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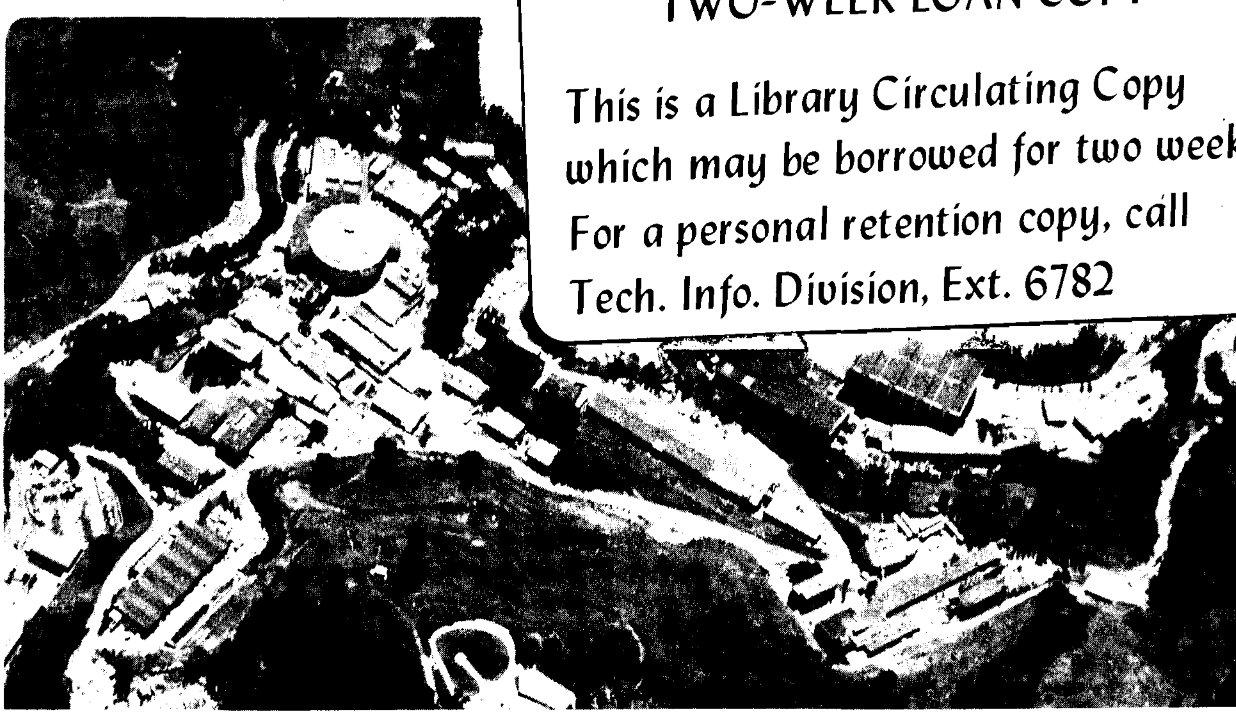

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DUAL TRANSFORMATIONS:
AN EQUIVALENCE BETWEEN THE CONTINUUM AND LATTICE VERSIONS OF THE TWO DIMENSIONAL ABELIAN HIGGS MODEL

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ABSTRACT

The topological excitations in the two dimensional Abelian Higgs model are studied in both the continuum and the Villain lattice version, and the results are compared. Using a singular gauge transformation, it is shown that there is an exact equivalence between the dual form of both versions. A fermionized form of the theory is useful in studying its confinement properties beyond the dilute gas limit. Also, the correct measure factor for the continuum version is derived.

I. INTRODUCTION

The existence of topological structures, instantons and solitons, has a significant impact on the theory in which they reside. Various attempts[1] have been made to construct effective field theories describing the modifications caused by these non-trivial topological structures. In particular, in the case of instantons, Polyakov[2] has introduced the notion of the dilute gas approximation as a calculational tool. Similar problems have been studied in solid state physics[3] utilizing the technique of duality transformations. The same methods were applied to lattice versions of field theories in the Villain[4] approximation and were successful in exposing the influence of topolgical structures on the properties of the theory. Both the continuum and the lattice versions involve various approximations and, although each gives important insight into the theory, there is a lack of unified approach.

In this paper, we present an exact procedure to treat the continuum limit by means of singular gauge transformations, and we check this method by showing that it is in exact correspondence with its lattice version. This is the main result of the paper. The method is applied to the two dimensional Abelian Higgs model, which has been studied extensively in both the continuum[5] and lattice versions.[6] After establishing this correspondence, we focus our attention on the structure of this particular system. By recasting the model into the equivalent fermionic representation, we are able to go beyond the dilute gas approximation and extract the long range structure of the theory exactly.

In particular, the Abelian Higgs model resembles the massive Schwinger model even for $m^2<0$.

The paper is organized as follows: We start by extracting identical forms for the partition function in both the continuum and lattice versions of the theory. We next discuss the infrared structure of the theory and rediscover the confinement of electric charge for a large range of parameters. Additional problems connected with the short distance singularities and renormalization are discussed in the last section.

II. TRANSFORMATION OF THE ABELIAN HIGGS MODEL IN (1+1) DIMENSIONS: THE CONTINUUM VERSION

In the following sections, starting from both the continuum and lattice versions of the Abelian Higgs model, a common form for the partition function will be obtained. Singular gauge transformations will be the main tool in extracting the topological structure of the model along the "continuum road." Along the "lattice road," dual transformations will play the key role. In both cases, a common form will be established, and the model, which started off as scalar fields interacting with themselves and with a massive vector boson, will be transformed into a model similar to a massive sine-Gordon theory. In this case, the existence of a non-vanishing mass term is a manifestation of confinement. We start by rewriting the Euclidean Lagrangian

$$\mathcal{L} = \frac{1}{4} F_{\mu\nu}^2 + |\partial_\mu\phi + ieA_\mu\phi|^2 + \beta^2(|\phi|^2 - \kappa^2)^2 \tag{2.1}$$

in the first order version:

$$\mathcal{L} = \frac{1}{2} H^2 + iFH + |\partial_\mu\phi + ieA_\mu\phi|^2 + \beta^2(|\phi|^2 - \kappa^2)^2 \quad , \tag{2.2}$$

where $F = \frac{1}{2} \varepsilon_{\mu\nu} F_{\mu\nu}$ and H is a real scalar field. Next, in order to extract the non-trivial topological structure of ϕ, we perform a singular gauge transformation, which maps ϕ into the unitary gauge $\mathrm{Im}(\phi) = 0$.

Under a regular gauge transformation

$$\phi \to e^{-i\Lambda}\phi, \quad A_\mu \to A_\mu + \frac{1}{e}\partial_\mu\Lambda = A_\mu' + \Delta A_\mu \quad , \tag{2.3}$$

the electric field $F_{\mu\nu}$ is invariant:

$$F_{\mu\nu} \to F_{\mu\nu} + \partial_\mu(\Delta A_\nu) - \partial_\nu(\Delta A_\mu) = F_{\mu\nu} \quad . \tag{2.4}$$

However, if ϕ has a non-trivial topological structure of the form

$$\phi = e^{in\theta}\,|\phi| = e^{in\theta}\,\bar{\phi} \tag{2.5}$$

where n is an integer and θ is the polar angle, the gauge transformation which maps ϕ into the unitary gauge is singular. As a result, there is an additional contribution to the electric flux given by[7]

$$\Delta F_{\mu\nu} = \frac{2\pi}{e}\,\varepsilon_{\mu\nu}\sum_\ell n_\ell\,\delta^2(x-x_\ell) \quad , \tag{2.6}$$

where x_ℓ are the locations of topological singularities of order n_ℓ. One can attempt to separate the functional integral into an integral over all topologically trivial "flat" configurations and a sum over all possible integer valued singularities integrated over all possible locations as follows:

$$Z = \int DH \int DA_\mu \int D\phi \, \exp[\, - \int d^2x \, \mathcal{L}(\phi, A_\mu, H)]$$

$$= \int DH \int DA_\mu \int D\bar{\phi} \int J_1 d^2y \sum_{n(y)=-\infty}^{\infty} \exp[\, - \int d^2x \, \bar{\mathcal{L}}(\bar{\phi}, n(y), A_\mu, H)] \quad , \tag{2.7}$$

where the new lagrangian $\bar{\mathcal{L}}$ is given by

$$\bar{\mathcal{L}}(\bar{\phi}) = \mathcal{L}(\bar{\phi}) + \frac{2\pi i}{e} H(x) n(y) \, \delta^2(x-y) \quad , \tag{2.8}$$

where $J_1 = \prod_i |\phi(x_i)|$ is the singular Jacobian needed in the gauge $\mathrm{Im}(\phi) = 0$, and the integration $D\bar{\phi}$ is only over flat (real) configurations of ϕ. We present this formula since it goes over, without any modification, to its lattice analogue to be derived in the next section. However, it is difficult to define an integer valued function $n(y)$ at each point of space in a continuum theory, whereas no such difficulty exists on a lattice. To avoid this problem, it is necessary to use a Fock space representation for the topological number, instead of the occupation number representation:

$$Z = \int DH \int DA_\mu \int D\bar{\phi} \; J_1 \sum_{N=0}^{\infty} \frac{\lambda^N}{N!} \prod_{n=1}^{N} \int d^2x_n$$

$$\sum_q \exp[\, -\int d^2x \, \mathcal{L}(x; x_1, \cdots, x_n, q_1, \cdots, q_n)] \quad , \tag{2.9}$$

where the q's are the integer topological charges and

$$\mathcal{L}(x;x_1,\cdots,x_n,q_1,\cdots,q_n) = \mathcal{L}(\bar{\phi})$$

$$+ \frac{2\pi i}{e} H(x) \sum_{\ell=1}^{n} q_\ell \, \delta^2(x-x_\ell) \quad . \tag{2.10}$$

In this new expression for Z, one first integrates over the positions of topological configurations and sums over their (integer) charges q_ℓ, keeping their total number N fixed, and finally N is summed over. λ is a measure factor which is singular for a zero size topological configuration; to have a non-singular λ, it is necessary to give the topological configurations temporarily a small but finite size and go to the zero size limit at the end. In Section V, a derivation of Eq. (2.9) will be given and the modifications due to finite size effects will be discussed.

There is a potential overcounting problem in connection with Eq. (2.9). When, say, two coordinates x_1 and x_2 coincide somewhere in their range of integration, a topological configuration of charge $q = q_1 + q_2$ is reproduced twice: once in the manner described above, and once as a single topological configuration of charge $q = q_1 + q_2$. It is easy to see, however, that this overcounting is proportional to the size of the configuration and goes to zero with the size.

We can now integrate over the vector potential A_μ, noticing that a Jacobian J_2 will arise from the change of variables $\bar{\phi}A_\mu \to A_\mu$ and will cancel J_1 introduced in Eq. (2.7). The resulting partition function is derived from the lagrangian

$$\mathcal{L} = \frac{1}{2} H^2 + \beta^2(\bar{\phi}^2 - \kappa^2)^2 + (\partial_\mu \bar{\phi})^2$$

$$+ \frac{1}{4e^2\bar{\phi}^2} (\partial_\mu H)^2 + \frac{2\pi i}{e} H(x)\, n(y)\, \delta^2(x-y) \quad . \tag{2.11}$$

After defining

$$m_v^2 = 2e^2\kappa^2, \quad m_s^2 = 4\lambda^2\kappa^2, \quad H = m_v E, \tag{2.12}$$

one can rewrite (2.11) to obtain

$$\mathcal{L} = \frac{1}{2} m_v^2 E^2 + \beta^2(\bar{\phi}^2 - \kappa^2)^2 + (\partial_\mu \bar{\phi})^2$$

$$+ \frac{m_v^2}{4e^2\bar{\phi}^2} (\partial_\mu E)^2 + \frac{2\pi i}{e} m_v\, E(x)\, n(y)\, \delta^2(x-y) \quad . \tag{2.13}$$

The continuum version of the Abelian Higgs model has now been cast into a form which is identical to the form to be derived from the lattice version in the next section. The transformation of Eq. (2.1) into Eq. (2.13) is the key result of this section. The physical consequences of Eq. (2.13) are not yet transparent; nevertheless, we postpone the discussion of the consequences of this equation and instead focus our attention on the lattice version of the same model in the next section.

III. ABELIAN HIGGS MODEL: LATTICE VERSION

In this paper, the lattice version of the continuum theory is used as a bookkeeping device to avoid various measure factors that arise in the continuum theory. In that spirit, we shall emphasize the similarities between the two versions instead of the technical differences. We start with the Higgs fields in their linear form coupled to a non-compact gauge field:

$$S = \frac{1}{4} \sum_p \theta_{\mu\nu}^2 - \sum_\ell \left[\rho(j+1)\rho(j)\left(\exp\left[i\left(\tilde{e}\theta_\mu(j) + \chi(j) - \chi(j+1)\right)\right] + h.c.\right]\right]$$
$$+ \sum_s \left[\rho^2(j) + \rho^2(j+1) + \tilde{\beta}^2\left(\rho^2(j) - \kappa^2\right)^2\right] \quad , \tag{3.1}$$

where the letters p, ℓ and s under the summation signs imply sums over plaquettes, links and sites respectively, θ_μ is the vector potential, $\theta_{\mu\nu}$ the electric field, and the Higgs field $\phi(j)$ is given by

$$\phi(j) = \rho(j)\, e^{i\chi(j)} \quad . \tag{3.2}$$

The correspondence between the continuum and lattice variables is $\tilde{e} \leftrightarrow ae$, $\tilde{\beta} \leftrightarrow a\beta$, a = lattice spacing,

$$A_\mu \leftrightarrow \theta_\mu, \quad F_{\mu\nu} \leftrightarrow \theta_{\mu\nu}, \quad \bar{\phi} \leftrightarrow \rho, \quad . \tag{3.3}$$

The generating functional of the theory is given by

$$Z = \int_0^\infty D\rho \; J_1 \int_0^{2\pi} D\chi \int_{-\infty}^{\infty} D\theta_\mu \; e^{-S} \quad , \tag{3.4}$$

where $J_1 = \prod_s \rho(j)$ is the usual volume factor in polar coordinates.

In the lattice version, it is the compact range of integration of the angular variable χ that introduces non-trivial topological structure, and so it is natural to try to remove the restriction on the range of the angular integration. This can be done by first performing a Fourier expansion at each link:

$$\exp\left[2\rho(j+1)\;\rho(j)\;\cos(\Delta_\mu\chi - \tilde{e}\theta_\mu)\right]$$

$$= \sum_{\ell_\mu=-\infty}^{\infty} \exp\left[i\ell_\mu(\Delta_\mu\chi - \tilde{e}\theta_\mu)\right] I_{\ell_\mu}\left[2\rho(j+1)\;\rho(j)\right] \quad , \tag{3.5}$$

where I is a Bessel function. The Villain approximation consists of replacing $I_{\ell_\mu}(z)$ by its large z asymptotic limit. If one wishes to avoid approximations, one could have started with the Villain form from the very beginning instead of the action given by Eq. (3.1). Letting

$$I_\ell(z) \underset{z\to\infty}{\to} \text{constant.}\; z^{-1/2} \exp\left(z - \frac{\ell^2}{2z}\right) \tag{3.6}$$

we arrive at

$$\exp\left[2\rho(j+1)\;\rho(j)\;\cos(\Delta_\mu\chi - \tilde{e}\theta_\mu) - \rho^2(j) - \rho^2(j+1)\right]$$

$$\to \sum_{\ell_\mu=-\infty}^{\infty} \left[2\rho(j)\;\rho(j+1)\right]^{-1/2} \exp\left[-(\Delta_\mu\rho)^2\right]$$

$$\exp\left(-\frac{\ell_\mu^2}{4\rho(j)\;\rho(j+1)}\right) \exp\left[i\ell_\mu(\Delta_\mu\chi - \tilde{e}\theta_\mu)\right] \quad . \tag{3.7}$$

Notice that the term $\prod_j \left[2\rho(j)\ \rho(j+1)\right]^{-1/2}$ cancels the measure factor J_1 introduced earlier. Substituting (3.7) back into (3.1) and using the first order formulation for the gauge field, we obtain

$$Z = \sum_{\{\ell_\mu\}} \int_{-\infty}^{\infty} DH \int_{-\infty}^{\infty} D\theta_\mu \int_0^{\infty} D\rho(j) \int_0^{2\pi} D\chi\ \exp(-\bar{S}) \quad , \tag{3.8}$$

where H is defined on the dual lattice sites and $\bar{S}$ is given by

$$\bar{S} = \sum \frac{1}{2} H^2 + \frac{i}{2} \sum \varepsilon_{\mu\nu}\theta_{\mu\nu} H + \sum (\Delta_\mu \rho)^2$$
$$- i \sum_\ell \ell_\mu (\Delta_\mu \chi - \tilde{e}\theta_\mu) + \tilde{\beta}^2 \sum_s \left[\rho^2(j) - \kappa^2\right]^2 \quad . \tag{3.9}$$

We now integrate over the compact variable χ, which results in a constraint on the integer valued link variable ℓ_μ:

$$Z = \sum_{\{\ell_\mu\}} \int D\rho \int DH \quad \delta(\Delta_\mu \ell_\mu)$$
$$\exp\left(- \frac{1}{2} \sum H^2 - \sum \frac{\ell_\mu^2}{4\rho(j)\ \rho(j+1)} - \sum (\Delta_\mu \rho)^2 - \tilde{\beta}^2 \sum \left(\rho^2(j) - \kappa^2\right)^2 \right.$$
$$\left. + i \sum \tilde{e}\ \ell_\mu \theta_\mu - \frac{i}{2} \sum \varepsilon_{\mu\nu}\theta_{\mu\nu}\ H \right) \quad . \tag{3.10}$$

The constraint can be solved explicitly in terms of a scalar variable ℓ:

$$\ell_\mu = \varepsilon_{\mu\nu}\ (\Delta_\nu \ell) \quad , \tag{3.11}$$

where ℓ, like H, is defined at the centers of the plaquettes or equivalently, on the dual lattice sites.

Since we wish to trade the compact variable χ for a non-compact variable, we make use of the Poisson summation formula:

$$\sum_{p=-\infty}^{\infty} f(p) = \sum_{\ell=-\infty}^{\infty} \int_{-\infty}^{\infty} d\phi \; f(\phi) \; \exp(2\pi i \ell \phi) \quad , \tag{3.12}$$

and obtain an unconstrained sum:

$$Z = \sum_{\{\ell\}} \int D\rho \int D\phi \int DH \int D\theta_\mu \; \exp\Bigg(- \frac{1}{2} \sum H^2 + 2\pi i \sum \ell \; \phi$$

$$- \sum (\Delta_\mu \rho)^2 - \tilde{\beta}^2 \sum \left[\rho^2(j) - \kappa^2\right]^2 - \sum \frac{(\Delta_\mu \phi)^2}{4\rho(j)\;\rho(j+1)}$$

$$+ i\tilde{e} \sum \epsilon_{\mu\nu} \theta_\mu (\partial_\nu \phi) - \frac{i}{2} \sum \epsilon_{\mu\nu} \theta_{\mu\nu} \; H \Bigg) \quad . \tag{3.13}$$

Next, we integrate over the vector potential θ_μ to obtain a constraint relating the ϕ and H fields:

$$Z = \sum_{\{\ell\}} \int D\rho \int DH \int D\phi \quad \delta\big(e(\Delta_\mu \phi) - (\Delta_\mu H)\big)$$

$$\exp\Bigg(- \frac{1}{2} \sum H^2 - \sum (\Delta_\mu \rho)^2 - \tilde{\beta}^2 \sum \left(\rho^2(j) - \kappa^2\right)^2$$

$$+ 2\pi i \sum \ell \phi - \sum \frac{(\Delta_\mu \phi)^2}{4\rho(j)\;\rho(j+1)} \Bigg) \quad . \tag{3.14}$$

The solution to the constraint equation is

$$\phi = \frac{1}{\tilde{e}} H + \theta \quad , \tag{3.15}$$

where θ is a constant.

After rescaling by $H = \tilde{m}_v E$, where $\tilde{m}_v = am_v = \sqrt{2}\, \tilde{e}\, \kappa$, we obtain

$$Z = \int D\rho \int DE \sum_{\{\ell\}} \exp\left(- \frac{1}{2} \tilde{m}_v^2 \sum E^2 - \sum (\Delta_\mu \rho)^2 - \tilde{\beta}^2 \sum \left(\rho^2(j) - \kappa^2\right)^2 - \frac{\tilde{m}_v^2}{4\tilde{e}^2} \sum \frac{(\Delta_\mu E)^2}{\rho(j)\, \rho(j+1)} + 2\pi i \sum \left(\frac{\tilde{m}_v}{\tilde{e}} E + \theta \right) \ell \right) . \tag{3.16}$$

This is the lattice version of Eq. (2.13) obtained in the continuum limit, and clearly there is full agreement between the two versions. We note that the parameter θ can be identified with the Coleman angle.[8] This term was neglected in the continuum derivation, and this neglect can be traced to the following step:

$$i\int d^2x\, FH = i\int d^2x\, A_\mu\, \varepsilon_{\mu\nu}(\partial_\nu H) \quad . \tag{3.17}$$

This step involves an integration by parts and a subsequent dropping of a possible surface term. Keeping this surface term, one easily recovers the Coleman angle.

We finally note that, had we started with the compact form for the gauge field, we would have obtained the same final result in the Villain version of the model, despite some additional complications.

In the next section, we shall study the physical properties of the theory in various approximations.

IV. PHYSICAL PROPERTIES

The Abelian Higgs theory was originally studied by Callen, Dashen and Gross[5] as a model in which instanton configurations lead to non-perturbative confinement of the electric charge. They have studied the model in the weak coupling regime, and the functional integral was saturated by a non-interacting gas of instantons and anti-instantons, corresponding to configurations of topological charge plus and minus one. The lattice version was also studied by Einhorn and Savit[6] in the limit of non-interacting instantons, and they obtained similar results for the partition function. In both versions, the question of interest is the confinement (or lack of it) of non-integer charge, and so the Wilson loop integral has to be evaluated in some approximation. If the field ρ is frozen at the value $\rho = \kappa$ by taking the limit $\beta \to \infty$, the partition function can be explicitly evaluated from Eq. (3.16):

$$Z = \sum_{\{\ell\}} \exp\left(- 4\pi^2\kappa^2 \sum_{\underset{\sim}{r},\underset{\sim}{r}'} \ell(\underset{\sim}{r})\, D(\underset{\sim}{r}-\underset{\sim}{r}',\tilde{m}_v)\, \ell(\underset{\sim}{r}') + 2\pi i\theta \sum_{\underset{\sim}{r}} \ell(\underset{\sim}{r}) \right) , \tag{4.1}$$

where D, the free propagator of mass $\tilde{m}_v$, is given by

$$D(\underset{\sim}{r},\tilde{m}_v) = \int_0^{2\pi/a} \int_0^{2\pi/a} d^2\underset{\sim}{k}\; \frac{e^{i\underset{\sim}{k}\cdot\underset{\sim}{r}}}{\tilde{m}_v^2+\underset{\sim}{k}^2} \quad . \tag{4.2}$$

Eq. (4.1) is still too complicated for direct use; however, it can be simplified considerably in the limit of large $\tilde{m}_V$ by making the following approximation to the propagator:

$$D(\underset{\sim}{r},\tilde{m}_V) \underset{\tilde{m}_V\to\infty}{\to} \delta_{\underset{\sim}{r},0} D(0,\tilde{m}_V) \quad . \tag{4.3}$$

This limit is similar to the dilute gas approximation, since only the self energy of each topological configuration is taken into account, and the interactions between configurations at different lattice sites are neglected. However, it goes one step beyond the conventional treatment by taking into account topological configurations of arbitrary charge, not just plus and minus one. In this approximation, the partition function is given by (V = volume of space)

$$Z = \left(\theta_3\left(\exp\left(-4\pi^2\kappa^2 D(0,\tilde{m}_V)\right),0\right)\right)^{V/a^2} \quad , \tag{4.4}$$

where, for simplicity, we have taken $\theta = 0$, and θ_3 is the Jacobi theta function. If now there is an external charge of q circulating through a Wilson loop of area A, we have

$$\frac{Z(J)}{Z(0)} = \left[\frac{\theta_3\left(\exp\left(-4\pi^2\kappa^2 D(0,\tilde{m}_V)\right),\pi q/e\right)}{\theta_3\left(\exp\left(-4\pi^2\kappa^2 D(0,\tilde{m}_V)\right),0\right)}\right]^{A/a^2} \quad . \tag{4.5}$$

The ratio of the theta functions is less than one when q is not an integer multiple of e, and the Wilson area law follows. If $\ell(\underset{\sim}{r})$ is restricted to 0, ±1, the expression for the partition function simplifies:

$$Z(0) = \left[1 + 2\cos(\theta)\exp\left(-4\pi^2\kappa^2 D(0,\tilde{m}_v)\right)\right]^{V/a^2} \quad . \tag{4.6}$$

Identifying the factor in the exponential with the classical action in this approximation,

$$4\pi^2\kappa^2 D(0,\tilde{m}_v)$$

$$= 2\pi^2\tilde{m}_v^2/\tilde{e}^2\, D(0,\tilde{m}_v) \to S_c \quad , \tag{4.7}$$

the well-known dilute gas result is obtained:

$$\frac{Z(J)}{Z(0)} = \exp\left(-\cos(\theta)\, e^{-S_c} A/a^2\right) \quad . \tag{4.8}$$

In the continuum version, similar results can be obtained by restricting the q's to 0 and ±1 in equation (2.9). The sum over q's can then easily be done:

$$Z = \int DH \int DA_\mu \int D\bar{\phi}\; J_1 \exp\left\{-\int d^2x \left(\frac{1}{2}H^2 + iFH\right.\right.$$
$$\left.\left. + (\partial_\mu\bar{\phi})^2 + e^2A_\mu^2\bar{\phi}^2 + \beta^2(|\phi|^2 - \kappa^2)^2 + 2\lambda\cos\left(\frac{2\pi}{e}H\right)\right)\right\} \quad . \tag{4.9}$$

We now integrate over A_μ and follow the steps that led to Eq. (2.13). This yields the Lagrangian

$$\mathcal{L} = \frac{1}{2} m_v^2 E^2 + \beta^2(\bar{\phi}^2 - \kappa^2)^2 + (\partial_\mu \bar{\phi})^2$$

$$+ \frac{m_v^2}{4e^2\bar{\phi}^2} (\partial_\mu E)^2 + 2\lambda \cos\left(2\pi \frac{m_v}{e} E \right) . \qquad (4.10)$$

For simplicity, if now $\bar{\phi}$ is frozen at the value κ by letting $\beta \to \infty$, a massive sine-Gordon equation is obtained:

$$\mathcal{L} = \frac{1}{2} (\partial_\mu E)^2 + \frac{1}{2} m_v^2 E^2 + 2\lambda \cos\left(2\pi \frac{m_v}{e} E \right) . \qquad (4.11)$$

A similar result was derived by Schaposnik,[5] who calculated the one loop correction to the semi-classical approximation. The lattice result given by Eq. (3.16) can also be cast into the above form by making the approximation

$$\sum_{\{\ell\}} \to \sum_{N=0}^{\infty} \frac{1}{N!} \prod_{n=1}^{N} \sum_{q=\pm 1} \sum_{x_n} , \qquad (4.12)$$

where the topological charge is again restricted to 0, and ±1, the overcounting due to the overlap of the positions of topological configurations is neglected. Then the lattice version of Eq. (4.11) follows, with $\lambda = 1/a^2$, where a is the lattice spacing.

In the dilute gas limit, when $m_v/e \ll 1$, it is reasonable to treat the last term in Eq. (4.11) as perturbation and expand in powers of λ. At first sight, it is not obvious that this can be done, since, $\lambda = 1/a^2$ is not necessarily small, and it seems that there is no small parameter of expansion available. The easiest way to generate such a parameter is to go over to the operator description in Minkowski space and consider the interaction representation Hamiltonian[9] corresponding the Lagrangian (4.11):

$$\mathcal{H}_I = 2\lambda \cos\left(2\pi \frac{m_v}{e} E_0 \right) \tag{4.13}$$

where E_0 is a free field of mass m_v. Notice that the interaction Hamiltonian is not normal ordered. This is because, when the Euclidean functional integral, which was our starting point, is translated into the operator language, the fields in the Lagrangian are time ordered. It is then convenient to normal order Eq. (4.13) before doing the perturbation expansion:

$$\mathcal{H}_I = 2\lambda \exp\left(- \frac{2\pi^2 m_v^2}{e^2} D(0,m_v) \right) : \cos\left(2 \frac{m_v}{e} E_0 \right) :$$

$$= 2\lambda \exp(-S_c) : \cos\left(2\pi \frac{m_v}{e} E_0 \right) : . \tag{4.14}$$

In the normal ordered form, the small parameter is the barrier penetration factor $\exp(-S_c)$, which serves as the expansion parameter. Of course, the same result can also be obtained without normal ordering; in that case, there will be additional graphs in the perturbation series including self contractions at a point.[10] Normal ordering conveniently sums these graphs into an overall barrier penetration factor, which is the exponential of the instanton self-action computed in weak coupling approximation. In fact, both in Polyakov's[2] original analysis and in some lattice versions of the model, when the barrier penetration factor explicitly appears multiplying the cosine term, this term should be normal ordered.

The perturbation expansion we have described can be used to evaluate the Wilson loop, and Eq. (4.8) is then recovered. However, we wish to go beyond the dilute gas limit and establish charge confinement also in the strong coupling regime $e/m_v \geqq 1$. Needless to say, we cannot carry out any explicit computations in this regime; however, we are able to present general physical arguments which we find persuasive. These arguments make use of the known properties of the massive sine-Gordon theory and in particular they rely on the correspondence between the sine-Gordon theory and a massive Schwinger-like model.[10-12] This correspondence is established by first scaling the field E by

$$E = e/m_v \;\; \pi^{-1/2} \, E' \qquad (4.15)$$

in order to cast the cosine term in Eq. (4.11) into the standard sine-Gordon form:

$$\mathcal{L} = \mathcal{L}_0 + \mathcal{L}_I ,$$

$$\mathcal{L}_0 = \frac{1}{2} (\partial_\mu E')^2$$

$$\mathcal{L}_I = \frac{1}{2} \left(\frac{e^2}{\pi m_v^2} - 1 \right) (\partial_\mu E')^2 + 2\lambda \cos\left(2\pi^{1/2} E' \right)$$

$$+ \frac{1}{2} e^2/\pi \, (E')^2 \quad . \tag{4.16}$$

The model can now be "fermionized" by translating $\mathcal{L}_I$ into the fermion language by the following dictionary:

$$j^\mu = \, : \bar{\psi} \gamma^\mu \psi : \, = \pi^{-1/2} \varepsilon^{\mu\nu} (\partial_\nu E') \quad ,$$

$$: \bar{\psi} \psi : \, = c \, : \cos(2\pi^{1/2} E') : \quad ,$$

$$\frac{1}{2} e^2/\pi \, (E')^2 \rightarrow \frac{1}{4} (\partial_\mu B_\nu - \partial_\nu B_\mu)^2 + e \, B_\mu j^\mu , \tag{4.17}$$

where ψ is the fermion field, c is a constant and B_μ is a (new) electromagnetic field. The Lagrangian of Eq. (4.11) reads as follows in the fermionic language (and in Minkowski metric):

$$\mathcal{L} = \bar{\psi} (i\partial\!\!\!/ - m) \psi - \frac{1}{4} (\partial_\mu B_\nu - \partial_\nu B_\mu)^2$$

$$+ e \, B_\mu \bar{\psi} \gamma^\mu \psi + \frac{1}{2} \left(\frac{e^2}{m_v^2} - \pi \right) \bar{\psi} \gamma^\mu \psi \, \bar{\psi} \gamma_\mu \psi \quad . \tag{4.18}$$

It is clear that there is a conserved fermionic charge which interacts with a "photon" field. In (1+1) dimensions, this gives rise to a confining Coulomb potential. Here, we are making the assumption that the additional short range four fermi contact interaction has no effect on the confining properties of the model, which depend solely on the long range Coulomb force. What is the physical meaning of this new charge? In fact, it is nothing but the (broken) charge of the Higgs field ϕ we started with, and therefore we have witnessed the restoration of a broken symmetry (charge conservation). To see this, introduce an external source in the original Lagrangian of Eq. (2.1):

$$\Delta\mathcal{L} = i\, A_\mu\, \varepsilon_{\mu\nu}(\partial_\nu K) = i\, F\, K \quad , \tag{4.19}$$

where K(x) equals the external charge Q over an area A and vanishes outside.[11] If charge conservation were not broken, we would expect the energy shift to behave like

$$\Delta E \approx \text{const. } A\, Q^2 \quad . \tag{4.20}$$

Now the steps that led to Eq. (4.11) can be repeated with this additional term to yield the following result:

$$\mathcal{L} + \Delta\mathcal{L} = \frac{1}{2}\left(\partial_\mu E + \frac{1}{m_v}\,\partial_\mu K\right)^2 + \frac{1}{2}\, m_v^2\, E^2 + 2\lambda \cos\left(\frac{2\pi m_v}{e}\, E + \frac{2\pi}{e}\, K\right) \quad . \tag{4.21}$$

The terms proportional to $\partial_\mu K$ go like the perimeter of A and can be neglected. In a perturbative treatment, E can be set equal to its zero vacuum expectation value in the cosine term and a contribution proportional to the area A is obtained. In fact, Coleman et al[12] show, by carrying the transformation (4.17), that K and hence Q can be identified with the charge of the fermion field ψ, without relying on any perturbation argument.

Let us now remove the restrictions we had to impose to deduce charge confinement. First, we can easily "unfreeze" the field $\bar{\phi}$. Defining

$$\bar{\phi} = \kappa + \frac{1}{\sqrt{2}} \phi_r \quad , \tag{4.22}$$

Eq. (4.18) gets replaced by the following:

$$\begin{aligned} \mathcal{L} = {} & \bar{\psi} (i\not{\partial} - m) \psi - \frac{1}{4} (\partial_\mu B_\nu - \partial_\nu B_\mu)^2 \\ & + e B_\mu \bar{\psi} \gamma^\mu \psi + V(\phi_r) \\ & + \frac{1}{2} \left(\frac{e^2}{m_v^2} \left[1 + \frac{e}{m_v} \phi_r\right]^{-2} - \pi \right) \bar{\psi} \gamma^\mu \psi \bar{\psi} \gamma_\mu \psi \quad , \end{aligned} \tag{4.23}$$

where $V(\phi_r)$ is the Higgs potential. The only change compared to (4.18) is in the coefficient of the four fermi contact term; in particular, the arguments about the existence of a conserved charge and a confining Coulomb interaction remain unchanged.

Finally, let us remove the restriction that q = 0, ±1 and allow only then do we have

$$E'\ (t, x \to \pm\infty) \to \pi^{1/2}\, n \quad , \tag{4.26}$$

where n is an integer, and we recover charge quantization, as we should. The mass term $\frac{1}{2}\, e^2/\pi\ (E')^2$ is again mapped into the (confining) electromagnetic interaction.

Although we have just constructed a general argument for charge confinement that does not rely on the dilute plasma (and hence the weak coupling) approximation, we cannot exclude the possibility that the short range interaction terms between the fermions may upset our argument. Although it appears unlikely on physical grounds, the contact interactions between the fermions may somehow succeed in screening the charge.

We conclude this section with one additional remark about the model. For a fixed κ, as β tends to infinity in the continuum theory, the classical action S_c also becomes infinite. The barrier penetration factor then entirely suppresses the cosine interaction term, and we are left with a free massive boson. We cannot shed new light on the claims of Kogut and Sinclair[11] that the lattice version of the theory with frozen Higgs field has another ((xy) like?) continuum limit. We do observe, however, that the Abelian Higgs model in the $m^2<0$ region is similar to the massive Schwinger model, and this makes the possibility of a phase transition as a function of m^2 unlikely.[5]

topological configurations of all possible charges. In the continuum version, this would lead to the following generalization of the interaction term in Eq. (4.10):

$$2\lambda \cos\left(2\pi \frac{m_v}{e} E\right) \rightarrow \sum_{n=1}^{\infty} 2\lambda_n \cos\left(2\pi n \frac{m_v}{e} E\right) \quad , \qquad (4.24)$$

where the λ's depend on n. In the lattice versions, all λ's are still equal to $1/a^2$, but one is faced with the overcounting problem due to the overlap of different topological configurations at the same lattice site. Therefore, it appears necessary to introduce short range repulsive interactions (hard core) to avoid this problem. We shall not attempt to solve this problem here, but instead we simply observe that so long as the potential term is a periodic function of E, as it is in Eq. (4.24), there is a conserved charge and a long range Coulomb interaction. This can be seen directly by carrying out the transformation given by (4.17); the terms with $n \geq 2$ in (4.24) are mapped into many fermion contact terms, and the essential features of Eq. (4.18) are unchanged. Or, alternatively, one can go back to the definition of charge Q for the massive Schwinger model given by (4.17):

$$Q = \int dx\ j_0\ (t,x) = \pi^{-1/2} \int dx\ (\partial_x E')$$

$$= \pi^{-1/2} \left[E'\ (t,x = +\infty) - E'\ (t,x = -\infty)\right] \quad . \qquad (4.25)$$

This definition works so long as the potential is periodic, since

V. DETERMINATION OF MEASURE IN THE CONTINUUM VERSION

In the continuum version of the Abelian Higgs model, the factor λ that appears in Eq. (4.10) was left undetermined. In this section, we shall rectify this omission and give a derivation of λ. It turns out that this factor is singular for a zero size topolgoical configuration, and to avoid this singularity, it is necessary to smooth out the singular gauge transformation given by Eq. (2.6). Consider a configuration of charges $q_1,\cdot\cdot,q_N$ located at points $x_1,\cdot\cdot\cdot,x_N$. The transformation of the field ϕ is the same as before; it is a product of singular phase transformations of the form

$$\prod_n \exp(i\ q_n\ \theta_n) \tag{5.1}$$

with the singularities located at points $x_1,\cdot\cdot,x_N$. In the transformation of the field A_μ however, the delta functions that appear in Eq. (2.6) are replaced by smooth functions:

$$A_\mu(x) = \bar{A}_\mu(x) + \sum_{n=1}^{N} q_n\ A_\mu^c(x - x_n), \tag{5.2}$$

where $A_\mu^c(x)$ is a smooth function localized at $x = 0$. The only other condition satisfied by A_μ^c is a normalization condition:

$$\int d^2x\ F^c(x) = \frac{2\pi}{e} \quad \text{where} \quad ,$$

$$F^c(x) = \varepsilon_{\mu\nu}\ \partial_\mu\ \left(A_\nu^c(x)\right) \quad . \tag{5.3}$$

A simple explicit example for A_μ^c is

$$A_\mu^c(x) = \frac{1}{e}\,\varepsilon_{\mu\nu}\,\frac{x_\nu}{x^2+\varepsilon^2} \quad , \tag{5.4}$$

where ε is a size parameter.

In the limit $\varepsilon\to 0$, the delta functions of Eq. (2.6) are recovered. We shall, of course, postpone taking this limit until the very end.

Now let us reconsider Eq. (4.12). This equation follows from the functional change of variable given by Eq. (5.2). The summations over N and q are straightforward; however, the points $x_1, x_2, \cdots, x_N$ are fixed and there is no integration over them.

At this point, we resort to a standard trick;[13] we impose a suitable constraint over the field A and eliminate this constraint by integrating over the x's. The identity needed is

$$1 = \int\int\cdots\int d^2x_1\cdots d^2x_N \prod_{n=1}^{N}\left\{\Delta(x_n)\prod_{\mu=1}^{2}\delta\left(\int d^2x\, A_\mu(x)\, F^c(x-x_n)\right)\right\} \quad , \tag{5.5}$$

where Δ is the Faddeev-Popov factor:

$$\Delta(x') = \det\left(\int d^2x\, A_\mu(x)\,\frac{\partial F^c(x-x')}{\partial x'_\nu}\right) \quad . \tag{5.6}$$

Reexpressed in terms of $\bar{A}_\mu$, Δ is given by

$$\Delta(x') = \det\left(\bar{\lambda}\ \varepsilon_{\mu\nu} - \int d^2x\ \bar{A}_\mu(x)\ \frac{\partial F^c(x-x')}{\partial x'_\nu}\right) , \tag{5.7}$$

where,

$$\bar{\lambda} = \frac{2}{3e^2\varepsilon^2} \tag{5.8}$$

if A^c_μ is given by (5.4). Clearly, the value of $\bar{\lambda}$ depends on the particular choice of A^c_μ; however, the $1/\varepsilon^2$ dependence is universal and follows by dimensional reasoning. If one now naively took the $\varepsilon \to 0$ limit, one would formally have

$$\lambda \to \bar{\lambda} = \frac{2}{3e^2\varepsilon^2} \to \infty \tag{5.9}$$

as $\varepsilon \to 0$, since the second term in Eq. (5.7) would be negligible compared to $\bar{\lambda}$. Although we cannot set $\varepsilon=0$, it should be taken much smaller than any natural constant with dimensions of length that appears in the Lagrangian, like the inverse of e or the inverse of any mass parameter. In this fashion, we assume that the contributions due to the overlap of configurations located at different points can be neglected. Then, everything proceeds as in the dilute gas case.

The next task is to carry out the functional integral over the field $\bar{A}_\mu$, as in Eq. (4.10), but now subject to the constraint of Eq. (5.5):

$$I_N = \int D\bar{A}_\mu \iiint \cdots \int d^2x_1 \cdots d^2x_N$$

$$\sum_q \prod_{n=1}^{N} \left\{ \Delta(x_n) \prod_\mu \delta\left[\int d^2x \; A_\mu(x) \; F^c(x-x_n)\right]\right\}$$

$$\exp\left\{- \int d^2x \; \mathcal{L} \; (\bar{A}_\mu ; \bar{\phi} ; q_1, \cdots, q_N ; x_1, \cdots, x_N)\right\} \quad . \qquad (5.10)$$

If the functional delta function is exponentiated by means of a lagrange multiplier, the integration over $\bar{A}_\mu$, being a gaussian, can easily be done. Also, because of the absence of overlap between configurations located at different points, the dependences on $x_1, \cdots, x_N$ and $q_1, \cdots, q_N$ nicely factorize. This enables one to do the integrals and the sums in closed form into an exponential. We skip the details, which are quite straightforward but tedious, and give the final answer for λ that appears in Eq. (4.10). Actually, λ turns out to be complex, which means that the cosine acquires a phase. If

$$\lambda = |\lambda| \; e^{i\psi}$$

then

$$2\lambda \cos\left(\frac{2\pi}{e} H\right) \to 2|\lambda| \cos\left(\frac{2\pi}{e} H + \psi\right) \; . \qquad (5.11)$$

To express the result in a compact form, it is convenient to define some auxilliary fields:

$$\chi_1^{\mu}(x) = \int d^2x' \frac{F^c(x-x')\partial_\mu\left(H(x')\right)}{\left(m_v+e\,\phi_r(x')\right)^2} ,$$

$$\chi_2^{\mu}(x) = \int d^2x' \frac{F^c(x-x')\partial_\mu\left(F^c(x-x')\right)}{\left(m_v+e\phi_r(x')\right)^2} ,$$

$$\chi^{\mu\nu}(x) = \int d^2x' \frac{\partial_\mu\left(F^c(x-x')\right)\partial_\nu\left(H(x')\right)}{\left(m_v+e\phi_r(x')\right)^2} ,$$

$$\chi(x) = \int d^2x' \frac{\left(F^c(x-x')\right)^2}{\left(m_v+e\phi_r(x')\right)^2} . \tag{5.12}$$

Then the expression for λ is

$$\lambda(x) = \det\left\{\frac{\pi}{e} F_c(x)\,\delta^{\mu\nu} - i\,\chi^{\mu\nu}(x) + \frac{i\chi_1^{\mu}(x)\chi_2^{\nu}(x)}{\chi(x)}\right\} \tag{5.13}$$

Finally, there are questions concerning the definition of terms like $\cos\left(\frac{2\pi}{e}H\right)$ or $\left(1+\frac{e}{m_v}\phi_r\right)^{-2}$. They are defined by power series expansion, and in a continuum theory a consistent prescription for their renormalization is needed. Although we have not investigated this question, it appears to us that dimensional regularization is the simplest method to try. There must then be a delicate interplay between the cutoff needed to define various operator products and the finite size of the instanton, so that one can remove both cutoffs together and obtain a finite answer. We hope to investigate some of these questions in the future.

VI. CONCLUDING REMARKS

In this paper, we have investiaged the topological structure of a much studied model, the Abelian Higgs theory in two dimensions, using two seemingly different approaches. The direct approach focuses on the topological singularities of the field configuration in a continuum theory, and singular gauge transformations are the main tool used in studying them. The second approach is suited to a lattice version of the theory; its basic tool is the lattice duality transformation, and there is no direct appeal to topological ideas. Both approaches lead to the same dual field theory, and that is the main result of this paper. We have also studied the dual theory in various approximations, and we have pointed out its relation to the massive sine-Gordon model and the massive Schwinger model. As a by product, we were able to give a general argument for charge confinement, relying on the fact that charge is confined in the massive Schwinger model.

Our methods should also apply to more physical models in higher dimensions. In particular, in the future we hope to study instantons in non-Abelian gauge theories.

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