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A Self-Triggered Control Strategy to Guarantee Forward Invariance and Uniform Positive Inter-Event Times for Constrained Systems

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A SELF-TRIGGERED CONTROL STRATEGY TO GUARANTEE FORWARD INVARIANCE AND UNIFORM POSITIVE INTER-EVENT TIMES FOR CONSTRAINED SYSTEMS

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by

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Abstract

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by

David W. Kooi

Self-triggered control algorithms have the key feature that plant measurements are only needed at isolated time instances. The information obtained at those time instances is used by self-triggered control algorithms to update the input to the plant and to determine the next sampling time event. In between these events, the input is usually held constant. In this thesis , a self-triggered control strategy is proposed for forward invariance of a set for constrained (controlled) differential inclusions. Using a (not necessarily periodic) zero-order hold control scheme, this thesis addresses two key issues: i) the computation of the next sampling time event, and ii) the assurance of a uniform lower bound on the inter-event times, both while guaranteeing forward invariance. Our results allow the sets to render forward invariant to be unbounded. Very importantly, the results impose mild regularity properties on the data defining the dynamics of the control system and of the forward invariance certificates, which are given in terms of barrier functions. Simulations showcase the proposed algorithms and provide comparisons with the literature.

1

Introduction

Given a continuous-time control system and a control law designed such that the resulting closed-loop system satisfies a prescribed control objective, it is not necessarily the case that a digital implementation, where the input is updated only after some fixed inter-event period, determined by a fixed sampling rate, will still guarantee the control objective, even if the inter-event period is small [1, 2]. Additionally, the assumption of fixed inter-event periods has limitations in the context of systems with shared resources, such as networked and embedded control systems [3].

In networked control systems, sensors, actuators, and controllers may be distributed spatially and connected over a communication network [4]. Fixing the inter-event period imposes a trade-off between responsiveness of the controller and the demand placed on the communication network. Shorter periods between data exchanges allow actuators to respond to disturbances quicker, but at the cost of network demand. On the other hand, longer periods between data exchanges can reduce demand on the communication network, but at the cost of controller responsiveness. In embedded control systems, an embedded processor may need to compute high priority control tasks while sharing computational resources with lower priority tasks [5]. There arises a similar trade-off between responsiveness of the controller and computational support for other tasks.

Generalizations of periodic controllers, where the control input is not necessarily periodically updated, find motivation in overcoming this trade-off. Recent advances in the design of controllers that use aperiodically available information can be decomposed into event-triggered (ET) and self-triggered (ST) control approaches, as initiated in [6] and [7], respectively. These methods show that the control input of a system can be updated on an 'as-needed' basis, while still preserving system properties such as stability, convergence, and safety. See [8] for an overview on the topic.

In ET control, continuous availability of the measurements is typically assumed. Hence, the control input is usually updated whenever the measurements reach a critical region that compromises the control objective. However, continuous availability of the measurements can be difficult to have in some applications.

Hence, in ST control, and in periodic ET strategies [9], state measurements are assumed to be available only at the same sequence of times when the input can be updated. In general, the challenges when studying ET and ST control systems are two fold:

1. What is the triggering condition that makes the control objective satisfied for the resulting (ET or ST) closed-loop system?

2. Under what conditions is the resulting inter-event period always larger than a positive constant?

Early results on ST and ET control focus on stability and convergence only. These results usually assume the existence of a feedback law that renders the (non-triggered) closed-loop system input to state stable with respect to input perturbations. The input to state stability property of the system is used to determine triggering conditions that guarantee stability and convergence. In ET methods, the triggering condition is a state dependent threshold that is checked continuously. In ST methods, the triggering condition is a timing threshold that is determined by the previously sampled state. That is, the current state is used to determine the duration of the next inter-event period. For both ET and ST strategies, it is important to show that inter-event periods are lower bounded by a positive constant. A positive lower bound on inter-event periods ensures that these periods cannot become arbitrarily small.

In [5] and [10], ET and ST control algorithms are proposed, respectively. In both works it is shown that the inter-event times are always larger than a positive constant. In [5], this constant is inversely proportional to the Lipschitz constant of the dynamics. In [10] this constant is determined in using the well-known comparison lemma [11], and further analyzed in [12] when the system is linear. While a ST method is analyzed only for homogeneous control systems in [10], an extension to general smooth control systems is given in [13], where practical asymptotic stability of the origin is established. In [14], a hybrid system framework is proposed to analyze and design ET control algorithms guaranteeing asymptotic stability of a compact set. The obtained inter-event times are shown to admit a semi-global positive lower bound.

In addition to using ST algorithms to apply static control laws, ST methods have also been applied to model predictive control (MPC) strategies because aperiodic sampling can be shown to reduce sensor measurements and control computations. For example, in [15], a ST MPC strategy is implemented to asymptotically stabilize a compact set for a specific non-holonomic system. The Lipschitz constant of the non-holonomic model is used to determine a ST sampling function which guarantees convergence to the compact set. In [16], a ST MPC strategy for linear systems is introduced. The inter-event period is added as a decision variable to the MPC objective function where longer sample periods are rewarded. Conditions on the system's data are given which render the origin asymptotically stable. In [17], a general ST MPC framework is proposed to render a compact set robustly asymptotically stable for constrained linear systems. In this strategy, a finite horizon optimal control problem is solved using a previously computed maximum feasible inter-event time as a parameter. Both [16] and [17] consider discretized linear systems. Recently, in [18], an ET and ST framework, based on reachable sets, is proposed for the stabilization of compact sets in the case of discretized linear control systems. ST MPC algorithms are an interesting area of research due to the computational requirements of MPC and the sampling time relaxations available in ST strategies.

Along with stability and convergence, safety is one of the most common control objectives encountered in applications. Such is the case, for example, in multiagent systems [19, 20], robotic manipulation [21, 22], and autonomous navigation [23]. Safety is the property that requires the system's solutions starting from a given set of initial conditions to remain in a desired safe region [24, 25]. Safety is equivalent to forward invariance of a set known as *inductive invariant* [26]. This set contains the set of initial conditions and does not intersect with the unsafe region [27]. Existing results guaranteeing forward invariance using ST and ET control algorithms often assume strong regularity properties on the data defining the system's dynamics or boundedness of the set to be rendered forward invariant. Indeed, in [28], a ST control algorithm is proposed for nonlinear systems while considering a compact set and using a Lyapunov-based technique. In [29], a ST control algorithm is proposed for perturbed linear systems while considering a convex and compact set. The resulting inter-event times are shown to admit a positive lower bound. See also [30], where nonlinear systems with a globally Lipschitz dynamics are considered. Finally, in [31], an ET control algorithm is proposed for nonlinear systems using continuously differentiable barrier functions. The resulting inter-event times are shown to admit a positive lower bound provided that the the right-hand side is globally bounded.

This thesis considers constrained control systems modeled by a control differential inclusions with a continuous-time feedback law that renders a given closed set forward invariant for the (non-triggered) closed-loop system. This thesis assumes the existence of a barrier function that certifies this invariance property with some robustness with respect to input perturbations. The thesis first proposes an approach that efficiently determines the next sampling time from a given initial condition. This yields a sampling sequence that guarantees the invariance task for the corresponding ST closed-loop system. Afterwards, the thesis investigates sufficient conditions for the existence of a sampling sequence for which the the inter-event times admit a positive lower bound. Finally, the two results are combined to provide a ST control algorithm that guarantees both the invariance task for the resulting ST closed-loop system and a strictly positive lower bound on the inter-event times. Note that the system's dynamics and the barrier function are not assumed to be smooth. Finally, via simulation, we show the effectiveness of our approach on two different examples. In the first one, the set to render forward invariant is a temporal funnel; hence, time-varying and unbounded. In the second example, we compare the performance of our method with respect to existing ones. To the best of our knowledge, this is the first time ST control is studied to guarantee forward invariance of general closed sets in the general context of constrained differential inclusions.

The remainder of this thesis is organized as follows. Background tools and notions are motivated and presented in Chapter 2. Forward invariance and Constrained differential inclusions are presented in Chapter 3. Two problem formulations are presented in Chapter 4. The main results are developed in Chapter 5. Finally, in Chapter 6 the results of Chapter 5 are applied to rendering a time-varying set forward invariant and are compared to existing ET and ST strategies.

Notation

- $\mathbb{R}^n$ denotes n-dimensional Euclidean space.
- $\mathbb{R}_{\geq 0} := [0, \infty)$ denotes the non negative real numbers.
- $\mathbb{N} := \{0, 1, \ldots\}$ denotes the natural numbers.

- $\mathbb{N}^* := \{1, 2, \ldots, \infty\}$ denotes the extended positive natural numbers.
- Given $x \in \mathbb{R}^n$, $x^\top$ denotes the transpose of x.
- Given $x \in \mathbb{R}^n$, $|x|$ denotes the Euclidean norm of x.
- Given $(x, y) \in \mathbb{R}^n \times \mathbb{R}^n$, and a set $K \subset \mathbb{R}^n$ $|x|_K := \inf_{y \in K} |x - y|$ defines the distance between x and the set K.
- Given $(x, y) \in \mathbb{R}^n \times \mathbb{R}^n$, $[x, y]$ denotes the line segment $[x, y]$ relating x to y
- Given $(x, y) \in \mathbb{R}^n \times \mathbb{R}^n$, $\langle x, y \rangle := x^\top y$ denotes the scalar product between x and y
- Given $x \in \mathbb{R}^n$, and a set $K \subset \mathbb{R}^n$, $\langle x, K \rangle = x^\top K := \left\{ x^\top z \ : \ z \in K \right\}$.
- Given a set $K \subset \mathbb{R}^n$, $\text{int}(K)$ denotes the interior of the set, ∂K its boundary, and $\text{cl}(K)$ its closure. $U(K)$ to denotes an open neighborhood around K.
- For a set $O \subset \mathbb{R}^n$, and a set $K \subset \mathbb{R}^n$, $K \backslash O$ denotes the subset of elements of K that are not in O.
- $F : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ denotes a set-valued map associating each element $x \in \mathbb{R}^n$ into a subset $F(x) \subset \mathbb{R}^n \cup \emptyset$.
- $\text{dom}\, F$ denotes the domain of definition of a set-valued map F.

- Given a set $K \in \mathbb{R}^n$ and a set-valued map F, $F(K) := \{\eta \in F(x) : x \in K\}$.

- For a continuously differentiable function $\rho : \mathbb{R}^n \to \mathbb{R}$, $\nabla\rho(x)$ denotes the gradient of ρ evaluated at x.

- For a symmetric matrix $A \in \mathbb{R}^{n\times n}$, $\lambda_{min}(A)$ and $\lambda_{max}(A)$ stand for the minimum and maximum eigenvalues of A, respectively.

2

Background

This background chapter provides an overview of tools and useful notions used in this thesis . This thesis 's results are applicable to systems modelled as constrained differential inclusions. Differential inclusions are a generalization of differential equations and analysis utilizing differential inclusions can allow for guarantees in the presence of external disturbances and modeling uncertainty. Additionally, since this thesis requires only mild regularity properties on the data defining the dynamics of the modeled system, as well as the barrier functions employed to certify forward invariance, tools from non-smooth analysis are also introduced.

2.1 Set-Valued Analysis

Many concepts from functional analysis have analogs in set-valued analysis. The following section defines several of these analogs. For example, a locally Lipschitz map is the set-valued analog of a locally Lipschitz function.

Definition 2.1.1 (Locally Lipschitz maps) *A set-valued map $F : \mathbb{R}^n \rightrightarrows \mathbb{R}^m$ is locally Lipschitz if, for each compact set $K \subset \mathbb{R}^n$, there exists $k > 0$ such that, for each $x \in K$ and $y \in K$,*

$$F(y) \subset F(x) + k|x - y|\mathbb{B}. \tag{2.1}$$

Concepts of continuity can also be extended to handle set-valued maps.

Definition 2.1.2 (Semicontinuous set-valued maps) *Consider a set-valued map $F : \mathbb{R}^m \rightrightarrows \mathbb{R}^n$.*

- *The map F is said to be outer semicontinuous at $x \in \mathbb{R}^m$ if, for all $\{x_i\}_{i=0}^{\infty} \subset \mathbb{R}^m$ and for all $\{y_i\}_{i=0}^{\infty} \subset \mathbb{R}^n$ with $x_i \to x$, $y_i \in F(x_i)$, and $y_i \to y \in \mathbb{R}^n$, we have $y \in F(x)$; see [32, Definition 5.9].*

- *The map F is said to be lower semicontinuous (or, equivalently, inner semicontinuous) at $x \in \mathbb{R}^m$ if, for each $\epsilon > 0$ and for each $y_x \in F(x)$, there exists $U(x)$ such that, for each $z \in U(x)$, there exists $y_z \in F(z)$ such that $|y_z - y_x| \leq \epsilon$; see [33, Proposition 2.1].*

- *The map F is said to be upper semicontinuous at $x \in \mathbb{R}^m$ if, for each $\epsilon > 0$, there exists $U(x)$ such that, for each $y \in U(x)$, $F(y) \subset F(x) + \epsilon\mathbb{B}$; see [34, Definition 1.4.1].*

Furthermore, the map F is said to be outer (lower, and upper, respectively) semicontinuous if it is outer (lower, and upper, respectively) for all $x \in \mathbb{R}^m$.

Finally, the map F is said to be uniformly upper semicontinuous on a set $K \subset \mathbb{R}^m$ if there exists $\delta > 0$ such that we can choose the neighborhood $U(x)$ in the third item to satisfy $x + \delta\mathbb{B} \subset U(x)$ for all $x \in K$.

Since single-valued maps are used in this thesis , the following definitions of continuity are recalled.

Definition 2.1.3 (Semicontinuous single-valued maps) *Consider a scalar function $B : K \to \mathbb{R}$, where $K \subset \mathbb{R}^m$.*

- *The scalar function B is said to be lower semicontinuous at $x \in K$ if, for every sequence $\{x_i\}_{i=0}^{\infty} \subset K$ such that $\lim_{i\to\infty} x_i = x$, we have $\liminf_{i\to\infty} B(x_i) \geq B(x)$.*

- *The scalar function B is said to be upper semicontinuous at $x \in K$ if, for every sequence $\{x_i\}_{i=0}^{\infty} \subset K$ such that $\lim_{i\to\infty} x_i = x$, we have $\limsup_{i\to\infty} B(x_i) \leq B(x)$.*

- *The scalar function B is said to be continuous at $x \in K$ if it is both upper and lower semicontinuous at x.*

Furthermore, B is said to be upper, lower semicontinuous, or continuous if it is upper, lower semicontinuous, or continuous for all $x \in K$, respectively.

Local boundedness allows conclusions to be made about points within a neighborhood of a point x. Roughly speaking, a singled-valued map f is locally bounded if f is bounded at every point. On the other hand, set-valued map F is locally bounded if the image of F is a bounded set at every point.

Definition 2.1.4 (Locally bounded maps) *A set-valued map $F : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ is said to be locally bounded if, for each $x \in \mathbb{R}^n$, there exist $U(x)$ and $K > 0$ such that*

$$|\zeta| \leq K \qquad \forall \zeta \in F(y) \quad \forall y \in U(x). \tag{2.2}$$

2.2 Nonsmooth Analysis

Nonsmooth behavior occurs commonly, for example when measuring the distance to a set, when a function is composed of min or max operators, or when dealing with hybrid systems [32]. Even if a locally Lipschitz function is non-differentiable at a point, the Clarke generalized gradient at that point always exists.

Definition 2.2.1 (Clarke generalized gradient) *Let $B : \mathbb{R}^n \to \mathbb{R}$ be locally Lipschitz. Let Ω be any subset of zero measure in $\mathbb{R}^n$, and let Ω_B be* *the set of points in $\mathbb{R}^n$ at which B fails to be differentiable. Then, the Clarke generalized gradient at x is defined as*

$$\partial_C B(x) := \operatorname{co}\left\{\lim_{i\to\infty} \nabla B(x_i) : x_i \to x,\ x_i \notin \Omega_B,\ x_i \notin \Omega\right\}. \tag{2.3}$$

Note that when the function B is differentiable, the Clarke generalized gradient $\partial_C B$ is equivalent to the gradient ∇B.

Nonsmooth functions such as min and max occur, for example, in multi-agent control strategies [20] and systems with multiple control objectives [35]. While the former reference uses smooth approximations of min and max operators, the latter reference uses tools similar to those presented in this thesis

.

Example 2.2.2 *Consider two sets, $A := \{x \in \mathbb{R} : \delta_1 - x > 0\}$ and $B := \{x \in \mathbb{R} : x - \delta_2\}$, with $\delta_1 > \delta_2$. Then, distance from a point x to $A \cup B$ is given by the function $B : \mathbb{R} \mapsto \mathbb{R}$ defined as*

$$B(x) := \min\{\delta_1 - x, x - \delta_2, 0\}, \forall \;\; x \in \mathbb{R}.$$

Note that the gradient is undefined for points on the boundary of $A \cup B$ as well as when $|x|_A = |x|_B$. The Clarke generalized gradient for B evaluated at *a point x is given by*

$$\partial_C B(x) = \begin{cases} \{0,-1\} & \text{if } x = \delta_1 \\ -1 & \text{if } x > (\delta_1 - \delta_2)/2 \\ \{-1,1\} & \text{if } x = (\delta_1 - \delta_2)/2 \\ 1 & \text{if } x < (\delta_1 - \delta_2)/2 \\ \{0,1\} & \text{if } x = \delta_2 \end{cases}.$$

In the following lemmas, some of the key properties of the Clarke generalized gradient ∂_C are recalled. The properties in the following lemma can be found in [36, Proposition 1.5, Proposition 3.1, and Theorem 5.7], [37, Proposition 2.1], and [38, Proposition 1.12], respectively.

Lemma 2.2.3 *Let $f, g : \mathbb{R}^n \to \mathbb{R}$ be two locally Lipschitz functions on $U(x)$ an open neighborhood of $x \in \mathbb{R}^n$.*

(P1) Homogeneity rule:

$$\partial_C(\lambda f)(x) = \lambda \partial_C f(x) \qquad \forall \lambda \in \mathbb{R}.$$

(P2) Nonsmooth sum rule:

$$\partial_C(f+g)(x) \subset \partial_C f(x) + \partial_C g(x). \tag{2.4}$$

(P3) Continuity: The set-valued map $\partial_C f$ is locally bounded and upper semi-continuous on $U(x)$.

(P4) Existence: For each $y \in U(x)$, $\partial_C f(y) \neq \emptyset$.

The following result is in [36, Theorem 2.5], and it extends the chain-rule property when differentiating the composition of nonsmooth functions.

Lemma 2.2.4 *[Nonsmooth Chain Rule] Let* $f : \mathbb{R}^m \mapsto \mathbb{R}^n$ *be locally Lipschitz near* $x \in \mathbb{R}^m$ *and* $g : \mathbb{R}^n \mapsto \mathbb{R}$ *be locally Lipschitz near* $f(x) \in \mathbb{R}^n$*. Then the function* $g \circ f$ *is locally Lipschitz near* x *and*

$$\partial_C(g \circ f)(x) \subset \overline{co}\{\partial_C\langle\gamma, f\rangle(x) : \gamma \in \partial_C g(f(x))\}, \tag{2.5}$$

where $\overline{co}$ *is the closed convex hull operation.*

Next, we recall from [36, Theorem 2.4] the following version of the mean-value theorem in the case of locally Lipschitz functions.

Lemma 2.2.5 *[Lebourg's Mean-Value Theorem] Let* $\rho : \mathbb{R}^n \mapsto \mathbb{R}$ *be locally Lipschitz on* $U([x, y])$*, an neighborhood of the line segment* $[x, y]$ *from* $x \in \mathbb{R}^n$ *to* $y \in \mathbb{R}^n$*. Then, there exists a point* $z \in [x, y]\backslash\{x, y\}$ *such that*

$$\rho(y) - \rho(x) \in \{\langle\eta, y - x\rangle : \eta \in \partial_C\rho(z)\}. \tag{2.6}$$

The following classical mean-value theorem for differentiable functions follows as a particular case of Lemma 2.2.5.

Lemma 2.2.6 *Let* $\rho : \mathbb{R}^n \mapsto \mathbb{R}$ *be continuous on* $U([x, y])$ *and differentiable on* $[x, y]\backslash\{x, y\}$*. Then, there exists* $z \in [x, y]\backslash\{x, y\}$ *such that*

$$\rho(y) - \rho(x) = \langle\nabla\rho(z), y - x\rangle. \tag{2.7}$$

Because the solutions of a constrained system might only evolve in a subset of Euclidean space, it is important to consider the contingent cone, which can be used to represent where solutions of a constrained system are allowed to evolve.

Definition 2.2.7 (Contingent Cone) *For a set $K \subset \mathbb{R}^n$, according to [39], the contingent cone of K at x is given by*

$$T_K(x) := \left\{ v \in \mathbb{R}^n : \liminf_{h \to 0^+} \frac{|x + hv|_K}{h} = 0 \right\}. \tag{2.8}$$

We also recall the equivalence (see [34, Page 122])

$$\begin{aligned} v \in & T_K(x) \iff \\ & \exists \{h_i\}_{i \in \mathbb{N}} \to 0^+ \text{ and } \{v_i\}_{i \in \mathbb{N}} \to v : x + h_i v_i \in K. \end{aligned} \tag{2.9}$$

Example 2.2.8 *Consider the set $C := [0, b] \subset \mathbb{R}$, with $b > 0$. Then the contingent cone of C at x is given by*

$$T_C(x) = \begin{cases} (-\infty, 0] & \text{if } x = b, \\ (-\infty, \infty) & \text{if } 0 < x < b, \\ [0, \infty) & \text{if } x = 0. \end{cases}$$

The next chapter introduces the modeling methodology used in this thesis as well as concepts related to reachability, barrier functions, and forward invariance.

3

Forward Invariance in Constrained Systems

3.1 Constrained Differential Inclusions

A constrained differential inclusion $\mathcal{H}_f := (C, F)$ is defined as the continuous-time system

$$\mathcal{H}_f : \qquad \dot{x} \in F(x) \quad x \in C \subset \mathbb{R}^n, \tag{3.1}$$

with the state variable $x \in \mathbb{R}^n$, the flow set $C \subset \mathbb{R}^n$ and the set-valued map $F : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$. Note that the set C in (3.1) is not necessarily open and does not necessarily correspond to $\mathbb{R}^n$. Next, we introduce the concept of a solution to $\mathcal{H}_f$.

Definition 3.1.1 *(Solution to $\mathcal{H}_f$) A function $x : \operatorname{dom} x \to \mathbb{R}^n$ with $\operatorname{dom} x \subset \mathbb{R}_{\geq 0}$ and $t \mapsto x(t)$ locally absolutely continuous is a* solution *to $\mathcal{H}_f$ if*

(S1) $x(0) \in \operatorname{cl}(C)$,

(S2) $x(t) \in C$ *for all* $t \in \mathit{int}(\operatorname{dom} x)$,

(S3) $\frac{dx}{dt}(t) \in F(x(t))$ *for almost all* $t \in \operatorname{dom} x$.

A solution x is complete if $\operatorname{dom} x$ is unbounded. It is maximal if there does not exist another solution y such that $\operatorname{dom} x$ is a proper subset of $\operatorname{dom} y$ and $x(t) = y(t)$ for all $t \in \operatorname{dom} x$. Note that each complete solution is maximal, but not all maximal solutions are complete.

Remark 3.1.2 *Condition (S1) allows solutions starting from $\partial C \backslash C$ to flow into C such that (S2) is satisfied. Furthermore, (S2) allows solutions starting from C to reach $\partial C \backslash C$. Hence, symmetry between forward and backward solutions is preserved.*

As customary in the literature, we assume that the set-valued map F satisfies the following mild conditions:

(A1) $F : \operatorname{cl}(C) \rightrightarrows \mathbb{R}^n$ is outer semicontinuous and locally bounded,

(A2) $F(x)$ is convex for all $x \in \operatorname{cl}(C)$.

Remark 3.1.3 *Assumptions (A1) and (A2) are used in the literature as the tightest requirements for the existence of solutions and adequate structural properties for the set of solutions to differential inclusions, see [40, 39, 36]. When F is single valued, (A1) and (A2) reduce to the classical continuity property of the right-hand side of a differential equation. In some of the existing literature, e.g. [36, 39], assumptions (A1) and (A2) are replaced by the equivalent assumption stating that F needs to be upper semicontinuous with compact and convex images on* $\mathrm{cl}(C)$*. Indeed, outer semicontinuous and locally bounded set-valued maps are upper semicontinuous with compact images [41, Theorem 5.19], the converse is also true using [32, Lemma 5.15] and the fact that upper semicontinuous set-valued maps with compact images are locally bounded.*

Since ST control strategies are predictive in nature, the following notions of reachability are defined.

Definition 3.1.4 (Reachability Map) *Given $x_o \in \mathrm{cl}(C)$ and $T > 0$, the reachability map $R : \mathbb{R}_{\geq 0} \times \mathrm{cl}(C) \rightrightarrows \mathrm{cl}(C)$ for $\mathcal{H}_f$ is given by*

$$R(T, x_o) := \{x(t) : x \in \mathcal{S}_{\mathcal{H}_f}(x_o),\ t \in \mathrm{dom}\, x \cap [0, T]\}, \tag{3.2}$$

where $\mathcal{S}_{\mathcal{H}_f}(x_o)$ is the set of maximal solutions to $\mathcal{H}_f = (C, F)$ starting from x_o.

At times it is useful to capture the set of points that are reached at the end of the horizon in a reachability map, rather than the points captured by R in definition 3.2. These points are given by a b-Reachability(branch-Reachability) map.

Definition 3.1.5 (b-Reachability Map) *Given $x_o \in \mathrm{cl}(C)$ and $T > 0$, the b-Reachability map is given by*

$$\begin{aligned} &R^b(T, x_0) := \{\phi(t) : \phi \in \mathcal{S}_{\mathcal{H}_f}(x_o), \\ &t \in \operatorname{dom} x \cap [0, T], \nexists t' \in [0, T] \cap \operatorname{dom} \phi : t' > t\}. \end{aligned} \tag{3.3}$$

where $\mathcal{S}_{\mathcal{H}_f}(x_o)$ is the set of maximal solutions to $\mathcal{H}_f = (C, F)$ starting from x_o.

3.2 Forward Invariance

Roughly speaking, a set $K \subset \mathbb{R}^n$ is forward invariant for a system $\mathcal{H}_f := (C, F)$ if solutions of H_f starting within K stay within K. The "pre" in forward pre-invariance is used to accommodate non complete maximal solutions. For example, if a solution x to a system $\mathcal{H}_f$ starts from X and stops evolving once on the boundary of C, then such a solution can still satisfy $x(t) \in X$ for all $t \in \operatorname{dom} x$, but with $\operatorname{dom} x$ bounded.

Definition 3.2.1 (Forward pre-Invariance) *The set X is forward pre-invariant for $\mathcal{H}_f$ if, for each initial condition $x_o \in X$ and for each solution $x \in \mathcal{S}_{\mathcal{H}_f}(x_o)$, $x(t) \in X$ for all $t \in \operatorname{dom} x$.*

Next, the notion of a barrier function candidate defining a set $X \subset \text{cl}(C)$ is introduced.

Definition 3.2.2 (Barrier Function Candidate) *Given $C \subset \mathbb{R}^n$, the function $\rho : C \to \mathbb{R}$ is a barrier function candidate defining the set $X \subset \text{cl}(C)$ if*

$$X = \{x \in \text{cl}(C) : \rho(x) \geq 0\}. \tag{3.4}$$

Barrier functions can be used to establish certificates of forward invariance. For example, consider given a set $C \subset \mathbb{R}^n$, a single valued function $f : \mathbb{R}^n \mapsto \mathbb{R}$, and a set $X := \{x \in \text{cl}(C) : \rho(x) \geq 0\}$ defined by a continuously differentiable barrier function candidate $\rho : \text{cl}(C) \mapsto \mathbb{R}$. If ρ satisfies

$$\langle \nabla \rho(x), f(x) \rangle > 0, \tag{3.5}$$

for all $x \in C$, then X is forward invariant for solutions of f [42]. The satisfaction of (3.5) by the barrier function candidate ρ is commonly referred to as a barrier function *certificate*.

The technique of guaranteeing forward invariance of a set without the need computing the solutions of a system originates from the seminal work of Nagumo [42] and is commonly applied in safety critical control [43].

We also introduce the following set that we use in some statements and proofs. For a set X given as in (3.4), we define

$$X_e := \{x \in \mathbb{R}^n : \rho(x) \geq 0\}. \tag{3.6}$$

Remark 3.2.3 *Definitions 3.2.2 and 3.2.1 can be extended to the case where the set* $X \subset \mathbb{R}_{\geq 0} \times \mathrm{cl}(C)$ *is time-varying and the barrier candidate* $\rho : \mathbb{R}_{\geq 0} \times \mathbb{R}^n \to \mathbb{R}$ *is also time dependent. In this case, it suffices to consider the augmented state variable* $x_a := [t \;\; x^\top]^\top \in \mathbb{R}_{\geq 0} \times \mathbb{R}^n$ *and the extended system*

$$\mathcal{H}_{f_a} : \qquad \dot{x}_a \in [1 \;\; F(x)^\top]^\top \quad x_a \in C_a := \mathbb{R}_{\geq 0} \times C. \tag{3.7}$$

In the next chapter two ST control problems are introduced and formalised.

4

Problem Formulation

Consider a constrained control system $\mathcal{H}_f^u$ given by

$$\mathcal{H}_f^u : \quad \dot{x} \in F(x,u) \quad x \in C \subset \mathbb{R}^n, \; u \in \mathbb{R}^m. \tag{4.1}$$

Assumption 4.0.1 *The set-valued map $F : C \times \mathbb{R}^m \rightrightarrows \mathbb{R}^n$ is outer semi-continuous and locally bounded with convex images. Additionally, C is closed.*

•

Remark 4.0.2 *We assume C to be closed without loss of generality. Indeed, when C is not closed, we can consider its closure, which can add solutions to the system, but the conditions we propose do not change.*

Given a continuous feedback law $\kappa : C \mapsto \mathbb{R}^m$, the resulting closed-loop system is given by

$$\mathcal{H}_f^{cl} : \qquad \dot{x} \in F^{cl}(x) := F(x, \kappa(x)) \qquad x \in C. \tag{4.2}$$

Note that F^{cl} satisfies (A1) and (A2).

In a ST control setting, the measurement of the state x is available only at sampling instants defining a sequence $\{t_i\}_{i=0}^{\infty} \subset \mathbb{R}_{\geq 0}$ such that $t_{i+1} > t_i$. Furthermore, we consider in this thesis the scenario where the control law κ remains constant between each two samples t_i and t_{i+1}; that is, κ is subject to a zero-order sample and hold. Hence, the actual control signal that is applied to the system is given by

$$u(t) = \kappa(x(t_i)) \quad \forall t \in [t_i, t_{i+1}] \quad \forall i \in \mathbb{N}. \tag{4.3}$$

Next, we define the concept of solutions to $\mathcal{H}_f^u$ in closed loop with (4.3).

Definition 4.0.3 (Solution to ST closed-loop system) *A locally absolutely continuous function $x : \operatorname{dom} x \to C$, $\operatorname{dom} x \subset \mathbb{R}_{\geq 0}$, starting from $x_o \in C$, is a solution to $\mathcal{H}_f^u$ in closed loop with (4.3) if, in addition to (S2), for all $i \in \mathbb{N}$,*

$$\dot{x}(t) \in F(x(t), \kappa(x(t_i))) \text{ for a.a } t \in [t_i, t_{i+1}] \cap \operatorname{dom} x. \tag{4.4}$$

Assume that the feedback law κ has been carefully designed such that a given set $X \subset C$ is forward pre-invariant for the system $\mathcal{H}_f^{cl}$. One objective in ST control problems is to use the state measurements available at each sampling instant t_i to deduce the largest possible next sampling time $t_{i+1} > t_i$ such that the closed loop of $\mathcal{H}_f^u$ using (4.3) still achieves forward pre-invariance of the set X. We call the resulting sequence $\{t_i\}_{i=0}^{\infty}$ a *feasible* sampling sequence. In the first part of this thesis , we propose a method to efficiently find the next sampling time to guarantee forward pre-invariance.

Problem 1 (Finding the next sampling time) *Consider the control system $\mathcal{H}_f^u$, a closed set $X \subset C$, and a continuous feedback law $\kappa : C \to \mathbb{R}^m$ such that X is forward pre-invariant for the closed-loop system* (4.2). *Find a function $T_s : C \mapsto \mathbb{R}_{\geq 0} \cup \{\infty\}$ such that, for each $x_o \in X$ and for each $t \mapsto x(t)$ solution to the system*

$$\dot{x} \in F(x, \kappa(x_o)) \qquad x \in C \tag{4.5}$$

starting from x_o, we have

$$x(t) \in X \quad \forall t \in [0, T_s(x_o)] \cap \operatorname{dom} x. \tag{4.6}$$

•

One solution to Problem 1 is provided in [44] where F is single valued and $C = \mathbb{R}^n$. Indeed, consider the system in (4.5) and assume that, for some $V > 0$,

$$|F(x, u)| \leq V \qquad \forall (x, u) \in \operatorname{dom} F.$$

Consider a closed set X admitting a barrier function candidate ρ as in Definition 3.2.2. Then, for each $x_o \in X$,

$$T_s(x_o) := \rho(x_o)/V. \tag{4.7}$$

Note that, although the sampling function in (4.7) solves Problem 1, it is conservative as we illustrate in the next example.

Example 4.0.4 *Consider the system in* (4.5) *with*

$$F(x, u) := [u \ \ 0]^\top$$

and $C = \mathbb{R}^2$. *Let* $X := \{x \in \mathbb{R}^2 : x_2 \leq 10\}$ *and note that the barrier function candidate* ρ *given by* $\rho(x) := 10 - x_2$ *defines the set* X *as in Definition 3.2.2. Now, for* $x_o := [0 \ \ 9]^\top$, *the sampling function in* (4.7) *provides* $T_s(x_o) = 1/10$. *However, the solution starting at* x_o *satisfies* $\rho(x(t)) > 0$ *for all* $t \geq 0$ *for any* u.

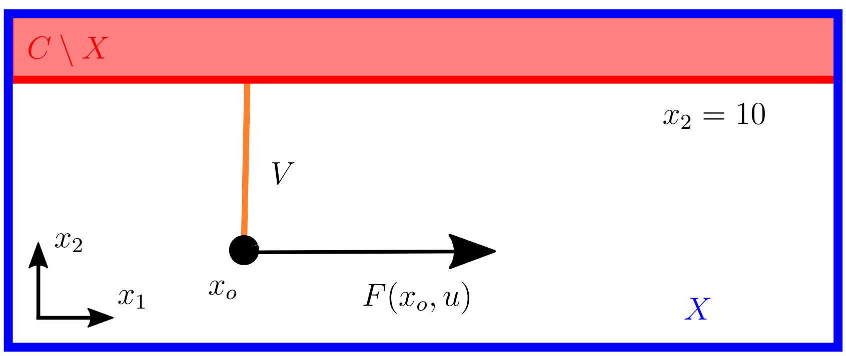

Figure 4.1: Example 4.0.4

A solution similar to (4.7) is proposed in [30], where T_s is proportional to the inverse of the Lipschitz constant of F. See also [28], where X is compact, F is smooth, and T_s is proportional to the inverse of a function upper bounding the decrease rate of a Lyapunov-like function. Our approach is compared to the one in [28] is in Chapter 6.

Solving Problem 1 allows us to recursively construct a feasible sampling sequence given by

$$t_{i+1} = t_i + T_s(x(t_i)).$$

However, such a sequence is not guaranteed to have the inter-event times $(t_{i+1} - t_i)$ uniformly larger than a positive constant. The absence of such a guarantee could lead to a Zeno behavior or to arbitrarily small inter-event times for initial conditions on the boundary of X.

For example, consider the scenario defined in Example 4.0.4 with $x_o := [0 \quad 10]^\top$ and notice the inter-event time provided by the sampling function given by (4.7) is not defined. Hence, we formulate the following problem.

Problem 2 (Uniformly non-vanishing inter-event times) *Consider the control system $\mathcal{H}_f^u$ in (4.1) and a continuous feedback law $\kappa : C \to \mathbb{R}^m$ such that X is forward pre-invariant for (4.2). Determine conditions guaranteeing the existence of $T_s^* > 0$, such that, for each initial condition $x_o \in X$, a sampling sequence $\{t_i\}_{i=0}^{\infty}$ can be constructed such that $t_{i+1} - t_i \geq T_s^*$ for all $i \in \mathbb{N}$, and each maximal solution to the ST closed-loop system starting from x_o remains in X.* •

Existing solutions to Problem 2 require smoothness of the barrier candidate ρ and either boundedness of X [28] or bounded variation of F [31, 30].

5

Main Results

5.1 Solutions to Problem 1

In the following result, we propose an answer to Problem 1. The construction of the function T_s at $x_o \in X$, proposed below, involves an approximation of how fast a solution x starting from x_o can move towards ∂X during a forward propagation interval $[0, \bar{T}]$. This speed is upper bounded by the supremum of the scalar product between the gradient of the barrier function candidate $\rho(\cdot)$ and $F(\cdot, \kappa(x_o))$ on the set that can be reached by the solution x to (4.5) over the interval $[0, \bar{T}]$. This set is denoted by $\hat{R}(\bar{T}, x_o)$, which is an overapproximation of the reachable set $R(\bar{T}, x_o)$ along the solution to (4.5).

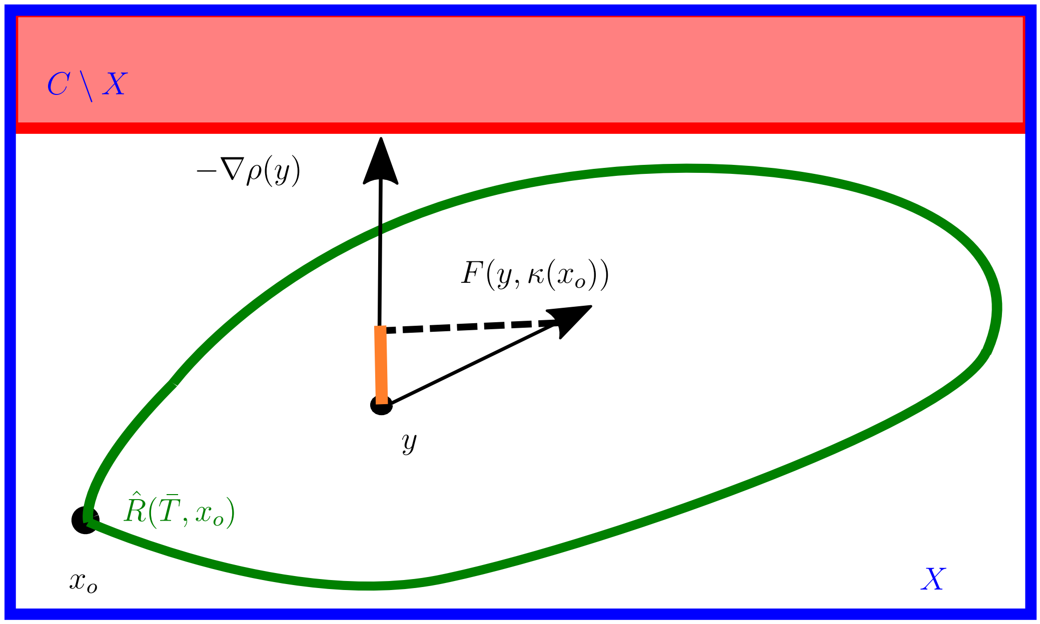

Figure 5.1: A positive speed towards ∂X

To illustrate this idea, consider the system $\mathcal{H}_f^u$ in (4.1) with F single valued, $C = \mathbb{R}^n$, and κ continuous such that X, defined by a smooth ρ, is forward pre-invariant for the system $\mathcal{H}_{cl}^f$. Now, the upper bound on the speed towards ∂X of a solution x to the system in (4.5) starting from $x_o \in X$ is given by

$$M(\bar{T}, x_o) := \sup\{\langle -\nabla\rho(y), F(y, \kappa(x_o))\rangle,\ y \in \hat{R}(\bar{T}, x_o))\}.$$

Indeed, the scalar product $\langle \nabla\rho(x(t)), F(x(t), \kappa(x_o)\rangle$ is merely the time derivative $\dot{\rho}(x(t))$ when $\kappa(x_o)$ is applied. Hence, evaluating the supremum of $\dot{\rho}$, over points in $\hat{R}(\bar{T}, x_o)$, provides a scalar value indicating whether the solution x evolves towards ∂X during $[0, \bar{T}]$ or not. When $M(\bar{T}, x_o) > 0$, the solution x flows towards ∂X; hence, the first time x reaches ∂X is lower bounded by $T_s(x_o) := \min\left\{\bar{T}, \frac{\rho(x_o)}{M(\bar{T}, x_o)}\right\}$. Figure 5.1 shows the 'speed' towards ∂X from a point $y \in \hat{R}(\bar{T}, x_o)$.

When $M(\bar{T}, x_o) \leq 0$, the solution x is not moving towards ∂X along the interval $[0, \bar{T}]$, and the next sampling time can be just $\bar{T}$. Figure 5.2 shows the 'speed' away from ∂X from a point $y \in \hat{R}(\bar{T}, x_o)$.

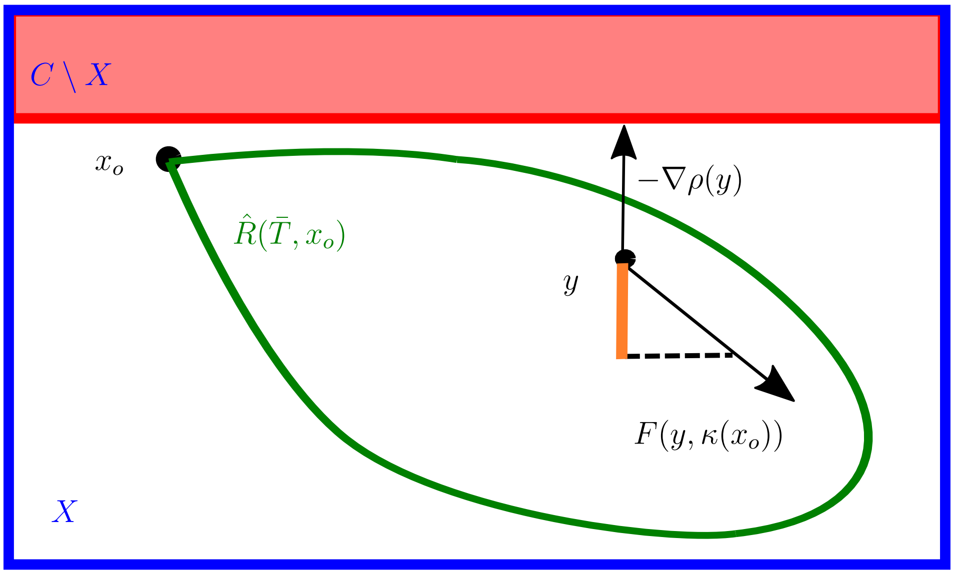

Figure 5.2: A negative speed towards ∂X

Requiring the barrier function candidate ρ to be continuously differentiable can be restrictive when, for example, ρ is the distance to the set $\mathbb{R}^n \setminus X$ or when ρ is the minimum or the maximum of multiple functions, in which case, ρ is only locally Lipschitz.

Theorem 5.1.1 *Consider the control system $\mathcal{H}_f^u$ such that Assumption 4.0.1 holds. Consider a locally Lipschitz barrier function candidate ρ defining a closed set X and a continuous feedback law $\kappa : C \to \mathbb{R}^m$ such that X is* *forward pre-invariant for* (4.2)*. Then, for any* $x_o \in C \cap X$ *and for any* $\bar{T} > 0$*,* (4.6) *is satisfied for*

$$T_s(x_o) := \begin{cases} \bar{T} & \text{if } M_s(\bar{T}, x_o) \leq 0 \\ \min\left\{\bar{T}, \frac{\rho(x_o)}{M_s(x_o,\bar{T})}\right\} & \text{otherwise.} \end{cases} \tag{5.1}$$

where,

$$\begin{aligned} M_s(x_o, \bar{T}) := \sup\{\langle -\gamma, \eta \rangle : \\ \gamma \in \partial_C \rho(y), \eta \in F(y, \kappa(x_o)) \cap T_C(y), \\ y \in \hat{R}(\bar{T}, x_o)\}, \end{aligned} \tag{5.2}$$

and $\hat{R} : \mathbb{R}_{\geq 0} \times C \rightrightarrows C$ *is an overestimation of the reachable set* $R(\bar{T}, x_o)$ *along the solutions to* (4.5)*.*

Remark 5.1.2 *When* ρ *is continuously differentiable,* (5.2) *reduces to*

$$\begin{aligned} M_s(x_o, \bar{T}) := \sup\{\langle -\nabla \rho(y), \eta \rangle : \\ \eta \in F(y, \kappa(x_o)) \cap T_C(y), y \in \hat{R}(\bar{T}, x_o)\}, \end{aligned} \tag{5.3}$$

Proof. We first prove the statement when ρ is continuously differentiable. Indeed, let x be a solution to (4.5) starting from $x_o \in C$ such that $\rho(x_o) > 0$. Since the solution x is locally absolutely continuous and ρ is continuously differentiable, it follows that the map $t \mapsto \rho(x(t))$ is locally absolutely continuous on $\operatorname{dom} x$. By definition of local absolute continuity, for each $T \in \operatorname{dom} x$, there exists a sequence $\{t_n\}_{n=0}^{N} \subset [0, T]$, where $N \in \mathbb{N}^* \cup \{\infty\}$ and $\lim_{n \to N} t_n = T$, such that $t_n - t_{n-1} > 0$ and the map $t \mapsto \rho(x(t))$ is differentiable on each interval (t_{n-1}, t_n) for all $n \in \mathbb{N}^*$. Next, we note that

$$\rho(x(T)) - \rho(x_o) = \sum_{n=1}^{N} \big[\rho(x(t_n)) - \rho(x(t_{n-1}))\big].$$

Furthermore, using Lemma 2.2.6, we conclude that, for each $n \in \{1, 2, ..., N\}$, there exists $c_n \in (t_{n-1}, t_n)$ such that

$$\begin{aligned} \rho(x(t_n)) - \rho(x(t_{n-1})) &= \frac{d}{dt}\rho(x(t))|_{t=c_n}(t_n - t_{n-1}) \\ &= \langle \nabla\rho(x(c_n)), \dot{x}(c_n)\rangle (t_n - t_{n-1}) \end{aligned} \tag{5.4}$$

Then,

$$\begin{aligned} \rho(x(0)) - \rho(x(T)) &= -\sum_{n=1}^{N} \big[\rho(x(t_n)) - \rho(x(t_{n-1}))\big] \\ &= -\sum_{n=1}^{N} \frac{d}{dt}\rho(x(t))|_{t=c_n}(t_n - t_{n-1}) \\ &\le \sup\left\{-\frac{d}{dt}\rho(x(t))|_{t=c_n} : n \in \{1, 2, ..., N\}\right\} T \\ &\le T \sup\{\langle -\nabla\rho(y), \eta\rangle : y \in \hat{R}(\bar{T}, x_o), \\ &\qquad \eta \in F(y, \kappa(x_o)) \cap T_C(y)\}, \end{aligned} \tag{5.5}$$

where $\bar{T} \ge T$. To obtain the last inequality in (5.5), we used Lemma A.0.3 to conclude that

$$\frac{d}{dt}x(t)\Big|_{t=c_n} \in T_C(x(c_n)) \qquad \forall n \in \{1, 2, ..., N\}.$$

Now, to guarantee that $\rho(x(t)) \ge 0$ for all $t \in [0, T]$, we choose T as

$$T \le \rho(x_o)/M_s(x_o, \bar{T}), \tag{5.6}$$

where $M_s(x_o, \bar{T})$ is introduced in (5.3). Since the last inequality in (5.5) is valid only when $T \leq \bar{T}$, the function T_s at x_o that solves Problem 1 can be chosen as in (5.1).

Similarly, when ρ is locally Lipschitz, we let x be a solution to (4.5) starting from $x_o \in C \cap X$. Since F is locally bounded, we conclude that the solution x is locally Lipschitz. Furthermore, since ρ is locally Lipschitz, it follows that the map $t \mapsto \rho(x(t))$ is also locally Lipschitz on $\operatorname{dom} x$. Using Lemma A.0.3, we conclude that, for each $T \in \operatorname{dom} x$, there exists a sequence $\{t_n\}_{n=0}^{N} \subset [0, T]$, where $N \in \mathbb{N}^* \cup \{\infty\}$ and $\lim_{n \to N} t_n = T$, such that $t_n - t_{n-1} > 0$ and $\dot{x}(t) \in T_C(x(t))$ for all $t \in (t_{n-1}, t_n)$ and for all $n \in \{1, 2, ..., N\}$.

Define the map $t \mapsto P(t) := \rho(x(t))$. Using Lemma 2.2.5, we conclude that, for each $n \in \{1, 2, ..., N\}$, there exists $s_n \in (t_{n-1}, t_n)$ and $\zeta_n \in \partial_C P(s_n)$ such that

$$P(t_{n-1}) - P(t_n) = -\zeta_n \cdot (t_n - t_{n-1}). \tag{5.7}$$

Furthermore, using Lemma 2.2.4, we know that

$$\partial_C P(s_n) \subset \overline{co}\{\partial_C \langle \gamma, x(s_n) \rangle : \gamma \in \partial_C \rho(x(s_n))\}. \tag{5.8}$$

Moreover, using Lemma A.0.1, we know that,

$$\partial_C \langle \gamma, x(s_n) \rangle \subset \langle \gamma, \partial_C x(s_n) \rangle \quad \forall n \in \{1, 2, ..., N\}. \tag{5.9}$$

Next, using Lemma A.0.2, we know that

$$\partial_C x(s_n) \subset F(x(s_n), \kappa(x_o)) \qquad \forall n \in \{1, 2, ..., N\}.$$

Further, using Lemma A.0.3, we know that

$$\partial_C x(s_n) \subset T_C(x(s_n)) \qquad \forall n \in \{1, 2, ..., N\}.$$

From these two facts and using (5.9), we conclude that, for each $n \in \{1, 2, ..., N\}$,

$$\begin{aligned} \partial_C \langle \gamma, x(s_n) \rangle &\subset \langle \gamma, \partial_C x(s_n) \rangle \\ &\subset \langle \gamma, F(x(s_n), \kappa(x_o)) \cap T_C(x(s_n))) \rangle. \end{aligned} \tag{5.10}$$

Thus,

$$\begin{aligned} \partial_C P(s_n) \subset \overline{co}\{ \langle \gamma, F(x(s_n), \kappa(x_o)) \cap T_C(x(s_n))) \rangle :& \\ \gamma \in \partial_C \rho(x(s_n)) \}&. \end{aligned} \tag{5.11}$$

Hence, the following upper bound follows for each $T \leq \bar{T}$:

$$\begin{aligned} \rho(x_o) - \rho(x(T)) &= - \sum_{n=1}^{N} (\rho(x(t_n)) - \rho(x(t_{n-1}))) \\ &= \sum_{n=1}^{N} (-\zeta_n) \cdot (t_n - t_{n-1}) \\ &\leq T \cdot \sup\{ \langle -\gamma, \eta \rangle : \gamma \in \partial_C \rho(y), \\ &\qquad \eta \in F(y, \kappa(x_o)) \cap T_C(y), \\ &\qquad y \in \hat{R}(\bar{T}, x_o) \}. \end{aligned} \tag{5.12}$$

The latter inequality holds because ζ_n is within the right-hand side of (5.11), $\{s_n\}_0^N \subset [0, T]$, and $x(s_n) \in \hat{R}(\bar{T}, x_o)$ since $\bar{T} \geq T$. To guarantee that $\rho(x(t)) \geq 0$ for all $t \in [0, T]$, we choose T as

$$T \leq \rho(x_o)/M_s(x_o, \bar{T}) \tag{5.13}$$

where $M_s(x_o, \bar{T})$ is introduced in (5.2).

Finally, since the last inequality in (5.12) is valid only when $T \leq \bar{T}$, the function T_s at x_o that solves Problem 1 can be chosen as in (5.1). ■

Remark 5.1.3 *Note that the over estimation $\hat{R}$ of the reachability map R can be always computed because under Assumption 4.0.1 and the continuity of κ, $(x, x_o) \mapsto F(x, \kappa(x_o))$ is locally bounded on $C \times C$. Thus, there always exists a neighborhood around x_o where an upper bound on $F(x, \kappa(x_o))$ can be used to over approximate R. However, the tighter this over approximation is, the larger the sampling function T_s will be. Methods to compute $\hat{R}$ are available in [45] and [46] for classes of nonlinear systems, in [47] for linear time-varying systems, and in [48] for Lipschitz differential inclusions.*

5.2 Solutions to Problem 2

Consider the constrained control system $\mathcal{H}_f^u$, a closed subset $X \subset C$, and a barrier function candidate ρ defining the set X. Let $\kappa : C \to \mathbb{R}^m$ be a continuous feedback law rendering the set X forward pre-invariant for (4.2). Having the set X forward pre-invariant for (4.2) does not necessarily imply that the same is true for the ST closed-loop system. Hence, we assume that the following robustness assumption holds:

Assumption 5.2.1 *There exists a barrier function candidate* ρ *that is locally Lipschitz and defines the sets* X *and* X_e *as in* (3.4) *and* (3.6), *respectively. Additionally, there exist locally Lipschitz functions* $\alpha : \mathbb{R}^n \to \mathbb{R}$ *and* $\gamma : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ *satisfying*

$$\alpha(x) > 0 \ \forall \ x \in \partial X_e \quad \text{and} \quad \gamma(x, x) = 0,$$

such that

$$\begin{aligned} &\langle \zeta, f \rangle \geq \alpha(x) - \gamma(x, \eta) \\ &\qquad \forall (\zeta, f) \in \partial_C \rho(x) \times (F(x, \kappa(\eta)) \cap T_C(x)) \\ &\qquad \forall (x, \eta) \in X \times X. \end{aligned} \tag{5.14}$$

•

Remark 5.2.2 *Condition* (5.14) *implies that the feedback law* κ *renders the set* X *pre-contractive for the closed-loop system* (4.2). *Pre-contractivity provides a robustness margin of the forward pre-invariance property; see [49].*

We also consider the following additional assumption.

Assumption 5.2.3 *There exist* $T_1 > 0$ *and* $\beta > 0$ *such that, for each solution* x *to* (4.5) *starting from* $K := \{x \in X : |x|_{\partial X_e} \geq \beta\}$, *we have* $x(t) \in X$ *for all* $t \in [0, T_1]$. •

Remark 5.2.4 *Note that Assumption 5.2.3 holds for free when the set* $K := \{x \in X : |x|_{\partial X} \geq \beta\}$ *is compact.*

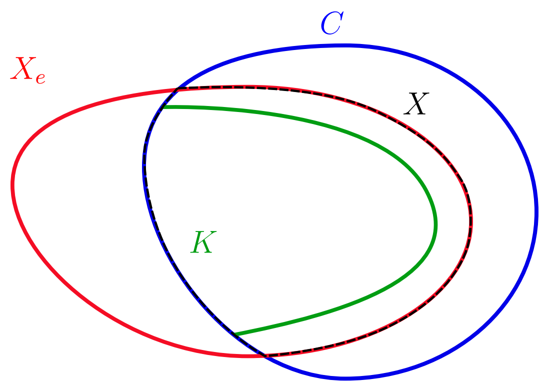

Figure 5.3: The sets X_e, X, and K

Indeed, for each neighborhood $U(K) \subset X$*, there exists* $\delta > 0$ *such that* $|x_1 - x_2| \geq \delta$ *for all* $(x_1, x_2) \in K \times (X \backslash U(K))$*. Next, since* F *is outer semicontinuous and locally bounded, it follows that*

$$\sup\{|\eta| : \eta \in F(x, \kappa(x_o)),\ (x, x_o) \in U(K) \times K\} < \infty.$$

Hence, it takes at least a time

$$T \geq \beta / \sup\{|\eta| : \eta \in F(x, \kappa(x_o)),\ (x, x_o) \in U(K) \times K\} > 0$$

for a solution to (4.5) *from* K *to reach* $X \backslash U(K)$*.*

In Lemma 5.2.7 below, we provide sufficient conditions to verify Assumption 5.2.3. To do so, we assume the following.

Assumption 5.2.5 *Given* $\beta > 0$ *and* $T_1 > 0$ *there exists a locally Lipschitz function* $V : \mathbb{R}^n \to \mathbb{R}$ *and* $\epsilon_1 > 0$ *such that*

$$V(x) \leq 0 \qquad \forall x \in \{x \in X : \rho(x) = 0\} \tag{5.15}$$

and, for each $x \in X$, $|x|_{\partial X} \geq \beta$ *implies* $V(x) \geq \epsilon_1$*. Furthermore, there exists a locally Lipschitz function* $\sigma : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$*, such that*

$$\begin{aligned} &\langle \zeta, \eta \rangle \geq \sigma(V(x), V(x_o)) \\ &\qquad \forall (\zeta, \eta) \in \partial_C V(x) \times (F(x, \kappa(x_o)) \cap T_C(x)) \\ &\qquad \forall (x, x_o) \in C \times C \end{aligned} \tag{5.16}$$

and

$$-\int_0^{\nu_o} \frac{d\nu}{\sigma(\nu, \nu_o)} \in (-\infty, 0] \cup [T_1, +\infty) \quad \forall \nu_o \geq \epsilon_1. \tag{5.17}$$

•

Remark 5.2.6 *Note that* (5.17) *holds for free if the function* σ *is linear on its arguments.*

Lemma 5.2.7 *Consider the control system* $\mathcal{H}_f^u$ *such that Assumption 4.0.1 holds. Consider a continuous barrier function candidate* ρ *defining a closed set* $X \subset C$ *and a continuous feedback law* $\kappa : C \to \mathbb{R}^m$*. Let* $\beta > 0$ *and* $T_1 > 0$ *be such that Assumption 5.2.5 holds. Then, Assumption 5.2.3 holds with such* T_1 *and* β*.*

Proof. Consider a solution x to (4.5) starting from $x_o \in K = \{x \in X : |x|_{\partial X_e} \geq \beta\}$; thus, $V(x_o) \geq \epsilon_1 > 0$. Under (5.15), we can show that Assumption 5.2.3

holds by showing that

$$V(x(t)) > 0 \qquad \forall t \in [0, T_1). \tag{5.18}$$

Now, to show (5.18), we first show that there exists a solution y to the scalar system

$$\dot{y} = \sigma(y, y_o) \tag{5.19}$$

starting from $y_o := V(x_o)$ such that

$$y(t) \leq V(x(t)) \qquad \forall t \in \operatorname{dom} x \tag{5.20}$$

and, second, we show that $y(t) > 0 \quad \forall t \in [0, T_1]$.

Indeed, since F is locally bounded, then the solution x starting from $x_o \in K$ is locally Lipschitz on $\operatorname{dom} x$. Furthermore, since V is locally Lipschitz, the mapping $t \mapsto V(x(t))$ is locally Lipchitz on $\operatorname{dom} x$. Next, using Lemma A.0.4 we can conclude that

$$\frac{d}{dt}V(x(t)) \in \{\langle \zeta, \dot{x}(t)\rangle : \zeta \in \partial_C V(x(t))\} \text{ for a.a. } t \in \operatorname{dom} x.$$

Furthermore, by Lemmas A.0.2 and A.0.3 we conclude that

$$\dot{x}(t)) \in F(x(t), \kappa(x_o)) \cap T_C(x(t)) \text{ for a.a. } t \in \operatorname{dom} x.$$

Hence, for almost all $t \in \operatorname{dom} x$,

$$\{\langle \zeta, \dot{x}(t)\rangle : \zeta \in \partial_C V(x(t))\} \subseteq$$

$$\{\langle \zeta, \eta\rangle : \zeta \in \partial_C V(x(t)), \eta \in F(x(t), \kappa(x_o)) \cap T_C(x(t))\}.$$

Thus, using (5.16), we conclude that

$$\frac{d}{dt}V(x(t)) \geq \sigma(V(x(t), V(x_o)) \text{ for a.a. } t \in \operatorname{dom} x.$$

Finally, using the comparison Lemma in [11, Lemma 3.4], we conclude that, for each solution solution y to (5.19) starting from $y_o := V(x_o)$, (5.20) holds.

Next, to complete the proof, we use (5.17) to conclude that

$$y(t) > 0 \qquad \forall t \in [0, T_1).$$

Indeed, since $y_o \geq \epsilon_1 > 0$, we show that either

$$y(t) > 0 \qquad \forall t \in \operatorname{dom} y,$$

otherwise, there exists $t_y > 0$ such that

$$y(t) > 0 \quad \forall t \in [0, t_y) \quad \text{and} \quad y(t_y) = 0.$$

Then by (5.17) we have that

$$t_y = \int_{y_o}^{y(t_y)} \frac{dy}{\sigma(y, y_o)} = \int_{y_o}^{0} \frac{dy}{\sigma(y, y_o)} \geq T_1.$$

■

In the following result, we provide sufficient conditions for the existence of $T_s^* > 0$ that solves Problem 2.

We consider the case where ρ, α, and γ in (5.14) are continuously differentiable as well as the case where they are only locally Lipschitz.

Theorem 5.2.8 *Consider the control system $\mathcal{H}_f^u$ such that Assumption 4.0.1 holds. Consider a closed subset $X \subset C$ and a corresponding barrier function candidate ρ. Assume that there exists a continuous feedback law $\kappa : C \to \mathbb{R}^m$ such that Assumption 5.2.1 holds. Furthermore, assume there exist $\beta > 0$ and $T_1 > 0$ such that Assumption 5.2.3 holds. Then, Problem 2 is solved with*

$$T_s^* := \min\{T_1, T_2\} \tag{5.21}$$

provided that

$$T_2 := \min\left\{T_r(x) : x \in X,\ |x|_{\partial X_e} \leq \beta\right\} > 0, \tag{5.22}$$

$$T_r(x) := \begin{cases} \bar{T} & \text{if } M_r(\bar{T}, x) \leq 0 \\ \min\left\{\bar{T}, \frac{2\alpha(x)}{M_r(\bar{T},x)}\right\} & \text{otherwise,} \end{cases} \tag{5.23}$$

$$M_r(\bar{T}, x) := M_\alpha(\bar{T}, x) + M_\gamma(\bar{T}, x), \tag{5.24}$$

for any $\bar{T} > 0$ and

$$\begin{aligned} M_\gamma(\bar{T}, x) := \sup\{&\langle \gamma_1, \eta \rangle : \gamma_1 \in \partial_C \gamma(y, x), \\ &\eta \in F(y, \kappa(x)) \cap T_C(y), y \in \hat{R}(\bar{T}, x)\}, \end{aligned} \tag{5.25}$$

$$\begin{aligned} M_\alpha(\bar{T}, x) := \sup\{&\langle -\gamma_2, \eta \rangle : \gamma_2 \in \partial_C \alpha(y), \\ &\eta \in F(y, \kappa(x)) \cap T_C(y),\ y \in \hat{R}(\bar{T}, x)\}. \end{aligned} \tag{5.26}$$

Remark 5.2.9 *When ρ, γ, and α are continuously differentiable,* (5.25) *and* (5.26) *reduce to*

$$\begin{aligned} M_\gamma(\bar{T}, x) &:= \sup\{\langle \nabla\gamma(y, x), \eta\rangle : \\ &\quad \eta \in F(y, \kappa(x)) \cap T_C(y),\ y \in \hat{R}(\bar{T}, x)\}, \end{aligned} \tag{5.27}$$

$$\begin{aligned} M_\alpha(\bar{T}, x) &:= \sup\{\langle -\nabla\alpha(y), \eta\rangle : \\ &\quad \eta \in F(y, \kappa(x)) \cap T_C(y),\ y \in \hat{R}(\bar{T}, x)\}. \end{aligned} \tag{5.28}$$

Proof. To prove the statement when ρ, γ, and α are continuously differentiable, we consider an initial condition $x_o \in G := \{x \in X : |x|_{\partial X_e} \leq \beta\}$. The prove follows if we show that, for each solution x to (4.5) starting from x_o, we have

$$x(t) \in X \qquad \forall t \in [0, T_2].$$

According to (5.14), we conclude that

$$\begin{aligned} &\langle \nabla\rho(x), f\rangle \geq \alpha(x) - \gamma(x, x_o) \\ &\qquad \forall f \in F(x, \kappa(x_o)) \cap T_C(x) \quad \forall x \in X. \end{aligned} \tag{5.29}$$

Let x be a solution, starting from x_o, to the system (4.5). Since the solution x is locally absolutely continuous and the functions α and γ in (5.29) are continuously differentiable, it follows that the maps $t \mapsto \alpha(x(t))$ and $t \mapsto \gamma(x(t), x_o)$ are both locally absolutely continuous. By definition of local absolute continuity, for each $t \in \operatorname{dom} x$, there exists a sequence $\{t_n\}_{n=0}^{N} \subset [0, t]$ such that:

1. $N \in \mathbb{N}^* \cup \infty$,

2. $\lim_{n\to N} t_n = t$ and $t_n - t_{n-1} > 0$, and

3. the maps $t \mapsto \alpha(x(t))$ and $t \mapsto \gamma(x(t), x_o)$ are differentiable on each interval (t_{n-1}, t_n) for all $n \in \{1, 2, ..., N\}$.

To avoid heavy notations, we use

$$\bar{\gamma}(x(t)) := \gamma(x(t), x_o) \quad \forall t \in \operatorname{dom} x.$$

Next, we note that

$$\bar{\gamma}(x(t)) - \bar{\gamma}(x_o) = \sum_{n=1}^{N} \left(\bar{\gamma}(x(t_n)) - \bar{\gamma}(t_{n-1})\right), \tag{5.30}$$

and

$$\alpha(x(t)) - \alpha(x_o) = \sum_{n=1}^{N} \left(\alpha(x(t_n)) - \alpha(x(t_{n-1}))\right). \tag{5.31}$$

By Lemma 2.2.6, we conclude that, for each $n \in \{1, 2, ..., N\}$, there exists $c_n \in (t_{n-1}, t_n)$ such that

$$\bar{\gamma}(x(t)) - \bar{\gamma}(x_o) = \sum_{n=1}^{N} \left(\frac{d}{dt}\bar{\gamma}(x(t))\Big|_{t=c_n} (t_n - t_{n-1}) \right),$$

and

$$\alpha(x(t)) - \alpha(x_o) = \sum_{n=0}^{N} \left(\frac{d}{dt}\alpha(x(t))\Big|_{t=c_n} (t_{n+1} - t_n) \right).$$

Now, we propose to find an upper bound on the map $t \mapsto \bar{\gamma}(x(t))$. Indeed, starting from (5.30), we have

$$\begin{aligned}\bar{\gamma}(x(t))-\bar{\gamma}(x_o) &= \sum_{n=1}^{N}\left(\frac{d}{dt}\bar{\gamma}(x(t))\Big|_{t=c_n}(t_n-t_{n-1})\right)\\ &\leq t\sup\{\langle\nabla\bar{\gamma}(x(c_n)),\dot{x}(c_n)\rangle :\\ &\qquad n\in\{1,2,...,N\}\}\\ &\leq t\sup\{\langle\nabla\bar{\gamma}(y),\eta\rangle : \eta\in F(y)\cap T_C(y),\\ &\qquad y\in\hat{R}(\bar{T},x_o)\},\end{aligned}\tag{5.32}$$

where $\bar{T}\geq t$. Note that the latter inequality in (5.32) is true using Lemmas A.0.2 and A.0.3. As result, the final upper bound for $t\mapsto\bar{\gamma}(x(t))$ is given by:

$$\bar{\gamma}(x(t))\leq tM_\gamma(\bar{T},x_o)\qquad \forall t\in[0,\bar{T}],\tag{5.33}$$

where $M_\gamma(\bar{T},x_o)$ is introduced in (5.27). Next, we propose to find a lower bound for the map $t\mapsto\alpha(x(t))$. Indeed, using (5.31), we conclude that

$$\begin{aligned}\alpha(x(t))-\alpha(x_o) &= \sum_{n=0}^{N}\left(\frac{d}{dt}\alpha(x(t))\Big|_{t=c_n}(t_{n+1}-t_n)\right)\\ &\geq t\inf\{\langle\nabla\alpha(x(c_n)),\dot{x}(c_n))\rangle :\\ &\qquad n\in\{1,2,...,N\}\\ &\geq t\inf\{\langle\nabla\alpha(y),\eta\rangle : \eta\in F(y)\cap T_C(y),\\ &\qquad y\in\hat{R}(\bar{T},x_o)\},\end{aligned}\tag{5.34}$$

where $\bar{T}\geq t$. The latter inequality in (5.34) is true using Lemma A.0.2. As result, the final lower bound of $t\mapsto\alpha(x(t))$ is given by:

$$\alpha(x(t))\geq\alpha(x_o)+tM_\alpha(\bar{T},x_o),\tag{5.35}$$

where $M_\alpha(\bar{T}, x_o)$ is introduced in (5.28). Now, combining (5.33) and (5.35), we will find an estimate of the time $T_r(x_o)$ such that,

$$\rho(x(t)) \geq 0 \quad \forall t \in [0, T_r(x_o)],\ \forall x \in \mathcal{S}_{\mathcal{H}_f}(x_o). \tag{5.36}$$

Note that the time derivative of ρ along a solution x to (5.29), starting from x_o, satisfies:

$$\frac{d}{dt}\rho(x(t)) = \langle \nabla\rho(x(t)), \dot{x}(t)\rangle \qquad \text{for a.a. } t \in \operatorname{dom} x.$$

Hence, if we integrate the previous equality from 0 to t, we get:

$$\rho(x(t)) - \rho(x_o) = \int_0^t \langle \nabla\rho(x(s)), \dot{x}(s)\rangle ds.$$

Furthermore, using (5.29) and the fact that

$$\dot{x}(s) \in F(x(s), \kappa(x_o)) \cap T_C(x(s)) \qquad \text{for a.a. } s \in [0, t],$$

we conclude that

$$\begin{aligned}
\rho(x(t)) - \rho(x_o) &\geq \int_0^t [\alpha(x(s)) - \gamma(x(s), x_o)]ds \\
&\geq \int_0^t [\alpha(x_o) + s(M_\alpha(\bar{T}, x_o) - M_\gamma(\bar{T}, x_o))]ds \\
&= \int_0^t [\alpha(x_o) + sM_r(\bar{T}, x_o)]ds \\
&= t\alpha(x_o) + \frac{t^2}{2}M_r(\bar{T}, x_o),
\end{aligned}$$

where $\bar{T} \geq t$ and $M_r(\bar{T}, x_o)$ is in (5.24). Furthermore, to provide the largest possible $T_r(x_o) \in (0, \bar{T}]$ such that (4.6) holds, we note that $t\alpha(x_o) + \frac{1}{2}t^2 M_r(\bar{T}, x_o)$

is non-negative for all $t \in [0, -2\alpha(x_o)/M_r(\bar{T}, x_o)]$ if $M_r(\bar{T}, x_o) < 0$ and for all $t \geq 0$ otherwise. Hence, we take $T_r(x_o)$ as in (5.23). Finally, it is easy to see that when (5.21) holds, Problem 2 is solved.

To prove the statement when ρ, γ, and α are locally Lipschitz, we consider an initial condition $x_o \in G := \{x \in X : |x|_{\partial X} \leq \beta\}$. The proof follows if we show that, for each solution x to (4.5) starting from x_o, we have

$$x(t) \in X \qquad \forall t \in [0, T_2].$$

According to (5.14), we conclude that

$$\begin{aligned} &\langle \zeta, f \rangle \geq \alpha(x) - \gamma(x, x_o) \\ &\qquad \forall (\zeta, f) \in \partial_C \rho(x) \times (F(x, \kappa(x_o)) \cap T_C(x)) \\ &\qquad \forall x \in X. \end{aligned} \tag{5.37}$$

Let x be a solution to (4.5) starting from x_o. Since F is locally bounded, we conclude that the solution x is locally Lipschitz. Furthermore, since the functions ρ, α, and γ are locally Lipschitz, it follows that the maps $t \mapsto \rho(x(t))$, $t \mapsto \alpha(x(t))$, and $t \mapsto \gamma(x(t), x_o)$ are locally Lipschitz on $\operatorname{dom} x$. To avoid heavy notations, we use

$$\bar{\gamma}(x(t)) := \gamma(x(t), x_o) \qquad \forall t \in \operatorname{dom} x.$$

Next, we use some non-smooth analysis tools to provide an upper and a lower bound for the maps $t \mapsto \bar{\gamma}(x(t))$ and $t \mapsto \alpha(x(t))$, respectively. Using Lemma A.0.3, for each $t \in \operatorname{dom} x$, there exists a sequence $\{t_n\}_{n=0}^{N} \subset [0, t]$ such that:

1. $N \in \{1, 2, ...\} \cup \infty$,
2. $\lim_{n \to N} t_n = t$ and $t_n - t_{n-1} > 0$, and
3. the derivative $\dot{x}(t) \in T_C(x(t))$ for each $t \in (t_{n-1}, t_n)$ for all $n \in \{1, 2, ..., N\}$.

Next, using Theorem 2.2.5, we conclude that, for each $n \in \{1, ..., N\}$, there exists $s_n \in [t_{n-1}, t_n]$, $\zeta_{\alpha n} \in \partial_C \alpha(x(s_n))$, and $\zeta_{\gamma n} \in \partial_C \bar{\gamma}(x(s_n))$ such that

$$\begin{aligned} &\bar{\gamma}(t_n) - \bar{\gamma}(t_{n-1}) = (t_n - t_{n-1})\zeta_{\gamma n} \qquad \text{and} \\ &\alpha(t_n) - \alpha(t_{n-1}) = (t_n - t_{n-1})\zeta_{\alpha n}. \end{aligned} \tag{5.38}$$

Furthermore, by Lemma 2.2.4 we conclude that

$$\partial_C \bar{\gamma}(s_n) \subset \overline{co}\{\partial_C \langle \gamma_1, x(s_n)\rangle : \gamma_1 \in \partial_C \bar{\gamma}(x(s_n))\}, \tag{5.39}$$

and

$$\partial_C \alpha(s_n) \subset \overline{co}\{\partial_C \langle \gamma_2, x(s_n)\rangle : \gamma_2 \in \partial_C \alpha(x(s_n))\}. \tag{5.40}$$

Next, using Proposition A.0.1, we conclude that

$$\partial_C \langle \gamma_1, x(s_n)\rangle \subset \langle \gamma_1, \partial_C x(s_n)\rangle, \tag{5.41}$$

and

$$\partial_C \langle \gamma_2, x(s_n)\rangle \subset \langle \gamma_2, \partial_C x(s_n)\rangle. \tag{5.42}$$

Further, using Lemmas A.0.2 and A.0.3 under the outer-semicontinuity of F, we conclude that

$$\langle \gamma_1, \partial_C x(s_n)\rangle \subset \langle \gamma_1, F(x(s_n) \cap T_C(x(s_n)))\rangle, \tag{5.43}$$

and

$$\langle \gamma_2, \partial x(s_n)\rangle \subset \langle \gamma_2, F(x(s_n) \cap T_C(x(s_n)))\rangle. \tag{5.44}$$

Combining these facts, we will be able formulate bounds on the maps $t \mapsto \gamma(x(t), x_o)$ and $t \mapsto \alpha(x(t))$. Indeed, using (5.43) and (5.39), we conclude that, for each $n \in \{1, 2, ..., N\}$,

$$\begin{aligned} \partial_C \bar{\gamma}(s_n) \subset \overline{co}\{&\partial_C \langle \gamma_1, \gamma_F\rangle : \\ &\gamma_1 \in \partial_C \bar{\gamma}(x(s_n)),\ \gamma_F \in F(x(s_n)) \cap T_C(x(s_n))\}. \end{aligned} \tag{5.45}$$

Hence,

$$\begin{aligned} \bar{\gamma}(x(t)) - \bar{\gamma}(x_o) &= \sum_{n=1}^{N} (\zeta_{\gamma n}(t_n - t_{n-1})) \\ &\leq t M_\gamma(\bar{T}, x_o), \end{aligned} \tag{5.46}$$

where $t \leq \bar{T}$, $M_\gamma(\bar{T}, x_o)$ is introduced in (5.25).

On the other hand, using (5.44) and (5.40), we conclude that, for each $n \in \{1, 2, ..., N\}$,

$$\begin{aligned} \partial_C \alpha(s_n) \subset \overline{co}\{\partial_C \langle \gamma_2, \gamma_F\rangle : &\gamma_2 \in \partial_C \alpha(x(s)), \\ &\gamma_F \in F(x(s))\}. \end{aligned} \tag{5.47}$$

Hence,

$$\begin{aligned} \alpha(x(t)) - \alpha(x_o) &= \sum_{n=1}^{N} (\zeta_{\alpha n}(t_n - t_{n-1})) \\ &\geq t M_\alpha(\bar{T}, x_o), \end{aligned} \tag{5.48}$$

where $t \leq \bar{T}$, and $M_\alpha(\bar{T}, x_o)$ is introduced in (5.26). Now, combining (5.46) and (5.48), we will find an estimate of the time $T_r(x_o) \in [0, \bar{T}]$ such that (4.6) holds. Note that the time derivatives $\frac{d}{dt}\rho(x(t))$ and $\dot{x}(t)$ exist for almost all $t \in \operatorname{dom} x$ and satisfy

$$\frac{d}{dt}\rho(x(t)) \in \{\langle \zeta, \dot{x}(t) \rangle : \zeta \in \partial_C \rho(x(t))\}. \tag{5.49}$$

The latter inclusion is true using Lemma A.0.4. Hence, for almost all $t \in \operatorname{dom} x$,

$$\frac{d}{dt}\rho(x(t)) \geq \inf\{\langle \zeta, \dot{x}(t) \rangle : \zeta \in \partial_C \rho(x(t))\}. \tag{5.50}$$

Furthermore, using Lemma A.0.3, we conclude that

$$\dot{x}(s) \in F(x(s), \kappa(x_o)) \cap T_C(x(s)) \qquad \text{for a.a. } s \in [0, t].$$

Hence, using (5.37), we conclude that

$$\begin{aligned}
\rho(x(t)) - \rho(x_o) &\geq \int_0^t [\alpha(x(s)) - \gamma(x(s), x_o)] ds \\
&\geq \int_0^t [\alpha(x_o) + s(M_\alpha(\bar{T}, x_o) - M_\gamma(\bar{T}, x_o))] ds \\
&= \int_0^t [\alpha(x_o) + s M_r(\bar{T}, x_o)] ds \\
&= t\alpha(x_o) + \frac{t^2}{2} M_r(\bar{T}, x_o),
\end{aligned}$$

where $\bar{T} \geq t$ and $M_r(\bar{T}, x_o)$ is introduced in (5.24). Furthermore, to provide the largest possible $T_r(x_o) \in (0, \bar{T}]$ such that (5.36) holds, we note that

$$t\alpha(x_o) + \frac{t^2}{2} M_r(\bar{T}, x_o) \geq 0 \quad \forall t \in \left[0, -2\frac{\alpha(x_o)}{M_r(\bar{T}, x_o)}\right] \tag{5.51}$$

provided that $M_r(\bar{T}, x_o) < 0$, and the inequality in (5.51) holds for all $t \geq 0$ when $M_r(\bar{T}, x_o) \geq 0$. Hence, we take $T_r(x_o)$ as in (5.23). Finally, it is easy to see that when (5.21) holds, Problem 2 is solved. ■

5.3 Combining the Solutions to Problems 1 and 2

In the following result, we combine the solutions to Problems 1 and 2. That is, we propose a ST control strategy for the system $\mathcal{H}_f^u$ in (4.1) that guarantees a strictly positive lower bound on the sampling period. Roughly speaking, when the system's states are deep inside the set X, the next sampling time is computed using Theorem 5.1.1. Since this result does not guarantee the existence of a uniform lower bound on inter-event times as the solutions approach ∂X_e, Theorem 5.2.8 is used to determine the next sampling time when x is close to ∂X_e.

Theorem 5.3.1 *Consider the control system $\mathcal{H}_f^u$ in (4.1) such that Assumption 4.0.1 holds. Assume that the solutions to $\mathcal{H}_f^u$ are unique for any piece-wise constant input signal $u : \mathbb{R}_{\geq 0} \to \mathbb{R}^m$. Consider a closed subset $X \subset C$ and assume that there exists a continuous feedback law $\kappa : C \to \mathbb{R}^m$ such that Assumption 5.2.1 holds. Furthermore, suppose there exist $\beta > 0$ and $T_1 > 0$ such that Assumption 5.2.3 holds. Then, given $\bar{T} > 0$, the sampling sequence $\{t_i\}_{i=0}^{\infty}$ designed recursively as*

$$t_{i+1} = t_i + \max\left\{T_s^*, T_r(x(t_i)), T_s(x(t_i))\right\}, \tag{5.52}$$

where T_s^*, T_r, and T_s are computed according to (5.21), (5.23), and (5.1), respectively, guarantees forward pre-invariance of the set X for the ST closed-

loop system in (4.4)*. Moreover, the inter-event times are always larger than a positive constant provided that* (5.22) *holds.*

Next, we relax Assumption 5.2.3 when, for some $\beta > 0$, the following assumption holds.

Assumption 5.3.2 *The set $K := \{x \in X : |x|_{\partial X_e} \geq \beta\}$ is compact.* •

We also need the following assumption.

Assumption 5.3.3 *The set-valued map $x_o \mapsto \hat{R}(\bar{T}, x_o)$, over estimating $x_o \mapsto R(\bar{T}, x_o)$ along the solutions to* (4.5)*, is outer semicontinuous and locally bounded on X.* •

Remark 5.3.4 *When the map $(x, x_o) \mapsto F(x, \kappa(x_o))$ is outer semicontinuous and locally bounded, we conclude, using [50, Proposition 1], that the map $x_o \mapsto R(\bar{T}, x_o)$ is also outer semicontinuous and locally bounded. Hence, assuming the same regularity properties to hold for the overestimated reachability map $x_o \mapsto \hat{R}(\bar{T}, x_o)$ is not very restrictive.*

Theorem 5.3.5 *Consider the control system $\mathcal{H}^u_f$ such that Assumption 4.0.1 hold. Assume that the solutions to $\mathcal{H}^u_f$ are unique for any piece-wise constant input signal $u : \mathbb{R}_{\geq 0} \to \mathbb{R}^m$. Consider a closed subset $X \subset C$ and assume that there exists a continuous feedback law $\kappa : C \to \mathbb{R}^m$ such that Assumptions 5.2.1* *and 5.3.3 holds. Then, given* $\bar{T} > 0$, *the sampling sequence* $\{t_i\}_{i=0}^{\infty}$ *designed recursively as:*

$$t_{i+1} = t_i + \max\left\{T_r(x(t_i)), T_s(x(t_i))\right\}, \tag{5.53}$$

where T_r *and* T_s *are computed according to* (5.23) *and* (5.1), *respectively, guarantees forward pre-invariance of the set* X *for the ST closed-loop system in* (4.4). *Moreover, the sampling period is always larger than a positive constant provided that there exists* $\beta > 0$ *such that Assumption 5.3.2 holds and provided that* (5.22) *holds.*

Proof. Without loss of generality, we establish the proof only for the case where the functions α, γ, and ρ are locally Lipschitz. Note that according to Theorem 5.2.8, the triggering sequence in (5.53) guarantees forward pre-invariance of the set X. Next, to complete the proof, it remains to show that, there exists $T_s^* > 0$, such that

$$\max\{T_r(x), T_s(x)\} \geq T_s^* \qquad \forall x \in X.$$

By assumption, we already know that

$$T_r(x) \geq T_2 > 0 \qquad \forall x \in G := \{x \in X : |x|_{\partial X_e} \leq \beta\}.$$

Hence, it remains to show the existence of $T_1 > 0$ such that

$$T_s(x) \geq T_1 \qquad \forall x \in K := \{x \in X : |x|_{\partial X_e} \geq \beta\}.$$

Note that the set K is compact by assumption and $T_s(x) > 0$ for all $x \in K$. Hence, to conclude the proof, we will show the existence of a function $\hat{T}_s : K \to \mathbb{R}_{>0}$ such that

$$\hat{T}_s(x) \leq T_s(x) \qquad \forall x \in X$$

and $\hat{T}_s$ is lower semicontinuous.

Claim 5.3.6 *The function*

$$\hat{T}_s(x) := \begin{cases} \bar{T} & \text{if } \hat{M}_s(\bar{T}, x_o) \leq 0 \\ \min\left\{\bar{T}, \rho(x_o)/\hat{M}_s(x_o, \bar{T})\right\} & \text{otherwise,} \end{cases}$$

where

$$\hat{M}_s(x_o, \bar{T}) := \sup\{\langle -\gamma, \eta\rangle :$$
$$\gamma \in \partial_C \rho(y),\ \eta \in F(y, \kappa(x_o)),\ y \in \hat{R}(\bar{T}, x_o)\},$$

is lower semicontinuous.

Proof. We will start showing upper semicontinuity and local boundedness of the map $x \mapsto \hat{M}(x, \bar{T})$ using Lemma A.0.6. Indeed, note that $\hat{M}$ can re-expressed as the following marginal function:

$$f(x_o) := \hat{M}_s(x_o, \bar{T}) :=$$
$$\sup\left\{ \langle -\gamma, \eta\rangle : y := \begin{bmatrix} \gamma \\ \eta \end{bmatrix} \in \begin{bmatrix} \partial_C \rho(\hat{R}(\bar{T}, x_o)) \\ F(\hat{R}(\bar{T}, x_o), \kappa(x_o)) \end{bmatrix} \right\}.$$

Next, we note that $g(y) := \langle -\gamma, \eta\rangle$ is continuous and

$$\Pi(x_o) := \begin{bmatrix} \partial_C \rho(\hat{R}(\bar{T}, x_o)) \\ F(\hat{R}(\bar{T}, x_o), \kappa(x_o)) \end{bmatrix}$$

is outer semicontinuous and locally bounded using (P4), Lemma A.0.6, and Remark 3.1.3. Now, since $x \mapsto \hat{M}_s(x, \bar{T})$ is upper semicontinuous and ρ is non-negative and continuous on X, we conclude that $x_o \mapsto \rho(x_o)/\hat{M}_s(x_o, \bar{T})$ is lower semicontinuous.

Finally, to show that $\hat{T}_s$ is lower semicontinuous, we consider a sequence $\{x_i\}_{i=0}^{\infty} \subset X$ that converges to $x_o \in X$. Since $x \mapsto \hat{M}_s(x, \bar{T})$ is upper semicontinuous, we conclude that $\limsup_{i \to \infty} \hat{M}_s(\bar{T}, x_i) \leq \hat{M}_s(\bar{T}, x_o)$. Furthermore, by passing to an adequate subsequence, we can assume without loss of generality that

$$\liminf_{i \to \infty} \hat{T}_s(x_i) = \lim_{i \to \infty} \hat{T}_s(x_i) = \alpha > 0.$$

Also, since $\hat{M}_s$ is locally bounded, by passing to another subsequence, we conclude the existence of $\beta \in \mathbb{R}$ such that

$$\lim_{i \to \infty} \hat{M}_s(x_i, \bar{T}) = \beta \leq \hat{M}_s(\bar{T}, x_o).$$

Next, we distinguish the following scenarios:

- When $\beta < 0$, we conclude that $\hat{M}_s(\bar{T}, x_i) < 0$ for all $i \in \mathbb{N}$ sufficiently large. In this case, $\hat{T}_s(x_i) = \bar{T}$ for all $i \in \mathbb{N}$ sufficiently large; thus, $\lim_{i \to \infty} \hat{T}_s(x_i) = \bar{T} \geq \hat{T}_s(x_o)$.

- When $\beta > 0$, we conclude that $\hat{M}_s(\bar{T}, x_i) > 0$ for all $i \in \mathbb{N}$ sufficiently large. In this case, we conclude that

$$\hat{T}_s(x_i) = \min\left\{\bar{T}, \rho(x_i)/\hat{M}_s(x_i, \bar{T})\right\}$$

for all $i \in \mathbb{N}$ sufficiently large. Hence,

$$\lim_{i\to\infty} \hat{T}_s(x_i) = \min\left\{\bar{T}, \lim_{i\to\infty} \rho(x_i)/\hat{M}_s(x_i,\bar{T})\right\}$$
$$\geq \min\left\{\bar{T}, \rho(x_o)/\hat{M}_s(x_o,\bar{T})\right\} \geq \hat{T}_s(x_o).$$

- When $\beta = 0$, we conclude that $|\rho(x_i)/\hat{M}_s(\bar{T},x_i)| \geq \bar{T}$ for all $i \in \mathbb{N}$ sufficiently large. Hence, $\hat{T}_s(x_i) = \bar{T}$ for all $i \in \mathbb{N}$ sufficiently large.

■

Next, we propose to relax some of the requirements in Theorem 5.3.5. That is, we would like to check an inequality similar to (5.22) only at points in the set $\partial X_e \cap X$. To do so, in addition to Assumptions 5.2.1, 5.3.2, and 5.3.3, for some $\beta^* > 0$ and for $G^* := \{x \in X : |x|_{\partial X_e} \leq \beta^*\}$, we consider the following assumption.

Assumption 5.3.7 *One of the following is true:*

1. *The set G^* is compact.*

2. *The set-valued maps*

$$\begin{aligned} &x_o \mapsto \partial_C \gamma(\hat{R}(\bar{T},x_o),x_o), \quad x_o \mapsto \partial_C \alpha(\hat{R}(\bar{T},x_o)), \\ &x_o \mapsto F(\hat{R}(\bar{T},x_o),\kappa(x_o)), \quad x_o \mapsto \alpha(x_o) \end{aligned} \tag{5.54}$$

are uniformly upper semicontinuous on G and bounded on $\partial X_e \cap X$. Furthermore,

$$T_3 := \inf\{\hat{T}_r(z) : z \in \partial X_e \cap X,\ \hat{M}_r(z, \bar{T}) > 0\} > 0, \tag{5.55}$$

$$\inf\{\alpha(z) : z \in \partial X_e \cap X\} > 0. \tag{5.56}$$

1) The set G^* is compact.

2) The set-valued maps

$$\begin{aligned} &x_o \mapsto \partial_C \gamma(\hat{R}(\bar{T}, x_o), x_o), \quad x_o \mapsto \partial_C \alpha(\hat{R}(\bar{T}, x_o)), \\ &x_o \mapsto F(\hat{R}(\bar{T}, x_o), \kappa(x_o)), \quad x_o \mapsto \alpha(x_o) \end{aligned} \tag{5.57}$$

are uniformly upper semicontinuous on G^* and bounded on $\partial X_e \cap X$. Furthermore,

$$T_3 := \inf\{\hat{T}_r(z) : z \in \partial X_e \cap X,\ \hat{M}_r(\bar{T}, z) > 0\} > 0, \tag{5.58}$$

$$\inf\{\alpha(z) : z \in \partial X_e \cap X\} > 0, \tag{5.59}$$

where $\hat{M}_r(\bar{T}, z) := \sup\{\langle \gamma_1, \eta \rangle : \gamma_1 \in \partial_C \gamma(y, z), \eta \in F(y, \kappa(z)), y \in \hat{R}(\bar{T}, z)\} + \sup\{\langle -\gamma_2, \eta \rangle : \gamma_2 \in \partial_C \alpha(y), \eta \in F(y, \kappa(z)),\ y \in \hat{R}(\bar{T}, z)\}$, $\hat{T}_r(z) := \min\left\{\bar{T}, \frac{2\alpha(z)}{\hat{M}_r(\bar{T}, z)}\right\}$.

•

For the second item in Assumption 5.3.7, the lower bound in (5.58) is key to build a uniform lower bound on the inter-event times, via the regularity properties in (5.57) and (5.59).

Remark 5.3.8 *When the sets G^* and $\partial X_e \cap X$ are bounded and Assumption 5.3.3 holds, the regularities of the maps in* (5.57) *assumed in the second item of Assumption 5.3.7 hold for free. Indeed, each upper semicontinuous set-valued map is uniformly upper semicontinuous on a compact set. Also, each continuous map is uniformly continuous on a compact set.*

Theorem 5.3.9 *Consider the control system $\mathcal{H}_f^u$ such that Assumption 4.0.1 holds. Assume that the solutions to $\mathcal{H}_f^u$ are unique for any piece-wise constant input signal $u : \mathbb{R}_{\geq 0} \to \mathbb{R}^m$. Consider a closed subset $X \subset C$ and assume that there exists a continuous feedback law $\kappa : C \to \mathbb{R}^m$ such that Assumptions 5.2.1 and 5.3.3 hold. Furthermore, let $\beta^* > 0$ such that Assumption 5.3.7 holds. Then, given $\bar{T} > 0$, the sampling sequence $\{t_i\}_{i=0}^{\infty}$ designed in* (5.53) *guarantees forward pre-invariance of X for the ST closed-loop system. Moreover, the inter-event times are always larger than a positive constant provided that Assumption 5.3.2 holds for some $\beta \in (0, \beta^*]$.*

Proof. To prove the statement, it is enough to show that, under Assumptions 5.3.3 and 5.3.7, there exists $\beta > 0$ such that (5.22) holds. The rest of the proof would follow using Theorem 5.3.5. To find a contradiction, we assume the existence of a sequence $\{x_i\}_{i=0}^{\infty}$ such that

$$\lim_{i \to \infty} |x_i|_{\partial X} = 0 \text{ and } \lim_{i \to \infty} T_r(x_i) = 0. \tag{5.60}$$

Let

$$z_i := \arg\min\{|x_i - z| : z \in \partial X\} \quad \forall i \in \mathbb{N}.$$

Note that $\lim_{i\to\infty} |x_i - z_i| = 0$. Furthermore, using (5.60), we conclude that

$$T_r(x_i) = 2\alpha(x_i)/M_r(x_i, \bar{T}) \tag{5.61}$$

for all $i \in \mathbb{N}$ sufficiently large. Furthermore, consider the function $\hat{T}_r : G^* \to \mathbb{R}_{\geq 0}$ given by

$$\hat{T}_r(x_o) := \begin{cases} \bar{T} & \text{if } \hat{M}_r(\bar{T}, x_o) \geq 0 \\ \min\left\{\bar{T}, \frac{2\alpha(x_o)}{\hat{M}_r(\bar{T}, x_o)}\right\} & \text{otherwise,} \end{cases}$$

where

$$\hat{M}_r(x_o, \bar{T}) := \hat{M}_\alpha(x_o, \bar{T}) + \hat{M}_\gamma(x_o, \bar{T}),$$

$$\hat{M}_\alpha(x_o, \bar{T}) := \sup\left\{ \langle -\gamma_1, \eta \rangle : \begin{bmatrix} \gamma_1 \\ \eta \end{bmatrix} \in \Pi_\alpha(x_o) := \begin{bmatrix} \partial_C \alpha(\hat{R}(\bar{T}, x_o)) \\ F(\hat{R}(\bar{T}, x_o), \kappa(x_o)) \end{bmatrix} \right\}.$$

$$\hat{M}_\gamma(x_o, \bar{T}) := \sup\left\{ \langle \gamma_2, \eta \rangle : \begin{bmatrix} \gamma_2 \\ \eta \end{bmatrix} \in \Pi_\gamma(x_o) := \begin{bmatrix} \partial_C \gamma(\hat{R}(\bar{T}, x_o), x_o) \\ F(\hat{R}(\bar{T}, x_o), \kappa(x_o)) \end{bmatrix} \right\}.$$

Note that

$$\hat{M}_r(x_o, \bar{T}) \geq M_r(x_o, \bar{T}) \qquad \forall x_o \in G^*$$

and, for all $i \in \mathbb{N}$ sufficiently large,

$$\hat{T}_r(x_i) \leq T_r(x_i).$$

Hence, $\lim_{i\to\infty} \hat{T}_r(x_i) = 0$ and, for all $i \in \mathbb{N}$ sufficiently large, $\hat{T}_r(x_i) > 0$.

The contradiction follows if we show the following property:

($\star$) There exists $\beta > 0$ such that, for each $\epsilon > 0$ sufficiently small, there exists $i_o \in \mathbb{N}$ such that, for all $i \geq i_o$,

$$\hat{T}_r(x_i) \geq \beta - \epsilon. \tag{5.62}$$

Each of the cases in Assumption 5.3.7 will now be considered separately.

1. When the set G^* is bounded, we use Claim 5.3.6 to conclude that $\hat{T}_r$ is lower semicontinuous and strictly positive on G^*. Hence,

$$\hat{T}_r(x_i) \geq \hat{T}_r(z_i) - \epsilon$$

and

$$\inf\{\hat{T}_r(z_i) : i \in \mathbb{N}\} \geq \beta > 0,$$

which proves ($\star$).

2. Now, when the set G^* is not bounded, the second item in Assumption 5.3.7 is considered and we conclude that the maps Π_γ and Π_α are uniformly upper semicontinuous on G^*. Hence, for every $\epsilon_1 > 0$, there exists $i_1 \in \mathbb{N}$ such that, for all $i \geq i_1$, we have

$$\Pi_\gamma(x_i) \subset \Pi_\gamma(z_i) + \epsilon_1\mathbb{B} \text{ and } \Pi_\alpha(x_i) \subset \Pi_\alpha(z_i) + \epsilon_1\mathbb{B}.$$

Hence, for all $i \geq i_1$,

$$\hat{M}_\alpha(x_i, \bar{T}) \leq \sup \left\{ \langle -\gamma_1, \eta \rangle : \begin{bmatrix} \gamma_1 \\ \\ \eta \end{bmatrix} \in \Pi_\alpha(z_i) + \epsilon_1 \mathbb{B} \right\}.$$

$$\hat{M}_\gamma(x_i, \bar{T}) \leq \sup \left\{ \langle \gamma_2, \eta \rangle : \begin{bmatrix} \gamma_2 \\ \\ \eta \end{bmatrix} \in \Pi_\gamma(z_i) + \epsilon_1 \mathbb{B} \right\}.$$

Furthermore, using Assumption 5.3.7, we conclude that the maps Π_γ and Π_α are bounded on $\partial X_e \cap X$. Hence, for all $\epsilon_2 > 0$, there exists $i_2 \in \mathbb{N}$ such that, for all $i \geq i_2$,

$$0 < \hat{M}_r(x_i, \bar{T}) \leq \hat{M}_r(z_i, \bar{T}) + \epsilon_2.$$

Finally, since $x_o \mapsto \alpha(x_o)$ is uniformly continuous, we conclude that, for all $\epsilon_2 > 0$ sufficiently small, there exists $i_3 \in \mathbb{N}$ such that, for all $i \geq i_3$,

$$\alpha(x_i) \geq \alpha(z_i) - \epsilon_2 \geq 0.$$

Hence, for each $\epsilon_2 > 0$ sufficiently small, there exists $i_4 \in \mathbb{N}$ such that, for all $i \geq i_4$,

$$\begin{aligned} \frac{\alpha(x_i)}{\hat{M}_r(x_i)} &\geq \frac{\alpha(z_i) - \epsilon_2}{\hat{M}_r(z_i) + \epsilon_2} \\ &= \frac{\alpha(z_i)}{\hat{M}_r(z_i)} - \epsilon_2 \frac{\alpha(z_i) + \hat{M}_r(z_i, \bar{T})}{\hat{M}_r(z_i, \bar{T})(\hat{M}_r(z_i, \bar{T}) + \epsilon_2)}. \end{aligned}$$

Note that under (5.59) and (5.61), $\hat{M}_r(z_i)$ needs to is always bigger than a positive constant. Hence, the term that $\frac{\alpha(z_i) + \hat{M}_r(z_i, \bar{T})}{\hat{M}_r(z_i, \bar{T})(\hat{M}_r(z_i, \bar{T}) + \epsilon_2)}$ is uniformly bounded. Finally, using (5.58) to conclude that

$$\inf\{\hat{T}_r(z_i) : i \in \mathbb{N}\} \geq \beta > 0.$$

and , which proves ($\star$).

$\blacksquare$

Remark 5.3.10 *Examples where Assumption 5.3.2 holds, even though the set X is unbounded, are when X is a temporal funnel that shrinks asymptotically in time towards a target set. Hence, we guarantee convergence to the target by guaranteeing forward pre-invariance of the funnel X, see Chapter 6.1.*

Remark 5.3.11 *Note that we can design the value of $\bar{T}$ in* (5.52) *and* (5.53) *as a function of $x(t_i)$. Indeed, although a large value of $\bar{T}$ seems to allow for a large sampling period when looking at* (5.52) *and* (5.53), *it increases the size of the overestimated reachable set $\hat{R}(x_o, \bar{T})$ and; thus, it can play a negative role on the size of the sampling period according to* (5.2) *or* (5.24). *Hence, an adequate scaling of $\bar{T}$ as function of $x(t_i)$ should encourage a large $\bar{T}$ when the system is slow and vice versa, see Example 6.2.*

6

Examples

In this chapter , we illustrate the ST control strategy proposed in Theorem 5.3.9 on two different examples[1]. In the first example, the set X is time dependent, as in Remark 5.3.10, and it represents a temporal funnel [51]. In funnel-control techniques, convergence properties with a prescribed transient behavior are recast into forward invariance of an appropriate temporal funnel X. In our case, we show, on a simple example, how to render a temporal funnel X forward invariant using Theorem 5.3.9. Note that the set X is unbounded in this case. In the second example, we consider a linear control system, where the set X is a sub-level set of a given quadratic function. The objective in this example is to compare the inter-event times obtained using Theorem 5.3.9 with respect to two other methods. One of the methods is an event-triggered

[1] Code at `https://github.com/HybridSystemsLab/SelfTriggeredFunnel`, and `https://github.com/HybridSystemsLab/SelfTriggeredSublevelSet`.

strategy. The other method, used for comparison, is based on the ST strategy proposed in [28, Theorem 4.3].

6.1 Example 1: ST Funnel Control

Consider the augmented control system

$$\mathcal{H}^{u}_{f_a} = (F_a, C_a)$$

with $x_a := (t, x) \in C_a := \mathbb{R}_{\geq 0} \times \mathbb{R}$ and

$$F(x_a, u) = (1, u)^{\top}.$$

Consider the temporal funnel delimited by the functions $\psi_1 : \mathbb{R}_{\geq 0} \mapsto \mathbb{R}$ and $\psi_2 : \mathbb{R}_{\geq 0} \mapsto \mathbb{R}$. with

$$\psi_1(t) := \left(-4t^2 + 2\right) \exp(-t) + 0.1,$$

$$\psi_2(t) := (-3.5 - \exp(-t)) \exp(-0.1t).$$

Note that $\psi_1(t) > \psi_2(t)$ for all $t \in \mathbb{R}_{\geq 0}$. Next, the temporal funnel is then given by

$$X := \{x_a \in C_a : x \in [\psi_2(t), \psi_1(t)]\}.$$

Hence, the set X admits the barrier function candidate $\rho(x_a) := \rho_1(x_a)\rho_2(x_a)$, where

$$\rho_1(x_a) := \psi_1(t) - x$$

$$\rho_2(x_a) := x - \psi_2(t).$$

The control law

$$\kappa(x_a) := \lambda(x_a)(\dot{\psi}_2(t) + d) + (1 - \lambda(x_a))(\dot{\psi}_1(t) - d)$$

is proposed to render X forward invariant for the non-triggered closed-loop system, where $d := 0.4$ and

$$\lambda(x_a) := (\psi_1(t) - x)/(\psi_1(t) - \psi_2(t))$$

for all $x_a \in \mathbb{R}_{\geq 0} \times \mathbb{R}$. Note that, along the solutions to (4.5),

$$\dot{\rho}(x_a) = \alpha(x_a) = d(\psi_1(t) - \psi_2(t)) > 0 \quad \forall x_a \in \partial X. \tag{6.1}$$

Hence, condition (5.14) is satisfied with

$$\alpha(x_a) := \dot{\psi}_1(t)\rho_2(x_a) - \dot{\psi}_2(t)\rho_1(x_a) + \kappa(x_a)(\rho_1(x_a) - \rho_2(x_a)),$$

and

$$\gamma(x_a, \eta) := (\rho_1(x_a) - \rho_2(x_a))(\kappa(\eta) - \kappa(x_a)).$$

In addition, the set $\hat{R}(\bar{T}, x_{ao})$, for $x_{ao} := (t_o, x_o)$, is computed as

$$\hat{R}(\bar{T}, x_{ao}) := R(\bar{T}, x_{ao}) + \epsilon\mathbb{B},$$

where

$$R(\bar{T}, x_{ao}) = \{(t_o + s, x_o + s\kappa(x_{ao})) : \forall s \in [0, \bar{T}]\}.$$

We consider $\epsilon := 0.25$ to capture possible errors in the modeling or the presence of perturbations.

Next, we will show that Assumptions 4.0.1, 5.3.2, 5.3.3, and 5.3.7 are satisfied for this example. Indeed, Assumption 4.0.1 is trivially satisfied since F_a is single valued and smooth. Assumption 5.3.2 is verified since the funnel X shrinks asymptotically in time towards a neighborhood of the set $x = 0$. Also, Assumption 5.3.3 is verified using Remark 5.3.4 under the fact that $\hat{R} \equiv R + \epsilon\mathbb{B}$. Finally, to verify the second item in Assumption 5.3.7, we use (6.1) to conclude that (5.59) holds. Furthermore, multiple values of $\bar{T}$ are simulated to determine the largest T_2 such that (5.58) holds.

According to Figure 6.1, we obtained $T_2 := 0.10$ for $\bar{T} := 1.5$.

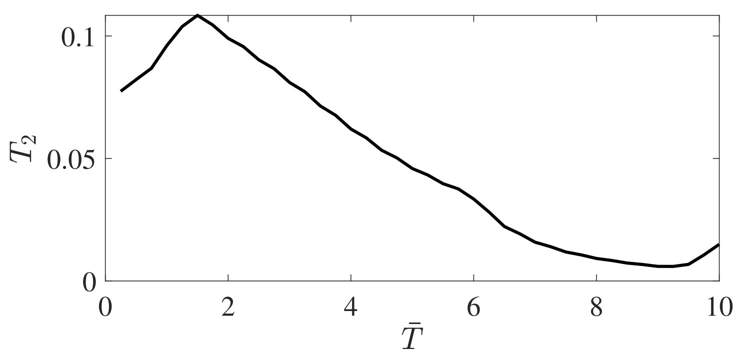

Figure 6.1: Dependence of T_2 on $\bar{T}$

Next, to verify the required regularity properties for the maps in (5.57), we let $\eta_a := (s, \eta) \in \mathbb{R}_{\geq 0} \times \mathbb{R}$ and $x_a := (t, x) \in \mathbb{R}_{\geq 0} \times \mathbb{R}$. Note that the map $(x_a, \eta_a) \mapsto F(\eta_a, \kappa(x_a))$ is continuously differentiable and it is bounded with a bounded derivative whenever x and η are bounded. Hence, the map $x_a \mapsto \hat{R}(\bar{T}, x_a)$ is bounded on X, globally Lipschitz on X; thus, uniformly upper semicontinuous on X. Finally, we note that the maps $(x_a, \eta_a) \mapsto \nabla\gamma(\eta_a, x_a)$, $x_a \mapsto \nabla\alpha(x_a)$, $x_a \mapsto \alpha(x_a)$ are continuously differentiable and bounded with a bounded derivative whenever x and η are bounded. Hence, they are uniformly upper semicontinuous whenever x and η are bounded. This implies that the maps in (5.57) verify the required properties.

For the initial condition $x_{ao} := (0, 0.5)$, the solution of the ST closed-loop system and the inter-event period are provided in Figure **??**. The average and minimum inter-event periods of the solution are 0.80 and 0.17, respectively.

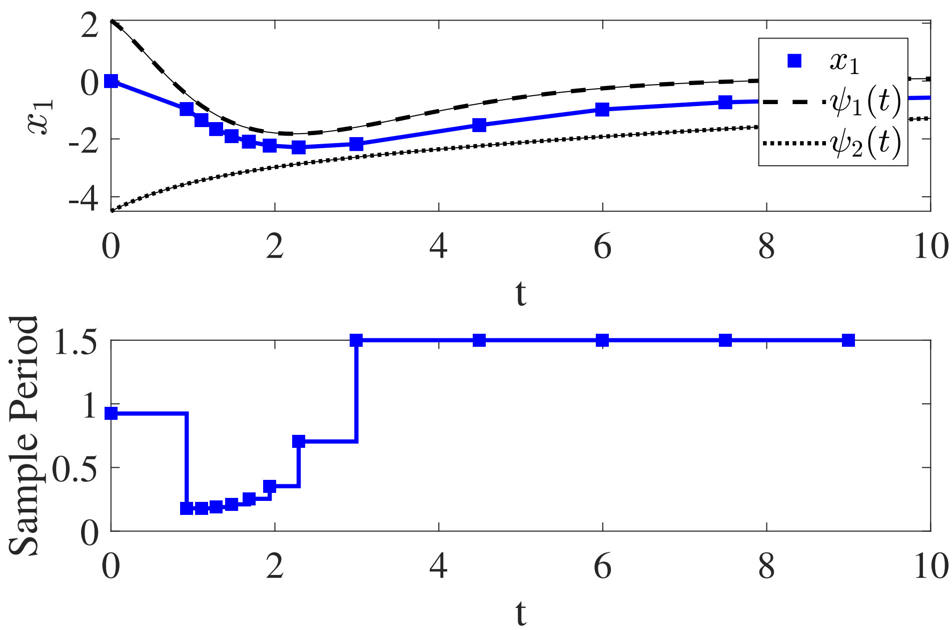

Figure 6.2: ST closed-loop solution with sampling instances marked.

6.2 Example 2: Comparison to Existing Methods

Consider the control system $\mathcal{H}_f^u = (F, \mathbb{R}^2)$, where

$$F(x,u) := \begin{bmatrix} 0 & 1 \\ -2 & 3 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u, \tag{6.2}$$

$x := (x_1, x_2) \in \mathbb{R}^2$, and $u \in \mathbb{R}$. Furthermore, consider the feedback law $\kappa(x) := Kx := [1 \quad -4]x$. The origin of the closed-loop system $\mathcal{H}_f^u$ using $u = \kappa(x)$, denoted $\mathcal{H}_f^{cl}$, is asymptotically stable. Indeed, using the Lyanpunov function

$$V(x) := x^\top P x,$$

with $P := \begin{bmatrix} 1 & 0.25 \\ 0.25 & 1 \end{bmatrix}$, we conclude that

$$\langle \nabla V(x), Ax + BKx \rangle = -x^\top Q x$$

and $Q := \begin{bmatrix} 0.5 & 0.25 \\ 0.25 & 1.5 \end{bmatrix}$.

Furthermore, we consider the set X given by

$$X := \{x \in \mathbb{R}^2 : V(x) \leq 0.1\}.$$

We propose to compare the following strategies.

1- Using an ET strategy: In this approach, we assume that the state x is measured all the time. Hence, the input is updated according to $u = \kappa(x)$

only when the solutions approach the boundary of X. More precisely, given $x_o \in X$, we design the triggering sequence $\{t_i\}_{i=0}^{\infty}$ recursively as follows:

$$t_{i+1} = t_i + \max\{t \geq 0 : V(x(s + t_i)) \leq 0.09 \ \ \forall s \in [0, t]\},$$

where x is the solution to ST closed-loop system starting from x_o using the triggering sequence $\{t_k\}_{k=0}^{i}$.

2- Using the strategy in [28, Theorem 4.3]: Consider the set $\mathcal{B}_\delta := \{x \in \mathbb{R}^2 : |x| < \delta\}$, and let $\delta := \max\{|x| : x \in X\} = 0.364$. Furthermore, let $d(x, x_o) := BK(x_o - x)$. Along the solution x to $\dot{x} = F(x, Kx_o)$ starting from x_o, we let $M_1(x_o) := |\dot{d}(x_o, x_o)|$ and

$$M_2(x_o) := \max\left\{|\ddot{d}(\bar{x}, x_o)| : \bar{x} \in X, \ V(\bar{x}) \leq V(x_o)\right\}. \tag{6.3}$$

Furthermore, note that

$$\lambda_{min}(P)|x|^2 \leq V(x) \leq \lambda_{max}(P)|x|^2,$$

$$\langle \nabla V(x), Ax + BKx \rangle \leq -\lambda_{min}(Q)|x|^2,$$

and

$$|\nabla V(x)| \leq \lambda_{max}(P)|x|.$$

Using [28, Theorem 4.3] and given $x_o \in X$, we can design the triggering sequence $\{t_i\}_{i=0}^{\infty}$ recursively to satisfy:

$$t_{i+1} = t_i + \max\{t \geq 0 :$$

$$M_1(x(t_i))t + M_2(x(t_i))t^2 \leq$$

$$\nu\delta\lambda_{min}(Q)\lambda_{min}(P)/\lambda_{max}(P)^2\},$$

for $\nu := 0.85$ tuned to provide the best result and where x is the solution to the ST closed-loop system starting from x_o using the triggering sequence $\{t_k\}_{k=0}^{i}$.

Remark 6.2.1 *One similarity between the strategy in Theorem 5.3.1 and the one in [28, Theorem 4.3] is that, from an initial condition $x_o \in X$, the next sampling time is computed based on upper bounding a function over a set including x_o. This set is $\{x \in X : V(x) \leq V(x_o)\}$ in [28, Theorem 4.3] and is $\hat{R}(\bar{T}, x_o)$ in Theorem 5.3.1, see* (6.3) *and* (5.2)*, respectively. In* (6.3)*, the upper bounded function is the norm of $\ddot{d}$ but, in* (5.2)*, it the scalar product between $\nabla \rho$ and F. When using such a scalar product, we account only for the component of F along directions pointing outside the set X. However, in* (6.3)*, we account for all the components of F. On the other hand, the evaluation along (an overestimated) reachable set is less restrictive than on the entire sub-level set.*

3- Using the strategy in Theorem 5.3.9:

Note that the set X admits the barrier function candidate

$$\rho(x) := 0.1 - V(x).$$

Also note that condition (5.14) is satisfied with

$$\alpha(x) := x^\top Q x$$

and

$$\gamma(x, \eta) := \frac{1}{2} x^\top P B K (x - \eta).$$

Next, the overestimation $\hat{R}$ from $x_o \in X$ along $[t_o, t_o + \bar{T}]$ is computed as

$$\hat{R}(\bar{T}, x_o) := R(\bar{T}, x_o) + r\mathbb{B},$$

where

$$R(\bar{T}, x_a) = \{y \in \mathbb{R}^2 : \exists t \in [0, \bar{T}] : y = x(t)\},$$

and x is the solution to (4.5) starting from x_o and $r = 0.025$ captures possible errors in the modeling or the presence of perturbations.

Note that Assumption 4.0.1 is trivially satisfied since F_a is single valued and smooth. Assumption 5.3.2 is satisfied since the set X is compact. Also, Assumption 5.3.3 is verified using Remark 5.3.4 and the fact that $\hat{R} \equiv R + \epsilon\mathbb{B}$. Finally, Assumption 5.3.7 is satisfied since Assumption 5.3.3 is satisfied and X is compact. As discussed in Remark 5.3.11, when computing $M(x_o, \bar{T})$ for (5.52), we propose two strategies for selecting the value of $\bar{T}$ as a function of x_o. To do so, we first re-express the functions T_s and T_r in (5.23) and (5.1), respectively, as functions of both x_o and $\bar{T}$; namely, $T_s(x_o) := T_s(x_o, \bar{T})$ and $T_r(x_o) := T_r(x_o, \bar{T})$.

a. Adapting $\bar{T}$ to the norm of $F(x_o, \kappa(x_o))$: We tested both linear and non-linear relationships between $\bar{T}$ and $|F(x_o, \kappa(x_o))|$. Indeed, consider the map $\bar{T} : \mathbb{R}^n \mapsto [T_{min}, T_{max}]$ given by

$$\bar{T}(x_o) := (T_{max} - T_{min})(1 - F_N(x_o))^{c_s} + T_{min}, \tag{6.4}$$

where $c_s \in (0, \infty)$, $T_{max} > T_{min} > 0$, and

$$F_N(x_o) := |F(x_o, \kappa(x_o))| \,/\, \sup\{|F(y, \kappa(y))| : y \in X\}.$$

For example, for $T_{max} = 2$ and $T_{min} = 0.25$, Figure 6.3 illustrates the nonlinear scaling when $c_s = 150$.

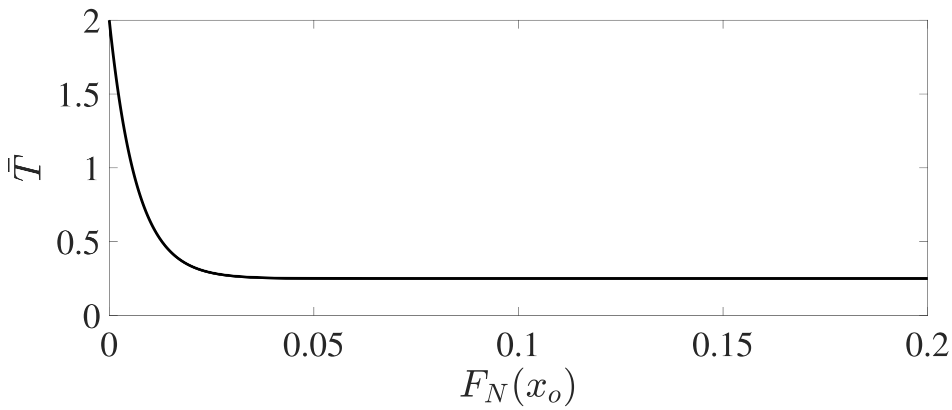

Figure 6.3: Nonlinear scaling of $\bar{T}$ when $c_s = 150$

b. Evaluating multiple values of $\bar{T}$ over a multiple-step receding horizon: Given $T_{max} > T_{min} > 0$, $N \in \mathbb{N}$, and $\Delta := (T_{max} - T_{min})/N$, we are interested in finding the value of $n \in \{0, 1, ..., N\}$ that maximizes the value function

$$J_n := c_h T_n^o + (1 - c_h) T_n^1,$$

where $c_h \in [0, 1]$, $T_n^o := f(n, x_o)$,

$$T_n^1 := \max\{f(m, x_1) : m \in \{0, ...N\},\ x_1 = x(T_{min} + n\Delta)\},$$

and

$$f(k, y) := \max\{T_r(T_{min} + k\Delta, y), T_s(T_{min} + k\Delta, y)\}.$$

The constant c_h adjusts the trade-off between the current sample time T_n^o and the best next sample time T_n^1.

This strategy is implemented according to Algorithm 1.

When $c_h = 1$ the value function J_n is equal to the next sampling period. However, maximizing the next sample period does not necessary lead to large sample periods in average. However, when $c_h \in (0, 1)$, a weighted average between the next two inter-events periods should yield to a better average sampling period.

Comparison: All the solutions are simulated from the initial condition $x_o := (-0.1, -0.3)$.

Figures 6.4 and 6.5 show the evolution of V along a ST closed-loop solution as well as the corresponding inter-events period. Both strategies use $T_{max} := 2$ and $T_{min} := 0.25$. In Figure 6.4, the value of $\bar{T}$ is adapted to the norm of $F(x_o, \kappa(x_o))$ as in (6.4). Furthermore, two different values of $c_s \in \{1, 150\}$ are tested. From Figure 6.4, we can see that the nonlinear scaling ($c_s = 150$) adapts the value of $\bar{T}$ better than the linear one $c_s = 1$. In Figure 6.5, the value of $\bar{T}$ is computed following Algorithm 1. Furthermore, two different values of $c_h \in \{0.5, 1\}$ are tested.

Figure 6.6 shows the evolution of V along a ST closed-loop solution as well as the corresponding inter-event times. Two strategies based on Theorem 5.3.9 are simulated. In the first one, the value of $\bar{T}$ is adapted to the norm of

$F(x_o, \kappa(x_o))$ as in (6.4), where the best performance is obtained for $c_s = 150$. Then, the value of $\bar{T}$ is computed following the two-steps receding horizon, where the best performance is obtained for $c_h = 0.5$. For these two methods, we took $T_{max} := 2$ and $T_{min} := 0.25$.

Furthermore, Figure 6.6 compares the strategies proposed in this thesis to an ET strategy, in which, we update the control input each time the solution reaches ∂X. We also compare to the ST strategy proposed in [28, Theorem 4.3]. The inter-event times obtained using [28, Theorem 4.3] are smaller than those obtained using Theorem 5.3.9. Remarkably, according to Table 6.1, our results are comparable to the ET strategy although our strategy does not require continuous availability of the measurements.

We observe that, for the strategy in [28, Theorem 4.3], the sampling periods are typically smaller compared to the other ones because of the reasons highlighted in Remark 6.2.1. Note that evaluating $\bar{T}$ over a two-step horizon yields to the best average sampling period compared to all the other methods, see Table 6.1.

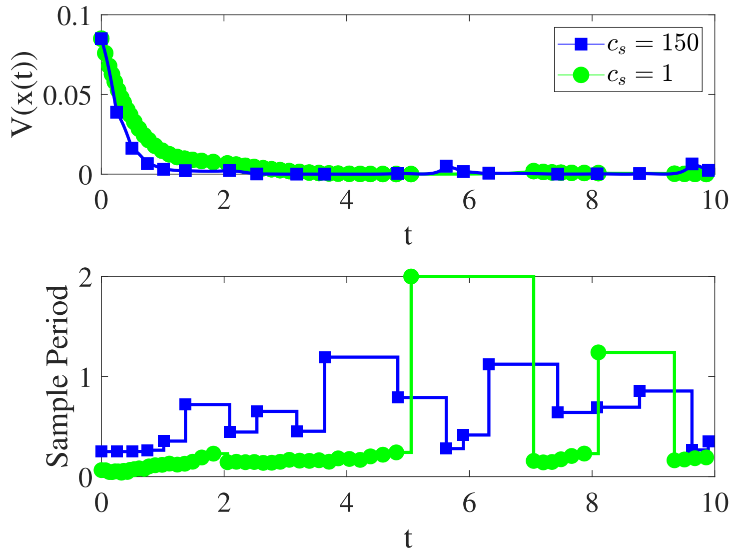

Figure 6.4: Computation of $\bar{T}$ using (6.4)

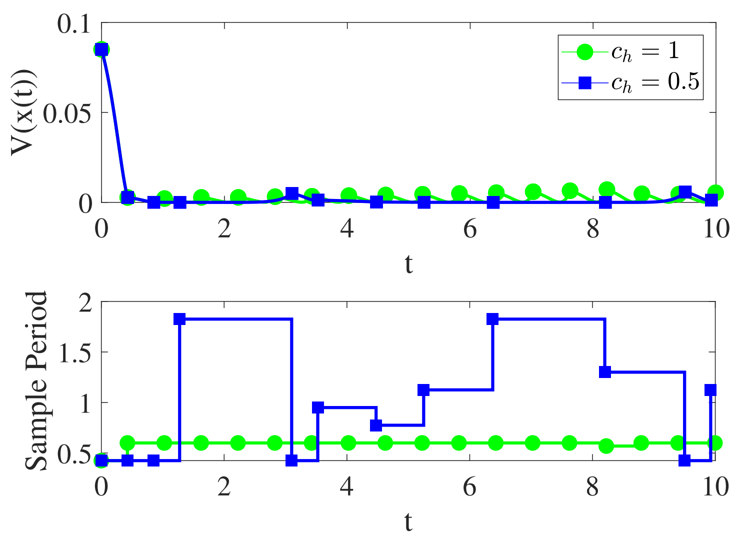

Figure 6.5: Computation of $\bar{T}$ using Algorithm 1.

Algorithm 1: Computing $\bar{T}$ over a two step horizon

```
input  : x_o, T_max, T_min, N
output: T̄
Δ = (T_max − T_min)/N;
for n ∈ {0, ..., N} do
    T_n^o = f(n, x_o);
    J_n = 0 ;
    x_1 = x(T_min + nΔ);
    for m ∈ {0, ..., N} do
        T_n^1 = f(m, x_1);
        J = c_h T_n^o + (1 − c) T_n^1;
        if J > J_n then
            J_n = J;
        end
    end
    n̄ = arg max{J_n : n ∈ {0, ..., N}};
    T̄ = T_min + n̄Δ;
end
```

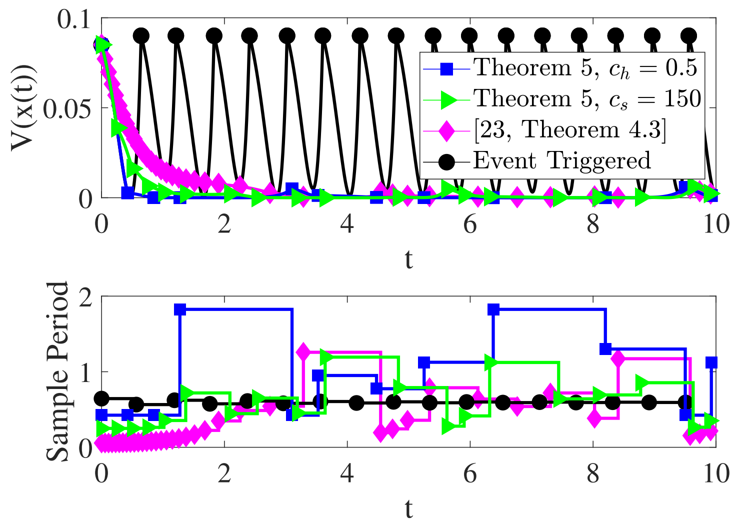

Figure 6.6: Closed-loop solutions using various strategies.

Theorem 5.3.9	Average Period	Minimum Period
Scaled		
$c_s = 1$	0.19	0.04
$c_s = 150$	0.53	0.25
Receding horizon		
$c_h = 1$	0.58	0.425
$c_h = 0.5$	0.92	0.425
[28, Theorem 4.3]	0.26	0.06
Event Triggered	0.59	0.56

Table 6.1: Summary of each inter-event properties

7

Conclusion

In this thesis , we present a self-triggered control strategy to guarantee forward pre-invariance of a closed set for a constrained differential inclusion. Sufficient conditions are derived such that the inter-event times are guaranteed to be always larger than a positive constant. Two numerical examples are provided to illustrate the effectiveness of this approach. For future work, we propose to use our results in the context of model predictive control (MPC). That is, given a continuous-time plant, existing MPC algorithms often only stabilize a discretized version of plant. However, it is still interesting to propose a framework where a digital MPC-based implementation stabilizes the continuous-time plant. One result in this direction is the ST MPC framework for linear systems proposed in [17]. Finally, a more thorough study of the balancing act that occurs between the computational demand required by the reachability computation and the increased inter-event time would be useful for practical application on resource limited platforms.

Appendix A

Auxiliary Results

Lemma A.0.1 *Consider a locally Lipschitz function* $x : \mathbb{R}_{\geq 0} \to \mathbb{R}^n$ *and a constant vector* $\gamma \in \mathbb{R}^n$*. Then,*

$$\partial_C \langle \gamma, x(s) \rangle \subset \langle \gamma, \partial_C x(s) \rangle \qquad \forall s \geq 0. \tag{A.1}$$

Proof.

For $\gamma \in \mathbb{R}^n$, we have

$$\partial_C \langle \gamma, x(s) \rangle = \partial_C \sum_{i=0}^{n} \gamma_i x_i(s) \tag{A.2}$$

By (P2) and (P1), we conclude that

$$\begin{aligned} \partial_C \langle \gamma, x(s) \rangle \subset \sum_{i=0}^{n} \partial_C (\gamma_i x_i)(s) = \sum_{i=0}^{n} \gamma_i \partial_C x_i(s) \\ = \langle \gamma, \partial_C x(s) \rangle. \end{aligned} \tag{A.3}$$

■

Lemma A.0.2 *Consider a system $\mathcal{H}_f = (C, F)$ such that (A1) and (A2) hold and let x be a solution to $\mathcal{H}_f$. The following hold.*

1. *For each $t \in \operatorname{dom} x$ such that $\dot{x}(t)$ exists, $\dot{x}(t) \in F(x(t))$.*

2. *For each $t \in \operatorname{dom} x$, $\partial_C x(t) \subset F(x(t))$.*

Proof. To prove item 1, we use the fact that the solution x is absolutely continuous. Hence, we can find a sequence $\{t_n\}_{n=0}^{\infty} \subset I \subset \operatorname{dom} x$, with I compact, such that $t_n \to t$ and $\dot{x}(t_n)$ exists and satisfies $\dot{x}(t_n) \in F(x(t_n))$ for all $n \in \mathbb{N}$. Next, since the map F is locally bounded, we conclude that the solution x is locally Lipschitz. Hence, using (P3) and (P4) under Heine-Cantor Theorem, we conclude that $\partial_C x$ is uniformly upper semicontinuous on I; namely, for each $\epsilon > 0$, there exists $\delta > 0$ such that, for each $t_1 \in t_o + \delta\mathbb{B}$, $\partial_C x(t_1) \subset \partial_C x(t_o) + \epsilon\mathbb{B}$ for all $t_o \in I$.

Furthermore, by definition of the time derivative, we conclude that, there exists sequence $\{t_m\}_{m=0}^{\infty} \subset \mathbb{R}_{>0}$ with $\lim_{m\to\infty} t_m = 0$, such that

$$\dot{x}(t_n) = \lim_{m\to\infty} (x(t_n + t_m) - x(t_n))/t_m \qquad \forall n \in \mathbb{N}.$$

Let us now consider the sequence

$$f(n, m) := (x(t_n + t_m) - x(t_n))/t_m.$$

We propose to show that

$$\lim_{n\to\infty} \dot{x}(t_n) = \dot{x}(t). \tag{A.4}$$

To this end, we need to show that

$$\lim_{n\to\infty}\lim_{m\to\infty} f(n,m) = \lim_{m\to\infty}\lim_{n\to\infty} f(n,m). \tag{A.5}$$

To show that (A.5) holds in our case, we apply Moore-Osgood Theorem [52]. Indeed, note that

$$\lim_{n\to\infty} f(n,m) = (x(t+t_m) - x(t))/t_m,$$

and the point-wise convergence is guaranteed for each $m \in \mathbb{N}$. Hence, it remains to show that the convergence in the following limit

$$\lim_{m\to\infty} f(n,m) = \dot{x}(t_n)$$

is uniform with respect to n; namely, for each $\epsilon > 0$, there exists $M \in \mathbb{N}$ such that, for each $n \in \mathbb{N}$ and for each $m \geq M$,

$$\begin{aligned} |f(n,m) - \lim_{m\to\infty} f(n,m)| &= |f(n,m) - \dot{x}(t_n)| \\ &= |f(n,m) - \partial_C x(t_n)| \leq \epsilon. \end{aligned} \tag{A.6}$$

Indeed, using Lemma 2.2.5, we conclude that, for each $m \in \mathbb{N}$, there exists $\alpha_m \in (t_n, t_n + t_m)$ and $\zeta_m \in \partial_C x(\alpha_m)$ such that $f(n,m) = \zeta_m$. Hence,

$$\begin{aligned} |f(n,m) - \partial_C x(t_n)| &= |\zeta_m - \partial_C x(t_n)| \\ &\leq \sup\{|\zeta_m - \partial_C x(t_n)| : \\ &\quad \zeta_m \in \partial_C(x(\alpha_m))\}. \end{aligned}$$

Finally, using the uniform upper semicontinuity of $\partial_C x$, we conclude the existence of $M \in \mathbb{N}$ and $\delta > 0$ such that $|t_m| \leq \delta$ for all $m \geq M$ and

$$\partial_C(x(\alpha_m)) \subset \partial_C x(t_n) + \epsilon\mathbb{B}.$$

Hence,

$$|f(n,m) - \partial_C x(t_n)| \leq \epsilon \qquad \forall m \geq M, \quad \forall n \in \mathbb{N}.$$

Finally, we know that we can find a sequence $\{t_n\}_0^N$ such that $t_n \to t$ and that $\dot{x}(t_n)$ exists and satisfies $\dot{x}(t_n) \in F(x(t_n))$. Then, since (A.4) holds, item 1 follows using outer semicontinuity of F.

To prove item 2, we consider a vector $\zeta \in \partial_C x(t)$, for some $t \in \operatorname{dom} x$, such that there exists a sequence $\{t_n\}_{n=0}^{\infty}$ such that $\dot{x}(t_n)$ exists for all $n \in \mathbb{N}$ and $\zeta = \lim_{n\to\infty} \dot{x}(t_n)$. Next, using item 1, we conclude that $\dot{x}(t_n) \in F(x(t_n))$ for all $n \in \mathbb{N}$. Then, since the map F is outer semicontinuous, we conclude that $\lim_{n\mapsto\infty} \dot{x}(t_n) = \zeta \in F(x(t))$. Finally, since $F(x)$ is convex, the convex hull of any set of vectors in $F(x)$ is also a subset of $F(x)$, which proves 2. ∎

Lemma A.0.3 *Consider a constrained system $\mathcal{H}_f = (C, F)$ and let x be a solution to $\mathcal{H}_f$. Then, for each $t \in \operatorname{dom} x$ such that $\dot{x}(t)$ exists, $\dot{x}(t) \in T_C(x(t))$.*

Proof. We consider $t \in \operatorname{dom} x$ such that $\dot{x}(t)$ exists. Moreover, let a sequence $\{t_n\}_{n\in\mathbb{N}} \subset \operatorname{dom} x$ such that $t_n \to t$. That is, for $v_n(t) := (x(t_n) - x(t))/(t_n - t)$, we have $\lim_n v_n(t) = \dot{x}(t)$ and at the same time $x(t) + t_n v_n(t) = x(t_n) \in \operatorname{cl}(C)$. Hence, using (2.9), we conclude that $\dot{x}(t) \in T_C(x(t))$. ∎

Lemma A.0.4 *Consider a system $\mathcal{H}_f = (C, F)$, with F locally bounded. Consider a locally Lipschitz function $\rho : \mathbb{R}^n \to \mathbb{R}$ and a solution x to $\mathcal{H}_f$. Then,* *for almost all* $t \in \operatorname{dom} x$, $\frac{d}{dt}\rho(x(t))$ *exists and the following inclusion is true:*

$$\frac{d}{dt}\rho(x(t)) \in \{\langle \zeta, \dot{x}(t) \rangle : \zeta \in \partial_C \rho(x(t))\}. \tag{A.7}$$

Proof. Consider a solution x to $\mathcal{H}$. Since F is locally bounded, it follows that x is locally Lipschitz. Furthermore, since ρ is also locally Lipschitz, it follows that the composition $\rho \circ x$ is locally Lipschitz and that the map $t \mapsto \rho(x(t))$ is absolutely continuous. Let $t \in \operatorname{dom} x$ such that $\frac{d}{dt}\rho(x(t))$ exists. Hence,

$$\frac{d}{dt}\rho(x(t)) = \lim_{h \to 0} \frac{\rho(x(t+h)) - \rho(x(t))}{h}.$$

Using Lemma 2.2.5, we conclude that there exists u_h belonging to the open line segment $(x(t), x(t+h))$ such that

$$\begin{aligned}(\rho(x(t+h)) - \rho(x(t)))/h \in & \\ & \left\{ \langle z, \frac{x(t+h) - x(t)}{h} \rangle : z \in \partial_C \rho(u_h) \right\}.\end{aligned} \tag{A.8}$$

Hence, there exists $w_h \in \partial_C \rho(u_h)$ such that

$$(\rho(x(t+h)) - \rho(x(t)))/h = \langle w_h, (x(t+h) - x(t))/h \rangle. \tag{A.9}$$

Furthermore, consider a sequence $\{h_n\}_{n=0}^{\infty} \subset (0, \bar{h}]$ with $h_n \to 0$ such that

$$\begin{aligned}&\lim_{n \to \infty} (\rho(x(t+h_n)) - \rho(x(t)))/h_n = \\ &\lim_{n \to \infty} \langle w_{h_n}, (x(t+h_n) - x(t))/h_n \rangle.\end{aligned} \tag{A.10}$$

Now, since $\partial_C \rho$ is locally bounded, we conclude that there exist $U(x(t))$ and $K > 0$ such that

$$|\zeta| \leq K \qquad \forall \zeta \in \partial_C \rho(y), \quad \forall y \in U(x(t)).$$

Furthermore, since the system's solutions are continuous, it follows that for $\bar{h}$ sufficiently small, both $x(t+h)$ and u_h belong to $U(x(t))$ for all $h \in [0, \bar{h}]$. Hence,

$$|w_{h_n}| \leq K \qquad \forall n \in \mathbb{N}. \tag{A.11}$$

Similarly, since F is locally bounded, then, there exists $U(x(t))$ and $K > 0$ such that $|\eta| \leq K$ for all $\eta \in F(y)$ and for all $y \in U(x(t))$. Furthermore, since the system's solutions are continuous, it follows that for $\bar{h} > 0$ sufficiently small, $x(s) \in U(x(t))$ for all $s \in [0, \bar{h}]$. Hence, in view of the integral

$$x(t+h) - x(t) = \int_0^h \dot{x}(t+s) ds \quad \dot{x}(s) \in F(x(s))$$
$$\text{for a.a. } s \in [0, h] \text{ and } \forall h \in [0, \bar{h}], \tag{A.12}$$

we conclude that

$$|(x(t+h_n) - x(t))/h_n| \leq K \qquad \forall n \in \mathbb{N}. \tag{A.13}$$

By passing to a subsequence, we conclude the existence of $w_o \in \mathbb{R}^n$ such that

$$w_{h_n} \to w_o \text{ and } (x(t+h_n) - x(t))/h_n \to \dot{x}(t).$$

Furthermore, since $w_{h_n} \in \partial_C \rho(u_{h_n})$, $u_{h_n} \to x(t)$ and $\partial_C \rho$ is upper semicontinuous, we conclude that $w_o \in \partial_C \rho(x(t))$. Hence, (A.7) follows. ■

The following lemma establishes how a marginal function

$$f(x) := \sup\{g(y) : y \in \Pi(x)\} \tag{A.14}$$

inherits the continuity properties of the set valued-map Π. The same statement can be found in [34, Theorem 1.4.16].

Lemma A.0.5 *Consider a locally bounded set-valued map* $\Pi : \mathbb{R}^m \rightrightarrows \mathbb{R}^n$, *with* $\Pi(x)$ *nonempty for all* $x \in \mathbb{R}^m$, *and a scalar function* $g : \mathbb{R}^n \to \mathbb{R}$ *and let the marginal function in* (A.14).

- *If* Π *is outer semicontinuous and* g *is continuous, then* f *is upper semicontinuous and locally bounded.*

Proof. In order to prove the statement, we use the definition of upper semicontinuity for scalar functions. That is, for every sequence $\{x_i\}_{i=0}^{\infty} \subset \mathbb{R}^m$ such that $\lim_{i\to\infty} x_i = x_o$, we will show that

$$\begin{aligned}\limsup_{i\to\infty} f(x_i) &= \limsup_{i\to\infty} \left(\sup\{g(y) : y \in \Pi(x_i)\}\right) \\ &\le \sup\{g(y) : y \in \Pi(x_o)\} = f(x_o)\end{aligned}$$

provided that g is continuous and the set-valued map Π is outer semicontinuous. Indeed, since the map Π is outer semicontinuous, we conclude that, for all

$y_i \in \Pi(x_i)$ such that $\lim_{i\to\infty} y_i = y_o \in \mathbb{R}^n$, we have $y_o \in \Pi(x_o)$. Furthermore, we choose $y_i \in \Pi(x_i)$ such that $g(y_i) = f(x_i)$. Hence,

$$\begin{aligned}\limsup_{i\to\infty} f(x_i) &= \limsup_{i\to\infty} \left(\sup\{g(y) : y \in \Pi(x_i)\}\right) \\ &= \limsup_{i\to\infty} g(y_i). \end{aligned} \tag{A.15}$$

Since g is continuous, we conclude that

$$\begin{aligned}\limsup_{i\to\infty} f(x_i) &= \limsup_{i\to\infty} \left(\sup\{g(y) : y \in \Pi(x_i)\}\right) \\ &= g(\limsup_{i\to\infty} y_i). \end{aligned}$$

Since Π is locally bounded, we conclude that the sequence $\{y_i\}_{i=0}^{\infty}$ is bounded; hence, $\limsup_{i\to\infty} y_i = y_o \in \mathbb{R}^n$. Moreover, by passing to a suitable subsequence $\{y_{i_k}\}_{k=0}^{\infty}$, we conclude that $\limsup_{i\to\infty} y_i = \lim_{k\to\infty} y_{i_k} = y_o$. Thus, since Π is outer semicontinuous, it follows that $y_o \in \Pi(x_o)$. Finally,

$$\limsup_{i\to\infty} f(x_i) = g(y_o) \leq \sup\{g(y) : y \in \Pi(x_o)\} = f(x_o).$$

Local boundedness of f is a straightforward consequence of local boundedness of Π and the continuity of g. ∎

Lemma A.0.6 *Consider two set-valued maps* $F : \operatorname{dom} F \rightrightarrows \mathbb{R}^m$ *and* $G : \operatorname{dom} G \rightrightarrows \operatorname{dom} F$ *such that* $\operatorname{dom} F$ *is closed,* F *and* G *are outer semicontinuous, and* G *is locally bounded. Then,* $F \circ G$ *is outer semicontinuous.*

Proof. Consider a sequence $\{x_i\}_{i=0}^{\infty} \subset \operatorname{dom} G$ and a sequence $\{y_i\}_{i=0}^{\infty} \subset \mathbb{R}^m$ such that $y_i \in F \circ G(x_i)$, $\lim_{i\to\infty} x_i = x_o \in \operatorname{dom} F$, and $\lim_{i\to\infty} y_i = y_o \in \mathbb{R}^m$.

We will show that $y_o \in F \circ G(x_o)$. Indeed, let a sequence $\{z_i\}_{i=0}^{\infty} \subset \operatorname{dom} F$ such that $z_i \in G(x_i)$ and $y_i \in F(z_i)$. Since the map G is locally bounded and $\operatorname{dom} F$ is closed, we conclude, along an adequate subsequence, the existence of $z_o \in \operatorname{dom} F$ such that $z_o = \lim_{i \to \infty} z_i$. Furthermore, since G outer semicontinuous, we conclude that $z_o \in G(x_o)$. Furthermore, since F is outer semicontinuous, we conclude that $y_o \in F(z_o) \in F \circ G(x_o)$. ■

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