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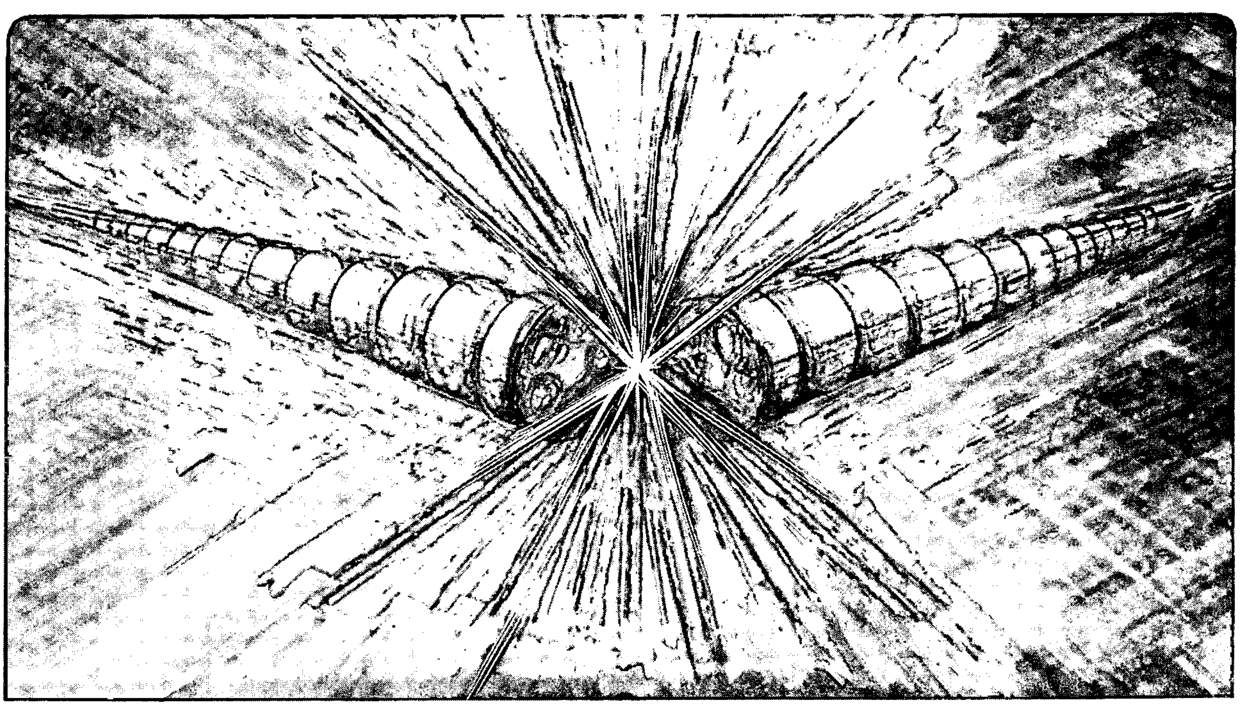

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Analytical expressions are obtained for fringe field multipoles of an N channel permanent magnet quadrupole array [1]. See figure 1 below. It is assumed that the system of magnetic wedges starts at some transverse (x, y) plane located at z=0, and it continues to a magnet length $z=\ell$, where it stops. The iron yoke continues to $z = \pm\infty$, but it will be shown that only a small overhang is actually required to maintain the quadrupole and translational symmetries. Recall the 2-d solution for the magnetic potential $\left(\underset{\sim}{H} = \underset{\sim}{\nabla}\phi\right)$:

$$\phi_2 = A\left[(x - x_i)^2 - (y - y_i)^2\right],$$

where $A = -M_o/4b$, M_o is the remnant field of the wedges, and (x_i, y_i) are the coordinates for the center of box (i). Boxes have dimensions 2b x 2b and alternate between vacuum fill (for beams) and magnetic wedge fill. The 2-d system looks like a portion of an infinite transverse lattice with periodicity lengthy = 4b in both the x and y directions. For the magnetic potential ϕ, the periodicity length is 2b.

3-d Magnetic potential for a semi-infinite system

More generally we have a 3-d situation where the potential ϕ is a function of z as well as (x, y). It still satisfies

$$\nabla^2 \phi = -\underset{\sim}{\nabla} \bullet \underset{\sim}{M} \quad ,$$

$$\nabla^2 = \nabla_\perp^2 + \frac{\partial^2}{\partial z^2} \quad ,$$

where $\underset{\sim}{M} = \underset{\sim}{B} - \underset{\sim}{H}$ is the local density of magnetization.

For now we take $\underset{\sim}{M}$, which is still transverse, to turn on at z=o and continue to z=+∞.

Hence

$$\underset{\sim}{\nabla} \bullet \underset{\sim}{M} = \Theta(z)\, \underset{\sim}{\nabla_\perp} \bullet \underset{\sim}{M_\perp} \ .$$

Where $\Theta(z)$ is the unit step function and $\underset{\sim}{\nabla_\perp} \bullet \underset{\sim}{M_\perp}$ is the source of the 2-d potential (ϕ_2) given above. The truncated 2-d potential $\Theta(z)\,\phi_2$ is clearly not a valid 3-d solution. However, we can add homogeneous 3-d potentials in the zones z<0 and z>0 to obtain the desired 3-d solution. Generally the homogeneous 3-d solutions consist of a sum of terms of the form

$$\left[\cos\left(\frac{n\pi x}{b}\right)\right] \bullet \left[\cos\left(\frac{m\pi y}{b}\right)\right] \bullet e^{\pm\sqrt{n^2+m^2}\,\frac{\pi z}{b}} \ .$$

These terms preserve the periodicity in x and y exhibited by the 2-d solution. In fact, since ϕ_2 is a pure quadrupole potential in each 2b X 2b box, we only use homogeneous combinations of the form

$$\left[\cos\left(\frac{n\pi x}{b}\right) - \cos\left(\frac{n\pi y}{b}\right)\right] e^{\pm\frac{n\pi z}{b}}.$$

Since $\underset{\sim}{M}$ is transverse, $\partial\phi / \partial z$ must be continuous at z = 0, and we obtain

$$\underline{\underline{z<0}}: \quad \phi = -\sum_1^\infty F_n \left[\cos\left(\frac{n\pi x}{b}\right) - \cos\left(\frac{n\pi y}{b}\right)\right] e^{\frac{n\pi z}{b}},$$

$$\underline{\underline{z>0}}: \quad \phi = \phi_2 + \sum_1^\infty F_n \left[\cos\left(\frac{n\pi x}{b}\right) - \cos\left(\frac{n\pi y}{b}\right)\right] e^{\frac{-n\pi z}{b}}.$$

Here $\{F_n\}$ is a set of coefficients to be determined (below).

We must also have ϕ continuous at z = 0; this gives

$$\sum_1^\infty F_n \left[\cos\left(\frac{n\pi x}{b}\right) - \cos\left(\frac{n\pi y}{b}\right)\right] = -\frac{1}{2}\phi_2.$$

We may restrict attention to a single 2b X 2b box centered at x=y=0; substituting for ϕ_2 we get (inside the box)

$$\sum_1^\infty F_n \left(\cos\frac{n\pi x}{b} - \cos\frac{n\pi y}{b}\right) = -\frac{A}{2}\left(x^2 - y^2\right).$$

To evaluate F_n we set y = 0; an elementary Fourier series on the interval - b < x < + b remains:

$$\sum_1^\infty F_n \left(\cos\frac{n\pi x}{b} - 1\right) = -\frac{Ax^2}{2}.$$

It follows that

$$\sum_{1}^{\infty} F_n = \frac{Ab^2}{6},$$

and the individual coefficients are

$$F_n = \frac{1}{b}\int_{-b}^{+b} dx \left(-\frac{Ax^2}{2}\right)\cos\left(\frac{n\pi x}{b}\right) = \frac{-2b^2 A}{n^2\pi^2}\cos(n\pi) = \frac{2b^2 A}{\pi^2} \bullet \left\{\frac{1}{1^2}, \frac{-1}{2^2}, \frac{1}{3^2}, \frac{-1}{4^2}, ..\right\}.$$

Evaluation of the quadrupole amptitude

Note that

$$\left(\cos\frac{n\pi x}{b} - \cos\frac{n\pi y}{b}\right) = \frac{-\pi^2}{2b^2} n^2 \left(x^2 - y^2\right) + \frac{\pi^4}{24b^4} n^4 \left(x^4 - y^4\right) - \frac{\pi^6}{720b^6} n^6 \left(x^6 - y^6\right) + \cdots.$$

So in the box centered at x = y = 0, the potential has the simple form

$$\phi = f(z)\left(x^2 - y^2\right) + g(z)\left(x^4 - y^4\right) + h(z)\left(x^6 - y^6\right) + \ldots$$

This is true for $z > 0$ as well as $z < 0$ since $\phi_2 \sim \Theta(z)\left(x^2 - y^2\right)$ and can be absorbed into f (z). For other boxes $(x-x_i)$ and $(y-y_j)$ must be inserted in this expression in place of x and y.

To obtain the quadrupole amplitude f(z) we gather terms of ϕ proportional to (x^2-y^2); for $z < 0$

$$f(z) = \sum_{1}^{\infty} F_n \left(\frac{\pi^2 n^2}{2b^2}\right) e^{\frac{n\pi z}{b}} = \frac{A}{1+e^{-\pi z/b}} = \frac{A}{2}\left(1+\tanh\frac{\pi z}{2b}\right).$$

The same formula is found for $z > 0$ by direct summation (or by invoking analyticity).

To obtain the field for a magnet of finite length ℓ we add a displaced semi-infinite solution with reverse $\underset{\sim}{M}$ to get

$$f = \frac{A}{2}\left[\tanh\frac{\pi z}{2b} - \tanh\frac{\pi(z-\ell)}{2b}\right].$$

Note that for the semi-infinite magnet the quadrupole amplitude has a "Fermi-Dirac" distribution form with amplitude A/2 at $z = 0$. We expect a profile of this general shape from elementary considerations. Outside the magnet ϕ falls off very rapidly with z, so it should actually be sufficient to continue the iron yoke to only about $\Delta z \approx 2b$ beyond the magnet ends; note the very low value of the quadrupole amplitude at this distance from the magnet:

$$f(z=-2b) \approx (.00186)\, A.$$

<u>Other fringe field components</u>

So far we have evaluated the lowest order quadrupole term:

$$\phi_{quad} = f(z)\left(x^2 - y^2\right) = fr^2 \cos 2\Theta,$$

with the expressions for f (z) derived in the previous section. For other fringe field features we will treat only the semi-infinite magnet, so

$$f(z) = \frac{A}{2}\left(1 + \tanh\frac{\pi z}{2b}\right).$$

A first result is a formula for H_Z in lowest order:

$$H_z = \frac{\partial\phi}{\partial z} \approx \frac{\partial f}{\partial z} r^2 \cos 2\Theta$$

$$= \left(\frac{A}{2}\right)\left(\frac{\pi}{2b}\right)\left(1 - \tanh^2\frac{\pi z}{2b}\right) r^2 \cos 2\Theta \quad .$$

This field component has magnitude similar to the transverse quadrupole components near z = 0. Higher order terms of ϕ can be extracted from the general expansion in the same way as was done for quadrupole component, however a short cut is available. We simply plug the expansion

$$\phi = f\left(x^2 - y^2\right) + g\left(x^4 - y^4\right) + h\left(x^6 - y^6\right) + \ldots$$

directly into $\nabla^2\phi = 0$ and equate coefficients of $(x^m - y^m)$:

$$0 = \nabla^2\phi = f''\left(x^2 - y^2\right) + g''\left(x^4 - y^4\right) + h''\left(x^6 - y^6\right) + \ldots$$

$$+ 12g\left(x^2 - y^2\right) + 30h\left(x^4 - y^4\right) + \ldots\ldots \quad .$$

We have immediately

$$g = \frac{-f''}{12},$$

$$h = \frac{-g''}{30} = \frac{f''''}{360},$$

and so forth. Hence

$$\phi = f\left(x^2 - y^2\right) - \frac{f''}{12}\left(x^4 - y^4\right) + \frac{f''''}{360}\left(x^6 - y^6\right) - \ldots .$$

After a little algebra this expansion may be cast in the form

$$\phi = fr^2 \cos 2\Theta - \frac{f''}{12} r^4 \cos 2\Theta$$

$$+ \frac{f''''}{360}\left(\frac{15}{16} r^6 \cos 2\Theta + \frac{1}{16} r^6 \cos 6\Theta\right) + \ldots \quad ,$$

displaying directly the allowed terms through sixth order (quadrupole, pseudo-octopole, pseudo dodecapole, and dodecapole).

Returning to the semi-infinite quadrupole amplitude

$$f(z) = \frac{A}{2}\left(1 + \tanh \frac{\pi z}{2b}\right),$$

we find by successive differentiations

$$f' = \frac{A}{2}\left(\frac{\pi}{2b}\right)\left(1 - \tanh^2 \frac{\pi z}{2b}\right),$$

$$f'' = \frac{A}{2}\left(\frac{\pi}{2b}\right)^2 (-2 \tanh)\left(1 - \tanh^2\right),$$

$$f''' = \frac{A}{2}\left(\frac{\pi}{2b}\right)^3 \left(-2 + 6 \tanh^2\right)\left(1 - \tanh^2\right),$$

$$f'''' = \frac{A}{2}\left(\frac{\pi}{2b}\right)^4 \left(16 \tanh - 24 \tanh^3\right)\left(1 - \tanh^2\right),$$

etc.

The pseudo octopole potential is

$$\phi_{PO} = \frac{-f''}{12} r^4 \cos 2\Theta$$

$$= \frac{A}{12}\left(\frac{\pi}{2b}\right)^2 \left(r^4 \cos 2\Theta\right)\left(\tanh \frac{\pi z}{2b}\right)\left(1 - \tanh^2 \frac{\pi z}{2b}\right).$$

There is a very small dodecapole potential

$$\phi_{DOD} = \frac{f''''}{(360)\cdot(16)} r^6 \cos 6\Theta$$

$$= \frac{A}{11520}\left(\frac{\pi}{2b}\right)^4 \left(r^6 \cos 6\Theta\right)\left(16 \tanh - 24 \tanh^3\right)\left(1 - \tanh^2\right).$$

Comparison of Multipole Fields

We calculate the relative strengths of the various multipoles near z = 0. Specifically, $B_r = \partial\phi/\partial r$ is computed and normalized to the 2-d maximum = $-M_0/2$:

$$\left(\frac{Br}{-M_0/2}\right) = \left[\frac{1+\tanh\left(\frac{\pi z}{2b}\right)}{2}\right]\left(\frac{r}{b}\right)\cos 2\Theta \qquad \text{(quadrupole)}$$

$$+\left[\frac{\pi^2}{24}\tanh\left(1-\tanh^2\right)\right]\left(\frac{r}{b}\right)^3\cos 2\Theta \qquad \text{(pseudo octopole)}$$

$$+\left[\frac{\pi^4}{240}\left(\text{tahn} - \frac{3}{2}\text{tahn}^3\right)\left(1-\tanh^2\right)\right]\left(\frac{r}{b}\right)^5\left[\frac{15}{16}\cos 2\Theta + \frac{1}{16}\cos 6\Theta\right] \qquad \left(\begin{matrix}\text{pseudo}\\ \text{dodecapole}\end{matrix} + \text{dodecapole}\right)$$

Values of the bracketed functions of z in the multipole field comparison are given in table 1. Note the antisymmetry of 3rd and 5th orders.

	1st order	3rd order	5th order
$\pi z/2b$	$\frac{1+\tanh\left(\frac{\pi z}{2b}\right)}{2}$	$\frac{\pi^2}{24}\tanh\left(1-\tanh^2\right)$	$\frac{\pi^4}{240}\left(\tanh-\frac{3}{2}\,\mathrm{tahn}^3\right)\left(1-\tanh^2\right)$
-3.0	.002473	-.004037	+.001933
-2.5	.006693	-.010789	+.004900
-2.0	.017986	-.028009	+.010892
-1.5	.047426	-.067264	+.015199
-1.0	.119203	-.131533	-.016871
-.7	.197816	-.157756	-.070393
-.5	.268941	-.149455	-.100256
-.4	.310026	-.133692	-.103376
-.3	.354344	-.109631	-.094428
-.2	.401312	-.078005	-.072489
-.1	.450166	-.040580	-.039454
0.0	.500000	.000000	.000000
.1	.549834	.040580	.039454
.2	.598688	.078005	.072489
.3	.645656	.109631	.094428
.4	.689974	.133692	.103376
.5	.731059	.149455	.100256
.7	.802184	.157756	.070393
1.0	.880797	.131533	.016871
1.5	.957574	.067264	-.015199
2.0	.982014	.028009	-.010892
2.5	.993307	.010789	-.004900
3.0	.997527	.004037	-.001933

Table 1

Relative strength of components of B_r at $r = b$, $\Theta = 0$.

Reference

1. E. Lee and M. Vella, LBL-38430, "Perfect 2-d Quadrupole Fields from Permanent Magnets".

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