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Representations of FI-Like Categories Associated to Subgroups of Wreath Products

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Representations of FI-Like Categories Associated to Subgroups of Wreath Products

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Dedicated to the memories of Robert Bolman and Anthony Muljat, Sr. God knows best.

ABSTRACT OF THE DISSERTATION

Representations of FI-Like Categories Associated to Subgroups of Wreath Products

by

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University of California, Riverside, June 2024
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The category FI_G has been studied in relation to the representation theory of the wreath products $G \wr S_n$, where G is a finite group. We take $G = A$ to be abelian and define the category FI_A^H associated to certain subgroups J_n of $A \wr S_n$, including the finite complex reflection groups $G(m, p, n)$ defined by Shepard and Todd. We also give axioms for "FI-like" categories $\mathcal{C}$, and prove the equivalence of the noetherian property for $\mathcal{C}$-modules over k with the noetherian property for $\mathcal{A}$-modules over k, where $\mathcal{A}$ is a suitably restricted subcategory of $\mathcal{C}$ and k is a commutative noetherian ring with unity. We apply this result to show that representation stability for a sequence of finite-dimensional J_n-representations over $\mathbb{C}$ is equivalent to finite generation of the corresponding FI_A^H-module. We also prove the analogous result for representations of the alternating groups. Lastly, we prove homological stability for the family $\{J_n\}$ with twisted coefficients from a finitely generated FI_A^H-module over $\mathbb{Z}$, as well as the equivalence of certain Serre quotient categories of locally noetherian $\mathcal{C}$- and $\mathcal{A}$-modules.

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Chapter 1

Introduction

We begin by recalling the notion of representation stability as originally defined by Church and Farb [CF]. Then we describe the categorical framework introduced by Church, Ellenberg, and Farb [CEF] and expanded upon by later works (such as [SS], [W2], and the present work) in terms of which many of the major known results concerning representation stability are formulated.

Throughout this work, all sequences $\{X_n\}$ are assumed to be indexed by the nonnegative integers, unless otherwise stated.

1.1 Representation stability

In general, stabilization results concerning a sequence $\{X_n\}$ are characterized by descriptions of how the objects in higher degrees (i.e. the objects X_n for $n \gg 0$) are determined by the objects in lower degrees in a manner which restricts the "growth" of the sequence. The authors of [CF] formulated *representation stability* in order to precisely describe this phenomenon as it pertains to certain naturally-arising sequences of group representations. We describe the sort of sequences for which representation stability is defined below.

Fix a field k of characteristic zero. Let $\{G_n\} : G_0 \hookrightarrow G_1 \hookrightarrow G_2 \hookrightarrow \cdots$ be an ascending chain of finite groups, and consider a sequence

$$\{V_n, \varphi_n\} : V_0 \xrightarrow{\varphi_0} V_1 \xrightarrow{\varphi_1} V_2 \xrightarrow{\varphi_2} \cdots$$

where, for each $n \geq 0$, V_n is a G_n-representation over k and $\varphi_n : V_n \to V_{n+1}$ is a homomorphism of k-vector spaces.

Definition 1.1 ([CF, p. 19])**.** The sequence $\{V_n, \varphi_n\}$ is *consistent* if both of the following hold for all $n \geq 0$:

(a) Each $v \in V_n$ is contained in a finite-dimensional G_n-representation $W_n \subseteq V_n$.

(b) The map φ_n is G_n-equivariant.

Condition (a) of Definition 1.1 guarantees that V_n decomposes into a direct sum of finite-dimensional irreducible representations of G_n with well-defined multiplicities, while condition (b) ensures that the map φ_n, considered as a map $V_n \to \operatorname{Res}_{G_n}^{G_{n+1}} V_{n+1}$, is a homomorphism of G_n-representations.

Definition 1.2. Let $\{V_n, \varphi_n\}$ be a consistent sequence of G_n-representations.

(a) We say $\{V_n, \varphi_n\}$ satisfies the *injectivity condition* if the map $\varphi_n : V_n \to V_{n+1}$ is injective for all $n \gg 0$.

(b) We say $\{V_n, \varphi_n\}$ satisfies the *surjectivity condition* if the k-span of the G_{n+1}-orbit of $\varphi_n(V_n)$ is all of V_{n+1} for all $n \gg 0$.

Definition 1.2 describes two of the three conditions imposed for the sequence $\{V_n, \varphi_n\}$ to be representation stable. The third condition, called *multiplicity stability*, may be formulated when the irreducible representations of G_n admit an indexing by some datum λ that does not depend on n. Such an indexing exists when G_n is the symmetric group S_n, as described in Notation 1.3 below.

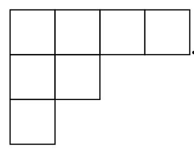
Notation 1.3. Let n be a positive integer. By a *partition of n*, we mean a sequence of positive integers

$$\lambda = (\lambda_1, \ldots, \lambda_\ell)$$

such that $\lambda_1 \geq \cdots \geq \lambda_\ell$ and

$$|\lambda| := \lambda_1 + \cdots + \lambda_\ell = n.$$

A partition $\lambda = (\lambda_1, \ldots, \lambda_\ell)$ may be visualized by the *Young diagram* $\mathcal{Y}_\lambda$, which is an array of boxes with λ_1 boxes in the top row, λ_2 boxes in the second row, and so on. For example, the partition $(4, 2, 1)$ of 7 corresponds to the Young diagram

$\mathcal{Y}_{(4,2,1)} =$  .

Denote by $\mathcal{P}$ the set of all partitions of positive integers. Given $\lambda \in \mathcal{P}$, define the *padded partition* $\lambda[n]$ for $n \geq |\lambda| + \lambda_1$ by setting

$$\lambda[n] := (n - |\lambda|, \lambda_1, \ldots, \lambda_\ell).$$

We have $|\lambda[n]| = n$; moreover, every partition of n may be written uniquely in this form. Recall that the (isomorphism classes of) irreducible representations of the symmetric group S_n over k are in one-to-one correspondence with all possible partitions of n. We write $V(\lambda)_n$ for the irreducible S_n-representation over k corresponding to $\lambda[n]$. If $n < |\lambda| + \lambda_1$, we set $V(\lambda)_n$ to be the trivial representation.

Let $\{V_n, \varphi_n\}$ be a consistent sequence of S_n-representations over k. Maschke's theorem guarantees that there exists a decomposition of V_n as a direct sum of the $V(\lambda)_n$ with multiplicities $0 \leq c_{\lambda,n} \leq \infty$:

$$V_n \cong \bigoplus_{\lambda \in \mathcal{P}} V(\lambda)_n^{\oplus c_{\lambda,n}} \tag{1.4}$$

We may now define mutliplicity stability (in the terminology of [CF], *uniform multiplicity stability*) for $\{V_n, \varphi_n\}$ as follows.

Definition 1.5 ([CF, Def. 2.6])**.** We say that $\{V_n, \varphi_n\}$ is *multiplicity stable* if there exists $N \geq 0$ not depending on λ such that, for all λ and all $n \geq N$, the multiplicity $c_{\lambda,n}$ in the decomposition (1.4) satisfies $c_{\lambda,n} = c_{\lambda,N}$.

Multiplicity stability $\{V_n, \varphi_n\}$ in the range $n \geq N$ for means that, for each $n \geq N$, the decomposition for V_{n+1} may be obtained by adding exactly one box to the top row of each Young diagram corresponding to each irreducible component in the decomposition of V_n.

Definition 1.6 ([CF, Def. 2.3])**.** We say that the sequence $\{V_n, \varphi_n\}$ is *representation stable* if the sequence $\{V_n, \varphi_n\}$ satisfies the injectivity and surjectivity conditions, and is multiplicity stable.

1.2 FI-modules

The authors of [CEF] introduced FI-*modules* as a means of encoding the information contained in certain consistent sequences of S_n-representations. In this section, we recapitulate the main structural results concerning FI-modules.

For this section, we set k to be a commutative noetherian ring with unity.

Definition 1.7. The category FI is defined by the following data:

- The objects of FI are the sets $\mathbf{n} = \{1, \ldots, n\}$ for all $n \geq 0$, where $\mathbf{0}$ is the empty set.
- For all $\mathbf{m}$ and $\mathbf{n}$, the set of morphisms $\mathrm{FI}(\mathbf{m}, \mathbf{n})$ consists of all injective functions $\mathbf{m} \hookrightarrow \mathbf{n}$. Composition of morphisms is defined as the usual composition of functions.

Originally, the authors of [CEF] defined FI to be the category of *all* finite sets and injective functions. However, the results cited below remain valid when stated in terms of Definition 1.7.

An endomorphism $\sigma : \mathbf{n} \to \mathbf{n}$ in FI is a permutation on n elements, so the set of endomorphisms $\mathrm{FI}(\mathbf{n}, \mathbf{n})$ together with the operation of composition forms the symmetric group S_n for all $n \geq 0$.

Definition 1.8. An FI-*module* (over k) is a covariant functor V from the category FI to the category of k-modules. An FI-module morphism $V \to W$ is a natural transformation of functors.

Fix an FI-module V. For each $n \geq 0$, we will denote the k-module $V(\mathbf{n})$ by V_n. Given an FI-morphism α, we will denote the k-module homomorphism $V(\alpha)$ by α_*.

The endomorphsisms $\mathrm{FI}(\mathbf{n}, \mathbf{n}) = S_n$ act on V_n by

$$\sigma v = \sigma_*(v) \qquad (\sigma \in S_n, \quad v \in V_n)$$

thus making V_n an S_n-representation over k, i.e. a kS_n-module.

For each pair of nonnegative integers $m \leq n$, let $\iota_{m,n}$ denote the inclusion $\mathbf{m} \hookrightarrow \mathbf{n}$. An FI-module V naturally determines the consistent sequence of S_n-representations $\{V_n, (\iota_{n,n+1})_*\}$. However, not every consistent sequence of S_n-representation may arise from an FI-module, as specified by the next proposition.

Proposition 1.9 (FI-module criterion, [CEF, Rem. 3.3.1])**.** *Let $\{W_n, \varphi_n\}$ be a consistent sequence of S_n-representations. Then there exists an* FI*-module W with $W(\mathbf{n}) = W_n$ and $\varphi_n = W(\iota_{n,n+1})$ if and only if the following condition is satisfied: for all $m < n$, the stabilizer of $\iota_{m,n}$ under the action of S_n by postcomposition acts trivially on the image $W(\iota_{m,n})(W_m)$.*

For instance, the consistent sequence of the regular representations kS_n with the inclusion maps $kS_n \hookrightarrow kS_{n+1}$ for all $n \geq 0$ cannot arise from an FI-module.

Definition 1.10. Let U be an FI-module. We say U is an FI-*submodule* of V if the following hold for all $n \geq 0$:

(a) U_n is an kS_n-submodule of V_n.

(b) For all $0 \le m \le n$ and all $\alpha \in \mathrm{FI}(\mathbf{m}, \mathbf{n})$, the map $U(\alpha)$ is the restriction of α_* to U_m. In other words, the following diagram commutes for all $0 \le m \le n$ and all $\alpha \in \mathrm{FI}(\mathbf{m}, \mathbf{n})$:

$$\begin{array}{ccc} U_m & \hookrightarrow & V_m \\ {\scriptstyle U(\alpha)}\downarrow & & \downarrow{\scriptstyle \alpha_*} \\ U_n & \hookrightarrow & V_n \end{array}$$

Remark 1.11. Any FI-morphism $\alpha : \mathbf{m} \to \mathbf{n}$ may be written

$$\alpha = \sigma \circ \iota_{n-1,n} \circ \iota_{n-2,n-1} \circ \cdots \iota_{m,m+1}$$

for some $\sigma \in S_n$. Thus, using the functorial properties of FI-modules, it suffices for condition (b) of Definition 1.10 that the map $U(\iota_{n,n+1})$ is the restriction of $(\iota_{n,n+1})_*$ for all $n \ge 0$. It follows from Proposition 1.9 that a consistent sequence $\{U_n, \varphi_n\}$, where U_n is a kS_n-submodule of V_n for all $n \ge 0$, may be promoted to an FI-submodule of V if and only if the following diagram commutes for all $n \ge 0$:

$$\begin{array}{ccc} U_n & \hookrightarrow & V_n \\ {\scriptstyle \varphi_n}\downarrow & & \downarrow{\scriptstyle (\iota_{n,n+1})_*} \\ U_{n+1} & \hookrightarrow & V_{n+1} \end{array}$$

where the horizontal arrows are the natural inclusions.

Definition 1.12. Let S be a subset of $\bigsqcup_{n \ge 0} V_n$. We say *S generates V* if the following equivalent conditions hold:

(a) The only FI-submodule U of V with $S \subseteq \bigsqcup_{n \ge 0} U_n$ is $U = V$.

(b) For all $n \ge 0$, the span of the S_n-orbit of $\bigcup_{m \le n} (\iota_{m,n})_*(S \cap V_m)$ is all of V_n.

If there is a subset of $\bigsqcup_{n \le d} V_n$ that generates V, we say V is *generated in degrees $\le d$*. If there is a finite set that generates V, we say V is *finitely generated*.

We recall the following two structural results concerning finitely generated FI-modules.

Theorem 1.13 (Noetherian property, [CEFN, Thm. A]). *Let k be a commutative noetherian ring with unity. If V is finitely generated, then any* FI*-submodule of V is also finitely generated.*

The statement of Theorem 1.13 is also expressed by saying that the category FI is *locally noetherian* (over k).

Theorem 1.14 ([CEF, Thm. 1.13]). *Let k be a field of characteristic zero. An* FI*-module V is finitely generated if and only if the consistent sequence $\{V_n, (\iota_{n,n+1})_*\}$ is representation stable and V_n is finite-dimensional for all $n \geq 0$.*

In chapters 2 and 3 of the present work, we will prove analogues of Theorems 1.13 and 1.14 for certain categories which resemble FI. The first of these is FA, which plays the role of FI for representations of the alternating groups A_n.

Definition 1.15. Let FA be the subcategory of FI defined by the following data.

- The objects of FA are all the objects of FI, i.e. the sets $\mathbf{n} = \{1, \ldots, n\}$ for all $n \geq 0$, where $\mathbf{0}$ is the empty set.
- If $m \neq n$, then $\mathrm{FA}(\mathbf{m}, \mathbf{n}) = \mathrm{FI}(\mathbf{m}, \mathbf{n})$.
- For all $n \geq 0$, the set of endomorphisms $\mathrm{FA}(\mathbf{n}, \mathbf{n})$ is the alternating subgroup A_n of $\mathrm{FI}(\mathbf{n}, \mathbf{n}) = S_n$.

Next we consider the category FI_G which generalizes FI. Analogous results to Theorems 1.13 and 1.14 are known for FI_G when G is a finite group ([SS, Theorem 1.2.1] and [GL, Theorem 1.12] respectively).

Definition 1.16 ([SS]). Let G be a finite group. The category FI_G is defined by the following data.

- The objects of FI_G are the sets $\mathbf{n} = \{1, \ldots, n\}$ for all $n \geq 0$, where $\mathbf{0}$ is the empty set.
- For each $\mathbf{m}, \mathbf{n} \in \mathrm{Ob}\,\mathrm{FI}_G$, the set of morphisms $\mathrm{FI}_G(\mathbf{m}, \mathbf{n})$ consists of pairs (α, f) where $\alpha : \mathbf{m} \to \mathbf{n}$ is an injection and $f : \mathbf{m} \to G$ is a map. The composition of $(\beta, g) \in$

$\mathrm{FI}_G(\mathbf{n}, \mathbf{p})$ and $(\alpha, f) \in \mathrm{FI}_G(\mathbf{m}, \mathbf{n})$ is defined by

$$(\beta, g) \circ (\alpha, f) = (\gamma, h)$$

where

$$\gamma(r) = \beta(\alpha(r)), \quad h(r) = g(\alpha(r))f(r) \qquad \text{for all } r \in \mathbf{m}.$$

Setting G to be the trivial group recovers the category FI. For every nonnegative integer n, the group of endomorphisms $\mathrm{FI}_G(\mathbf{n}, \mathbf{n})$ is isomorphic to the wreath product

$$G \wr S_n = G^n \rtimes S_n$$

via the map

$$(\alpha, f) \mapsto (f(1), \ldots, f(m), \alpha) \qquad ((\alpha, f) \in \mathrm{FI}_G(\mathbf{n}, \mathbf{n})).$$

For each positive integer m, we denote the cyclic group of m-th roots of unity by C_m. Following Shephard and Todd [ST], we will denote the wreath product $C_m \wr S_n$ by $G(m, 1, n)$. This notation is defined in general below.

Notation 1.17. [LT] Given positive integers m, p, n where $p|m$, let

$$A(m, p, n) = \{(r_1, \ldots, r_n) \in C_m^n \mid (r_1 \cdots r_n)^{m/p} = 1\}$$

and

$$G(m, p, n) = A(m, p, n) \rtimes S_n.$$

The group $G(m, p, n)$ is a normal subgroup of $G(m, 1, n) = C_m \wr S_n$ with index p. Some of these groups are commonly known by other names:

- $G(m, p, 1)$ is the cyclic group of order m/p.
- $G(1, 1, n)$ is the symmetric group S_n.

- $G(2,1,n)$ is the Coxeter group (or Weyl group) B_n.
- $G(2,2,n)$ is the Coxeter group (or Weyl group) D_n.
- $G(m,m,2)$ is the dihedral group of order $2m$.

In fact, any irreducible complex reflection group either belongs to the family $G(m,p,n)$ or is one of 34 exceptional cases. This family also arises from certain subcategories of FI_{C_m} that we will define below. First we will fix the following notation.

Notation 1.18. Let A be a finite abelian group. For each $n \geq 0$, we write G_n for the wreath product $A \wr S_n$. In particular, G_0 is the trivial group. Fix a subgroup H of A and $n \geq 0$. We define the subgroup H' of A^n by setting

$$H' := \{(a_1, \ldots, a_n) \in A^n \mid a_1 \cdots a_n \in H\}.$$

For each $n \geq 0$, we set J_n to be the subgroup $H' \rtimes S_n$ of G_n. In particular, $J_0 = G_0$ is the trivial group and

$$J_1 = H' \rtimes S_1 = H \rtimes \langle (1) \rangle \cong H.$$

In the case where $H = \langle e \rangle$ is the trivial subgroup of A, we will write $K_n = J_n = \langle e \rangle' \rtimes S_n$ for all $n \geq 0$.

The groups J_n and K_n may also be described as follows. Define the map

$$\pi_n : A \wr S_n \to A$$

by setting

$$\pi_n(a_1, \ldots, a_n, \sigma) = a_1 \cdots a_n \qquad (a_1, \ldots, a_n \in A, \quad \sigma \in S_n).$$

Then $J_n = \pi^{-1}(H)$ and $K_n = \ker \pi_n$. Some special cases are noteworthy:

- If $H = A$, then $H' = A^n$ and $J_n = A \wr S_n = G_n$ for all $n \geq 0$.
- If $A = C_m$ and $H = C_{m/p}$, then $J_n \cong G(m,p,n)$ and $K_n \cong G(m,m,n)$ for all $n \geq 0$.

Definition 1.19. Let A be a finite abelian group and H a subgroup of A. Define the subcategory FI_A^H of FI_A by the following data.

- The objects of FI_A^H are the sets $\mathbf{n} = \{1, \ldots, n\}$ for all $n > 0$ and $\mathbf{0} = \emptyset$.
- If $m \neq n$, then $\mathrm{FI}_A^H(\mathbf{m}, \mathbf{n}) = \mathrm{FI}_A(\mathbf{m}, \mathbf{n})$.
- For all $n \geq 0$, the set of endomorphisms $\mathrm{FI}_A^H(\mathbf{n}, \mathbf{n})$ is the subgroup J_n of $\mathrm{FI}_A(\mathbf{n}, \mathbf{n}) = G_n$.

Setting $H = A$ recovers the category FI_A. In particular, the category $\mathrm{FI}_{C_2}^{\langle e \rangle}$ is equivalent to the category FI_D defined by Wilson [W2, Definition 1.1] and which is associated to the Coxeter groups $D_n = G(2,2,n)$. Indeed, much of chapters 2 and 3 of the present work is based on Wilson's proofs of analogues of Theorems 1.13 and 1.14 for FI_D ([W2, Theorem 4.21] and [W2, Theorem 4.22] respectively).

The plan of the rest of this paper is as follows. In chapter 2, we define FI-*like* categories and say that a subcategory $\mathcal{A}$ of an FI-like category $\mathcal{C}$ is *almost-full* if $\mathcal{A}$ has all the objects of $\mathcal{C}$ and all of the morphisms of $\mathcal{C}$ which are not endomorphisms. We prove structural results on FI-like categories and their almost-full subcategories, the main result being the following:

Theorem A. *The category $\mathcal{C}$ is locally noetherian if and only if $\mathcal{A}$ is locally noetherian.*

In particular, the categories FA and FI_A^H are locally noetherian.

In chapter 3, we define representation stability for consistent sequences of A_n- (resp. J_n-) representations and prove the following analogue of Theorem 1.14 for the category FA (resp. FI_A^H):

Theorem B/C. *Let W be an* FA*-module (resp.* FI_A^H*-module) over a field k of characteristic* 0 *(resp. over $\mathbb{C}$). Then W is finitely generated if and only if the sequence $\{W_n, (\iota_{n,n+1})_*\}$ is representation stable and W_n is finite-dimensional for all $n \geq 0$.*

In the case where W is an FI_A^H-module, the map $(\iota_{n,n+1})_* : W_n \to W_{n+1}$ is the one induced by the inclusion $\mathbf{n} \hookrightarrow \mathbf{n+1}$ and the trivial map $\mathbf{n} \to A$.

Chapter 4 concerns stabilization for the homology groups of the family $\{J_n\}$ with trivial coefficients. We obtain explicit bounds for the homology groups to stabilize in the following sense:

Theorem D. *If $n \geq 2m+2$, then the map $\iota_* : H_m(J_{n-1}) \to H_m(J_n)$ is an isomorphism.*

Here the map $\iota_* : H_m(J_{n-1}) \to H_m(J_n)$ is induced by the inclusion $J_{n-1} \hookrightarrow J_n$. As a corollary, we deduce homological stability for the family $\{J_n\}$ with twisted coefficients $\{W_n\}$, where W is a finitely-generated FI_A^H-module over $\mathbb{Z}$.

Finally, chapter 5 deals with the Serre quotient categories

$$Q_{\mathcal{C}} = \frac{\mathcal{C}-\mathrm{fgMod}}{\mathcal{C}-\mathrm{fdMod}} \quad \text{and} \quad Q_{\mathcal{A}} = \frac{\mathcal{A}-\mathrm{fgMod}}{\mathcal{A}-\mathrm{fdMod}}$$

where $\mathcal{A}$ is an almost-full subcategory of an FI-like category $\mathcal{C}$, and $\mathcal{C}-\mathrm{fgMod}$ (resp. $\mathcal{C}-\mathrm{fdMod}$) is the functor category of finitely generated (resp. *finite-dimensional*) $\mathcal{C}$-modules. We prove that these Serre quotients are equivalent when $\mathcal{C}$ is locally noetherian:

Theorem E. *Let $\mathcal{C}$ be a locally noetherian* FI*-like category and $\mathcal{A}$ an almost-full subcategory of $\mathcal{C}$. The restriction functor* $\mathrm{Res} : \mathcal{C}-\mathrm{fgMod} \to \mathcal{A}-\mathrm{fgMod}$ *induces an equivalence of categories*

$$Q_{\mathcal{C}} \xrightarrow{\sim} Q_{\mathcal{A}}.$$

In the case where $\mathcal{C} = \mathrm{FI}_A$ and $\mathcal{A} = \mathrm{FI}_A^H$, we may apply a theorem of [GLX] to deduce an equivalence

$$\frac{\mathrm{FI}_A^H-\mathrm{fgMod}}{\mathrm{FI}_A^H-\mathrm{fdMod}} \cong \mathrm{FI}_A-\mathrm{fdMod}$$

which is independent of the choice of the subgroup $H \leq A$.

Chapter 2

Noetherian property for $\mathcal{C}$-modules

The settings of the present investigations are modules over a certain class of categories which we call FI-*like*. Throughout this chapter, we set k to be a commutative noetherian ring with unity.

2.1 FI-Like Categories

Definition 2.1. A category $\mathcal{C}$ is said to be FI-*like* if it satisfies the following axioms:

(1) The objects of $\mathcal{C}$ are the sets $\mathbf{n} = \{1, \ldots, n\}$ for all $n \geq 0$, where $\mathbf{0}$ is the empty set.

(2) The category $\mathcal{C}$ is *hom-finite*, that is, the class of morphisms $\mathcal{C}(\mathbf{m}, \mathbf{n})$ is a finite set for all $\mathbf{m}, \mathbf{n} \in \operatorname{Ob} \mathcal{C}$.

(3) If $m > n$, then $\mathcal{C}(\mathbf{m}, \mathbf{n})$ is the empty set.

Example 2.2. The categories FI, FA, FI_G, and FI_A^H are all FI-like, and the general results in this chapter will be applied to these particular cases in subsequent chapters.

Definition 2.3. Let $\mathcal{C}$ be an FI-like category. A subcategory $\mathcal{A}$ of $\mathcal{C}$ is called *almost-full* if it satisfies the following properties.

(1) The objects of $\mathcal{A}$ are all the objects of $\mathcal{C}$.

(2) If $m \neq n$, then $\mathcal{A}(\mathbf{m}, \mathbf{n}) = \mathcal{C}(\mathbf{m}, \mathbf{n})$.

Remark 2.4. The categories FA and FI_A^H are almost-full subcategories of FI and FI_A respectively. Note that any almost-full subcategory of an FI-like category is also FI-like.

2.2 $\mathcal{C}$-modules

For the remainder of this chapter, we fix an FI-like category $\mathcal{C}$ and an almost full subcategory $\mathcal{A}$ of $\mathcal{C}$. For each pair of nonnegative integers m, n, let $k\mathcal{C}(\mathbf{m}, \mathbf{n})$ be the free k-module with basis $\mathcal{C}(\mathbf{m}, \mathbf{n})$.

Definition 2.5. A $\mathcal{C}$*-module* (over k) is a covariant functor $V : \mathcal{C} \to k$-Mod. A homomorphism of $\mathcal{C}$-modules $V \to W$ is a natural transformation of functors.

Denote by $\mathcal{C}$-Mod (resp. $\mathcal{A}$-Mod) the functor category of $\mathcal{C}$-modules (resp. $\mathcal{A}$-modules). Fix a $\mathcal{C}$-module V. For each $n \geq 0$, we will denote the k-module $V(\mathbf{n})$ by V_n, and given a $\mathcal{C}$-morphism $\alpha : \mathbf{m} \to \mathbf{n}$, we will denote the k-module homomorphism $V(\alpha) : V_m \to V_n$ by α_*. Each V_n is naturally a $k\mathcal{C}(\mathbf{n}, \mathbf{n})$-module via the action defined by

$$\alpha \cdot v = \alpha_*(v) \qquad (\alpha \in \mathcal{C}(\mathbf{n}, \mathbf{n}), \quad v \in V_n).$$

Definition 2.6. Let U be a $\mathcal{C}$-module. We say that U is a $\mathcal{C}$*-submodule* of V if U_n is a $k\mathcal{C}(\mathbf{n}, \mathbf{n})$-submodule of V_n for all $n \geq 0$ and the following diagram commutes for all $0 \leq m \leq n$ and all $\alpha \in \mathcal{C}(\mathbf{m}, \mathbf{n})$:

$$\begin{array}{ccc} U_m & \hookrightarrow & V_m \\ {\scriptstyle U(\alpha)}\downarrow & & \downarrow{\scriptstyle \alpha_*} \\ U_n & \hookrightarrow & V_n \end{array}$$

Definition 2.7. Let S be a subset of $\bigsqcup_{n \geq 0} V_n$. Define the $\mathcal{C}$*-span of* S as the set

$$\mathrm{Span}_{\mathcal{C}}(S) := \left\{ v \in \bigsqcup_{n \geq 0} V_n \;\middle|\; v = \sum_i c_i (\alpha_i)_*(s_i), \quad c_i \in k, \quad \alpha_i \text{ is a } \mathcal{C}\text{-morphism}, \quad s_i \in S \right\}.$$

Remark 2.8. It is clear from Definition 2.3 that $\mathrm{Span}_{\mathcal{A}}(S) \subseteq \mathrm{Span}_{\mathcal{C}}(S)$ for any subset $S \subseteq \bigsqcup_{n\geq 0} V_n$.

Definition 2.9. Let V be a $\mathcal{C}$-module, and let $S \subseteq \bigsqcup_{n\geq 0} V_n$ be a subset.

(a) We say S *generates* V if the only $\mathcal{C}$-submodule W of V with $S \subseteq \bigsqcup_{n\geq 0} W_n$ is $W = V$. Equivalently, the set S generates V if $\bigsqcup_{n\geq 0} V_n = \mathrm{Span}_{\mathcal{C}}(S)$.

(b) We say V is *generated in degrees* $\leq d$ if there is a subset $S \subseteq \bigsqcup_{m\leq d} V_m$ that generates V as a $\mathcal{C}$-module.

(c) We say V is *finitely generated* if there exists a finite set S that generates V as a $\mathcal{C}$-module.

Lemma 2.10. *Let V, W be $\mathcal{C}$-modules with V finitely generated, and suppose there exists a surjective homomorphism of $\mathcal{C}$-modules $f : V \to W$. Then W is finitely generated.*

Proof. Choose a finite set of generators S for V. We claim that the finite set $T = f(S)$ generates W as a $\mathcal{C}$-module. Let $\widetilde{T}$ be the $\mathcal{C}$-submodule of W generated by T, and let $U = f^{-1}(\widetilde{T})$. Since U is a $\mathcal{C}$-submodule of V containing S, we have $U = V$. Therefore $W = f(U) = \widetilde{T}$, which proves the claim. □

Definition 2.11. We say V is *noetherian* if every $\mathcal{C}$-submodule of V is finitely generated. If every finitely generated $\mathcal{C}$-module over k is noetherian, we say that $\mathcal{C}$ is *locally noetherian* over k.

Remark 2.12. If V is an $\mathcal{C}$-module, we may regard V as an $\mathcal{A}$-module via the restriction functor $\mathrm{Res} : \mathcal{C}\text{-Mod} \to \mathcal{A}\text{-Mod}$ where $\mathrm{Res}\, V = V|_{\mathcal{A}}$. Suppose $S \subseteq \bigsqcup_{n\geq 0} V_n$ generates $\mathrm{Res}\, V$ as an $\mathcal{A}$-module. By axiom (1) of Definition 2.3, we have $(\mathrm{Res}\, V)_n = V_n$ for all $n \geq 0$. Then by Remark 2.8, we have

$$\bigsqcup_{n\geq 0} V_n = \bigsqcup_{n\geq 0} (\mathrm{Res}\, V)_n = \mathrm{Span}_{\mathcal{A}}(S) \subseteq \mathrm{Span}_{\mathcal{C}}(S) \subseteq \bigsqcup_{n\geq 0} V_n.$$

So S generates V as a $\mathcal{C}$-module. In particular, if $\mathrm{Res}\, V$ is finitely generated as an $\mathcal{A}$-module then V is finitely generated as a $\mathcal{C}$-module.

Definition 2.13. Given $m \geq 0$, the *free $\mathcal{C}$-module $M_{\mathcal{C}}(\mathbf{m})$* is defined by setting

$$M_{\mathcal{C}}(\mathbf{m})_n = k\mathcal{C}(\mathbf{m}, \mathbf{n}) \qquad (n \geq 0)$$

with the action of $\mathcal{C}$-morphisms by postcomposition.

The next proposition is proved similarly to the analogous [CEF, Prop. 2.3.5] and [W2, Prop. 3.15].

Proposition 2.14. *The $\mathcal{C}$-module V is finitely generated in degree $\leq d$ if and only if there is a surjection of $\mathcal{C}$-modules $\bigoplus_i M_{\mathcal{C}}(\mathbf{m_i}) \to V$ for some finite sequence of integers $m_1, \ldots, m_\ell \leq d$.*

Proof. Assume V is finitely generated in degree $\leq d$. Then there exist vectors

$$v_1, \ldots, v_\ell \in \bigsqcup_{m=1}^{d} V_m$$

that generate V as a $\mathcal{C}$-module. Suppose $v_i \in V_{m_i}$ for each $1 \leq i \leq \ell$. Then $m_1, \ldots, m_\ell \leq d$, and the maps

$$\bigoplus_{i=1}^{\ell} M_{\mathcal{C}}(\mathbf{m_i})_n \to V_n$$

$$\alpha \in \mathcal{C}(\mathbf{m_i}, \mathbf{n}) \mapsto \alpha_*(v_i)$$

define a surjective homomorphism of $\mathcal{C}$-modules.

Now suppose there is a finite sequence of integers $m_1, \ldots, m_\ell \leq d$ such that there exists a surjective $\mathcal{C}$-module homomorphism $\bigoplus_i M_{\mathcal{C}}(\mathbf{m_i}) \to V$. The maps $\mathrm{id}_{\mathbf{m_1}}, \ldots, \mathrm{id}_{\mathbf{m}_\ell}$ generate $\bigoplus_i M_{\mathcal{C}}(\mathbf{m_i})$ as a $\mathcal{C}$-module, so Lemma 2.10 implies that V is finitely generated. □

Definition 2.15. Given an $\mathcal{A}$-module W, define the $\mathcal{C}$-module $\operatorname{Ind} W$ by the following data. Let U_n be the k-submodule of $\bigoplus_{r \leq n} M_{\mathcal{C}}(\mathbf{r})_n \otimes W_r$ generated by elements of the form

$$\alpha_2\alpha_1 \otimes w - \alpha_2 \otimes \alpha_1 w \qquad (w \in W_r, \quad \alpha_2 \in \mathcal{C}(\mathbf{s}, \mathbf{n}), \quad \alpha_1 \in \mathcal{A}(\mathbf{r}, \mathbf{s}), \quad r \leq s \leq n).$$

We define the k-module $(\operatorname{Ind} W)_n$ by setting

$$(\operatorname{Ind} W)_n = \left(\bigoplus_{r \le n} M_{\mathcal{C}}(\mathbf{r})_n \otimes W_r \right) \Big/ U_n$$

with the action of $\beta \in \mathcal{C}(\mathbf{n}, \mathbf{m})$ given by

$$\beta \cdot (\alpha \otimes w) = \beta\alpha \otimes w \qquad (\alpha \in \mathcal{C}(\mathbf{r}, \mathbf{n}), \quad w \in W_r).$$

Remark 2.16. We may regard Ind as a functor $\mathcal{A}$-Mod $\to$ $\mathcal{C}$-Mod in the following manner. Let $f : V \to W$ be a homomorphism of $\mathcal{A}$-modules, i.e. a collection of k-linear maps

$$\{f_n : V_n \to W_n \mid n \ge 0\}$$

such that, given any $m, n \ge 0$, one has $f_n V(\beta) = W(\beta) f_m$ for every $\beta \in \mathcal{A}(\mathbf{m}, \mathbf{n})$. Define the corresponding collection of k-linear maps

$$\begin{aligned} (\operatorname{Ind} f)_n : (\operatorname{Ind} V)_n &\to (\operatorname{Ind} W)_n \\ \alpha \otimes v &\mapsto \alpha \otimes f_r(v) \qquad (\alpha \in \mathcal{C}(\mathbf{r}, \mathbf{n}), \quad v \in V_r). \end{aligned}$$

Given $\mathbf{m}, \mathbf{n} \in \operatorname{Ob} \mathcal{C}$, one can verify that the following diagram commutes for every $\beta \in \mathcal{A}(\mathbf{m}, \mathbf{n})$:

$$\begin{array}{ccc} (\operatorname{Ind} V)_m & \xrightarrow{(\operatorname{Ind} V)(\beta)} & (\operatorname{Ind} V)_n \\ \big\downarrow{\scriptstyle (\operatorname{Ind} f)_m} & & \big\downarrow{\scriptstyle (\operatorname{Ind} f)_n} \\ (\operatorname{Ind} W)_m & \xrightarrow{(\operatorname{Ind} W)(\beta)} & (\operatorname{Ind} W)_n \end{array}$$

Proposition 2.17. *The functor* Ind *is left adjoint to* Res. *The unit* $\eta : \mathrm{id}_{\mathcal{A}\text{-Mod}} \to \operatorname{Res} \operatorname{Ind}$ *of this adjunction is given by the collection of* $\mathcal{A}$*-module homomorphisms*

$$\{\eta_W : W \to \operatorname{Res} \operatorname{Ind} W \mid W \in \mathcal{A}\text{-Mod}\}$$

where, given $W \in \mathcal{A}$-Mod and $n \geq 0$, the map η_W is given by $w \in W_n \mapsto \mathrm{id}_{\mathbf{n}} \otimes w$.

Remark 2.18. As k-modules, we have $V_n = (\mathrm{Res}\, V)_n$ for every $V \in \mathcal{C}$-Mod and $n \geq 0$. In particular, given a homomorphism $f : V \to W$ of $\mathcal{C}$-modules, the maps

$$f_n : V_n \to W_n \quad \text{and} \quad (\mathrm{Res}\, f)_n : \mathrm{Res}\, V_n \to \mathrm{Res}\, W_n$$

have the same kernel and cokernel. Thus Res is exact. The functor Ind is right exact because it has a right adjoint.

The next result is analogous to [W2, Prop. 3.26].

Proposition 2.19. *For each $m \geq 0$, there is a natural isomorphism of $\mathcal{C}$-modules*

$$\mathrm{Ind}\, M_{\mathcal{A}}(\mathbf{m}) \cong M_{\mathcal{C}}(\mathbf{m}).$$

Proof. Let m, r, s, n be nonnegative integers such that $m \leq r \leq s \leq n$. Define the k-modules

$$A_r := M_{\mathcal{C}}(\mathbf{r})_n \otimes_k M_{\mathcal{A}}(\mathbf{m})_r \quad \text{and} \quad A := \bigoplus_{m \leq r \leq n} A_r.$$

Let $\Phi : A \to M_{\mathcal{C}}(\mathbf{m})_n$ be the k-linear map given by $\Phi(\alpha \otimes \beta) = \alpha\beta$. Given $\gamma \in \mathcal{C}(\mathbf{m}, \mathbf{n})$, the map Φ sends $\gamma \otimes \mathrm{id}_{\mathbf{m}} \in A_m$ to γ. Every generator of $M_{\mathcal{C}}(\mathbf{m})_n$ may be obtained in this way, so Φ is surjective. Furthermore, we have

$$\Phi(\alpha \otimes \tau\beta) = \alpha(\tau\beta) = (\alpha\tau)\beta = \Phi(\alpha\tau \otimes \beta) \qquad (\alpha \in \mathcal{C}(\mathbf{s}, \mathbf{n}), \quad \beta \in \mathcal{A}(\mathbf{m}, \mathbf{r}), \quad \tau \in \mathcal{A}(\mathbf{r}, \mathbf{s})).$$

Therefore $\ker \Phi$ contains the submodule U_n in Definition 2.15. Let $\widetilde{\Phi}$ be the (surjective) k-linear map induced by Φ on the quotient $A/U = \mathrm{Ind}\, M_{\mathcal{A}}(\mathbf{m})$. Suppose $\alpha \in \mathcal{C}(\mathbf{s}, \mathbf{n})$, $\beta \in \mathcal{A}(\mathbf{m}, \mathbf{r})$ satisfy $\widetilde{\Phi}(\alpha \otimes \beta) = \gamma$. Then $\alpha\beta = \gamma$, and by definition of the quotient we have $\alpha \otimes \beta = \alpha\beta \otimes \mathrm{id}_{\mathbf{m}}$.

Define a k-linear map $\Psi : M_{\mathcal{C}}(\mathbf{m})_n \to A$ by $\Psi(\gamma) = \gamma \otimes \mathrm{id}_m$. Clearly $\Phi(\Psi(\gamma)) = \gamma$. Furthermore, for all $\alpha \in \mathcal{C}(\mathbf{r}, \mathbf{n})$ and $\beta \in \mathcal{A}(\mathbf{m}, \mathbf{r})$, we have

$$\Psi(\Phi(\alpha \otimes \beta)) = \Psi(\alpha\beta) = \alpha\beta \otimes \mathrm{id}_m = \alpha \otimes \beta.$$

where the last equality follows from the relations in the quotient A/U. $\square$

Definition 2.20. Suppose V is finitely generated. We say V is *finitely presented* with *generator degree* $\leq g$ and *relation degree* $\leq r$ if there exists an exact sequence

$$\bigoplus_{m=0}^{r} M_{\mathcal{C}}(\mathbf{m})^{\oplus \ell_m} \longrightarrow \bigoplus_{n=0}^{g} M_{\mathcal{C}}(\mathbf{n})^{\oplus c_n} \longrightarrow V \longrightarrow 0.$$

Remark 2.21. Note that if V is generated in degrees $\leq d$ and has generator degree $\leq g$, then $g \leq d$. If the category $\mathcal{C}$ is locally noetherian, then all finitely generated $\mathcal{C}$-modules are finitely presented.

The following proposition is proved similarly to [W2, Cor. 3.28].

Proposition 2.22. *Suppose W is a finitely presented $\mathcal{A}$-module with generator degree $\leq g$ and relation degree $\leq r$. Then* Ind W *is finitely presented with generator degree $\leq g$ and relation degree $\leq r$.*

Proof. By Definition 2.20, there is an exact sequence

$$\bigoplus_{m=0}^{r} M_{\mathcal{A}}(\mathbf{m})^{\oplus \ell_m} \longrightarrow \bigoplus_{n=0}^{g} M_{\mathcal{A}}(\mathbf{n})^{\oplus c_n} \longrightarrow W \longrightarrow 0.$$

Applying the right exact functor Ind produces the exact sequence

$$\bigoplus_{m=0}^{r} \mathrm{Ind}\, M_{\mathcal{A}}(\mathbf{m})^{\oplus \ell_m} \longrightarrow \bigoplus_{n=0}^{g} \mathrm{Ind}\, M_{\mathcal{A}}(\mathbf{n})^{\oplus c_n} \longrightarrow \mathrm{Ind}\, W \longrightarrow 0.$$

Using Proposition 2.19, we may write this sequence as

$$\bigoplus_{m=0}^{r} M_{\mathcal{C}}(\mathbf{m})^{\oplus \ell_m} \longrightarrow \bigoplus_{n=0}^{g} M_{\mathcal{C}}(\mathbf{n})^{\oplus c_n} \longrightarrow \operatorname{Ind} W \longrightarrow 0$$

which is what we wanted to show. $\square$

Lemma 2.23. *If $\mathcal{C}$ is locally noetherian over k, then every $\mathcal{A}$-submodule of $M_{\mathcal{A}}(\mathbf{m})$ is finitely generated.*

Proof. Let V be an $\mathcal{A}$-submodule of $M_{\mathcal{A}}(\mathbf{m})$, and let $\widetilde{V}$ be the $\mathcal{C}$-submodule of $M_{\mathcal{C}}(\mathbf{m})$ generated by $\bigsqcup_{n\geq 0} V_n$. By assumption, $\widetilde{V}$ is finitely generated as a $\mathcal{C}$-module, say by $v_1, \ldots, v_r \in \bigsqcup_{m=1}^{d} \widetilde{V}_m$ where $v_i \in \widetilde{V}_{m_i}$ for each i. Since $\widetilde{V}$ is generated as a $\mathcal{C}$-module by V, for $i = 1, \ldots, r$ we may choose $u_{i,j} \in \bigsqcup_{i\geq 0} V_i$ and $\mathcal{C}$-morphisms $\alpha_{i,j} : \mathbf{r}_{i,j} \to \mathbf{m}_i$ such that

$$v_i = \sum_{j=1}^{s} (\alpha_{i,j})_*(u_{i,j}).$$

Fix $n > d$ and let $v \in V_n$. Then $v \in \widetilde{V}_n$, and we may write

$$v = \sum_{i=1}^{r} b_i(\beta_i)_*(v_i) \qquad (b_i \in k, \quad \beta_i \in \mathcal{C}(\mathbf{m_i}, \mathbf{n})).$$

Hence

$$v = \sum_{i,j=1}^{rs} b_i(\beta_i \circ \alpha_{i,j})_*(u_{i,j}).$$

Since $|\mathbf{r}_{i,j}| \leq d < n$ for all $i = 1, \ldots, r$ and $j = 1, \ldots, s$, each $\mathcal{C}$-morphism $\beta_i \circ \alpha_{i,j} : \mathbf{r}_{i,j} \to \mathbf{n}$ is an $\mathcal{A}$-morphism. Therefore $\bigsqcup_{i\geq d} V_i$ is in the $\mathcal{A}$-span of the $u_{i,j}$, meaning V is generated as an $\mathcal{A}$-module in degree $\leq d$. By assumption, the k-module V_i is finitely generated for all $i \geq 0$, so it follows that V is finitely generated as an $\mathcal{A}$-module. $\square$

Proposition 2.24. *Let W be an $\mathcal{A}$-module finitely presented with generator degree $\leq g$ and relation degree $\leq r$. Then the natural map*

$$(\eta_W)_n : W_n \to (\operatorname{Res}\operatorname{Ind} W)_n$$

is a bijection for all $n > \max\{g, r\}$.

Proof. By assumption, there is an exact sequence

$$\bigoplus_{j=1}^{q} M_{\mathcal{A}}(\mathbf{n_j}) \longrightarrow \bigoplus_{i=1}^{p} M_{\mathcal{A}}(\mathbf{m_i}) \longrightarrow W \longrightarrow 0$$

for some integers $m_1, \ldots, m_p \leq g$ and $n_1, \ldots, n_q \leq r$. Applying the right exact functor Ind and then the exact functor Res, we obtain the exact sequence

$$\bigoplus_{j=1}^{q} \operatorname{Res}\operatorname{Ind} M_{\mathcal{A}}(\mathbf{n_j}) \longrightarrow \bigoplus_{i=1}^{p} \operatorname{Res}\operatorname{Ind} M_{\mathcal{A}}(\mathbf{m_i}) \longrightarrow \operatorname{Res}\operatorname{Ind} W \longrightarrow 0 \qquad (2.25)$$

Using the identification in Proposition 2.19, we may write (2.25) as

$$\bigoplus_{j=1}^{q} \operatorname{Res} M_{\mathcal{C}}(\mathbf{n_j}) \longrightarrow \bigoplus_{i=1}^{p} \operatorname{Res} M_{\mathcal{C}}(\mathbf{m_i}) \longrightarrow \operatorname{Res}\operatorname{Ind} W \longrightarrow 0$$

Thus for $n \geq 0$, we have the following commuting diagram with exact rows:

$$\begin{array}{ccccccc} \bigoplus_{j=1}^{q} M_{\mathcal{A}}(\mathbf{n_j})_n & \longrightarrow & \bigoplus_{i=1}^{p} M_{\mathcal{A}}(\mathbf{m_i})_n & \longrightarrow & W_n & \longrightarrow & 0 \\ \downarrow & & \downarrow & & \downarrow{\scriptstyle (\eta_W)_n} & & \\ \bigoplus_{j=1}^{q} \operatorname{Res} M_{\mathcal{C}}(\mathbf{n_j})_n & \longrightarrow & \bigoplus_{i=1}^{p} \operatorname{Res} M_{\mathcal{C}}(\mathbf{m_i})_n & \longrightarrow & \operatorname{Res}\operatorname{Ind} W_n & \longrightarrow & 0 \end{array} \qquad (2.26)$$

Suppose $n > \max\{m_1, \ldots, m_p, n_1, \ldots, n_q\}$. Note that if $n > m$, the k-modules $M_{\mathcal{A}}(\mathbf{m})_n$ and $M_{\mathcal{C}}(\mathbf{m})_n$ are freely generated on the same basis. It follows that the two vertical maps on the left in (2.26) are bijective. This implies the vertical map on the right is bijective. □

Proposition 2.27. *The $\mathcal{C}$-module V is finitely generated if and only if* Res V *is finitely generated as an $\mathcal{A}$-module.*

Proof. The "if" statement is established by Remark 2.12; we will prove the "only if" statement. Using Proposition 2.14, we will reduce to the case where $V = M_{\mathcal{C}}(\mathbf{m})$. In particular, we claim that Res $M_{\mathcal{C}}(\mathbf{m})$ is generated as an $\mathcal{A}$-module by the finite set $\mathcal{C}(\mathbf{m}, \mathbf{m})$.

Let $v \in M_{\mathcal{C}}(\mathbf{m})_n$ and write

$$v = \sum_{i=1}^{r} c_i(\alpha_i \circ \mathrm{id}_{\mathbf{m}}) \qquad (c_1, \ldots, c_r \in k, \quad \alpha_1, \ldots, \alpha_r \in \mathcal{C}(\mathbf{m}, \mathbf{n})). \tag{2.28}$$

If $n > m$, then $\mathcal{C}(\mathbf{m}, \mathbf{n}) = \mathcal{A}(\mathbf{m}, \mathbf{n})$, so equation (2.28) means that v is in the $\mathcal{A}$-span of $\{\mathrm{id}_{\mathbf{m}}\} \subseteq \mathcal{C}(\mathbf{m}, \mathbf{m})$. Now suppose $n = m$. Then equation (2.28) implies

$$v = \sum_{i=1}^{r} c_i(\mathrm{id}_{\mathbf{m}} \circ \alpha_i). \tag{2.29}$$

Since $\mathrm{id}_{\mathbf{m}}$ is an $\mathcal{A}$-morphism, equation (2.29) means that v is in the $\mathcal{A}$-span of $\{\alpha_1, \ldots, \alpha_r\} \subseteq \mathcal{C}(\mathbf{m}, \mathbf{m})$. This proves the claim.

Now suppose V is any finitely generated $\mathcal{C}$-module. By Proposition 2.14, there is an exact sequence

$$\bigoplus_{i=1}^{\ell} M_{\mathcal{C}}(\mathbf{m}_i) \longrightarrow V \longrightarrow 0.$$

Since Res is exact, we thereby obtain an exact sequence

$$\bigoplus_{i=1}^{\ell} \operatorname{Res} M_{\mathcal{C}}(\mathbf{m}_i) \longrightarrow \operatorname{Res} V \longrightarrow 0.$$

By the claim, $\bigoplus_{i=1}^{\ell} \operatorname{Res} M_{\mathcal{C}}(\mathbf{m}_i)$ is finitely generated. It follows from Lemma 2.10 that $\operatorname{Res} V$ is finitely generated. □

We are now prepared to prove Theorem A:

Theorem A. *The category $\mathcal{C}$ is locally noetherian if and only if $\mathcal{A}$ is locally noetherian.*

Proof. (⇒) Suppose $\mathcal{C}$ is locally Noetherian. Let W be a finitely generated $\mathcal{A}$-module over k. By Proposition 2.14, there exists a surjection of $\mathcal{A}$-modules

$$f : \bigoplus_{i=1}^{\ell} M_{\mathcal{A}}(\mathbf{m}_i) \to W.$$

Given an $\mathcal{A}$-submodule U of W, Lemma 2.23 implies that its preimage $f^{-1}(U)$ is finitely generated as an $\mathcal{A}$-module. Then Lemma 2.10 implies $f(f^{-1}(U)) = U$ is finitely generated.

(⇐) Now suppose $\mathcal{A}$ is locally noetherian. Let V be a finitely generated $\mathcal{C}$-module over k, and let U be a $\mathcal{C}$-submodule of V. By Proposition 2.27, $\operatorname{Res} V$ is finitely generated as an $\mathcal{A}$-module. Then by assumption, $\operatorname{Res} U$ is finitely generated. Thus U is finitely generated as a $\mathcal{C}$-module by Remark 2.12. □

Corollary 2.30. *The categories* FA *and* FI_A^H *are locally noetherian.*

Proof. The categories FI and FI_A are locally noetherian (Theorem 1.13; [SS, Thm 1.2.3]). Since FA and FI_A^H are almost-full subcategories of FI and FI_A respectively, Theorem A implies that FA and FI_A^H are locally noetherian. □

Chapter 3

Representation stability

Our goal in this chapter is to prove analogues of Theorem 1.14 for $\mathcal{A}$-modules, where $\mathcal{A}$ is one of the categories FA or FI_A^H. First we define representation stability for consistent sequences of representations of each of the families of groups $\{A_n\}$ and $\{J_n\}$ over a suitably chosen field. We then prove the equivalence of representation stability for such sequences to finite generation of the corresponding $\mathcal{A}$-modules.

3.1 Alternating groups

Throughout this subsection, we work over a field k with characteristic zero, and fix a FA-module W over k.

Lemma 3.1. *Let m, n be positive integers such that $n \geq m + 2$. The group $A_n = \mathrm{FA}(\mathbf{n}, \mathbf{n})$ acts transitively on the set $\mathrm{FA}(\mathbf{m}, \mathbf{n})$ by postcomposition.*

Proof. Let α and β be elements of $\mathrm{FA}(\mathbf{m}, \mathbf{n})$. Choose $\sigma \in S_n$ such that $\beta = \sigma\alpha$. If σ is even, then $\sigma \in A_n$, as desired. Suppose σ is odd. By assumption, there are distinct positive integers $a, b \in \mathbf{n}$ that are not contained in $\alpha(\mathbf{m})$. Denote the product of σ with the transposition (a, b) by τ. Then τ is in A_n and satisfies $\beta = \tau\alpha$. □

Note that if W is finitely generated, then the sequence $\{W_n, (\iota_{n,n+1})_*\}$ is consistent.

Lemma 3.2. *Suppose W is finitely generated as an* FA*-module. Then $\{W_n, (\iota_{n,n+1})_*\}$ satisfies the injectivity and surjectivity conditions of Definition 1.2.*

Proof. First we show that $\{W_n, (\iota_{n,n+1})_*\}$ satisfies the surjectivity condition. Assume W is generated by S in degrees $\leq d$. Let $n > d$ and $w \in W_{n+1}$. We may write

$$w = \sum_i c_i (\alpha_i)_*(s_i) \qquad (c_i \in \mathbb{C}, \quad \alpha_i \in \mathrm{FA}(\mathbf{m_i}, \mathbf{n+1}), \quad s_i \in S) \tag{3.3}$$

where $m_i \leq d < n$ for all i. This last inequality implies $(n+1) - m_i \geq 2$, so (Lemma 3.1) A_{n+1} acts transitively on $\mathrm{FA}(\mathbf{m_i}, \mathbf{n+1})$. For each i, choose $\beta_i \in A_{n+1}$ such that $\alpha_i = \beta_i \cdot \iota_{m_i,n+1}$. Then we may write equation (3.3) as

$$w = \sum_i c_i(\beta_i \cdot (\iota_{m_i,n+1})_*(s_i)) = \sum_i c_i(\beta_i \cdot (\iota_{n,n+1})_*((\iota_{m_i,n})_*(s_i)))$$

So w is in the span of the A_{n+1}-orbit of $(\iota_{n,n+1})_*(W_n)$, as desired.

Now we will show that $\{W_n, (\iota_{n,n+1})_*\}$ satisfies the injectivity condition. Let $U_n = \ker(\iota_{n,n+1})_*$ for each $n \geq 0$. We claim that U_n is trivial for sufficiently large n. By an argument similar to Remark 1.11, the consistent sequence consisting of the U_n together with the zero maps $U_n \to U_{n+1}$ determines an FA-submodule U of W. By the noetherian property for FA (Corollary 2.30), U is finitely generated, say in degrees $\leq d$. Let $n > d$; since the surjectivity property holds in the range $n > d$, the span of the A_{n+1} orbit of $(\iota_{n,n+1})_*(U_n) = \{0\}$ is all of U_{n+1}. Therefore U_{n+1} is trivial, as claimed.

□

Let $\mathcal{P}$ denote the set of all partitions. Given a partition $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_\ell)$ of n, let $\mathcal{Y}_\lambda$ denote the Young diagram corresponding to λ and let V_λ denote the irreducible representation of S_n corresponding to λ. Denote by λ' the *conjugate* of λ, i.e. the partition corresponding to the Young diagram obtained by transposing the rows and columns of $\mathcal{Y}_\lambda$. We say λ is *symmetric* if $\lambda = \lambda'$.

Proposition 3.4 ([FH, 5.1])**.** *Let $n \geq 2$ and let λ be a partition of n. If λ is non-symmetric, then the restrictions of V_λ and $V_{\lambda'}$ to A_n are isomorphic irreducible representations of A_n.*

On the other hand, if λ is symmetric then the restriction of $V_\lambda = V_{\lambda'}$ to A_n decomposes as the direct sum of two non-isomorphic irreducible representations of A_n. Moreover, every irreducible representation of A_n may be obtained in one of these two ways.

Notation 3.5. Let $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_\ell)$ be a partition of m. Define the A_n-representation $W(\lambda)_n$ to be the restriction of $V(\lambda)_n$ to A_n.

Remark 3.6. Note that $W(\lambda)_n$ is reducible precisely when $\lambda[n]$ is symmetric, which for fixed λ occurs for at most one value of n. Furthermore, the padded partition $\lambda[n]$ is non-symmetric, and thus $W(\lambda)_n$ is irreducible, whenever $n > 2m + 1$.

For the remainder of this section, let $\{W_n, \psi_n\}$ denote a consistent sequence of A_n-representations.

Definition 3.7. We say $\{W_n, \psi_n\}$ is *multiplicity stable* with *stable range* $n \geq N$ if there exists a positive integer N such that the following hold for all $n \geq N$:

(a) There exists a decomposition of W_n into irreducible A_n-representations as

$$W_n \cong \bigoplus_{\lambda \in \mathcal{P}} W(\lambda)_n^{\oplus c_{\lambda,n}}.$$

In particular, $W(\lambda)_n$ is irreducible for all λ.

(b) $c_{\lambda,n} = c_{\lambda,N}$ for all $\lambda \in \mathcal{P}$.

Definition 3.8. We say $\{W_n, \psi_n\}$ is *representation stable* if it satisfies the injectivity and surjectivity conditions of Definition 1.2, and is multiplicity stable in the sense of Definition 3.7.

The following theorem is proved in a similar manner as [W2, Thm. 4.27].

Theorem B. *Let k be a field of characteristic zero. An* FA*-module W is finitely generated if and only if $\{W_n, (\iota_{n,n+1})_*\}$ is representation stable and W_n is finite-dimensional for all $n \geq 0$.*

Proof. To prove the "if" direction, suppose $\{W_n, (\iota_{n,n+1})_*\}$ is representation stable and W_n is finite-dimensional for all $n \geq 0$. The surjectivity property for $\{W_n, (\iota_{n,n+1})_*\}$ implies the existence of a generating set S of W. Since the W_n are finite-dimensional, the generating set S must be finite.

To prove the "only if" direction, suppose W is finitely generated. Then $\{W_n, (\iota_{n,n+1})_*\}$ satisfies the injectivity and surjectivity properties (Lemma 3.2). It remains to show that $\{W_n, (\iota_{n,n+1})_*\}$ is multiplicity stable.

The FA-module W is finitely presented (Remark 2.21), say with generator degree $\leq g$ and relation degree $\leq r$. It follows (Proposition 2.22) that $\operatorname{Ind} W$ is finitely presented, hence finitely generated as an FI-module. Therefore (Theorem 1.14) the consistent sequence $\{(\operatorname{Ind} W)_n, (\iota_{n,n+1})_*\}$ is multiplicity stable, meaning that there exists a positive integer N not depending on λ such that, in the decomposition

$$(\operatorname{Ind} W)_n \cong \bigoplus_{\lambda \in \mathcal{P}} V(\lambda)_n^{\oplus c_{\lambda,n}} \tag{3.9}$$

the multiplicities $c_{\lambda,n}$ do not depend on n for all $n \geq N$ and all λ. Furthermore, since each V_n is finite-dimensional, there are at most finitely many λ such that $c_{\lambda,n} \neq 0$. Let m be the maximum of $|\lambda|$ over all such λ.

Let $n > \max\{N, 2m+1, g, r\}$. By Proposition 2.24, there is an isomorphism $W_n \cong (\operatorname{Res} \operatorname{Ind} W)_n$, and by Remark 3.6, the A_n-representation $W(\lambda)_n$ is irreducible. So, restricting the decomposition 3.26 to A_n, we obtain

$$(\operatorname{Res} \operatorname{Ind} W)_n \cong W_n \cong \bigoplus_{\lambda \in \mathcal{P}} W(\lambda)_n^{\oplus c_{\lambda,n}}$$

where the $W(\lambda)_n$ are irreducible and the multiplicities $c_{\lambda,n}$ do not depend on n for all λ. Hence W is multiplicity stable. □

3.2 Wreath products

The exposition in this section proceeds in similar order as the previous section, although we now assume $k = \mathbb{C}$.

Lemma 3.10. *Let m, n be positive integers such that $n \geq m$. The group $J_n = \mathrm{FI}_A^H(\mathbf{n}, \mathbf{n})$ acts transitively on the set $\mathrm{FI}_A^H(\mathbf{m}, \mathbf{n})$ by postcomposition.*

Proof. The result is trivial in the case $m = n$. Suppose $n > m$. Let (α, f) and (β, g) be elements of $\mathrm{FI}_A^H(\mathbf{m}, \mathbf{n})$. Choose $\sigma \in S_n$ such that $\beta = \sigma\alpha$, and choose $t \in \mathbf{n} \setminus \alpha(\mathbf{m})$. Define $h : \mathbf{n} \to A$ by setting

$$h(s) = \begin{cases} g(\alpha^{-1}(s))f(\alpha^{-1}(s))^{-1} & \text{if } s \in \alpha(\mathbf{m}) \\ \prod_{i \in \mathbf{m}} g(i)^{-1} f(i) & \text{if } s = t \\ e & \text{otherwise.} \end{cases}$$

Then

$$h(1) \cdots h(n) = \left(\prod_{s \in \alpha(\mathbf{m})} h(s) \right) h(t) = \left(\prod_{i \in \mathbf{m}} g(i) f(i)^{-1} \right) \left(\prod_{i \in \mathbf{m}} g(i)^{-1} f(i) \right) = e \in H.$$

Therefore (σ, h) is in J_n. Moreover, we have $(\sigma, h) \circ (\alpha, f) = (\beta, g')$ where

$$g'(i) = h(\alpha(i))f(i) = g(\alpha^{-1}(\alpha(i)))f(\alpha^{-1}(\alpha(i)))^{-1}f(i) = g(i)f(i)^{-1}f(i) = g(i)$$

for all $i \in \mathbf{m}$. □

For the remainder of this section, let $\mu_n : \mathbf{n} \to \mathbf{n+1}$ denote the the FI_A^H-morphism consisting of the inclusion $\iota_{n,n+1} : \mathbf{n} \hookrightarrow \mathbf{n+1}$ and the trivial map $T : \mathbf{n} \to A$. Given $(\sigma, f) \in J_n$, we may regard (σ, f) as an element of J_{n+1} where $\sigma(n+1) = n+1$ and $f(n+1) = e$. Thus,

$$(\sigma, f) \circ \mu_n = (\sigma \circ \iota_{n,n+1}, (f \circ \iota_{n,n+1})T) = (\iota_{n,n+1} \circ \sigma, (T \circ \sigma)f) = \mu_n \circ (\sigma, f).$$

It follows that, given an FI_A^H-module W, the sequence $\{W_n, (\mu_n)_*\}$ is consistent.

The following lemma corresponds to Lemma 3.2, and is proved in a similar manner.

Lemma 3.11. *Suppose W is finitely generated as an FI_A^H-module. Then $\{W_n, (\mu_n)_*\}$ satisfies the injectivity and surjectivity conditions of Definition 1.2.*

Proof. First we show that $\{W_n, (\mu_n)_*\}$ satisfies the surjectivity condition. Assume W is generated by S in degrees $\leq d$. Let $n \geq d$ and $w \in W_{n+1}$. We may write

$$w = \sum_i c_i(\eta_i)_*(s_i) \qquad (c_i \in \mathbb{C}, \quad \eta_i \in \mathrm{FI}_A^H(\mathbf{m_i}, \mathbf{n+1}), \quad s_i \in S) \tag{3.12}$$

where $m_i \leq d \leq n$ for all i. Lemma 3.10 implies that J_{n+1} acts transitively on $\mathrm{FI}_A^H(\mathbf{m_i}, \mathbf{n+1})$. For each i, choose $\theta_i \in J_{n+1}$ such that $\eta_i = \theta_i \cdot (\iota_{m_i,n+1}, T)$. Then we may write equation (3.12) as

$$w = \sum_i c_i(\theta_i \cdot (\iota_{m_i,n+1}, T)_*(s_i)) = \sum_i c_i(\theta_i \cdot (\mu_n)_*((\iota_{m_i,n}, T)_*(s_i)))$$

So w is in the span of the J_{n+1}-orbit of $(\mu_n)_*(W_n)$, as desired.

Now we will show that $\{W_n, (\mu_n)_*\}$ satisfies the injectivity condition. Let $U_n = \ker(\mu_n)_*$ for each $n \geq 0$. We claim that U_n is trivial for sufficiently large n. By an argument similar to Remark 1.11, the consistent sequence consisting of the U_n together with the zero maps $U_n \to U_{n+1}$ determines an FA-submodule U of W. By the noetherian property for FI_A^H (Corollary 2.30), U is finitely generated, say in degrees $\leq d$. Let $n \geq d$; since the surjectivity property holds in the range $n \geq d$, the span of the J_{n+1} orbit of $(\mu_n)_*(U_n) = \{0\}$ is all of U_{n+1}. Therefore U_{n+1} is trivial, as claimed. □

Recall from Notation 1.18 the definition of the groups $K_n \leq J_n \leq G_n$. We now move towards defining multiplicity stability for consistent sequences of $\mathbb{C}$-linear representations of the groups J_n. First, we recall the representation theory of the wreath products $G_n = A \wr S_n$, where A is a finite abelian group, over the complex numbers. For a general treatment, see [S, 8.2].

Notation 3.13. Let $\mathrm{Irr}(A)$ be a complete system of distinct irreducible characters of A. We write

$$\mathrm{Irr}(A) = \{\, \psi_i : A \to \mathbb{C}^* \mid i \in I\}$$

where $|I| = |A|$.

Let $\{c_i\}_{i\in I}$ be a collection of nonnegative integers such that $\sum_{i\in I} c_i = n$. Then the map $\chi : A^n \to \mathbb{C}^*$ given by

$$\chi = \bigotimes_{i\in I} \psi_i^{\otimes c_i} \tag{3.14}$$

is an irreducible character of A^n. Moreover, any irreducible character of A^n may be written uniquely (up to equivalence) in this form. Let $\mathrm{Irr}(A^n)$ denote the complete system of pairwise inequivalent irreducible characters of A^n obtained in this manner.

The symmetric group S_n acts on $\chi \in \mathrm{Irr}(A^n)$ by permuting the order of the ψ_i in the decomposition (3.14). Let $\mathrm{Stab}(\chi)$ be the stabilizer in S_n of χ; we have

$$\mathrm{Stab}(\chi) \cong \prod_{i\in I} S_{c_i}.$$

Let $\underline{\lambda} : I \to \mathcal{P}$ be a partition-valued function such that $|\underline{\lambda}(i)| = c_i$ for all i, so that $|\underline{\lambda}| = n$, where

$$|\underline{\lambda}| := \sum_{i\in I} |\underline{\lambda}(i)|.$$

As before, $V_{\underline{\lambda}(i)}$ denotes the irreducible representation of S_{c_i} corresponding to the partition $\underline{\lambda}(i)$.

The following is a special case of [S, Prop. 25]:

Proposition 3.15. *Let $\underline{\lambda} : I \to \mathcal{P}$ be a partition-valued function such that $|\underline{\lambda}| = n$. Then $\underline{\lambda}$ determines an irreducible representation of G_n given by*

$$\mathrm{Ind}_{A^n \rtimes \mathrm{Stab}(\chi)}^{G_n} \left(\chi \otimes \bigotimes_{i\in I} V_{\underline{\lambda}(i)} \right)$$

where $c_i = |\underline{\lambda}(i)|$ *for all* $i \in I$ *and* $\chi = \bigotimes_{i \in I} \psi_i^{\otimes c_i}$. *Moreover, every irreducible representation of* G_n *(up to equivalence) has this form for some unique* $\underline{\lambda}$.

In order to define multiplicity stability for consistent sequences of G_n-representations over $\mathbb{C}$, it is necessary to define some labelling of the irreducible representations of G_n that does not depend on n. This is done in [GL, p. 3], which we recall below.

Notation 3.16. Let $\underline{\lambda} : I \to \mathcal{P}$ be a partition-valued function with $|\underline{\lambda}| = m$. Suppose that ψ_t is the trivial character of A, and write $\underline{\lambda}(t) = (\lambda_1, \ldots, \lambda_\ell)$. For $n \geq m + \lambda_1$, we define the "padded" partition-valued function $\underline{\lambda}[n] : I \to \mathcal{P}$ by setting

$$\underline{\lambda}[n](i) = \begin{cases} (n-m, \lambda_1, \ldots, \lambda_\ell) & \text{if } i = t \\ \underline{\lambda}(i) & \text{if } i \neq t. \end{cases}$$

We have

$$\begin{aligned} |\underline{\lambda}[n]| &= \sum_{i \neq t} |\underline{\lambda}(i)| + |\underline{\lambda}[n](t)| \\ &= (m - |\underline{\lambda}(t)|) + (n - m + |\underline{\lambda}(t)|) \\ &= n. \end{aligned}$$

Therefore (Proposition 3.15) the function $\underline{\lambda}[n]$ determines a unique (up to equivalence) irreducible representation of G_n. Denote this representation by $V(\underline{\lambda})_n$. Given n such that $n < m + \lambda_1$, we set $V(\underline{\lambda})_n$ to be the trivial representation.

Definition 3.17. [GL, Def. 1.10] Let $\{V_n, \varphi_n\}$ be a consistent sequence of finite-dimensional G_n-representations. We say $\{V_n, \varphi_n\}$ is *multiplicity stable* if there exists a positive integer N such that, in the decomposition

$$V_n \cong \bigoplus_{\underline{\lambda}: I \to \mathcal{P}} V(\underline{\lambda})_n^{\oplus c_{\underline{\lambda},n}}$$

the multiplicities $c_{\underline{\lambda},n}$ do not depend on n for all $\underline{\lambda}$ and all $n \geq N$.

We may similarly define multiplicity stability for a consistent sequence $\{J_n, \varphi_n\}$ of the subgroups $J_n = H' \rtimes S_n$ of G_n. Given one of the functions $\underline{\lambda} : I \to \mathcal{P}$, let $W(\underline{\lambda})_n$ denote the restriction of the G_n-representation $V(\underline{\lambda})_n$ to J_n. Note that $W(\underline{\lambda})_n$ is not necessarily irreducible as a J_n-representation.

Definition 3.18. Let $\{W_n, \varphi_n\}$ be a consistent sequence of finite-dimensional J_n-representations. We say $\{W_n, \varphi_n\}$ is *multiplicity stable* if there exists a positive integer N such that the following hold for all $n \geq N$:

(a) We may decompose W_n as

$$W_n \cong \bigoplus_{\underline{\lambda}: I \to \mathcal{P}} W(\underline{\lambda})_n^{\oplus c_{\underline{\lambda},n}}$$

where $W(\underline{\lambda})_n$ is irreducible for all $\underline{\lambda}$ such that $c_{\underline{\lambda},n} \neq 0$.

(b) $c_{\underline{\lambda},n} = c_{\underline{\lambda},N}$ for all $\underline{\lambda}$.

Given $\chi \in \mathrm{Irr}(A^n)$, we will denote the restriction of χ to the subgroup $\langle e \rangle' \leq A^n$ by χ'. Note that since χ has degree 1, its restriction χ' also has degree 1, hence is irreducible.

Lemma 3.19. *Let $\chi_1 = \psi_{i_1} \otimes \cdots \otimes \psi_{i_n}$ and $\chi_2 = \psi_{j_1} \otimes \cdots \otimes \psi_{j_n}$ be characters of A^n, where $n \geq 2$. The characters χ_1' and χ_2' of $\langle e \rangle'$ are identical if and only if*

$$\psi_{i_1}\psi_{j_1}^{-1} = \psi_{i_2}\psi_{j_2}^{-1} = \cdots = \psi_{i_n}\psi_{j_n}^{-1}.$$

Proof. To prove the "if" direction, suppose that $\psi_{i_1}\psi_{j_1}^{-1} = \psi_{i_2}\psi_{j_2}^{-1} = \cdots = \psi_{i_n}\psi_{j_n}^{-1}$. This is an irreducible character of A, say ψ_k. We have

$$\chi_1\chi_2^{-1} = (\psi_{i_1} \otimes \cdots \otimes \psi_{i_n})(\psi_{j_1} \otimes \cdots \otimes \psi_{j_n})^{-1} = \psi_k^n.$$

In particular, for any $(a_1, \ldots, a_n) \in \langle e \rangle'$ we have

$$(\chi_1\chi_2^{-1})(a_1, \ldots, a_n) = \psi_k(a_1) \cdots \psi_k(a_n) = \psi_k(a_1 \cdots a_n) = \psi_k(e) = 1.$$

Hence $\chi'_1 = \chi'_2$.

To prove the "only if" direction, suppose $\chi'_1 = \chi'_2$ and choose any element $a \in A$. For each positive integer $m \leq n-1$, let $x_m = (e, \dots, e, a, a^{-1}, e, \dots, e)$ be the element of $\langle e \rangle'$ with a in the m-th position, a^{-1} in the $(m+1)$-th position, and e in every other position. Then the hypothesis implies

$$\begin{aligned} \chi_1(x_m) &= \chi_2(x_m) \\ \psi_{i_m}(a)\psi_{i_{m+1}}(a)^{-1} &= \psi_{j_m}(a)\psi_{j_{m+1}}(a)^{-1} \\ \psi_{i_m}(a)\psi_{j_m}(a)^{-1} &= \psi_{i_{m+1}}(a)\psi_{j_{m+1}}(a)^{-1} \end{aligned}$$

for every $1 \leq m \leq n-1$. □

Lemma 3.20. *Let $\chi = \psi_{i_1} \otimes \cdots \otimes \psi_{i_n} \cong \bigotimes_{i \in I} \psi_i^{\otimes c_i}$ be an irreducible character of A^n. If there exists $j \in I$ such that the multiplicity c_j of ψ_j in χ is nonzero and distinct from c_i for all $i \neq j$, then* $\mathrm{Stab}(\chi) = \mathrm{Stab}(\chi')$.

Proof. Suppose $j \in I$ satisfies the hypothesis. We will show that $\mathrm{Stab}(\chi')$ is contained in $\mathrm{Stab}(\chi)$. Let σ be an element of $\mathrm{Stab}(\chi')$, so that $\chi' = (\sigma \cdot \chi)'$. By Lemma 3.19, we have

$$\psi_{i_1}\psi_{i_{\sigma(1)}}^{-1} = \psi_{i_2}\psi_{i_{\sigma(2)}}^{-1} = \cdots = \psi_{i_n}\psi_{i_{\sigma(n)}}^{-1}. \tag{3.21}$$

This is an irreducible character of A, say ψ_k. Equation (3.21) yields the system of equations

$$\psi_{i_1} = \psi_{i_{\sigma(1)}}\psi_k, \quad \dots, \quad \psi_{i_n} = \psi_{i_{\sigma(n)}}\psi_k. \tag{3.22}$$

By assumption, ψ_j is the only component of χ that occurs exactly c_j times. On the other hand, (3.22) shows that $\psi_j\psi_k$ also occurs exactly c_j times. Therefore ψ_k is the trivial representation, so (3.22) reduces to

$$\psi_{i_1} = \psi_{i_{\sigma(1)}}, \quad \dots, \quad \psi_{i_n} = \psi_{i_{\sigma(n)}}.$$

Hence $\sigma \cdot \chi = \chi$, meaning that σ is in $\mathrm{Stab}(\chi)$. □

Remark 3.23. The hypothesis of Lemma 3.20 is satisfied in the following instance. Let $\underline{\lambda} : I \to \mathcal{P}$ be a partition-valued function with $|\underline{\lambda}| = m$ and $\underline{\lambda}(t) = (\lambda_1, \ldots, \lambda_\ell)$, where ψ_t is the trivial character of A. Write $c_i = |\underline{\lambda}(i)|$ for all $i \in I$. Suppose $n > 2m$, and consider the "padded" partition-valued function $\underline{\lambda}[n]$. Since $\lambda_1 \leq m$, we have $n \geq m + \lambda_1$. Hence by Notation 3.16, the irreducible character of A^n associated to $\underline{\lambda}[n]$ is given by

$$\chi \cong \psi_t^{\otimes(n-m+c_t)} \otimes \bigotimes_{i \neq t} \psi_i^{\otimes c_i}.$$

We have $n > m$, so $n - m + c_t > 0$. Furthermore, since $c_i - c_t \leq m$ for all $i \in I$, we have

$$\begin{aligned} n &> m + c_i - c_t \\ n - m + c_t &> c_i \end{aligned}$$

for all $i \in I$. So Lemma 3.20 implies that $\mathrm{Stab}(\chi) = \mathrm{Stab}(\chi')$.

Theorem 3.24. *Let $V(\underline{\lambda})_n$ be an irreducible representation of G_n, where $|\underline{\lambda}| = m$. If $n > 2m$, then the restriction $W(\underline{\lambda})_n$ of $V(\underline{\lambda})_n$ to J_n is irreducible.*

Proof. It suffices to show that the restriction of $V(\underline{\lambda})_n$ to the subgroup K_n of J_n is irreducible. Let χ be the irreducible character of A^n associated to $\underline{\lambda}[n]$. Then the representation $V(\underline{\lambda})_n$ is given by

$$\begin{aligned} V(\underline{\lambda})_n &= \mathrm{Ind}_{A^n \rtimes \mathrm{Stab}(\chi)}^{G_n} \left(\chi \otimes \bigotimes_{i \in I} V_{\underline{\lambda}[n](i)} \right) \\ &= (A^n \rtimes S_n) \otimes_{A^n \rtimes \mathrm{Stab}(\chi)} \left(\chi \otimes \bigotimes_{i \in I} V_{\underline{\lambda}[n](i)} \right) \\ &\cong S_n \otimes_{\mathrm{Stab}(\chi)} \left(\chi \otimes \bigotimes_{i \in I} V_{\underline{\lambda}[n](i)} \right). \end{aligned}$$

By Remark 3.23, the hypothesis implies $\mathrm{Stab}(\chi) = \mathrm{Stab}(\chi')$. Hence

$$\begin{aligned}\mathrm{Res}^{G_n}_{K_n} V(\underline{\lambda})_n &\cong S_n \otimes_{\mathrm{Stab}(\chi')} \left(\chi' \otimes \bigotimes_{i \in I} V_{\underline{\lambda}[n](i)} \right) \\ &\cong (\langle e \rangle' \rtimes S_n) \otimes_{\langle e \rangle' \rtimes \mathrm{Stab}(\chi')} \left(\chi' \otimes \bigotimes_{i \in I} V_{\underline{\lambda}[n](i)} \right) \\ &= \mathrm{Ind}^{K_n}_{\langle e \rangle' \rtimes \mathrm{Stab}(\chi')} \left(\chi' \otimes \bigotimes_{i \in I} V_{\underline{\lambda}[n](i)} \right).\end{aligned}$$

Thus by [S, Prop. 25], $W(\underline{\lambda})_n = \mathrm{Res}^{G_n}_{K_n} V(\underline{\lambda})_n$ is irreducible. □

The proof of Theorem C is similar to the proof of Theorem B. First, let us recall the analogous result for FI_A-modules:

Theorem 3.25 ([GL, Theorem 1.12])**.** *An* FI_A*-module* V *over* $\mathbb{C}$ *is finitely generated if and only if* $\{V_n, (\mu_n)_*\}$ *is representation stable and* V_n *is finite-dimensional for all* $n \geq 0$*.*

Theorem C. *Let* W *be an* FI_A^H*-module over* $\mathbb{C}$*. Then* W *is finitely generated if and only if* $\{W_n, (\mu_n)_*\}$ *is representation stable and* W_n *is finite-dimensional for all* $n \geq 0$*.*

Proof. To prove the "if" direction, suppose $\{W_n, (\mu_n)_*\}$ is representation stable and W_n is finite-dimensional for all $n \geq 0$. The surjectivity property for $\{W_n, (\mu_n)_*\}$ implies the existence of a generating set S of W. Since the W_n are finite-dimensional, the generating set S must be finite.

To prove the "only if" direction, suppose W is finitely generated. Then $\{W_n, (\mu_n)_*\}$ satisfies the injectivity and surjectivity properties (Lemma 3.11). It remains to show that $\{W_n, (\mu_n)_*\}$ is multiplicity stable.

The FI_A^H-module W is finitely presented (Remark 2.21), say with generator degree $\leq g$ and relation degree $\leq r$. It follows (Proposition 2.22) that the induced FI_A-module $\mathrm{Ind}\, W$ is finitely presented, hence finitely generated as an FI_A-module. Therefore (Theorem 3.25) the consistent sequence $\{(\mathrm{Ind}\, W)_n, (\mu_n)_*\}$ is multiplicity stable, meaning that there exists a positive integer N not depending on $\underline{\lambda}$ such that, in the decomposition

$$(\operatorname{Ind} W)_n \cong \bigoplus_{\underline{\lambda}: I \to \mathcal{P}} V(\underline{\lambda})_n^{\oplus c_{\underline{\lambda},n}}, \tag{3.26}$$

the multiplicities $c_{\underline{\lambda},n}$ do not depend on n for all $n \geq N$ and all $\underline{\lambda}$. Furthermore, since each V_n is finite-dimensional, there are at most finitely many $\underline{\lambda}$ such that $c_{\underline{\lambda},n} \neq 0$. Let m be the maximum of $|\underline{\lambda}|$ over all such $\underline{\lambda}$.

Let $n > \max\{N, 2m, g, r\}$. By Proposition 2.24, there is an isomorphism $W_n \cong (\operatorname{Res} \operatorname{Ind} W)_n$. By Theorem 3.24, the J_n-representation $W(\underline{\lambda})_n$ is irreducible. So, restricting the decomposition (3.26) to J_n, we obtain

$$(\operatorname{Res} \operatorname{Ind} W)_n \cong W_n \cong \bigoplus_{\underline{\lambda}: I \to \mathcal{P}} W(\underline{\lambda})_n^{\oplus c_{\underline{\lambda},n}}$$

where the $W(\underline{\lambda})_n$ are irreducible and the multiplicities $c_{\underline{\lambda},n}$ do not depend on n for all $\underline{\lambda}$. Hence W is multiplicity stable. □

Chapter 4

Homological stability

We begin this chapter by recalling the construction of the homology groups $H_i(G, X)$ of a group G with nontrivial coefficients in X and the corresponding notion of stabilization. We also state some basic facts about induced G-modules. In section 4.2, we generalize homological stability for the groups $G_n = A \wr S_n$ with trivial coefficients (shown independently by [HW] and [G]) to the family of subgroups $J_n \leq G_n$ described in Notation 1.18. An easy corollary extends this result to twisted coefficients V_n arising from a finitely generated FI_A^H-module V over $\mathbb{Z}$.

4.1 G-modules

Definition 4.1. Let G be a group. The category G-Mod is defined by the following data:

- An object of G-Mod is a *G-module*, which is an abelian group X equipped with an additive left action of G.
- A G-module morphism $X \to Y$ is a G-equivariant group homomorphism.

The category G-Mod may be identified with $\mathbb{Z}G$-Mod, the category of left modules over the group ring $\mathbb{Z}G$. Given G-modules X and Y, we write $X \otimes_G Y$ for their tensor product in G-Mod.

Definition 4.2. Let G be a group and $F_\bullet$ a projective resolution of the trivial G-module $\mathbb{Z}$. For each integer $i \geq 0$, we define $H_i(G, X)$ by setting

$$H_i(G, X) = H_i(F_\bullet \otimes_G X).$$

The abelian group $H_i(G, X)$ is called the i-th *homology group of G with coefficients in X*.

The groups $H_i(G, X)$ are well-defined up to canonical isomorphism (see e.g. [W1, Lemma 2.4.1]). We write $H_i(G) = H_i(G, \mathbb{Z})$ when $\mathbb{Z}$ is the trivial G-module.

For notational convenience, we will write $H_*(G, X)$ when the index $* \geq 0$ is arbitrary.

Remark 4.3 ([B, p.78f]). It is useful to consider $H_*(-,-)$ as a functor of two variables in the following way. Let $\mathcal{C}$ be the category consisting of the following data.

- An object of $\mathcal{C}$ is a pair (G, X), where G is a group and X is a G-module.
- A $\mathcal{C}$-morphism $(G, X) \to (G', X')$ is a pair (α, f) of group homomorphisms $\alpha : G \to G'$ and $f : X \to X'$ which satisfy $f(g \cdot x) = \alpha(g) \cdot f(x)$ for all $g \in G$ and $x \in X$. In other words, f is compatible with the G-action via α. Composition of morphisms is defined by $(\beta, g) \circ (\alpha, f) = (\beta \circ \alpha, g \circ f)$.

Let $F_\bullet$ and $F'_\bullet$ be projective resolutions of the trivial G-module $\mathbb{Z}$ over $\mathbb{Z}G$ and $\mathbb{Z}G'$ respectively. We may regard $F'_\bullet$ as an (acyclic) complex of G-modules where G acts via α. By [B, Lemma I.7.4], the identity map on $\mathbb{Z}$ lifts to a chain map $\tau : F_\bullet \to F'_\bullet$ that is compatible with the G-action. Then the chain map $\tau \otimes f : F_\bullet \otimes_G X \to F'_\bullet \otimes_G X'$ induces a map on homology $(\alpha, f)_* : H_*(G, X) \to H_*(G', X')$. The assignments $(G, X) \mapsto H_*(G, X)$ and $(\alpha, f) \mapsto (\alpha, f)_*$ make $H_*(-,-)$ a covariant functor $\mathcal{C} \to \mathrm{Ab}$.

In the case where G is a subgroup of G' and $\iota : G \hookrightarrow G'$ is the inclusion map, we will write $f_* = (\iota, f)_*$. Similarly, in the case where X is a G-submodule of X' and $\iota : X \hookrightarrow X'$ is the inclusion map, we will write $\alpha_* = (\alpha, \iota)_*$.

Suppose H is a subgroup of G and X is a H-module. The *induced G-module* $\operatorname{Ind}_H^G(X)$ is defined by setting

$$\operatorname{Ind}_H^G X = \mathbb{Z}G \otimes_H X.$$

Remark 4.4 ([B, p.67ff]). We have $\operatorname{Ind}_H^G X = \bigoplus_{g \in G/H} gX$ where g ranges over a set of representatives of the cosets of H in G. Thus, in the case where X is the trivial H-module $\mathbb{Z}$, there is a G-module isomorphism $\operatorname{Ind}_H^G \mathbb{Z} \cong \mathbb{Z}[G/H]$ where G acts on the family of cosets G/H by left translation.

We recall the following well-known result:

Lemma 4.5 (Shapiro's lemma). *The inclusion map $\iota : H \hookrightarrow G$ and the map $X \to \operatorname{Ind}_H^G X$ defined by $x \mapsto e \otimes x$ for $x \in X$ induce an isomorphism*

$$H_*(H, X) \xrightarrow{\simeq} H_*(G, \operatorname{Ind}_H^G X).$$

Definition 4.6. Let $\{G_n\} : G_0 \hookrightarrow G_1 \hookrightarrow G_2 \hookrightarrow \cdots$ be an ascending chain of groups. We say $\{G_n\}$ is *homologically stable* if, for each $i \geq 0$, there exists $N_i \geq 0$ such that the map $H_i(G_n) \to H_i(G_{n+1})$ induced by the inclusion $G_n \hookrightarrow G_{n+1}$ is an isomorphism for all $n \geq N_i$.

4.2 Homological stability

For the remainder of this chapter, we set G_n to be the wreath product $G_n = A \wr S_n$. Unless otherwise stated, we regard $\mathbb{Z}$ as the trivial G_n-module. Most of the following discussion follows the proof in [G] of homological stability of arbitrary wreath product groups with trivial coefficients.

Definition 4.7. Let r be a nonnegative integer $\leq n$. An *injective word of length r* on G_n is a finite sequence $j_1, \ldots, j_r, a_1, \ldots, a_r$, where $j_1, \ldots, j_r$ are pairwise distinct positive integers $\leq n$ and $a_1, \ldots, a_r$ are any elements of A. In particular, an injective word of length 0 is the empty sequence.

Notation 4.8. The set of all injective words of length r on G_n is denoted $\Delta_r(\mathbf{n})$. For $r \leq n$, we set $C_r(\mathbf{n})$ to be the free abelian group with generating set $\Delta_r(\mathbf{n})$. We define an additive action of G_n on $C_r(\mathbf{n})$ where, given

$$x = (j_1, \dots j_r, a_1, \dots, a_r) \in \Delta_r(\mathbf{n}) \quad \text{and} \quad g = (a'_1, \dots, a'_n, \sigma) \in G_n,$$

we set $g \cdot x \in \Delta_r(\mathbf{n})$ to be

$$g \cdot x = (\sigma(j_1), \dots, \sigma(j_r), a'_{j_1} a_1, \dots, a'_{j_r} a_r).$$

Thus $C_r(\mathbf{n})$ is endowed with the structure of a G_n-module. For $r > n$, we set $C_r(\mathbf{n}) = \mathbb{Z}$.

For $i \leq r$, let $d_{r,i-1} : C_r(\mathbf{n}) \to C_{r-1}(\mathbf{n})$ be the abelian group homomorphism defined by

$$d_{r,i-1}(j_1, \dots, j_r, a_1, \dots, a_r) = (j_1, \dots, \widehat{j_i}, \dots, j_r, a_1, \dots, \widehat{a_i}, \dots, a_r)$$

that is, $d_{r,i-1}$ acts on $(j_1, \dots j_r, a_1, \dots, a_r) \in \Delta_r(\mathbf{n})$ by deleting the elements j_i and a_i. The $d_{r,i-1}$ are compatible with the G_n-action, so the G_n-modules $C_r(\mathbf{n})$ assemble to a G_n-chain complex $C_\bullet(\mathbf{n})$ with differentials $\partial_r : C_r(\mathbf{n}) \to C_{r-1}(\mathbf{n})$ defined by the alternating sum

$$\partial_r = \sum_{i=1}^{r} (-1)^{i-1} d_{r,i-1}.$$

A key result concerning the complex $C_\bullet(\mathbf{n})$ is the following:

Proposition 4.9 ([G, Theorem 3])**.** *If $i < n$ then $H_i(C_\bullet(\mathbf{n})) = 0$.*

Fix one of the subgroups $J_n \leq G_n$ and regard $C_\bullet(\mathbf{n})$ as a J_n-chain complex. The identity map on J_n and the map $d_{r,i-1}$ induce a map

$$(d_{r,i-1})_* : H_*(J_n, C_r(\mathbf{n})) \to H_*(J_n, C_{r-1}(\mathbf{n})).$$

Similarly, the identity map on J_n and the differentials ∂_r induce a map

$$(\partial_r)_* = \sum_{i=1}^{r} (-1)^{i-1} (d_{r,i-1})_* : H_*(J_n, C_r(\mathbf{n})) \to H_*(J_n, C_{r-1}(\mathbf{n})).$$

Lemma 4.10. *If $r < n$, then J_n acts transitively on $C_r(\mathbf{n})$.*

Proof. It suffices to show that J_n acts transitively on $\Delta_r(\mathbf{n})$. The case where $r = 0$ is trivial. Assume $0 < r < n$, and let x and y be elements of $\Delta_r(\mathbf{n})$, where

$$x = (j_1, \ldots, j_r, a_1, \ldots, a_r) \quad \text{and} \quad y = (k_1, \ldots, k_r, b_1, \ldots, b_r).$$

By assumption, the set $\mathbf{n} \setminus \{j_1, \ldots, j_r\}$ is nonempty; let c be an element of this set. For each $\ell \in \{1, \ldots, n\}$, define the element $\gamma_\ell \in A$ by setting

$$\gamma_\ell = \begin{cases} a_i^{-1} b_i & \text{if } \ell = j_i \\ \prod_{i=1}^{r} a_i b_i^{-1} & \text{if } \ell = c \\ e & \text{otherwise.} \end{cases}$$

Note that $\gamma_1 \cdots \gamma_n = e$. Choose $\sigma \in S_n$ such that $\sigma(j_i) = k_i$ for $i = 1, \ldots, r$. Then the element $t = (\gamma_1, \ldots, \gamma_n, \sigma)$ in J_n satisfies

$$t \cdot x = (\sigma(j_i), \ldots, \sigma(j_r), \gamma_{j_1} a_1, \ldots, \gamma_{j_r} a_r) = (k_1, \ldots, k_n, b_1, \ldots, b_n) = y.$$

□

Notation 4.11. Fix $r < n$ and let x_r be the element of $C_r(\mathbf{n})$ given by

$$x_r = (n - r + 1,\ n - r + 2,\ \ldots,\ n,\ e,\ \ldots,\ e).$$

Let $\mathbb{Z}$ be the trivial J_n-module. Denote by J_{n-r} the stabilizer of x_r in J_n. By Remark 4.4, we may identify $\operatorname{Ind}_{J_{n-r}}^{J_n} \mathbb{Z}$ with $\mathbb{Z}[J_n/J_{n-r}]$. Furthermore, the group J_n acts transitively on

$C_r(\mathbf{n})$ by Lemma 4.10, so the J_n-module map $f : \mathbb{Z}[J_n/J_{n-r}] \to C_r(\mathbf{n})$ which sends each coset representative $g \in J_n/J_{n-r}$ to $g \cdot x_r \in C_r(\mathbf{n})$ is an isomorphism.

Define the J_n-module map $\alpha(x_r) : \mathbb{Z} \to C_r(\mathbf{n})$ by the assignment $\lambda \mapsto \lambda x_r$. Note that $\alpha(x_r)$ factors through the isomorphism f via the map $\mathbb{Z} \to \mathrm{Ind}_{J_{n-r}}^{J_n} \mathbb{Z}$ given by $\lambda \mapsto e \otimes \lambda$. Hence for $r < n$, Shapiro's lemma implies that the inclusion map $J_{n-r} \hookrightarrow J_n$ and the map $\alpha(x_r)$ induce an isomorphism

$$\alpha(x_r)_* : H_*(J_{n-r}) \xrightarrow{\sim} H_*(J_n, C_r(\mathbf{n})).$$

The following Lemma 4.12 is proved in a similar manner as [G, Lemma 7].

Lemma 4.12. *Let $r < n$ and let $\iota : J_{n-r} \hookrightarrow J_{n-r+1}$ be the inclusion map. For $i = 1, \ldots, r$, there is a commuting diagram*

$$\begin{array}{ccc} H_*(J_{n-r}) & \xrightarrow{\iota_*} & H_*(J_{n-r+1}) \\ \downarrow{\scriptstyle \alpha(x_r)_*} & & \downarrow{\scriptstyle \alpha(x_{r-1})_*} \\ H_*(J_n, C_r(\mathbf{n})) & \xrightarrow{(d_{r,i-1})_*} & H_*(J_n, C_{r-1}(\mathbf{n})) \end{array}$$

Proof. For the case $i = 1$, it suffices to observe that the map $d_{r,0} : C_r(\mathbf{n}) \to C_{r-1}(\mathbf{n})$ satisfies

$$d_{r,0}(x_r) = (\widehat{n-r+1}, n-r+2, \ldots, n, e, \ldots, e) = x_{r-1}$$

so that the diagram commutes. Suppose $i > 1$. Write $y = d_{i-1}(x_r)$ and let $j : J_{n-r} \hookrightarrow \mathrm{Stab}(y)$ be the inclusion map. Then we have the commuting diagram

$$\begin{array}{ccc} H_*(J_{n-r}) & \xrightarrow{j_*} & H_*(\mathrm{Stab}(y)) \\ \downarrow{\scriptstyle \alpha(x_r)_*} & & \downarrow{\scriptstyle \alpha(y)_*} \\ H_*(J_n, C_r(\mathbf{n})) & \xrightarrow{(d_{r,i-1})_*} & H_*(J_n, C_{r-1}(\mathbf{n})) \end{array} \tag{4.13}$$

Let $\mu \in S_n$ be the cyclic permutation $(n-r+1, \ldots, n-r+i)$ and define $t \in J_n$ by setting $t = (e, \ldots, e, \mu)$. Let κ be the inner automorphism on J_n given by conjugation by t. The map

$f : C_{r-1}(\mathbf{n}) \to C_{r-1}(\mathbf{n})$ defined by $f(x) = t \cdot x$ is compatible with the J_n-action via κ, so $\delta = (\kappa, f)$ induces a map $\delta_* : H_*(J_n, C_{r-1}(\mathbf{n})) \to H_*(J_n, C_{r-1}(\mathbf{n}))$. Moreover, by [B, Prop. III.8.1], the map δ_* is the identity on $H_*(J_n, C_{r-1}(\mathbf{n}))$. Now consider the diagram

$$\begin{array}{ccccc} H_*(J_{n-r}) & \xrightarrow{j_*} & H_*(\mathrm{Stab}(y)) & \xrightarrow{\kappa_*} & H_*(J_{n-r+1}) \\ \downarrow{\scriptstyle \alpha(x_r)_*} & & \downarrow{\scriptstyle \alpha(y)_*} & & \downarrow{\scriptstyle \alpha(x_{r-1})_*} \\ H_*(J_n, C_r(\mathbf{n})) & \xrightarrow{(d_{r,i-1})_*} & H_*(J_n, C_{r-1}(\mathbf{n})) & \xrightarrow{\delta_*} & H_*(J_n, C_{r-1}(\mathbf{n})) \end{array} \tag{4.14}$$

We just showed that the left square of (4.14) commutes. Furthermore, we have

$$\begin{aligned} t \cdot y &= (\mu(n-r+1), \dots, \mu(\widehat{n-r+i}), \dots, \mu(n), e, \dots, e) \\ &= (n-r+2, \dots, n, e, \dots, e) \\ &= x_{r-1} \end{aligned}$$

so the right square of (4.14) also commutes. Therefore all of (4.14) commutes. Since $\kappa_* \circ j_* = \iota_*$ and $\delta_* \circ (d_{r,i-1})_* = (d_{r,i-1})_*$, the diagram (4.14) reduces to the diagram in the statement of the lemma. $\square$

Remark 4.15. The upshot of Lemma 4.12 is that the map $\iota_* : H_*(J_{n-r}) \to H_*(J_{n-r+1})$ may be identified with $(d_{r,i-1})_* : H_*(J_n, C_r(\mathbf{n})) \to H_*(J_n, C_{r-1}(\mathbf{n}))$ for any choice of $i \leq r < n$. Hence, if $r < n$, then

$$(\partial_r)_* = \sum_{i=1}^{r} (-1)^{i-1} (d_{r,i-1})_* = \iota_* - \iota_* + \iota_* - \dots = \begin{cases} \iota_* & \text{if } r \text{ is odd} \\ 0 & \text{if } r \text{ is even.} \end{cases}$$

Remark 4.16. Choose a free resolution $F_\bullet$ of $\mathbb{Z}$ over $\mathbb{Z}J_n$. Define a first-quadrant double complex D by setting $D_{r,s} = F_s \otimes_{J_n} C_r(\mathbf{n})$, and denote the total complex of D by $\mathrm{Tot}_\bullet(\mathrm{D})$.

Consider the spectral sequence associated to the vertical filtration of $\mathrm{Tot}_\bullet(\mathrm{D})$. The E^1 terms of this sequence are given by

$$E^1_{r,s} = H_s(F_r \otimes_{J_n} C_\bullet(\mathbf{n})).$$

Since $F_r \otimes_{J_n} -$ is exact, it follows from Proposition 4.9 that $E^1_{r,s} = 0$ for $s < n$. This spectral sequence converges to $H_{r+s}(\mathrm{Tot}_\bullet(\mathrm{D}))$, so we conclude that $H_i(\mathrm{Tot}_\bullet(\mathrm{D})) = 0$ for $i < n$.

Now consider the spectral sequence associated to the horizontal filtration of $\mathrm{Tot}_\bullet(\mathrm{D})$. This sequence also converges to $H_{r+s}(\mathrm{Tot}_\bullet(\mathrm{D}))$, so we have

$$E^\infty_{r,s} = 0 \quad \text{if} \quad r + s < n. \tag{4.17}$$

The E^1 terms are

$$E^1_{r,s} = H_s(F_\bullet \otimes_{J_n} C_r(\mathbf{n})) = H_s(J_n, C_r(\mathbf{n})).$$

By Remark 4.15, for $r < n$, the differential $d^1 : E^1_{r,s} \to E^1_{r-1,s}$ is

$$d^1 = (\partial_r)_* = \begin{cases} \iota_* : H_s(J_{n-r}) \to H_s(J_{n-r+1}) & \text{if } r \text{ is odd} \\ 0 & \text{if } r \text{ is even.} \end{cases} \tag{4.18}$$

Theorem D. *If $n \geq 2m + 2$, then the map $\iota_* : H_m(J_{n-1}) \to H_m(J_n)$ is an isomorphism.*

Proof. By equation (4.18), the map $\iota_* : H_m(J_{n-1}) \to H_m(J_n)$ is the differential $d^1 : E^1_{1,m} \to E^1_{0,m}$. We will show that $E^2_{1,m} = E^2_{0,m} = 0$, so that d^1 is an isomorphism.

We proceed by strong induction on m. For the base case $m = 0$, suppose that $n \geq 2$. Then the map $d^1 : E^1_{1,m} \to E^1_{0,m}$ is the identity on $\mathbb{Z}$, and we are done.

Suppose $m \geq 1$ and $n \geq 2m + 2$. Thus $n \geq 4$, and we have

$$m + 1 \leq \frac{n}{2} < n - 1$$

which by equation (4.17) implies $E^{\infty}_{0,m} = 0$ and $E^{\infty}_{1,m} = 0$. We will show that $E^2_{0,m} = E^{\infty}_{0,m}$ and $E^2_{1,m} = E^{\infty}_{1,m}$. To do this, we will use the following claim:

Claim: If $r + s \leq m + 2$ and $s < m$, then $E^2_{r,s} = 0$.

Pf. of Claim: Suppose $r \leq 3$. Then $r < n$, so equation (4.18) implies that row $m - 1$ on the E^1 page of the spectral sequence looks like

$$\cdots \longleftarrow 0 \longleftarrow H_{m-1}(J_n) \underset{\iota_*}{\longleftarrow} H_{m-1}(J_{n-1}) \underset{0}{\longleftarrow} H_{m-1}(J_{n-2}) \underset{\iota_*}{\longleftarrow} H_{m-1}(J_{n-3}) \longleftarrow \cdots \tag{4.19}$$

Furthermore, if $r \leq 2$ then

$$n - r \geq 2(m-1) + 2$$

so the induction hypothesis implies that the maps ι_* in the diagram (4.19) are isomorphisms. Then it is clear that $E^2_{r,s} = 0$ for $r \leq 3$ and $s = m - 1$.

The preceding is sufficient to prove the claim for $m = 1$; now suppose $m \geq 2$, $s \leq m - 2$ and $r + s \leq m + 2$. We have

$$r + 1 \leq m + 3 \leq n - m + 1 \leq n - 1$$

so equation (4.18) implies that row s on the E^1 page of the spectral sequence looks like

$$\cdots \longleftarrow H_s(J_{n-r+1}) \underset{d^1_{r,s}}{\longleftarrow} H_s(J_{n-r}) \underset{d^1_{r+1,s}}{\longleftarrow} H_s(J_{n-r-1}) \longleftarrow \cdots \tag{4.20}$$

where $d^1_{r,s} = \iota_*$ and $d^1_{r+1,s} = 0$ if r is odd and $d^1_{r,s} = 0$ and $d^1_{r+1,s} = \iota_*$ if r is even. Furthermore, we have

$$\begin{aligned} n - r &\geq n - m - 2 + s \\ &\geq 2m + 2 - m - 2 + s \\ &= m + s \\ &\geq s + 2 + s \\ &= 2s + 2 \end{aligned}$$

so the induction hypothesis implies that the maps ι_* in the diagram (4.20) are isomorphisms. Then it is clear that $E^2_{r,s} = 0$, and we have proved the claim.

Suppose $k \geq 2$. The claim implies that $E^k_{r,s} = 0$ if $r + s \leq m + 2$ and $s < m$. In particular, $E^k_{k,m-k+1} = 0$ and $E^k_{k+1,m-k+1} = 0$. Considering the differentials on the E^k page

$$E^k_{k,m-k+1} \to E^k_{0,m} \to 0 \qquad \text{and} \qquad E^k_{k+1,m-k+1} \to E^k_{1,m} \to 0,$$

it is clear that $E^k_{0,m} = E^{k+1}_{0,m}$ and $E^k_{1,m} = E^{k+1}_{1,m}$. Hence $E^2_{0,m} = E^\infty_{0,m}$ and $E^2_{1,m} = E^\infty_{1,m}$, as desired. □

Let V be an FI^H_A-module over $\mathbb{Z}$. Recall that, for each $n \geq 0$, the inclusion map $\mathbf{n} \hookrightarrow \mathbf{n+1}$ and the trivial map $\mathbf{n} \to A$ induce a map of J_n-modules $V_n \to V_{n+1}$. Hence the functor $H_*(-,-)$ provides a well-defined map $H_*(J_n, V_n) \to H_*(J_{n+1}, V_{n+1})$.

Lemma 4.21. *Let $M(\mathbf{m})$ be the free FI^H_A-module generated on $\mathbf{m}$. For $n \geq 2i+2$, the induced map $H_i(J_n, M(\mathbf{m})_n) \to H_i(J_{n+1}, M(\mathbf{m})_{n+1})$ is an isomorphism.*

Proof. By Lemma 3.23, there is an isomorphism $M(\mathbf{m})_n \cong \mathrm{Ind}^{J_n}_{J_{n-m}}(\mathbb{Z})$, where J_{n-m} is the stabilizer in J_n of the FI^H_A-morphism consisting of the inclusion $\mathbf{m} \hookrightarrow \mathbf{n}$ and the trivial map $\mathbf{m} \to A$. By Shapiro's lemma, there is an isomorphism

$$H_i(J_n, M(\mathbf{m})_n) \cong H_i(J_{n-m}).$$

If $n \geq 2i + 2$, then Theorem D implies that the induced map

$$H_i(J_{n-m}) \to H_i(J_{n+1-m})$$

is an isomorphism. Thus, the induced map $H_i(J_n, M(\mathbf{m})_n) \to H_i(J_{n+1}, M(\mathbf{m})_{n+1})$ is an isomorphism, as desired. $\square$

Corollary 4.22. *Let V be a finitely generated FI_A^H-module over $\mathbb{Z}$. For $n \gg i$, the natural inclusion $J_n \to J_{n+1}$ and the induced map $V_n \to V_{n+1}$ induce an isomorphism $H_i(J_n, V_n) \to H_i(J_{n+1}, V_{n+1})$.*

Proof. Set $H_{-1}(J_n, V_n) = 0$. We proceed by induction on i. The case $i = -1$ is trivial. Let $i \geq 0$. There exists an exact sequence of FI_A^H-modules

$$0 \longrightarrow W \longrightarrow P \longrightarrow V \longrightarrow 0$$

where P is finitely generated and projective. By the noetherian property (Corollary 2.30), W is also finitely generated. This sequence gives rise to the following commuting diagram with exact columns:

$$\begin{array}{ccc}
\vdots & & \vdots \\
\downarrow & & \downarrow \\
H_i(J_n, W_n) & \xrightarrow{f_1} & H_i(J_{n+1}, W_{n+1}) \\
\downarrow & & \downarrow \\
H_i(J_n, P_n) & \xrightarrow{f_2} & H_i(J_{n+1}, P_{n+1}) \\
\downarrow & & \downarrow \\
H_i(J_n, V_n) & \xrightarrow{f_3} & H_i(J_{n+1}, V_{n+1}) \\
\downarrow & & \downarrow \\
H_{i-1}(J_n, W_n) & \xrightarrow{f_4} & H_{i-1}(J_{n+1}, W_{n+1}) \\
\downarrow & & \downarrow \\
H_{i-1}(J_n, P_n) & \xrightarrow{f_5} & H_{i-1}(J_{n+1}, P_{n+1}) \\
\downarrow & & \downarrow \\
\vdots & & \vdots
\end{array}$$

By Lemma 4.21, the maps f_2 and f_5 are isomorphisms for $n \gg i$, and by the induction hypothesis f_4 is an isomorphism for $n \gg i$. The five lemma implies that f_3 is surjective for $n \gg i$. Since W is also finitely generated, we may apply the same argument to deduce that f_1 is surjective for $n \gg i$. Then the five lemma implies that f_3 is injective for $n \gg i$. Hence f_3 is an isomorphism. $\square$

Chapter 5

Serre quotients

We recall the following definitions.

Definition 5.1. Let $\mathcal{A}$ be an abelian category. A nonempty full subcategory $\mathcal{B}$ of $\mathcal{A}$ is called a *Serre subcategory* of $\mathcal{A}$ if it satisfies the following property: for any exact sequence in $\mathcal{A}$

$$X \longrightarrow Y \longrightarrow Z$$

with $X, Z \in \mathrm{Ob}(\mathcal{B})$, then also $Y \in \mathrm{Ob}(\mathcal{B})$.

Lemma 5.2 ([SP, Lemma 02MS])**.** *Let $\mathcal{A}$ be an abelian category with Serre subcategory $\mathcal{B}$. There is an abelian category $\mathcal{A}/\mathcal{B}$ and an exact functor $P : \mathcal{A} \to \mathcal{A}/\mathcal{B}$ which is essenitally surjective and whose kernel is $\mathcal{B}$. Moreover, the category $\mathcal{A}/\mathcal{B}$ and the functor P are characterized by the following universal property: given an exact functor $F : \mathcal{A} \to \mathcal{C}$ and a subcategory $\mathcal{B} \subset \ker(F)$, there is a unique exact functor $G : \mathcal{A}/\mathcal{B} \to \mathcal{C}$ such that $F = G \circ P$.*

Such a category $\mathcal{A}/\mathcal{B}$ is called the *Serre quotient* of $\mathcal{A}$ with $\mathcal{B}$. We may realize $\mathcal{A}/\mathcal{B}$ as the category whose objects are the objects of $\mathcal{A}$ and whose morphisms $X \to Y$ are given by

$$\mathrm{Hom}_{\mathcal{A}/\mathcal{B}}(X, Y) = \mathrm{colim}_{(X', Y')} \mathrm{Hom}_{\mathcal{A}}(X', Y/Y')$$

where the limit is taken over pairs of subobjects $X' \subset X$ and $Y' \subset Y$ such that $X/X', Y' \in \mathrm{Ob}(\mathcal{B})$, with ordering $\leq$ defined by $(X', Y') \leq (X'', Y'')$ if and only if $X'' \subset X'$ and $Y' \subset Y''$. In particular, if X is an object of $\mathcal{B}$, then X is isomorphic (in $\mathcal{A}/\mathcal{B}$) to 0.

For the remainder of this chapter, fix a field k and let $\mathcal{C}$ be a locally noetherian FI-like category.

Definition 5.3. We say that a $\mathcal{C}$-module V is *finite-dimensional* if $\bigoplus_{n\geq 0} V_n$ is finite-dimensional as a k-vector space.

Remark 5.4. Suppose V is finitely generated. Then V is finite-dimensional if and only if there is a nonnegative integer N such that $V_n = 0$ for all $n \geq N$. When the latter condition holds, we say that V *vanishes in degrees* $\geq N$.

Denote by $\mathcal{C}-\mathrm{fgMod}$ the functor category of finitely generated $\mathcal{C}$-modules over k. Since $\mathcal{C}$ is locally noetherian, $\mathcal{C}-\mathrm{fgMod}$ is abelian. Denote by $\mathcal{C}-\mathrm{fdMod}$ the full subcategory of $\mathcal{C}-\mathrm{fgMod}$ whose objects are the finite-dimensional $\mathcal{C}$-modules.

Lemma 5.5. *The category* $\mathcal{C}-\mathrm{fdMod}$ *is a Serre subcategory of* $\mathcal{C}-\mathrm{fgMod}$*.*

Proof. Consider an exact sequence in $\mathcal{C}-\mathrm{fgMod}$

$$U \longrightarrow V \longrightarrow W$$

where U vanishes in degrees $\geq N_1$ and W vanishes in degrees $\geq N_2$. For each $n \geq 0$, the induced sequence of vector spaces

$$U_n \longrightarrow V_n \longrightarrow W_n$$

is exact. If $n \geq \max\{N_1, N_2\}$, then $U_n = W_n = 0$, hence $V_n = 0$ by exactness. Hence V is finite-dimensional. $\square$

Recall from Definition 2.3 that a subcategory $\mathcal{A}$ of $\mathcal{C}$ is *almost-full* if $\mathrm{Ob}\,\mathcal{A} = \mathrm{Ob}\,\mathcal{C}$ and $\mathcal{A}(\mathbf{m}, \mathbf{n}) = \mathcal{C}(\mathbf{m}, \mathbf{n})$ whenever $m \neq n$. Fix an almost-full subcategory $\mathcal{A}$ of $\mathcal{C}$. Since $\mathcal{C}$ is locally noetherian, the subcategory $\mathcal{A}$ is also locally noetherian (Theorem A). In particular, every finitely generated $\mathcal{C}$-module (resp. finitely generated $\mathcal{A}$-module) is finitely presented. We denote

$$Q_{\mathcal{C}} = \frac{\mathcal{C}-\mathrm{fgMod}}{\mathcal{C}-\mathrm{fdMod}} \quad \text{and} \quad Q_{\mathcal{A}} = \frac{\mathcal{A}-\mathrm{fgMod}}{\mathcal{A}-\mathrm{fdMod}}.$$

Let us recall the following proposition (Proposition 2.24):

Proposition 5.6. *Let V be a finitely generated $\mathcal{C}$-module generated in degrees $\geq d$ and related in degrees $\geq r$. Let $\eta_V : V \to \mathrm{Res}\,\mathrm{Ind}\,V$ denote the unit map of the adjunction $\mathrm{Ind} \dashv \mathrm{Res}$. In degrees $n > \max\{d, r\}$, the map*

$$(\eta_V)_n : V_n \to (\mathrm{Res}\,\mathrm{Ind}\,V)_n$$

is an isomorphism of vector spaces.

Descending to the quotient $Q_{\mathcal{C}}$ yields the following:

Lemma 5.7. *Let P be the canonical functor $\mathcal{C}-\mathrm{fgMod} \to Q_{\mathcal{C}}$. Given $V \in \mathrm{Ob}(Q_{\mathcal{C}})$, the $Q_{\mathcal{C}}$-morphism*

$$P(\eta) : V \to \mathrm{Res}\,\mathrm{Ind}\,V$$

is an isomorphism.

Proof. Consider the exact sequence (in $\mathcal{C}-\mathrm{fgMod}$)

$$0 \longrightarrow \ker\eta \longrightarrow V \xrightarrow{\eta} \mathrm{Res}\,\mathrm{Ind}\,V \longrightarrow \mathrm{coker}\,\eta \longrightarrow 0.$$

Exactness of the canonical functor $P : \mathcal{C}-\mathrm{fgMod} \to Q_{\mathcal{C}}$ ensures that the sequence in $Q_{\mathcal{C}}$

$$0 \longrightarrow \ker\eta \longrightarrow V \xrightarrow{P(\eta)} \mathrm{Res}\,\mathrm{Ind}\,V \longrightarrow \mathrm{coker}\,\eta \longrightarrow 0 \qquad (*)$$

is also exact. Suppose V is generated in degrees $\leq d$ and related in degrees $\leq r$. Then Proposition 5.6 implies $(\ker\eta)_n = (\mathrm{coker}\,\eta)_n = 0$ for all $n > \max\{d, r\}$. Hence $\ker\eta$ and $\mathrm{coker}\,\eta$ are finite-dimensional, and moreover $\ker\eta \cong \mathrm{coker}\,\eta \cong 0$ in $Q_{\mathcal{C}}$. So we obtain from

$(*)$ the exact sequence

$$0 \longrightarrow V \xrightarrow{P(\eta)} \operatorname{Res}\operatorname{Ind} V \longrightarrow 0$$

which ensures that $P(\eta)$ is an isomorphism. $\square$

Proposition 5.8. *Let V be a $\mathcal{C}$-module finitely generated in degrees $\leq d$ and related in degrees $\leq r$. In degrees $n > \max\{d, r\}$, the k-linear map*

$$T_n : (\operatorname{Ind}\operatorname{Res} V)_n \to V_n$$

$$\alpha \otimes v \mapsto \alpha v \qquad (\alpha \in M_{\mathcal{C}}(\mathbf{m})_n, \quad v \in V_m)$$

is an isomorphism.

Proof. Given $v \in V_n$, the element $\mathrm{id}_{\mathbf{n}} \otimes v \in (\operatorname{Ind}\operatorname{Res} V)_n$ satisfies $T_n(\mathrm{id}_{\mathbf{n}} \otimes v) = \mathrm{id}_{\mathbf{n}} v = v$. So T_n is surjective for all $n > 0$. We claim that T_n is injective for $n > \max\{d, r\}$.

First consider the case $V = M_{\mathcal{C}}(\mathbf{m})$. We will show that, given $n > m$, the map

$$U_n : M_{\mathcal{C}}(\mathbf{m})_n \to (\operatorname{Ind}\operatorname{Res} M_{\mathcal{C}}(\mathbf{m}))_n$$

$$\alpha \mapsto \alpha \otimes \mathrm{id}_{\mathbf{m}}$$

is an inverse to T_n. Given $\alpha \in M_{\mathcal{C}}(\mathbf{m})_n$, we compute

$$(T_n \circ U_n)(\alpha) = T_n(\alpha \otimes \mathrm{id}_{\mathbf{m}}) = \alpha \mathrm{id}_{\mathbf{m}} = \alpha$$

Hence $T_n \circ U_n$ is the identity map on $M_{\mathcal{C}}(\mathbf{m})_n$. Now, given $\alpha \in M_{\mathcal{C}}(\mathbf{r})_n$ and $\beta \in \operatorname{Res} M_{\mathcal{C}}(\mathbf{m})_r$, we have

$$(U_n \circ T_n)(\alpha \otimes \beta) = U_n(\alpha\beta) = \alpha\beta \otimes \mathrm{id}_{\mathbf{m}}.$$

If $r > m$ then β is an $\mathcal{A}$-morphism, so the relations of $(\operatorname{Ind}\operatorname{Res} M_{\mathcal{C}}(\mathbf{m}))_n$ (Definition 2.15) imply

$$\alpha\beta \otimes \mathrm{id}_{\mathbf{m}} = \alpha \otimes \beta.$$

Now suppose $r = m$. Then $n > r$, so α and $\alpha\beta$ are $\mathcal{A}$-morphisms. The relations of $(\operatorname{Ind}\operatorname{Res} M_{\mathcal{C}}(\mathbf{m}))_n$ then imply

$$\alpha\beta \otimes \mathrm{id}_{\mathbf{m}} = \mathrm{id}_{\mathbf{m}} \otimes \alpha\beta = \alpha \otimes \beta.$$

Hence $U_n \circ T_n$ is the identity map on $M_{\mathcal{C}}(\mathbf{m})_n$. This proves the claim in the case $V = M_{\mathcal{C}}(\mathbf{m})$.

Now suppose V is any $\mathcal{C}$-module finitely generated in degrees $\leq d$ and related in degrees $\leq r$. Then there exists $g \leq d$ such that, for all $n > \max\{r, g\}$, the rows of the diagram

$$\begin{array}{ccccccc}
\bigoplus_{m=1}^{r} (\operatorname{Ind}\operatorname{Res} M_{\mathcal{C}}(\mathbf{m}))_n^{\oplus \ell_m} & \longrightarrow & \bigoplus_{m=1}^{g} (\operatorname{Ind}\operatorname{Res} M_{\mathcal{C}}(\mathbf{m}))_n^{\oplus c_m} & \longrightarrow & (\operatorname{Ind}\operatorname{Res} V)_n & \longrightarrow & 0 \\
\downarrow \oplus T_n & & \downarrow \oplus T_n & & \downarrow T_n & & \\
\bigoplus_{m=1}^{r} M_{\mathcal{C}}(\mathbf{m})_n^{\oplus \ell_m} & \longrightarrow & \bigoplus_{m=1}^{g} M_{\mathcal{C}}(\mathbf{m})_n^{\oplus c_m} & \longrightarrow & V_n & \longrightarrow & 0
\end{array}$$

are exact. The two vertical arrows on the left are isomorphisms. Thus the third vertical arrow is injective by the four lemma. □

The maps T_n comprise a natural transformation $T : \operatorname{Ind}\operatorname{Res} V \to V$. By an argument similar to Lemma 5.7, we obtain the following:

Lemma 5.9. *Let P be the canonical functor $\mathcal{C}{-}\mathrm{fgMod} \to Q_{\mathcal{C}}$. Given $V \in \mathrm{Ob}(Q_{\mathcal{C}})$, the $Q_{\mathcal{C}}$-morphism*

$$P(T) : \operatorname{Ind}\operatorname{Res} V \to V$$

is an isomorphism.

Lemmas 5.7 and 5.9 provide the following:

Theorem E. *Let $\mathcal{C}$ be a locally noetherian* FI*-like category and $\mathcal{A}$ an almost-full subcategory of $\mathcal{C}$. The restriction functor* $\mathrm{Res} : \mathcal{C}-\mathrm{fgMod} \to \mathcal{A}-\mathrm{fgMod}$ *induces an equivalence of categories*

$$Q_{\mathcal{C}} \xrightarrow{\sim} Q_{\mathcal{A}}.$$

Corollary 5.10. *Let A be a finite abelian group and H a subgroup of A. Assume k has characteristic 0. There is an equivalence of categories*

$$\frac{\mathrm{FI}_A^H-\mathrm{fgMod}}{\mathrm{FI}_A^H-\mathrm{fdMod}} \cong \mathrm{FI}_A-\mathrm{fdMod}.$$

Proof. Taking $\mathcal{C} = \mathrm{FI}_A$ and $\mathcal{A} = \mathrm{FI}_A^H$ in Theorem E, we obtain an equivalence of categories

$$\frac{\mathrm{FI}_A^H-\mathrm{fgMod}}{\mathrm{FI}_A^H-\mathrm{fdMod}} \cong \frac{\mathrm{FI}_A-\mathrm{fgMod}}{\mathrm{FI}_A-\mathrm{fdMod}}.$$

Furthermore, by [GLX, Theorem 4.2], the Nakayama functor $\nu : \mathrm{FI}_A-\mathrm{fgMod} \to \mathrm{FI}_A-\mathrm{fdMod}$ induces an equivalence of categories

$$\frac{\mathrm{FI}_A-\mathrm{fgMod}}{\mathrm{FI}_A-\mathrm{fdMod}} \cong \mathrm{FI}_A-\mathrm{fdMod}.$$

□

It is notable that the Serre quotient in Corollary 5.10 does not depend on the choice of the subgroup H of A.

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