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A statistical framework for sensitivity analysis of structural dynamic characteristics

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ABSTRACT

The uncertainty involved in the structural parameters inevitably leads to uncertainty in predicting the resulting structural dynamic characteristics, and this relationship is important to quantify. There is a large volume of work dealing with quantifying the overall uncertainty propagated from parameters to the structural dynamic responses, while little work has been done in measuring the contributions of the individual parameters and groups of parameters to the overall uncertainty, which is corresponding to the variance-based global sensitivity analysis (GSA). The variance-based GSA allows for providing a robust assessment of the relative influences of individual parameters on the structural dynamic characteristics. Although it is a powerful tool, the variance-based GSA suffers from the limitation of high computational cost, especially when applied to the expensive-to-run complex systems such as the long-span cable-stayed bridge in this study. To alleviate the computational burden, a fast-running nonparametric Gaussian process model (GPM), a

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17 fully specified statistical model, is employed as a surrogate model. The highlights of the
18 developed metamodel-based approach for the variance-based GSA are: (1) we adopt the
19 full GPM rather than the mean of GPM, where the latter drops the valuable uncertainty
20 information of the prediction variance; (2) we enable the full GPM-based method not to
21 be restricted to the calculation of the first-order sensitivity index and to be applicable for
22 the computation of sensitivity indices of groups of parameters; (3) we generalize this ap-
23 proach to be suitable for the cases whose uncertain parameters are arbitrarily distributed.
24 Then this full GPM-based approach is applied for sensitivity analysis of structural dynamic
25 characteristics of a long-span cable-stayed bridge.

26 **Keywords:** Uncertainty; Structural dynamics; Variance-based global sensitivity analysis;
27 Gaussian process model

NOTATION

Symbol	Meaning
d	Number of model inputs
$\mathcal{D}$	Training data
c	Constant
Θ	Hyperparameters of covariance function
x_k	Single parameter
$\mathbf{x}$	Parameter vector
$\mathbf{x}_Q$	Groups of parameters
$\mathbf{X}$	Output component of training data
$\mathbf{Y}$	Input component of training data
$C(\bullet, \bullet)$	Covariance function
$\mathcal{L}(\bullet)$	Negative logarithmic marginal likelihood
$ \bullet $	Determinant operator
$\text{tr}(\bullet)$	Trace operator
$(\bullet)^\top$	Transpose operator
$\mathbb{E}(\bullet)$	Expectation operator
$\mathbb{V}(\bullet)$	Variance operator
$\mathbb{E}_*(\bullet)$	Expectation operator with respect to the process
$\mathbb{C}_*(\bullet, \bullet)$	Covariance operator with respect to the process
$\mathcal{N}(\bullet, \bullet)$	Normal distribution
$\mathcal{LN}(\bullet, \bullet)$	Log-normal distribution
$\mathcal{U}(\bullet, \bullet)$	Uniform distribution
$\text{Erf}(\bullet)$	Error function
$\Phi(\bullet)$	Cumulative distribution function of the standard normal distribution
$\hat{y}, \hat{y}_*, \hat{y}'$	Mean of model outputs
$z_i(x_i)$	Main effect of input x_i
$z_{i,j}(\mathbf{x}_{i,j})$	Interaction effect of inputs x_i and x_j
$z_{i,j,\dots,p}(\mathbf{x}_{i,j,\dots,p})$	Interaction effect of larger sets of inputs linked to the index set $\{i, j, \dots, p\}$
V	Total variance
$V_i, V_{i,j}, V_{i,j,k}$	Partial variances
S_u	Sensitivity index associated with coordinate u
S_i	First-order sensitivity index for individual parameters
S_{Ti}	Total sensitivity index for individual parameters
S_Q	First-order sensitivity index for groups of parameters
S_{TQ}	Total sensitivity index for groups of parameters
$p(x_k)$	Probability density function of parameter x_k
$I_i^a, I_k^a, I_k^b, I_k^{ab}$	Single-dimensional integrals
$\tilde{I}_k$	Two-dimensional integral
x_i, w_i	Nodes and weights of the Gaussian quadrature rule
μ	Mean vector
Σ	Covariance matrix

INTRODUCTION

Engineering structures are inevitably subjected to various sources of parameter uncertainty, such as geometric dimension variability due to construction process, inherent random variation of materials. These unavoidable sources of uncertainty have great impact on dynamic characteristics such as natural frequencies, which have a close relationship with safety, functionality, and reliability of the structure, since they are frequently used for a variety of engineering design activities, such as wind- and earthquake-resistant design, structural vibration control, and structural dynamic optimization. Thus, it is of great importance to take into account the uncertainty in structural parameters in order to characterize structural dynamic characteristics accurately.

The significance of uncertainty modeling and mitigation in structural dynamics is recognized by a surge of work in recent years. [Modares et al. \(2006\)](#) present an interval analysis-based approach to determine the bounds of natural frequencies of a structure with bounded uncertainty. [Moens and Vandepitte \(2007\)](#) employ fuzzy set theory to calculate the total envelope frequency response functions (FRFs) of structures with uncertain parameters. [Jacquelin et al. \(2014\)](#) compute the statistics (mean and standard deviation) of FRFs of random structural systems based on polynomial chaos expansion. [Chowdhury and Adhikari \(2012\)](#) focus on high dimensional model representation surrogate model in combined with fuzzy uncertainty analysis to characterize uncertainty of natural frequencies of structural with parametric uncertainty. [DiazDelaO et al. \(2013\)](#) and [Kundu et al. \(2014\)](#) explore the application of GPM in the uncertainty quantification of FRFs propagated from uncertain parameters. [Mao and Todd \(2013\)](#) quantify the uncertainty of FRFs induced by internal and external uncertainties in physical system using a direct forward-propagating

53 statistical modeling approach. [Wan et al. \(2014\)](#) develop an analytical method using GPM
54 to estimate the uncertainty bound of natural frequencies of two full-scale bridges. We can
55 see that a large volume of work is concerned with uncertainty analysis (UA), which refers
56 to the quantification of the uncertainty in model outputs propagated from uncertain inputs.
57 As opposed to UA, sensitivity analysis (SA) seeks to determine the contributions of indi-
58 vidual parameters to the total uncertainty of model responses. UA and SA are essential
59 parts of the modeling and risk assessment of physical systems subject to uncertainty. How-
60 ever, to the best of authors' knowledge, the research regarding SA in structural dynamics
61 is very limited. By performing SA, one is able to evaluate the effects of each uncertain
62 structural parameter, either in isolation or through interaction effects, on the dynamic char-
63 acteristics, and then to rank them in order of quantitative importance measurement. The
64 SA results shed light on the internal mechanism of how structural parameters influence
65 the dynamic characteristics. As a consequence, with the aid of SA results, it is likely to
66 design a more robust product without dynamic issues, such as dangerously high vibration.
67 The focus of this study is on the contribution quantification of individual parameters and
68 subsets of parameters to overall uncertainty in structural dynamic characteristics through
69 SA.

70 In general, the SA techniques are either local or global. Local SA (LSA) concentrates
71 on investigating the local effects of the inputs on model outputs. LSA is computed by
72 partial derivatives of model outputs with respect to individual inputs or by finite difference
73 approximation when analytical derivatives are too difficult to obtain for complex model-
74 s. LSA belongs to the class of one-factor-at-a-time (OAT) methods. That is, sensitivity
75 indices are measured by perturbing each single input slightly in turn while keeping all

other inputs constant at their nominal value (usually, mean) (see [Moaveni et al. 2009](#) for a good application of LSA). In contrast to LSA, GSA assesses the impacts of individual inputs over their entire space on model outputs. LSA is not appropriate for the cases where models are nonlinear or inputs involve sizeable uncertainty, whereas GSA can well address high uncertainty in inputs and is also model-free ([Marino et al. 2008](#); [Yang 2011](#)). GSA techniques include variance-based method, screening-based technique, regression-based approach, to name a few. Among these GSA techniques, the variance-based method presents a number of desirable properties, including model independence, the capability to measure interaction effects, the capacity to cope with the influence of the whole input space, and being able to consider groups of inputs ([Saltelli et al. 2008](#)). As a result, the variance-based GSA has attracted a lot of modelers, which can be evidenced by its extensive applications in a variety of areas ([Helton et al. 2006](#); [Sumner et al. 2012](#); [Rahman 2011](#); [Yadav and Rahman 2014](#); [Kala 2011](#); [Rohmer and Foerster 2011](#); [Wan and Ren 2015a,b](#); [Zhang et al. 2015](#); [Yan et al. 2017](#)). This paper exploits the application of this powerful variance-based GSA tool to separate the contributions of individual parameters and groups of parameters to the total uncertainty in structural dynamic characteristics. Therefore, this paper and our previous paper ([Wan et al. 2014](#)) may be seen as two companion papers, which refer to the work of SA and UA, respectively.

Although powerful for assessing the impact of model inputs on model outputs, the variance-based GSA remains computationally intensive, especially when applied to the large-scale complex systems such as the long-span cable-stayed bridge in this paper. The high computational cost involved in the variance-based GSA restricts the usage of traditional methods such as Monte Carlo simulation due to the expensive-to-run model. This

provides us strong motivation to seek a more efficient method to circumvent the issue of the high computational expense. One effective means of addressing this problem is through the use of a metamodel. GPM, one particular type of surrogate model, is widely used to perform variance-based GSA. The popular use of GPM in variance-base GSA is mainly because of its modeling flexibility and high expressive power, as well as its analytical capability that ensures the efficient implementation of the variance-based GSA. The methods for the analytical implementation of variance-based GSA within GPM framework are typically classified into two categories. The first focuses on the use of the mean of GPM, as proposed by [Chen et al. \(2005\)](#). The method proposed by [Chen et al. \(2005\)](#) is not restricted to the use of GPM, and is suitable for other types of metamodels (e.g., polynomial regression model and Gaussian radial basis function) that can be expressed as a tensor product basis function. The second is concerned with the use of the global stochastic model in statistical context, which can be traced back to the work of [Oakley and O'Hagan \(2004\)](#) and later is detailed in [Morris et al. \(2008\)](#). [Marrel et al. \(2009\)](#) provide a more comprehensive analysis of these two approaches and point out that the latter is superior to the former in terms of convergence and robustness, especially when the constructed GPM is not accurate enough in the sense that its prediction variance is not zero (or not close to zero) everywhere in parameter space, or when the limited design points are used to train the GPM. However, their methods in references ([Oakley and O'Hagan 2004](#); [Morris et al. 2008](#); [Marrel et al. 2009](#)) remain restricted to calculation of the first-order sensitivity index and the derivation for the total sensitivity index is not given. It is also important to note that the mentioned analytical implementation of variance-based GSA based on either the full GPM or only its mean is only available under the special assumption that model

parameters are uniformly or normally distributed. Hence, there is an increasing need to extend the second approach in order to make it more general.

To summarize, our main contributions in this paper are twofold. First, we apply the powerful variance-based GSA to separate the contributions of structural parameters to uncertainty in structural dynamic characteristics. The importance of uncertain parameters can be ranked based on their contributions to the induced total uncertainty. Second, we extend the full GPM-based approach for variance-based GSA to compute the sensitivity indices coping with clusters of parameters, and to be generally applicable for the cases with arbitrary probability distributed parameters. The remainder of this paper is organized as follows. Next section presents an overview and the formulation of the GPM. Afterward the theoretical fundamentals of the variance-based GSA are given. Subsequently, we elaborate on how to implement the variance-based GSA using the full GPM, followed by a test function to verify the feasibility of the developed GPM method. After that, a long-span cable-stayed bridge is provided to demonstrate the application of the full GPM-based method in sensitivity analysis of structural dynamic characteristics. Finally, the conclusions of our work are drawn.

GAUSSIAN PROCESS MODEL

Brief review

The probabilistic, non-parametric GPM is developed based on concepts of Bayesian statistics. The study on Gaussian process (GP) estimator has been a couple of decades of history. Initially, GPM has been extensively used in geostatistics field (Matheron 1973), where it is well known as Kriging named after the South African mining engineer D. G. Krige. The GPM is first introduced by Sacks et al. (1989) to be applied to the field of the

deterministic computer simulator (noise-free), which returns the same outputs every time if it is given the same inputs. In the machine learning literature, GPM is widely exploited to model the noise-contaminated physical system (Neal 1999; Rasmussen and Williams 2006). See references (Rasmussen and Williams 2006; Santner et al. 2003) for the state of the art in GPM. The data-driven property of a GPM enables it not to be restricted to a certain functional form and guarantees the flexibility in capturing the complexity of a physical system. Moreover, GPM presents an appealing feature that it provides not just a predicted value but also the associated prediction variance at an untried point. The conceptual simplicity, flexibility of GPM makes it a promising modeling tool for a wide range of problems.

Construction of Gaussian process model

The GPM is derived from a Bayesian framework. It treats model outputs as a random function with the associated probability distribution modeled through a GP prior; a GP prior in conjunction with Gaussian likelihood give rises to a posterior GP over prediction at a new point using the maximum likelihood estimate. GPM is fully characterized by its mean function and covariance function rather than the algebraic structure of input-output relationship. In this paper, the mean function is set to zero because we lack prior knowledge of the overall trend of the latent function (Neal 1999). However, zero mean function does not mean that the actual function is expected to be a constant zero. On the other hand, we employ the popularly used squared exponential covariance function expressed below, which presents a good property in that it leads to a smooth, infinitely

differentiable function (Santner et al. 2003).

$$C(\mathbf{x}, \mathbf{x}') = \eta^2 \exp \left[-\frac{1}{2} \sum_{k=1}^d \left(\frac{x_k - x'_k}{\ell_k} \right)^2 \right] \quad (1)$$

where x_k is the k -th component of $\mathbf{x}$, and d is the dimension of input space. The covariance function parameters $\Theta = \{\ell_1, \ell_2, \dots, \ell_d, \eta\}$ are often called hyperparameters.

Assume we have a training set with n observations, $\mathcal{D} = (\mathbf{X}, \mathbf{Y})$, where $\mathbf{X} = [\mathbf{x}_1^\top, \mathbf{x}_2^\top, \dots, \mathbf{x}_n^\top]^\top$ and $\mathbf{Y} = [y_1, y_2, \dots, y_n]^\top$. The predictive distribution of the target y_* at an unobserved point $\mathbf{x}_*$ is also Gaussian with mean and variance given by

$$\hat{y}_* = \boldsymbol{\alpha}^\top \mathbf{C}_* \quad (2)$$

$$\nu_{y_*} = \eta^2 - \mathbf{C}_*^\top \mathbf{C}^{-1} \mathbf{C}_* \quad (3)$$

where $\mathbf{C}_* = [C(\mathbf{x}_*, \mathbf{x}_1), C(\mathbf{x}_*, \mathbf{x}_2), \dots, C(\mathbf{x}_*, \mathbf{x}_n)]^\top$; $\mathbf{C} = C(\mathbf{X}, \mathbf{X})$; $\boldsymbol{\alpha} = \mathbf{C}^{-1} \mathbf{Y}$.

As described above, the GPM is completely governed by the hyperparameters Θ , which can be inferred through the standard conjugate gradient optimization routine by maximizing the marginal likelihood (i.e., minimizing negative logarithmic marginal likelihood (NLML)) of the training set $\mathcal{D}$, formulated as

$$\hat{\Theta} = \arg \min_{\Theta} \mathcal{L}(\Theta). \quad (4)$$

Both the expressions of NLML $\mathcal{L}(\Theta)$ and its partial derivatives with respect to the hyperparameters are analytically tractable (Rasmussen and Williams 2006), which are as

follows

$$\mathcal{L}(\Theta) = -\log p(\mathbf{Y}|\mathbf{X}, \Theta) = \frac{1}{2}\mathbf{Y}^\top \mathbf{C}^{-1}\mathbf{Y} + \frac{1}{2}\log |\mathbf{C}| + \frac{n}{2}\log(2\pi) \quad (5)$$

$$\frac{\partial \mathcal{L}(\Theta)}{\partial \Theta_i} = \frac{1}{2}\text{tr} \left(\mathbf{C}^{-1} \frac{\partial \mathbf{C}}{\partial \Theta_i} \right) - \frac{1}{2}\mathbf{Y}^\top \mathbf{C}^{-1} \frac{\partial \mathbf{C}}{\partial \Theta_i} \mathbf{C}^{-1} \mathbf{Y} \quad (6)$$

where $|\bullet|$, $\text{tr}(\bullet)$, and $(\bullet)^\top$ represent the determinant, trace, and transpose operators, respectively.

Here we do not go into details on the construction of GPM, such as design of experiments (DOE), hyperparameters estimation, model validation, but refer interested readers to our previous work (Wan et al. 2014).

VARIANCE-BASED GLOBAL SENSITIVITY ANALYSIS

The basic idea behind the variance-based GSA is to apportion the total variance of the model output into a sequence of partial variances induced by main effects of individual inputs as well as their interaction effects. In the context of the probabilistic analysis, the model output is typically represented by a function of multiple random inputs. Let $y(\mathbf{x})$ denote a model with d random inputs $\mathbf{x} = \{x_1, x_2, \dots, x_d\}$, which can be decomposed into main effects and interaction effects of increasing dimensionality, such that all the summands are mutually orthogonal (Efron and Stein 1981), given as follows

$$y(\mathbf{x}) = z_0 + \sum_{1 \leq i \leq d} z_i(x_i) + \sum_{1 \leq i < j \leq d} z_{i,j}(\mathbf{x}_{i,j}) + \sum_{1 \leq i < j < k \leq d} z_{i,j,k}(\mathbf{x}_{i,j,k}) + \dots + z_{1,2,\dots,d}(\mathbf{x}) \quad (7)$$

where

$$z_0 = \mathbb{E}(y)$$

$$z_i(x_i) = \mathbb{E}(y | x_i) - z_0$$

$$z_{i,j}(\mathbf{x}_{i,j}) = \mathbb{E}(y | \mathbf{x}_{i,j}) - z_i(x_i) - z_j(x_j) - z_0$$

$$z_{i,j,k}(\mathbf{x}_{i,j,k}) = \mathbb{E}(y | \mathbf{x}_{i,j,k}) - z_{i,j}(\mathbf{x}_{i,j}) - z_{i,k}(\mathbf{x}_{i,k}) - z_{j,k}(\mathbf{x}_{j,k}) - z_i(x_i) - z_j(x_j) - z_k(x_k) - z_0 \quad (8)$$

and similarly for higher order terms. Here z_0 is the global mean of the model $y(\mathbf{x})$ (zeroth-order), $z_i(x_i)$ is the main effect of input x_i (first-order), $z_{i,j}(\mathbf{x}_{i,j})$ is the interaction effect of inputs x_i and x_j (second-order), and so on.

It is obvious that main effects are all functions of the model inputs. Accordingly, depicting the relationship plot of main effects versus inputs can enable us to gain insight into how model outputs respond to inputs. Specifically, for example, **if the model output is sensitive to an input**, then its main effect plot curve experiences large variation as the input value varies. Although the main effect plot provides an effective way to determine whether an input is influential or not qualitatively, we still need the special index that characterizes the relative importance of individual inputs in a quantitative manner. This need motivates the following description of the variance-based quantitative indices that measure the relative importance of inputs. Due to the pairwise orthogonality of the terms of Eq. (7), the following relationship of total variance decomposition can be obtained (Sobol 1993)

$$V = \sum_{1 \leq i \leq d} V_i + \sum_{1 \leq i < j \leq d} V_{i,j} + \sum_{1 \leq i < j < k \leq d} V_{i,j,k} + \cdots + V_{1,2,\dots,d} \quad (9)$$

where

$$V = \mathbb{V}(y)$$

$$V_i = \mathbb{V}(\mathbb{E}(y|x_i))$$

$$V_{i,j} = \mathbb{V}(\mathbb{E}(y|\mathbf{x}_{i,j})) - V_i - V_j$$

$$V_{i,j,k} = \mathbb{V}(\mathbb{E}(y|\mathbf{x}_{i,j,k})) - V_{i,j} - V_{i,k} - V_{j,k} - V_i - V_j - V_k$$

and so on, in which $\mathbb{E}(\bullet)$ and $\mathbb{V}(\bullet)$ denote the expectation and variance operators, respectively. The Eq. (9) demonstrates that the total variance of the model outputs can be decomposed as the sum of partial variances induced by the individual and joint effects of inputs, which is the essence of the variance-based GSA. The variance-based sensitivity indices are defined as the ratios of the partial variances to the total variance, expressed as follows

$$S_u = \frac{V_u}{V} \quad (11)$$

where the subscript $u \in \left\{ \binom{i}{1 \leq i \leq d}, \binom{i,j}{1 \leq i < j \leq d}, \dots, (1, 2, \dots, d) \right\}$. Apparently, the variance-based GSA has an attractive mathematical property that the sum of all variance-based sensitivity indices will be equal to one, that is, $\sum_{u \neq \emptyset} S_u = 1$ holds. The first-order sensitivity index S_i quantifies the main effect of input x_i itself; the second-order sensitivity index $S_{i,j}$ measures the interaction effect between inputs x_i and x_j ; and the higher-order sensitivity index $S_{i,j,\dots,p}$ characterizes the joint influences from larger sets of inputs. For

the i -th input, there exist two types of sensitivity indices defined as

$$S_i = \frac{V_i}{V} \quad (12)$$

$$S_{T_i} = \sum_{i' \in \mathbf{v}} \frac{V_{i'}}{V} \quad (13)$$

where $\mathbf{v}$ is all the subsets of indices including index i . The total sensitivity index S_{T_i} is defined to be the sum of all the sensitivity indices involving the i -th input. For example, when consider a model with three inputs, then $S_{T_1} = S_1 + S_{1,2} + S_{1,3} + S_{1,2,3}$. Notice that the total sensitivity index S_{T_i} has a more simplified expression in the form

$$S_{T_i} = 1 - \frac{V_{-i}}{V} \quad (14)$$

where $V_{-i} = \mathbb{V}(\mathbb{E}(y|\mathbf{x}_{-i}))$ is the total contribution to V due to all inputs except the input x_i . The first-order sensitivity index S_i provides a measure of the contribution to the total variance of model outputs accounted for by i -th input only, while the total sensitivity index S_{T_i} quantifies the total contribution owing to all terms related to i -th input. The difference between S_i and S_{T_i} is that the latter takes into account the joint effects with other inputs while the former does not. If S_i and S_{T_i} are nearly equal, then we can infer that interaction effects between i -th input with other inputs are negligible. The combination of S_i and S_{T_i} provides a reliable way to measure its impact on model outputs.

Apart from powerful for measuring the importance of the individual inputs, the variance-based GSA can also be applied on clusters of inputs. Consider a set of inputs denoted as $\mathbf{x}_Q = \{x_{m_1}, x_{m_2}, \dots, x_{m_k}\}$, in which the subscript $Q = \{m_1, m_2, \dots, m_k\} \subseteq \{1, 2, \dots, d\}$. In this case, considering a group as an input, the variance-based sensitivity

indices for the subset of inputs $\mathbf{x}_Q$ are defined similarly through

$$S_Q = \frac{V_Q}{V} \quad (15)$$

$$S_{TQ} = 1 - \frac{V_{-Q}}{V} \quad (16)$$

where $V_Q = \mathbb{V}(\mathbb{E}(y|\mathbf{x}_Q))$ and $V_{-Q} = \mathbb{V}(\mathbb{E}(y|\mathbf{x}_{-Q}))$, in which $\mathbf{x}_{-Q}$ denotes a collection of inputs $\mathbf{x}$ with coordinates $\{\mathbf{x}_i\}_{i \in Q}$ removed.

From the theory of GSA described above, the evaluations of the main effects and sensitivity indices are to solve the high-dimensional integration problems. Monte Carlo simulation (MCS) is a general approach for the evaluations of the high-dimensional integrals, but it is not computationally affordable for large-scale engineering problems where the number of random parameters is large. The MCS implementation of GSA can be found in Saltelli et al. (2008). In the subsequent section, we will detail on how to analytically perform GSA within the framework of the full GPM.

BAYESIAN INFERENCE FOR SENSITIVITY MEASURES USING GPM

Before going into the expansion of the full GPM-based approach that within the GPM framework, a global stochastic model rather than a deterministic one is used for variance-based GSA, we need to introduce some basic materials that will be used for the subsequent derivations. Throughout the paper, y and y' represents two different observations that are yielded at $\mathbf{x}$ and $\mathbf{x}'$, respectively; $\mathbb{E}_*(\bullet)$ and $\mathbb{C}_*(\bullet, \bullet)$ denote the expectation and covariance with respect to the process. In light of the property associated with GPM that the posterior predictive distribution over observations are joint Gaussian, thus observations (y, y')

246 follow a bivariate norm distribution, such that

$$247 \quad p(y, y') \sim \mathcal{N} \left(\begin{bmatrix} \hat{y} \\ \hat{y}' \end{bmatrix}, \begin{bmatrix} \eta^2 - \mathbf{C}_x^\top \mathbf{C}^{-1} \mathbf{C}_x & \tilde{C} - \mathbf{C}_x^\top \mathbf{C}^{-1} \mathbf{C}_{x'} \\ \tilde{C} - \mathbf{C}_{x'}^\top \mathbf{C}^{-1} \mathbf{C}_x & \eta^2 - \mathbf{C}_{x'}^\top \mathbf{C}^{-1} \mathbf{C}_{x'} \end{bmatrix} \right) \quad (17)$$

where

$$\tilde{C} = C(\mathbf{x}, \mathbf{x}') = c \prod_{k=1}^d \mathcal{N}_{x_k}(x'_k, \ell_k^2) \quad (18)$$

$$\hat{y} = \boldsymbol{\alpha}^\top \mathbf{C}_x = \boldsymbol{\alpha}^\top C(\mathbf{x}, \mathbf{X}) = c \sum_{a=1}^n \alpha_a \prod_{k=1}^d \mathcal{N}_{x_k}(x_k^a, \ell_k^2) \quad (19)$$

$$\hat{y}' = \boldsymbol{\alpha}^\top \mathbf{C}_{x'} = \boldsymbol{\alpha}^\top C(\mathbf{x}', \mathbf{X}) = c \sum_{a=1}^n \alpha_a \prod_{k=1}^d \mathcal{N}_{x'_k}(x_k^a, \ell_k^2) \quad (20)$$

248 where $c = (2\pi)^{\frac{d}{2}} \prod_{k=1}^d \ell_k$; α_a is the a -th element of $\boldsymbol{\alpha}$; x_k^a is the (a, k) -th element of
 249 the sampling matrix $\mathbf{X}$; $\mathcal{N}_{x_k}(x'_k, \ell_k^2) = \frac{1}{\ell_k \sqrt{2\pi}} \exp \left[-\frac{1}{2\ell_k^2} (x_k - x'_k)^2 \right]$ and analogously for
 250 $\mathcal{N}_{x_k}(x_k^a, \ell_k^2)$ and $\mathcal{N}_{x'_k}(x_k^a, \ell_k^2)$.

251 The posterior expectation calculation of main effects

252 Bear in mind that the underlying function $y(\mathbf{x})$ is approximated by a GPM and hence
 253 model output y is a Gaussian process with mean and variance given in Eqs. (2) and (3),
 254 respectively. Making use of the definition of main effects, one can calculate their posterior

255 expectation from

$$\begin{aligned}
\mathbb{E}_* (z_i(x_i)) &= \mathbb{E}_* (\mathbb{E}(y|x_i) - z_0) \\
&= \mathbb{E} (\mathbb{E}_*(y)|x_i) - \mathbb{E} (\mathbb{E}_*(y)) \\
&= \mathbb{E} (\hat{y}|x_i) - \mathbb{E} (\hat{y}) .
\end{aligned}
\tag{21}$$

257 Substituting Eq. (19) into Eq. (21), the above equation can now be expressed as

$$\begin{aligned}
\mathbb{E}_* [z_i(x_i)] &= \mathbb{E} \left[c \sum_{a=1}^n \alpha_a \prod_{k=1}^d \mathcal{N}_{x_k} (x_k^a, \ell_k^2) | x_i \right] - \mathbb{E} \left[c \sum_{a=1}^n \alpha_a \prod_{k=1}^d \mathcal{N}_{x_k} (x_k^a, \ell_k^2) \right] \\
&= \sum_{a=1}^n \alpha_a \left\{ \mathcal{N}_{x_i} (x_i^a, \ell_i^2) - \mathbb{E}_{x_i} [\mathcal{N}_{x_i} (x_i^a, \ell_i^2)] \right\} \prod_{\substack{k=1 \\ k \neq i}}^d \mathbb{E}_{x_k} [\mathcal{N}_{x_k} (x_k^a, \ell_k^2)] \\
&= \sum_{a=1}^n \alpha_a [\mathcal{N}_{x_i} (x_i^a, \ell_i^2) - I_i^a] \prod_{\substack{k=1 \\ k \neq i}}^d I_k^a
\end{aligned}
\tag{22}$$

259 where the computations of I_i^a and I_k^a are presented in Appendix B.

260 To draw the main effect plot, we just have to evaluate the $\mathbb{E}_* (z_i(x_i))$ over a grid of
 261 parameter values. Then the main effect plot can be naturally obtained. Note that for
 262 the sake of a better visualization, all parameters are commonly scaled to the same range
 263 (usually bounds $[0, 1]$) when drawing the main effect plot.

264 The posterior expectation calculation of sensitivity indices

Following the work of Oakley and O'Hagan (2004), the posterior expectation of sensitivity indices is defined as the ratio of the expectation of the partial variances to the

expectation of the total variance; that is,

$$\mathbb{E}_*(S_Q) = \frac{\mathbb{E}_*(V_Q)}{\mathbb{E}_*(V)} \quad (23)$$

$$\mathbb{E}_*(S_{TQ}) = 1 - \frac{\mathbb{E}_*(V_{-Q})}{\mathbb{E}_*(V)}. \quad (24)$$

265 Next the derivations for Eqs. (23) and (24), which are formulas for sensitivity indices of
 266 input subsets, are our focus. It is worth noting that the derivations for the expectation of
 267 the sensitivity indices associated with the single input are similar by replacing subscript
 268 Q with i . To compute the expectation of sensitivity indices, one has to calculate $\mathbb{E}_*(V_Q)$,
 269 $\mathbb{E}_*(V_{-Q})$ and $\mathbb{E}_*(V)$. By definition and using probability theory, we have

$$\begin{aligned} \mathbb{E}_*(V_Q) &= \mathbb{E}_* \{ \mathbb{V} [\mathbb{E}(y|\mathbf{x}_Q)] \} \\ &= \mathbb{E}_* \{ \mathbb{E} [\mathbb{E}^2(y|\mathbf{x}_Q)] - \mathbb{E}^2 [\mathbb{E}(y|\mathbf{x}_Q)] \} \\ &= \mathbb{E}_* \{ \mathbb{E} [\mathbb{E}^2(y|\mathbf{x}_Q)] \} - \mathbb{E}_* [\mathbb{E}^2(y)] \\ &= \mathbb{E} \{ \mathbb{E}_* [\mathbb{E}(y|\mathbf{x}_Q) \mathbb{E}(y'|\mathbf{x}'_Q)] \} - \mathbb{E}_* [\mathbb{E}(y) \mathbb{E}(y')] \\ &= \mathbb{E}_{\mathbf{x}_Q} \left\{ \mathbb{E}_{\mathbf{x}_{-Q} \mathbf{x}'_{-Q}} [\mathbb{E}_*(yy')] \right\} - \mathbb{E}_{\mathbf{x} \mathbf{x}'} [\mathbb{E}_*(yy')]. \end{aligned} \quad (25)$$

271 Similarly one can obtain $\mathbb{E}_*(V_{-Q})$ from

$$\begin{aligned} \mathbb{E}_*(V_{-Q}) &= \mathbb{E}_* \{ \mathbb{V} [\mathbb{E}(y|\mathbf{x}_{-Q})] \} \\ &= \mathbb{E} \{ \mathbb{E}_* [\mathbb{E}(y|\mathbf{x}_{-Q}) \mathbb{E}(y'|\mathbf{x}'_{-Q})] \} - \mathbb{E}_* [\mathbb{E}(y) \mathbb{E}(y')] \\ &= \mathbb{E}_{\mathbf{x}_{-Q}} \left\{ \mathbb{E}_{\mathbf{x}_Q \mathbf{x}'_Q} [\mathbb{E}_*(yy')] \right\} - \mathbb{E}_{\mathbf{x} \mathbf{x}'} [\mathbb{E}_*(yy')], \end{aligned} \quad (26)$$

273 and $\mathbb{E}_*(V)$ from

$$\begin{aligned}
 \mathbb{E}_*(V) &= \mathbb{E}_* [\mathbb{E}(y^2) - \mathbb{E}^2(y)] \\
 &= \mathbb{E}_x [\mathbb{E}_*(y^2)] - \mathbb{E}_{\mathbf{x}\mathbf{x}'} [\mathbb{E}_*(yy')].
 \end{aligned}
 \tag{27}$$

275 Using the property of covariance and then combining Eqs. (17)-(20), two terms $\mathbb{E}_*(yy')$
 276 and $\mathbb{E}_*(y^2)$ in Eqs. (25)-(27) are

$$\begin{aligned}
 \mathbb{E}_*(yy') &= \mathbb{C}_*(y, y') + \mathbb{E}_*(y)\mathbb{E}_*(y') \\
 &= \tilde{C} - \mathbf{C}_x^\top \mathbf{C}^{-1} \mathbf{C}_{x'} + \hat{y} \hat{y}' \\
 &= \tilde{C} - \mathbf{C}_x^\top \mathbf{C}^{-1} \mathbf{C}_{x'} + (\boldsymbol{\alpha}^\top \mathbf{C}_x) (\boldsymbol{\alpha}^\top \mathbf{C}_{x'}) \\
 &= \tilde{C} - \mathbf{C}_x^\top \mathbf{C}^{-1} \mathbf{C}_{x'} + \mathbf{C}_x^\top \boldsymbol{\alpha} \boldsymbol{\alpha}^\top \mathbf{C}_{x'} \\
 &= \tilde{C} + \mathbf{C}_x^\top \Xi \mathbf{C}_{x'}
 \end{aligned}
 \tag{28}$$

$$\begin{aligned}
 \mathbb{E}_*(y^2) &= \mathbb{C}_*(y, y) + \mathbb{E}_*^2(y) \\
 &= \eta^2 - \mathbf{C}_x^\top \mathbf{C}^{-1} \mathbf{C}_x + \hat{y}^2 \\
 &= \eta^2 - \mathbf{C}_x^\top \mathbf{C}^{-1} \mathbf{C}_x + (\boldsymbol{\alpha}^\top \mathbf{C}_x)^2 \\
 &= \eta^2 + \mathbf{C}_x^\top \Xi \mathbf{C}_x
 \end{aligned}
 \tag{29}$$

280 where $\Xi = \boldsymbol{\alpha} \boldsymbol{\alpha}^\top - \mathbf{C}^{-1}$.

281 By inserting Eqs. (28) and (29), the expressions of $\mathbb{E}_*(V_Q)$, $\mathbb{E}_*(V_{-Q})$ and $\mathbb{E}_*(V)$ become

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$$\begin{aligned}
\mathbb{E}_* (V_Q) &= \mathbb{E}_{\mathbf{x}_Q} \left[\mathbb{E}_{\mathbf{x}_{-Q}\mathbf{x}'_{-Q}} \left(\tilde{C} + \mathbf{C}_{\mathbf{x}}^\top \Xi \mathbf{C}_{\mathbf{x}'} \right) \right] - \mathbb{E}_{\mathbf{xx}'} \left(\tilde{C} + \mathbf{C}_{\mathbf{x}}^\top \Xi \mathbf{C}_{\mathbf{x}'} \right) \\
&= \left\{ \mathbb{E}_{\mathbf{x}_Q} \left[\mathbb{E}_{\mathbf{x}_{-Q}\mathbf{x}'_{-Q}} \left(\tilde{C} \right) \right] + \mathbb{E}_{\mathbf{x}_Q} \left[\mathbb{E}_{\mathbf{x}_{-Q}\mathbf{x}'_{-Q}} \left(\mathbf{C}_{\mathbf{x}}^\top \Xi \mathbf{C}_{\mathbf{x}'} \right) \right] \right\} \\
&\quad - \left[\mathbb{E}_{\mathbf{xx}'} \left(\tilde{C} \right) + \mathbb{E}_{\mathbf{xx}'} \left(\mathbf{C}_{\mathbf{x}}^\top \Xi \mathbf{C}_{\mathbf{x}'} \right) \right] \\
&= (\mathbf{I}_Q + \mathbf{I}'_Q) - (\mathbf{I} + \mathbf{I}')
\end{aligned} \tag{30}$$

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$$\begin{aligned}
\mathbb{E}_* (V_{-Q}) &= \mathbb{E}_{\mathbf{x}_{-Q}} \left[\mathbb{E}_{\mathbf{x}_Q\mathbf{x}'_Q} \left(\tilde{C} + \mathbf{C}_{\mathbf{x}}^\top \Xi \mathbf{C}_{\mathbf{x}'} \right) \right] - \mathbb{E}_{\mathbf{xx}'} \left(\tilde{C} + \mathbf{C}_{\mathbf{x}}^\top \Xi \mathbf{C}_{\mathbf{x}'} \right) \\
&= \left\{ \mathbb{E}_{\mathbf{x}_{-Q}} \left[\mathbb{E}_{\mathbf{x}_Q\mathbf{x}'_Q} \left(\tilde{C} \right) \right] + \mathbb{E}_{\mathbf{x}_{-Q}} \left[\mathbb{E}_{\mathbf{x}_Q\mathbf{x}'_Q} \left(\mathbf{C}_{\mathbf{x}}^\top \Xi \mathbf{C}_{\mathbf{x}'} \right) \right] \right\} \\
&\quad - \left[\mathbb{E}_{\mathbf{xx}'} \left(\tilde{C} \right) + \mathbb{E}_{\mathbf{xx}'} \left(\mathbf{C}_{\mathbf{x}}^\top \Xi \mathbf{C}_{\mathbf{x}'} \right) \right] \\
&= (\mathbf{I}_{-Q} + \mathbf{I}'_{-Q}) - (\mathbf{I} + \mathbf{I}')
\end{aligned} \tag{31}$$

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$$\begin{aligned}
\mathbb{E}_* (V) &= \mathbb{E}_{\mathbf{x}} \left(\eta^2 + \mathbf{C}_{\mathbf{x}}^\top \Xi \mathbf{C}_{\mathbf{x}} \right) - \mathbb{E}_{\mathbf{xx}'} \left(\tilde{C} + \mathbf{C}_{\mathbf{x}}^\top \Xi \mathbf{C}_{\mathbf{x}'} \right) \\
&= \left[\eta^2 + \mathbb{E}_{\mathbf{x}} \left(\mathbf{C}_{\mathbf{x}}^\top \Xi \mathbf{C}_{\mathbf{x}} \right) \right] - \left[\mathbb{E}_{\mathbf{xx}'} \left(\tilde{C} \right) + \mathbb{E}_{\mathbf{xx}'} \left(\mathbf{C}_{\mathbf{x}}^\top \Xi \mathbf{C}_{\mathbf{x}'} \right) \right] \\
&= \left(\eta^2 + \tilde{\mathbf{I}} \right) - (\mathbf{I} + \mathbf{I}') .
\end{aligned} \tag{32}$$

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The calculations of the integral terms $\mathbf{I}$, $\mathbf{I}'$, $\mathbf{I}_Q$, $\mathbf{I}'_Q$, $\mathbf{I}_{-Q}$, $\mathbf{I}'_{-Q}$, and $\tilde{\mathbf{I}}$ involved in Eqs. (30)-

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(32) are detailed in the [Appendix A](#).

290

TEST FUNCTION VERIFICATION

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In this section, we focus on verifying the feasibility of the method presented previous

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section. To this end, a test function, which is commonly used as benchmark for investigat-

ing the feasibility of SA methods Saltelli et al. (2008), is given by

$$y(\mathbf{x}) = \sum_{i=1}^5 x_i x_{i+5} \quad (33)$$

Table 1 summarizes the distributions of a total of 10 uncertain parameters in the test function. This test function is provided to investigate the capability of the developed GPM method in sensitivity analysis of individual parameters as well as group of parameters. Thus, the scenario of the parameter clusters is introduced. Specifically, herein the parameters with the same distributions are classified as one group. As a result, there are 3 groups of parameters in total, that is, $\mathbf{x}_{1-3} = \{x_1, x_2, x_3\}$, $\mathbf{x}_{4-7} = \{x_4, \dots, x_7\}$, and $\mathbf{x}_{8-10} = \{x_8, x_9, x_{10}\}$. Obviously, for this test function with the given uncertain parameters, the sensitivity indices of individual parameters and groups of parameters can be analytically evaluated through the direct integration method by following the theory described before, since their calculations finally become the evaluations of a series of two kinds of single-dimensional integrals, namely, $\int x dx$ and $\int x^2 dx$.

To have a visual understanding of the discrepancy between the results of the developed GPM method and analytical results, the bar plot is presented. The comparison of the results is shown in Figs. 1 and 2. The sensitivity indices derived by the present GPM method present an excellent match with the analytical ones, which verifies the high accuracy of the GPM method for sensitivity analysis of both individual parameters and a set of parameters. It is worth mentioning here that the performance comparison of the full GPM-based method and the GPM mean-based method at different sample sizes is extensively discussed (Marrel et al. 2009), and it is found that the former has advantages over the latter in terms of convergence and robustness. Thus the comparison of these two GPM-based

methods is not given here, but refer interested readers to Marrel et al. (2009).

APPLICATION: A LONG-SPAN CABLE-STAYED BRIDGE

Bridge description

The Qingzhou Bridge (Fig. 3) under investigation is a long-span cable-stayed bridge, which is located on the Luochang highway over the Ming River in Fuzhou city in Fujian province, China. The bridge has a composite-deck system consisting of five spans with a total length of 1185 m (40 m + 250 m + 605 m + 250 m + 40 m). At that time, its 605 m main span was the longest span length among the completed composite-deck cable-stayed bridges, and ranks the fifth among the all types of completed cable-stayed bridges all over the world. The bridge was completed in the year of 2000, but it was officially opened to the traffic in the year of 2002 due to the construction delay of approach spans. Fig. 4 shows the schematic elevation and deck cross-section views of the Qingzhou Bridge.

The composite-deck system of the bridge has an open-section consisting of two main I-type steel girders, steel floor beams, and 25 cm thickness concrete slab. Each steel girder is 2.45 m high and its maximum plate thickness reaches 80 mm. The ratio of girder height to span length is about 1/202. The bridge, having six lanes, carries two roadways with 29 m wide. There are two diamond-shaped pylons, as shown in Fig. 5, with a height of 175.5 m. The pylons are constructed on group piles with a diameter of 2 m, among which the longest pile has a length of 71.6 m. The pylons and bridge deck system are connected via a total of 168 stay cables, each having multiple 15 mm diameter steel wires, ranging from 27 to 85 in total per cable.

Finite element modeling

The three-dimensional linear elastic finite element model (FEM) for the Qingzhou Bridge is developed based on the design drawings using ANSYS® (ANSYS 2003). The modeling details are: the steel girders, stringers, floor beams, and pylons are modeled by two-node beam elements (BEAM4); the 3-D tension-only truss elements (LINK10) are used to model all stay cables since they are primarily designed to sustain tension forces; the concrete slab is modeled with shell elements (SHELL63); the solid elements (SOLID45) are adopted to simulate piers and platforms; the mass elements (MASS21) are utilized to model the non-structural components (such as railings) to just consider their contributions to inertia forces (masses) while neglecting their stiffness contribution that is sufficiently slight. The whole model has a total of 1840 nodes and 3238 elements. The resulting FEM and four mode shapes are shown in Fig. 6. For more details about the finite element modeling of the Qingzhou Bridge, we refer readers to references (Ren and Peng 2005; Ren et al. 2005).

Results of sensitivity analysis

A total of 10 structural parameters of different structural components, such as bridge pylon, stay cable, and main girder, are assumed as uncertain variables, whose statistical characteristics are summarized in Table 2. In practice, such parameter uncertainty may arise due to the lack of knowledge and incomplete information in physical modeling, the inherent variability of the physical systems (e.g., changes in operating conditions), etc. The selection of the probability distributions for the parameters should embody our judgment of how plausible it is that the parameters should have certain values. In this paper, the choice of the distribution types and their governing parameters (COV) mainly follows

the engineering common sense that the structural parameters have the physical meaning of positive definiteness and in addition geometrical parameter generally has lower level of uncertainty than material parameter. However, the choice of which parameters are uncertain and the corresponding probability distributions is somewhat arbitrary.

Four natural frequencies of the Qingzhou Bridge corresponding to the mode shapes in Fig. 6 are our quantities of interest (QoIs). To conduct sensitivity analysis of QoIs by using GPM, first we need to build a total of four GPMs, one for each of QoIs. Then sensitivity analysis is carried out within each GPM framework separately. The main effect plots for these parameters are shown in Fig. 7, and the sensitivity index plots are presented in Fig. 8 and the detailed magnitudes of sensitivity indices are tabulated in Table 3. It should be pointed out that the main effect plot is totally different from the local sensitivity analysis plot, since the latter is plotted by varying parameters one at a time with the rest being constant, while the former is a plot averaged over variations in the other parameters. Thus the main effect plot provides a global estimate as it takes into account the effects of other parameters.

As seen from results of GSA given in Figs. 7 and 8 and Table 3, the observations are reported as follows:

- For the first vertical natural frequency, all parameters present the linear relationships between main effects and their entire input space except that the parameter E_{sc} presents nonlinearity over the small range [0.8, 1.0], which indicates that the interaction effects are not evident. This notion is confirmed by the fact shown in Fig. 8 that their first-order and total sensitivity indices are nearly equal, implying little interaction effects involved. The phenomenon that among these parameter, only parameters E_{sc} , ρ_d and T_d

display the obvious variation of the main effect plot reveals that this natural frequency is sensitive to these three parameters. The conclusion of relative sensitivity of these parameters drawn with main effect plots can be easily validated through the Fig. 8.

- For the second vertical natural frequency, the main effect plots of parameters E_{sc} and E_d show most obvious nonlinear features, followed by parameters E_{mg} and T_d , and the remaining parameters have strongly linear features, which means that their mutual effects of these four parameters are pronounced, whereas the joint effects of the rest are negligible. The variation of the main effect plots of parameters E_{sc} and ρ_d is most noticeable, followed by parameters E_{mg} , E_d and T_d , which discloses that the former two parameters are most influential and the latter three parameters are second most influential. It should be noted that how natural frequency responds to these sensitive parameters is significantly distinguishing; for example, an increase in parameter E_{sc} produces an increase in natural frequency, while the effect of ρ_d is opposite, that is, the increase of ρ_d leads to the decrease of the natural frequency instead. Also, observations from the main effect plots regarding interaction effects and importance ranking of parameters are consistent with their quantitative importance measures, i.e., the variance-based sensitivity indices, demonstrated in Fig. 8.

- For the first torsional natural frequency, unlike the previous two natural frequencies, more parameters including E_p , ρ_p , E_{sc} , ρ_d and T_d have nonlinear features in their main effect plots, indicating that interactions involving these parameters account for a certain proportion of their total influences on natural frequency. The values of the interaction effects involved are measured by the difference between the first-order and total sensitivity indices, that is, $S_{Ti} - S_i$. Similarly, how main effect varies with the increase in

different parameters is different. For instance, increasing parameters, says E_p , acts to increase the natural frequency, while increasing parameters, says ρ_p , acts to decrease the natural frequency. According to the rule that the larger the variation of the main effect plot, the greater the sensitivity (importance) of that parameter to the target response, one can obtain their importance ranking. In particular, parameter E_{sc} has the most significant influence on this natural frequency; the influence of parameter E_p comes second, followed by parameters ρ_p , ρ_d and T_d ; the other parameters have minimal influence. The obtained importance ranking of parameters can be easily verified through their total sensitivity indices of parameters summarized in [Table 3](#).

- For the second torsional natural frequency, all parameters except E_{sc} , which presents weak nonlinearity, maintain the strongly linear relationship of main effects and each parameter space. This observation tells us that interaction effects among parameters do not exist obviously. This behavior is similar to that of the first vertical natural frequency. Likewise, in light of how much variation in main effect over the parameter space, one can gain insight into the relative importance of individual parameters as: the effect of parameter E_{sc} is dominant; the impact of parameter ρ_d comes second, followed by parameter T_d ; the rest are non-influential. It should be mentioned that the main effect plot does not always easily identify the relative importance (sensitivity) of all parameters, especially when the main effect plots of multiple parameters are quite identical, in this situation it is reliable to rank relative importance of parameters via their sensitivity indices that measure the effects of parameters on the target response in a quantitative fashion.

- In summary, two common features are: (1) certain parameters such as E_{sc} , ρ_d and T_d

always present high sensitivity to all the natural frequencies of interest, while some parameters such as E_{cg} and ρ_{cg} hardly influence all the natural frequencies, and (2) for a set of parameters such as E_{sc} and E_p , all of natural frequencies increase as their values increase, and conversely for another set of parameters such as ρ_d and T_d , their increase results in the decrease of all natural frequencies. In addition, it is interesting to point out that the influences of parameters on the natural frequencies are quite different or even opposite. That is to say, the effect of the different parameters on natural frequencies of different modes varies considerably. For example, parameter E_p that has little impact on the first vertical natural frequency influences the first torsional natural frequency significantly.

Following the sensitivity analysis of individual parameters to natural frequencies, we then move on to that of groups of parameters. Sensitivity analysis of parameter clusters is especially useful for problems with large number of parameters and also for the model with several components, each being characterized by a group of parameters. Herein, based on the different structural components, these 10 parameters are divided into 5 groups, which are detailed in Table 4. Then at the next stage the developed GPM-based method is adopted to conduct GSA for this parameter grouping case. The results of GSA are presented in Fig. 9 and Table 5, from which it can be found that all natural frequencies are sensitive to the parameters of both stay cables and bridge deck while parameters of cross girder are opposite, and the each of the remaining two groups has noticeable impact on only one of the natural frequencies, that is, parameters of bridge pylon greatly affect the first torsional natural frequency and parameters of main girder have a certain amount of impact on the second vertical natural frequency.

Discussion

Apparently, for this long-span, complex cable-stayed bridge, only using engineering common sense for the interpretation of the underlying input-output relationship is not enough and is unable to quantitatively assess the relative importance of model parameters. By performing the variance-based GSA, we can gain insight into intrinsic relationship between the structural parameters and the natural frequencies of the Qingzhou Bridge. In particular, the main effect plot, which is a powerful graphical tool, sheds light on how the structural parameters influence the natural frequencies. The main effect plot not only enables modelers to qualitatively identify critical structural parameters, but also provides valuable information regarding how natural frequencies respond to the change in structural parameters. On the other hand, the variance-based sensitivity indices provide an effective way to accurately measure the contributions of individual structural parameters to the total variance of natural frequencies, thus giving a quantitative measure of the relative importance of structural parameters.

Based on the results of sensitivity analysis, it can be found that the effects of the different parameters on natural frequencies of different modes vary considerably. Some parameters that have great impact on one mode may hardly influence other modes. For example, parameter E_p has a strong effect on the first torsional natural frequency but has negligible effect on the remaining natural frequencies, as does the parameter ρ_p . And interaction effects among parameters play an essential role in the second vertical natural frequency and the first torsional natural frequency, but not in other two natural frequencies. Ratto et al. (2001) divide factors into three categories: (I) factors with high main effect, (II) factors with small main effect but high total effect and (III) factors with small main and

total effect. We should pay close attention to the type I factors because these dominant factors take more responsibility for the uncertainty of model outputs, such as parameter E_{sc} for the first vertical natural frequency. Special care must be taken to avoid underestimating the effects of type II factors because their joint effects account for the large contribution for model outputs and are predominant over their main effects. For the type III factors, it is feasible to fix them at their constant nominal values, such as parameters E_{cg} and ρ_{cg} for all natural frequencies. Considering the type III factors as deterministic ones and setting them to fixed values would lead to a reduced-order model that maintains the rationale of the original model, but surpasses it in simplicity. As a result, sensitivity analysis enables to offer valuable guidance on these activities, including product design, model verification and validation, and decision making.

CONCLUDING REMARKS

In this work, an efficient statistical framework for performing GSA is presented by using the full GPM. To be more specific, when conducting GSA, the model output is characterized by a Gaussian process rather than by the mean of GPM, that is, the prediction variance of GPM is also taken into consideration in carrying out GSA. This concept of using the full GPM for GSA is originally introduced by Oakley and O'Hagan (2004). However, in reference (Oakley and O'Hagan 2004), only the derivations of the first-order sensitivity index are given and the derivations of total sensitivity index are not covered. No detailed derivations for the total sensitivity index appear to have been published to date, and this paper attempts to fill this gap. In addition, we enable the full GPM-based approach to carry out GSA of groups of parameters. Furthermore, we extend the full GPM-based method to efficiently deal with the situation that model parameters are neither normal nor

uniform distributions because in this case the analytical implementation of GSA is not available.

The present full GPM-based methodology is applied to perform GSA of natural frequencies of a long-span cable-stayed bridge. The GSA enables us to gain insight into the underlying processes and to characterize the dynamic behaviors of the cable-stayed bridge. Specifically, on one hand, not only does the main effect plot reveal the fundamental mechanism of how structural parameters influence the natural frequencies, but also provides a global estimate of their relative influences by the criterion that the larger the variation of the main effect plot, the greater the effect of that parameter. On the other hand, the variance-based sensitivity indices offer a global and quantitative assessment of parameter importance, and with the aid of sensitivity indices, we can clearly know how big the contribution of individual parameters and groups of parameters to the total variance in natural frequencies. Overall, the combination of the main effect plot and the variance-based sensitivity indices enables a good understanding of how structural parameters affect the natural frequencies and how big the relative influences of individual parameters as well. The variance-based GSA results can provide modelers with helpful guidance on many tasks such as product design, model verification and validation, and decision making.

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520 and JZ2015HGBZ0098).

521 **APPENDIX A. COMPUTATIONS OF $\mathbf{I}$, $\mathbf{I}'$, $\mathbf{I}_Q$, $\mathbf{I}'_Q$, $\mathbf{I}_{-Q}$, $\mathbf{I}'_{-Q}$, AND $\tilde{\mathbf{I}}$**

522 According to probability theory, the terms of $\mathbf{I}$, $\mathbf{I}'$, $\mathbf{I}_Q$, $\mathbf{I}'_Q$, $\mathbf{I}_{-Q}$, $\mathbf{I}'_{-Q}$, and $\tilde{\mathbf{I}}$ can be
 523 evaluated as follows

$$\begin{aligned}
 \mathbf{I} &= \mathbb{E}_{\mathbf{xx}'} \left(\tilde{C} \right) \\
 &= \mathbb{E}_{\mathbf{xx}'} \left[c \prod_{k=1}^d \mathcal{N}_{x_k} (x'_k, \ell_k^2) \right] \\
 &= c \prod_{k=1}^d \mathbb{E}_{x_k x'_k} [\mathcal{N}_{x_k} (x'_k, \ell_k^2)] \\
 &= c \prod_{k=1}^d \tilde{I}_k
 \end{aligned} \tag{A.1}$$

$$\begin{aligned}
 \mathbf{I}' &= \mathbb{E}_{\mathbf{xx}'} \left(\mathbf{C}_{\mathbf{x}}^\top \Xi \mathbf{C}_{\mathbf{x}'} \right) \\
 &= \mathbb{E}_{\mathbf{xx}'} \left[c^2 \sum_{b=1}^n \sum_{a=1}^n \Xi_{ab} \prod_{k=1}^d \mathcal{N}_{x_k} (x_k^a, \ell_k^2) \mathcal{N}_{x'_k} (x_k^b, \ell_k^2) \right] \\
 &= c^2 \sum_{b=1}^n \sum_{a=1}^n \Xi_{ab} \prod_{k=1}^d \mathbb{E}_{x_k x'_k} [\mathcal{N}_{x_k} (x_k^a, \ell_k^2) \mathcal{N}_{x'_k} (x_k^b, \ell_k^2)] \\
 &= c^2 \sum_{b=1}^n \sum_{a=1}^n \Xi_{ab} \prod_{k=1}^d \mathbb{E}_{x_k} [\mathcal{N}_{x_k} (x_k^a, \ell_k^2)] \mathbb{E}_{x'_k} [\mathcal{N}_{x'_k} (x_k^b, \ell_k^2)] \\
 &= c^2 \sum_{b=1}^n \sum_{a=1}^n \Xi_{ab} \left(\prod_{k=1}^d I_k^a I_k^b \right)
 \end{aligned} \tag{A.2}$$

$$\begin{aligned}
\mathbf{I}_Q &= \mathbb{E}_{\mathbf{x}_Q} \left[\mathbb{E}_{\mathbf{x}_{-Q} \mathbf{x}'_{-Q}} \left(\tilde{C} \right) \right] \\
&= \mathbb{E}_{\mathbf{x}_Q} \left[\mathbb{E}_{\mathbf{x}_{-Q} \mathbf{x}'_{-Q}} \left(c \prod_{k=1}^d \mathcal{N}_{x_k} (x'_k, \ell_k^2) \right) \right] \\
&= c \left(\prod_{k \in Q} \frac{1}{\sqrt{2\pi} \ell_k} \right) \left[\prod_{k \notin Q} \mathbb{E}_{x_k x'_k} [\mathcal{N}_{x_k} (x'_k, \ell_k^2)] \right] \\
&= c \left(\prod_{k \in Q} \frac{1}{\sqrt{2\pi} \ell_k} \right) \left(\prod_{k \notin Q} \tilde{I}_k \right)
\end{aligned} \tag{A.3}$$

$$\begin{aligned}
\mathbf{I}'_Q &= \mathbb{E}_{\mathbf{x}_Q} \left[\mathbb{E}_{\mathbf{x}_{-Q} \mathbf{x}'_{-Q}} \left(\mathbf{C}_x^\top \Xi \mathbf{C}_{x'} \right) \right] \\
&= \mathbb{E}_{\mathbf{x}_Q} \left[\mathbb{E}_{\mathbf{x}_{-Q} \mathbf{x}'_{-Q}} \left(c^2 \sum_{b=1}^n \sum_{a=1}^n \Xi_{ab} \prod_{k=1}^d \mathcal{N}_{x_k} (x_k^a, \ell_k^2) \mathcal{N}_{x'_k} (x_k^b, \ell_k^2) \right) \right] \\
&= c^2 \sum_{b=1}^n \sum_{a=1}^n \Xi_{ab} \left\{ \prod_{k \notin Q} \mathbb{E}_{x_k x'_k} [\mathcal{N}_{x_k} (x_k^a, \ell_k^2) \mathcal{N}_{x'_k} (x_k^b, \ell_k^2)] \right\} \left\{ \prod_{k \in Q} \mathbb{E}_{x_k x'_k} [\mathcal{N}_{x_k} (x'_k, \ell_k^2)] \right\} \\
&= c^2 \sum_{b=1}^n \sum_{a=1}^n \Xi_{ab} \left\{ \prod_{k \notin Q} \mathbb{E}_{x_k} [\mathcal{N}_{x_k} (x_k^a, \ell_k^2)] \mathbb{E}_{x'_k} [\mathcal{N}_{x'_k} (x_k^b, \ell_k^2)] \right\} \left\{ \prod_{k \in Q} \mathbb{E}_{x_k x'_k} [\mathcal{N}_{x_k} (x'_k, \ell_k^2)] \right\} \\
&= c^2 \sum_{b=1}^n \sum_{a=1}^n \Xi_{ab} \left(\prod_{k \notin Q} I_k^a I_k^b \right) \left(\prod_{k \in Q} \tilde{I}_k \right)
\end{aligned} \tag{A.4}$$

$$\begin{aligned}
\mathbf{I}_{-Q} &= \mathbb{E}_{\mathbf{x}_{-Q}} \left[\mathbb{E}_{\mathbf{x}_Q \mathbf{x}'_Q} \left(\tilde{C} \right) \right] \\
&= \mathbb{E}_{\mathbf{x}_{-Q}} \left\{ \mathbb{E}_{\mathbf{x}_Q \mathbf{x}'_Q} \left[c \prod_{k=1}^d \mathcal{N}_{x_k} (x'_k, \ell_k^2) \right] \right\} \\
&= c \left(\prod_{k \notin Q} \frac{1}{\sqrt{2\pi} \ell_k} \right) \left[\prod_{k \in Q} \mathbb{E}_{x_k x'_k} [\mathcal{N}_{x_k} (x'_k, \ell_k^2)] \right] \\
&= c \left(\prod_{k \notin Q} \frac{1}{\sqrt{2\pi} \ell_k} \right) \left(\prod_{k \in Q} \tilde{I}_k \right)
\end{aligned} \tag{A.5}$$

$$\begin{aligned}
\mathbf{I}'_{-Q} &= \mathbb{E}_{\mathbf{x}_{-Q}} \left[\mathbb{E}_{\mathbf{x}_Q \mathbf{x}'_Q} \left(\mathbf{C}_{\mathbf{x}}^\top \Xi \mathbf{C}_{\mathbf{x}'} \right) \right] \\
&= \mathbb{E}_{\mathbf{x}_{-Q}} \left[\mathbb{E}_{\mathbf{x}_Q \mathbf{x}'_Q} \left(c^2 \sum_{b=1}^n \sum_{a=1}^n \Xi_{ab} \prod_{k=1}^d \mathcal{N}_{x_k} (x_k^a, \ell_k^2) \mathcal{N}_{x'_k} (x_k^b, \ell_k^2) \right) \right] \\
&= c^2 \sum_{b=1}^n \sum_{a=1}^n \Xi_{ab} \left\{ \prod_{k \in Q} \mathbb{E}_{x_k x'_k} [\mathcal{N}_{x_k} (x_k^a, \ell_k^2) \mathcal{N}_{x'_k} (x_k^b, \ell_k^2)] \right\} \left\{ \prod_{k \notin Q} \mathbb{E}_{x_k x'_k} [\mathcal{N}_{x_k} (x'_k, \ell_k^2)] \right\} \\
&= c^2 \sum_{b=1}^n \sum_{a=1}^n \Xi_{ab} \left\{ \prod_{k \in Q} \mathbb{E}_{x_k} [\mathcal{N}_{x_k} (x_k^a, \ell_k^2)] \mathbb{E}_{x'_k} [\mathcal{N}_{x'_k} (x_k^b, \ell_k^2)] \right\} \left\{ \prod_{k \notin Q} \mathbb{E}_{x_k x'_k} [\mathcal{N}_{x_k} (x'_k, \ell_k^2)] \right\} \\
&= c^2 \sum_{b=1}^n \sum_{a=1}^n \Xi_{ab} \left(\prod_{k \in Q} I_k^a I_k^b \right) \left(\prod_{k \notin Q} \tilde{I}_k \right)
\end{aligned} \tag{A.6}$$

534

$$\begin{aligned}
\tilde{\mathbf{I}} &= \mathbb{E}_{\mathbf{x}} \left(\mathbf{C}_{\mathbf{x}}^{\top} \Xi \mathbf{C}_{\mathbf{x}} \right) \\
&= \mathbb{E}_{\mathbf{x}} \left[c^2 \sum_{b=1}^n \sum_{a=1}^n \Xi_{ab} \prod_{k=1}^d \mathcal{N}_{x_k} (x_k^a, \ell_k^2) \mathcal{N}_{x_k} (x_k^b, \ell_k^2) \right] \\
&= c^2 \sum_{b=1}^n \sum_{a=1}^n \Xi_{ab} \prod_{k=1}^d \mathbb{E}_{x_k} \left[\mathcal{N}_{x_k} (x_k^a, \ell_k^2) \mathcal{N}_{x_k} (x_k^b, \ell_k^2) \right] \\
&= c^2 \sum_{b=1}^n \sum_{a=1}^n \Xi_{ab} \prod_{k=1}^d \mathbb{E}_{x_k} \left[\mathcal{N}_{x_k^a} (x_k^b, 2\ell_k^2) \mathcal{N}_{x_k} \left(\frac{x_k^a + x_k^b}{2}, \frac{\ell_k^2}{2} \right) \right] \tag{A.7} \\
&= c^2 \sum_{b=1}^n \sum_{a=1}^n \Xi_{ab} \prod_{k=1}^d \mathcal{N}_{x_k^a} (x_k^b, 2\ell_k^2) \mathcal{N}_{x_k} \mathbb{E}_{x_k} \left[\left(\frac{x_k^a + x_k^b}{2}, \frac{\ell_k^2}{2} \right) \right] \\
&= c^2 \sum_{b=1}^n \sum_{a=1}^n \Xi_{ab} \prod_{k=1}^d \mathcal{N}_{x_k^a} (x_k^b, 2\ell_k^2) \mathcal{N}_{x_k} I_k^{ab}
\end{aligned}$$

536 The involved single-dimensional integrals I_i^a , I_k^a , I_k^b , and I_k^{ab} , and the two dimensional
537 integral $\tilde{I}_k$ are detailed in [Appendix B](#).

538 **APPENDIX B. COMPUTATIONS OF I_i^a , I_k^a , I_k^b , I_k^{ab} , AND $\tilde{I}_k$**

539 (1) The first four terms I_i^a , I_k^a , I_k^b and I_k^{ab} , which are all the single-dimensional integrals,
540 can be expressed as an unified expression of the form

$$541 \quad \mathcal{I} = \int \mathcal{N}_x (\zeta, \vartheta^2) p(x) dx. \tag{B.1}$$

- When $p(x)$ is normal distribution, the one-dimensional integral $\mathcal{I}$ is estimated by

$$\begin{aligned}\mathcal{I} &= \int \mathcal{N}_x(\zeta, \vartheta^2) \mathcal{N}_x(\xi, \theta^2) dx \\ &= \int \mathcal{N}_\zeta(\xi, \vartheta^2 + \theta^2) \mathcal{N}_x(\mu_3, \Sigma_3) dx \\ &= \mathcal{N}_\zeta(\xi, \vartheta^2 + \theta^2),\end{aligned}\tag{B.2}$$

whose derivation makes use of the identity of Gaussian multiplication (see [Appendix C](#)) and $\int \mathcal{N}_x(\mu_3, \Sigma_3) dx = 1$. Note that it is not necessary to make every effort to compute out the μ_3 and Σ_3 , since the integral of any normal density function over the entire space is equal to one.

- When $p(x)$ is uniform distribution, one can compute $\mathcal{I}$ as follows

$$\begin{aligned}\mathcal{I} &= \frac{1}{\bar{x} - \underline{x}} \int_{\underline{x}}^{\bar{x}} \mathcal{N}_x(\zeta, \vartheta^2) dx \\ &= \frac{1}{\bar{x} - \underline{x}} \left[\Phi\left(\frac{\bar{x} - \zeta}{\vartheta}\right) - \Phi\left(\frac{\underline{x} - \zeta}{\vartheta}\right) \right]\end{aligned}\tag{B.3}$$

where $\underline{x}$ and $\bar{x}$ are corresponding to lower bound and upper bound of the parameter, respectively; $\Phi(\bullet)$ denotes the cumulative distribution function of the standard normal distribution.

- When $p(x)$ is other types of distributions, $\mathcal{I}$ can be efficiently calculated via the effective Gaussian quadrature technique

$$\mathcal{I} = \sum_{i=1}^m w_i \mathcal{N}_{x_i}(\zeta, \vartheta^2)\tag{B.4}$$

where x_i and w_i are abscissae (nodes) and weights of the Gaussian quadrature rule,

respectively. The details of the computation of $\mathcal{I}$ using the Gaussian quadrature solution are omitted here and interested readers are referred to authors' previous work (Wan et al. 2017).

- (2) The last term $\tilde{I}_k$, which is a two-dimensional integrals, can be written as (the subscript k is removed for simplicity)

$$\tilde{\mathcal{I}} = \iint \mathcal{N}_x(x', \ell^2) p(x) p(x') dx dx'. \quad (\text{B.5})$$

- When $p(x)p(x')$ is normal distribution, the computation of $\tilde{\mathcal{I}}$ is

$$\begin{aligned} \tilde{\mathcal{I}} &= \iint \mathcal{N}_x(x', \ell^2) \mathcal{N}_x(\xi, \theta^2) \mathcal{N}_{x'}(\xi, \theta^2) dx dx' \\ &= \iint \mathcal{N}_{x'}(\xi, \ell^2 + \theta^2) \mathcal{N}_x(\tilde{\mu}_3, \tilde{\Sigma}_3) \mathcal{N}_{x'}(\xi, \theta^2) dx dx' \\ &= \int \mathcal{N}_{x'}(\xi, \ell^2 + \theta^2) \mathcal{N}_{x'}(\xi, \theta^2) dx' \\ &= \int \mathcal{N}_\xi(\xi, \ell^2 + 2\theta^2) \mathcal{N}_{x'}(\tilde{\mu}'_3, \tilde{\Sigma}'_3) dx' \\ &= \mathcal{N}_\xi(\xi, \ell^2 + 2\theta^2). \end{aligned} \quad (\text{B.6})$$

Again, the above derivation makes use of the identity of Gaussian multiplication and $\int \mathcal{N}_x(\tilde{\mu}_3, \tilde{\Sigma}_3) dx = 1$, $\int \mathcal{N}_{x'}(\tilde{\mu}'_3, \tilde{\Sigma}'_3) dx' = 1$. Likewise, The computations of $\tilde{\mu}_3$, $\tilde{\Sigma}_3$, $\tilde{\mu}'_3$ and $\tilde{\Sigma}'_3$ are not necessary.

- When $p(x)p(x')$ is uniform distribution, one can evaluate $\tilde{\mathcal{I}}$ from

$$\begin{aligned}
\tilde{\mathcal{I}} &= \frac{1}{(\bar{x} - \underline{x})^2} \int_{\underline{x}}^{\bar{x}} \int_{\underline{x}}^{\bar{x}} \mathcal{N}_x(x', \ell^2) dx dx' \\
&= \frac{1}{(\bar{x} - \underline{x})^2} \int_{\underline{x}}^{\bar{x}} \int_{\underline{x}}^{\bar{x}} \frac{1}{\sqrt{2\pi}\ell} \exp\left(-\frac{(x - x')^2}{2\ell^2}\right) dx, dx' \\
&= \frac{1}{(\bar{x} - \underline{x})^2} \left\{ \left[\exp\left(-\frac{(\bar{x} - \underline{x})^2}{2\ell^2}\right) - 1 \right] \sqrt{\frac{2}{\pi}}\ell + (\bar{x} - \underline{x}) \operatorname{Erf}\left(\frac{\bar{x} - \underline{x}}{\sqrt{2}\ell}\right) \right\}
\end{aligned} \tag{B.7}$$

where $\operatorname{Erf}(\bullet)$ is the error function such that $\operatorname{Erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2} dt$. The derivation of Eq. (A.7) uses integration by parts, and the details are given in the [Supplementary Material](#).

- When $p(x)p(x')$ is neither normal nor uniform distribution, the Gaussian quadrature technique can be adopted to evaluate $\tilde{\mathcal{I}}$, such that

$$\tilde{\mathcal{I}} = \sum_{j=1}^m \sum_{i=1}^m \omega_j \omega_i \mathcal{N}_{x_i}(x_j, \ell^2), \tag{B.8}$$

which uses Gaussian quadrature technique twice.

APPENDIX C. IDENTITY OF GAUSSIAN MULTIPLICATION

The multiplication of two Gaussian distributions is another Gaussian distribution (but no longer normalized) expressed as ([Petersen and Pedersen 2008](#))

$$\mathcal{N}_{\mathbf{x}}(\boldsymbol{\mu}_1, \boldsymbol{\Sigma}_1) \mathcal{N}_{\mathbf{x}}(\boldsymbol{\mu}_2, \boldsymbol{\Sigma}_2) = z \mathcal{N}_{\mathbf{x}}(\boldsymbol{\mu}_3, \boldsymbol{\Sigma}_3) \tag{C.1}$$

where $\boldsymbol{\Sigma}_3 = (\boldsymbol{\Sigma}_1^{-1} + \boldsymbol{\Sigma}_2^{-1})^{-1}$; $\boldsymbol{\mu}_3 = \boldsymbol{\Sigma}_3 (\boldsymbol{\Sigma}_1^{-1} \boldsymbol{\mu}_1 + \boldsymbol{\Sigma}_2^{-1} \boldsymbol{\mu}_2)$; $z = \mathcal{N}_{\boldsymbol{\mu}_1}(\boldsymbol{\mu}_2, \boldsymbol{\Sigma}_1 + \boldsymbol{\Sigma}_2)$.

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Table 1. Characteristics of parameters of the test function.

Parameter	Distribution	Description
x_1	Normal	$\mathcal{N}(1, 0.10)$
x_2	Normal	$\mathcal{N}(1, 0.15)$
x_3	Normal	$\mathcal{N}(1, 0.20)$
x_4	Uniform	$\mathcal{U}(0.9, 1.1)$
x_5	Uniform	$\mathcal{U}(0.8, 1.2)$
x_6	Uniform	$\mathcal{U}(0.7, 1.3)$
x_7	Uniform	$\mathcal{U}(0.6, 1.4)$
x_8	Log-normal	$\mathcal{LN}(1, 0.10)$
x_9	Log-normal	$\mathcal{LN}(1, 0.15)$
x_{10}	Log-normal	$\mathcal{LN}(1, 0.20)$

Table 2. Characteristics of parameters of the Qingzhou Bridge.

Parameter	Distribution	Mean	COV
Elastic moduli of bridge pylon (E_p)	Log-normal	3.45E+10 (Pa)	0.15
Density of bridge pylon (ρ_p)	Weibull	2500 (kg/m ³)	0.10
Elastic moduli of stay cable (E_{sc})	Log-normal	1.95E+11 (Pa)	0.15
Elastic moduli of main girder (E_{mg})	Log-normal	2.10E+11 (Pa)	0.15
Density of main girder (ρ_{mg})	Weibull	7850 (kg/m ³)	0.10
Elastic moduli of cross girder (E_{cg})	Log-normal	2.1E+11 (Pa)	0.15
Density of cross girder (ρ_{cg})	Weibull	7850 (kg/m ³)	0.10
Elastic moduli of bridge deck (E_d)	Log-normal	3.45E+10 (Pa)	0.15
Density of bridge deck (ρ_d)	Weibull	2500 (kg/m ³)	0.10
Thickness of bridge deck (T_d)	Uniform	0.30 (m)	0.10

Table 3. Lists of posterior expectation of sensitivity indices for individual parameters.

Parameter	f_1		f_2		f_3		f_4	
	S_i	S_{Ti}	S_i	S_{Ti}	S_i	S_{Ti}	S_i	S_{Ti}
E_p	0.0173	0.0179	0.0186	0.0209	0.2339	0.3554	0.0076	0.0174
ρ_p	0.0056	0.0060	0.0020	0.0049	0.1000	0.1806	0.0038	0.0128
E_{sc}	0.6137	0.6184	0.2672	0.3415	0.3361	0.4497	0.5789	0.5912
E_{mg}	0.0036	0.0046	0.1018	0.1275	0.0050	0.0150	0.0464	0.0538
ρ_{mg}	0.0043	0.0047	0.0064	0.0082	0.0038	0.0090	0.0120	0.0140
E_{cg}	0.0001	0.0005	0.0001	0.0014	0.0001	0.0039	0.0001	0.0016
ρ_{cg}	0.0047	0.0052	0.0052	0.0068	0.0014	0.0052	0.0034	0.0049
E_d	0.0005	0.0016	0.0862	0.1424	0.0004	0.0042	0.0072	0.0122
ρ_d	0.1831	0.1847	0.3002	0.3062	0.0861	0.1295	0.2010	0.2070
T_d	0.1613	0.1631	0.1211	0.1417	0.0685	0.1119	0.1134	0.1218

Table 4. Summary of the clusters of parameters.

Group no	Structural component	Parameter
G_1	Bridge pylon	E_p, ρ_p
G_2	Stay cable	E_{sc}
G_3	Main girder	E_{mg}, ρ_{mg}
G_4	Cross girder	E_{cg}, ρ_{cg}
G_5	Bridge deck	E_d, ρ_d, T_d

Table 5. Lists of posterior expectation of sensitivity indices for groups of parameters.

Group no	f_1		f_2		f_3		f_4	
	S_i	S_{Ti}	S_i	S_{Ti}	S_i	S_{Ti}	S_i	S_{Ti}
G_1	0.0229	0.0239	0.0206	0.0257	0.3478	0.4801	0.0142	0.0258
G_2	0.6137	0.6184	0.2672	0.3415	0.3361	0.4497	0.5789	0.5912
G_3	0.0078	0.0092	0.1083	0.1354	0.0088	0.0238	0.0585	0.0672
G_4	0.0047	0.0056	0.0053	0.0081	0.0016	0.0090	0.0036	0.0063
G_5	0.3450	0.3492	0.5121	0.5818	0.1560	0.2359	0.3229	0.3374

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Fig. 1. Comparison of the evaluations of sensitivity indices for individual parameters with analytical values by the GPM method.

Fig. 2. Comparison of the evaluations of sensitivity indices for groups of parameters with analytical values by the GPM method.

Fig. 3. Qingzhou Bridge.

Fig. 4. Elevation (top) and cross-section (bottom) of Qingzhou Bridge (unit: m).

Fig. 5. Configuration of bridge pylon (unit: cm).

(a) Finite element model

(b) First vertical mode

(c) Second vertical mode

(d) First torsional mode

(e) Second torsional mode

Fig. 6. Finite element model and mode shapes of the Qingzhou Bridge.

Fig. 7. Plots of posterior expectation of main effects for each parameter. All parameters are normalized to the unit interval for a better visualization.

(a) f_1

(b) f_2

(c) f_3

(d) f_4

Fig. 8. Plots of posterior expectation of sensitivity indices for individual parameters.

Fig. 9. Plots of posterior expectation of sensitivity indices for groups of parameters.