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Recursive Cramer-Rao Lower Bound for Random Parameters

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I thank my professors, without whose help, I would not have been here.

I thank my advisor for his patience.

To my parents.

ABSTRACT OF THE THESIS

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In many cases an estimator is needed to estimate a certain quantity from an observation. The estimators can take different forms and so a method of comparison or a method of assessment will prove to be useful. The Cramer Rao Lower Bound (CRLB) is one such method. The CRLB lower bounds the variance of an estimator; however, the CRLB assumes the parameter to be estimated to be deterministic. The Posterior Cramer Rao Lower Bound (PCRLB) bounds the variance of an estimate for stochastic parameters. In this thesis the PCRLB is thoroughly discussed and is also applied to the Autoregressive moving average (ARMA) model.

Contents

Chapter 1

Estimators and bounds

Many well established estimators exist, and it is part of the designer's job to choose or come up with an appropriate estimator. The designer must make sure that the estimator chosen complies justly with the application at hand, and more importantly, that the estimator is feasible and efficient. First, the designer must assess the feasibility of the estimator design. Feasibility aids the designer in checking if the initial design can be implemented. Second, assessing the efficiency of the estimator will enable the designer to know if the design can be enhanced, which in turn reduces the error of the estimator.

The classic Cramer Rao Lower Bound (CRLB) is well known and a large literature exists. The CRLB lower bounds the covariance of an esitmator. This allows for making practical goals and provides a point of reference when designing an efficient esitmator. The CRLB is the inverse of the Fisher Information Matrix (FIM). The FIM is a measure of how much information the observation variable carries about the unknown state variable. When the CRLB is to be calculated, the FIM makes use of a probability distibution function

(PDF) for the observation vector conditioned on the unknown state state variable (also called unknown parameters). It is worth noting that for the most classical CRLB, only the observation data (which will also be referred to as observation variable or observation vector) is considered to be stochastic.

In many cases, not only is the observation variable stochastic but so is the unknown state variable to be estimated. In such cases the CRLB needs to be modified to take the randomness of the unknown state variable into account. One such modified CRLB is known as the Posterior Cramer Rao Lower Bound (PCRLB). The PCRLB also uses the FIM. However, in the case of a stochastic state variable, the FIM uses a joint PDF of the observation variable and the unknown state variable. The PCRLB is a method (and an extension of the CRLB) of lower bounding the covariance of an estimator when the state variable to be estimated is stochastic.

In chapter 2 , the mathematical tools used in this Thesis are stated. This chapter will also briefly describe the CRLB and PCRLB. In chapter 3, the PCRLB is thouroghly discussed. Chapter 3 is divided into three main sections: Non-linear and non-singular case (this generalization includes the Linear and non-singular case), Linear and singular case, and Non-linear and singular case. The Non-linear and singular case will only be discussed briefly. In chapter 4, the PCRLB will be applied to an ARMA process. The ARMA process will be considered to have slowly varying parameters. Chapter 5 will contain some proofs referenced in this Thesis.

An acknowledgement to [1] must be mentioned. Chapter 3 is based upon [1]. Chapter 4 applies chapter 3 to an ARMA model.

Chapter 2

PCRLB properties and models

2.1 PCRLB and CRLB

The CRLB (and PCRLB) is the inverse of the FIM. The FIM is given by [2]

$$\Phi(p) = E[(\nabla_p logl)(\nabla_p logl)^T] \tag{2.1}$$

Here p is the parameter to be estimated of size $n_p \times 1$ and ∇_p is the gradient operator with respect to vector p. l represents the likelihood function of the parameter (or the PDF of the observation) to be estimated (in the case of the PCRLB, the likelihood function is the joint probability distribution function). If certain regularity conditions are met, the FIM can have the alternate form [3]

$$\Phi(p) = -E[\Delta_p^p logl] \tag{2.2}$$

where

$$\Delta_{p_i}^{p_j} = \nabla_{p_i} \nabla_{p_j}^T \tag{2.3}$$

In some instances finding the CRLB (or PCRLB) for only a subset of the parameters may be required. In such a case, partitioning of the unknown state vector by grouping the elements in question together is effective

$$p = [p_a^T | p_b^T]^T \tag{2.4}$$

where p_b is the subset of elements that need to be bounded by the CRLB (or PCRLB). According to this partitioning scheme the FIM can be partitioned as

$$\Phi(p) = \begin{bmatrix} \Phi_{aa} & \Phi_{ab} \\ \Phi_{ba} & \Phi_{bb} \end{bmatrix} \tag{2.5}$$

It is the bottom right $n_{p_b} \times n_{p_b}$ submatrix of $\Phi(p)^{-1}$ that contains the information about p_b. Since the CRLB (and PCRLB) is the inverse of the FIM, it is useful to find the submatrix block of the inverse. Using properties of block matrix inversion

$$\Phi^{-1}(p) = \begin{bmatrix} \Phi_{aa}^{-1} + \Phi_{aa}^{-1}\Phi_{ab}(\Phi_{bb} - \Phi_{ba}\Phi_{aa}^{-1}\Phi_{ab})^{-1} & -\Phi_{aa}^{-1}\Phi_{ab}(\Phi_{bb} - \Phi_{ba}\Phi_{aa}^{-1}\Phi_{ab})^{-1} \\ -(\Phi_{bb} - \Phi_{ba}\Phi_{aa}^{-1}\Phi_{ab})^{-1}\Phi_{ba}\Phi_{aa}^{-1} & (\Phi_{bb} - \Phi_{ba}\Phi_{aa}^{-1}\Phi_{ab})^{-1} \end{bmatrix} \tag{2.6}$$

Extracting the bottom right $n_{p_b} \times n_{p_b}$ submatrix

$$\Phi^{-1}(p_b) = (\Phi_{bb} - \Phi_{ba}\Phi_{aa}^{-1}\Phi_{ab})^{-1} \leq E[(\hat{p_b} - p_b)(\hat{p_b} - p_b)^T] \tag{2.7}$$

Here, $\hat{p_b}$ is the estimate of of p_b. It can easily be seen that

$$\Phi(p_b) = (\Phi_{bb} - \Phi_{ba}\Phi_{aa}^{-1}\Phi_{ab}) \tag{2.8}$$

2.2 Stationary ARMA model

An autoregressive moving average (ARMA) model has the following form [4]

$$
\begin{aligned}
X_{k+1} - \sum_{i=0}^{p} \phi_i X_{k-i} &= c + \epsilon_{k+1} - \sum_{i=1}^{q} \theta_i \epsilon_{k-i} \\
X_{k+1} - \phi^T \underline{X}_k &= c + \epsilon_{k+1} - \theta^T \underline{\epsilon}_k
\end{aligned} \tag{2.9}
$$

where $\phi = [\phi_1 ... \phi_p]$ and $\theta = [\theta_1 ... \theta_q]$, $\underline{\epsilon}_k = [\epsilon_k ... \epsilon_{k-q+1}]$, $\underline{X}_k = [X_k ... X_{k-p+1}]$, c is a constant, and ϵ_k is white noise with $\mathcal{N}(0, \sigma_\epsilon^2)$. This is an ARMA process with p autoregressive terms and q moving average terms, or ARMA(p,q). If the absolute value of the roots of the characteristic polynomial of the AR term are all strictly greater than one, then the ARMA process is stationary. In the case of stationarity

$$
\begin{aligned}
E[X_{k+1}] = E[X_{k-i}] = E[X] &= \frac{c}{1 - \phi_{k+1}^T \underline{1}} \\
cov(X_r, X_s) = \gamma(t) &= \sigma^2 \sum_{j=0}^{\infty} \psi_j \psi_{j+|t|}
\end{aligned} \tag{2.10}
$$

$\underline{1}$ is a vector of ones with dimension p x 1 and $E[X] = 0$ if $c = 0$. $\gamma(t)$ is the autocovariance function of the ARMA process. $t = |r - s|$ is known as the lag. ψ is given by

$$\psi_j - \sum_{0<k\leq p} \phi_k \psi_{j-k} = 0, \qquad j \geq max(p, q+1) \tag{2.11}$$

The initial conditions are found using the following

$$\psi_j - \sum_{0<k\leq j} \phi_k \psi_{j-k} = \theta_j, \qquad 0 \leq j < max(p, q+1) \tag{2.12}$$

where $\theta_0 = 1$, $\theta_j = 0$ for $j > q$, $\phi_j = 0$ for $j > p$.

2.3 Symplectic matrices

A symplectic matrix is a $2n \times 2n$ real matrix, Z, such that

$$Z^T J Z = J \tag{2.13}$$

where J is a $2n \times 2n$ nonsingular, skew symmetric matrix, usually chosen to be

$$J = \begin{bmatrix} 0 & -I_n \\ I_n & 0 \end{bmatrix} \tag{2.14}$$

Another property of symplectic matrices is that eigen values occur in reciprocal pairs [5].

Chapter 3

PCRLB

3.1 Bounding a Nonlinear and Nonsingular Model [1]

Let e_k be the observation random variable and let it also be a function of an unknown state parameter, p_k, to be estimated. e_k is also affected by noise, v_k. The parameter p_k is itself a random variable. In general, this model can be non-linear and time variant.

$$\begin{aligned} e_k &= h_k(p_k, v_k) \\ p_{k+1} &= g_k(p_k, u_k) \end{aligned} \tag{3.1}$$

Here, v_k and u_k are both independent and identically distributed (IID), white noise random variables. Let

$$f(P_k, \varepsilon_k) = f_k = f(p_0) \prod_{i=1}^{k} [f(p_i|p_{i-1}) f(e_i|p_i)] \tag{3.2}$$

where $f(p_0)$ is assumed to be known. Let

$$\begin{aligned} P_k &= [p_0^T ... p_k^T]^T \\ \epsilon_k &= [e_0^T ... e_k^T]^T \end{aligned} \tag{3.3}$$

and let f_{k+1} denote the joint probability distribution of P_{k+1} and ϵ_{k+1}. In other words, $f_{k+1} = f(P_{k+1}, \epsilon_{k+1})$. f_{k+1} can be written as

$$f_{k+1} = f_k \cdot f(p_{k+1}|p_k) \cdot f(e_{k+1}|p_{k+1}) \tag{3.4}$$

It can be shown that the recursive PCRLB has the form [1]

$$\begin{aligned} \Phi(p_{k+1}) &= \Phi_{33} - \Phi_{32}(\Phi(p_k) - E[\Delta_{p_k}^{p_k} logf(p_{k+1}|p_k)])^{-1}\Phi_{23} \\ &= \Phi_{33} - \Phi_{32}(\Phi(p_k) + \Phi_R)^{-1}\Phi_{23} \end{aligned} \tag{3.5}$$

where

$$\begin{aligned} \Phi_{23} &= -E[\Delta_{p_k}^{p_{k+1}} logf(p_{k+1}|p_k)] = \Phi_{32}^T \\ \Phi_{33} &= -E[\Delta_{p_{k+1}}^{p_{k+1}} (logf(p_{k+1}|p_k) + logf(e_{k+1}|p_{k+1}))] \\ \Phi_R &= -E[\Delta_{p_k}^{p_k} logf(p_{k+1}|p_k)] \end{aligned} \tag{3.6}$$

The recursion can be initiated with

$$\Phi(p_0) = E[-\Delta_{p_0}^{p_0} logf(p_0)] \tag{3.7}$$

A proof for this case is shown in section 6.1. It can also be shown that if the system is time invariant then (3.5) takes the form of the DARE [1]

$$\begin{aligned} \Phi(p_\infty) = \Phi_\infty &= \Phi_{33} - \Phi_{32}(\Phi_\infty + \Phi_R)^{-1}\Phi_{23} \\ &= \Phi_{33} + \Phi_{32}\Phi_R^{-1}\Phi_\infty\Phi_R^{-1}\Phi_{23} - \Phi_{32}\Phi_R^{-1}\Phi_\infty(\Phi_\infty + \Phi_R)^{-1}\Phi_\infty\Phi_R^{-1}\Phi_{23} - \Phi_{32}\Phi_R^{-1}\Phi_{23} \end{aligned} \tag{3.8}$$

A proof of (3.8) is provided in section 6.3. To solve the DARE, the use of symplectic matrices and schur form will prove to be useful. A method for solving such an equation makes use of symplectic matrices and Schur Decomposition. If A is invertible then the following symplectic matrix can be formed

$$Z = \begin{bmatrix} (\Phi_R^{-1}\Phi_{23})^{-1} & (\Phi_R^{-1}\Phi_{23})^{-1}R^{-1} \\ (\Phi_{33} - \Phi_{32}\Phi_R^{-1}\Phi_{23})(\Phi_R^{-1}\Phi_{23})^{-1} & \Phi_{32}\Phi_R^{-1} + (\Phi_{33} - \Phi_{32}\Phi_R^{-1}\Phi_{23})(\Phi_R^{-1}\Phi_{23})^{-1}R^{-1} \end{bmatrix} \tag{3.9}$$

Applying Schur decomposition and ordering the spectrum of the Schur form of Z such that the eigenvalues inside the unit circle are grouped together

$$Z = \begin{bmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{bmatrix} \begin{bmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{bmatrix} \begin{bmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{bmatrix}^{-1} \tag{3.10}$$

Where λ is a diagonal matrix containing the eigenvalues of Z greater than one. Σ will have the solution [5]

$$\Phi_\infty = T_{21}T_{11}^{-1} \tag{3.11}$$

3.2 Bounding a Linear and Singular Model [1]

The computation of the FIM fails when the PDF being used is singular. In the case of a normal distribution, this occurs when the covariance matrix is not invertible. Assume the vector of parameters can be partitioned as

$$p_k = \begin{bmatrix} p_k^{(1)} \\ p_k^{(2)} \end{bmatrix} \tag{3.12}$$

where $p_k^{(j)}$ has a size of $n_{p^{(j)}} \times 1$, $j = 1, 2$ and $n_{p^{(1)}} + n_{p^{(2)}} = n_p$. The vector, p_k, is partitioned such that

$$\begin{aligned} p_{k+1}^{(1)} &= g_k^{(1)}(p_k, u_k) \\ p_{k+1}^{(2)} &= g_k^{(2)}(p_k, p_{k+1}^{(1)}) \\ e_k &= h_k(p_k, v_k) \end{aligned} \tag{3.13}$$

is satisfied. With $g_k^{(2)}(p_k, p_{k+1}^{(1)})$ assumed to be linear, such that

$$p_{k+1}^{(2)} = G_k^{(1)} p_k^{(1)} + G_k^{(2)} p_k^{(2)} + G_k^{(3)} p_{k+1}^{(1)} \tag{3.14}$$

Where it is also assumed that $G_k^{(2)}$ is invertible for all k. Under such conditions, the recursive PRCLB can be shown to be

$$\Phi(p_{k+1}) = \begin{bmatrix} \Phi_{d22} & \Phi_{d23} \\ \Phi_{d32} & \Phi_{d33} \end{bmatrix} - \begin{bmatrix} \Phi_{d21} \\ \Phi_{d31} \end{bmatrix} \Phi_{d11}^{-1} \begin{bmatrix} \Phi_{d12} & \Phi_{d13} \end{bmatrix} \tag{3.15}$$

where

$$\Phi(p_k^{(1)}, p_{k+1}^{(2)}, p_{k+1}^{(1)}) = M_k^{-T}\Phi(p_k^{(1)}, p_k^{(2)}, p_{k+1}^{(1)})M_k^{-1} \triangleq \begin{bmatrix} \Phi_{d11} & \Phi_{d12} & \Phi_{d13} \\ \Phi_{d21} & \Phi_{d22} & \Phi_{d23} \\ \Phi_{d31} & \Phi_{d32} & \Phi_{d33} \end{bmatrix}$$

$$M_k = \begin{bmatrix} I & 0 & 0 \\ G_k^{(1)} & G_k^{(2)} & G_k^{(3)} \\ 0 & 0 & I \end{bmatrix}$$

$$\Phi(p_k^{(1)}, p_k^{(2)}, p_{k+1}^{(1)}) =$$

$$\begin{bmatrix} \Phi_{b11} - E[\Delta_{p_k^{(1)}}^{p_k^{(1)}}(B_1 + B_2)] & \Phi_{b12} - E[\Delta_{p_k^{(1)}}^{p_k^{(2)}}(B_1 + B_2)] & -E[\Delta_{p_k^{(1)}}^{p_{k+1}^{(1)}}(B_1 + B_2)] \\ \Phi_{b21} - E[\Delta_{p_k^{(2)}}^{p_k^{(1)}}(B_1 + B_2)] & \Phi_{b22} - E[\Delta_{p_k^{(2)}}^{p_k^{(2)}}(B_1 + B_2)] & -E[\Delta_{p_k^{(2)}}^{p_{k+1}^{(1)}}(B_1 + B_2)] \\ -E[\Delta_{p_{k+1}^{(1)}}^{p_k^{(1)}}(B_1 + B_2)] & -E[\Delta_{p_{k+1}^{(1)}}^{p_k^{(2)}}(B_1 + B_2)] & -E[\Delta_{p_{k+1}^{(1)}}^{p_{k+1}^{(1)}}(B_1 + B_2)] \end{bmatrix}$$

$$\Phi(p_k) = \begin{bmatrix} \Phi_{b11} & \Phi_{b12} \\ \Phi_{b21} & \Phi_{b22} \end{bmatrix}$$

$$B_1 \triangleq logf(p_{k+1}^{(1)}|p_k)$$

$$B_2 \triangleq logf(e_{k+1}|p_k, p_{k+1}^{(1)}) \tag{3.16}$$

A proof for this case is shown in section 6.4.

3.3 Bounding a Nonlinear and Singular Model [1]

In the case where the system model has nonlinearities and a singular distribution, regularization will prove to be useful. Assume $g_k^{(2)}(p_k, p_{k+1}^{(1)})$ of equation (3.13) to be nonlinear and affected by white Gaussian noise. Using this modification, (3.13) becomes

$$\begin{aligned} p_{k+1}^{(1)} &= g_k^{(1)}(p_k, u_k) \\ p_{k+1}^{(2)} &= g_k^{(2)}(p_k, p_{k+1}^{(1)}) + w_n^{(2)} \\ e_k &= h_k(p_k, v_k) \end{aligned} \tag{3.17}$$

where $w_n^{(2)}$ is a Gaussian random variable with zero mean and a covariance matrix δI, such that $w_n^{(2)}$ is independent of v_n and u_k, and such that δ is close to 0. (3.17) will be shown to have a similar PCRLB to that of (3.5). In this case the PDF can be shown to be

$$f_{k+1} = f_k \cdot f(p_{k+1}^{(2)} | p_k, p_{k+1}^{(1)}) \cdot f(p_{k+1}^{(1)} | p_k) \cdot f(e_{k+1} | p_{k+1}) \tag{3.18}$$

The new PDF results in a modified version of the equations in (3.6). The new equations take the following form

$$\begin{aligned} \overline{\Phi}_{23} &= \widetilde{\Phi}_{23} + K_{23} = \overline{\Phi}_{32}^T \\ \overline{\Phi}_{33} &= \widetilde{\Phi}_{33} + K_{33} \\ \overline{\Phi}_R &= \widetilde{\Phi}_R + K_R \end{aligned} \tag{3.19}$$

where

$$\begin{aligned} K_{23} &= \frac{1}{\delta} E[(\nabla_{p_k} g^T)(\nabla_{p_k} g^T)^T] \\ K_R &= \frac{1}{\delta} \begin{bmatrix} E[(\nabla_{p_k} g^T)(\nabla_{p_{k+1}^{(1)}} g^T)^T] & -E[(\nabla_{p_k} g^T)] \end{bmatrix} \\ K_{33} &= \frac{1}{\delta} \begin{bmatrix} E[(\nabla_{p_{k+1}^{(1)}} g^T)(\nabla_{p_{k+1}^{(1)}} g^T)^T] & -E[(\nabla_{p_{k+1}^{(1)}} g^T)] \\ E[(-\nabla_{p_{k+1}^{(1)}} g^T)^T] & I \end{bmatrix} \end{aligned} \tag{3.20}$$

and $\widetilde{\Phi}_{23}$, $\widetilde{\Phi}_{33}$, $\widetilde{\Phi}_R$ are solved the same way (3.6) is solved except using the PDF $f(p_{k+1}^{(1)} | p_k)$. The recursive PCRLB is then

$$\Phi(p_{k+1}) = \overline{\Phi}_{33} - \overline{\Phi}_{32}(\Phi(p_k) + \overline{\Phi}_R)^{-1} \overline{\Phi}_{23} \tag{3.21}$$

where once again the recursion can be initiated with (3.7). A proof for this case is shown in section 6.5.

Chapter 4

ARMA PCRLB

4.1 ARMA process

Given a time series, the Autoregressive Moving Average (ARMA) model describes the future value (future observable) of the series as a linear combination of p previous values (previously observed variables) of the series and q past error terms (previous unknown error terms). The model is usually assumed to be a stationary process, as is the case in this thesis. ARMA models are used in many fields such as econometrics to forcast certain quantities (such as stock prices) [6], machine learning to obtain training sets for artificial neural networks [7], or even in structural dynamics applications (such as estimating earthquake ground motion [8]). In some cases the parameters of the ARMA model are unknown and need to be estimated in order to fully describe a process. Furthermore, in some cases the unknown parameters are themselves stochastic. Estimators for such parameters may be required in some applications and thus lower bounding such an estimator is necessary. The bound for such an estimator is discussed in this section. Specifically, the bound for an ARMA process with slowly varying parameters.

4.2 ARMA process with slowly varying parameters

Let z_{k+1} be the observation random variable that obeys the recursion

$$
\begin{aligned}
z_{k+1} &= \phi_{k+1}^T \underline{z}_k - \theta_{k+1}^T \underline{\epsilon}_k + \epsilon_{k+1} \\
\phi_{k+1} &= \phi_k + v_k \\
\theta_{k+1} &= \theta_k + w_k
\end{aligned}
\tag{4.1}
$$

where $\underline{z}_k = [z_k, \ldots, z_{k-q+1}]^T$ and $\underline{\epsilon}_k = [\epsilon_k, \ldots, \epsilon_{k-q+1}]^T$ and where $\epsilon_k \overset{iid}{\sim} \mathcal{N}(0, \sigma^2)$, $v_k \overset{iid}{\sim} \mathcal{N}(0, Q_1)$, $w_k \overset{iid}{\sim} \mathcal{N}(0, Q_2)$ such that ϵ_k, v_k, w_k are all mutually independent. This is the ARMA(p,q) model from section 2.2, modified to have stochastic parameters. Here ϕ_{k+1} and θ_{k+1} are state parameter vectors at time instance $k+1$. Furthermore define the state vector as

$$
x_{k+1} = \begin{bmatrix} \phi_{k+1} \\ \theta_{k+1} \end{bmatrix} \tag{4.2}
$$

For such a state partition, the joint PDF for X_{k+1} and Z_{k+1} such that $X_k = [x_0^T, \ldots, x_k^T]^T$ and $Z_k = [z_0, \ldots, z_k]^T$ can be represented as

$$
f(X_{k+1}, Z_{k+1}) = f(z_{k+1}|\phi_{k+1}, \theta_{k+1}, \underline{z}_k) \cdot f(\phi_{k+1}|\theta_{k+1}, \phi_k, \theta_k) \cdot f(\theta_{k+1}|\phi_k, \theta_k) \cdot f(X_k, Z_k) \tag{4.3}
$$

If $\underline{\epsilon}_k$ is assumed to be equal to its mean 0, of size q x 1, then ϵ_{k+i} for $i > 0$ can be calculated iteratively [9]. This assumption allows for (4.3) to be written as

$$
f(X_{k+1}, Z_{k+1}) = f(z_{k+1}|\phi_{k+1}, \theta_{k+1}, \underline{z}_k, \underline{\epsilon}_k) \cdot f(\phi_{k+1}|\theta_{k+1}, \phi_k, \theta_k) \cdot f(\theta_{k+1}|\phi_k, \theta_k) \cdot f(X_k, Z_k) \tag{4.4}
$$

To find the PCRLB, equations (3.6) must be found and used in (3.5) to solve for $\Phi(x_{k+1})$. The conditional means and conditional covariances required for equations (3.6) are found to be

$$\begin{aligned} E[\phi_{k+1}|\theta_{k+1},\phi_k,\theta_k] &= \phi_k, \quad cov(\phi_{k+1}|\theta_{k+1},\phi_k,\theta_k) = Q_1 \\ E[\theta_{k+1}|\phi_k,\theta_k] &= \theta_k, \quad cov(\theta_{k+1}|\phi_k,\theta_k) = Q_2 \\ E[z_{k+1}|\phi_{k+1},\theta_{k+1},\underline{z}_k,\underline{\epsilon}_k] &= \phi_{k+1}^T\underline{z}_k - \theta_{k+1}^T\underline{\epsilon}_k, \quad cov(z_{k+1}|\phi_{k+1},\theta_{k+1},\underline{z}_k,\underline{\epsilon}_k) = \sigma^2 \end{aligned} \tag{4.5}$$

Using the corresponding moments from (4.5), Φ_R is found to be

$$\begin{aligned} \Phi_R &= E[-\Delta_{x_k}^{x_k} f(x_{k+1}|x_k)] \\ &= E[-\Delta_{x_k}^{x_k} logf(\phi_{k+1}|\theta_{k+1},\phi_k,\theta_k)] + E[-\Delta_{x_k}^{x_k} logf(\theta_{k+1}|\phi_k,\theta_k)] \\ &= A_R + B_R \end{aligned} \tag{4.6}$$

such that

$$\begin{aligned} A_R &= \frac{1}{2}E\begin{bmatrix} \Delta_{\phi_k}^{\phi_k}(\phi_{k+1}-\phi_k)^TQ_1^{-1}(\phi_{k+1}-\phi_k) & \Delta_{\phi_k}^{\theta_k}(\phi_{k+1}-\phi_k)^TQ_1^{-1}(\phi_{k+1}-\phi_k) \\ \Delta_{\theta_k}^{\phi_k}(\phi_{k+1}-\phi_k)^TQ_1^{-1}(\phi_{k+1}-\phi_k) & \Delta_{\theta_k}^{\theta_k}(\phi_{k+1}-\phi_k)^TQ_1^{-1}(\phi_{k+1}-\phi_k) \end{bmatrix} \\ B_R &= \frac{1}{2}E\begin{bmatrix} \Delta_{\phi_k}^{\phi_k}(\theta_{k+1}-\theta_k)^TQ_2^{-1}(\theta_{k+1}-\theta_k) & \Delta_{\phi_k}^{\theta_k}(\theta_{k+1}-\theta_k)^TQ_2^{-1}(\theta_{k+1}-\theta_k) \\ \Delta_{\theta_k}^{\phi_k}(\theta_{k+1}-\theta_k)^TQ_2^{-1}(\theta_{k+1}-\theta_k) & \Delta_{\theta_k}^{\theta_k}(\theta_{k+1}-\theta_k)^TQ_2^{-1}(\theta_{k+1}-\theta_k) \end{bmatrix} \end{aligned} \tag{4.7}$$

solving for A_R and B_R and substituting into (4.6)

$$\Phi_R = \begin{bmatrix} Q_1^{-1} & 0 \\ 0 & Q_2^{-1} \end{bmatrix} \tag{4.8}$$

Using the corresponding moments from (4.5), Φ_{23} is found to be

$$\begin{aligned} \Phi_{23} &= E[-\Delta_{x_k}^{x_{k+1}} logf(x_{k+1}|x_k)] \\ &= E[-\Delta_{x_k}^{x_{k+1}} logf(\phi_{k+1}|\theta_{k+1},\phi_k,\theta_k)] + E[-\Delta_{x_k}^{x_{k+1}} logf(\theta_{k+1}|\phi_k,\theta_k)] \\ &= A_{23} + B_{23} \end{aligned} \tag{4.9}$$

such that

$$A_{23} = \frac{1}{2}E\begin{bmatrix}\Delta_{\phi_k}^{\phi_{k+1}}(\phi_{k+1}-\phi_k)^T Q_1^{-1}(\phi_{k+1}-\phi_k) & \Delta_{\phi_k}^{\theta_{k+1}}(\phi_{k+1}-\phi_k)^T Q_1^{-1}(\phi_{k+1}-\phi_k)\\ \Delta_{\theta_k}^{\phi_{k+1}}(\phi_{k+1}-\phi_k)^T Q_1^{-1}(\phi_{k+1}-\phi_k) & \Delta_{\theta_k}^{\theta_{k+1}}(\phi_{k+1}-\phi_k)^T Q_1^{-1}(\phi_{k+1}-\phi_k)\end{bmatrix}$$

$$B_{23} = \frac{1}{2}E\begin{bmatrix}\Delta_{\phi_k}^{\phi_{k+1}}(\theta_{k+1}-\theta_k)^T Q_2^{-1}(\theta_{k+1}-\theta_k) & \Delta_{\phi_k}^{\theta_{k+1}}(\theta_{k+1}-\theta_k)^T Q_2^{-1}(\theta_{k+1}-\theta_k)\\ \Delta_{\theta_k}^{\phi_{k+1}}(\theta_{k+1}-\theta_k)^T Q_2^{-1}(\theta_{k+1}-\theta_k) & \Delta_{\theta_k}^{\theta_{k+1}}(\theta_{k+1}-\theta_k)^T Q_2^{-1}(\theta_{k+1}-\theta_k)\end{bmatrix} \tag{4.10}$$

solving for A_{23} and B_{23} and substituting into (4.9)

$$\Phi_{23} = -\begin{bmatrix}Q_1^{-1} & 0\\ 0 & Q_2^{-1}\end{bmatrix} = \Phi_{32}^T \tag{4.11}$$

Using the corresponding moments from (4.5), Φ_{33} is found to be

$$\begin{aligned}\Phi_{33} &= E[-\Delta_{x_{k+1}}^{x_{k+1}} log f(x_{k+1}|x_k)] + E[-\Delta_{x_{k+1}}^{x_{k+1}} log f(z_{k+1}|x_{k+1},\underline{z}_k)]\\ &= E[-\Delta_{x_{k+1}}^{x_{k+1}} log f(\phi_{k+1}|\theta_{k+1},\phi_k,\theta_k)] + E[-\Delta_{x_{k+1}}^{x_{k+1}} log f(\theta_{k+1}|\phi_k,\theta_k)]\\ &\quad + E[-\Delta_{x_{k+1}}^{x_{k+1}} log f(z_{k+1}|\phi_{k+1},\theta_{k+1},\underline{z}_k,\underline{\epsilon}_k)]\\ &= A_{33}+B_{33}+C_{33}\end{aligned} \tag{4.12}$$

such that

$$A_{33} = \frac{1}{2}E\begin{bmatrix}\Delta_{\phi_{k+1}}^{\phi_{k+1}}(\phi_{k+1}-\phi_k)^T Q_1^{-1}(\phi_{k+1}-\phi_k) & \Delta_{\phi_{k+1}}^{\theta_{k+1}}(\phi_{k+1}-\phi_k)^T Q_1^{-1}(\phi_{k+1}-\phi_k)\\ \Delta_{\theta_{k+1}}^{\phi_{k+1}}(\phi_{k+1}-\phi_k)^T Q_1^{-1}(\phi_{k+1}-\phi_k) & \Delta_{\theta_{k+1}}^{\theta_{k+1}}(\phi_{k+1}-\phi_k)^T Q_1^{-1}(\phi_{k+1}-\phi_k)\end{bmatrix}$$

$$B_{33} = \frac{1}{2}E\begin{bmatrix}\Delta_{\phi_{k+1}}^{\phi_{k+1}}(\theta_{k+1}-\theta_k)^T Q_2^{-1}(\theta_{k+1}-\theta_k) & \Delta_{\phi_{k+1}}^{\theta_{k+1}}(\theta_{k+1}-\theta_k)^T Q_2^{-1}(\theta_{k+1}-\theta_k)\\ \Delta_{\theta_{k+1}}^{\phi_{k+1}}(\theta_{k+1}-\theta_k)^T Q_2^{-1}(\theta_{k+1}-\theta_k) & \Delta_{\theta_{k+1}}^{\theta_{k+1}}(\theta_{k+1}-\theta_k)^T Q_2^{-1}(\theta_{k+1}-\theta_k)\end{bmatrix}$$

$$C_{33} = \frac{1}{2\sigma^2}E\begin{bmatrix}\Delta_{\phi_{k+1}}^{\phi_{k+1}}(z_{k+1}-\mu)^T(z_{k+1}-\mu) & \Delta_{\phi_{k+1}}^{\theta_{k+1}}(z_{k+1}-\mu)^T(z_{k+1}-\mu)\\ \Delta_{\theta_{k+1}}^{\phi_{k+1}}(z_{k+1}-\mu)^T(z_{k+1}-\mu) & \Delta_{\theta_{k+1}}^{\theta_{k+1}}(z_{k+1}-\mu)^T(z_{k+1}-\mu)\end{bmatrix} \tag{4.13}$$

Where $\mu = E[z_{k+1}|\phi_{k+1}, \theta_{k+1}, \underline{z}_k, \underline{\epsilon}_k] = \phi_{k+1}^T \underline{z}_k - \theta_{k+1}^T \underline{\epsilon}_k$.

With methods similar to the ones used to solve Φ_R and Φ_{23}, A_{33} and B_{33} can be found to be

$$A_{33} + B_{33} = \begin{bmatrix} Q_1^{-1} & 0 \\ 0 & Q_2^{-1} \end{bmatrix} \tag{4.14}$$

C_{33} can be found to be

$$C_{33} = \frac{1}{\sigma^2} E \begin{bmatrix} \underline{z}_k \underline{z}_k^T & \underline{z}_k \underline{\epsilon}_k^T \\ \underline{\epsilon}_k \underline{z}_k^T & \underline{\epsilon}_k \underline{\epsilon}_k^T \end{bmatrix} \tag{4.15}$$

Here, $E[\underline{z}_k \underline{z}_k^T]$, can be found using the methods described in section 2.2. $E[\underline{z}_k \underline{\epsilon}_k^T]$ will take the following form

$$E[\underline{z}_k \underline{\epsilon}_k^T] = \begin{bmatrix} z_k \epsilon_k & z_k \epsilon_{k-1} & \cdots & z_k \epsilon_{k-q+1} \\ \vdots & \vdots & \vdots & \vdots \\ z_{k-p+1} \epsilon_k & z_{k-p+1} \epsilon_{k-1} & \cdots & z_{k-p+1} \epsilon_{k-q+1} \end{bmatrix} \tag{4.16}$$

The matrix elements can be found to be

$$E[z_{k-i} \epsilon_{k-j}^T] = \begin{cases} 0, & i > j \\ \sigma^2, & i = j \\ f(i,j), & i < j \end{cases} \tag{4.17}$$

where $0 < i \leq p$, $0 < j \leq q$, and $f(i,j)$ can be found using the recursive equation

$$f(i,j) = \sum_{l=1}^{j-i} E[\tilde{\phi}_{k-i}^{(l)}] f(i+l, j) + E[\tilde{\theta}_{k-i}^{(j-i)}] E[\epsilon_k \epsilon_k] \tag{4.18}$$

where $\tilde{\phi}_k^{(l)}$ and $\tilde{\theta}_k^{(l)}$ represent the l'th elements of vectors $\tilde{\phi}_k$ and $\tilde{\theta}_k$ respectively. $\tilde{\phi}_k^{(l)}$ and

$\tilde{\theta}_k^{(l)}$ are defined to be

$$\tilde{\phi}_k^{(l)} = \begin{cases} \phi_k^{(l)}, & l \leq p \\ 0, & l > p \end{cases} \tag{4.19}$$

and

$$\tilde{\theta}_k^{(l)} = \begin{cases} \theta_k^{(l)}, & l \leq q \\ 0, & l > q \end{cases} \tag{4.20}$$

$f(j,j)$ is the base case of the recursive equation and is defined to be equal to $E[\epsilon_k \epsilon_k]$. If ϕ_k and θ_k are assumed to be slowly varying around their means and if $\epsilon_k \overset{iid}{\sim} \mathcal{N}(0, \sigma^2)$ then $f(i,j)$ can be written as

$$f(i,j) = \sum_{l=1}^{j-i} E[\tilde{\phi}^{(l)}] f(i+l, j) + E[\tilde{\theta}^{(j-i)}] E[\epsilon_k \epsilon_k] \tag{4.21}$$

A proof is shown in section 6.1

4.3 ARMA(1,1) process example

To further analyze and understand the ARMA(p,q) PCRLB, let $p = q = 1$ so that

$$\begin{aligned} z_{k+1} &= \phi_{k+1} z_k - \theta_{k+1} \epsilon_k + \epsilon_{k+1} \\ \phi_{k+1} &= \phi_k + v_k \\ \theta_{k+1} &= \theta_k + w_k \end{aligned} \tag{4.22}$$

where $\epsilon_k \overset{iid}{\sim} \mathcal{N}(0, \sigma^2)$, $v_k \overset{iid}{\sim} \mathcal{N}(0, Q_1)$, $w_k \overset{iid}{\sim} \mathcal{N}(0, Q_2)$ such that ϵ_k, v_k, w_k are all mutually independent. However, as suggested in the previous section, let the parameters to be estimated be slowly varying upon their mean values, $\overline{\phi}$ and $\overline{\theta}$. To calculate the FIM, and hence the PCRLB, (3.6) must be found for this specific case. The general form for

(3.6) for an ARMA(p,q) process has been found in the previous section. Now, (3.6) will be evaluated for an ARMA(1,1) process

$$\Phi_R = \begin{bmatrix} Q_1^{-1} & 0 \\ 0 & Q_2^{-1} \end{bmatrix} \tag{4.23}$$
$$\Phi_{23} = -\Phi_R = \Phi_{32}^T$$

From (4.12), Φ_{33} is found to be

$$\Phi_{33} = \Phi_R + \frac{1}{\sigma^2} E \begin{bmatrix} \underline{z}_k \underline{z}_k^T & \underline{z}_k \underline{\epsilon}_k^T \\ \underline{\epsilon}_k \underline{z}_k^T & \underline{\epsilon}_k \underline{\epsilon}_k^T \end{bmatrix} \tag{4.24}$$

To find the submatrices of C_{33}, $E[\underline{z}_k \underline{\epsilon}_k^T]$ and $E[\underline{z}_k \underline{z}_k^T]$ must first be found. Since $p = q = 1$, $E[\underline{z}_k \underline{\epsilon}_k^T]$ and $E[\underline{z}_k \underline{z}_k^T]$ are reduced to $E[z_k^2]$ and $E[z_k \epsilon_k]$ respectively. Using sections 2.2 and 4.2

$$E[\underline{z}_k \underline{z}_k^T] = E[z_k^2] = \sigma^2 \frac{1 + \theta_{k+1}^2 - 2\phi_{k+1}\theta_{k+1}}{1 - \phi_{k+1}^2}$$
$$E[\underline{z}_k \underline{\epsilon}_k^T] = E[z_k \epsilon_k] = \sigma^2 \tag{4.25}$$

Finally, Φ_{33} is found to be

$$\Phi_{33} = \Phi_R + \begin{bmatrix} \frac{1+\overline{\theta}_{k+1}^2 - 2\overline{\phi}_{k+1}\overline{\theta}_{k+1}}{1-\overline{\phi}_{k+1}^2} & 1 \\ 1 & 1 \end{bmatrix} \tag{4.26}$$

Now that Φ_R, Φ_{23} and Φ_{33} have been found, (3.5) can be used to solve for the PCRLB. In the case where the system is time invariant, the DARE form of (3.5) can be used to calculate Φ_∞. Using either the method described in 3.1 or existing techniques such as the MATLAB function dare(), the following plot can be achieved The x-axis sweeps the AR parameter, ϕ, and the y-axis sweeps the MA parameter, θ. Both, ϕ and θ are taken to be between -1 to 1 to ensure stationarity and invertibility respectively. Invertibility allows the estimator to be achieved using simpler techniques.

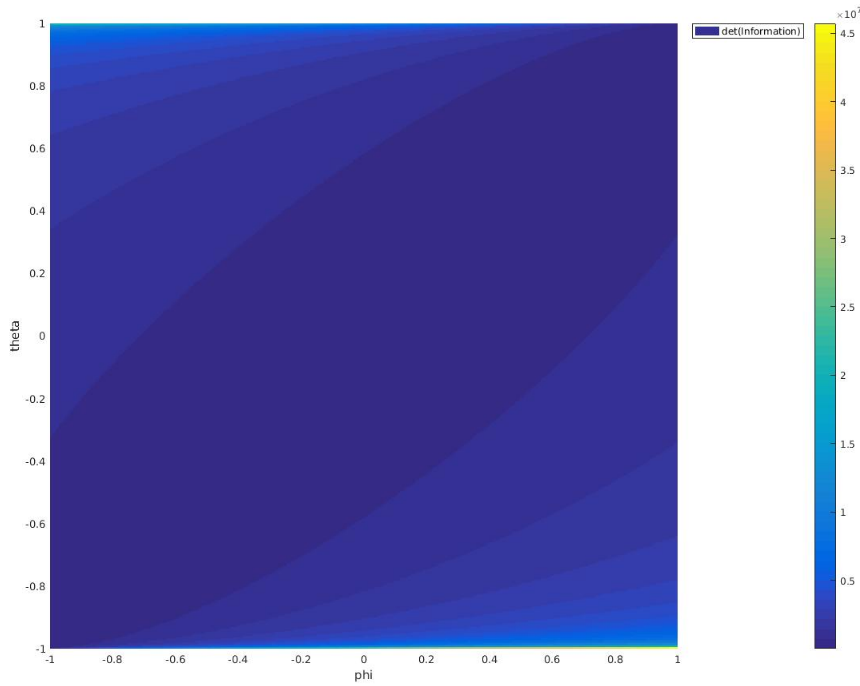

Figure 4.1: A plot of the solution to (3.8) for an ARMA(1,1) process using MATLAB. Q_1 and Q_2 are set to 10^{-6} and 10^{-6}. The x and y axis represent the mean of the MA and AR parameters respectively.

Chapter 5

Conclusion

The recursive Cramer Rao Lower Bound for stochastic parameters (better known as the Posterior Cramer Rao Lower Bound or the PCRLB) was thoroughly discussed. Proofs were provided for three cases: non-linear and non-singular, linear and singular, and non-linear and singular. The PCRLB was then applied to an ARMA(p,q) process with slowly varying random parameters. A small analysis was performed on an ARMA(1,1) process with slowly varying random parameters. The results for the ARMA(1,1) are shown in the figure of section 4.3.

Chapter 6

Proofs

6.1 Expectation of $\underline{X}_k\underline{\epsilon}_k^T$

This proof uses the following model

$$
\begin{aligned}
X_{k+1} &= \tilde{\phi}_{k+1}^T \underline{X}_k - \tilde{\theta}_{k+1}^T \underline{\epsilon}_k + \epsilon_{k+1} \\
\phi_{k+1} &= \phi_k + v_{k+1} \\
\theta_{k+1} &= \theta_k + w_{k+1}
\end{aligned}
\tag{6.1}
$$

Let $\tilde{\phi}$ and $\tilde{\theta}$ be defined as

$$
\begin{aligned}
\tilde{\phi}_k^{(l)} &= \begin{cases} \phi_k^{(l)}, & l \le p \\ 0, & l > p \end{cases} \\
\tilde{\theta}_k^{(l)} &= \begin{cases} \theta_k^{(l)}, & l \le q \\ 0, & l > q \end{cases}
\end{aligned}
\tag{6.2}
$$

and $\epsilon_k \overset{iid}{\sim} \mathcal{N}(0, \sigma^2)$, $v_k \overset{iid}{\sim} \mathcal{N}(0, Q_1)$, $w_k \overset{iid}{\sim} \mathcal{N}(0, Q_2)$. ϵ, v and w are assumed to be independent of each other. The purpose of this proof is to show how the elements of the matrix $E[\underline{X}_k\underline{\epsilon}_k^T]$ are found. This matrix has the following form

$$E[\underline{X}_k\underline{\epsilon}_k^T] = \begin{bmatrix} x_k\epsilon_k & x_k\epsilon_{k-1} & \dots & x_k\epsilon_{k-q+1} \\ \vdots & \vdots & \vdots & \vdots \\ x_{k-p+1}\epsilon_k & x_{k-p+1}\epsilon_{k-1} & \dots & x_{k-p+1}\epsilon_{k-q+1} \end{bmatrix} \tag{6.3}$$

In the case where $j < i$, ϵ_{k-j} is a newer instance than all instances of X in $\underline{X}_{k-i+1}$, instances of ϵ in $\underline{\epsilon}_{k-i+1}$ and ϵ_{k-i}, then ϵ_{k-j} is also independent of them, thus

$$\begin{aligned} E[X_{k-i}\epsilon_{k-j}] &= E[\tilde{\phi}_{k-i}^T\underline{X}_{k-i-1}\epsilon_{k-j}] + E[\tilde{\theta}_{k-i}^T\underline{\epsilon}_{k-i-1}\epsilon_{k-j}] + E[\epsilon_{k-i}\epsilon_{k-j}] \\ &= E[\tilde{\phi}_{k-i}^T\underline{X}_{k-i-1}]E[\epsilon_{k-j}] + E[\tilde{\theta}_{k-i}^T\underline{\epsilon}_{k-i-1}]E[\epsilon_j] + E[\epsilon_{k-i}]E[\epsilon_{k-j}] \\ &= 0 \end{aligned} \tag{6.4}$$

In the case where $i = j$, ϵ_{k-j} is a newer instance than all instances of X in $\underline{X}_{k-i+1}$ and instances of ϵ in $\underline{\epsilon}_{k-i+1}$, then ϵ_{k-j} is also independent of them, thus

$$\begin{aligned} E[X_{k-i}\epsilon_{k-j}] &= E[\tilde{\phi}_{k-i}^T\underline{X}_{k-i-1}\epsilon_{k-j}] + E[\tilde{\theta}_{k-i}^T\underline{\epsilon}_{k-i-1}\epsilon_{k-j}] + E[\epsilon_{k-i}\epsilon_{k-j}] \\ &= E[\tilde{\phi}_{k-i}^T\underline{X}_{k-i-1}]E[\epsilon_{k-j}] + E[\tilde{\theta}_{k-i}^T\underline{\epsilon}_{k-i-1}]E[\epsilon_{k-j}] + E[\epsilon_{k-i}\epsilon_{k-j}] \\ &= E[\epsilon_{k-i}\epsilon_{k-j}] \\ &= \sigma^2 \end{aligned} \tag{6.5}$$

In the case where $j > i$, ϵ_{k-j} is only gaurenteed to be independent of ϵ_{k-i}. Depending on the lag, $j - i$, ϵ_{k-j} may be independent of some, all or none of the instances of X in $\underline{X}_{k-i+1}$

and instances of ϵ in $\underline{\epsilon}_{k-i+1}$, thus

$$\begin{aligned}
E[X_{k-i}\epsilon_{k-j}] = f(i,j) &= E[\tilde{\phi}_{k-i}^T \underline{X}_{k-i-1}\epsilon_{k-j}] + E[\tilde{\theta}_{k-i}^T \underline{\epsilon}_{k-i-1}\epsilon_{k-j}] + E[\epsilon_{k-i}\epsilon_{k-j}] \\
&= E[\tilde{\phi}_{k-i}^T]E[\underline{X}_{k-i-1}\epsilon_{k-j}] + E[\tilde{\theta}_{k-i}^T]E[\underline{\epsilon}_{k-i-1}\epsilon_{k-j}] + E[\epsilon_{k-i}]E[\epsilon_{k-j}] \\
&= E[\phi_{k-i}^{(1)}]E[X_{k-i-1}\epsilon_{k-j}] + E[\phi_{k-i}^{(2)}]E[X_{k-i-2}\epsilon_{k-j}] + ... + E[\phi_{k-i}^{(j-i)}]E[X_{k-j}\epsilon_{k-j}] \\
&\quad + E[\theta_{k-i}^{(j-i)}]E[\epsilon_{k-j}\epsilon_{k-j}] \\
&= \sum_{l=1}^{j-i} E[\tilde{\phi}_{k-i}^{(l)}]f(i+l,j) + E[\tilde{\theta}_{k-i}^{(j-i)}]E[\epsilon_{k-j}\epsilon_{k-j}] \\
&= \sum_{l=1}^{j-i} E[\tilde{\phi}_{k-i}^{(l)}]f(i+l,j) + E[\tilde{\theta}_{k-i}^{(j-i)}]\sigma^2
\end{aligned} \tag{6.6}$$

6.2 Nonlinear and Nonsingular PCRLB [1]

The objective is to find a recursive PRCLB for the esitmation of p_{k+1} where e_{k+1} is the observation and where the system is assumed to be nonlinear and nonsingular. The joint PDF for the observation and parameter is assumed to be nonsingular. Let the time sequences of e and p be denoted as ϵ_k and P_k respectively. Such that

$$\begin{aligned}
P_k &= [p_0^T ... p_k^T]^T \\
\epsilon_k &= [e_0^T ... e_k^T]^T
\end{aligned} \tag{6.7}$$

The joint probability distribution of ϵ_k and P_k can be shown to have the following form

$$f(P_k, \epsilon_k) = f_k = f(p_0)\prod_{i=1}^{k}[f(p_i|p_{i-1})f(e_i|p_i)] \tag{6.8}$$

where $f(p_0)$ is assumed to be known. Let f_{k+1} denote the joint probability distribution of P_{k+1} and ϵ_{k+1}, in other words, $f_{k+1} = f(P_{k+1}, \epsilon_{k+1})$. f_{k+1} can be written as

$$f_{k+1} = f_k \cdot f(p_{k+1}|p_k) \cdot f(e_{k+1}|p_{k+1}) \tag{6.9}$$

For reference purposes

$$logf_{k+1} = logf_k + logf(p_{k+1}|p_k) + logf(e_{k+1}|p_{k+1}) \tag{6.10}$$

where

$$\begin{aligned} A_1 &\triangleq logf_k \\ A_2 &\triangleq logf(p_{k+1}|p_k) \\ A_3 &\triangleq logf(e_{k+1}|p_{k+1}) \end{aligned} \tag{6.11}$$

Partition P_{k+1} as $P_{k+1} = [P_{k-1}^T|p_k^T|p_{k+1}^T]^T$, so that the FIM for P_{k+1} is

$$\begin{aligned} \Phi(P_{k+1}) &= - \begin{bmatrix} E[\Delta_{P_{k-1}}^{P_{k-1}} logf_{k+1}] & E[\Delta_{P_{k-1}}^{p_k} logf_{k+1}] & E[\Delta_{P_{k-1}}^{p_{k+1}} logf_{k+1}] \\ E[\Delta_{p_k}^{P_{k-1}} logf_{k+1}] & E[\Delta_{p_k}^{p_k} logf_{k+1}] & E[\Delta_{p_k}^{p_{k+1}} logf_{k+1}] \\ E[\Delta_{p_{k+1}}^{P_{k-1}} logf_{k+1}] & E[\Delta_{p_{k+1}}^{p_k} logf_{k+1}] & E[\Delta_{p_{k+1}}^{p_{k+1}} logf_{k+1}] \end{bmatrix} \\ &= \begin{bmatrix} \Phi_{11} & \Phi_{12} & \Phi_{13} \\ \Phi_{21} & \Phi_{22} & \Phi_{23} \\ \Phi_{31} & \Phi_{32} & \Phi_{33} \end{bmatrix} \end{aligned} \tag{6.12}$$

Using the fact that each submatrix in (6.12) is differentiated with respect to a different combination of two parameters and using logarithmic properties, some terms go to zero, thus allowing the submatrices to be simplified

$$
\begin{aligned}
&\Phi_{11} = -E[\Delta_{P_{k-1}}^{P_{k-1}} A_1] \\
&\Phi_{12} = -E[\Delta_{P_{k-1}}^{p_k} A_1] \\
&\Phi_{13} = 0 \\
&\Phi_{22} = -E[\Delta_{p_k}^{p_k} (A_1 + A_2)] \\
&\Phi_{23} = -E[\Delta_{p_k}^{p_{k+1}} A_2] \\
&\Phi_{33} = -E[\Delta_{p_{k+1}}^{p_{k+1}} (A_2 + A_3)]
\end{aligned}
\tag{6.13}
$$

Substituting (6.13) into (6.12) gives

$$
\Phi(P_{k+1}) = \begin{bmatrix} -E[\Delta_{P_{k-1}}^{P_{k-1}} A_1] & -E[\Delta_{P_{k-1}}^{p_k} A_1] & 0 \\ -E[\Delta_{p_k}^{P_{k-1}} A_1] & -E[\Delta_{p_k}^{p_k} (A_1 + A_2)] & -E[\Delta_{p_k}^{p_{k+1}} A_2] \\ 0 & -E[\Delta_{p_{k+1}}^{p_k} A_2] & -E[\Delta_{p_{k+1}}^{p_{k+1}} (A_2 + A_3)] \end{bmatrix} \tag{6.14}
$$

To find $\Phi^{-1}(p_{k+1})$, the lower right $n_p \times n_p$ submatrix of $\Phi^{-1}(P_{k+1})$ must be found. Following the partition scheme of (6.14) appropriately and using (2.6)

$$
\begin{aligned}
\Phi(p_{k+1}) &= \Phi_{33} - \Phi_{32}(\Phi(p_k) - E[\Delta_{p_k}^{p_k} A_2])^{-1} \Phi_{23} \\
&= \Phi_{33} - \Phi_{32}(\Phi(p_k) - \Phi_R)^{-1} \Phi_{23}
\end{aligned}
\tag{6.15}
$$

where the recursion can be initiated with

$$
\Phi(p_0) = E[-\Delta_{p_0}^{p_0} log f(p_0)] \tag{6.16}
$$

6.3 DARE form of time invariant PCRLB

The Woodbury matrix identity or the matrix inversion lemma can be written as [5]

$$
(A + UCV)^{-1} = A^{-1} - A^{-1} U (C^{-1} + V A^{-1} U)^{-1} V A^{-1} \tag{6.17}
$$

Under the conditions where the system is assumed to be time invariant, the recursive PCRLB given in (3.5) can be shown to take the form of the DARE. Using the matrix inversion lemma

$$\begin{aligned}\Phi(p_\infty) = \Phi_\infty &= \Phi_{33} - \Phi_{32}(\Phi(p_\infty) + \Phi_R)^{-1}\Phi_{23} \\ &= \Phi_{33} + \Phi_{32}\Phi_R^{-1}\Phi_\infty\Phi_R^{-1}\Phi_{23} - \Phi_{32}\Phi_R^{-1}\Phi_\infty(\Phi_\infty + \Phi_R)^{-1}\Phi_\infty\Phi_R^{-1}\Phi_{23} - \Phi_{32}\Phi_R^{-1}\Phi_{23}\end{aligned} \tag{6.18}$$

By inspection of (6.18), it can be concluded that it is $(\Phi_\infty + \Phi_R)^{-1}$ that is being manipulated. Applying the matrix inversion lemma such that $\Phi_R = A$, $\Phi_\infty = C$ and $U = V = I$, results in

$$(\Phi_\infty + \Phi_R)^{-1} = \Phi_R^{-1} - \Phi_R^{-1}(\Phi_\infty^{-1} + \Phi_R^{-1})^{-1}\Phi_R^{-1} \tag{6.19}$$

Using the lemma again, but this time on $(\Phi_\infty^{-1} + \Phi_R^{-1})^{-1}$ such $\Phi_R^{-1} = C$, $\Phi_\infty^{-1} = A$ and $U = V = I$, results in

$$(\Phi_\infty + \Phi_R)^{-1} = \Phi_R^{-1} - \Phi_R^{-1}(\Phi_\infty - \Phi_\infty(\Phi_\infty + \Phi_R)\Phi_\infty)^{-1}\Phi_R^{-1} \tag{6.20}$$

Substituting (6.20) into the recursive PRCLB shown in (3.5), the DARE form is obtained.

6.4 Linear and Singular PCRLB [1]

The objective, as before, is to find the recursive PCRLB for the estimation of p_{k+1} or, $p_{k+1}^{(1)}$ and $p_{k+1}^{(2)}$. In this case however, the joint PDF of the observations and parameters is singular.

Finding the PCRLB in such a case requires two primary steps. The first step requires that

$\Phi(P^{(1)}_{k+1}, p^{(2)}_k)$ be found using the PDF, $f(P^{(1)}_{k+1}, p^{(2)}_k)$. In order for the FIM to be calculated recursively, $\Phi(P^{(1)}_{k+1}, p^{(2)}_k)$ must be stated in terms of $\Phi(P^{(1)}_k, p^{(2)}_k)$. The second step requires the use of a transformation matrix to get the FIM from $\Phi(P^{(1)}_{k+1}, p^{(2)}_k)$ to $\Phi(P^{(1)}_{k+1}, p^{(2)}_{k+1})$.

For the first step, $\Phi(p_k)$ is found from $\Phi(P^{(1)}_{k-1}, p^{(1)}_k, p^{(2)}_k)$.

$$\begin{aligned}\Phi(P^{(1)}_{k-1}, p^{(1)}_k, p^{(2)}_k) &= -\begin{bmatrix} E[\Delta^{P^{(1)}_{k-1}}_{P^{(1)}_{k-1}} log f_k] & E[\Delta^{p^{(1)}_k}_{P^{(1)}_{k-1}} log f_k] & E[\Delta^{p^{(2)}_k}_{P^{(1)}_{k-1}} log f_k] \\ E[\Delta^{P^{(1)}_{k-1}}_{p^{(1)}_k} log f_k] & E[\Delta^{p^{(1)}_k}_{p^{(1)}_k} log f_k] & E[\Delta^{p^{(2)}_k}_{p^{(1)}_k} log f_k] \\ E[\Delta^{P^{(1)}_{k-1}}_{p^{(2)}_k} log f_k] & E[\Delta^{p^{(1)}_k}_{p^{(2)}_k} log f_k] & E[\Delta^{p^{(2)}_k}_{p^{(2)}_k} log f_k] \end{bmatrix} \\ &\triangleq \begin{bmatrix} \Phi_{a11} & \Phi_{a12} & \Phi_{a13} \\ \Phi_{a21} & \Phi_{a22} & \Phi_{a23} \\ \Phi_{a31} & \Phi_{a32} & \Phi_{a33} \end{bmatrix}\end{aligned} \tag{6.21}$$

where

$$f_k \triangleq f(P^{(1)}_k, p^{(2)}_k) = f(P^{(1)}_{k-1}, p^{(1)}_k, p^{(2)}_k) \tag{6.22}$$

Using (2.6), $\Phi(p_k)$ can be found from (6.21)

$$\begin{aligned}\Phi(p_k) &= \begin{bmatrix} \Phi_{a22} - \Phi_{a21}\Phi^{-1}_{a11}\Phi_{a12} & \Phi_{a23} - \Phi_{a21}\Phi^{-1}_{a11}\Phi_{a13} \\ \Phi_{a32} - \Phi_{a31}\Phi^{-1}_{a11}\Phi_{a12} & \Phi_{a33} - \Phi_{a31}\Phi^{-1}_{a11}\Phi_{a13} \end{bmatrix} \\ &\triangleq \begin{bmatrix} \Phi_{b11} & \Phi_{b12} \\ \Phi_{b21} & \Phi_{b22} \end{bmatrix}\end{aligned} \tag{6.23}$$

f_{k+1} can be shown to be

$$\begin{aligned} f_{k+1} &= f_k \cdot f(p^{(1)}_{k+1} | P^{(1)}_k, p^{(2)}_k, \epsilon_k) \cdot f(e_{k+1} | P^{(1)}_k, p^{(2)}_k, p^{(1)}_{k+1}, \epsilon_k) \\ &= f_k \cdot f(p^{(1)}_{k+1} | p_k) \cdot f(e_{k+1} | p_k, p^{(1)}_{k+1}) \end{aligned} \tag{6.24}$$

For reference purposes

$$log f_{k+1} = log f_k + log f(p^{(1)}_{k+1} | p_k) + log f(e_{k+1} | p_k, p^{(1)}_{k+1}) \tag{6.25}$$

where

$$B_1 \triangleq logf(p_{k+1}^{(1)}|p_k)$$
$$B_2 \triangleq logf(e_{k+1}|p_k, p_{k+1}^{(1)}) \tag{6.26}$$

Using (6.21) and (6.24), $\Phi(P_{k-1}^{(1)}, p_k^{(1)}, p_{k+1}^{(1)}, p_k^{(2)})$ can be written as

$$\Phi(P_{k-1}^{(1)}, p_k^{(1)}, p_k^{(2)}, p_{k+1}^{(1)}) =$$
$$-\begin{bmatrix} E[\Delta_{P_{k-1}^{(1)}}^{P_{k-1}^{(1)}} logf_{k+1}] & E[\Delta_{P_{k-1}^{(1)}}^{p_k^{(1)}} logf_{k+1}] & E[\Delta_{P_{k-1}^{(1)}}^{p_k^{(2)}} logf_{k+1}] & E[\Delta_{P_{k-1}^{(1)}}^{p_{k+1}^{(1)}} logf_{k+1}] \\ E[\Delta_{p_k^{(1)}}^{P_{k-1}^{(1)}} logf_{k+1}] & E[\Delta_{p_k^{(1)}}^{p_k^{(1)}} logf_{k+1}] & E[\Delta_{p_k^{(1)}}^{p_k^{(2)}} logf_{k+1}] & E[\Delta_{p_k^{(1)}}^{p_{k+1}^{(1)}} logf_{k+1}] \\ E[\Delta_{p_k^{(2)}}^{P_{k-1}^{(1)}} logf_{k+1}] & E[\Delta_{p_k^{(2)}}^{p_k^{(1)}} logf_{k+1}] & E[\Delta_{p_k^{(2)}}^{p_k^{(2)}} logf_{k+1}] & E[\Delta_{p_k^{(2)}}^{p_{k+1}^{(1)}} logf_{k+1}] \\ E[\Delta_{p_{k+1}^{(1)}}^{P_{k-1}^{(1)}} logf_{k+1}] & E[\Delta_{p_{k+1}^{(1)}}^{p_k^{(1)}} logf_{k+1}] & E[\Delta_{p_{k+1}^{(1)}}^{p_k^{(2)}} logf_{k+1}] & E[\Delta_{p_{k+1}^{(1)}}^{p_{k+1}^{(1)}} logf_{k+1}] \end{bmatrix} \tag{6.27}$$
$$\triangleq \begin{bmatrix} \Phi_{c11} & \Phi_{c12} & \Phi_{c13} & \Phi_{c14} \\ \Phi_{c21} & \Phi_{c22} & \Phi_{c23} & \Phi_{c24} \\ \Phi_{c31} & \Phi_{c32} & \Phi_{c33} & \Phi_{c34} \\ \Phi_{c41} & \Phi_{c42} & \Phi_{c43} & \Phi_{c44} \end{bmatrix}$$

Solving for the submatrices similarly to (6.13)

$$
\begin{aligned}
\Phi_{c11} &= \Phi_{a11} \\
\Phi_{c12} &= \Phi_{a12} \\
\Phi_{c13} &= \Phi_{a13} \\
\Phi_{c14} &= 0 \\
\Phi_{c22} &= \Phi_{a22} - E[\Delta^{p_k^{(1)}}_{p_k^{(1)}}(B_1 + B_2)] \\
\Phi_{c23} &= \Phi_{a23} - E[\Delta^{p_k^{(2)}}_{p_k^{(1)}}(B_1 + B_2)] \\
\Phi_{c24} &= -E[\Delta^{p_{k+1}^{(1)}}_{p_k^{(1)}}(B_1 + B_2)] \\
\Phi_{c33} &= \Phi_{a33} - E[\Delta^{p_k^{(2)}}_{p_k^{(2)}}(B_1 + B_2)] \\
\Phi_{c34} &= -E[\Delta^{p_{k+1}^{(1)}}_{p_k^{(2)}}(B_1 + B_2)] \\
\Phi_{c44} &= -E[\Delta^{p_{k+1}^{(1)}}_{p_{k+1}^{(1)}}(B_1 + B_2)]
\end{aligned}
\tag{6.28}
$$

Substituting (6.28) into (6.27)

$$
\Phi(P_{k-1}^{(1)}, p_k^{(1)}, p_k^{(2)}, p_{k+1}^{(1)}) =
\begin{bmatrix}
\Phi_{a11} & \Phi_{a12} & \Phi_{a13} & 0 \\
\Phi_{a21} & \Phi_{a22} - E[\Delta^{p_k^{(1)}}_{p_k^{(1)}}(B_1 + B_2)] & \Phi_{a23} - E[\Delta^{p_k^{(2)}}_{p_k^{(1)}}(B_1 + B_2)] & -E[\Delta^{p_{k+1}^{(1)}}_{p_k^{(1)}}(B_1 + B_2)] \\
\Phi_{a31} & \Phi_{a32} - E[\Delta^{p_k^{(1)}}_{p_k^{(2)}}(B_1 + B_2)] & \Phi_{c33} - E[\Delta^{p_k^{(2)}}_{p_k^{(2)}}(B_1 + B_2)] & -E[\Delta^{p_{k+1}^{(1)}}_{p_k^{(2)}}(B_1 + B_2)] \\
0 & -E[\Delta^{p_k^{(1)}}_{p_{k+1}^{(1)}}(B_1 + B_2)] & -E[\Delta^{p_k^{(2)}}_{p_{k+1}^{(1)}}(B_1 + B_2)] & -E[\Delta^{p_{k+1}^{(1)}}_{p_{k+1}^{(1)}}(B_1 + B_2)]
\end{bmatrix}
\tag{6.29}
$$

Now using (2.6) to find the submatrix $\Phi(p_k^{(1)}, p_k^{(2)}, p_{k+1}^{(1)})$ from (6.29)

$$\Phi(p_k^{(1)}, p_k^{(2)}, p_{k+1}^{(1)}) =$$
$$\begin{bmatrix} \Phi_{a22} - E[\Delta_{p_k^{(1)}}^{p_k^{(1)}}(B_1+B_2)] & \Phi_{a23} - E[\Delta_{p_k^{(1)}}^{p_k^{(2)}}(B_1+B_2)] & -E[\Delta_{p_k^{(1)}}^{p_{k+1}^{(1)}}(B_1+B_2)] \\ \Phi_{a32} - E[\Delta_{p_k^{(2)}}^{p_k^{(1)}}(B_1+B_2)] & \Phi_{c33} - E[\Delta_{p_k^{(2)}}^{p_k^{(2)}}(B_1+B_2)] & -E[\Delta_{p_k^{(2)}}^{p_{k+1}^{(1)}}(B_1+B_2)] \\ -E[\Delta_{p_{k+1}^{(1)}}^{p_k^{(1)}}(B_1+B_2)] & -E[\Delta_{p_{k+1}^{(1)}}^{p_k^{(2)}}(B_1+B_2)] & -E[\Delta_{p_{k+1}^{(1)}}^{p_{k+1}^{(1)}}(B_1+B_2)] \end{bmatrix} \tag{6.30}$$
$$- \begin{bmatrix} \Phi_{a21} \\ \Phi_{a31} \\ 0 \end{bmatrix} (\Phi_{a11})^{-1} \begin{bmatrix} \Phi_{a12} & \Phi_{a13} & 0 \end{bmatrix}$$

Solving (6.30) and substituting (6.23) into (6.30)

$$\Phi(p_k^{(1)}, p_k^{(2)}, p_{k+1}^{(1)}) =$$
$$\begin{bmatrix} \Phi_{b11} - E[\Delta_{p_k^{(1)}}^{p_k^{(1)}}(B_1+B_2)] & \Phi_{b12} - E[\Delta_{p_k^{(1)}}^{p_k^{(2)}}(B_1+B_2)] & -E[\Delta_{p_k^{(1)}}^{p_{k+1}^{(1)}}(B_1+B_2)] \\ \Phi_{b21} - E[\Delta_{p_k^{(2)}}^{p_k^{(1)}}(B_1+B_2)] & \Phi_{b22} - E[\Delta_{p_k^{(2)}}^{p_k^{(2)}}(B_1+B_2)] & -E[\Delta_{p_k^{(2)}}^{p_{k+1}^{(1)}}(B_1+B_2)] \\ -E[\Delta_{p_{k+1}^{(1)}}^{p_k^{(1)}}(B_1+B_2)] & -E[\Delta_{p_{k+1}^{(1)}}^{p_k^{(2)}}(B_1+B_2)] & -E[\Delta_{p_{k+1}^{(1)}}^{p_{k+1}^{(1)}}(B_1+B_2)] \end{bmatrix} \tag{6.31}$$

a FIM for the parameters $[p_k^{(1)}, p_k^{(2)}, p_{k+1}^{(1)}]$ is achieved in a recursive form. The reason for extracting the submatrix $\Phi(p_k^{(1)}, p_k^{(2)}, p_{k+1}^{(1)})$ will become evident in the second step. It is worth noting that with $\Phi(p_k^{(1)}, p_k^{(2)}, p_{k+1}^{(1)})$, it is not possible to extract $\Phi(p_{k+1})$. This is because $p^{(2)}$ still needs to be incremented (or promoted) from $p_k^{(2)}$ to $p_{k+1}^{(2)}$. The promotion of $p^{(2)}$ is the second step, which follows directly from the following lemma [10], let

$$\Phi(x) = E[-\Delta_x^x logf(x,z)] \tag{6.32}$$

be the FIM for the state vector x with probability density $f(x,z)$. If $y = Mx$ and M is invertible, then $f(y,z)$ exists and the FIM for y is given by

$$\Phi(y) = M^{-T}\Phi(x)M^{-1} \tag{6.33}$$

The objective for this step is to find the appropriate M matrix. In other words, relating the vector $[(p_k^{(1)})^T(p_k^{(2)})^T(p_{k+1}^{(1)})^T]^T$ (analogous to x in the lemma above) to $[(p_k^{(1)})^T(p_{k+1}^{(2)})^T(p_{k+1}^{(1)})^T]^T$ (analogous to y in the lemma above) via a transformation matrix M. Using (3.14), the relationship can be formed

$$\begin{bmatrix} p_k^{(1)} \\ p_{k+1}^{(2)} \\ p_{k+1}^{(1)} \end{bmatrix} = \begin{bmatrix} I & 0 & 0 \\ G_k^{(1)} & G_k^{(2)} & G_k^{(3)} \\ 0 & 0 & I \end{bmatrix} \begin{bmatrix} p_k^{(1)} \\ p_k^{(2)} \\ p_{k+1}^{(1)} \end{bmatrix} = M_k \begin{bmatrix} p_k^{(1)} \\ p_k^{(2)} \\ p_{k+1}^{(1)} \end{bmatrix} \tag{6.34}$$

It should now be evident why the matrix $\Phi(p_k^{(1)}, p_k^{(2)}, p_{k+1}^{(1)})$ was extracted. It is that submatrix that can be transformed by the matrix M_k. Finally, using (6.34) and (6.31)

$$\begin{aligned} \Phi(p_k^{(1)}, p_{k+1}^{(2)}, p_{k+1}^{(1)}) &= M_k^{-T}\Phi(p_k^{(1)}, p_k^{(2)}, p_{k+1}^{(1)})M_k^{-1} \\ &\triangleq \begin{bmatrix} \Phi_{d11} & \Phi_{d12} & \Phi_{d13} \\ \Phi_{d21} & \Phi_{d22} & \Phi_{d23} \\ \Phi_{d31} & \Phi_{d32} & \Phi_{d33} \end{bmatrix} \end{aligned} \tag{6.35}$$

then extracting the submatrix for $\Phi(p_{k+1})$ or $\Phi(p_{k+1}^{(1)}, p_{k+1}^{(2)})$ from (6.35) using (2.6)

$$\Phi(p_{k+1}) = \begin{bmatrix} \Phi_{d22} & \Phi_{d23} \\ \Phi_{d32} & \Phi_{d33} \end{bmatrix} - \begin{bmatrix} \Phi_{d21} \\ \Phi_{d31} \end{bmatrix} \Phi_{d11}^{-1} \begin{bmatrix} \Phi_{d12} & \Phi_{d13} \end{bmatrix} \tag{6.36}$$

Using the two steps shown, the FIM for p_{k+1} was achieved while bypassing the singular PDF that would have emerged if the regular method mentioned in section 3.1 had been used.

6.5 Nonlinear and Singular PCRLB [1]

Finding the recursive PCRLB of the parameter p_{k+1} in the case where the system is nonlinear and the joint PDF is singular, can be found using similar methods used in section 3.1. Due to the noise added to $p_{k+1}^{(2)}$ in (3.17), the PDF is now

$$\begin{aligned}
f_{k+1} &= f_k \cdot f(p_{k+1}|p_k) \cdot f(e_{k+1}|p_{k+1}) \\
&= f_k \cdot f(p_{k+1^{(1)}}, p_{k+1^{(2)}}|p_k) \cdot f(e_{k+1}|p_{k+1}) \\
&= f_k \cdot \frac{f(p_{k+1}^{(1)}, p_{k+1}^{(2)}, p_k)}{f(p_n)} \cdot f(e_{k+1}|p_{k+1}) \\
&= f_k \cdot f(p_{k+1}^{(2)}|p_k, p_{k+1}^{(1)}) \frac{f(p_{k+1}^{(1)}, p_k)}{f(p_k)} \cdot f(e_{k+1}|p_{k+1}) \\
&= f_k \cdot f(p_{k+1}^{(2)}|p_k, p_{k+1}^{(1)}) \cdot f(p_{k+1}^{(1)}|p_k) \cdot f(e_{k+1}|p_{k+1})
\end{aligned} \tag{6.37}$$

where, as in section 3.1, e_k is the obvserved variable. For reference purposes, let

$$log f_{k+1} = log f_k + log f(p_{k+1}^{(2)}|p_k, p_{k+1}^{(1)}) + log f(p_{k+1}^{(1)}|p_k) + log f(e_{k+1}|p_{k+1}) \tag{6.38}$$

where

$$\begin{aligned}
C_1 &\triangleq log f_k \\
C_2 &\triangleq log f(p_{k+1}^{(2)}|p_k, p_{k+1}^{(1)}) \\
C_3 &\triangleq log f(p_{k+1}^{(1)}|p_k) \\
C_4 &\triangleq log f(e_{k+1}|p_{k+1})
\end{aligned} \tag{6.39}$$

Since $w_n^{(2)}$ is a Gaussian random variable, $f(p_{k+1}^{(2)}|p_k, p_{k+1}^{(1)})$ will be a Gaussian distribution with a mean of $g_k^{(2)}(p_k, p_{k+1}^{(1)})$ and a covariance of δI. Thus

$$-C_2 = c + \frac{1}{2\delta}||p_{k+1}^{(2)} - g_k^{(2)}(p_k, p_{k+1}^{(1)})||^2 = c + \frac{1}{2\delta}||t(p_{k+1}^{(2)}, p_{k+1}^{(1)}, p_k)||^2 = c + \frac{1}{2\delta}||t||^2 \tag{6.40}$$

where c is a constant and $t = p_{k+1}^{(2)} - g_k^{(2)}(p_k, p_{k+1}^{(1)})$. Finding the analogous submatrices to (3.6) using (6.37)

$$\begin{aligned}
\Phi_{e23} &= -E[\Delta_{p_k}^{p_{k+1}} C_3] - E[\Delta_{p_k}^{p_{k+1}} C_2] = A_{e23} - E[\Delta_{p_k}^{p_{k+1}} C_2] \\
\Phi_{e33} &= -E[\Delta_{p_{k+1}}^{p_{k+1}} C_3] - E[\Delta_{p_{k+1}}^{p_{k+1}} C_2] = A_{e33} - E[\Delta_{p_{k+1}}^{p_{k+1}} C_2] \\
\Phi_{eR} &= -E[\Delta_{p_k}^{p_k}(C_3 + C_4)] - E[\Delta_{p_k}^{p_k} C_2] = A_{eR} - E[\Delta_{p_k}^{p_k} C_2]
\end{aligned} \tag{6.41}$$

The terms A_{e23}, A_{e33} and A_{eR} can be found the same way as (6.13). Solving for $-E[\Delta_{p_{k+1}}^{p_{k+1}} C_2]$

$$\begin{aligned}
E[\Delta_{p_{k+1}}^{p_{k+1}}(-C_2)] &= E[\Delta_{p_{k+1}}^{p_{k+1}}(\frac{1}{2\delta}||p_{n+1}^{(2)} - g_k^{(2)}(p_k, p_{k+1}^{(1)})||^2)] \\
&= \frac{1}{2\delta}\begin{bmatrix} E[\Delta_{p_{k+1}^{(1)}}^{p_{k+1}^{(1)}}(||t||^2)] & E[\Delta_{p_{k+1}^{(1)}}^{p_{k+1}^{(2)}}(||t||^2)] \\ E[\Delta_{p_{k+1}^{(2)}}^{p_{k+1}^{(1)}}(||t||^2)] & E[\Delta_{p_{k+1}^{(2)}}^{p_{k+1}^{(2)}}(||t||^2)] \end{bmatrix} \\
&= \frac{1}{2\delta}\begin{bmatrix} E[\nabla_{p_{k+1}^{(1)}} \nabla_{p_{k+1}^{(1)}}^T (||t||^2)] & E[\nabla_{p_{k+1}^{(1)}} \nabla_{p_{k+1}^{(2)}}^T (||t||^2)] \\ E[\nabla_{p_{k+1}^{(2)}} \nabla_{p_{k+1}^{(1)}}^T (||t||^2)] & E[\nabla_{p_{k+1}^{(2)}} \nabla_{p_{k+1}^{(2)}}^T (||t||^2)] \end{bmatrix} \\
&= \frac{1}{\delta}\begin{bmatrix} E[(\nabla_{p_{k+1}^{(1)}} g^T)(\nabla_{p_{k+1}^{(1)}} g^T)^T] & -E[(\nabla_{p_{k+1}^{(1)}} g^T)] \\ -E[(\nabla_{p_{k+1}^{(1)}} g^T)^T] & I \end{bmatrix}
\end{aligned} \tag{6.42}$$

where the product rule of the gradient operator was used to get from the second to last line in (6.42) to the last line. Using the same method used in (6.42), $E[\Delta_{p_k}^{p_{k+1}} C_2]$ and $E[\Delta_{p_k}^{p_k} C_2]$ can be found to be

$$\begin{aligned}
E[\Delta_{p_k}^{p_k} C_2] &= \frac{1}{\delta} E[(\nabla_{p_k} g^T)(\nabla_{p_k} g^T)^T] \\
E[\Delta_{p_k}^{p_{k+1}} C_2] &= \frac{1}{\delta}\begin{bmatrix} E[(\nabla_{p_k} g^T)(\nabla_{p_{k+1}^{(1)}} g^T)^T] & -E[(\nabla_{p_k} g^T)] \end{bmatrix}
\end{aligned} \tag{6.43}$$

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