arrow[r] & E_1 \arrow[r] \arrow[d] & E_2 \arrow[r] \arrow[d] & \cdots \arrow[r] & E_{n-1} \arrow[r] \arrow[d] & E_n = E \arrow[d] \\
 & A_1 \arrow[ul, dashed] & A_2 \arrow[ul, dashed] & \cdots \arrow[ul, dashed] & A_{n-1} \arrow[ul, dashed] & A_n \arrow[ul, dashed]
\end{tikzcd}$$

with $A_i \in \mathcal{A}[k_i]$.

The objects A_i obtained in the filtration above are often referred to as the *cohomology objects* of E (with respect to the t-structure $\mathcal{T}$), written as $\mathcal{H}^{-k_i}_{\mathcal{A}}(E) := A_i[-k_i]$. In addition, the second condition tells us that there is an isomorphism $K(\mathcal{D}) \cong K(\mathcal{A})$ since each class $[E] \in K(\mathcal{D})$ can now be expanded as $[A_1]+\cdots+[A_n]$ using the above filtration.

It is also easy to see that a bounded t-structure is uniquely determined by its heart: given a heart $\mathcal{A} \subset \mathcal{D}$, we may simply take $\mathcal{T} := \bigcup_{i\geq 0} \mathcal{A}[i]$ to reobtain $\mathcal{A} = \mathcal{T} \cap \mathcal{T}^{\perp}[1]$.

Lemma 2.12. *Let $\mathcal{T}, \mathcal{T}'$ be bounded t-structures on $\mathcal{D}$ admitting hearts $\mathcal{A}, \mathcal{A}'$, respectively. If $\mathcal{A} \subset \mathcal{A}'$ then $\mathcal{A} = \mathcal{A}'$.*

Proof. Fix some $F \in \mathcal{T}$. By the previous lemma, there exists a filtration by cohomology objects $A_i \in \mathcal{A}[k_i]$; we first wish to show that $k_i \geq 0$. It suffices to show that $k_n \geq 0$; assume without loss of generality that $k_n = -1$ so that $A_n \in \mathcal{A}[-1] = \mathcal{T}[-1] \cap \mathcal{T}^{\perp}$. However, since $A_n \in \mathcal{T}^{\perp}$ the map $F \to A_n$ is trivial so that $E_{n-1} \cong E_n$. An identical argument shows that for any object $F \in \mathcal{T}^{\perp}[1]$, one must have $k_i \leq 0$ in the cohomology filtration.

Next fix some $E \in \mathcal{A}'$. By the previous paragraph, any $F \in \mathcal{T}$ has a finite filtration by cohomology objects in $\mathcal{A}[k_i] \subset \mathcal{A}'[k_i]$ for $k_i \geq 0$. By part (i) of the previous lemma, this implies $\mathrm{Hom}_{\mathcal{D}}(F, E[-1]) = 0$ so that $E \in \mathcal{T}^{\perp}[1]$. Similarly, any object $G \in \mathcal{D}^{\perp}$ has a cohomology filtration by objects in $\mathcal{A}[k_i] \subset \mathcal{A}'[k_i]$ with $k_i < 0$ so that $\mathrm{Hom}_{\mathcal{D}}(E, G) = 0$. Thus $E \in {}^{\perp}(\mathcal{T}^{\perp}) = \mathcal{T}$ implying $E \in \mathcal{A} = \mathcal{T} \cap \mathcal{T}^{\perp}[1]$. □

Remark 2.13. It is *not* true in general that there is an equivalence $\mathcal{D} \cong D^{\flat}(\mathcal{A})$ whenever $\mathcal{A}$ is the heart of a bounded t-structure on $\mathcal{D}$. A straightforward example is given in [MS17, Exercise 5.3], where one can show that $\mathcal{A} := \langle \mathcal{O}_{\mathbb{P}^1}[2], \mathcal{O}_{\mathbb{P}^1}(1)\rangle$ is the heart of a bounded t-structure on $\mathbb{P}^1$; however, as the objects $\{\mathcal{O}_{\mathbb{P}^1}, \mathcal{O}_{\mathbb{P}^1}(1)\}$ form strong exceptional collection (see Definition 2.36), the derived category of $\mathcal{A}$ is equivalent to two copies of the derived category of a point.

For the standard example of the bounded derived category of an abelian category $\mathcal{A}$, there is a natural t-structure coming from the obvious grading on the objects of $D^{\flat}(\mathcal{A})$ as complexes: $\mathcal{D}^{\leq 0} = \{E \in \mathcal{D} \mid H^i(E) = 0 \text{ for } i > 0\}$ where $H^i(E)$ denotes the cohomology of the complex E. However, producing other examples of t-structures — and thus hearts — turns out to be a non-trivial task; one of the main techniques was discovered [HRS96] through the process of *tilting*.

Definition 2.14. Given an abelian category $\mathcal{A}$, a *torsion pair* on $\mathcal{A}$ is a pair of full additive subcategories $(\mathcal{T}, \mathcal{F})$ satisfying the following:

(1) $\mathrm{Hom}_{\mathcal{A}}(T, F) = 0$ for all $F \in \mathcal{F}$ and $T \in \mathcal{T}$.

(2) For all $E \in \mathcal{A}$, there exists a short exact sequence

$$0 \to T \to E \to F \to 0$$

where $T \in \mathcal{T}$ and $F \in \mathcal{F}$.

Proposition 2.15 ([HRS96, Proposition 2.1])**.** *Let $\mathcal{A} \subset \mathcal{D}$ be the heart of a bounded t-structure and $(\mathcal{T}, \mathcal{F})$ a torsion pair on $\mathcal{A}$. Then*

$$\mathcal{A}^\sharp := \{E \in \mathcal{D} \mid \mathcal{H}^{-1}_{\mathcal{A}}(E) \in \mathcal{F},\ \mathcal{H}^0_{\mathcal{A}}(E) \in \mathcal{T},\ \mathcal{H}^i_{\mathcal{A}}(E) = 0 \text{ otherwise}\}$$

is the heart of a bounded t-structure on $\mathcal{D}$.

The heart $\mathcal{A}^\sharp$ obtained from the proposition above is often referred to as the *tilt* of $\mathcal{A}$ at the torsion pair $(\mathcal{T}, \mathcal{F})$. In a certain sense, the new objects of the tilted heart can be thought of as extensions of $\mathcal{T}$ by $\mathcal{F}[1]$, while the second axiom of a torsion pair implies objects of the original heart $\mathcal{A}$ are extensions of $\mathcal{F}$ by $\mathcal{T}$.

In light of Lemma 2.11 above, one can refine the notion of the heart of a bounded t-structure by further splitting the heart into a collection of full subcategories.

Definition 2.16. A *slicing* of $\mathcal{D}$ is a collection of full additive subcategories $\{\mathcal{P}(\phi)\}_{\phi\in\mathbb{R}}$ such that:

(1) $\mathcal{P}(\phi + 1) = \mathcal{P}(\phi)[1]$

(2) If $\phi_1 > \phi_2$ and $A_i \in \mathcal{P}(\phi_i)$ for $i = 1, 2$, then $\mathrm{Hom}_{\mathcal{D}}(A_1, A_2) = 0$.

(3) For each nonzero $E \in \mathcal{D}$, there exists a decreasing sequence of real numbers

$$\phi_1 > \phi_2 > \cdots > \phi_n$$

and a sequence of distinguished triangles

$$\begin{tikzcd}
0 = E_0 \arrow[r] & E_1 \arrow[r] \arrow[d] & E_2 \arrow[r] \arrow[d] & \cdots \arrow[r] & E_{n-1} \arrow[r] \arrow[d] & E_n = E \arrow[d] \\
& A_1 \arrow[ul, dashed] & A_2 \arrow[ul, dashed] & \cdots \arrow[ul, dashed] & A_{n-1} \arrow[ul, dashed] & A_n \arrow[ul, dashed]
\end{tikzcd}$$

with $A_i \in \mathcal{P}(\phi_i)$.

The filtration obtained from condition (3) is often referred to as the *Harder-Narasimhan filtration* of E (with respect to $\mathcal{P}$), and the A_i are called the *Harder-Narasimhan factors* of E. Additionally, given a nonzero object $E \in \mathcal{D}$, it will be convenient to use $\phi^+_{\mathcal{P}}(E) := \phi_1$ and $\phi^-_{\mathcal{P}}(E) := \phi_n$ to refer to the maximal and minimal phases with respect to the Harder-Narasimhan factors of E. When the slicing is clear from context (i.e. in the case of a fixed Bridgeland stability condition), the subscript will be omitted.

Before concluding this section, it will be useful to introduce some additional notation. Given an interval I, let $\mathcal{P}(I)$ denote the extension-closed subcategory generated by objects of $\mathcal{P}(\phi)$ for any $\phi \in I$. We will say that a slicing $\mathcal{P}$ is *locally-finite* if there exists some uniform $\epsilon > 0$ such that the categories $\mathcal{P}(\phi - \epsilon, \phi + \epsilon) \subset \mathcal{D}$ are of finite-length for all $\phi \in \mathbb{R}$.

Remark 2.17. The technicality of local-finiteness will be revisited in Chapter 4 when considering how 'nice' slicings deform into degenerate cases. For example, fix some w in the strict upper half-plane and $t > 0$; moreover, for each $n \in \mathbb{Z}$, define $\phi_{n,t} = \mathrm{Arg}(w - n/t) \in (0, \pi)$. We may constructively define a slicing $\mathcal{P}_w$ on $D^\flat(\mathbb{P}^1)$ by $\mathcal{P}_w(\phi_{n,t}/\pi) = \langle \mathcal{O}_{\mathbb{P}^1}(n) \rangle$ and extend in such a way that $\mathcal{P}_w((0,1]) = \mathrm{Coh}(\mathbb{P}^1)$. However, as we take $t \to \infty$ all $\phi_{n,t}$ will converge to $\overline{\phi} = \mathrm{Arg}(w)$ so that $\mathcal{P}((0,1]) = \mathcal{P}(\overline{\phi}) = \mathrm{Coh}(\mathbb{P}^1)$ — that is, the entire ablian category is concentrated at a single phase. As we will see in the next section, this is an edge case we will want to avoid to guarantee that certain decompositions into 'simple' objects always yield finite filtrations.

Theorem 2.18 ([Bri07, Section 6]). *Denote the space of locally-finite slicings on a triangulated category $\mathcal{D}$ as* $\mathrm{Slice}(\mathcal{D})$. *Then there exists a generalized metric on* $\mathrm{Slice}(\mathcal{D})$

defined by

$$d_{\text{Slice}}(\mathcal{P}, \mathcal{Q}) := \sup_{0\neq E\in\mathcal{D}} \left\{ |\phi^+_{\mathcal{P}}(E) - \phi^+_{\mathcal{Q}}(E)|, |\phi^-_{\mathcal{P}}(E) - \phi^-_{\mathcal{Q}}(E)| \right\}$$

2.3 Bridgeland Stability Conditions

In the previous section slope stability was defined using the ratio of two functions on $\mathrm{Coh}(C)$ which are additive on short exact sequences. It is easy to see that, up to rescaling, this is equivalent to the data of a complex-valued function $Z = -\deg + i\,\mathrm{rk}$ defined on the Grothendieck group $K(\mathrm{Coh}(X))$ by taking $\mu_H = -\Re Z/\Im Z$. With this precedent, we recall the construction of stability conditions on a triangulated category $\mathcal{D}$ as introduced in [Bri07].

Definition 2.19. A *stability condition* on $\mathcal{D}$ is a pair $\sigma = (Z, \mathcal{P})$ consisting of a group homomorphism $Z : \Lambda \to \mathbb{C}$ (known as the *central charge*) together with a slicing $\{\mathcal{P}(\phi)\}_{\phi\in\mathbb{R}}$, which are compatible in the sense that

(1) If $0 \neq E \in \mathcal{P}(\phi)$ then $Z(E) = m(E)\exp(i\pi\phi)$ for some $m(E) \in \mathbb{R}_{>0}$.

In this case, the nonzero objects of $\mathcal{P}(\phi)$ are said to be *semistable* of phase ϕ, and the simple objects are said to be *stable*.

Proposition 2.20 ([Bri07, Proposition 5.3])**.** *Giving a stability condition $\sigma = (Z, \mathcal{P})$ on $\mathcal{D}$ is equivalent to giving the heart of a bounded t-structure $\mathcal{A} \subset \mathcal{D}$ together with a group homomorphism $\widetilde{Z} : K(\mathcal{A}) \to \mathbb{C}$ satisfying:*

(1) For every non-zero $E \in \mathcal{A}$, $\widetilde{Z}(E)$ lies in the half-closed upper half plane $\overline{\mathbb{H}}$. In addition, $\Re(\widetilde{Z}(E)) < 0$ whenever $\Im(\widetilde{Z}(E)) = 0$.

(2) (Harder-Narashiman Property) Let $\mu_{\widetilde{Z}} = -\Im\widetilde{Z}/\Re\widetilde{Z}$ denote the generalized slope function induced by $\widetilde{Z}$, where we again adopt the convention $\mu_{\widetilde{Z}} = +\infty$ whenever $\Re\widetilde{Z} = 0$. Then for every nonzero $E \in \mathcal{A}$ there exists a finite chain of subobjects

$$0 = E_0 \subset E_1 \subset \cdots \subset E_{n-1} \subset E_n = E$$

such that $F_i = E_i/E_{i-1}$ are $\mu_{\widetilde{Z}}$-semistable (in the sense that there do not exist any subobjects with strictly larger $\mu_{\widetilde{Z}}$) and

$$\mu_{\widetilde{Z}}(F_1) > \mu_{\widetilde{Z}}(F_2) > \cdots > \mu_{\widetilde{Z}}(F_n).$$

The filtration obtained in condition (2) is also (as one would expect) referred to as the *Harder-Narasimhan filtration*, and the objects F_i are referred to as the Harder-Narasimhan factors. It is shown in [Bri07, Proposition 2.4] and [MS17, Proposition 4.10] that the second condition almost always holds in Noetherian categories under some additional mild assumptions. We refer the interested reader to the original paper for a complete proof; however, it is important to reiterate several key details since we will be switching between notations somewhat freely in the sequel. By taking the bounded t-structure $\mathcal{P}((0,\infty)) \subset \mathcal{D}$, it is easy to see that $\mathcal{A}^\sharp := \mathcal{P}((0,1])$ is the heart obtained. Conversely, one may define a slicing $\mathcal{P}$ on $\mathcal{D}$ by setting $\mathcal{P}(\phi)$ to be the full subcategory consisting of semi-stable objects of phase ϕ (with respect to $\widetilde{Z}$).

Lemma 2.21. *The Harder-Narasimhan filtration for any nonzero object is unique up to isomorphism.*

The proof of the above lemma is nearly identical to the usual induction argument given for slope stability (see, for example, [Rei08, Lemma 4.7]) along with the octahedral axiom. As a result , we can construct several well-defined real value functions on the non-zero objects of $\mathcal{D}$ using their Harder-Narasimhan filtrations.

Definition 2.22. Given a stability condition $\sigma = (Z, \mathcal{P})$ on $\mathcal{D}$, define the *mass* $m_\sigma(E)$ of any nonzero object E as

$$m_\sigma(E) \coloneqq \sum_{i=1}^{n} |Z(A_i)|$$

where A_i are the Harder-Narasimhan factors of E.

Remark 2.23. The condition $m_\sigma(E) = |Z(E)|$ does not necessarily imply that the object E is σ-semistable. Indeed, for any σ-semistable object E we can consider the destabilizing sequence

$$E[2] \to E \oplus E[2] \to E$$

so that $m_\sigma(E\oplus E[2]) = |Z(E[2])|+|Z(E)| = |Z(E\oplus E[2])|$. A standard triangle-inequality argument in fact shows that $m_\sigma(E) = |Z(E)|$ only implies the Harder-Narasimhan factors A_i must satisfy $\phi(A_i) - \phi(A_j) \in 2\mathbb{Z}$.

One of the main results of [Bri07] is that the "compatibility" between the slicing and central charge of a stability condition allows local deformations of the central charge to induce deformations of the stability condition itself. To make this precise, let $\mathrm{Stab}_\Lambda(\mathcal{D})$ denote the space of locally-finite [3] stability conditions on $\mathcal{D}$, and consider the forgetful map

$$\mathcal{Z} : \mathrm{Stab}_\Lambda(\mathcal{D}) \subset \mathrm{Slice}(\mathcal{D}) \times \mathrm{Hom}(\Lambda, \mathbb{C}) \to \mathrm{Hom}(\Lambda, \mathbb{C}).$$

By [Bri07, Theorem 1.2], the restriction of $\mathcal{Z}$ to any connected component of $\mathrm{Stab}_\Lambda(\mathcal{D})$ is a local homeomorphism onto some linear subspace $\mathcal{V} \subseteq \mathrm{Hom}(\Lambda, \mathbb{C})$, thus endowing $\mathrm{Stab}_\Lambda(\mathcal{D})$ with a well-defined linear topology. Consequently, each connected component of $\mathrm{Stab}_\Lambda(\mathcal{D})$ has the structure of a complex manifold of dimension $\mathrm{rk}(\Lambda)$. Bridgeland showed in the same article that

$$d(\sigma_1, \sigma_2) := \sup_{0\neq E\in\mathcal{D}} \left\{ |\phi^\pm_{\mathcal{P}_1}(E) - \phi^\pm_{\mathcal{P}_2}(E)|,\ |\log \frac{m_{\sigma_1}(E)}{m_{\sigma_2}(E)}| \right\}$$

(where $\mathcal{P}_i$ denotes the slicing of σ_i) defines a generalized metric structure on $\mathrm{Stab}_\Lambda(\mathcal{D})$ inducing the same topology. It was pointed out by [Huy12] that on each connected component of $\mathrm{Stab}(\mathcal{D})$ the above generalized metric is simply equivalent to the product metric

$$d(\sigma_1, \sigma_2) = \max\{d_{\mathrm{Slice}}(\mathcal{P}_1, \mathcal{P}_2), |Z_1 - Z_2|\}.$$

By construction, this is equivalent to giving $\mathrm{Stab}_\Lambda(\mathcal{D})$ the weakest topology in which $\phi^\pm(E)$ and $m(E)$ are continuous for fixed $0 \neq E \in \mathcal{D}$. Under the additional mild assumption of fullness (i.e. that the linear subspace $\mathcal{V}$ has maximal dimension), [Woo12] also showed that each connected component of $\mathrm{Stab}_\Lambda(\mathcal{D})$ is a complete metric space.

[3] Local-finiteness guarantees that every semistable object $E \in \mathcal{P}(\phi)$ has only finitely many subobjects, and thus has a finite Jordan-Hölder filtration with stable factors of the same phase.

Theorem 2.24 ([Woo12, Theorem 3.6]). *Let* $\mathrm{Stab}^*(\mathcal{D})$ *be a connected component of* $\mathrm{Stab}_\Lambda(\mathcal{D})$ *with associated linear space* $\mathcal{V}$. *If* $\mathcal{V}$ *has maximal dimension then* $\mathrm{Stab}^*(\mathcal{D})$ *is complete with respect to the metric* d. *In fact, the limit point* $\sigma_\infty = (\mathcal{P}_\infty, Z_\infty)$ *of a Cauchy sequence* $\{\sigma_n\} \subset \mathrm{Stab}^*(\mathcal{D})$ *is given by:*

$$Z_\infty := \lim_{n\to\infty} Z_n,$$
$$\mathcal{P}_\infty(\phi) := \langle 0 \neq E \in \mathcal{D} \mid \lim_{n\to\infty} \phi^{\pm}_{\sigma_n}(E) = \phi \rangle \cup \{0\}.$$

Many of the technical conditions, such as local-finiteness and fullness, are often implicitly assumed for stability conditions by requiring an additional axiom known as the 'support property' (as introduced in [KS08]). Fixing a norm $\|\cdot\|$ on $\Lambda_\mathbb{R} := \Lambda \otimes \mathbb{R}$ [4], a stability condition $\sigma = (Z, \mathcal{P})$ is said to have the *support property* if there exists a constant $C_\sigma > 0$ such that

$$\|\nu(E)\| \leq C_\sigma |Z(\nu(E))|$$

for all semi-stable objects E (i.e. $0 \neq E \in \mathcal{P}(\phi)$ for some ϕ). Though a seemingly small detail, it was shown by [KS08], [BM11, Appendix B] that the support property ensures the following:

(1) When $K(\mathcal{D})$ has finite rank, the stability condition $\sigma = (Z, \mathcal{P})$ is locally-finite,

(2) Any connected component $\mathrm{Stab}^*(\mathcal{D}) \subset \mathrm{Stab}(\mathcal{D})$ containing a stability condition satisfying the support property is of maximal dimension (i.e. full). This additionally guarantees that wall crossing is well-behaved in the sense of [Bri08, Section 9].

(3) The set $\{Z(E) \in \mathbb{C} \mid E \text{ is semistable}\}$ is a discrete subset of $\mathbb{C}$ with at most polynomial growing density at infinity [KS08, Section 2].

Thus, in most recent literature, the definition given above in Definition 2.19 is known as a *pre-stability condition* and is not called a stability condition unless the support property is met.

Remark 2.25. The support property as given above is equivalent to the existence of a

[4] As all norms on finite dimensional vector spaces are equivalent, the choice of norm $\|\cdot\|$ on $\Lambda_\mathbb{R}$ is irrelevent.

symmetric bilinear form Q on $\Lambda_{\mathbb{R}}$ that is positive semidefinite on all semistable objects and negative definite on the kernel of $Z_{\mathbb{R}}$.

Example 2.26. Let C be a smooth projective curve. Take $\Lambda = (H^0 \oplus H^2)(X, \mathbb{Z})$ together with the surjective homomorphism $\nu(-) = \mathrm{ch}(-) = (\mathrm{rk}(-), \deg(-))$. Slope stability in this case corresponds to the slope function $Z = -\deg(E) + i \cdot \mathrm{rk}(E)$ with heart $\mathcal{A} = \mathrm{Coh}(X)$. In this case we may simply take $Q = 0$ to satisfy the support property.

It should be pointed out that standard slope stability only defines a Bridgeland stability condition on $\mathrm{Coh}(X)$ in the case of $\dim X = 1$. For higher-dimensional varieties, torsion sheaves supported in codimension ≥ 2 will always map to the origin. In fact, it was shown by [YT08] that for $\dim(X) \geq 2$ there never exist stability conditions with heart $\mathcal{A} = \mathrm{Coh}(X)$ whose central charge factors through the Chern character — this will be addressed in subsection 2.3.1.

We next recall the two natural actions on $\mathrm{Stab}_{\Lambda}(\mathcal{D})$: first, let $\widetilde{\mathrm{GL}}_2^+(\mathbb{R})$ denote the universal cover of $\mathrm{GL}_2^+(\mathbb{R})$. Recall the presentation

$$\widetilde{\mathrm{GL}}_2^+(\mathbb{R}) = \left\{ (M, f) \;\middle|\; \begin{array}{l} f : \mathbb{R} \to \mathbb{R} \text{ increasing}, \\ f(\phi + 1) = f(\phi) + 1 \\ M \in \mathrm{GL}_2^+(\mathbb{R}) \\ M \exp(i\pi\phi) \in \mathbb{R}_{>0} \exp(i\pi f(\phi)) \end{array} \right\}.$$

Then $\widetilde{\mathrm{GL}}_2^+(\mathbb{R})$ acts on the right of $\mathrm{Stab}(\mathcal{D})$ via

$$(Z, \mathcal{P}) \cdot (M, f) = (M^{-1} \circ Z, \mathcal{P}')$$

where $\mathcal{P}'(\phi) = \mathcal{P}(f(\phi))$.

Second, note that $\Phi \in \mathrm{Aut}(\mathcal{D})$ induces a group automorphism $\Phi_* \in \mathrm{Aut}(K(\mathcal{D}))$. We say $\Phi \in \mathrm{Aut}(\mathcal{D})$ is compatible with $\nu : K(\mathcal{D}) \to \Lambda$ if there exists an automorphism

$\phi : \Lambda \to \Lambda$ such that the following diagram commutes

$$\begin{tikzcd} K(\mathcal{D}) \arrow[r, "\Phi_*"] \arrow[d, "\nu"] & K(\mathcal{D}) \arrow[d, "\nu"] \\ \Lambda \arrow[r, "\phi"] & \Lambda \end{tikzcd}.$$

Letting $\mathrm{Aut}_\Lambda(\mathcal{D})$ denote the collection of pairs (Φ, ϕ) that are compatible with $\nu : K(\mathcal{D}) \to \Lambda$, there is a left action of $\mathrm{Aut}_\Lambda(\mathcal{D})$ on $\mathrm{Stab}_\Lambda(\mathcal{D})$ via

$$\Phi \cdot (Z, \mathcal{P}) = (Z \circ \phi^{-1},\, \Phi(\mathcal{P})).$$

It will be useful in the sequel to describe how the actions of $\widetilde{\mathrm{GL}}_2^+(\mathbb{R})$ and $\mathrm{Aut}_\Lambda(\mathcal{D})$ interact with the functions $\phi_\sigma^\pm$ and m_σ.

Lemma 2.27. *Fix $\sigma \in \mathrm{Stab}_\Lambda(\mathcal{D})$. Then for any $\Phi \in \mathrm{Aut}_\Lambda(\mathcal{D})$ and nonzero $E \in \mathcal{D}$*

(i) $\phi^\pm_{\Phi\cdot\sigma}(\Phi(E)) = \phi^\pm_\sigma(E)$,

(ii) $m_{\Phi\cdot\sigma}(\Phi(E)) = m_\sigma(E)$,

Additionally, for $(M, f) \in \widetilde{\mathrm{GL}}_2^+(\mathbb{R})$ we have $\phi^\pm_{\sigma\cdot(M,f)}(E) = f(\phi^\pm_\sigma(E))$.

Proof. Let $\{A_i\}_{i=1}^n$ denote the Harder-Narasimhan factors of E with respect to $\sigma = (Z, \mathcal{P})$, so that $A_i \in \mathcal{P}(\phi_i)$ for some $\phi_1 > \cdots > \phi_n$. Since Φ is exact we obtain a sequence of triangles

$$\begin{tikzcd}[column sep=small] 0 = \Phi(E_0) \arrow[rr] & & \Phi(E_1) \arrow[rr] \arrow[d] & & \Phi(E_2) \arrow[r] \arrow[d] & \dots \arrow[rr] & & \Phi(E_n) = \Phi(E) \arrow[d] \\ & & \Phi(A_1) \arrow[ull, dashed] & & \Phi(A_2) \arrow[ull, dashed] & & & \Phi(A_n) \arrow[ull, dashed] \end{tikzcd}$$

with $\Phi(A_i) \in \Phi(\mathcal{P})(\phi_i)$; by uniqueness, $\{\Phi(A_i)\}$ must be the Harder-Narasimhan filtration of $\Phi(E)$ with respect to $\Phi \cdot \sigma$ so that (1) immediately follows. The second result follows from the fact that the central charge of $\Phi \cdot \sigma$ is given by $Z \circ \phi^{-1}$ (where ϕ is the automorphism of Λ compatible with Φ).

The last result follows from the fact that $\widetilde{\mathrm{GL}}_2^+(\mathbb{R})$ does not change the semistable objects of a stability condition, but merely relabels the phase via the increasing function f ([Bri07, Lemma 8.2]). $\square$

Definition 2.28. A stability condition $\sigma = (Z, \mathcal{P})$ is said to be *degenerate* if the image of the central charge is contained in a real line, and *non-degenerate* otherwise.

Following the notation of [Li17], we will refer to the subset of non-degenerate stability conditions as $\mathrm{Stab}^{\mathrm{nd}}(\mathcal{D}) \subset \mathrm{Stab}_\Lambda(\mathcal{D})$.

Remark 2.29. The action of $\widetilde{\mathrm{GL}}_2^+(\mathbb{R})$ on $\mathrm{Stab}_\Lambda(\mathcal{D})$ is not free whenever $\mathrm{Stab}^{\mathrm{nd}}(\mathcal{D}) \neq \mathrm{Stab}_\Lambda(\mathcal{D})$. However, the subgroup $\mathbb{C} \hookrightarrow \widetilde{\mathrm{GL}}_2^+(\mathbb{R})$ acts freely on the entirety $\mathrm{Stab}_\Lambda(\mathcal{D})$ via

$$\zeta \cdot (Z, \mathcal{P}) = (\exp(\zeta) Z, \mathcal{P}'),$$

where $\mathcal{P}'(\phi) = \mathcal{P}(\phi - \Im(\zeta)/\pi)$. In particular, when $\zeta = n\pi i$ for $n \in \mathbb{Z}$ the action of ζ is identical to the action of the shift functor $[n]$. This action has closed orbits and no stabilizers, yielding a well-behaved quotient — in particular, it preserves the connected components of $\mathrm{Stab}_\Lambda(\mathcal{D})$ and respects the metric on $\mathrm{Stab}_\Lambda(\mathcal{D})$ so that $\mathrm{Stab}_\Lambda(\mathcal{D})/\mathbb{C}$ is also a complete complex manifold.

For later sections, it will also be useful to make the notion of *walls* and *chambers* in the stability manifold precise.

Definition 2.30. Fix a connected component $\mathrm{Stab}^*(\mathcal{D}) \subset \mathrm{Stab}_\Lambda(\mathcal{D})$ and let $w, v \in \Lambda$ be linearly-independent nonzero vectors. Define the numerical wall for v with respect to w as

$$\mathcal{W}_w(v) = \{\sigma = (\mathcal{P}, Z) \in \mathrm{Stab}^*(\mathcal{D}) \mid \mu_Z(v) = \mu_Z(w)\ \}$$

(where $\mu_Z = -\Im Z / \Re Z$).

As shown in Bridgeland's original article, every numerical wall is a submanifold of real codimension one. In particular, when there exists some $\phi \in \mathbb{R}$ and inclusion $E \hookrightarrow F$ in $\mathcal{P}(\phi)$ with $\nu(E) = w, \nu(F) = v$, we refer to $\mathcal{W}_w(v)$ as an *actual wall* for the object $E \in \mathcal{D}$. Bridgeland originally showed that, for K3 surfaces, the actual walls effectively control the regions — known as chambers — where an object remains stable. The result was later generalized to any finite-dimensional stability manifold by [BM11]:

Proposition 2.31 ([BM11, Proposition 3.3])**.** *Assume $K(\mathcal{D})$ is finite-dimensional and $v \in \Lambda$ is a primitive element. Let $\mathcal{S}$ be a set of objects in $\mathcal{D}$ with $\nu(S) = v$ for all $S \in \mathcal{S}$. Then there is a set of actual walls $\mathcal{W}_w^{\mathcal{S}}(v)$ in a connected component $\operatorname{Stab}^*(\mathcal{D}) \subset \operatorname{Stab}(\mathcal{D})$ such that:*

(i) The set $\mathcal{W}_w^{\mathcal{S}}(v)$ is locally-finite in the sense that every compact set $K \subset \operatorname{Stab}^*(\mathcal{D})$ intersects only finitely many walls.

(ii) If $\mathcal{C} \subset \operatorname{Stab}^*(\mathcal{D})$ is a connected component in the complement of $\bigcup_{w\in\Lambda} \mathcal{W}_w^{\mathcal{S}}(v)$ and $\sigma_1, \sigma_2 \in \mathcal{C}$, then any object of $\mathcal{S}$ is σ_1-stable if and only if it is σ_2-stable.

2.3.1 Stability Conditions on Surfaces

While stability manifolds are defined for arbitrary triangulated categories, significant motivation in physics and geometry stems from the derived category of a smooth projective variety. In his introductory paper [Bri07], Bridgeland computed the first example of a stability manifold by taking the triangulated category to be $D^\flat(X)$ where X is an elliptic curve; the $\dim X = 1$ case was then later extended to $\mathbb{P}^1$ by Okada [Oka06] and, more generally, all smooth projective curves in [Mac07a]. Unfortunately, the explicit construction of stability conditions in one dimension higher becomes significantly more non-trivial — for example, it was noted above that it is impossible to define a slope function on $\operatorname{Coh}(X)$ that factors through the Chern character. In light of this, we will briefly spend this chapter summarizing the general approach for constructing stability conditions on (smooth projective) surfaces, as given in [ABL07].

For the first part of this section, let X denote a smooth projective surface over $\mathbb{C}$, and fix $\omega, B \in \operatorname{NS}_{\mathbb{R}}(X)$ where ω is an ample class. For any $E \in D^\flat(X)$, we define the B-twisted slope function $\mu_{\omega,B}$ by

$$\mu_{\omega,B}(E) := \frac{\omega^{n-1} \cdot \operatorname{ch}_1(E)}{\omega^n \cdot \operatorname{ch}_0(E)} - \frac{\omega^{n-1} \cdot B}{\omega^n}$$

Using the above slope, we can construct new hearts of $D^\flat(X)$ for every pair $(\omega, B) \in$

$\mathrm{Amp}(X)\times\mathrm{NS}_R(X)$ using the tilting construction from Proposition 2.15 above. Explicitly, define the torsion pair $(\mathcal{T}_{\omega,B}, \mathcal{F}_{\omega,B})$ by

$$\mathcal{T}_{\omega,B} := \{E \in \mathrm{Coh}(X) \mid \text{any semi-stable factor } F \text{ of } E \text{ satisfies } \mu_{\omega,B}(F) > 0\}$$
$$\mathcal{F}_{\omega,B} := \{E \in \mathrm{Coh}(X) \mid \text{any semi-stable factor } F \text{ of } E \text{ satisfies } \mu_{\omega,B}(F) \leq 0\}$$

and let $\mathrm{Coh}_{\omega,B}(X)$ denote the heart obtained by tilting $\mathrm{Coh}(X)$ at this torsion pair.

Remark 2.32. When $\mathrm{Pic}(X) = \mathbb{Z}[H]$ for some ample generator H, we will often write $\omega = \alpha H$ and $B = \beta H$ for some $\alpha > 0$ and $\beta \in \mathbb{R}$. In this case, it is easy to see that

$$\mu_{\omega,B}(E) := \frac{1}{\alpha}(\mu_H(E) - \beta)$$

Thus, $\mathcal{T}_{\omega,B}$ consists of coherent sheaves whose slope-stable factors have slope strictly larger than β and $\mathcal{F}_{\omega,B}$ consists of coherent sheaves whose slope stable factors have slope less than or equal to β — thus we will often refer to $\mathrm{Coh}^{\sharp(\beta)}(X) := \mathrm{Coh}_{\omega,B}(X)$ as the heart obtained by tilting at $\beta \in \mathbb{R}$, and the corresponding torsion pair as $(\mathcal{T}_\beta, \mathcal{F}_\beta)$.

Definition 2.33. Let $\mathcal{D}$ be a geometric triangulated category — that is, it is an admissible [5] subcategory of $D^\flat(X)$ for some smooth variety X. A stability condition $\sigma = (Z, \mathcal{P})$ on $\mathcal{D}$ is said to be *geometric* if the skyscraper sheaves $\Bbbk(x)$ are all stable of the same phase. [6]

The subset $\mathrm{Stab}^{\mathrm{Geo}}(\mathcal{D}) \subset \mathrm{Stab}(\mathcal{D})$ consisting of all geometric stability conditions is often referred to as the *geometric chamber*. It follows from Remark 2.29 above that $\mathrm{Stab}^{\mathrm{Geo}}(\mathcal{D})$ is the $\widetilde{\mathrm{GL}}_2^+(\mathbb{R})$-orbit of all stability conditions $\sigma = (Z, \mathcal{P})$ satisfying $\Bbbk(x) \in \mathcal{P}(1)$.

By Proposition 2.15 above, to produce a stability condition on X it suffices to give the heart of a bounded t-structure $\mathcal{A} \subset D^\flat(X)$ together with a group homomorphism $\overline{Z} : K(\mathcal{A}) \to \mathbb{C}$ (subject to certain conditions). For the hearts $\mathrm{Coh}_{\omega,B}(X)$ constructed

[5]Recall a triangulated subcategory $\mathcal{C} \subset \mathcal{D}$ is said to be left (resp. right) admissible if the inclusion functor $i : \mathcal{C} \hookrightarrow \mathcal{D}$ has a left (resp. right) adjoint. When both exist, the subcategory $\mathcal{C}$ is simply referred to as admissible.

[6]It is worth mentioning that some authors will additionally require that the connected component of $\mathrm{Stab}(\mathcal{D})$ containing such a stability condition is full — however, this will be automatically satisfied if we assume our stability condition satisfies the support property.

above, it was shown in [ABL07], [MS17] that the central charge

$$Z_{\omega,B}(E) = -\int_Z e^{(B+i\omega)}\operatorname{ch}(E)$$

yields a continuous embedding $\operatorname{Amp}(X) \times N^1(X) \to \operatorname{Stab}^{\mathrm{Geo}}(D^\flat(X))$ (given by $(\omega, B) \mapsto (\operatorname{Coh}_{\omega,B}(X), Z_{\omega,B})$) into the subset consisting of stability conditions satisfying $\Bbbk(x)$ is stable of phase 1. In particular, [ABL07, Corollary 2.1] shows that the existence of Harder-Narasimhan filtrations with respect to the above central charges follows from the Hodge index theorem and the Bogomolov-Gieseker Inequality.

Example 2.34. Let X denote a very general Abelian surface of Picard rank 1; that is, a complex torus with $\operatorname{NS}(X) = \mathbb{Z}[\mathcal{O}_X]$. The extended cohomology lattice $H^*(X,\mathbb{Z}) = (H^0 \oplus H^2 \oplus H^4)(X,\mathbb{Z})$ comes equipped with a weight two Hodge structure obtained by declaring $(H^0 \oplus H^4)(X,\mathbb{C})$ be of type (1,1). Let

$$\operatorname{NS}(X) = H^{1,1}(X) \cap H^2(X,\mathbb{Z})$$

denote the Néron-Severi group, and define the *Mukai lattice* as

$$\Lambda = H^0(X,\mathbb{Z}) \oplus \operatorname{NS}(X) \oplus H^4(X,\mathbb{Z})$$

equipped with the pairing $\langle (r_1, D_1, s_1), (r_2, D_2, s_2)\rangle := D_1.D_2 - r_1s_2 - r_2s_1$. The Chern character gives a surjective group homomorphism $\operatorname{ch}(-) : K(D^\flat(X)) \to \Lambda$.

Let $\operatorname{Stab}^\dagger(X)$ denote the connected component of $\operatorname{Stab}(X)$ containing the geometric chamber. By [Bri08], every stability condition in $\operatorname{Stab}^\dagger(X)$ is constructed from the embedding $(\omega, B) \mapsto (\operatorname{Coh}_{\omega,B}(X), Z_{\omega,B})$ up to the $\widetilde{\mathrm{GL}}_2^+(\mathbb{R})$-action. In fact, by [BB17, Theorem 2.2] the forgetful map $\mathcal{Z} : \operatorname{Stab}^\dagger(X) \to \Lambda$ given by sending a pair $(Z, \mathcal{P})$ to $Z^\vee$ induces a $\mathbb{C}$-equivariant isomorphism

$$\operatorname{Stab}^\dagger(X) \cong \mathbb{C} \times \mathbb{H}$$

In this regard, the stability manifold of an abelian surface is very similar to that of elliptic curves.

Remark 2.35. For almost all smooth projective varieties X with $\dim(X) > 1$, it is an open problem as to whether the stability manifold $\mathrm{Stab}(X)$ is connected. In most literature, the connected component containing the geometric chamber is denoted $\mathrm{Stab}^{\dagger}(X)$ and is sometimes referred to as the *principal connected component*.

While the above tilting construction gives an open subset of (a connected component of) the stability manifold $\mathrm{Stab}(D^{\flat}(X))$, the geometric chamber is not expected to cover the entire connected component $\mathrm{Stab}^{\dagger}(D^{\flat}(X))$ except in a handful of cases. In particular, it has only been shown by [FLZ22] that a sufficient condition for $\mathrm{Stab}^{\dagger}(D^{\flat}(X)) = \mathrm{Stab}^{\mathrm{Geo}}(D^{\flat}(X))$ (in any dimension) is when X admits a finite Albenese morphism. For projective K3 surfaces, Bridgeland showed in [Bri08] that all stability conditions (within the principal connected component) are obtained by the above construction up to some "spherical twist" — in other words, there is a wall-and-chamber decomposition of $\mathrm{Stab}^{\dagger}(X)$ with respect to $\Bbbk(x)$ such that every chamber is identical to $\mathrm{Stab}^{\mathrm{Geo}}(X)$. This may initially lead one to believe that all stability conditions on $D^{\flat}(X)$ for a general surface X are obtained from the above construction up to the action of $\mathrm{Aut}(D^{\flat}(X))$ and $\widetilde{\mathrm{GL}}_2^+(\mathbb{R})$ — however, it was shown in [Mac07b] that when the canonical bundle K_X of the surface is anti-ample, the stability manifold $\mathrm{Stab}(X)$ often yield "quivery stability conditions" which are not obtained from the above construction.

2.4 Algebraic Stability Conditions induced by Exceptional Collections

Originally studied by [GR87] in the context of vector bundles, exceptional collections are particularly well-behaved collections of "simple" objects that heavily influence the structure of the underlying triangulated category. For example, the existence of an exceptional collection $\mathfrak{E}$ of length n on $\mathcal{D}$ allows one to construct a decomposition of $K(\mathcal{D})_{\mathbb{C}}$ into $(K(\mathfrak{E}^{\perp})_{\mathbb{C}}) \oplus \mathbb{C}^n$, where $\mathfrak{E}^{\perp}$ is a certain orthogonal admissible subcategory consisting of 'residual' objects not spanned by the exceptional collection [7]. In a certain sense, this allows us to think of exceptional collections as categorical analogs of well-behaved "bases" for subcategories of $\mathcal{D}$.

[7]This follows from the fact that a semiorthogonal decompositions of a triangulated category induces a decomposition of its K-groups, and every exceptional collection induces a non-trivial semiorthogonal decomposition.

Definition 2.36. Let $\mathcal{D}$ denote a $\Bbbk$-linear category. An object $\mathcal{E} \in \mathcal{D}$ is said to be *exceptional* if

$$\mathrm{Hom}^i_{\mathcal{D}}(\mathcal{E},\mathcal{E}) = \begin{cases} \Bbbk & i = 0 \\ 0 & \text{otherwise} \end{cases}$$

Moreover, an ordered sequence of exceptional objects $\mathfrak{E} = \{\mathcal{E}_0, \mathcal{E}_1, \ldots, \mathcal{E}_{n-1}\}$ is said to be an *exceptional collection* if

$$\mathrm{Hom}^\bullet_{\mathcal{D}}(\mathcal{E}_i, \mathcal{E}_j) = 0$$

whenever $i > j$. The exceptional collection is said to be

(i) *Ext* if $\mathrm{Hom}^{\leq 0}_{\mathcal{D}}(\mathcal{E}_i, \mathcal{E}_j) = 0$ for $i \neq j$

(ii) *Strong* if $\mathrm{Hom}^{\neq 0}_{\mathcal{D}}(\mathcal{E}_i, \mathcal{E}_j) = 0$ for $i \neq j$

(iii) *Full* (or *complete*) if $\mathfrak{E}$ generates $\mathcal{D}$ under shifts, direct summands, and cones.

(iv) *Regular* if for all $i < j$, $\mathrm{Hom}^k_{\mathcal{D}}(\mathcal{E}_i, \mathcal{E}_j) \neq 0$ for at most one $k \geq 0$.

Lastly, in the case that $\mathcal{D} = D^\flat(X)$ for some smooth variety X with canonical bundle ω_X, we say the exceptional collection $\mathfrak{E}$ is *simple*[8] if it is full and for any integers $p \leq 0$ and $0 \leq i, j < n$

$$\mathrm{Hom}^k_{\mathcal{D}}(\mathcal{E}_i, \mathcal{E}_j \otimes \omega_X^p) = 0 \quad \text{unless } k = 0.$$

Continuing the above analogy, full strong exceptional collections can be thought of as categorical analogs of orthonormal bases for vector spaces. They will also be desirable for later purposes due to the fact that whenever a $\Bbbk$-linear category $\mathcal{D}$ admits strong exceptional collection $\mathfrak{E} = \{\mathcal{E}_0, \ldots, \mathcal{E}_{n-1}\}$, the object $T = \bigoplus_{i=0}^{n-1} \mathcal{E}_i$ is a *classical generator*[9] of $\mathcal{D}$.

It was also shown by [GR87] and [Bon90] that the existence of one exceptional collection leads to infinitely more through the process of *mutation*:

[8]An equivalent definition arises much more naturally from the theory of helices, as introduced by [BP94], to guarantee that all mutations of strong exceptional collections remain strong. However, to avoid the additional terminology, we will simply use the definition provided above. In either case, it should be noted that all simple exceptional collections are strong, and any mutation of a simple collection is again simple.

[9]Recall T is said to be a classical generator if the smallest full, saturated, triangulated category containing T is equal to $\mathcal{D}$.

Definition 2.37. Let $\mathcal{D}$ be a $\Bbbk$-linear category $\mathcal{D}$, $\mathcal{C} \subset \mathcal{D}$ an admissible subcategory, and denote by i^* and $i^!$ the left and right adjoints, respectively. Define the *left mutation functor* $\mathcal{L}_{\mathcal{C}}$ by the functorial triangle

$$\mathcal{L}_{\mathcal{C}} \to ii^! \to \mathrm{id}\,.$$

Similarly, define the *right mutation functor* $\mathcal{R}_{\mathcal{C}}$ by the triangle

$$\mathrm{id} \to ii^* \to \mathcal{R}_{\mathcal{C}}$$

When $\mathcal{E}$ is an exceptional object, $\langle \mathcal{E} \rangle$ is an admissible subcategory and the left and right mutations take the form

$$\mathcal{L}_{\mathcal{E}} \to \mathrm{Hom}^\bullet_{\mathcal{D}}(\mathcal{E}, -) \otimes \mathcal{E} \to \mathrm{id}$$

and

$$\mathrm{id} \to \mathrm{Hom}^\bullet_{\mathcal{D}}(-, \mathcal{E})^\vee \otimes \mathcal{E} \to \mathcal{R}_{\mathcal{E}}\,.$$

Moreover, when $\mathfrak{E} = \{\mathcal{E}_0, \ldots, \mathcal{E}_n\}$ is an exceptional collection, the collections

$$\mathcal{L}_{\mathcal{E}_i}\,\mathfrak{E} := \{\mathcal{E}_0, \ldots, \mathcal{E}_{i-1}, \mathcal{L}_{\mathcal{E}_i}\,\mathcal{E}_{i+1}, \mathcal{E}_i, \mathcal{E}_{i+2}, \ldots, \mathcal{E}_{n-1}\}$$

and

$$\mathcal{R}_{\mathcal{E}_i}\,\mathfrak{E} := \{\mathcal{E}_0, \ldots, \mathcal{E}_{i-2}, \mathcal{E}_i, \mathcal{R}_{\mathcal{E}_i}\,\mathcal{E}_{i-1}, \mathcal{E}_{i+1}, \ldots, \mathcal{E}_{n-1}\}.$$

are both exceptional collections as well, and known as the *left* (resp. *right*) mutation of $\mathfrak{E}$ through $\mathcal{E}_i$.

It should be noted that the definitions of left and right mutation given above differ in from the usual convention by a homological shift of ± 1; in particular, the left mutation is sometimes defined as a cofiber of the natural map as opposed to its fiber.

Lemma 2.38 ([Bon90, Assertion 2.3])**.** *Let $\mathfrak{E} = \{\mathcal{E}_0, \ldots, \mathcal{E}_{n-1}\}$ be an exceptional collection. Then*

$$\mathcal{L}_{\mathcal{E}_i} \circ \mathcal{L}_{\mathcal{E}_{i+1}} \circ \mathcal{L}_{\mathcal{E}_i} \cong \mathcal{L}_{\mathcal{E}_{i+1}} \circ \mathcal{L}_{\mathcal{E}_i} \circ \mathcal{L}_{\mathcal{E}_{i+1}}$$

and

$$\mathcal{R}_{\mathcal{E}_i} \circ \mathcal{R}_{\mathcal{E}_{i+1}} \circ \mathcal{R}_{\mathcal{E}_i} \cong \mathcal{R}_{\mathcal{E}_{i+1}} \circ \mathcal{R}_{\mathcal{E}_i} \circ \mathcal{R}_{\mathcal{E}_{i+1}} .$$

Moreover, $\mathcal{L}_{\mathcal{E}_i}$ and $\mathcal{R}_{\mathcal{E}_i}$ are mutually inverse in the sense that

$$\mathcal{R}_{\mathcal{E}_i} \circ \mathcal{L}_{\mathcal{E}_i} \cong \mathrm{id} .$$

Let A_n denote the n-string Artin braid group — that is, the fundamental group of the configuration space $C_n(\mathbb{C})/\operatorname{Sym}_n$ up to permutation of the points. It is generated by elements σ_i for $1 \leq i \leq n$ with relations

$$\begin{aligned} \sigma_i \sigma_{i+1} \sigma_i &= \sigma_{i+1} \sigma_i \sigma_{i+1} \\ \sigma_i \sigma_j &= \sigma_j \sigma_i \qquad \text{for } |i-j| \neq 1 \end{aligned}$$

By the above lemma, the Artin braid group A_n acts on the set of full exceptional collections in $D^\flat(X)$ by mutation, i.e.

$$\sigma_i \mathfrak{E} = \mathcal{L}_{\mathcal{E}_i} \mathfrak{E}.$$

Most important for our purposes is the fact that full, strong exceptional collections on a triangulated category $\mathcal{D}$ produce "quivery hearts". To make this explicit, it was shown by [Ric89] that whenever $\mathfrak{E} = \{\mathcal{E}_0, \ldots, \mathcal{E}_{n-1}\}$ is a full, strong exceptional collection on X and A is the endomorphism algebra $A = \operatorname{End}_X(\bigoplus_{i=0}^{n-1} \mathcal{E}_i)$, the functor

$$\mathcal{F}_{\mathfrak{E}} := \operatorname{Hom}_X^\bullet\Big(\bigoplus_{i=0}^{n-1} \mathcal{E}_i, -\Big) \; : D^\flat(S) \to D(\operatorname{Mod} A)$$

gives an equivalence of categories. In particular, the algebra A can also be described as the path algebra of the quiver

$$\bullet \xrightarrow{d_0} \bullet \xrightarrow{d_1} \ldots \xrightarrow{d_{n-3}} \bullet \xrightarrow{d_{n-2}} \bullet$$

with $d_i = \dim \operatorname{Hom}_S^0(\mathcal{E}_i, \mathcal{E}_{i+1})$ ([Bon90]). It is well known that the derived category of the path algebra of any finite quiver comes equipped with a standard t-structure; pulling back this t-structure under the equivalence $\mathcal{F}_{\mathfrak{E}}$ yields a t-structure, and thus heart, on $D^\flat(S)$ which we will denote by $\mathcal{A}_{\mathfrak{E}}$. We will refer to any heart $\mathcal{A} \subset D^\flat(S)$ as *quivery* if it is of the form $\Phi(\mathcal{A}_{\mathfrak{E}})$ for some $\Phi \in \mathrm{Aut}(D^\flat(S))$ and simple exceptional collection $\mathfrak{E}$.

As one would expect, the process of tilting at simple objects also produces an action on the set of quivery hearts; in particular, it was shown by Bridgeland in [Bri05] that if $\mathcal{A} \subset \mathcal{D}$ is the heart of bounded t-structure and $S \in \mathcal{A}$ is a simple object, one can produce a torsion-pair $(\mathcal{T}_S, \mathcal{F}_S)$ by

$$\mathcal{T}_S = \{E \in \mathcal{A} \mid \ \mathrm{Hom}(S, E) = 0\}$$
$$\mathcal{F}_S = \{E \in \mathcal{A} \mid \ \mathrm{Hom}(E, S) = 0\}.$$

Then similar to the construction of [HRS96], we may define tilted subcategories of $D^\flat(X)$

$$\mathbb{L}_S\mathcal{A} := \left\{ E \in D^\flat(X) \left| \begin{array}{l} \bullet\ \mathcal{H}^i(E) = 0, \text{ for all } i \neq 0, 1 \\ \bullet\ \mathcal{H}^0(E) \in \mathcal{F}_S \\ \bullet\ \mathcal{H}^{-1}(E) \in \langle S \rangle \end{array} \right. \right\}$$

and

$$\mathbb{R}_S\mathcal{A} := \left\{ E \in D^\flat(X) \left| \begin{array}{l} \bullet\ \mathcal{H}^i(E) = 0, \text{ for all } i \neq 0, -1 \\ \bullet\ \mathcal{H}^{-1}(E) \in \langle S \rangle \\ \bullet\ \mathcal{H}^0(E) \in \mathcal{T}_S \end{array} \right. \right\}$$

known as the *left* (resp. *right*) *mutations of* $\mathcal{A}$ *through* S. It should also be noted that tilting commutes with the action of $\mathrm{Aut}(D^\flat(X))$ in the sense that

$$\mathcal{L}_{\Phi(S)}\, \Phi(\mathcal{A}) = \Phi(\mathbb{L}_S\mathcal{A})$$

for any $\Phi \in \mathrm{Aut}(D^\flat(X))$, heart $\mathcal{A} \subset D^\flat(X)$ and simple object $S \in \mathcal{A}$ ([Bri05, Lemma 2.6]).

The above construction descends to an action of the Artin braid group A_n on the set of quivery hearts of $D^\flat(S)$:

Lemma 2.39 ([Bri05, Proposition 3.5])**.** *Let* $\mathfrak{E} = \{\mathcal{E}_0, \ldots, \mathcal{E}_{n-1}\}$ *be a simple exceptional* *collection and S_i denote the i^{th} simple object in $\mathcal{A}_{\mathfrak{E}}$ corresponding to the i^{th} vertex of the quiver under $\mathcal{F}_{\mathfrak{E}}$. Then there is an identification of subcategories*

$$\mathcal{L}_{S_i}\,\mathcal{A}_{\mathfrak{E}} = \mathcal{A}_{\sigma_i \mathfrak{E}}$$

for each $0 \leq i < n$.

Remark 2.40. The condition that $\mathfrak{E}$ is a *simple* exceptional collection is necessary and sufficient for the action of A_n to be well-defined. In particular, this requirement guarantees the mutation of full strong exceptional collections will remain both full and strong so that $\mathcal{F}_{\mathcal{L}_i\,\mathfrak{E}}$ is an equivalence. However, as we will see in the next chapter, the existence of simple exceptional collections turns out to be a restrictive condition which does not generally hold for del Pezzo surfaces; for example, it is not true that a mutation of the strong exceptional collection $\{\mathcal{O}(0,0), \mathcal{O}(0,1), \mathcal{O}(1,0), \mathcal{O}(1,1)\}$ on $\mathbb{P}^1 \times \mathbb{P}^1$ remains strong.

We next turn our attention to the stability conditions that arise from these quivery hearts, as described in [Mac07b]. Continue to let $\mathfrak{E} = \{\mathcal{E}_0, \dots, \mathcal{E}_{n-1}\}$ denote a full, strong exceptional collection on $D^\flat(X)$ (such that $K(X) \cong \mathbb{Z}^{\oplus n}$). Fix integers $p_0, \dots, p_{n-1}$ with $p_{i+1} - p_i \geq 1$ so that $\{\mathcal{E}_0[p_0], \dots, \mathcal{E}_{n-1}[p_{n-1}]\}$ is an Ext-exceptional collection; additionally, fix complex numbers $z_0, \dots, z_{n-1} \in \overline{\mathbb{H}}$. As long as the z_i are chosen appropriately (which will be explained in the following paragraph), there is a unique, locally-finite stability condition σ with heart $\mathcal{A}_{\mathfrak{E}}$ and central charge Z satisfying

$$Z(\mathcal{E}_i[p_i]) = z_i.$$

Let $\Theta_{\mathfrak{E}}$ denote the set of all such stability conditions obtained from the construction up to the $\widetilde{\mathrm{GL}}_2^+(\mathbb{R})$-action. While the above construction initially indicates that the central charges of exceptional objects may be chosen freely, it was shown in [Mac07b, Lemma 3.18] that there is a subtle restriction on the phases of the exceptional objects arising from the underlying K-theory.

Given any exceptional sub-collection $\mathfrak{E}' := \{\mathcal{E}_{\ell_0}, \dots, \mathcal{E}_{\ell_s}\} \subseteq \mathfrak{E}$ with $0 < s < n$ and

$\{\ell_i\}_{i=0}^s \subseteq \{0,\dots,n-1\}$ strictly increasing, define

$$k_{i,j}^{\mathfrak{E}'} := \begin{cases} +\infty & \mathcal{E}_{\ell_i} \text{ and } \mathcal{E}_{\ell_j} \text{ are orthogonal} \\ \min_k\{\mathrm{Hom}_X^k(\mathcal{E}_{\ell_i},\mathcal{E}_{\ell_j}) \neq 0\} & \text{otherwise} \end{cases}$$

for $i < j$, where two objects $\mathcal{E},\mathcal{F}$ are said to be orthogonal if $\mathrm{Hom}_X^k(\mathcal{E},\mathcal{F}) = 0$ for all k. Set $\alpha_s^{\mathfrak{E}'} = 0$, and inductively define $\alpha_i^{\mathfrak{E}'} \in \mathbb{Z} \cup \{+\infty\}$ for $i < s$ by

$$\alpha_i^{\mathfrak{E}'} := \min_{j>i}\{k_{i,j}^{\mathfrak{E}'} + \alpha_j^{\mathfrak{E}'}\} - (s-i-1).$$

Then the main result of [Mac07b, Section 3] is that $\Theta_{\mathfrak{E}}$ is an open, connected, and simply connected $(n+1)$-dimensional manifold homeomorphic to

$$\left\{ (m_0,\dots,m_{n-1},\phi_0,\dots,\phi_{n-1}) \in \mathbb{R}^{2n} \;\middle|\; \begin{array}{l} \bullet\ m_i > 0 \\ \bullet\ \phi_{\ell_0} < \phi_{\ell_s} + \alpha_0^{\mathfrak{E}'} \text{ for all } \{\mathcal{E}_{\ell_0},\dots,\mathcal{E}_{\ell_s}\} \subseteq \mathfrak{E} \end{array} \right\}.$$

Though a cumbersome construction, this turns out to give examples of stability conditions in which no object of class $[\Bbbk(x)]$ is stable. For example, the objects $\mathcal{E}_0,\dots,\mathcal{E}_{n-1}$ are always stable within $\Theta_{\mathfrak{E}}$; however, when the $z_0,\dots,z_n$ are chosen such that the image of the central charge lies in a line, they are the *only* stable objects.

Remark 2.41. As the definition suggests, the existence of orthogonal pairs plays a large role in the relations between exceptional objects. For example, let $\pi_k : X_k \to \mathbb{P}^2$ denote the blow-up of $\mathbb{P}^2$ at k distinct points $\{p_1,\dots,p_k\}$, and let $\ell_1,\dots,\ell_k$ denote the exceptional (-1)-curves. As shown in [Orl93], [KO95], if $\{\mathcal{E}_0,\mathcal{E}_1,\mathcal{E}_2\}$ is an exceptional collection in $D^\flat(\mathbb{P}^2)$, then $\mathfrak{F} = \{\pi^*\mathcal{E}_0, \pi^*\mathcal{E}_1, \pi^*\mathcal{E}_2, \mathcal{O}_{\ell_1},\dots,\mathcal{O}_{\ell_k}\}$ is a full, strong exceptional collection in $D^\flat(X_k)$. In particular, the structure sheaves of the (-1)-curves are all mutually orthogonal so that, for any sub-exceptional collection $\mathfrak{F}' \subset \mathfrak{F}$ consisting only of the $\mathcal{O}_{\ell_i}$, the $\alpha_0^{\mathfrak{F}'}$ are always $+\infty$. In particular, there are no necessary relations between $\phi(\mathcal{O}_{\ell_i})$ and $\phi(\mathcal{O}_{\ell_j})$ for $i \neq j$ in $\Theta_{\mathfrak{F}} \subset \mathrm{Stab}(D^\flat(X_k))$.

When $\mathfrak{E}$ is regular and all iterated mutations of $\mathfrak{E}$ are regular, one may define

$$\Sigma_{\mathfrak{E}} := \bigcup_{\mathfrak{F} \in A_n \cdot \mathfrak{E}} \Theta_{\mathfrak{F}}$$

where $A_n \cdot \mathfrak{E}$ denotes that A_n-orbit of $\mathfrak{E}$ consisting of all exceptional collections $\mathfrak{F}$ obtained by a series of mutations from $\mathfrak{E}$. Macrì showed in [Mac07b, Corollary 3.19] that $\Sigma_{\mathfrak{E}}$ is in fact a connected, $(n+1)$-dimensional open sub-manifold of $\mathrm{Stab}(\mathcal{D})$; however, it is not immediately clear when the manifold $\Sigma_{\mathfrak{E}}$ is simply-connected. It was later shown by [Che18] that under the additional assumption $\mathfrak{E}$ is a full strong exceptional collection with (i) no exceptional pairs and (ii) all iterations mutations of $\mathfrak{E}$ are also strong, then the open submanifold $\Sigma_{\mathfrak{E}}$ is simply connected, and the closure of any $\Theta_{\mathfrak{F}} \subset \Sigma_{\mathfrak{E}}$ is contained in $\Sigma_{\mathfrak{E}}$.

Remark 2.42. In the case that $\mathcal{D}$ is *constructible* — that is, all exceptional collections can be obtained (up to shifts) from a single exceptional collection by a series of mutations — there is only a single A_n-orbit which will be referred to as the *algebraic stability manifold* and denoted $\mathrm{Stab}^{\mathrm{Alg}}(\mathcal{D}) \subset \mathrm{Stab}(\mathcal{D})$. An important class of examples, as described in [Mac07b], is the bounded derived category of del Pezzo surfaces.

2.5 Algebraic Stability Conditions on Local Calabi-Yau Varieties

As Bridgeland's notion of stability conditions was originally meant to give a rigorous formalism to Π-stability in string theory, a primary objective in the field has been to construct the stability manifold of a projective Calabi-Yau threefold Y. However, this has proven to be a challenging problem, as the construction given in Section 2.3.1 only produces *weak* stability conditions — in particular, skyscraper sheaves become massless when considering the central charge obtained from a single tilt. By instead considering 'local models' of the form $\mathfrak{Y} = \mathrm{Tot}\,\omega_X$ for some del Pezzo surface X, we obtain a family of non-compact Calabi-Yau threefolds that admit stability conditions which are easily obtained from the underlying base.

For the remainder of this section, let us consider the general case that X is a Fano variety. By abuse of notation, let ω_X denote both (i) the total space of the canonical bundle of X, which we can think of as a quasi-projective Calabi-Yau variety of dimension $\dim X+1$ fibred over X and (ii) the invertible $\mathcal{O}_X$-module consisting of holomorphic top-dimensional forms. Moreover, Let $\pi : \omega_X \to X$ denote the fibration and $i : X \hookrightarrow \omega_X$ the inclusion of the zero-section. Following [Bri05] define $\mathcal{D}_{0,X}$ to be the full subcategory of $D^\flat(\omega_X)$ consisting of objects whose cohomology sheaves are supported on the zero section $X \subset \omega_X$.

Definition 2.43. Let $\mathcal{D}$ denote an algebraic $\Bbbk$-linear category. An object $S \in \mathcal{D}$ is said to be *n-spherical* if

$$\mathrm{Hom}^i_{\mathcal{D}}(S,S) = \begin{cases} \Bbbk & i = 0, n \\ 0 & \text{otherwise.} \end{cases}$$

In other words, the endomorphism algebra of S is identical to the singular cohomology of the n-sphere.

It immediately follows from the definition that for any spherical object S and automorphism Φ, the object $\Phi(S)$ is also spherical. More importantly, spherical objects themselves define a rich class of automorphisms that heavily contribute to the geometry of stability manifolds for our Calabi-Yau categories [10]:

Definition 2.44 ([ST01])**.** Let $\mathcal{D}$ be a $\Bbbk$-linear category and $S \in \mathcal{D}$ an n-spherical object. Then there is an autoequivalence $\mathrm{Tw}_S \in \mathrm{Aut}(\mathcal{D})$ defined functorially by the exact triangle

$$\mathrm{Hom}^\bullet_{\mathcal{D}}(S,E) \otimes S \to E \to \mathrm{Tw}_S\, E$$

for any object $E \in \mathcal{D}$. Moreover, $\mathrm{Tw}_{S[1]}\, E \cong \mathrm{Tw}_S\, E$ for any object $E \in \mathcal{D}$, $\mathrm{Tw}_S\, S \cong S[-n+1]$ and

$$\mathrm{Tw}_{S_1} \circ \mathrm{Tw}_{S_2} \circ \mathrm{Tw}^{-1}_{S_1} \cong \mathrm{Tw}_{\mathrm{Tw}_{S_1} S_2}$$
$$\mathrm{Tw}^{-1}_{S_1} \circ \mathrm{Tw}_{S_2} \circ \mathrm{Tw}_{S_1} \cong \mathrm{Tw}_{\mathrm{Tw}^{-1}_{S_1} S_2}$$

[10]Recall a $\Bbbk$-linear category $\mathcal{D}$ is said to be *n-Calabi-Yau* if it admits a Serre functor which is equivalent to the shift functor $[n]$.

for any spherical S_1, S_2.

The inverse spherical twist may be defined similarly to the right mutation by using the functorial exact triangle

$$\mathrm{Tw}_S^{-1} E \to E \to (\mathrm{Hom}^\bullet_{\mathcal{D}}(E,S))^\vee \otimes S.$$

Lemma 2.45 ([Bri05, Lemma 4.6])**.** *If $\mathcal{E}$ is an exceptional object in $D^\flat(X)$, then $i_*\mathcal{E}$ is $(\dim X+1)$-spherical in $\mathcal{D}_{0,X}$.*

To see this, recall that for any smooth divisor $i : D \hookrightarrow Y$ (where Y is an arbitrary smooth projective variety) there is a functorial triangle

$$i^* i_* \to \mathrm{id} \to - \otimes \mathcal{O}_D(-D)[2].$$

When X is Fano, ω_X has no nontrivial global sections so that the zero section is always isomorphic to X — in particular, we may always consider X as a principal divisor on ω_X. By adjunction, $\mathcal{O}_X(-X) \cong \mathcal{N}^\vee_{X,\omega_X} \cong \omega_X^\vee$ so that the above triangle takes the form

$$i^* i_* \mathcal{E} \to \mathcal{E} \to \mathcal{E} \otimes \omega_X^\vee[2]$$

for every $\mathcal{E} \in D^\flat(X)$. The general argument of the proof follows from Serre duality, adjunction, and applying contravariant $\mathrm{Hom}^\bullet_X(-,\mathcal{E})$ to the above triangle.

Similar to the case of the underlying Fano base X, one is able to produce quivery hearts on $D^\flat(\omega_X)$ by examining the endomorphism algebra of a collection of objects obtained from some full, strong exceptional collection. Following [Bri05], let $\mathfrak{E} = \{\mathcal{E}_0, \ldots, \mathcal{E}_{n-1}\}$ be a simple exceptional collection on the Fano base X. Then we may define a functor

$$\widetilde{\mathcal{F}_{\mathfrak{E}}} := \mathrm{Hom}^\bullet_{\omega_X}\left(\bigoplus_{i=0}^{n-1} \pi^*\mathcal{E}_i, -\right) : D^\flat(\omega_X) \to D^\flat(\mathrm{Mod}\ \ B)$$

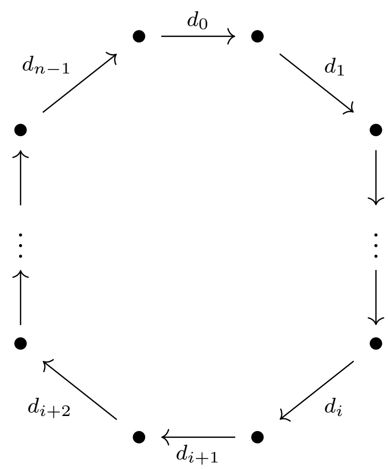
where B is the endomorphism algebra $B = \mathrm{End}_{\omega_X}\left(\bigoplus_{i=0}^{n-1} \pi^*\mathcal{E}_i\right)$. The requirement that $\mathfrak{E}$ is a *simple* exceptional collection is necessary due to the fact that $\pi_*(\mathcal{O}_{\omega_X}) = \bigoplus_{p\le 0} \omega_X^p$

and thus

$$\mathrm{Hom}^k_{\omega_X}(\pi^*\mathcal{E}_i, \pi^*\mathcal{E}_j) = \bigoplus_{p \leq 0} \mathrm{Hom}^k_X(\mathcal{E}_i, \mathcal{E}_j \otimes \omega_X^p)$$

as shown in the proof of [Bri05, Proposition 4.1].

Again, the algebra B can be described as the path algebra of a certain quiver; however, due to the fact that $D^\flat(\omega_X)$ is n-Calabi-Yau, the quiver takes the form



where $d_i = \dim \mathrm{Hom}_X(\mathcal{E}_i, \mathcal{E}_{i+1})$ for $i \in \mathbb{Z}_n$.

Let $D^\flat_0(\mathrm{Mod}\ \ B) \subset D^\flat(\mathrm{Mod}\ \ B)$ denote the full subcategory consisting of objects whose cohomology modules are nilpotent. As shown in [Bri05, Lemma 4.4], the equivalence $\widetilde{\mathcal{F}_{\mathfrak{E}}}$ restricts to $\mathcal{D}_{0,X}$ to again give an equivalence of full subcategories

$$\widetilde{\mathcal{F}_{\mathfrak{E},0}} : \mathcal{D}_{0,X} \to D^\flat_0(\mathrm{Mod}\ \ B).$$

As before, one may pull back the induced t-structure on $D^\flat_0(\mathrm{Mod}\ \ B)$ to produce a heart on $\mathcal{D}_{0,X}$, which we will denote $\mathcal{A}_{0,\mathfrak{E}}$.

Remark 2.46. The simple objects of $\mathcal{A}_{0,\mathfrak{E}}$ are not necessarily the objects $\pi^*\mathcal{E}_i$. Continue to let A_n denote the n-string Artin braid group; it was shown by Bondal in [Bon90] that if we define the element $\delta \in A_n$ by

$$\delta = (\sigma_1 \ldots \sigma_{n-1})(\sigma_1 \ldots \sigma_{n-2}) \ldots (\sigma_1\sigma_2)\sigma_1$$

then the exceptional collection $\{\mathcal{F}_0, \ldots, \mathcal{F}_{n-1}\} := \delta\{\mathcal{E}_0, \ldots, \mathcal{E}_{n-1}\}$ is said to be the *(left) dual collection* to $\{\mathcal{E}_0, \ldots, \mathcal{E}_{n-1}\}$ in the sense that

$$\operatorname{Hom}^k_X(\mathcal{E}_i, \mathcal{F}_{n-1-j}) = \mathbb{C}$$

whenever $i = j$ for $k = 0$, and is 0 otherwise. Then by [Bri05], the simple objects of $\mathcal{A}_{0,\mathfrak{E}}$ are precisely $S_j := i_*\mathcal{F}_{n-1-j}[j]$ — in particular, the S_j map under $\widetilde{\mathfrak{F}_{\mathfrak{E},0}}$ to the simple representations of the above quiver.

We will refer to any ordered collection of distinct spherical objects $\{S_0, \ldots, S_{n-1}\}$ in $\mathcal{D}_{0,X}$ as a *spherical collection*. By the previous remark, for any simple exceptional collection $\mathfrak{E} := \{\mathcal{E}_0, \ldots, \mathcal{E}_{n-1}\}$ with dual collection $\{\mathfrak{F}_0, \ldots, \mathfrak{F}_{n-1}\}$ the sequence

$$\{i_*\mathfrak{F}_{n-1}, i_*\mathfrak{F}_{n-2}[1], \ldots, i_*\mathfrak{F}_0[n-1]\}$$

is a spherical collection. Similar to the case of exceptional collections on Fano varieties, there is a certain braid group action on the set of ordered spherical collections in $\mathcal{D}_{0,X}$; however, due to the triviality of the Serre functor, we examine a slightly different braid group.

Let CB_n denote the n-string annular braid group for $n > 2$ — that is, the fundamental group of the configuration space $C_n(\mathbb{C}^*)/\operatorname{Sym}_n$ up to permutation of the points. It is generated by elements τ_i for $i \in \mathbb{Z}_n$ and an element r, subject to the relations

$$\begin{aligned} r\tau_i r^{-1} &= \tau_{i+1} \\ \tau_i\tau_{i+1}\tau_i &= \tau_{i+1}\tau_i\tau_{i+1} \\ \tau_i\tau_j &= \tau_j\tau_i \qquad \text{for } |i-j| \neq 1. \end{aligned}$$

Let B_n denote the quotient of the n-string annular braid group by the subgroup $\langle r^n \rangle$. By [Bri05, Lemma 4.7], B_n acts on the set of spherical collections as follows: if $\mathfrak{S} = \{S_0, \ldots, S_{n-1}\}$ is a spherical collection, then

$$r\mathfrak{S} := \{S_{n-1}, S_0, S_1, \ldots, S_{n-2}\}$$

and

$$\tau_i\mathfrak{S} := \{S_0, S_1, \ldots, S_{i-2}, S_i[-1], \operatorname{Tw}_{S_i} S_{i-1}, S_{i+1}, \ldots, S_{n-1}\}$$

are both spherical collections (where the indices are taken in $i \in \mathbb{Z}_n$).

Remark 2.47. We require that $n > 2$ in the construction above since the action is not well-defined when considering spherical collections of length two. To make this explicit, let $\mathfrak{S} = \{S_0, S_1\}$ be a spherical collection on some $\mathcal{D}_{0,X}$ with $\dim X + 1 = 2 = \dim K(X) \otimes \mathbb{C}$ — as discussed in the next chapter, this effectively reduces to the case $X = \mathbb{P}^1$. One may directly check from the construction above that

$$\tau_0\tau_1\tau_0\mathfrak{S} = \{\operatorname{Tw}^3_{S_0} S_1, S_0[-3]\}.$$

In particular, $r\tau_0\tau_1\tau_0$ is the same as applying $\operatorname{Tw}^3_{S_0}$ to the spherical objects S_i. However,

$$\tau_1\tau_0\tau_1\mathfrak{S} = \{S_1[-3], \operatorname{Tw}^3_{S_1} S_0\}$$

so that the action does not respect the braid group relation $\tau_0\tau_1\tau_0 = \tau_1\tau_0\tau_1$.

When $\mathfrak{S}$ is a spherical collection arising from some exceptional collection $\mathfrak{E}$ on X, it turns out that the collections $\tau_i\mathfrak{S}$ are closely related to mutations of $\mathfrak{E}$.

Lemma 2.48. *Let $\mathfrak{S} = \{i_*\mathcal{E}_{n-1}, \ldots, i_*\mathcal{E}_0[n-1]\}$ be a spherical collection in $\mathcal{D}_{0,X}$ arising from a simple exceptional collection $\mathfrak{E} = \{\mathcal{E}_0, \ldots, \mathcal{E}_{n-1}\}$ on X. Then*

$$\tau_i\mathfrak{S} = \{i_*\mathcal{E}'_{n-1}, \ldots, i_*\mathcal{E}'_0[n-1]\}$$

where $\mathcal{E}'_j$ is the j^{th} exceptional object of the simple exceptional collection $\mathcal{L}_{\mathcal{E}_{n-1-i}} \mathfrak{E}$.

Proof. It is easy to verify that the objects and shifts coincide for all indices except the i^{th} place. Thus, it suffices to show that

$$\operatorname{Tw}_{i_*\mathcal{E}_{n-1-i}} i_*\mathcal{E}_{n-1-(i-1)}[i-1] \cong i_* \mathcal{L}_{\mathcal{E}_{n-1-i}} \mathcal{E}_{n-1-(i-1)}[i].$$

Since $i_* : D^\flat(X) \to \mathcal{D}_{0,X}$ is a triangluated functor, this will follow from the defining triangles of the above two objects if we are able to show

$$\mathrm{Hom}_{\omega_X}(i_*\mathcal{E}_{n-1-i}, i_*\mathcal{E}_{n-1-(i-1)}) \cong \mathrm{Hom}_X(\mathcal{E}_{n-1-i}, \mathcal{E}_{n-1-(i-1)}).$$

Using the adjunction $i^* \dashv i_*$ as well as the distinguished triangle

$$i^*i_*\mathcal{E}_{n-1-i} \to \mathcal{E}_{n-1-i} \to \mathcal{E}_{n-1-i} \otimes \omega_X^\vee[2]$$

the above isomorphism holds if and only if $\mathrm{Hom}(\mathcal{E}_{n-1-i} \otimes \omega_X^\vee[2], \mathcal{E}_{n-1-(i-1)}) = 0$. However, this follows from Serre duality and the fact that $\mathfrak{E}$ is an exceptional collection. $\square$

The above lemma is intentionally phrased in terms of spherical collections arising from exceptional collections, as opposed to the hearts $\mathcal{A}_{0,\mathfrak{E}}$, to avoid dealing with dual exceptional collections. However, when examining the stability manifold of $\mathcal{D}_{0,X}$ it will be important to extend the action of B_n to certain finite-length hearts.

Definition 2.49. A heart of a bounded t-structure in $\mathcal{D}_{0,X}$ is said to be *quivery* if it is of the form $\Phi(\mathcal{A}_{0,\mathfrak{E}})$ for some simple exceptional collection $\mathfrak{E} = \{\mathcal{E}_0, \ldots, \mathcal{E}_n\}$ in $D^\flat(X)$ and $\Phi \in \mathrm{Aut}(\mathcal{D}_{0,X})$. Moreover, the quivery heart is said to be *ordered* if its spherical objects $S_0, \ldots, S_{n-1}$ satisfy

$$\mathrm{Hom}^k_{\omega_X}(S_i, S_j) = 0 \text{ unless } 0 \le k \le n \text{ and } i - j \equiv k \pmod{n}.$$

By [Bri05, Proposition 4.10, Theorem 4.11], the braid group B_n acts on the set of ordered quivery hearts as follows: if $\mathcal{A}$ is an ordered quivery heart with simple objects obtained from the spherical collection $\{S_0, \ldots, S_{n-1}\}$, then the element τ_i acts by

$$\tau_i \cdot \mathcal{A} := \mathcal{L}_{S_i}\,\mathcal{A}$$

and its simple objects are given by the spherical collection $\tau_i\{S_0, \ldots, S_{n-1}\}$ — this can be thought of as a generalization of the previous lemma.

By letting $\mathrm{Aut}(\mathcal{D}_{0,X})$ act on spherical collections in the obvious way, i.e.

$$\Phi \cdot \{S_0, S_1, \dots, S_{n-1}\} := \{\Phi(S_0), \Phi(S_1), \dots, \Phi(S_{n-1})\}$$

one obtains a direct analog between spherical collections and ordered quivery hearts. In practice, the class of exceptional collections in $D^\flat(X)$ is invariant under automorphisms of X, shifts, and twists by line bundles — thus, the only significant automorphisms Φ used to translate the quivery hearts $\mathcal{A}_{0,\mathfrak{E}}$ arise from compositions of spherical twists.

Lemma 2.50 ([Bri05, Lemma 4.8])**.** *Let $\mathfrak{S} = \{S_0, S_1, \dots, S_{n-1}\}$ be a spherical collection. Then for $0 \leq i \leq n-1$,*

$$\alpha_i\{S_0, S_1, \dots, S_{n-1}\} = \mathrm{Tw}_{S_i} \cdot \{S_0, S_1, \dots, S_{n-1}\}$$

where $\alpha_i := r^i(\tau_1 \dots \tau_{n-1})r^{(n-1)-i} \in B_n$.

We next examine the stability conditions arising from these quivery hearts. Following the notation of [Bri06], [BM11], let $\mathfrak{E} = \{\mathcal{E}_0, \dots, \mathcal{E}_{n-1}\}$ be a simple exceptional collection on $D^\flat(X)$ with dual exceptional collection $\{\mathcal{F}_0, \dots, \mathcal{F}_{n-1}\}$ and let $\mathcal{A}_{0,\mathfrak{E}}$ denote the quivery heart in $\mathcal{D}_{0,X}$ with spherical collection $\{S_0, \dots, S_{n-1}\} = \{i_*\mathcal{F}_{n-1}, i_*\mathcal{F}_{n-2}[1], \dots, i_*\mathcal{F}_0[n-1]\}$. Choose complex numbers $z_0, \dots, z_{n-1} \in \overline{\mathbb{H}}$. Since the heart $\mathcal{A}_{0,\mathfrak{E}}$ has finite length, the Harder-Narasimhan property (c.f. [BM11, Appendix B, Proposition B.2]) is always satisfied, and thus there exists a stability condition σ on $\mathcal{D}_{0,X}$ with heart $\mathcal{A}_{0,\mathfrak{E}}$ and central charge Z satisfying

$$Z(S_i) = z_i.$$

By abuse of notation, we will again use $\Theta_{\mathfrak{E}}$ to denote the set of all stability conditions obtained from the above construction up to the $\widetilde{\mathrm{GL}}_2^+(\mathbb{R})$-action. In general, for any ordered quivery heart $\mathcal{A}_{\mathfrak{S}}$ with spherical collection $\mathfrak{S} = \{S_0, \dots, S_{n-1}\}$ we may define $\Theta_{\mathfrak{S}}$ in an analogous way — such stability conditions will be called *algebraic*. Similar to the previous chapter, when B_n acts transitively on the set of spherical collections in $\mathcal{D}_{0,X}$, we will refer to the union $\mathrm{Stab}^{\mathrm{Alg}}(\mathcal{D}_{0,X}) := \bigcup_{\mathfrak{S}} \Theta_{\mathfrak{S}}$ as the *algebraic stability manifold*.

Using an identical argument to [BM11, Lemma 6.5], one can show that the region

$\Theta_{\mathfrak{S}}$ may be characterized as the subset where the simple objects $\{S_0, \ldots, S_{n-1}\}$ are all stable and where their phases $\phi_j = \phi(S_j)$ satisfy

$$\phi_0, \ldots, \phi_{n-1} \in (t, t+1] \tag{1}$$

for some $t \in \mathbb{R}$. It is homeomorphic to the region

$$\{(m_0, \ldots, m_{n-1}, \phi_0, \ldots, \phi_{n-1}) \in \mathbb{R}^{2n} \mid m_j > 0 \text{ and } (1) \text{ holds}\}. \tag{2}$$

Definition 2.51. Let $\mathfrak{S} = \{S_0, \ldots, S_{n-1}\}$ be an ordered quivery collection and consider the symmetry group on n letters Sym_n. For each element $g \in \mathrm{Sym}_n$ we define the $\widetilde{\mathrm{GL}}_2^+(\mathbb{R})$-*orbit* $\Theta_{g,\mathfrak{S}}$ as follows: for the identity $g = \mathrm{id}$, $\Theta_{\mathrm{id},\mathfrak{S}}$ is the open subset of $\Theta_{\mathfrak{S}}$ where

$$\phi(S_0) < \cdots < \phi(S_{n-1})$$

More generally, if $g\{S_0, \ldots, S_{n-1}\} = \{S'_0, \ldots, S'_{n-1}\}$, $\Theta_{g,\mathfrak{S}}$ is the open region in $\Theta_{\mathfrak{S}}$ where

$$\phi(S'_0) < \cdots < \phi(S'_{n-1})$$

One of the main reasons we wish to distinguish these open sets is the following:

Lemma 2.52. *Let $\Theta_{g_1,\mathfrak{S}}$ and $\Theta_{g_2,\mathfrak{S}}$ be distinct $\widetilde{\mathrm{GL}}_2^+(\mathbb{R})$-orbits in $\Theta_{\mathfrak{S}}$. Then there is a spherical twist $\mathrm{Tw}_{S_{i+1}} S_i$ of the simple objects in $\Theta_{\mathfrak{S}}$ which is stable in one of the $\widetilde{\mathrm{GL}}_2^+(\mathbb{R})$-orbits and unstable in the other.*

Proof. This is a simple consequence of a fact given in [BM11, Section 6]: the object $\mathrm{Tw}_{S_{i+1}} S_i$ is stable whenever $\phi(S_i) < \phi(S_{i+1})$ and unstable whenever the reverse inequality holds. Without loss of generality we may take g_1 to be the identity and $g := g_2$ to be any non-trivial element in Sym_n. It is an easy inductive argument on n to show that non-trivial elements in Sym_n necessarily swaps the order of two consecutive letters i and $i+1$ (taken in $\mathbb{Z}_n$). The result then directly follows from our definition of the $\widetilde{\mathrm{GL}}_2^+(\mathbb{R})$-orbit of $\Theta_{g,\mathfrak{S}}$ above. $\square$

Lemma 2.53. *Let $\mathfrak{S} = \{S_0, \dots, S_{n-1}\}$ be an ordered quivery collection on $\mathcal{D}_{0,X}$. Then precisely $(n-1)!$ $\widetilde{\mathrm{GL}}_2^+(\mathbb{R})$-orbits are contained in the intersection $\Theta_{\mathfrak{S}} \cap \Theta_{\tau_i^{\pm}\mathfrak{S}}$.*

Proof. Recall that the simple objects of $\tau_i\mathfrak{S}$ are given by

$$\{S_0, S_1, \dots, S_{i-2}, S_i[-1], \mathrm{Tw}_{S_i} S_{i-1}, S_{i+1}, \dots, S_{n-1}\}.$$

In particular, for any stability condition $\sigma \in \Theta_{\mathfrak{S}} \cap \Theta_{\tau_i\mathfrak{S}}$ the above objects are all stable within phase 1 of one another, as well as the objects $\{S_0, \dots, S_{n-1}\}$. Since the objects S_j must be within phase 1 of both S_i and $S_i[-1]$ for $j \notin \{i, i-1\}$, we have that $S_j \in \mathcal{P}((t, t+1])$ where $t = \phi(S_i) - 1$. Moreover, $\phi(S_{i-1}) < \phi(\mathrm{Tw}_{S_i} S_{i-1}) < \phi(S_i)$ since $\mathrm{Tw}_{S_i} S_{i-1}$ is stable; as S_{i-1} is required to be within phase 1 of S_i to be in $\Theta_{\mathfrak{S}}$, we obtain $S_j, \mathrm{Tw}_{S_i} S_{i-1} \in \mathcal{P}((t, t+1])$ for $j \neq i$. Therefore, there are precisely $(n-1)!$ possibilities for the order of the phases $\phi_j := \phi(S_j)$ for $j \neq i$ (which, in turn, induce orderings on the simple objects of $\mathcal{A}_{\tau_i\mathfrak{S}}$). An identical argument holds for τ_i^{-1}. $\square$

Note that for words $\gamma = \gamma_1 \dots \gamma_n \in B_n$ with $\gamma_i \in \{\tau_0^{\pm}, \dots, \tau_{n-1}^{\pm}\}$ of length $n > 1$, it need not be true that the sets $\Theta_{\mathfrak{S}}$ and $\Theta_{\gamma\mathfrak{S}}$ intersect non-trivially. For example, by taking $\gamma = \tau_i\tau_{i+1}$, the ordered quivery collection $\gamma\mathfrak{S}$ is given by

$$\{S_0, S_1, \dots, S_{i+1}[-2], \mathrm{Tw}_{S_{i+1}} S_{i-1}, \mathrm{Tw}_{S_{i+1}} S_i, S_{i+2}, \dots, S_{n-1}\}.$$

However, in order for a stability condition σ to lie in $\Theta_{\mathfrak{S}} \cap \Theta_{\gamma\mathfrak{S}}$ we must have that the simple objects S_j for $j \neq i+1$ must be within phase 1 of both $\phi(S_{i+1})$ and $\phi(S_{i+1}) - 2$ which is impossible. A similar argument holds for $\gamma = \tau_i\tau_{i-1}^{-1}$.

Remark 2.54. As described in the proof of [BM11, Proposition 6.6], we may describe the boundary $\partial\Theta_{\mathfrak{S}}$ explicitly using the characterization from (1) above. In particular, two things can happen for a stability condition $\overline{\sigma} = (\overline{Z}, \overline{\mathcal{P}}) \in \partial\Theta_{\mathfrak{S}}$:

(a) $\overline{\sigma}$ is degenerate in the sense that the central charge $\overline{Z}$ lies on a line, and at least two of the simple objects S_i, S_j lie exactly phase 1 away from one another.

(a) The central charge $\overline{Z}$ does not lie on a line, and at least two of the simple objects

S_i, S_j lie exactly phase 1 away from one another.

In the case that no strong exceptional collection $\mathfrak{E}$ on X admits orthogonal pairs (for example, this is always the case for $\mathbb{P}^n$), we may apply the same argument as [Mac07b, Theorem 4.7] to show that $\overline{\sigma}$ is contained in the intersection of the closure of two $\widetilde{\mathrm{GL}}_2^+(\mathbb{R})$-orbits in some $\Theta_{g\mathfrak{S}}$ for $g \in B_n$. Notably, case (a) above always corresponds to a stability condition in the intersection of the closure of *all* $\widetilde{\mathrm{GL}}_2^+(\mathbb{R})$-orbits in $\Theta_{g\mathfrak{S}}$ — all such stability conditions in this intersection are equivalent up to the $\mathbb{C}$-action.

3 Stability Manifolds for Two Local Projective Spaces

A recurrent issue in the modern theory of projective Calabi-Yau varieties is that their derived categories tend to host vast numbers of spherical objects, leading to many simultaneous A_n-configurations (in the sense of [Bri09]) that overlap with one another. In particular, this makes the group of automorphisms much more complicated than a standard braid group, and in general, the whole category will never be generated by some finite collection of objects. However, under the additional assumption that the Calabi-Yau variety in question contains certain Fano divisors, it is possible to create a 'local model' of the derived category that is often significantly better behaved.

In Section 2.5 we saw that given any smooth projective variety X admitting a *simple* exceptional collection, one could always construct quivery stability conditions in a 'localized' subcategory $\mathcal{D}_{0,X} \subset D^\flat(\omega_X)$. However, the existence of simple exceptional collections turns out to be a very restrictive condition; in particular, it was shown by [BP94] that the rank of the Grothendieck group completely controls the existence of such collections:

Proposition 3.1 ([BP94, Proposition 3.3])**.** *A full exceptional collection of coherent sheaves of length m on a smooth projective variety X is simple if and only if $m = \dim(X) + 1$.*

As a direct result, the functor $\widetilde{\mathcal{F}_{\mathfrak{E}}}$ only allows one to produce quivery hearts when the variety X satisfies

$$\dim K(X)_{\mathbb{C}} = \dim(X) + 1. \qquad (\dagger)$$

While the full category $\mathcal{D}_{0,X} \subset D^\flat(\omega_X)$ may still be defined for any Fano variety X to give a 'local Calabi-Yau model', tilting at simple objects does not necessarily produce bounded t-structures when $\dim K(X)_{\mathbb{C}} > \dim(X) + 1$. In other words, one can no longer mimic representation theory by using the group B_n to determine the geometry of $\mathrm{Stab}(\mathcal{D}_{0,X})$.

In lieu of the above proposition, a natural question is to determine which varieties X satisfy (†) in low dimensions; we may significantly refine the search by also requiring that X is Fano to ensure that it is isomorphic to the zero section of $\pi : \omega_X \to X$. In the case of smooth projective curves of genus g, it is well known that

$$K(C) \cong \mathrm{Pic}(C) \times \mathbb{Z} \cong \mathbb{C}^g/\Lambda \times \mathbb{Z}^2$$

so that the techniques of the previous chapter only apply when $g = 0$ — that is, $C \cong \mathbb{P}^1$.

For del Pezzo surfaces, slightly more effort is required when considering the rank of the Grothendieck group. By Remark 2.41, the blow-up of $\mathbb{P}^2$ at $1 \leq k \leq 8$ points has numerical Grothendieck group of rank $3 + k$, leaving only the deformation classes of $\mathbb{P}^2$ and the quadric $\mathbb{P}^1 \times \mathbb{P}^1$. However, it was shown by [Kap88] that on any smooth quadric Q with $n = \dim(X)$ even, $\{\mathcal{S}_+, \mathcal{S}_-, \mathcal{O}_Q, \mathcal{O}_Q(1), \ldots, \mathcal{O}_Q(n-1)\}$ is a full exceptional collection (where $S_\pm$ denote the spinor bundles) — in the case $Q = \mathbb{P}^1 \times \mathbb{P}^1$, the spinor bundles are simply $\mathcal{O}(-1, 0)$, $\mathcal{O}(0, -1)$ and thus $\mathrm{rk}\, K(Q) = 4$. Ultimately, we again find that the condition (†) restricts to the rational case $\mathbb{P}^2$.

The threefold case has slightly more variety, as the condition (†) no implies X is rational. Aside from the obvious $X \cong \mathbb{P}^3$, it was shown by [Kap88] that for a three dimensional quadric $Q \subset \mathbb{P}^4$ the collection $\{\mathcal{O}_Q(-1), \mathcal{S}(-1), \mathcal{O}_Q, \mathcal{O}_Q(1)\}$ gives a full strong exceptional collection (where $\mathcal{S}$ denotes the spinor bundle). Moreover, it was shown by Orlov [Orl91] that the Fano threefold V_5, obtained as a linear section $\mathrm{Gr}(2, 5) \cap \mathbb{P}^6 \subset \mathbb{P}^9$ of the Plücker embedding, admits a full strong exceptional collection of length 4; it was later shown by Kuznetsov [Kuz96] that a similar statement holds for the Fano variety $V_{22} \subset \mathbb{P}^{13}$ obtained as the zero section of $\left(\bigwedge^2 \mathcal{U}^\vee\right)^{\oplus 3}$ (where $\mathcal{U}$ denotes the tautological bundle of $\mathrm{Gr}(3, 7)$).

As the hearts of bounded t-structures on threefolds is generally much more arduous to construct, this chapter will focus on the stability conditions of $\mathbb{P}^1$ and $\mathbb{P}^2$, as well as their respective local Calabi-Yau varieties $\omega_{\mathbb{P}^1}$ and $\omega_{\mathbb{P}^2}$.

3.1 The case of the Projective Line and $\omega_{\mathbb{P}^1}$

The stability manifold of $\mathbb{P}^1$ was originally studied by [Oka06] and turns out to have a much more complicated structure than the stability manifold of higher genus curves due to the construction of "quivery" hearts from the previous section. As is the case for any smooth projective curve, there is a well-defined notion of geometric stability conditions on $D^\flat(\mathbb{P}^1)$; however, as we will see later in this section, it is not necessary to define the geometric chamber to describe the stability manifold $\mathrm{Stab}(\mathbb{P}^1)$.

Recall from [Mac07a] that for any smooth projective curve C and w in the half-closed upper half-plane $\overline{\mathbb{H}}$, one is able to construct a geometric stability condition

$$\sigma_w = (\mathrm{Coh}(C), Z_w)$$

where $Z_w(-) = -\deg(-) + w \cdot \mathrm{rk}(-)$; moreover, all geometric stability conditions are of this form up to the $\mathbb{C}$-action. In the case of higher genus curves, these are in fact the only stability conditions.

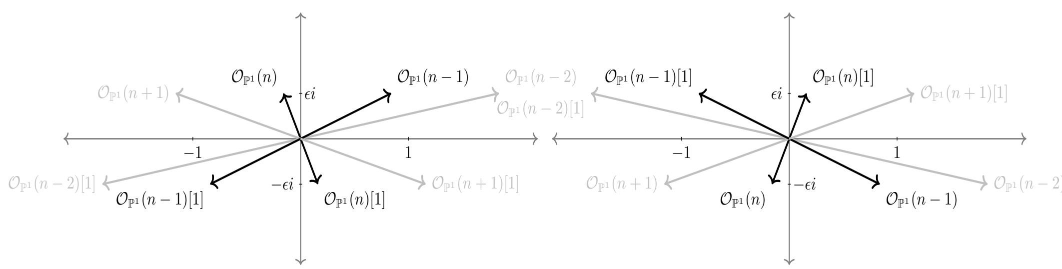
The main difference in the case of the projective line is that exceptional collections of the form $\{\mathcal{O}_{\mathbb{P}^1}(k-1), \mathcal{O}_{\mathbb{P}^1}(k)\}$ allow us to construct hearts that are distinct from $\mathrm{Coh}(\mathbb{P}^1)$ using the techniques of Section 2.4. For example, suppose that we tilt at an angle μ along a line between $\mathcal{O}_{\mathbb{P}^1}(k)$ and $\mathcal{O}_{\mathbb{P}^1}(k-1)$ — this produces a heart $\mathcal{A}_\mu = \langle \mathcal{O}_{\mathbb{P}^1}(k), \mathcal{O}_{\mathbb{P}^1}(k-1)[1]\rangle$. The central charge in this case may be taken by choosing z_1, z_2 in the upper half plane with $\mathrm{Arg}(z_1) < \mathrm{Arg}(z_2)$ and assigning $Z(\mathcal{O}_{\mathbb{P}^1}(k)) = z_1$ and $Z(\mathcal{O}_{\mathbb{P}^1}(k-1)[1]) = z_2$. By [GKR04], there are triangles

$$\begin{aligned} \mathcal{O}_{\mathbb{P}^1}(k)^{\oplus n-k+1} \to \mathcal{O}_{\mathbb{P}^1}(n) \to \mathcal{O}_{\mathbb{P}^1}(k-1)[1]^{\oplus n-k} \quad &\text{for } n > k \\ \mathcal{O}_{\mathbb{P}^1}(k)[-1]^{\oplus k-1-n} \to \mathcal{O}_{\mathbb{P}^1}(n) \to \mathcal{O}_{\mathbb{P}^1}(k-1)^{\oplus k-n} \quad &\text{for } n < k-1 \\ \mathcal{O}_{\mathbb{P}^1}(k) \to \Bbbk(x) \to \mathcal{O}_{\mathbb{P}^1}(k-1)[1] \quad & \end{aligned} \tag{$**$}$$

so that the stable objects are skyscraper sheaves, $\mathcal{O}_{\mathbb{P}^1}(n)$ for $n \geq k$, and $\mathcal{O}_{\mathbb{P}^1}(n)[1]$ for $n \leq k-1$. As described in [BMW15], such stability conditions may be deformed so that $\phi(\mathcal{O}_{\mathbb{P}^1}(k)) > \phi(\mathcal{O}_{\mathbb{P}^1}(k-1)[1])$, in which case the only stable objects in $\mathcal{A}_\mu$ are shifts of the two stable objects $\mathcal{O}_{\mathbb{P}^1}(k)$ and $\mathcal{O}_{\mathbb{P}^1}(k-1)[1]$. The authors show that one may again tilt the heart $\mathcal{A}_\mu$ at the vertical axis to produce a heart $\langle \mathcal{O}_{\mathbb{P}^1}(k), \mathcal{O}_{\mathbb{P}^1}(k-1)[2]\rangle$; this process can continue to produce hearts $\langle \mathcal{O}_{\mathbb{P}^1}(k), \mathcal{O}_{\mathbb{P}^1}(k-1)[\ell]\rangle$ for $\ell \geq 2$ such that the only stable objects are shifts of $\mathcal{O}_{\mathbb{P}^1}(k)$ and $\mathcal{O}_{\mathbb{P}^1}(k-1)$. These are exactly the degenerate stability conditions used in [KKO22] to show that the Thurston compactification does not yield an embedding for $\mathrm{Stab}(\mathbb{P}^1)$.

Remark 3.2. Defining the region $\mathrm{Stab}^{\mathrm{Geo}}(\mathbb{P}^1)$ is somewhat redundant since any σ_w can be rotated by the $\mathbb{C}$-action to an algebraic stability condition with heart $\mathcal{A}_\mu$ for some $\mu \in \mathbb{R}$. In particular, fix some w in the upper half-plane and take $\sigma = \exp(-\pi i/2) \cdot \sigma_w$ so that $Z_\sigma(\Bbbk(x)) = i$. Let $n \in \mathbb{Z}$ be such that $\Re(w) \in [n-1, n)$ and choose $z_1 = \exp(-\pi i/2) \cdot (w - n)$ and $z_2 = \exp(-\pi i/2) \cdot ((n-1) - w)$. If we define a stability condition σ' with heart $\mathcal{A}_{\Re(w)}$ such that $Z_{\sigma'}(\mathcal{O}_{\mathbb{P}^1}(n)) = z_1$ and $Z_{\sigma'}(\mathcal{O}_{\mathbb{P}^1}(n-1)[1]) = z_2$, it is easy to see that $\sigma = \sigma'$ using the triangles above. Thus, in most literature the description of $\mathrm{Stab}(\mathbb{P}^1)$ is usually given by classifying stability conditions in terms of objects $\mathcal{O}_{\mathbb{P}^1}(k)[j]$ and $\mathcal{O}_{\mathbb{P}^1}(k-1)[p+j]$ for $j, k, p \in \mathbb{Z}$ and $p \geq 1$.

In order to see how the non-geometric (algebraic) stability conditions connect to $\mathrm{Stab}^{\mathrm{Geo}}(\mathbb{P}^1)$, we follow the approach of [Bay11]. Up to the $\mathbb{C}$-action, only consider stability conditions of the form $\sigma_w = (\mathrm{Coh}(\mathbb{P}^1), Z_w)$. Fix $\epsilon > 0$ and take $w = x + i\epsilon$ with $n - 1 < x < n$. The central charge Z_w can be visualized as the diagram on the left:



(a) Z_w (b) $Z_{\overline{w}}$

By deforming Z_w along the vertical line in $\mathbb{C}$ from w to $\overline{w}$, we see that the order of $\phi(\mathcal{O}_{\mathbb{P}^1}(n))$ and $\phi(\mathcal{O}_{\mathbb{P}^1}(n-1)[1])$ flips — in particular, $\mathbb{R}\backslash\mathbb{Z}$ becomes a wall for $\Bbbk(x)$ using the third triangle of $(**)$, where the integer points must be removed as $Z_n(\mathcal{O}_{\mathbb{P}^1}(n))$ vanishes.

The above construction produces an open subset $U_1 \subset \operatorname{Stab}(\mathbb{P}^1)$ identical to $\mathbb{C}\backslash\mathbb{Z}$; however, the inclusion is strict as this does not account for stability conditions with hearts $\langle \mathcal{O}_{\mathbb{P}^1}(d), \mathcal{O}_{\mathbb{P}^1}(d-1)[\ell]\rangle$ for $\ell \geq 2$. Following [Oka06], define

$$S^-_{p,i,j} := \{(Z,\mathcal{P}) \in \operatorname{Stab}(\mathbb{P}^1) \mid 0 < \phi(\mathcal{O}_{\mathbb{P}^1}(i)[j]) \leq \phi(\mathcal{O}_{\mathbb{P}^1}(i-1)[p+j]) \leq 1\}$$
$$S^+_{p,i,j} := \{(Z,\mathcal{P}) \in \operatorname{Stab}(\mathbb{P}^1) \mid 0 < \phi(\mathcal{O}_{\mathbb{P}^1}(i-1)[p+j]) \leq \phi(\mathcal{O}_{\mathbb{P}^1}(i)[j]) \leq 1\}$$

and say that two of these regions are *neighbors* if the intersection of their closures is a codimension one submanifold.

Lemma 3.3 ([Oka06, Lemmas 5.4, 5.5])**.**

(i) $S^-_{1,i,j}$ *has neighbors* $S^-_{1,i+1,j}$, $S^-_{1,i-1,j}$, *and* $S^+_{1,i,j}$.

(ii) For $p > 1$, $S^-_{p,i,j}$ *has neighbors* $S^+_{p,i,j}$, $S^+_{p-1,i,j+1}$, *and* $S^+_{p-1,i,j}$.

(iii) For $p \geq 1$, $S^+_{p,i,j}$ *has neighbors* $S^-_{p,i,j}$, $S^-_{p+1,i,j-1}$, *and* $S^-_{p+1,i,j}$.

One of the main results of the above author is that this provides a complete decomposition of $\operatorname{Stab}(\mathbb{P}^1)$ into regions where the stable objects can be completely described.

3.1.1 Local $\mathbb{P}^1$

Since the stability manifold of the projective line (as well as its Thurston compactification) have been completely described, we instead wish to turn our attention to the local projective line $\omega_{\mathbb{P}^1}$. As before, let $\pi : \omega_{\mathbb{P}^1} \to \mathbb{P}^1$ denote the fibration and $i : \mathbb{P}^1 \hookrightarrow \omega_{\mathbb{P}^1}$ the inclusion of the zero section. We will continue to let $\mathcal{D}_{0,\mathbb{P}^1}$ denote the full subcategory of $D^\flat(\omega_{\mathbb{P}^1})$ consisting of objects whose cohomology sheaves are supported on $\mathbb{P}^1$, and let Coh_0 denote the full subcategory of $\mathrm{Coh}(\omega_{\mathbb{P}^1})$ consisting of sheaves whose reduced support is contained in the zero section $\mathbb{P}^1$. It is well-known that the space $\omega_{\mathbb{P}^1}$ is the minimal resolution of the Kleinian singularity $\mathbb{C}^2/\mathbb{Z}_2$, and thus by the derived McKay correspondence the category $\mathcal{D}_{0,\mathbb{P}^1}$ corresponds to the full subcategory of $D^\flat_{\mathbb{Z}_2}(\mathrm{Coh}\,\mathbb{C}^2)$ consisting of complexes of $\mathbb{Z}_2$-equivariant coherent sheaves whose cohomology objects are supported at the origin [Bri09]. In a certain sense, the category $\mathcal{D}_{0,\mathbb{P}^1}$ behaves much like like the derived category of a non-generic K3 surface near a rational curve $C \cong \mathbb{P}^1$.

Unlike the case of K3 surfaces, the stability manifold of $\mathcal{D}_{0,\mathbb{P}^1}$ inherits much of its structure from the base-scheme $\mathbb{P}^1$. For example, the inclusion

$$i_* : K(\mathbb{P}^1) \to K(\mathcal{D}_{0,\mathbb{P}^1})$$

is surjective, and allows us to identify $\mathrm{ch}(E)$ with $\mathrm{ch}(\pi_* E)$ for any object $E \in \mathcal{D}_{0,\mathbb{P}^1}$. By [BMW15, Lemma 4.1], the geometric chamber of $\mathcal{D}_{0,\mathbb{P}^1}$ is identical to the geometric chamber of $D^\flat(\mathbb{P}^1)$; the stability conditions are those with central charge $Z_w(\mathcal{E}) = -\deg(\pi_*\mathcal{E}) + w \cdot \mathrm{rk}(\pi_*\mathcal{E})$ (with w in the upper half-plane) and heart Coh_0, and the stable objects are simply of the form $i_*\mathcal{O}_{\mathbb{P}^1}(n)$ or $\Bbbk(x)$ for $x \in \mathbb{P}^1$. Similar to the case of $\mathbb{P}^1$, defining the geometric chamber for $\mathcal{D}_{0,\mathbb{P}^1}$ is somewhat redundant as it can be completely covered by quivery stability conditions.

Using the results of Section 2.5, any simple exceptional collection of sheaves on $\mathbb{P}^1$ takes the form $\mathfrak{E}_k = \{\mathcal{O}_{\mathbb{P}^1}(k-1), \mathcal{O}_{\mathbb{P}^1}(k)\}$ and thus we obtain spherical collections $\{i_*\mathcal{O}_{\mathbb{P}^1}(k), i_*\mathcal{O}_{\mathbb{P}^1}(k-1)[1]\}$ in $\mathcal{D}_{0,\mathbb{P}^1}$ for each $k \in \mathbb{Z}$. Let $\mathcal{A}_k$ denote the heart $\langle i_*\mathcal{O}_{\mathbb{P}^1}(k), i_*\mathcal{O}_{\mathbb{P}^1}(k-1)[1]\rangle$, and continue to let Θ_k denote the subset of of all stability conditions with heart is equivalent to $\mathcal{A}_k$ up to the $\widetilde{\mathrm{GL}}_2^+(\mathbb{R})$-action. There are of course two $\widetilde{\mathrm{GL}}_2^+(\mathbb{R})$-orbits, which we will denote by Θ_k^- and Θ_k^+, corresponding to when $\phi(i_*\mathcal{O}_{\mathbb{P}^1}(k)) < \phi(i_*\mathcal{O}_{\mathbb{P}^1}(k-1)[1])$ and $\phi(i_*\mathcal{O}_{\mathbb{P}^1}(k)) > \phi(i_*\mathcal{O}_{\mathbb{P}^1}(k-1)[1])$, respectively. By considering the pushforwards of the third triangle in $(**)$, we see that $\Theta_k^- = \mathrm{Stab}^{\mathrm{Geo}}(\mathcal{D}_{0,\mathbb{P}^1})$ by an argument similar to Remark 3.2 above.

One of the main differences between the stability manifold of $D^\flat(\mathbb{P}^1)$ and $\mathcal{D}_{0,\mathbb{P}^1}$ is that there is always a semi-stable object of class $[\Bbbk(x)]$ for every stability condition in $\mathrm{Stab}(\mathcal{D}_0)$. Even in the simplest case of Θ_k^+ which is adjacent to the geometric chamber, it takes a non-trivial amount of diagram chasing to produce stable objects using spherical twists. For brevity, we will denote the automorphism $\mathrm{Tw}_{i_*\mathcal{O}_{\mathbb{P}^1}(k)}$ by Tw_k. We first want to find when the objects $\mathrm{Tw}_a\, i_*\mathcal{O}_{\mathbb{P}^1}(b)$ still lie in the heart $\mathcal{A}_k$, and determine their respective Jordan-Holder filtrations.

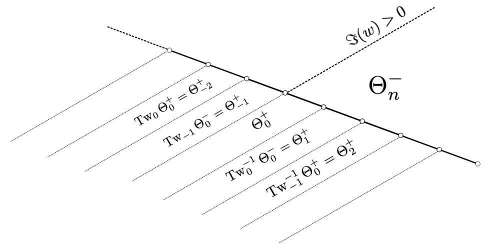


Figure 2: How crossing the wall $\Im(w) = 0$ yields hearts which, up to the $\mathbb{C}$-action, are equivalent to the action of certain spherical twists

Lemma 3.4. *Fix $a, b \in \mathbb{Z}$, and consider the following cases:*

(i) When $b - a \leq -1$, $\mathrm{Tw}_a\, i_*\mathcal{O}_{\mathbb{P}^1}(b)$ fits into the following distinguished triangle:

$$i_*\mathcal{O}_{\mathbb{P}^1}(a-1)^{\oplus a-b} \to \mathrm{Tw}_a\, i_*\mathcal{O}_{\mathbb{P}^1}(b) \to i_*\mathcal{O}_{\mathbb{P}^1}(a)[-1]^{\oplus a-b+1}$$

(ii) When $b - a \geq 1$, $\mathrm{Tw}_a\, i_*\mathcal{O}_{\mathbb{P}^1}(b)$ fits into the following distinguished triangle:

$$i_*\mathcal{O}_{\mathbb{P}^1}(a-1)[1]^{\oplus b-a} \to \mathrm{Tw}_a\, i_*\mathcal{O}_{\mathbb{P}^1}(b) \to i_*\mathcal{O}_{\mathbb{P}^1}(a)^{\oplus b-a-1}$$

in particular, when $b - a = 1$ we have $\mathrm{Tw}_a\, i_*\mathcal{O}_{\mathbb{P}^1}(a+1) \cong i_*\mathcal{O}_{\mathbb{P}^1}(a-1)[1]$

Similarly for the inverse spherical twists

(iii) When $b - a \leq -1$, $\mathrm{Tw}_a^{-1}\, i_*\mathcal{O}_{\mathbb{P}^1}(b)$ fits into the following distinguished triangle

$$i_*\mathcal{O}_{\mathbb{P}^1}(a)^{\oplus a-b-1} \to \mathrm{Tw}_a^{-1}\, i_*\mathcal{O}_{\mathbb{P}^1}(b) \to i_*\mathcal{O}_{\mathbb{P}^1}(a+1)^{\oplus a-b}[-1]$$

in particular, when $b - a = -1$ we have $\mathrm{Tw}_a^{-1}\, i_*\mathcal{O}_{\mathbb{P}^1}(a-1) \cong i_*\mathcal{O}_{\mathbb{P}^1}(a+1)[-1]$.

(iv) When $b - a \geq 1$, $\mathrm{Tw}_a^{-1}\, i_*\mathcal{O}_{\mathbb{P}^1}(b)$ fits into the following distinguished triangle

$$i_*\mathcal{O}(a)[1]^{\oplus b-a+1} \to \mathrm{Tw}_a^{-1}\, i_*\mathcal{O}(b) \to i_*\mathcal{O}_{\mathbb{P}^1}(a+1)^{\oplus b-a}$$

Proof. First note that the last morphism in the distinguished triangle

$$i^*i_*\mathcal{E} \to \mathcal{E} \to \mathcal{E} \otimes \mathcal{O}_{\mathbb{P}^1}(2)[2]$$

is necessarily zero for every object $\mathcal{E}$, so that the triangle splits in $D^\flat(\mathbb{P}^1)$. As a consequence, this also forces

$$\mathrm{RHom}^\bullet_{\mathbb{P}^1}(\mathcal{E}, \mathcal{O}_{\mathbb{P}^1}(b)) \to \mathrm{RHom}^\bullet_{\mathbb{P}^1}(i^* i_* \mathcal{E}, \mathcal{O}_{\mathbb{P}^1}(b)) \to \mathrm{RHom}^\bullet_{\mathbb{P}^1}(\mathcal{E}, \mathcal{O}_{\mathbb{P}^1}(b-2))[-1]$$

to split. Taking $\mathcal{E} = \mathcal{O}_{\mathbb{P}^1}(a)$, we obtain $\mathrm{RHom}^\bullet_{\mathbb{P}^1}(i^* i_* \mathcal{O}_{\mathbb{P}^1}(a), \mathcal{O}_{\mathbb{P}^1}(b)) \cong H^\bullet(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(b-a)) \oplus H^\bullet(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(b-a-2))[-1]$ so that by adjointness of $i^* \dashv i_*$ we obtain the straightforward decomposition

$$\mathrm{RHom}^\bullet_{\mathcal{D}_{0,\mathbb{P}^1}}(i_*\mathcal{O}_{\mathbb{P}^1}(a), i_*\mathcal{O}_{\mathbb{P}^1}(b)) = \begin{cases} \mathbb{C}^{a-b-1}[-1] \oplus \mathbb{C}^{a-b+1}[-2] & b-a \leq -1 \\ \mathbb{C} \oplus \mathbb{C}[-2] & a = b \\ \mathbb{C}^{b-a-1}[-1] \oplus \mathbb{C}^{b-a+1} & b-a \geq 1 \end{cases}$$

First consider the case that $b - a \leq -2$. By considering the pushforward of the second triangle in $(**)$ under i_*, we obtain a commuting diagram

$$\begin{tikzcd}
i_*\mathcal{O}_{\mathbb{P}^1}(a)[-1]^{\oplus a-b-1} \arrow[r, "i_* \mathrm{ev}_{\mathbb{P}^1}"] \arrow[d] & i_*\mathcal{O}_{\mathbb{P}^1}(b) \arrow[r] \arrow[d, equal] & i_*\mathcal{O}_{\mathbb{P}^1}(a-1)^{\oplus a-b} \arrow[d] \\
i_*\mathcal{O}_{\mathbb{P}^1}(a)[-1]^{\oplus a-b-1} \oplus i_*\mathcal{O}_{\mathbb{P}^1}(a)[-2]^{\oplus a-b+1} \arrow[r, "\mathrm{ev}_{\omega_{\mathbb{P}^1}}"] \arrow[d] & i_*\mathcal{O}_{\mathbb{P}^1}(b) \arrow[r] \arrow[d] & \mathrm{Tw}_a\, i_*\mathcal{O}_{\mathbb{P}^1}(b) \arrow[d] \\
i_*\mathcal{O}_{\mathbb{P}^1}(a)[-2]^{\oplus a-b+1} \arrow[r] & 0 \arrow[r] & i_*\mathcal{O}_{\mathbb{P}^1}(a)[-1]^{\oplus a-b+1}
\end{tikzcd}$$

so that by (TR4) the last column gives us the desired distinguished triangle in $\mathcal{D}_{0,\mathbb{P}^1}$.

In the case that $b - a = -1$, shifting the defining triangle of $\mathrm{Tw}_a\, i_*\mathcal{O}_{\mathbb{P}^1}(a-1)$ gives the desired result. When $b - a = 1$, we simply have that the spherical twist through $i_*\mathcal{O}_{\mathbb{P}^1}(a)$ is the same as the pushforward of the left mutation through $\mathcal{O}_{\mathbb{P}^1}(a)$ under i_* due to (TR3):

$$\begin{tikzcd}
i_*\mathcal{O}_{\mathbb{P}^1}(a)^{\oplus 2} \arrow[r] \arrow[d, equal] & i_*\mathcal{O}_{\mathbb{P}^1}(a+1) \arrow[r] \arrow[d, equal] & i_*\mathcal{O}_{\mathbb{P}^1}(a-1)[1] \arrow[d] \\
i_*\mathcal{O}_{\mathbb{P}^1}(a)^{\oplus 2} \arrow[r] & i_*\mathcal{O}_{\mathbb{P}^1}(a+1) \arrow[r] & \mathrm{Tw}_a\, i_*\mathcal{O}_{\mathbb{P}^1}(a+1)
\end{tikzcd}$$

When $b-a \geq 2$, we may use the first distinguished triangle of $(**)$ to again construct a commuting diagram

$$\begin{tikzcd}
i_*\mathcal{O}_{\mathbb{P}^1}(a)^{\oplus b-a+1} \arrow[r, "i_*\,\mathrm{ev}_{\mathbb{P}^1}"] \arrow[d] & i_*\mathcal{O}_{\mathbb{P}^1}(b) \arrow[r] \arrow[d, equal] & i_*\mathcal{O}_{\mathbb{P}^1}(a-1)[1]^{\oplus b-a} \arrow[d] \\
i_*\mathcal{O}_{\mathbb{P}^1}(a)^{\oplus b-a+1} \oplus i_*\mathcal{O}_{\mathbb{P}^1}(a)^{\oplus b-a-1}[-1] \arrow[r, "\mathrm{ev}_{\omega_{\mathbb{P}^1}}"] \arrow[d] & i_*\mathcal{O}_{\mathbb{P}^1}(b) \arrow[r] \arrow[d] & \mathrm{Tw}_a\, i_*\mathcal{O}_{\mathbb{P}^1}(b) \arrow[d] \\
i_*\mathcal{O}_{\mathbb{P}^1}(a)^{\oplus b-a-1}[-1] \arrow[r] & 0 \arrow[r] & i_*\mathcal{O}_{\mathbb{P}^1}(a)^{\oplus b-a-1}
\end{tikzcd}$$

The computation of the inverse spherical twists $\mathrm{Tw}_a^{-1}\, i_*\mathcal{O}_{\mathbb{P}^1}(b)$ is similar and therefore omitted. $\square$

We omit the trivial cases of $\mathrm{Tw}_a\, i_*\mathcal{O}_{\mathbb{P}^1}(a)$ from the lemma above since the objects $i_*\mathcal{O}_{\mathbb{P}^1}(a)$ are 2-spherical. It is also possible to find decompositions of $\mathrm{Tw}_k^{\pm}\, \mathbb{k}(x)$ in the heart $\langle i_*\mathcal{O}_{\mathbb{P}^1}(k), i_*\mathcal{O}_{\mathbb{P}^1}(k-1)[1]\rangle$ by following the technique of [BMW15]; by applying Tw_k to the triangle

$$i_*\mathcal{O}_{\mathbb{P}^1}(k+1) \to \mathbb{k}(x) \to i_*\mathcal{O}_{\mathbb{P}^1}(k)[1]$$

we obtain

$$i_*\mathcal{O}_{\mathbb{P}^1}(k-1)[1] \to \mathrm{Tw}_k\, \mathbb{k}(x) \to i_*\mathcal{O}_{\mathbb{P}^1}(k).$$

Similarly, we can apply Tw_{k-1}^{-1} to the triangle

$$i_*\mathcal{O}_{\mathbb{P}^1}(k-1) \to \mathbb{k}(x) \to i_*\mathcal{O}_{\mathbb{P}^1}(k-2)[1]$$

to give

$$i_*\mathcal{O}_{\mathbb{P}^1}(k-1)[1] \to \mathrm{Tw}_{k-1}^{-1}\, \mathbb{k}(x) \to i_*\mathcal{O}_{\mathbb{P}^1}(k).$$

By [BMW15, Lemma 4.2], $\mathrm{Tw}_k\, \mathbb{k}(x)$ is necessarily stable in Θ_k^+, and an identical argument (along with the above lemma) shows that $\mathrm{Tw}_k\, i_*\mathcal{O}_{\mathbb{P}^1}(n)[1-\varepsilon]$ is stable in the heart $\mathcal{A}_k$, where $\varepsilon = 1$ when $n > k$ and $\varepsilon = 0$ when $n < k$. Consequently, we can explicitly describe the intersection of $\Theta_k \cap \mathrm{Tw}_k\, \Theta_k$ in an analogous way to the proof of [BM11, Lemma 7.7].

As the object $\mathrm{Tw}_k\, i_*\mathcal{O}_{\mathbb{P}^1}(k-1)[1]$ is stable in $\mathrm{Tw}_k\,\Theta_k$ and fits into a distinguished triangle

$$i_*\mathcal{O}_{\mathbb{P}^1}(k-1)[1] \to \mathrm{Tw}_k\, i_*\mathcal{O}_{\mathbb{P}^1}(k-1)[1] \to i_*\mathcal{O}_{\mathbb{P}^1}(k)^{\oplus 2}$$

by the above lemma, we immediately have $\phi(i_*\mathcal{O}_{\mathbb{P}^1}(k-1)[1]) < \phi(i_*\mathcal{O}_{\mathbb{P}^1}(k))$ so that $\Theta_k \cap \mathrm{Tw}_k\,\Theta_k = \Theta_k^+$. As one would expect, by deforming such a stability condition in $\mathrm{Tw}_k\,\Theta_k$ to one with $\phi(i_*\mathcal{O}_{\mathbb{P}^1}(k)[-1]) > \phi(\mathrm{Tw}_k\, i_*\mathcal{O}_{\mathbb{P}^1}(k-1)[1])$, the objects $\mathrm{Tw}_k\, i_*\mathcal{O}_{\mathbb{P}^1}(n)[1-\varepsilon]$ (for $n \neq k$) are now destabilized. This can be seen in the case that $n > k$ by applying Tw_k to the triangle

$$i_*\mathcal{O}_{\mathbb{P}^1}(k)^{\oplus n-k+1} \to i_*\mathcal{O}_{\mathbb{P}^1}(n) \to i_*\mathcal{O}_{\mathbb{P}^1}(k-1)[1]^{\oplus n-k}.$$

An identical argument shows that $\Theta_k \cap \mathrm{Tw}_{k-1}^{-1}\,\Theta_k = \Theta_k^+$.

Let $\mathfrak{S} = \{S_0, S_1\}$ denote an arbitrary spherical collection in $\mathcal{D}_{0,\mathbb{P}^1}$. Using Lemma 3.4 and the fact that

$$\begin{aligned} \mathrm{Tw}_{S_1} \circ \mathrm{Tw}_{S_2} \circ \mathrm{Tw}_{S_1}^{-1} &\cong \mathrm{Tw}_{\mathrm{Tw}_{S_1} S_2} \\ \mathrm{Tw}_{S_1}^{-1} \circ \mathrm{Tw}_{S_2} \circ \mathrm{Tw}_{S_1} &\cong \mathrm{Tw}_{\mathrm{Tw}_{S_1}^{-1} S_2} \end{aligned}$$

it immediately follows that the entire subgroup of $\mathrm{Aut}_{\mathrm{sph}}(\mathcal{D}_{0,\mathbb{P}^1}) \subset \mathrm{Aut}(\mathcal{D}_{0,\mathbb{P}^1})$ consisting of compositions spherical twists $\mathrm{Tw}_{k_1}^{\pm\alpha_1} \circ \cdots \circ \mathrm{Tw}_{k_n}^{\pm\alpha_n}$ can be generated by $\langle \mathrm{Tw}_k^{\pm}, \mathrm{Tw}_{k+1}^{\pm} \rangle$ for any choice of $k \in \mathbb{Z}$. As a consequence, we can always obtain the heart $\mathcal{A}_k$ from $\mathcal{A}_0$ using only the twists $\mathrm{Tw}_{-1}^{\pm}, \mathrm{Tw}_0^{\pm}$; for example, when $k < 0$ we can use the fact that $\mathrm{Tw}_a\, i_*\mathcal{O}_{\mathbb{P}^1}(a+1) \cong i_*\mathcal{O}_{\mathbb{P}^1}(a-1)[1]$ to show $\mathcal{A}_k \cong \mathrm{Tw}_k \ldots \mathrm{Tw}_{-2}\,\mathrm{Tw}_{-1}\,\mathcal{A}_0$. By applying

$$\begin{aligned} \mathrm{Tw}_k\,\mathrm{Tw}_{k+1} &\cong \mathrm{Tw}_{\mathrm{Tw}_{k+1} i_*\mathcal{O}_{\mathbb{P}^1}(k+2)}\,\mathrm{Tw}_{k+1} \\ &\cong \mathrm{Tw}_{k+1}\,\mathrm{Tw}_{k+2}\,\mathrm{Tw}_{k+1}^{-1}\,\mathrm{Tw}_{k+1} \\ &\cong \mathrm{Tw}_{k+1}\,\mathrm{Tw}_{k+2} \end{aligned}$$

a finite number of times, we may always replace adjacent twists $\mathrm{Tw}_i\,\mathrm{Tw}_{i+1}$ with $\mathrm{Tw}_{-1}\,\mathrm{Tw}_0$.

Thus,

$$\mathrm{Tw}_k \dots \mathrm{Tw}_{-2} \mathrm{Tw}_{-1}$$

is equivalent to a length-k word of the form $\mathrm{Tw}_{-1} \mathrm{Tw}_0 \mathrm{Tw}_{-1} \dots \mathrm{Tw}_{-\epsilon}$ where $0 \leq \epsilon \leq 1$ and $\epsilon \equiv k \mod 2$. The case $k > 0$ follows from taking the inverse of the above automorphism. As a result, to describe the stability manifold $\mathrm{Stab}(\mathcal{D}_{0,\mathbb{P}^1})$ it will suffice to look only at hearts obtained from $\mathcal{A}_0$ by the action of $\langle \mathrm{Tw}_{-1}^{\pm}, \mathrm{Tw}_0^{\pm} \rangle$ [11].

By starting with the orbit Θ_0, we may consecutively piece together adjacent orbits starting with the fact that $\Theta_0 \cap \mathrm{Tw}_0 \Theta_0 = \Theta_0^+ = \Theta_0 \cap \mathrm{Tw}_{-1}^{-1} \Theta_0$ as shown above. After taking the quotient under the $\mathbb{C}$-action, a naïve attempt at visualizing the stability manifold may look something like that in Figure 3.

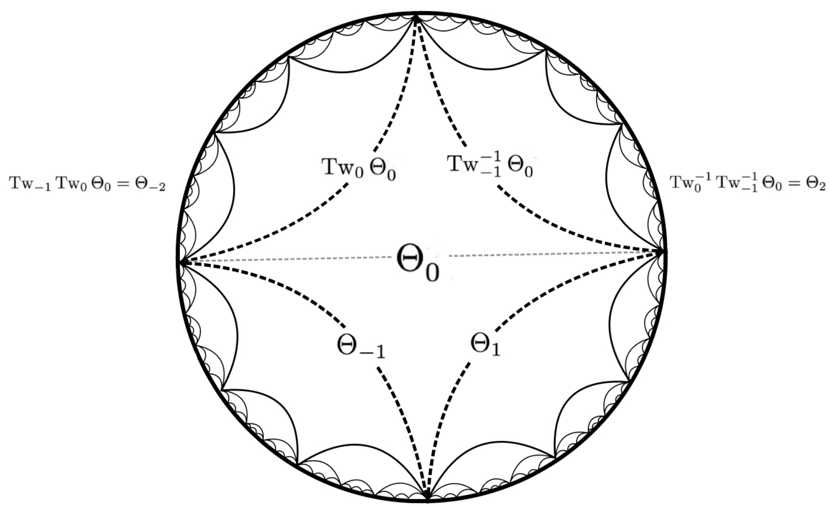


Figure 3: The orbit Θ_0 consists of the inner diamond figure. The orbits $\mathrm{Tw}_i^{\pm} \Theta_0$ all contain exactly one-half of the diamond figure, along with the respective chamber with solid boundary.

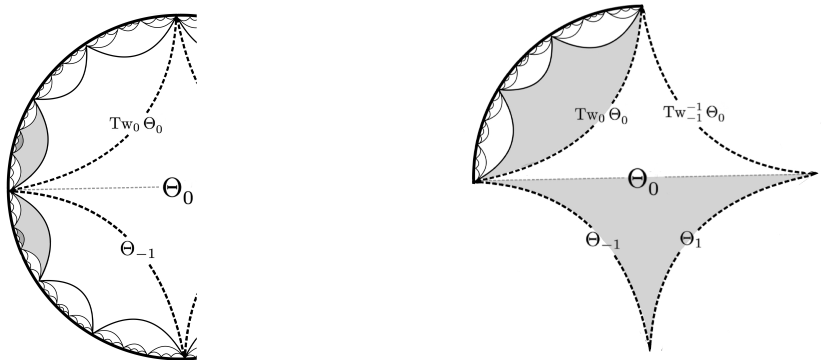
Unfortunately, the figure shown above is far from accurate due to the braid relations

$$\mathrm{Tw}_{-1} \mathrm{Tw}_0 \mathrm{Tw}_{-1} \cong \mathrm{Tw}_0 \mathrm{Tw}_{-1} \mathrm{Tw}_0$$

as well as the identities $\mathrm{Tw}_{-1} \mathrm{Tw}_0 \Theta_0^- = \Theta_{-2}^-$, $\mathrm{Tw}_{-1} \mathrm{Tw}_0 \mathrm{Tw}_{-1} \Theta_0^- = \Theta_{-3}^-$, etc. In par-

[11] This also heavily depends on the fact that all exceptional collections of vector bundles on $\mathbb{P}^1$ can be obtained by a sequence of mutations from $\mathcal{A}_0$ — namely, that $D^b(\mathbb{P}^1)$ is constructible

ticular, infinitely many of the chambers in the above picture actually overlap to make a single $\widetilde{\mathrm{GL}}_2^+(\mathbb{R})$-orbit, as pictured in the figure below.



(a) Identification under braid relations

(b) Identification under $\Theta_n^- = \Theta_m^-$

Instead, it was shown by the authors of [BDL20] that by only using a single $\widetilde{\mathrm{GL}}_2^+(\mathbb{R})$-orbit as the tile and slightly changing the generators of the tiling, there is a much more faithful representation of $\mathrm{Stab}^\dagger(\mathcal{D}_{0,\mathbb{P}^1})$. Since $i_*\mathcal{O}_{\mathbb{P}^1}(1) = \mathrm{Tw}_0^{-1}\, i_*\mathcal{O}_{\mathbb{P}^1}(-1)[1]$, we may consider the braid

$$\gamma = \mathrm{Tw}_{-1}\,\mathrm{Tw}_0 = \mathrm{Tw}_0\,\mathrm{Tw}_1 = \mathrm{Tw}_1\,\mathrm{Tw}_{-1}$$

(where the last equality follows from the braid relations and $\mathrm{Tw}_1 \cong \mathrm{Tw}_0^{-1}\,\mathrm{Tw}_{-1}\,\mathrm{Tw}_0$) which fixes the geometric chamber. By tiling with respect to the closure of the geometric chamber instead of the entire set Θ_0 yields the following diagram:

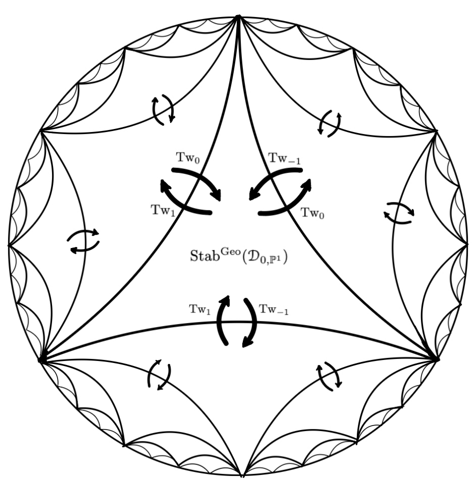


Figure 5: A faithful representation of $\mathrm{Stab}(\mathcal{D}_{0,\mathbb{P}^1})/\mathbb{C}$ adapted from [BDL20, Figure 6]

Theorem 3.5 ([BMW15]). *There is a single connected component*

$$\mathrm{Stab}(\mathcal{D}_{0,\mathbb{P}^1}) = \mathrm{Stab}^{\dagger}(\mathcal{D}_{0,\mathbb{P}^1}) = \mathrm{Stab}^{\mathrm{Alg}}(\mathcal{D}_{0,\mathbb{P}^1})$$

and it is simply connected.

3.2 The case of the Projective Plane and $\omega_{\mathbb{P}^2}$

In the previous section, it was noted that although exceptional collections need only be full to define quivery stability conditions on a del Pezzo surface S, a much stronger property was required to define analogous conditions on their Calabi-Yau counterparts $\mathcal{D}_{0,S} \subseteq D^{\flat}(\omega_S)$ — the exceptional collections had to be simple. In the case of del Pezzo surfaces, the condition (†) from the beginning of this chapter restricts us to exactly one example — the projective plane $\mathbb{P}^2$. The stability manifolds of $\mathbb{P}^2$ and $\mathcal{D}_{0,\mathbb{P}^2}$ were extensively studied in [Li17] and [BM11], respectively. In both cases, the construction of stability conditions is heavily dependent on the classical theory of Chern characters of Geiseker-stable sheaves on $\mathbb{P}^2$ — thus, the beginning portion of this section will focus on the results of [DLP85], while the latter half will review the results of [BM11],[Li17].

Let H denote the hyperplane class and, by abuse of notation, let us identify ch_i with its degree $\mathrm{ch}_i \cdot H^{2-i}$. Consider the real plane with coordinates (μ, Δ), and define

$$\mathfrak{D} := \{(\mu_H(\mathcal{E}), \Delta(\mathcal{E})) \in \mathbb{Q}^2 \mid \mathcal{E} \text{ slope-stable, torsion-free sheaf on } \mathbb{P}^2\}$$

where $\Delta(\mathcal{E}) = \frac{\mathrm{ch}_1(\mathcal{E})^2}{2\,\mathrm{ch}_0(\mathcal{E})^2} - \frac{\mathrm{ch}_2(\mathcal{E})}{\mathrm{ch}_0(\mathcal{E})}$ denotes the discriminant of $\mathcal{E}$. Similarly, let $\mathfrak{D}_E$ denote $\{(\mu(\mathcal{E}), \Delta(\mathcal{E})) \in \mathbb{Q}^2 \mid \mathcal{E} \text{ exceptional vector bundle on } \mathbb{P}^2\}$. It was shown by [DLP85] that the discriminant of an exceptional vector bundle can be uniquely determined by its slope; consequently, if we let $\mathfrak{M}_E$ denote the projection of $\mathfrak{D}_E$ onto the first coordinate, for any $\alpha \in \mathfrak{M}_E$ there exists a unique $\Delta_\alpha \in \mathbb{Q}$ such that $(\alpha, \Delta_\alpha) \in \mathfrak{D}_E$. The main result of [DLP85] is that if we consider the functions

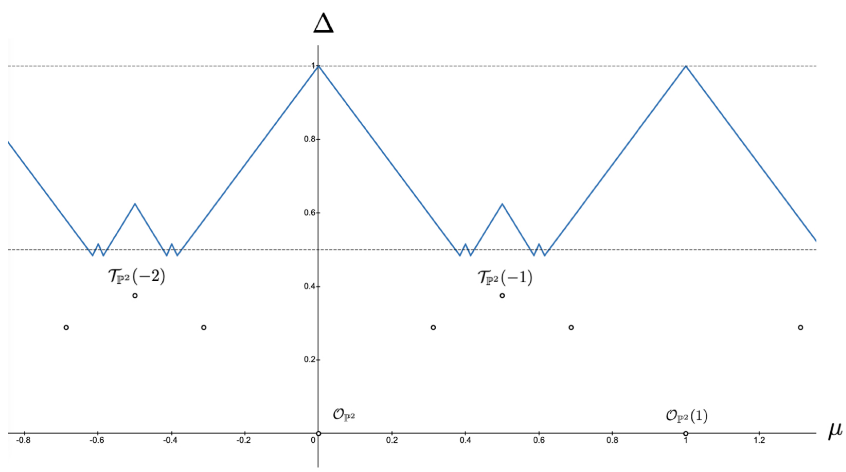
$$P(X) = 1 + \frac{3}{2}X + \frac{1}{2}X^2$$

$$p(x) = \begin{cases} P(-|x|) & |x| < 3 \\ 0 & \text{otherwise} \end{cases}$$

and define $\delta_\infty^{DP} : \mathbb{R} \to [1/2, 1]$ (often referred to as the *Drézet-Le Potier curve*) by

$$\delta_\infty^{DP}(x) = \sup\{p(x-\alpha) - \Delta_\alpha \mid \alpha \in \mathfrak{M}_E\}$$

there exists a non-exceptional slope-stable sheaf $\mathcal{E}$ of slope $\mu(\mathcal{E})$ and discriminant $\Delta(\mathcal{E})$ if and only if $\Delta(\mathcal{E}) \geq \delta_\infty^{DP}(\mu(\mathcal{E}))$. While we will not go into any of the details of the original proof, this result is fundamental to the construction of stability conditions on $\mathbb{P}^2$.



Define a family of central charges $Z_{a,b}$ parameterized by $a, b \in \mathbb{C}$ via

$$Z_{a,b}(\mathcal{E}) = -\operatorname{ch}_2(\mathcal{E}) + a \cdot \operatorname{ch}_1(\mathcal{E}) + b \cdot \operatorname{ch}_0(\mathcal{E}).$$

It should be noted that although this family of central charges is different from those given by [ABL07] in the previous section, they are equivalent up to the $\widetilde{\mathrm{GL}}_2^+(\mathbb{R})$-action (c.f. [BM11, Section 6]). By [Bri07, Proposition 5.3], to define a stability condition whose central charge is $Z_{a,b}$, it suffices to find a heart $\mathcal{A}$ such that $Z_{a,b}(K(\mathcal{A}))$ lies in the upper half-plane (and satisfies a Harder-Narasimhan property). Letting $\mathrm{Coh}^{\sharp(B)}(\mathbb{P}^2)$ denote the heart obtained by tilting $\mathrm{Coh}(\mathbb{P}^2)$ at some $B \in \mathbb{R}$, in order for $Z_{a,b}(\mathcal{E})$ to be contained in the upper half-plane for some $\mathcal{E} \in \mathcal{T}_B$ (where $\mathcal{T}_B$ denotes the subcategory in the torsion pair $(\mathcal{T}_B, \mathcal{F}_B)$) we must have

$$(\Im a)\ \mathrm{ch}_1(\mathcal{E}) + (\Im b)\ \mathrm{ch}_0(\mathcal{E}) \geq 0.$$

In the case that $\Im a \neq 0$, this is equivalent to requiring $\mu_H(\mathcal{E}) \geq -\Im b/\Im a$; as a consequence, we take $B = -\Im b/\Im a$ whenever $\Im a \neq 0$.

In addition, the support property requires any semi-stable vector bundle $\mathcal{E} \in \mathrm{Coh}^{\sharp(B)}(\mathbb{P}^2)$ to satisfy $Z_{a,b}(\mathcal{E}) \neq 0$. Assuming $r(\mathcal{E}) > 0$, one can easily show $Z_{a,b}(\mathcal{E}) = 0$ if and only if

$$\begin{aligned}
0 &= (\Im a)\mu_H(\mathcal{E}) + \Im b \\
\Re b &= -(\Re a)\mu_H(\mathcal{E}) + \frac{\mathrm{ch}_2(E)}{\mathrm{ch}_0(E)} \\
&= -(\Re a)\mu_H(\mathcal{E}) - \Delta(\mathcal{E}) + \frac{1}{2}\mu_H(\mathcal{E})^2.
\end{aligned}$$

Using the characterization of slope-stable vector bundles as described above, the authors of [BM11] define the region $G \subset \mathbb{C}^2$ as the set of pairs $(a, b) \in \mathbb{C}^2$ such that

(i) $\Im a > 0$

(ii) Letting $B := -\Im b/\Im a$, we require

$$\Re b > -B \cdot \Re a - \Delta_B + \frac{1}{2}B^2 \tag{3}$$

if there exists an exceptional bundle of slope B (in which case we let Δ_B denote the discriminant), and

$$\Re b > -B \cdot \Re a - \delta_\infty^{DP}(B) + \frac{1}{2}B^2 \tag{4}$$

otherwise.

Proposition 3.6 ([BM11, Corollary 4.6], [Li17, Proposition 1.12])**.** *The map $G \to$*

$\mathrm{Stab}^{\mathrm{Geo}}(\mathbb{P}^2)$ *given by*

$$(a,b) \mapsto \sigma_{a,b} = \big(Z_{a,b}, \mathrm{Coh}^{\sharp(B)}(\mathbb{P}^2)\big)$$

gives a continuous embedding into the set of stability conditions such that each $\Bbbk(x)$ *is stable of phase 1. In fact, any such stability condition with this property can be written in the above form.*

As an immediate consequence, [Li17] showed that this implies $\mathrm{Stab}^{\mathrm{Geo}}(\mathbb{P}^2) \subset \mathrm{Stab}^{\mathrm{nd}}(\mathbb{P}^2)$ so that $\widetilde{\mathrm{GL}}_2^+(\mathbb{R})$ acts freely on the set of geometric stability conditions, giving

$$\mathrm{Stab}^{\mathrm{Geo}}(\mathbb{P}^2)/\widetilde{\mathrm{GL}}_2^+(\mathbb{R}) \cong G$$

([Li17, Corollary 1.15]).

While the central charges $Z_{a,b}$ will be necessary later when building stability conditions on $\mathcal{D}_{0,\mathbb{P}^2}$, it will also be useful for us to re-parameterize the geometric stability conditions with respect to the $\{\frac{\mathrm{ch}_1}{\mathrm{ch}_0}, \frac{\mathrm{ch}_2}{\mathrm{ch}_0}\}$-plane to obtain a better picture of the quivery stability conditions intersecting $\mathrm{Stab}^{\mathrm{Geo}}(\mathbb{P}^2)$. Let $\widetilde{\nu} : \mathrm{Coh}(\mathbb{P}^2) \to \mathbb{RP}^2$ denote the map $\mathcal{F} \mapsto [\mathrm{ch}_0(\mathcal{F}), \mathrm{ch}_1(\mathcal{F}), \mathrm{ch}_2(\mathcal{F})]$, and let $\nu(\mathcal{F})$ denote the projection of the affine chart $\{\mathrm{ch}_0(\mathcal{F}) \neq 0\}$ to $\mathbb{R}^2$ given by

$$\nu(\mathcal{F}) := \left(\frac{\mathrm{ch}_1(\mathcal{F})}{\mathrm{ch}_0(\mathcal{F})}, \frac{\mathrm{ch}_2(\mathcal{F})}{\mathrm{ch}_0(\mathcal{F})}\right) = (\mu_H(\mathcal{F}), \frac{1}{2}\mu_H(\mathcal{F})^2 - \Delta(\mathcal{F})).$$

For brevity, let (s,q) denote the coordinates on the $\{\frac{\mathrm{ch}_1}{\mathrm{ch}_0}, \frac{\mathrm{ch}_2}{\mathrm{ch}_0}\}$-plane; moreover, let Δ_a refer to the parabola $q = \frac{1}{2}s^2 - a$. It is clear from the construction of the map ν that the parabola Δ_a contains all coherent sheaves whose discriminant is precisely a. By Riemann-Roch, it is also easy to see that any exceptional vector bundle $\mathcal{E}$ has discriminant $\Delta(\mathcal{E}) = \frac{1}{2}(1 - 1/r(\mathcal{E})^2)$; thus, the image of all exceptional vector bundles under ν lies on or below the parabola Δ_0 and strictly above $\Delta_{1/2}$. Lastly, for any exceptional object $\mathcal{E}$, let $\mathcal{E}^x$ denote the kernel of the natural map $\mathcal{E}^{\oplus \mathrm{rk}\,\mathcal{E}} \twoheadrightarrow \Bbbk(x)$, and denote by $\mathcal{W}_{\mathcal{E}}$ the vertical wall in the $\{\frac{\mathrm{ch}_1}{\mathrm{ch}_0}, \frac{\mathrm{ch}_2}{\mathrm{ch}_0}\}$-plane connecting $\nu(\mathcal{E})$ to $\nu(\mathcal{E}^x)$. Using the obvious diffeomorphism between the $\{\mu, \Delta\}$-plane and the $\{\frac{\mathrm{ch}_1}{\mathrm{ch}_0}, \frac{\mathrm{ch}_2}{\mathrm{ch}_0}\}$-plane, we obtain the following picture

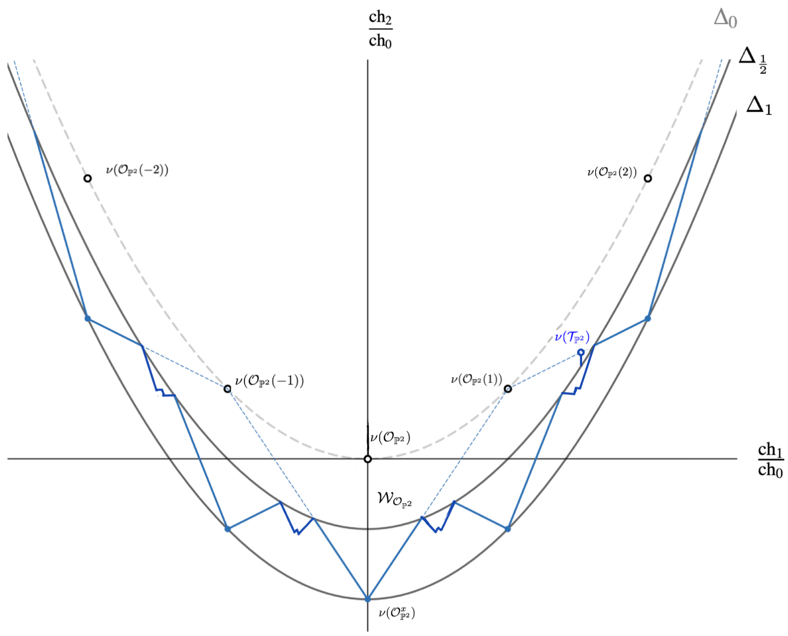


Figure 6: An embedding of the Drézet-Le Potier curve (blue) in the $\{\frac{\mathrm{ch}_1}{\mathrm{ch}_0}, \frac{\mathrm{ch}_2}{\mathrm{ch}_0}\}$-plane.

There is an obvious diffeomorphism between the (s, q) plane and the (μ, Δ)-plane given by $\frac{\mathrm{ch}_1}{\mathrm{ch}_0} = \mu_H$ and $\frac{\mathrm{ch}_2}{\mathrm{ch}_0} = \frac{1}{2}\mu_H^2 - \Delta$. Let $\widetilde{\delta}_\infty^{DP}$ denote the embedding of the Drézet-Le Potier curve in the (s, q)-plane.

Proposition 3.7 ([Li17, Proposition 1.10])**.** *The central charge of the geometric stability condition $\sigma_{a,b}$ can be written of the form*

$$Z_{s,q}(E) := (-\,\mathrm{ch}_2(E) + q\,\mathrm{ch}_0(E)) + i(\mathrm{ch}_1(E) - s\,\mathrm{ch}_0(E))$$

where $s = \Im(b)/\Im(a)$ and $q = \frac{\Re(a)\Im(b)}{\Im(a)} + \Re(b)$. In this case, $(a, b) \in G$ is equivalent to (s, q) lying above the curve $\widetilde{\delta}_\infty^{DP}(\mu) := \frac{1}{2}\mu^2 - \delta_\infty^{DP}(\mu)$ and not on any of the walls $\mathcal{W}_\mathcal{E}$ (for $\mathcal{E}$ exceptional).

One benefit of the parametrization above is that it gives a straightforward description of when a stability condition admits a massless exceptional object: the point (s, q) must not equal $(\frac{\mathrm{ch}_1(\mathcal{E})}{\mathrm{ch}_0(\mathcal{E})}, \frac{\mathrm{ch}_2(\mathcal{E})}{\mathrm{ch}_0(\mathcal{E})})$ for any slope stable $\mathcal{E} \in \mathrm{Coh}(\mathbb{P}^2)$ (i.e. lies on the holes $\nu(\mathcal{E})$). In the case that a stability condition $\sigma_{s,q}$ lies on the vertical line $s = \mu(\mathcal{E})$ where $\mathcal{E}$ is some exceptional vector bundle, we must have $\Re(Z_{s,q}(\mathcal{E})) < 0$ in order for $\mathcal{E}$ to be in the heart $\mathrm{Coh}^{\sharp s}(\mathbb{P}^2)$. It is easy to see that this is equivalent to $q < \frac{\mathrm{ch}_2(\mathcal{E})}{\mathrm{ch}_0(\mathcal{E})}$, thus giving the wall $\mathcal{W}_{\mathcal{E}}$ where the skyscraper sheaves $\Bbbk(x)$ become strictly semi-stable. In the case that $q > \frac{\mathrm{ch}_2(\mathcal{E})}{\mathrm{ch}_0(\mathcal{E})}$, it will turn out that the object $\mathcal{E}[1]$ will become one of the simple objects in an 'algebraic' heart as constructed in Section 2.4.

Let $\mathfrak{E} = \{\mathcal{E}_0, \mathcal{E}_1, \mathcal{E}_2\}$ be a full strong exceptional collection of coherent sheaves on $\mathbb{P}^2$. As a direct result of [Mac07b, Lemma 3.18], the set $\Theta_{\mathfrak{E}}$ can be described as the set of all σ such that such that $\mathcal{E}_i$ is stable of phase ϕ_i, where the phases satisfy

(i) $\phi_0 < \phi_1 < \phi_2$, and

(ii) $\phi_0 + 1 < \phi_2$.

By [Mac07b, Lemma 4.9], this is homeomorphic to the open region in $\mathbb{R}^6$ given by

$$\{(m_0, m_1, m_2, \phi_0, \phi_1, \phi_2) \mid m_i > 0 \text{ and } \phi_i \text{ satisfy } (i), (ii)\}$$

where the homeomorphism is simply given by assigning $Z(E_j) = m_j e^{\pi i \phi_j}$.

One of the main results of [Li17, Section 2.2] is describing the intersection of $\Theta_{\mathfrak{E}}$ with $\mathrm{Stab}^{\mathrm{Geo}}(\mathbb{P}^2)$ (up to the $\widetilde{\mathrm{GL}}_2^+(\mathbb{R})$ action) using the stability conditions of the above form. Adopting the same notation as the author, we first decompose $\Theta_{\mathfrak{E}}$ into several distinct regions as follows:

(a) $\Theta_{\mathfrak{E}}^{\nabla} := \{\sigma \in \Theta_{\mathfrak{E}} \mid \phi_1 - \phi_0 \leq 1, \phi_2 - \phi_1 \leq 1, \phi_2 - \phi_0 \neq 2\}$

(b) $\Theta_{\mathfrak{E}}^{\mathrm{Geo}} := \Theta_{\mathfrak{E}} \cap \mathrm{Stab}^{\mathrm{Geo}}(\mathbb{P}^2)$

(c) $\Theta_{\mathfrak{E}}^{\mathrm{pure}} := \{\sigma \in \Theta_{\mathfrak{E}} \mid \phi_1 - \phi_0 \geq 1, \phi_2 - \phi_1 \geq 1\}$

(d) $\Theta_{\mathfrak{E},\mathcal{E}_0}^{+} := \{\sigma \in \Theta_{\mathfrak{E}} \mid \phi_2 - \phi_1 < 1\} \backslash \Theta_{\mathfrak{E}}^{\mathrm{Geo}}$

(e) $\Theta_{\mathfrak{E},\mathcal{E}_2}^{-} := \{\sigma \in \Theta_{\mathfrak{E}} \mid \phi_1 - \phi_0 < 1\} \backslash \Theta_{\mathfrak{E}}^{\mathrm{Geo}}$

Each of these regions produces polygonal set in the (s, q)-plane whose walls dictate the phases of ϕ_0, ϕ_1, ϕ_2:

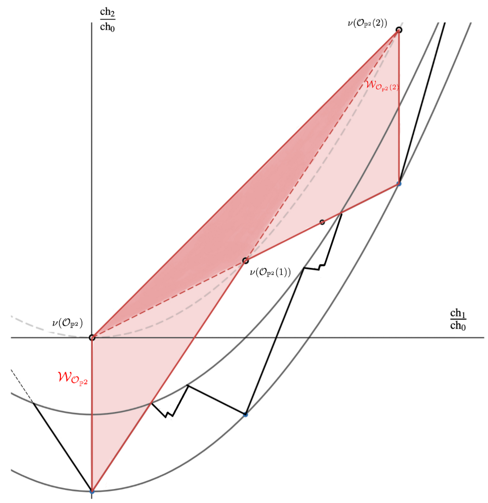


Figure 7: The intersection $\Theta_{\mathfrak{E}} \cap \mathrm{Stab}^{\mathrm{Geo}}(\mathbb{P}^2)$ in the $\{\frac{\mathrm{ch}_1(\mathcal{E})}{\mathrm{ch}_0(\mathcal{E})}, \frac{\mathrm{ch}_2(\mathcal{E})}{\mathrm{ch}_0(\mathcal{E})}\}$-plane for $\mathfrak{E} = \{\mathcal{O}_{\mathbb{P}^2}, \mathcal{O}_{\mathbb{P}^2}(1), \mathcal{O}_{\mathbb{P}^2}(2)\}$. The darker region represents the intersection with $\Theta_{\mathfrak{E}}^{\nabla}$

By [Li17, Proposition 2.5], given a full strong exceptional collection $\mathfrak{E} = \{\mathcal{E}_0, \mathcal{E}_1, \mathcal{E}_2\}$, any stability condition in $\Theta_{\mathfrak{E}}^{\mathrm{Geo}}$ is equivalent up to the $\widetilde{\mathrm{GL}}_2^+(\mathbb{R})$-action to a stability condition $\sigma_{s,q}$ falling into one of the polygonal regions between $\mathcal{W}_{\mathcal{E}_0}$ and $\mathcal{W}_{\mathcal{E}_2}$ as shown above. An explicit description of how to construct the polygonal regions can be found at the beginning of [Li17, Section 2.2].

Lemma 3.8 ([Li17, Lemma 2.4])**.** *Let $\mathfrak{E} = \{\mathcal{E}_0, \mathcal{E}_1, \mathcal{E}_2\}$ be a full strong exceptional collection of coherent sheaves on $\mathbb{P}^2$. Then the only stable objects for any stability condition in $\Theta_{\mathfrak{E}}^{\mathrm{pure}}$ are $\mathcal{E}_i[n]$ for $n \in \mathbb{Z}$.*

As a direct consequence, $\Theta_{\mathfrak{E}}^{\mathrm{pure}}$ cannot intersect the geometric chamber since all skyscraper sheaves $\mathbb{k}(x)$ must destabilize. It is not possible to give a visualization of the region $\Theta_{\mathfrak{E}}^{\mathrm{pure}}$ in terms of the (s,q)-plane as above since, even up to the $\widetilde{\mathrm{GL}}_2^+(\mathbb{R})$-action, no stability condition $\sigma_{s,q}$ will give $\phi(\mathcal{E}_0[2]) \leq \phi(\mathcal{E}_1[1]) \leq \phi(\mathcal{E}_2)$.

Continuing the notation of [Li17], note that for any exceptional object $\mathcal{E} := \mathcal{E}_0$, there are infinitely many exceptional collections on $\mathbb{P}^2$ of the form $\{\mathcal{E}, \widetilde{\mathcal{E}}_1, \widetilde{\mathcal{E}}_2\}$ due to (left and right) mutations on the latter two exceptional objects.

Proposition 3.9 ([Li17, Proposition 3.1])**.** *Let $\mathcal{E}$, $\mathcal{F}$ be exceptional coherent sheaves on $\mathbb{P}^2$, and let $\mathfrak{E} = \{\mathcal{E}, \mathcal{E}_1, \mathcal{E}_2\}$, $\mathfrak{E}' = \{\mathcal{E}, \mathcal{E}_1', \mathcal{E}_2'\}$, $\mathfrak{F} = \{\mathcal{F}_0, \mathcal{F}_1, \mathcal{F}\}$ and $\mathfrak{F}' = \{\mathcal{F}_0', \mathcal{F}_1', \mathcal{F}\}$ be full* *strong exceptional collections. Then*

1. $\Theta^+_{\mathfrak{E},\mathcal{E}} = \Theta^+_{\mathfrak{E}',\mathcal{E}}$
2. $\Theta^-_{\mathfrak{F},\mathcal{F}} = \Theta^-_{\mathfrak{F}',\mathcal{F}}$

Consequently, the subsets $\Theta^+_{\mathfrak{E},\mathcal{E}}$ and $\Theta^-_{\mathfrak{F},\mathcal{F}}$ may be defined more concisely in terms of a single exceptional object; specifically, writing $\Theta^+_{\mathcal{E}}$ (resp. $\Theta^-_{\mathcal{F}}$) instead of $\Theta^+_{\mathfrak{E},\mathcal{E}}$ (resp. $\Theta^-_{\mathfrak{F},\mathcal{F}}$) is perfectly well-defined by the previous proposition. The last result we need for the purposes of our Thurston compactification is a relation between these non-geometric orbits for distinct exceptional objects, as shown at the end of the previous author's article.

Proposition 3.10 ([Li17, Proposition 3.4])**.** *Let $\mathcal{E}$ and $\mathcal{F}$ be two exceptional bundles.*

(i) $\Theta^+_{\mathcal{E}} \cap \Theta^+_{\mathcal{F}} \neq \emptyset$ *if and only if* $\mathcal{E} = \mathcal{F}$

(ii) $\Theta^-_{\mathcal{E}} \cap \Theta^-_{\mathcal{F}} \neq \emptyset$ *if and only if* $\mathcal{E} = \mathcal{F}$

(iii) $\Theta^+_{\mathcal{E}} \cap \Theta^-_{\mathcal{F}} = \emptyset$

In particular, the author shows that in cases (i) and (ii), if $\mathcal{E} \neq \mathcal{F}$ we may assume without loss of generality $\mu(\mathcal{E}) < \mu(\mathcal{F})$. In this case, $\mathcal{E}$ will not be stable in $\Theta^+_{\mathcal{F}}$ and similarly $\mathcal{F}$ will not be stable in $\Theta^-_{\mathcal{E}}$ ([Li17, Lemma 3.3].

This ultimately gives a complete description of the subset

$$\mathrm{Stab}^{\mathrm{Alg}}(\mathbb{P}^2) := \bigcup_{\mathfrak{E}\text{ exc. col.}} \Theta_{\mathfrak{E}}$$

in terms of stable objects. By Remark 2.35, it is still ultimately unknown whether the entire stability manifold $\mathrm{Stab}(\mathbb{P}^2)$ is connected. Continuing to let $\mathrm{Stab}^\dagger(\mathbb{P}^2)$ refer to the principal connected component containing the geometric chamber, the fact that $\mathrm{Stab}^{\mathrm{Geo}}(\mathbb{P}^2)$ and $\mathrm{Stab}^{\mathrm{Alg}}(\mathbb{P}^2)$ intersect nontrivially shows that $\mathrm{Stab}^{\mathrm{Alg}}(\mathbb{P}^2) \subset \mathrm{Stab}^\dagger(\mathbb{P}^2)$. The main result of [Li17] is that these two open subsets encompass the entire connected component $\mathrm{Stab}^\dagger(\mathbb{P}^2)$:

Theorem 3.11 ([Li17, Theorem 3.9])**.** *The connected component* $\mathrm{Stab}^\dagger(\mathbb{P}^2)$ *containing* *all geometric stability conditions decomposes as*

$$\mathrm{Stab}^{\dagger}(\mathbb{P}^2) = \mathrm{Stab}^{\mathrm{Geo}}(\mathbb{P}^2) \cup \mathrm{Stab}^{\mathrm{Alg}}(\mathbb{P}^2).$$

3.2.1 Local $\mathbb{P}^2$

To conclude this chapter, we lastly examine the principal connected component of the stability manifold of $\omega_{\mathbb{P}^2}$ (i.e. the "local projective plane"), as described by [Bri06], [BM11]. For the sake of brevity, we will let Coh_0 denote the full subcategory of $\mathrm{Coh}(\omega_{\mathbb{P}^2})$ consisting of sheaves whose reduced support is contained in the zero section $\mathbb{P}^2$.

Similar to the case of the projective line, it was shown by [BM11, Theorem 2.5] and [Li17, Proposition 1.12] that the geometric chambers $\mathrm{Stab}^{\mathrm{Geo}}(\mathbb{P}^2)$ and $\mathrm{Stab}^{\mathrm{Geo}}(\mathcal{D}_0)$ are identical to one another. Thus, we will continue to denote by $\sigma_{a,b}$ the stability condition on $\mathcal{D}_0$ with central charge

$$Z_{a,b}(E) = -c(E) + a \cdot d(E) + b \cdot r(E)$$

(where $(r(E), d(E), c(E))$ now denotes the Chern character of $\pi_*(E)$ for $\pi : \omega_{\mathbb{P}^2} \to \mathbb{P}^2$) with heart $\mathrm{Coh}_0^{\sharp(B)}$ for any $a, b \in \mathbb{C}$ (and $B := -\Im b / \Im a$ for $\Im a > 0$) satisfying the same inequalities from (3), (4) above.

One notable topological difference between $\mathrm{Stab}(\mathcal{D}_{0,\mathbb{P}^1})$ and $\mathrm{Stab}(\mathcal{D}_{0,\mathbb{P}^2})$ is that the connectedness of $\mathrm{Stab}(\mathcal{D}_{0,\mathbb{P}^2})$ remains conjectural (c.f. Remark 2.35) — thus, we will continue to let $\mathrm{Stab}^{\dagger}(\mathcal{D}_{0,\mathbb{P}^2})$ denote the principal connected component of $\mathrm{Stab}(\mathcal{D}_{0,\mathbb{P}^2})$ containing $\mathrm{Stab}^{\mathrm{Geo}}(\mathcal{D}_{0,\mathbb{P}^2})$.

Lemma 3.12 ([BM11, Lemma 3.1])**.** *A sheaf $\mathcal{F} \in \mathrm{Coh}_0$ is a pure slope-stable sheaf if and only if it is the push-forward $\mathcal{F} = i_*\mathcal{F}_0$ of a slope-stable sheaf $\mathcal{F}_0 \in \mathrm{Coh}(\mathbb{P}^2)$.*

As a consequence, we arrive at an analog of [Li17] for $\mathcal{D}_{0,\mathbb{P}^2}$: specifically, if $\mathcal{E}$ is an exceptional bundle on $\mathbb{P}^2$ and $\Im(b)/\Im a < \mu(\mathcal{E})$ for some $a, b \in \mathbb{C}^2$ satisfying (3) and (4) above, then $i_*\mathcal{E}$ is a $\sigma_{a,b}$-stable spherical object. In the following chapter, this will turn out to be of crucial importance by allowing us to always find stable spherical objects at any given point $\sigma \in \mathrm{Stab}^{\dagger}(\mathcal{D}_{0,\mathbb{P}^2})$.

Since the geometric chambers $\mathrm{Stab}^{\mathrm{Geo}}(\mathbb{P}^2)$ and $\mathrm{Stab}^{\mathrm{Geo}}(\mathcal{D}_{0,\mathbb{P}^2})$ are identical, we obtain identical boundaries. As shown in [BM11, Section 5], up to the $\mathbb{C}$-action the only stability conditions in the boundary of $\mathrm{Stab}^{\mathrm{Geo}}(\mathcal{D}_0)$ that can be obtained from the inequalities (3) and (4) occur when

(a) $\Im a = 0$

(b) $\Im a > 0$, there exists an exceptional vector bundle $\mathcal{E}$ on $\mathbb{P}^2$ of slope $\mu_H(\mathcal{E}) = B = -\Im b/\Im a$ and discriminant Δ_B and $a, b \in \mathbb{C}$ satisfy

$$\delta_{\mathrm{DP}}^{\infty}(B) > -\Re b - (\Re a)B + \frac{1}{2}B^2 > \Delta_B.$$

By [BM11, Lemma 5.5], stability conditions satisfying (a) yield a single codimension 2 locus on the boundary $\partial\,\mathrm{Stab}^{\mathrm{Geo}}(\mathcal{D}_0)$ whose central charge lies on the real line — in particular, these stability conditions fall into $\mathrm{Stab}^{\dagger}(\mathcal{D})\setminus \mathrm{Stab}^{\mathrm{nd}}(\mathcal{D})$, and we will later see these intersect the set of algebraic stability conditions.

On the other hand, for a fixed exceptional vector bundle $\mathcal{E} \in \mathrm{Coh}\,\mathbb{P}^2$, the $\mathbb{C}$-orbit of stability conditions of the form $\sigma_{a,b}$ where a, b satisfy (b) produces a codimension one wall $\mathcal{W}_{\mathcal{E}}^{+}$. In particular, on the wall $\mathcal{W}_{\mathcal{E}}^{+}$ each skyscraper sheaf becomes strictly semistable and has a Jordan-Hölder filtration given by

$$i_*\mathcal{E}^{\mathrm{rk}(\mathcal{E})} \to \Bbbk(x) \to \mathcal{E}^x[1] \tag{5}$$

where $\mathcal{E}^x$ is the kernel of $\mathrm{Hom}^0(\mathcal{E}, \Bbbk(x)) \otimes \mathcal{E} \to \Bbbk(x)$. Similarly, the wall $\mathcal{W}_{\mathcal{E}}^{-}$ can be described as the region in $\overline{\mathrm{Stab}^{\mathrm{Geo}}(\mathcal{D}_0)}$ in which the skyscraper sheaves $\Bbbk(x)$ destabilize with Jordan-Hölder filtration

$$\mathrm{Tw}_{i_*\mathcal{E}}^{-1}(\mathcal{E}^x) \to \Bbbk(x) \to i_*\mathcal{E}^{\mathrm{rk}(\mathcal{E})}[2]. \tag{6}$$

Similar to the cases of $\mathrm{Stab}(\mathbb{P}^1)$ and $\mathrm{Stab}(\mathcal{D}_{0,\mathbb{P}^1})$ from the previous section, crossing a wall $\mathcal{W}_{\mathcal{E}}^{\pm}$ corresponding to an exceptional vector bundle $\mathcal{E}$ leads to very different behaviors between $\mathrm{Stab}(\mathbb{P}^2)$ and $\mathrm{Stab}(\mathcal{D}_{0,\mathbb{P}^2})$. For example, the wall $\mathcal{W}_{\mathcal{E}}^{+}$ lies in the intersection of $\partial\,\mathrm{Stab}^{\mathrm{Geo}}(\mathcal{D}_{0,\mathbb{P}^2}) \cap \partial(\mathrm{Tw}_{i_*\mathcal{E}}\,\mathrm{Stab}^{\mathrm{Geo}}(\mathcal{D}_{0,\mathbb{P}^2}))$ — upon crossing the wall, the phases of $i_*\mathcal{E}$ and $i_*\mathcal{E}^x$ swap so that all skyscraper sheaves destabilize by (5) and the objects $\mathrm{Tw}_{i_*\mathcal{E}}\,\Bbbk(x)$ become stable by (6). Much like the stability manifold of a generic projective K3 surface, the action of (compositions of) spherical twists $\widetilde{\Phi} = \mathrm{Tw}^{\pm}_{i_*\mathcal{E}_1} \circ \cdots \circ \mathrm{Tw}^{\pm}_{i_*\mathcal{E}_n}$ produces an infinite set of open chambers identical to $\mathrm{Stab}^{\mathrm{Geo}}(\mathcal{D}_{0,\mathbb{P}^2})$ where the objects $\widetilde{\Phi}(\Bbbk(x))$ are stable of the same phase.

Corollary 3.13 ([BM11, Corollary 5.2])**.** *The translates of* $\overline{\mathrm{Stab}^{\mathrm{Geo}}(\mathcal{D}_{0,\mathbb{P}^2})}$ *under autoequivalences of the form* $\mathrm{Tw}_{i_*\mathcal{E}}$ *cover the entire connected component* $\mathrm{Stab}^{\dagger}(\mathcal{D}_{0,\mathbb{P}^2})$*.*

As before, the stability conditions lying on the boundary of $\mathrm{Stab}^{\mathrm{Geo}}(\mathcal{D}_{0,\mathbb{P}^2})$ can be completely described using the $\widetilde{\mathrm{GL}}_2^+(\mathbb{R})$-orbits of the spherical collections defined in Section 2.5. Recall that given an exceptional collection of vector bundles $\mathfrak{E} = \{\mathcal{E}_0, \mathcal{E}_1, \mathcal{E}_2\}$ in $\mathbb{P}^2$, we obtain a spherical collection $\mathfrak{S}_{\mathfrak{E}} = \{i_*\mathcal{E}_2, i_*\mathcal{E}_1[1], i_*\mathcal{E}_0[2]\}$ in $\mathcal{D}_{0,\mathbb{P}^2}$. By abuse of notation, we will use the same $\Theta_{\mathfrak{E}}$ (as in the case of $\mathrm{Stab}(\mathbb{P}^2)$) to denote the set of all stability conditions in $\mathrm{Stab}(\mathcal{D}_{0,\mathbb{P}^2})$ with heart equivalent to $\mathcal{A}_{\mathfrak{E}} := \langle i_*\mathcal{E}_2, i_*\mathcal{E}_1[1], i_*\mathcal{E}_0[2]\rangle$ up to the $\widetilde{\mathrm{GL}}_2^+(\mathbb{R})$-action. By [BM11, Lemma 6.5], $\Theta_{\mathfrak{E}}$ is characterized as the subset of $\mathrm{Stab}(\mathcal{D}_{0,\mathbb{P}^2})$ where the $i_*\mathcal{E}_j[2-j]$ are stable and their phases $\phi_j = \phi(i_*\mathcal{E}_j[2-j])$ satisfy $|\phi_j - \phi_{j+1}| < 1$ for $j \in \mathbb{Z}_3$.

Let B_3 denote the quotient of the annular braid group given by the presentation

$$\left\langle \tau_0, \tau_1, \tau_2, r \;\middle|\; \begin{array}{l} r^3 = 1 \\ \tau_i\tau_{i+1}\tau_i = \tau_{i+1}\tau_i\tau_{i+1} \quad \text{for } i \in \mathbb{Z}_3 \\ r\tau_i r^{-1} = \tau_{i+1} \end{array} \right\rangle.$$

As shown in the previous chapter, there is a well-defined group action of B_3 on the set of spherical collections — for example, if $\mathfrak{S} = \{S_0, S_1, S_2\}$ is a spherical collection in $\mathcal{D}_{0,\mathbb{P}^2}$ then the elements $\tau_2^{\pm}$ act via

$$\tau_2\mathfrak{S} = \{S_0, S_2[-1], \mathrm{Tw}_{S_2} S_1\}$$
$$\tau_2^{-1}\mathfrak{S} = \{S_0, \mathrm{Tw}_{S_1}^{-1} S_2, S_1[1]\}.$$

By Remark 2.47, this makes the algebraic stability manifold of local $\mathbb{P}^2$ notably 'larger' than its local $\mathbb{P}^1$ counterpart — the only way to construct new quivery hearts in $\mathcal{D}_{0,\mathbb{P}^1}$ is by applying the action of $\mathrm{Aut}_{\mathrm{sph}}(\mathcal{D}_{0,\mathbb{P}^1})$. However, since the action $\mathrm{Aut}_{\mathrm{sph}}(\mathcal{D}_{0,\mathbb{P}^2})$ is the same as that of the subgroup

$$\langle r^i \tau_1 \tau_2 r^{2-i} \mid i = 0, 1, 2\rangle \subset B_3$$

by Lemma 2.50, we obtain significantly more open sets $\Theta_{\mathfrak{S}}$ in the algebraic stability manifold which, in a sense, connect geometric stability conditions to translates of geometric stability conditions under $\mathrm{Aut}_{\mathrm{sph}}(\mathcal{D}_{0,\mathbb{P}^2})$.

Example 3.14. Consider the exceptional collection $\mathfrak{E} = \{\mathcal{O}_{\mathbb{P}^2}(-1), \mathcal{O}_{\mathbb{P}^2}, \mathcal{O}_{\mathbb{P}^2}(1)\}$ in $D^\flat(\mathbb{P}^2)$ and the corresponding spherical collection $\mathfrak{S} = \{i_*\mathcal{O}_{\mathbb{P}^2}(1), i_*\mathcal{O}_{\mathbb{P}^2}[1], i_*\mathcal{O}_{\mathbb{P}^2}(-1)[2]\}$ in $\mathcal{D}_{0,\mathbb{P}^2}$. For brevity, we will continue to denote spherical twists around line bundles by $\mathrm{Tw}_n := \mathrm{Tw}_{i_*\mathcal{O}_{\mathbb{P}^2}(n)}$. Applying τ_1 to $\mathfrak{S}$ yields the spherical collection

$$\{i_*\mathcal{O}_{\mathbb{P}^2}, \mathrm{Tw}_0\, i_*\mathcal{O}_{\mathbb{P}^2}(1), i_*\mathcal{O}_{\mathbb{P}^2}(-1)[2]\}$$

which is precisely the spherical collection associated to $\mathcal{L}_{\mathcal{O}_{\mathbb{P}^2}}\mathfrak{E}$ by Lemma 2.48. As a consequence, we immediately see that $\mathrm{Tw}_n\, i_*\mathcal{O}_{\mathbb{P}^2}(n+1) \cong i_*\Omega_{\mathbb{P}^2}(n+1)[1]$ for each $n \in \mathbb{Z}$.

Since $i_* : K(\mathbb{P}^2) \to K(\mathcal{D}_{0,\mathbb{P}^2})$ identifies the Chern characters of $\mathcal{E}$ and $i_*\mathcal{E}$, we may consider the slice of our geometric stability conditions parameterized by the same (s,q)-plane as constructed for $\mathbb{P}^2$. By an identical argument to that of [Li17, Section 2], the intersection of the orbits $\Theta_{\mathfrak{S}}$ and $\Theta_{\tau_1\mathfrak{S}}$ with the (s,q)-plane may be visualized as follows:

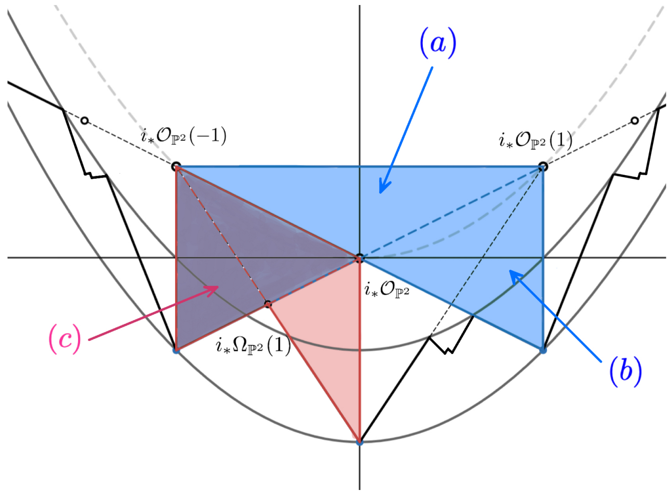


Figure 8: The blue / purple polygonal region represents the intersection of the (s, q)-plane with $\Theta_{\mathfrak{S}}$; similarly, the red / purple polygonal region represents the intersection with $\Theta_{\tau_1\mathfrak{S}}$

Letting $\{S_0, S_1, S_2\}$ denote the simple objects $\{i_*\mathcal{O}_{\mathbb{P}^2}(1), i_*\mathcal{O}_{\mathbb{P}^2}[1], i_*\mathcal{O}_{\mathbb{P}^2}(-1)[2]\}$, we see that the (s, q)-plane intersects exactly three $\widetilde{\mathrm{GL}}_2^+(\mathbb{R})$-orbits of $\Theta_{\mathfrak{S}}$: (a) $\phi(S_0) < \phi(S_1) < \phi(S_2)$, (b) $\phi(S_1) < \phi(S_0) < \phi(S_2)$, and (c) $\phi(S_0) < \phi(S_2) < \phi(S_1)$. As a consequence, the object $i_*\Omega_{\mathbb{P}^2}(1)[1] \cong \mathrm{Tw}_{S_1} S_0$ is stable in the regions (a) and (c), as well as the light red-colored $\widetilde{\mathrm{GL}}_2^+(\mathbb{R})$-orbit of $\Theta_{\tau_1\mathfrak{S}}$. However, in region (b) the (pushforward of the) Euler sequence

$$i_*\mathcal{O}_{\mathbb{P}^2}(1) \to i_*\Omega_{\mathbb{P}^2}(1)[1] \to i_*\mathcal{O}_{\mathbb{P}^2}[1]^{\oplus 3}$$

destabilizes $\mathrm{Tw}_{S_1} S_0$. Since the action of Tw_{S_1} on spherical collections corresponds to the element $r\tau_1\tau_2 r = (r\tau_1)^2$ of B_3 and r does not change the stable objects, we see that the geometric chamber $\mathrm{Stab}^{\mathrm{Geo}}(\mathcal{D}_{0,\mathbb{P}^2})$ and $\mathrm{Tw}_0 \,\mathrm{Stab}^{\mathrm{Geo}}(\mathcal{D}_{0,\mathbb{P}^2})$ connect via $\Theta_{\tau_1\mathfrak{S}}$.

Letting $\{R_0, R_1, R_2\}$ denote the simple objects $\{i_*\mathcal{O}_{\mathbb{P}^2}, i_*\Omega_{\mathbb{P}^2}(1)[1], i_*\mathcal{O}_{\mathbb{P}^2}(-1)[2]\}$ of $\Theta_{\tau_1\mathfrak{S}}$, R_0 must have the smallest phase in the intersection with $\Theta_{\mathfrak{S}}$ since $R_0[1]$ is also in the heart; by the same logic, R_0 must have the largest phase in the intersection with $\mathrm{Tw}_{S_1} \Theta_{\mathfrak{S}}$ since $R_0[-1]$ is in the heart. This leaves two remaining $\widetilde{\mathrm{GL}}_2^+(\mathbb{R})$-orbits: the red region in the figure above where $\phi(R_1) < \phi(R_0) < \phi(R_2)$ and one remaining chamber contained in $\mathrm{Tw}_{S_1} \mathrm{Stab}^{\mathrm{Geo}}(\mathcal{D}_{0,\mathbb{P}^2})$ where $\phi(R_2) < \phi(R_0) < \phi(R_1)$.

While the example above only demonstrated how the wall $\mathcal{W}^+_{i_*\mathcal{O}_{\mathbb{P}^2}}$ connecting $\mathrm{Stab}^{\mathrm{Geo}}(\mathcal{D}_{0,\mathbb{P}^2})$ to its neighboring chamber is contained in the orbit $\Theta_{\tau_1\mathfrak{S}}$, a much stronger statement was shown in [BM11, Sections 5,6]:

Theorem 3.15 ([BM11])**.** *The algebraic stability submanifold* $\mathrm{Stab}^{\mathrm{Alg}}(\mathcal{D}_{0,\mathbb{P}^2}) = B_3 \cdot \Theta_{\mathfrak{S}_1}$ *is simply connected and*

$$\partial \bigcup_{\Phi} \Phi(\mathrm{Stab}^{\mathrm{Geo}}(\mathcal{D}_{0,\mathbb{P}^2})) \subset \mathrm{Stab}^{\mathrm{Alg}}(\mathcal{D}_{0,\mathbb{P}^2})$$

where $\Phi \in \mathrm{Aut}_{\mathrm{sph}}(\mathcal{D}_{0,\mathbb{P}^2})$ *ranges over all compositions of spherical twists.*

Recall for $\mathrm{Stab}(\mathcal{D}_{0,\mathbb{P}^1})$, it was shown in Section 3.1.1 that any heart of the form $\mathcal{A}_k = \langle i_*\mathcal{O}_{\mathbb{P}^1}(k), i_*\mathcal{O}_{\mathbb{P}^1}(k-1)[1]\rangle$ could be obtained from $\mathcal{A}_0 = \langle i_*\mathcal{O}_{\mathbb{P}^1}, i_*\mathcal{O}_{\mathbb{P}^1}(-1)[1]\rangle$ by applying a composition of spherical twists

$$\mathrm{Tw}_{-1} \dots \mathrm{Tw}_0 \, \mathrm{Tw}_{-1} \, \mathrm{Tw}_0 \, .$$

In a sense, this means that one may start at a geometric stability condition in $\mathrm{Stab}(\mathcal{D}_{0,\mathbb{P}^1})$ and pass through a finite number of *distinct* walls exactly once and wind up back in the geometric chamber.

To show that the same phenomenon occurs for local $\mathbb{P}^2$, we heavily rely on the fact that the open sets $\Theta_{\mathfrak{S}}$ intersect the geometric chamber whenever $\mathfrak{S}$ arises from an exceptional collection. By [BM11, Proposition 8.4], any automorphism which sends geometric stability conditions to geometric stability conditions is isomorphic to a composition of an automorphism of the formal completion of $\omega_{\mathbb{P}^2}$ along $\mathbb{P}^2$ with $- \otimes \mathcal{O}_{\mathbb{P}^2}(n)[k]$. By an almost identical argument to that of $\mathbb{P}^2$, any automorphism of the formal completion of $\omega_{\mathbb{P}^2}$ along $\mathbb{P}^2$ must send $i_*\mathcal{O}_{\mathbb{P}^2}(1)$ to $i_*\mathcal{O}_{\mathbb{P}^2}(1)$ — thus, the problem reduces to showing that some composition of spherical twists takes some $\Theta_{\mathfrak{S}_m}$ to $\Theta_{\mathfrak{S}_n}$, where $\mathfrak{S}_m = \{i_*\mathcal{O}_{\mathbb{P}^2}(m), i_*\mathcal{O}_{\mathbb{P}^2}(m-1)[1], i_*\mathcal{O}(m-2)_{\mathbb{P}^2}[2]\}$.

Example 3.16. Continuing with the previous example, consider the exceptional collection $\mathfrak{E} = \{\mathcal{O}_{\mathbb{P}^2}(-1), \mathcal{O}_{\mathbb{P}^2}, \mathcal{O}_{\mathbb{P}^2}(1)\}$ on $\mathbb{P}^2$, along with its mutation

$$\mathcal{L}_{\mathcal{O}}\,\mathfrak{E} = \{\mathcal{O}_{\mathbb{P}^2}(-1), \Omega_{\mathbb{P}^2}(1), \mathcal{O}_{\mathbb{P}^2}\}.$$

Mutating again at $\mathcal{O}_{\mathbb{P}^2}(-1)$ gives the exceptional collection $\{\mathcal{L}_{\mathcal{O}(-1)}\,\Omega_{\mathbb{P}^2}(1), \mathcal{O}_{\mathbb{P}^2}(-1), \mathcal{O}_{\mathbb{P}^2}\}$ where $\mathcal{L}_{\mathcal{O}(-1)}\,\Omega_{\mathbb{P}^2}(1)$ fits into the distinguished triangle

$$\mathcal{L}_{\mathcal{O}(-1)}\,\Omega_{\mathbb{P}^2}(1) \to \mathrm{Hom}(\mathcal{O}_{\mathbb{P}^2}(-1), \Omega_{\mathbb{P}^2}(1)) \otimes \mathcal{O}_{\mathbb{P}^2}(-1) \to \Omega_{\mathbb{P}^2}(1). \tag{7}$$

Twisting the Euler sequence and the Koszul complex

$$0 \to \mathcal{O}_{\mathbb{P}^2}(-3) \to \mathcal{O}_{\mathbb{P}^2}^{\oplus 3}(-2) \to \Omega_{\mathbb{P}^2} \to 0$$

by $- \otimes\, \mathcal{O}_{\mathbb{P}^2}(2)$ and then taking the long exact sequence in cohomology gives

$$\mathrm{Hom}(\mathcal{O}_{\mathbb{P}^2}(-1), \Omega_{\mathbb{P}^2}(1)) = \mathbb{C}^3.$$

But then (7) is precisely the triangle obtained by twisting the Koszul resolution above by $- \otimes\, \mathcal{O}_{\mathbb{P}^2}(1)$ so that $\mathcal{L}_{\mathcal{O}(-1)}\,\Omega_{\mathbb{P}^2}(1) \cong \mathcal{O}_{\mathbb{P}^2}(-2)$. On the level of spherical objects in $\mathcal{D}_{0,\mathbb{P}^2}$ the above computations show that $\mathrm{Tw}_{-1}\,\mathrm{Tw}_0\, i_*\mathcal{O}_{\mathbb{P}^2}(1) \cong i_*\mathcal{O}_{\mathbb{P}^2}(-2)[2]$. Slightly more generally, we always have $\mathrm{Tw}_n\, i_*\Omega_{\mathbb{P}^2}(n+2) \cong i_*\mathcal{O}_{\mathbb{P}^2}(n-1)[1]$.

Letting $\mathfrak{S} = \{i_*\mathcal{O}_{\mathbb{P}^2}(1), i_*\mathcal{O}_{\mathbb{P}^2}[1], i_*\mathcal{O}_{\mathbb{P}^2}(-1)[2]\}$ denote the spherical collection associated to $\mathfrak{E}$, we obtain

$$\mathrm{Tw}_{-1}\,\mathrm{Tw}_0\,\mathfrak{S} = \{i_*\mathcal{O}_{\mathbb{P}^2}(-2)[2], i_*\Omega_{\mathbb{P}^2}, E\}$$

where $E := \mathrm{Tw}_{-1}\,\mathrm{Tw}_0\, i_*\mathcal{O}_{\mathbb{P}^2}(-1)[2]$. By applying Tw_{-1} to the defining triangle of $\mathrm{Tw}_0\, i_*\mathcal{O}_{\mathbb{P}^2}(-1)$

$$i_*\mathcal{O}_{\mathbb{P}^2}^{\oplus 3}[-3] \to i_*\mathcal{O}_{\mathbb{P}^2}(-1) \to \mathrm{Tw}_0\, i_*\mathcal{O}_{\mathbb{P}^2}(-1)$$

we see that E fits into a distinguished triangle of the form

$$i_*\Omega_{\mathbb{P}^2}^{\oplus 3} \to i_*\mathcal{O}_{\mathbb{P}^2}(-1) \to E$$

Applying Tw_{-2} to the above triangle, the identities $\mathrm{Tw}_n\, i_*\mathcal{O}_{\mathbb{P}^2}(n+1) \cong i_*\Omega_{\mathbb{P}^2}(n+1)[1]$ and $\mathrm{Tw}_n\, i_*\Omega_{\mathbb{P}^2}(n+2) \cong i_*\mathcal{O}_{\mathbb{P}^2}(n-1)[1]$ together imply that $\mathrm{Tw}_{-2}\, E$ sits in the distinguished triangle

$$i_*\mathcal{O}_{\mathbb{P}^2}(-3)^{\oplus 3}[1] \to i_*\Omega_{\mathbb{P}^2}(-1)[1] \to \mathrm{Tw}_{-2}\, E$$

Using the Koszul resolution of $\Omega_{\mathbb{P}^2}(-1)$, we see that $\mathrm{Tw}_{-2}\, E \cong i_*\mathcal{O}_{\mathbb{P}^2}(-4)[2]$. In particular, letting $\Psi := \mathrm{Tw}_{-2}\, \mathrm{Tw}_{-1}\, \mathrm{Tw}_0$, we have

$$\Psi\{i_*\mathcal{O}_{\mathbb{P}^2}(1), i_*\mathcal{O}_{\mathbb{P}^2}[1], i_*\mathcal{O}_{\mathbb{P}^2}(-1)[2]\} = \{i_*\mathcal{O}_{\mathbb{P}^2}(-2), i_*\mathcal{O}_{\mathbb{P}^2}(-3)[1], i_*\mathcal{O}(-4)[2]\}$$

so that Ψ is equivalent to $- \otimes i_*\omega_{\mathbb{P}^2}$ on spherical collections.

Corollary 3.17. *Let $\Lambda = \overline{\mathrm{Stab}^{\mathrm{Geo}}(\mathcal{D}_{0,\mathbb{P}^2})}$ and consider the tiling of $\mathrm{Stab}^\dagger(\mathcal{D}_{0,\mathbb{P}^2})$ obtained from the action of $\mathrm{Aut}_{sph}(\mathcal{D}_{0,\mathbb{P}^2})$ on Λ. Then for any integer n, $\Psi_n = \mathrm{Tw}_n\, \mathrm{Tw}_{n+1}\, \mathrm{Tw}_{n+2}$ is in the stabilizer of this action.*

The above statement reflects the observation of [Asp04] that the monodromy around the orbifold point in the mirror model of the $\mathbb{C}^3/\mathbb{Z}_3$ orbifold should preserve the quantum $\mathbb{Z}_3$-symmetry. In particular, Aspinwall showed that by taking the Fourier-Mukai kernel

$$\mathcal{G} := \left(i_*\mathcal{O}^\vee_{\mathbb{P}^2}(1) \boxtimes i_*\mathcal{O}_{\mathbb{P}^2} \longrightarrow \mathcal{O}_\Delta\right) \in D^\flat(\omega_{\mathbb{P}^2} \times \omega_{\mathbb{P}^2})$$

corresponding to the transform $\Phi_{\mathcal{G}}(-) = \mathrm{Tw}_{i_*\mathcal{O}_{\mathbb{P}^2}}(- \otimes \pi^*\mathcal{O}_{\mathbb{P}^2}(1))$, Beilinson's resolution of the diagonal yields $\mathcal{G}^{\star 3} \cong \mathcal{O}_\Delta$ [12] so $\Phi_{\mathcal{G}}$ has order 3. More recently, [HLS16],[KP21] have shown that whenever $i : Z \hookrightarrow X$ is a smooth projective Fano divisor and $\mathfrak{E} = \{\mathcal{E}_0, \mathcal{E}_1, \ldots, \mathcal{E}_n\}$ is a full exceptional collection, there is a relation

$$- \otimes \pi^*(\omega_Z) \cong \mathrm{Tw}_{i_*\mathcal{E}_n} \circ \cdots \circ \mathrm{Tw}_{i_*\mathcal{E}_0}$$

under the condition that the triangulated subcategories $\langle \mathcal{E}_i \rangle$ are invariant with respect to the functor

$$\mathrm{Cone}\left(i^*i_*(-) \to \mathrm{id}\right) \otimes \omega_Z.$$

[12] here $\mathcal{B} \star \mathcal{A}$ denotes the standard convolution of Fourier-Mukai kernels $\mathcal{B} \star \mathcal{A} = p_{13*}(p_{12}^*\mathcal{A} \otimes p_{23}^*\mathcal{B})$

As a consequence, one can show that non-trivial relations (aside from the standard braid group relations) exist whenever $\mathfrak{C}$ is simple.

4 Thurston Compactifications of Stability Manifolds

As described in the introduction, the subtle connection between Teichmüller spaces and stability manifolds led the authors of [BDL20] to define a categorical analog of Thurston's compactification for triangulated categories. Despite the naming convention, it was shown in [KKO22] that the categorical and topological compactifications are distinct even in the original case of Riemann surfaces — this is somewhat expected, however, as the Teichmuller space of a Riemann surface X is only loosely related to the space of stability conditions on a *different* triangulated $\mathcal{D}$ (modulo a subgroup of autoequivalences) as shown by [BS14]. Regardless, generalizing Thurston's approach to the compactification of Teichmuller space provides a useful means of compactifying connected components of stability manifolds in certain settings — most notably in the case of our local Calabi-Yau categories $\mathcal{D}_{0,X}$ as constructed in Section 2.5 above.

Before explicitly describing the compactifications for several of our examples, we first recall several general facts originally shown or used by the authors of [BDL20] to ensure the Thurston compactification does, in fact, compactify a connected component of the stability manifold. As the key structure of triangulated categories is how objects fit into distinguished triangles, a natural first question is how the mass map m_σ distributes across triangles. For example, suppose that we have a distinguished triangle

$$A \to B \to C$$

in some triangulated category $\mathcal{D}$, and moreover suppose that B is semi-stable with respect to some locally-finite stability condition σ. In this case $m_\sigma(B) = |Z_\sigma(B)|$, and so the triangle inequality immediately gives us

$$\begin{aligned} m_\sigma(B) &= |Z_\sigma(B)| \\ &= |Z_\sigma(A) + Z_\sigma(C)| \\ &\leq |Z_\sigma(A)| + |Z_\sigma(C)| \\ &\leq m_\sigma(A) + m_\sigma(C). \end{aligned}$$

It was shown in [Ike21] that this inequality always holds regardless of semi-stability of the middle term:

Lemma 4.1 ([Ike21, Prop. 3.3])**.** *Suppose that $A \to B \to C$ is a distinguished triangle in $\mathcal{D}$, and σ is a stability condition on $\mathcal{D}$. Then*

$$m_\sigma(B) \leq m_\sigma(A) + m_\sigma(C).$$

While we often refer to Ikeda's result as a "triangle inequality", showing when equality holds is a significantly more involved problem than in the standard case. Since the mass of an object $E \in \mathcal{D}$ is defined purely in terms of its Harder-Narasimhan filtration, it is natural to expect that

$$m_\sigma(B) = m_\sigma(A) + m_\sigma(C)$$

whenever A and C are semistable with $\phi(A) > \phi(C)$. More generally, [BDL20] showed that equality holds whenever there is no overlap between the phases of the semistable factors:

Proposition 4.2 ([BDL20, Proposition 3.2])**.** *Suppose that $A \to B \to C$ is a distinguished triangle in $\mathcal{D}$ and $\phi^-_\sigma(A) > \phi^+_\sigma(C)$. Then*

$$m_\sigma(B) = m_\sigma(A) + m_\sigma(C).$$

In fact, the proof of the above proposition relies on the fact that the Harder-Narasimhan filtration of B can be constructed directly from the filtrations of A and C using the octahedral axiom.

Definition 4.3. Let $\mathcal{S}$ be a collection of nonzero objects in $\mathcal{D}$, which we will refer to as our *coordinate set*, and fix a connected component $\mathrm{Stab}^*(\mathcal{D}) \subset \mathrm{Stab}(\mathcal{D})$. We define $m : \mathrm{Stab}^*(\mathcal{D}) \to \mathbb{R}^{\mathcal{S}}_{\geq 0}$ via

$$\sigma \mapsto (m_\sigma(E))_{E \in \mathcal{S}}.$$

Note that $m : \mathrm{Stab}(\mathcal{D}) \to \mathbb{R}^{\mathcal{S}}_{\geq 0}$ is continuous with respect to the product topology on $\mathbb{R}^{\mathcal{S}}_{\geq 0}$. Since we do not consider weak stability conditions, the image of the mass map is contained in $\mathbb{R}^{\mathcal{S}}_{\geq 0} \setminus \{0\}$; thus, we obtain a continuous map

$$\mathbb{P}m : \mathrm{Stab}^*(\mathcal{D}) \to \mathbb{RP}^{\mathcal{S}}$$

Lemma 4.4. *The morphism* $\mathbb{P}m : \mathrm{Stab}^*(\mathcal{D}) \to \mathbb{RP}^{\mathcal{S}}$ *factors through the* $\mathbb{C}$*-action on* $\mathrm{Stab}(\mathcal{D})$

Proof. Suppose that $\sigma_1 = (P_1, Z_1), \sigma_2 = (P_2, Z_2) \in \mathrm{Stab}(\mathcal{D})$ lie in the same $\mathbb{C}$-orbit in $\mathrm{Stab}(\mathcal{D})$ so that there exists $\omega \in \mathbb{C}$ with $Z_2 = \exp(w)Z_1$. Since the $\mathbb{C}$-action (and more generally the $\widetilde{\mathrm{GL}}_2^+(\mathbb{R})$-action) does not affect the stability objects, for any object E that is σ_1-semistable

$$m_{\sigma_1}(E) = |Z_1(E)| = |\exp(w)||Z_2(E)| = |\exp(w)|\, m_{\sigma_2}(E)$$

$\square$

The natural question to ask is whether the map $\mathbb{P}m$ does in fact yield some means of compactifying $\mathrm{Stab}^*(\mathcal{D})$. Motivated by Theorem 1.1, the authors of [BDL20] posed the following question(s):

Question 4.5.

(1) Is $\mathbb{P}m$ injective?

(2) Is $\mathbb{P}m$ a homeomorphism onto its image?

(3) Is $\overline{\mathbb{P}m(\mathrm{Stab}(\mathcal{D})/\mathbb{C})}$ compact?

(4) In the case that the above are true, is there a morphism $i : \mathcal{S} \times \mathcal{S} \to \mathbb{R}_{\geq 0}$ such that

$$\partial\overline{\mathbb{P}m(\mathrm{Stab}(\mathcal{D})/\mathbb{C})} = \overline{i_*(\mathcal{S})}?$$

It is clear that Question 4.5(1-2) depend on the choice of coordinate set $\mathcal{S}$ — for higher dimensional objects, one must be able to choose objects that can distinguish between chambers in our stability manifold, which itself is a non-trivial task. The authors of

[BDL20] initially provided a positive answer to the question above in a few cases of 2-Calabi-Yau categories associated with finite connected quivers. In these cases, the authors' expectation was that

$$i(x, y) = \sum_{i=0} \hom(x, y[i]).$$

provided a positive answer to Question 4.5(4) above. More recently, [KKO22] were able to provide positive answers for (1), (2), and (3) in the geometric case of (smooth projective) curves of positive genus. Additionally, the authors give a positive answer to (4) in the case of elliptic curves by requiring $\mathcal{S}$ to contain all spherical objects and considering the functional $i : \mathcal{S} \times \mathcal{S} \to \mathbb{R}_{\geq 0}$ given by $|\chi(-, -)|$.

The authors of [BDL20] later showed that (3) having a positive answer is a fairly weak assumption which is always true for smooth quasi-projective varieties (over an algebraically closed field) that admit an ample line bundle:

Theorem 4.6 ([BDL20, Proposition 4.1]). *Suppose that $\mathcal{S}$ contains objects $p_1, \ldots, p_n$ such that $\bigoplus_{i=1}^n p_i$ is a classical generator of $\mathcal{D}$. Then the closure of the image of $\mathbb{P}m$ in $\mathbb{RP}^{\mathcal{S}}_{\geq 0}$ is compact.*

As shown by [Rou08], whenever X is an n-dimensional smooth quasi-projective variety with ample line bundle $\mathcal{L}$, the coproduct $\bigoplus_{i=0}^n \mathcal{L}^{\otimes i}$ is a classical generator of $D^\flat(X)$. Since we are also interested in the categories $\mathcal{D}_{0,X}$ constructed in Section 2.5, it is worth noting that when $\mathcal{D}$ admits a full strong exceptional collection $\{\mathcal{E}_0, \ldots, \mathcal{E}_n\}$, the object $\bigoplus \mathcal{E}_i$ is a classical generator due to the equivalence $\mathcal{D} \xrightarrow{\sim} D^\flat(\mathrm{Mod}\, A)$ from Section 2.4. This argument also clearly applies to the equivalence $\mathcal{D}_{0,X} \xrightarrow{\sim} D^\flat_0(\mathrm{Mod}\, B)$ so that the coproduct of objects in any spherical collection in $\mathcal{D}_{0,X}$ constructs a classical generator.

The two most interesting prospects of categorical Thurston compactifications are when (2) has a positive answer and whether there is an explicit description of (4). As noted above, the injectivity of the map $\mathbb{P}m$ clearly depends on the choice of the coordinate set $\mathcal{S}$; however, we are also able to examine a slightly more general question of whether there exist choices of coordinate set $\mathcal{S}$ for a particular triangulated category $\mathcal{D}$ that make the mass map injective. It turns out that this is a subtle question that is ultimately related to whether every stability condition (in a connected component of the stability manifold) admits "enough" stable objects. In the next section, we generalize the ideas of [KKO22] to describe a large class of categories that never admit a valid choice of coordinate set $\mathcal{S}$ for the mass map. On the other hand, we are also able to describe a class of categories relevant to our local Calabi-Yau varieities above which always admit some choice of coordinate set.

For the remainder of this section, let $\mathcal{D}$ be a $\Bbbk$-linear category and suppose that there is a connected component $\mathrm{Stab}^\dagger(\mathcal{D}) \subseteq \mathrm{Stab}(\mathcal{D})$ consisting of numerical stability conditions. Let E be an object with primitive character $\nu(E) \in \mathcal{N}(\mathcal{D})$ such that the open region

$$U_E = \{\sigma \in \mathrm{Stab}^\dagger(\mathcal{D}) \mid E \ \text{ is } \ \sigma\text{-stable}\}$$

is nonempty. Moreover, suppose that G is a subgroup of $\mathrm{Aut}(\mathcal{D})$ preserving the connected component $\mathrm{Stab}^\dagger(\mathcal{D})$ such that

$$\mathrm{Stab}^\dagger(\mathcal{D}) = \bigcup_{\Phi \in G} \overline{\Phi(U_E)}.$$

The motivating examples that one should consider for the above setup are the stability manifolds of the local Calabi-Yau varieties (described in the previous section), the stabiliy manifolds of K3 surfaces (as described in [Bri08]), and the stability manifolds of finite acyclic quivers. In the majority of cases, the open set U_E will be the geometric chamber where the forgetful map $\mathcal{Z} : \mathrm{Stab}^\dagger(\mathcal{D}) \to \mathrm{Hom}(K(\mathcal{D}), \mathbb{C})$ is a homeomorphism onto its image; however, this map fails to be a covering map for local $\mathbb{P}^2$ ([BM11]), so we aren't able to explicitly define G as some group of deck-transformations.

Theorem 4.7. *Suppose that $n = \mathrm{rk}\,\mathcal{N}(\mathcal{D}) < \infty$. Let Σ denote a class of objects in $\mathcal{D}$ such that for every two $\sigma_1, \sigma_2 \in U_E$, one can find at least $\binom{n+2}{2} - 1$ elements of σ that are semi-stable with respect to both σ_1, σ_2. If*

$$m_\sigma(\Phi(E)) > m_\sigma(E) \qquad\qquad \forall \sigma \in U_E \tag{$\star$}$$

for every Φ such that $\Phi(U_E) \cap U_E = \emptyset$, then for generic choice of Σ the map $\mathbb{P}m$: $\mathrm{Stab}^\dagger(\mathcal{D})/\mathbb{C} \to \mathbb{RP}^{\mathcal{S}}_{\geq 0}$ is injective with respect to the coordinate set

$$\mathcal{S} = G \cdot (\{E\} \cup \Sigma).$$

Proof. Fix distinct $\sigma_1, \sigma_2 \in \mathrm{Stab}^\dagger(\mathcal{D})/\mathbb{C}$ and first consider the case that $\sigma_1, \sigma_2 \in U_E$. For stable inputs, the square of the mass function $(m_\sigma(-))^2 = Z_\sigma(-)\overline{Z_\sigma(-)}$ is a real-valued quadratic form. In particular, the generic choice of stable objects gives $\binom{n+2}{n} - 1$ independent linear conditions on the quadratic form, which uniquely recovers the central charge.

Next, suppose that $\sigma_2 \in \Phi(U_E)$ for $\Phi \in G$ with $\Phi(U_E) \cap U_E = \emptyset$. Up to the $\mathbb{C}$-action, we may assume that σ_1 is a "normalized" stability condition such that $m_{\sigma_1}(E) = 1$; similarly, assume σ_2 is a translate of a normalized stability condition under Φ. By Lemma 2.27 and our assumption,

$$m_{\sigma_2}(E) = m_{\Phi^{-1}\sigma_2}(\Phi^{-1}E) > 1 = m_{\sigma_1}(E).$$

An identical argument shows that $m_{\sigma_2}(\Phi(E)) < m_{\sigma_1}(\Phi(E))$. If we restrict the image of the mass map $\mathbb{P}m : \mathrm{Stab}^\dagger(\mathcal{D})/\mathbb{C} \to \mathbb{RP}^{\mathcal{S}}$ to the two coordinates corresponding to E and $\Phi(E)$, this implies $\mathbb{P}m(\sigma_1) = [m_{\sigma_1}(E) : m_{\sigma_1}(\Phi(E))] = [1 : r]$ and $\mathbb{P}m(\sigma_2) = [m_{\sigma_2}(E) : m_{\sigma_2}(\Phi(E))] = [s : 1]$ with $r, s > 1$. $\square$

The set $\mathcal{S}$ constructed in the theorem above will be referred to as the *naïve coordinates* of $\mathrm{Stab}^\dagger(\mathcal{D})$ *with respect to* G since its ad-hoc construction is intended to differentiate between any two stability conditions by reducing the more nuanced mass with the central charge — in other words, this construction effectively reduces injectivity to a linear algebra problem on the Grothendieck group.

Remark 4.8. When the open chamber U_E covers the entire connected component $\mathrm{Stab}^\dagger(\mathcal{D})$, the set G is taken to be the trivial subgroup. In this case, the proof of injectivity effectively mimics the argument of [KKO22]; for example, on smooth projective curves of positive genus one can take $E = \Bbbk(x)$ so that $U_E = \mathrm{Stab}^{\mathrm{Geo}}(D^\flat(C))$. This region is the

$\widetilde{\mathrm{GL}}_2^+(\mathbb{R})$-orbit of classical slope stability by [Mac07a], and in fact accounts for the entire stability manifold. Thus, one can take $\Bbbk(x)$ along with any two other slope-stable objects with distinct characters to obtain injectivity.

The above example may be slightly generalized to *non-commutative varieties*, e.g. admissible subcategories of some $D^\flat(X)$ whose Hoschild cohomology mimics that of a smooth projctive variety. For example, if $X \subset \mathbb{P}^5$ is a very general cubic fourfold, then the *Kuznetsov Component* $\mathrm{Ku}(X) := \langle \mathcal{O}_X, \mathcal{O}_X(1), \mathcal{O}_X(2)\rangle^\perp$ has a rank two numerical Grothendieck group. For a general line $L \subset X$, the characters of $\mathcal{O}_L(1)$ and $\mathcal{O}_L(2)$ give a basis for $\mathcal{N}(\mathrm{Ku}(X))$, and the intersection pairing takes the form

$$\begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}$$

Since the Serre-functor of $\mathrm{Ku}(X)$ satisfies $S_{\mathrm{Ku}(X)} = [2]$, the category $\mathrm{Ku}(X)$ is more specifically referred to as a *non-commutative K3 surface*. However, by [FP23, Corollary 3.9] the stability manifold of $\mathrm{Ku}(X)$ much more closely resembles that of a positive genus curve since it is a single $\widetilde{\mathrm{GL}}_2^+(\mathbb{R})$-orbit. By [BLMS23, Lemma A.4, A.5] it is always possible to objects with specific primitive characters that are stable in the entire connected component.

Remark 4.9. In lieu of the previous remark, it is also easy to show that any smooth projective variety X with a finite Albanese morphism allows a Thurston compactification whenever Σ contains suficiently many slope-stable bundles, since $\mathrm{Stab}^{\mathrm{Geo}}(D^\flat(X))/\mathbb{C}$ accounts for an entire connected component by [FLZ22]. In these cases, arguing $\mathbb{P}m$ is a homeomorphism onto its image should be a fairly immediate consequence of the fact that the forgetful map $\mathcal{Z} : \mathrm{Stab}^\dagger(D^\flat(X)) \to \mathrm{Hom}(K(X), \mathbb{C})$ is a homeomorphism (not just local homeomorphism).

When X does admit a finite Albanese morphism, it is an interesting question as to whether 4.5(4) always has a positive answer. For example, when X is a Picard rank 1 abelian surface, [Bri08] showed that $\mathrm{Geo}(D^\flat(X)) \cong \mathcal{V} \times \mathbb{C}$ where $\mathcal{V}$ is often referred to as the (α, β)-plane. On rational points of the boundary $\alpha = 0$, the (modulus of the) central charge $|Z_{\alpha,\beta}(-)|$ approaches the dual character of some semi-homogeneous vector bundle $E_{p/q}$. However, it is unknown to the author whether a homological mirror symmetry argument can be used to generalize the argument of [KKO22] for elliptic curves.

Corollary 4.10. *If $\mathcal{D}$ is a K3-category and $\nu(E)$ is in the stabilizer of the action of G on the numerical Grothendieck group $\mathcal{N}(\mathcal{D})$, then $\mathbb{P}m : \mathrm{Stab}^{\dagger}(\mathcal{D})/\mathbb{C} \to \mathbb{RP}^{\mathcal{S}}$ is injective with respect to the naive coordinate set $\mathcal{S}$.*

Proof. By the previous theorem, it suffices to show that $m_\sigma(\Phi(E)) > m_\sigma(E)$ holds for every $\sigma \in U_E$ whenever the chambers $\Phi(U_E)$ and U_E are distinct. By construction of the set U_E, we have $\Phi(E)$ is only stable in $\Phi(U_E)$ and thus unstable in U_E. Fix some $\sigma \in U_E$ and let $A_1, \ldots, A_n$ denote the semi-stable factors of $\Phi(E)$. Since $\nu(E) = \nu(\Phi(E))$ by assumption,

$$m_\sigma(E) = |Z(E)| = \Big|\sum_{i=1}^{n} Z(A_i)\Big| \leq \sum_{i=1}^{n} |Z(A_i)| = m_\sigma(\Phi(E))$$

where equality holds if and only if $\phi(A_{i-1}) - \phi(A_i) \in 2\mathbb{Z}$ for each $1 < i \leq n$. To see that the strict inequality must hold, suppose to the contrary that $\Phi(E)$ is unstable with all semistable factors satisfying $\phi(A_{i-1}) - \phi(A_i) \in 2\mathbb{Z}_{>0}$. In particular, we may rewrite the semi-stable filtration as

$$\begin{tikzcd}
0 = E_0 \arrow[r] & E_1 \arrow[r] \arrow[d] & E_2 \arrow[r] \arrow[d] & \cdots \arrow[r] & E_{n-1} \arrow[r] \arrow[d] & E_n = \Phi(E) \arrow[d] \\
& A_1'[2m_1] \arrow[ul, dashed] & A_2'[2m_2] \arrow[ul, dashed] & \cdots & A_{n-1}'[2m_{n-1}] \arrow[ul, dashed] & A_n'[2m_n] \arrow[ul, dashed]
\end{tikzcd}$$

where the objects A_i' are in the heart of σ and $m_1, ..., m_n$ is a strictly decreasing sequence of integers. Without loss of generality, we may take $m_i = n - i$. Since $\mathcal{D}$ is assume to be a K3-category,

$$\mathrm{RHom}_{\mathcal{D}}(A_n', E_{n-1}[1]) \cong \mathrm{RHom}_{\mathcal{D}}(E_{n-1}, A_n'[1])^{\vee}$$

by Serre duality. However, $\phi^-(E_{n-1}) = \phi(A_{n-1}'[2]) > \phi(A_n'[1])$ so that the above RHom-spaces are trivial and thus $\Phi(E)$ is the trivial extension $\Phi(E) \cong E_{n-1} \oplus A_n'$. Proceeding inductively, the same Serre-duality argument shows that

$$E_k \cong \Phi(E) \oplus \bigoplus_{j=0}^{n-k-1} A'_{n-j}[2j]$$

But then $E_1 \cong A'_1[2n] \cong \Phi(E) \oplus A'_2[2n-2] \oplus \cdots \oplus A'_n$ splits into stable factors with distinct phases and is thus unstable, which is impossible.

Alternatively, if all semi-stable factors have same phase, the object $\Phi(E)$ would in fact be semi-stable — in turn, this would imply that E is $\Phi^{-1}(\sigma)$-semi-stable, so that $\sigma \in \partial U_E$ contradicting our assumption. Therefore, the strict inequality must hold within U_E. □

4.1 Thurston Compactification for Varieties admitting Strong Exceptional Collections

It was originally pointed out in [KKO22, Remark 6.4] that the mass map $\mathbb{P}m$ should fail to be injective whenever $\mathcal{D}$ admits a full strong exceptional collection $\mathfrak{E} = \{\mathcal{E}_0, \ldots, \mathcal{E}_{n-1}\}$. An explicit argument was given in the case of the projective line which relied on the construction of stability conditions with "degenerate hearts" where the only stable objects are the generating exceptional objects. Using the results of [Mac07b, Lemma 3.18], this is always possible under the technical assumption that a strong exceptional collection exists. For the sake of completeness, we provide proof of this fact; however, it should be noted the argument is practically identical to the proof of [KKO22, Proposition 6.3].

Before giving an argument for general varieties admitting exceptional collections, we first use the results of [Li17] outlined in Section 3.2 to provide an example argument

Proposition 4.11. *If $\mathcal{S}$ contains all exceptional vector bundles, the mass map $\mathbb{P}m$: $\mathrm{Stab}^\dagger(\mathbb{P}^2)/\mathbb{C} \to \mathbb{RP}^{\mathcal{S}}_{\geq 0}$ restricted to $\mathrm{Stab}^{\mathrm{Geo}}(\mathbb{P}^2)/\mathbb{C}$ is injective.*

Proof. Suppose that $\mathbb{P}m(\sigma_1) = \mathbb{P}m(\sigma_2)$ for $\sigma_1, \sigma_2 \in \mathrm{Stab}^{\mathrm{Geo}}(\mathbb{P}^2)$. Without loss of generality, we may write $\sigma_1 = \sigma_{a,b}$ and $\sigma_2 = \sigma_{a',b'} \cdot (M, f)$ for some $(a,b), (a',b') \in G$ and $(M, f) \in \widetilde{\mathrm{GL}}_2^+(\mathbb{R})$. We first show that M is necessarily induced by the $\mathbb{C}$-action. Since geometric stability conditions are non-degenerate, we may choose two exceptional objects $\mathcal{E}, \mathcal{F}$ whose image under the central charge of both σ_1 and σ_2 do not lie on a real line. Moreover, we may choose $\mathcal{E}$ and $\mathcal{F}$ of sufficiently large slope so that $\mathcal{E}, \mathcal{F}$, and $\mathcal{L}_{\mathcal{E}}\,\mathcal{F}$ are stable with respect to both stability conditions. Thus, the assumption $\mathbb{P}m(\sigma_1) = \mathbb{P}m(\sigma_2)$ implies

$$|Z_{a,b}(\mathcal{E})| = c|M^{-1} \circ Z_{a',b'}(\mathcal{E})|$$
$$|Z_{a,b}(\mathcal{F})| = c|M^{-1} \circ Z_{a',b'}(\mathcal{F})|$$
$$|Z_{a,b}(\mathcal{L}_{\mathcal{E}}\,\mathcal{F})| = c|M^{-1} \circ Z_{a',b'}(\mathcal{L}_{\mathcal{E}}\,\mathcal{F})|$$

for some $c > 0$ (which we may in fact assume $c = 1$). However, since $\mathcal{L}_{\mathcal{E}}\,\mathcal{F}$ fits into a distinguished triangle

$$\mathcal{F}[-1] \to \mathcal{L}_{\mathcal{E}}\,\mathcal{F} \to \mathrm{RHom}(\mathcal{E}, \mathcal{F}) \otimes \mathcal{E}$$

the magnitude of (the central charge of) $\mathcal{L}_{\mathcal{E}}\,\mathcal{F}$ depends directly on the difference in phases between $\mathcal{F}$ and $\mathcal{E}$ (resp. $\mathcal{F}$ and $\mathcal{E}[1]$) when $\chi(\mathcal{E}, \mathcal{F}) < 0$ (resp. when $\chi(\mathcal{E}, \mathcal{F}) > 0$). Therefore, the third equality immediately implies that M is induced by the $\mathbb{C}$-action.

Since we may factor out the action of $\mathbb{C}$ under the quotient $\mathrm{Stab}^\dagger(\mathbb{P}^2)/\mathbb{C}$, we may now assume σ_2 is simply of the form $\sigma_{a',b'}$. Again choose exceptional bundles $\mathcal{E}_1, \ldots, \mathcal{E}_n$ of sufficiently large slope such that they are all stable with respect to both σ_1 and σ_2. Then the condition that

$$|Z_{a,b}(\mathcal{E}_i)|^2 = |Z_{a',b'}(\mathcal{E}_i)|^2$$

yields a quadratic system of n-equations in the variables $\Re(a), \Re(b), \Im(a), \Im(b)$. Since we may always choose n and $\mathcal{E}_i$ in such a way that the characters $\mathrm{ch}(\mathcal{E}_i) = (r(\mathcal{E}_i), d(\mathcal{E}_i), c(\mathcal{E}_i))$ give a non-degenerate system of equations over $\mathbb{R}$, we immediately obtain $(a, b) = (a', b')$.

□

Proposition 4.12. *The mass map* $\mathbb{P}m : \mathrm{Stab}^\dagger(\mathbb{P}^2)/\mathbb{C} \to \mathbb{RP}^{\mathcal{S}}_{\geq 0}$ *restricted to* $\mathrm{Stab}^a/\mathbb{C}$ *is* not *injective for any choice of objects* $\mathcal{S} \subset \mathrm{Ob}(D^\flat(\mathbb{P}^2))$.

Proof. We need only consider the region $\Theta^{\mathrm{pure}}_{\mathfrak{E}}$ for some (full strong) exceptional collection

$\mathfrak{E} = \{\mathcal{E}_0, \mathcal{E}_1, \mathcal{E}_2\}$. By Lemma 3.8 (i.e. [Li17, Lemma 2.4]), the only stable objects for any stability condition $\sigma \in \Theta_{\mathfrak{E}}^{\text{pure}}$ are shifts of the $\mathcal{E}_i$. Thus, we may use an identical argument to [KKO22, Proposition 6.3]: fix $m_0, m_1, m_2 > 0$, ϕ_0, ϕ_1 satisfying $\phi_1 - \phi_0 \geq 1$, and lastly $\phi_2 \neq \phi_2'$ with both $\phi_2 - \phi_1 \geq 1$ and $\phi_2' - \phi_1 \geq 1$. Define two stability conditions $\sigma, \sigma' \in \Theta_{\mathfrak{E}}^{\text{pure}}$ by

$$Z_\sigma(\mathcal{E}_j) = Z_{\sigma'}(\mathcal{E}_j) = m_j e^{\pi i \phi_j}$$

for $j = 0, 1$, $Z_\sigma(\mathcal{E}_2) = m_2 e^{\pi i \phi_2}$ and $Z_{\sigma'}(\mathcal{E}_2) = m_2 e^{\pi i \phi_2'}$. By construction, σ and σ' are not equivalent up to the $\mathbb{C}$-action. However, the Harder-Narasimhan filtration of any object $E \in \mathcal{S}$ is the same for both σ and σ', and the Harder-Narasimhan factors are all shifts of the exceptional objects $\mathcal{E}_i$. Since $|Z_\sigma(\mathcal{E}_i)| = |Z_{\sigma'}(\mathcal{E}_i)|$, we immediately obtain $m_\sigma(E) = m_{\sigma'}(E)$ for any object E and thus $\mathbb{P}m(\sigma) = \mathbb{P}m(\sigma')$. □

A slightly less general argument can be made for the regions $\Theta_{\mathcal{E}}^+$ and $\Theta_{\mathcal{F}}^-$ (for arbitrary exceptional objects $\mathcal{E}$, $\mathcal{F}$). For example, if we complete $\mathcal{E}$ to any exceptional collection $\{\mathcal{E}_0 = \mathcal{E}, \mathcal{E}_1, \mathcal{E}_2\}$ so that a stability condition in $\Theta_{\mathcal{E}}^+$ can be defined in terms of $m(\mathcal{E}_i)$ and $\phi(\mathcal{E}_i)$, then by definition of $\Theta_{\mathcal{E}}^+$ we must have $\phi_2 - \phi_1 < 1$ so that $\mathcal{L}_{\mathcal{E}_1} \mathcal{E}_2$ is a stable (non-trivial) extension of $\mathcal{E}_2$ and $\mathcal{E}_1[1]$. In particular, whenever the coordinate set $\mathcal{S}$ contains a non-trivial extension $\mathcal{B}$ of $\mathcal{E}_1$ and $\mathcal{E}_2$, $m_\sigma(\mathcal{B}) = m_{\sigma'}(\mathcal{B})$ forces $\phi_2 - \phi_1 = \phi_2' - \phi_1'$ for any two stability conditions $\sigma, \sigma' \in \Theta_{\mathcal{E}}^+$. Up to the $\mathbb{C}$-action, simply assume $Z_\sigma(\mathcal{E}_j) = Z_{\sigma'}(\mathcal{E}_j)$ for $j = 1, 2$. Since ϕ_0 satisfies $\phi_1 - \phi_0 > 1$, we can slightly deform ϕ_0 to some ϕ_0' such that $\phi_1 - \phi_0' > 1$ still holds; as before, this will not change the Harder-Narasimhan filtration of any object in $\mathcal{S}$ so that $\mathbb{P}m(\sigma) = \mathbb{P}m(\sigma')$. An identical argument holds for the $\Theta_{\mathcal{F}}^+$.

The key detail in the above proof for $\mathbb{P}^2$ was [Li17, Lemma 2.4], which showed that when $\phi(\mathcal{E}_0) + 2 < \phi(\mathcal{E}_1) + 1 < \phi(\mathcal{E}_2)$ for some exceptional collection $\mathfrak{E} = \{\mathcal{E}_0, \mathcal{E}_1, \mathcal{E}_2\}$, the only stable objects were shifts of $\mathcal{E}_i$. In general, given a full strong exceptional collection $\mathcal{E} = \{\mathcal{E}_0, \ldots, \mathcal{E}_n\}$ in some triangulated category $\mathcal{D}$, [Mac07b, Lemma 3.18] allows one to construct a connected open region of stability conditions such that $\phi(\mathcal{E}_0) + n < \phi(\mathcal{E}_1) + (n-1) < \cdots < \phi(\mathcal{E}_n)$. For such stability conditions, Li's argument directly generalizes: the heart must be of the form $\mathcal{A}_{\mathfrak{E}} = \langle \mathcal{E}_0[a_0], \ldots, \mathcal{E}_1[a_1], \ldots, \mathcal{E}_{n-1}[a_{n-1}], \mathcal{E}_n \rangle$ where $a_{n-1} \geq 1$

and $a_i - a_{i+1} \geq 1$ for all $0 \leq i < n-1$. While an inductive argument can be used to reduce to the case $a_i - a_{i+1} = 1$, this is more than we will need to prove non-injectivity.

Theorem 4.13. *Suppose that $\mathcal{D}$ admits a full strong exceptional collection $\mathfrak{E} = \{\mathcal{E}_0, \ldots, \mathcal{E}_n\}$. Then for any choice of objects $\mathcal{S} \subset \mathrm{Ob}(\mathcal{S})$, the mass map $\mathbb{P}m : \mathrm{Stab}^\dagger(\mathcal{D})/\mathbb{C} \to \mathbb{RP}^{\mathcal{S}}_{\geq 0}$ fails to be injective.*

Proof. Consider the modified exceptional collection $\mathfrak{E}' = \{\mathcal{E}_0[n], \mathcal{E}_1[n-1], \ldots, \mathcal{E}_n\}$. Since the objects of $\mathfrak{E}$ form a strong exceptional collection, the graded-hom spaces satisfy $\mathrm{Hom}^{\leq 0}(\mathcal{E}_i[n-i], \mathcal{E}_j[n-j]) = 0$ for $i \neq j$; by [Mac07b, Lemma 3.14], $\langle \mathcal{E}_0[n], \ldots, \mathcal{E}_n \rangle$ is the heart of a bounded t-structure. Consider the $\widetilde{\mathrm{GL}}_2^+(\mathbb{R})$-orbit of stability conditions in $\Theta_{\mathfrak{E}'}$ with $\phi(\mathcal{E}_0[n]) < \cdots < \phi(\mathcal{E}_n)$, and suppose that F is a nonzero stable object in the above heart. By the previous chapter, the heart $\mathcal{A}_{\mathfrak{E}}$ is obtained by pulling back the standard t-structure mod-A for $A = \mathrm{End}(\oplus_i \mathcal{E}_i)$; in particular, every object in the heart is quasi-isomorphic to a complex of the form

$$\mathcal{E}_0^{\oplus b_0} \to \mathcal{E}_1^{\oplus b_1} \to \cdots \to \mathcal{E}_n^{\oplus b_n}$$

concentrated in degrees $-n$ through 0. The t-structure on this heart is obtained via the truncation functors $\mathrm{tr}^{\leq k}$ and $\mathrm{tr}^{\geq k+1}$. Letting k be the smallest number such that $b_{n-k} \neq 0$, we obtain a short exact sequence in the heart

$$0 \to \mathrm{tr}^{\leq k} F \to F \to \mathrm{tr}^{\geq k+1} F \to 0$$

where $\mathrm{tr}^{\leq k} F = \mathcal{E}_{n-k}^{\oplus b_{n-k}}[k]$ by assumption and $\mathrm{tr}^{\geq k+1} \in \langle \mathcal{E}_0[n], \ldots, \mathcal{E}_{n-k-1}[k+1] \rangle$. If $\mathrm{tr}^{\geq k+1} F$ is nonzero, then $\phi(\mathrm{tr}^{\leq k}) > \phi(F)$, a contradiction; thus, the stable objects of the above $\widetilde{\mathrm{GL}}_2^+(\mathbb{R})$-orbit are precisely shifts of the simple objects $\mathcal{E}_i[n-i]$.

The remainder of the proof closely mimics [KKO22]: let σ_1, σ_2 be two distinct stability conditions in the above $\widetilde{\mathrm{GL}}_2^+(\mathbb{R})$-orbit such that

$$[\ |Z_1(\mathcal{E}_0[n])| : \cdots : |Z_1(\mathcal{E}_n)|\ ] = [\ |Z_2(\mathcal{E}_0[n])| : \cdots : |Z_2(\mathcal{E}_n)|\ ]$$

in $\mathbb{RP}^n$. Moreover, assume there exists some $1 \leq i, j \leq n$ such that $\phi_1(\mathcal{E}_i[n-i]) - \phi_1(\mathcal{E}_j[n-j]) \neq \phi_2(\mathcal{E}_i[n-i]) - \phi_2(\mathcal{E}_j[n-j])$ — in particular, this ensures that the two stability conditions are in distinct $\mathbb{C}$-orbits. As the only stable objects for both stability conditions are the objects $\mathcal{E}_i[n-i]$, the Harder-Narasimhan filtrations of any object in $\mathcal{T}$ will be the same with respect to both stability conditions. Thus, $\mathbb{P}m(\sigma_1) = \mathbb{P}m(\sigma_2)$ for any choice of $\mathcal{S}$. $\square$

4.2 Thurston Compactification for Local $\mathbb{P}^1$

Due to the derived McKay correspondence explained in [Bri09], the category $\mathcal{D}_{0,\mathbb{P}^1}$ is nearly identical to the 2-Calabi-Yau category $\mathcal{C}_\Gamma$ associated to the A_2 quiver. Thus, the arguments of [BDL20, Section 7] apply almost verbatim to provide an algebraic approach to the compactification of $\mathrm{Stab}(\mathcal{D}_{0,\mathbb{P}^1})/\mathbb{C}$. However, since our argument will almost directly generalize to $\mathcal{D}_{0,\mathbb{P}^2}$ in the next section, we still provide a geometric argument.

As described in Section 3.1, the stability manifold $\mathrm{Stab}(\mathcal{D}_{0,\mathbb{P}^1})/\mathbb{C}$ *locally* resembles that of a projective curve of positive genus; the wall-and-chamber decomposition with respect to $[\mathbb{k}(x)]$ allows us to represent the stability manifold as a union of open regions homeomorphic to the upper half-plane. With this in mind, it would not be amiss to assume a reasonable coordinate set $\mathcal{S}$ should mimic the coordinates introduced by the authors of [KKO22] — specifically, $\mathcal{S} = \{\mathbb{k}(x), i_*\mathcal{O}_{\mathbb{P}^1}, i_*\mathcal{O}_{\mathbb{P}^1}(-1)[1]\}$. The main technicality is now whether the map $\mathbb{P}m$ can distinguish between distinct chambers; for example, when $\Theta_0^- \neq \Phi(\Theta_0^{-1})$ for $\Phi \in \langle \mathrm{Tw}_0^{\pm}, \mathrm{Tw}_{-1}^{\pm} \rangle$, why should $\mathbb{P}m(\Theta_0^-)$ and $\mathbb{P}m(\Phi(\Theta_0^-))$ not overlap? When the length of the word Φ is small, it turns out that the map $\sigma \to [m_\sigma(\mathbb{k}(x)) : m_\sigma(i_*\mathcal{O}_{\mathbb{P}^1}) : m_\sigma(i_*\mathcal{O}_{\mathbb{P}^1}(-1))]$ suffices to distinguish Θ_0^- from neighboring chambers.

Proposition 4.14. *Fix $\sigma \in \Theta_0^+$, and assume $\Phi \in \langle \mathrm{Tw}_{-1}, \mathrm{Tw}_0^{-1} \rangle$ is such that*

$$\phi_\sigma^-(\Phi(i_*\mathcal{O}_{\mathbb{P}^1})) > \phi_\sigma^+(\Phi(i_*\mathcal{O}_{\mathbb{P}^1}(-1)[1])).$$

Then

(i) If $\phi_\sigma^+(\Phi(i_*\mathcal{O}_{\mathbb{P}^1})) - \phi_\sigma^-(\Phi(i_*\mathcal{O}_{\mathbb{P}^1})) < 1$ *then by taking* $\widetilde{\Phi} = \Phi \circ \mathrm{Tw}_0^{-1}$*, one obtains*

$$\phi_\sigma^-(\widetilde{\Phi}(i_*\mathcal{O}_{\mathbb{P}^1})) > \phi_\sigma^+(\widetilde{\Phi}(i_*\mathcal{O}_{\mathbb{P}^1}(-1)[1]))$$

(ii) If $\phi_\sigma^+(\Phi(i_*\mathcal{O}_{\mathbb{P}^1}(-1)[1])) - \phi_\sigma^-(\Phi(i_*\mathcal{O}_{\mathbb{P}^1}(-1)[1])) < 1$ *then by taking* $\widetilde{\Phi} = \Phi \circ \mathrm{Tw}_{-1}$*, one obtains*

$$\phi_\sigma^-(\widetilde{\Phi}(i_*\mathcal{O}_{\mathbb{P}^1})) > \phi_\sigma^+(\widetilde{\Phi}(i_*\mathcal{O}_{\mathbb{P}^1}(-1)[1]))$$

In both cases, the Harder-Narasimhan filtration of $\widetilde{\Phi}(\Bbbk(x))$ *can be obtained directly from the Harder-Narasimhan filtrations of* $\Phi(i_*\mathcal{O}_{\mathbb{P}^1})$ *and* $\Phi(i_*\mathcal{O}_{\mathbb{P}^1}(-1)[1])$*.*

Proof. This is almost an immediate consequence of Proposition 4.2. For case (i), apply Φ to the triangle

$$i_*\mathcal{O}_{\mathbb{P}^1}^{\oplus 2} \to \mathrm{Tw}_0^{-1} i_*\mathcal{O}_{\mathbb{P}^1}(-1)[1] \to i_*\mathcal{O}_{\mathbb{P}^1}(-1)[1]$$

to obtain

$$\Phi(i_*\mathcal{O}_{\mathbb{P}^1}^{\oplus 2}) \to \widetilde{\Phi}(i_*\mathcal{O}_{\mathbb{P}^1}(-1)[1]) \to \Phi(i_*\mathcal{O}_{\mathbb{P}^1}(-1)[1])$$

Since $\phi_\sigma^-(\Phi(i_*\mathcal{O}_{\mathbb{P}^1})) > \phi_\sigma^+(\Phi(i_*\mathcal{O}_{\mathbb{P}^1}(-1)[1]))$, the Harder-Narasimhan factors of $\widetilde{\Phi}(i * \mathcal{O}_{\mathbb{P}^1}(-1)[1])$ are simply the Harder-Narasimhan factors of $\Phi(i_*\mathcal{O}_{\mathbb{P}^1}^{\oplus 2})$ concatenated to the factors of $\Phi(i_*\mathcal{O}_{\mathbb{P}^1}(-1)[1])$. In particular, $\phi_\sigma^+(\widetilde{\Phi}(i_*\mathcal{O}_{\mathbb{P}^1}(-1)[1]) = \phi_\sigma^+(\Phi(i_*\mathcal{O}_{\mathbb{P}^1}))$. Since $\phi_\sigma^-(\widetilde{\Phi}(i_*\mathcal{O}_{\mathbb{P}^1}) = \phi_\sigma^-(\Phi(i_*\mathcal{O}_{\mathbb{P}^1}) + 1$ the claim follows by assumption. The proof of case (ii) is almost identical and therefore omitted. □

Though the above lemma is somewhat specific to a small handful of cases, it shows that within the neighboring chambers Θ_0^+ and $\Phi(\Theta_0^+)$ for

$$\Phi \in \{\mathrm{id}, \mathrm{Tw}_0^{>0}, \mathrm{Tw}_{-1}^{<0}, \mathrm{Tw}_0 \mathrm{Tw}_{-1}^{-1}, \mathrm{Tw}_{-1}^{-1} \mathrm{Tw}_0\}$$

one has

$$m_{\Phi\sigma}(\Bbbk(x)) = m_\sigma(\Phi^{-1}\Bbbk(x)) = m_\sigma(\Phi^{-1} i_*\mathcal{O}_{\mathbb{P}^1}) + m_\sigma(\Phi^{-1} i_*\mathcal{O}_{\mathbb{P}^1}(-1)[1])$$

Thus, we may apply a straightforward injectivity argument: if $\sigma_1 \in \Theta_0^-$ and $\sigma_2 \in \Theta_0^+$, then $\mathbb{P}m(\sigma_1) = [1, a, b]$ and (after rescaling) $\mathbb{P}m(\Phi(\sigma_2)) = [1, c, d]$ where $1 < a + b$ and $1 = c + d$. It is worth noting that there is an obvious injectivity argument for $\sigma_1, \sigma_2 = \Theta_0^- = \mathrm{Stab}^{\mathrm{Geo}}(\mathbb{P}^1)$; thus we choose $\sigma_2 \in \Theta_0^+$ since $\mathrm{Tw}_0 \Theta_0^- = \Theta_0^+ = \mathrm{Tw}_{-1}^{-1} \Theta_0^-$.

Unfortunately the above argument breaks down when considering

$$\widetilde{\Phi} = \mathrm{Tw}_{-1} \mathrm{Tw}_0^{-1} \mathrm{Tw}_{-1} .$$

by fixing $\sigma \in \Theta_0^+$, we first note that finding an HN-filtration of any object $E \in \mathcal{D}_{0,\mathbb{P}^1}$ with respect to $\widetilde{\Phi}^{-1}(\sigma)$ is the same as finding an HN-filtration of $\widetilde{\Phi}(E)$ with respect to σ. Consecutively applying the argument of [BDL20, Proposition 3.2], we can stitch together HN-filtrations of $\widetilde{\Phi}(i_*\mathcal{O}_{\mathbb{P}^1})$ and $\widetilde{\Phi}(i_*\mathcal{O}_{\mathbb{P}^1}(-1)[1])$ by applying spherical twists to the triangles introduced in Lemma 3.4 above. In this case, we find that

$$\begin{aligned} \phi_\sigma^-(\widetilde{\Phi}(i_*\mathcal{O}_{\mathbb{P}^1})) &= \phi(i_*\mathcal{O}_{\mathbb{P}^1}(-1)[1]) - 1 \\ \phi_\sigma^+(\widetilde{\Phi}(i_*\mathcal{O}_{\mathbb{P}^1}(-1)[1])) &= \phi(i_*\mathcal{O}_{\mathbb{P}^1}) - 1 \end{aligned}$$

so that the HN-filtration of $\widetilde{\Phi}(\mathbb{k}(x))$ is no longer obtained by concatenating the filtrations of $\widetilde{\Phi}(i_*\mathcal{O}_{\mathbb{P}^1})$ and $\widetilde{\Phi}(i_*\mathcal{O}_{\mathbb{P}^1}(-1)[1])$.

While it is an interesting question to ask whether there exists a finite set $\mathcal{S}$ such that $\mathbb{P}m : \mathrm{Stab}(\mathcal{D}_{0,\mathbb{P}^1}) \to \mathbb{RP}^{\mathcal{S}}$ is injective, we instead provide an argument for the naïve compactification introduced in the beginning of this chapter.

Definition 4.15. Suppose that X is a smooth projective Fano variety admitting a simple exceptional collection. Whenever the annular braid group B_n acts transitively on the set of spherical collections in $\mathcal{D}_{0,X}$, we define the *naïve coordinates of* $\mathcal{D}_{0,X}$ to be the set

$$\mathcal{S}' := \big(B_n \cdot \mathfrak{S}\big) \cup \langle \mathrm{Tw}_{S_0}^{\pm}, \dots, \mathrm{Tw}_{S_{n-1}}^{\pm} \rangle \cdot \{\mathbb{k}(x)\},$$

where $\mathfrak{S} = \{S_0, \dots, S_{n-1}\}$ is some spherical collection and $x \in X$ is a closed point.

For example, when $X = \mathbb{P}^1$ the naive compactification coordinates of $\mathcal{D}_{0,\mathbb{P}^1}$ are

$$\mathcal{S}' = \langle \mathrm{Tw}_0^{\pm}, \mathrm{Tw}_{-1}^{\pm} \rangle \cdot \{\Bbbk(x), i_*\mathcal{O}_{\mathbb{P}^1}, i_*\mathcal{O}_{\mathbb{P}^1}(-1)[1]\}.$$

For the following theorem, we continue to let $\mathrm{Stab}^{\mathrm{Alg}}(\mathcal{D}_{0,X})$ denote the union of the open sets $\Theta_{\mathfrak{S}}$ (constructed in Section 2.5) whenever B_n acts transitively on the set of spherical collections in $\mathcal{D}_{0,X}$.

Theorem 4.16. *If the map $\mathbb{P}m : \mathrm{Stab}^{\mathrm{Alg}}(\mathcal{D}_{0,X}) \to \mathbb{RP}^{\mathcal{S}}$ is injective with respect to the naïve coordinates of $\mathcal{D}_{0,X}$, then it is a homeomorphism onto its image.*

Proof. Fix some spherical collection $\mathfrak{S} = \{S_0, \ldots, S_{n-1}\}$. By [BM11, Lemma 6.5], the set $\Theta_{\mathfrak{S}}$ is homeomorphic to an open subset of $\mathbb{R}^{2n}$ via the map

$$\sigma \mapsto \{\big(m(S_0), \ldots, m(S_{n-1}), \phi(S_0), \ldots, \phi(S_{n-1})\big) \mid m(S_i) > 0 \text{ and } |\phi(S_i) - \phi(S_j)| < 1\}.$$

Taking the quotient under the $\mathbb{C}$ action, this map descends to a homeomorphism from $\Theta_{\mathfrak{S}}/\mathbb{C}$ to an open subset of $\mathbb{R}^{2n-2}$ given by

$$[\sigma] \mapsto \Big(\frac{m(S_1)}{m(S_0)}, \ldots, \frac{m(S_{n-1})}{m(S_0)}, \phi(S_1) - \phi(S_0), \ldots, \phi(S_{n-1}) - \phi(S_{n-2})\Big).$$

One can easily verify that the open subset of $\mathbb{R}^{2n-2}$ is defined in such a way that certain sums of the last $(n-1)$-coordinates are equivalent to $|\phi(S_i) - \phi(S_j)| < 1$.

Consider the map $\widetilde{m} : \Theta_{\mathfrak{S}} \to \mathbb{R}^N$ given by

$$\sigma \mapsto \big(m_\sigma(S_0), \ldots, m_\sigma(S_{n-1}), m_\sigma(\mathrm{Tw}_{S_0}^{\pm} S_1), \ldots, m_\sigma(\mathrm{Tw}_{S_{n-1}}^{\pm} S_{n-2})\big).$$

This map clearly factors through the $\mathbb{C}$-action to give a continuous map $\mathbb{P}\widetilde{m} : \Theta_{\mathfrak{S}}/\mathbb{C} \to \mathbb{RP}^{N-1}$. On each $\widetilde{\mathrm{GL}}_2^+(\mathbb{R})$-orbit $U \subset \Theta_{\mathfrak{S}}$, we can always find stable extensions of the spherical objects of the form $\mathrm{Tw}_{S_i}^{\pm} S_j$; thus we obtain a homeomorphism ϕ_U between the image of $\mathbb{P}m(U/\mathbb{C})$ and an open subset of $\mathbb{R}^{2n-2}$ since the masses of stable extensions of the $\{S_0, \ldots, S_{n-1}\}$ control the difference in phases $\phi(S_{j+1}) - \phi(S_j)$. The homeomorphisms ϕ_U agree on the intersection of closures of $\widetilde{\mathrm{GL}}_2^+(\mathbb{R})$-orbits since stable extensions $\mathrm{Tw}_{S_i}^{\pm} S_j$

become at worst semi-stable; in particular, one still has an equality $m_\sigma(\mathrm{Tw}_{S_i} S_j) = |Z_\sigma(\mathrm{Tw}^{\pm}_{S_i} S_j)|$ so that the data of $|Z(S_i)|, |Z(S_j)|, |Z(\mathrm{Tw}^{\pm}_{S_i} S_j)|$ still uniquely recovers $\phi(S_i) - \phi(S_j)$. It follows that the restriction of the composition $\mathbb{RP}^{\mathcal{S}} \to \mathbb{RP}^{N-1} \to \Theta_{\mathfrak{S}}/\mathbb{C}$ to the image $\mathbb{P}m(\Theta_{\mathfrak{S}}/\mathbb{C})$ is a homeomorphism. Since bijective local homeomorphisms are homeomorphisms, the result follows. $\square$

Corollary 4.17. *The map $\mathbb{P}m : \mathrm{Stab}(\mathcal{D}_{0,\mathbb{P}^1})/\mathbb{C} \to \mathbb{RP}^{\mathcal{S}}$ is a homeomorphism onto its image with respect to the naïve coordinates*

$$\mathcal{S}' = \langle \mathrm{Tw}^{\pm}_0, \mathrm{Tw}^{\pm}_{-1} \rangle \cdot \{\Bbbk(x), i_*\mathcal{O}_{\mathbb{P}^1}, i_*\mathcal{O}_{\mathbb{P}^1}(-1)[1]\}.$$

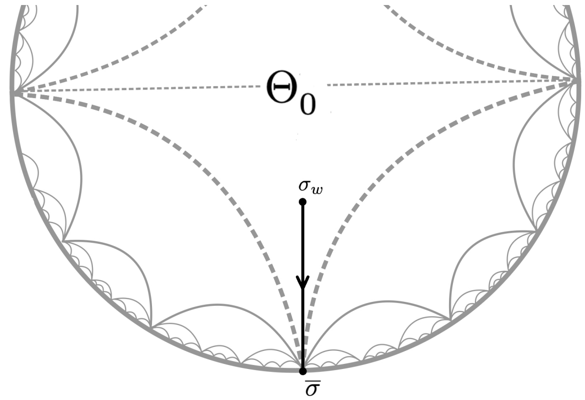
Proof. Using the notation of Theorem 4.7, take U_E to be the open set where $\Bbbk(x)$ is stable; by the results of Section 3.1.1, we may simply identify this with the $\widetilde{\mathrm{GL}}^+_2(\mathbb{R})$ orbit Θ^-_0 where $\phi(i_*\mathcal{O}) < \phi(i_*\mathcal{O}(-1)[1])$. In particular, all line bundles $i_*\mathcal{O}(n)$ are stable in Θ^-_0, and $G = \langle \mathrm{Tw}^{\pm}_0, \mathrm{Tw}^{\pm}_{-1} \rangle$ generates the subgroup of $\mathrm{Aut}(\mathcal{D}_{0,\mathbb{P}^1})$ consisting of spherical twists. Since Tw_0 and Tw_{-1} act numerically on $\mathbb{Z}[i_*\mathcal{O}] \oplus \mathbb{Z}[i_*\mathcal{O}(-1)[1]]$ via

$$[\mathrm{Tw}_0] = \begin{pmatrix} -1 & 2 \\ 0 & 1 \end{pmatrix} \qquad [\mathrm{Tw}_{-1}] = \begin{pmatrix} 1 & 0 \\ 2 & -1 \end{pmatrix},$$

the subgroup G fixes the character $[\Bbbk(x)] = [i_*\mathcal{O}] + [i_*\mathcal{O}(-1)[1]]$. By Corollary 4.10, the map $\mathbb{P}m : \mathrm{Stab}(\mathcal{D}_{0,\mathbb{P}^1})/\mathbb{C} \to \mathbb{RP}^{\mathcal{S}}$ is injective. By Theorem 3.5 the algebraic submanifold of $\mathrm{Stab}(\mathcal{D}_{0,\mathbb{P}^1})$ is, in fact, the entire stability manifold so that $\mathbb{P}m$ is a homeomorphism onto its image by the previous theorem. $\square$

To motivate an approach to Question 4.5(4), let us return to our visualization of $\mathrm{Stab}(\mathcal{D}_{0,\mathbb{P}^1})/\mathbb{C}$ from the previous chapter.

Suppose that we start with some geometric stability condition $\sigma \in \Theta^-_0$; up to the $\mathbb{C}$-action, we may write σ as σ_w for some w in the strict upper half-plane. Consider the path $\{\sigma_{tw} : t > 0\}$ in Θ^-_0; for each t, the phase of the stable objects $i_*\mathcal{O}_{\mathbb{P}^1}(n)$ is $\phi_{\sigma_{tw}}(i_*\mathcal{O}_{\mathbb{P}^1}(n)) = \mathrm{Arg}(tw - n)/\pi$. By taking $t \to \infty$, it becomes clear that the slicing $\mathcal{P}_\infty$ is no longer locally-finite (compare with [Bri07, Example 5.6]). However, since the



objects $i_*\mathcal{O}_{\mathbb{P}^1}(n)$ are always stable in Θ_0^-, we are able to describe the corresponding point in $\overline{\mathbb{P}m(\mathrm{Stab}(\mathcal{D}_{0,\mathbb{P}^1})/\mathbb{C}}$. Suppose that we write the first several coordinates of $\mathbb{P}m(\sigma_w)$ as

$$\begin{aligned}\mathbb{P}m(\sigma_{tw}) = [&m_{\sigma_{tw}}(\Bbbk(x)) : m_{\sigma_{tw}}(i_*\mathcal{O}_{\mathbb{P}^1}) : m_{\sigma_{tw}}(i_*\mathcal{O}_{\mathbb{P}^1}(-1)[1] : \\ &m_{\sigma_{tw}}(\mathrm{Tw}_0\,\Bbbk(x)) : m_{\sigma_w}(\mathrm{Tw}_0\, i_*\mathcal{O}_{\mathbb{P}^1}) : m_{\sigma_{tw}}(\mathrm{Tw}_0\, i_*\mathcal{O}_{\mathbb{P}^1}(-1)[1]) : \dots]\end{aligned}$$

$$= [1 : |w| : |1+tw| : |tw| + |1+tw| : |tw| : 2|tw| + |1+tw| : \dots]$$

Dividing all the coordinates by $1/|tw|$ and taking $t \to \infty$, this yields the point

$$\overline{\sigma} = [0 : 1 : 1 : 1+1 : 1 : 2+1 : \dots]$$

in particular, each coordinate of $m_\sigma(E)$ is precisely the sum of the ranks of the Harder-Narasimhan factors. For semi-stable objects, this would suggest the point $\overline{\sigma}$ corresponds to the map $i(-, \Bbbk(x)) = \dim \mathrm{Hom}^0(-, \Bbbk(x))$.

Another notable difficulty occurs when we approach the horizontal corners of the chamber Θ_0. Recall that the wall connecting Θ_0^- to Θ_0^+ corresponds to stability conditions σ_w such that $\Im(w) = 0$ and $\Re(w) \in (-1, 0)$. As pointed out at the beginning of Section 3.1, the objects $i_*\mathcal{O}_{\mathbb{P}^1}(-1)[1]$ and $i_*\mathcal{O}_{\mathbb{P}^1}$ become massless at the boundary points $w = -1$ and $w = 0$, respectively — in particular, these do not correspond to stability conditions in $\mathrm{Stab}(\mathcal{D}_{0,\mathbb{P}^1})$ as the support property fails. However, we may still describe the corresponding point in $\overline{\mathbb{P}m(\mathrm{Stab}(\mathcal{D}_{0,\mathbb{P}^1})/\mathbb{C}}$ as a certain 'occurrence' function. For any object $E \in \mathcal{S}$, let $\{A_i\}$ denote the semistable factor(s) of E. Up to a sign, we may write each $[A_i] = a_i[i_*\mathcal{O}_{\mathbb{P}^1}] + b_i[i_*\mathcal{O}_{\mathbb{P}^1}(-1)[1]]$ with $a_i, b_i \geq 0$. Then coordinates of the point in $\overline{\mathbb{P}m(\mathrm{Stab}(\mathcal{D}_{0,\mathbb{P}^1}/\mathbb{C})}$ corresponding to $w = -1$ are precisely given by taking the sum of the a_i over all Harder-Narasimhan factors of the object corresponding to that coordinate, and similarly the coordinates of the point corresponding to $w = 0$ are given by taking the sum over the b_i.

4.3 Thurston Compactification for Local $\mathbb{P}^2$

A notable difference between the stability manifold of the local projective line and local projective plane is the existence of non-algebraic stability conditions. Explicitly, it was shown by [BM11] that the stability conditions $\sigma_{s,q} = (\mathcal{A}_{s,q}, Z_{s,q})$ on $\mathcal{D}_{0,\mathbb{P}^2}$ are non-algebraic above the parabola $q = \frac{s^2+3}{2}$. The main consequence of this is that $\mathbb{P}m : \mathrm{Stab}^\dagger(\mathcal{D}_{0,\mathbb{P}^2})/C \to \mathbb{RP}^{\mathcal{S}}$ will not immediately follow from Theorem 4.16. However, we will still be able to follow the general principle that continuous bijections which are local homeomorphisms on an open cover are homeomorphisms.

Lemma 4.18. *Let $\mathfrak{S} = \{S_0, S_1, S_2\}$ denote the ordered quivery collection*

$$\{i_*\mathcal{O}_{\mathbb{P}^2}, i_*\Omega^1(1)[1], i_*\mathcal{O}_{\mathbb{P}^2}(-1)[2]\}.$$

The map $\mathbb{P}m : \mathrm{Stab}^\dagger(\mathcal{D}_{0,\mathbb{P}^2})/\mathbb{C} \to \mathbb{RP}^{\mathcal{S}}$ is injective with respect to the naïve coordinates

$$\mathcal{S} = \Big(B_3 \cdot \mathfrak{S}\Big) \cup \langle \mathrm{Tw}^{\pm}_{S_0}, \mathrm{Tw}^{\pm}_{S_1}, \mathrm{Tw}^{\pm}_{S_2}\rangle \cdot \{\Bbbk(x)\}.$$

Proof. Similar to the proof of Corollary 4.17, we consider $U_E := \mathrm{Stab}^{\mathrm{Geo}}(\mathcal{D}_{0,\mathbb{P}^2})$ and take

$$G = \langle \mathrm{Tw}^{\pm}_{S_0}, \mathrm{Tw}^{\pm}_{S_1}, \mathrm{Tw}^{\pm}_{S_2}\rangle \subset \mathrm{Aut}(\mathcal{D}_{0,\mathbb{P}^2}).$$

By [BM11, Lemma 3.1] the stable coherent sheaves in $\mathrm{Coh}(\omega_{\mathbb{P}^2})$ that are supported on the zero divisor are precisely the pushforward of stable sheaves on $\mathbb{P}^2$. It follows from [Li17, Corollary 1.19] that given any two geometric stability conditions $\sigma_1, \sigma_2 \in \mathrm{Stab}^{\mathrm{Geo}}(\mathcal{D}_{0,\mathbb{P}^2})$, there exist infinitely many objects of the form $i_*\mathcal{E}$ of sufficiently large slope that are stable with respect to σ_1, σ_2.

Fixing the basis $([S_0], [S_1], [S_2])$ for $K(\mathcal{D}_{0,\mathbb{P}^2})$, it was shown by [Bri06] that the twists $\mathrm{Tw}_{S_0}, \mathrm{Tw}_{S_1}, \mathrm{Tw}_{S_2}$ act numerically via

$$[\mathrm{Tw}_{S_0}] = \begin{pmatrix} 1 & 3 & -3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad [\mathrm{Tw}_{S_1}] = \begin{pmatrix} 1 & 0 & 0 \\ -3 & 1 & 3 \\ 0 & 0 & 1 \end{pmatrix} \qquad [\mathrm{Tw}_{S_2}] = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 3 & -3 & 1 \end{pmatrix}.$$

In particular, G fixes the numerical character of $[\Bbbk(x)] = [S_0] + [S_1] + [S_2]$. Fix some non-trivial $\Phi \in G$ such that Φ does not fix the geometric chamber, and consider the semi-stable filtration

$$\begin{tikzcd}
0 = E_0 \arrow[r] & E_1 \arrow[r] \arrow[d] & \dots \arrow[r] & E_{n-1} \arrow[r] \arrow[d] & E_n = \Phi(\Bbbk(x)) \arrow[d] \\
& A_1 \arrow[ul, dashed] & & A_{n-1} \arrow[ul, dashed] & A_n \arrow[ul, dashed]
\end{tikzcd}$$

Since $|Z(E)| = |Z(E[1])|$ for any object E, we may write each class $[A_i] = a_i[S_0] + b_i[S_i] + c_i[S_2]$. As $n \geq 2$ and $[\Phi(\Bbbk(x))] = \sum_{i=1}^{n}[A_i] = [\Bbbk(x)]$, it is easy to see that the smallest possible filtrations (in terms of the mass of $\Phi(\Bbbk(x))$) are when either $n = 2$ and exactly one of the semi-stable factors are the S_j, or $n = 3$ and all the semi-stable factors are the distinct S_0, S_1, S_2. In the first case, it is easy to show that the semi-stable filtration is precisely the decomposition in [BM11, Theorem 5.1] occurring on walls of the geometric chamber. In both cases, we may use a similar triangle-inequality argument as in the proof of Corollary 4.10 to show that $m_\sigma(\Phi(\Bbbk(x))) \geq m_\sigma(\Bbbk(x))$ with equality only occuring on the boundary of the geometric chamber. $\square$

We conclude this section by showing that the mass map $\mathbb{P}m$ gives an effective topological compactification of the stability manifold; in particular, this involves showing that $\mathbb{P}m$ is a homeomorphism onto it's image. Similar to the case of local $\mathbb{P}^1$, this is immediate for algebraic stability conditions in $\mathcal{D}_{0,\mathbb{P}^2}$.

Theorem 4.19. *The mass map $\mathbb{P}m : \mathrm{Stab}^\dagger(\mathcal{D}_{0,\mathbb{P}^2})/\mathbb{C} \to \mathbb{RP}^{\mathcal{S}}$ is a homeomorphism onto its image.*

Proof. First consider the mass map $\mathbb{P}m$ restricted to the quotient of the geometric chamber $\mathrm{Stab}^{\mathrm{Geo}}(\mathcal{D}_{0,\mathbb{P}^2})/\mathbb{C}$. Fixing some $\sigma \in \mathrm{Stab}^{\mathrm{Geo}}(\mathcal{D}_{0,\mathbb{P}^2})/\mathbb{C}$, we may assume up to the $\mathbb{C}$-action that $\sigma = \sigma_{s,q}$. By the proof of the previous lemma, there exist infinitely many stable (pushforwards of) exceptional vector bundles on $\mathbb{P}^2$ in a neighborhood of σ. In particular, the central charge $Z_{s,q}$ can be uniquely determined in this neighborhood by a finite number of objects in $\mathcal{S}$ from the quadric $Q_\sigma(-) = m_\sigma(-)^2 = Z_\sigma(-)\overline{Z_\sigma(-)}$. Since the forgetful map $\mathcal{Z} : \mathrm{Stab}^\dagger(\mathcal{D}_{0,\mathbb{P}^2}) \to \mathrm{Hom}(K(\mathcal{D}_{0,\mathbb{P}^2}), \mathbb{C})$ is a homeomorphism when restricted to $\mathrm{Stab}^{\mathrm{Geo}}(\mathcal{D}_{0,\mathbb{P}^2})$, it follows that $\mathbb{P}m$ restricted to the geometric chamber is an open map and thus a homeomorphism.

By construction of the action of $\mathrm{Aut}(\mathcal{D}_{0,\mathbb{P}^2})$ on $\mathrm{Stab}^\dagger(\mathcal{D}_{0,\mathbb{P}^2})$, $\mathbb{P}m$ is a homeomorphism when restricted to every chamber $\Phi(\mathrm{Stab}^{\mathrm{Geo}}(\mathcal{D}_{0,\mathbb{P}^2}))$. Since

$$\mathrm{Stab}^\dagger(\mathcal{D}_{0,\mathbb{P}^2}) = \bigcup_{\Phi} \overline{\Phi(\mathrm{Stab}^{\mathrm{Geo}}(\mathcal{D}_{0,\mathbb{P}^2}))}$$

and the boundary of each chamber is contained in $\mathrm{Stab}^{\mathrm{Alg}}(\mathcal{D}_{0,\mathbb{P}^2})$, by [BM11, Corollary 6.8], the result again follows from the fact that continuous bijections which are local homeomorphisms are homeomorphisms. $\square$

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