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Integral and Euclidean Ramsey Theory

A dissertation submitted in partial satisfaction of the
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by

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The dissertation of Eric Tressler is approved, and it is acceptable in quality and form for publication on microfilm:

Chair

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2010

DEDICATION

To my parents.

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Chapter 2, Section 2.2 is based on the paper “The first nontrivial Hales-Jewett number is four,” written by the author together with Neil Hindman, accepted to *Ars Combinatoria*.

Chapter 4 is based on the paper “An intersection theorem about domino tilings,” written by the author together with Steve Butler and Paul Horn, to be published in *The Fibonacci Quarterly*.

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Ron Graham and Eric Tressler, "Open problems in Euclidean Ramsey theory," submitted, 2009.

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ABSTRACT OF THE DISSERTATION

Integral and Euclidean Ramsey Theory

by

Eric Tressler
Doctor of Philosophy in Mathematics

University of California San Diego, 2010

Professor Ronald Graham, Chair

Ramsey theory is the study of unavoidable structure within a system. This idea is very broad, and also useful in many applications, so the theory is vast. The original theorem of Ramsey [32] states that given k, there is n such that for any graph G on n vertices, either G or its complement contain K_k as a subgraph. Statements like this can be made about any mathematical structure, but this dissertation will focus on sets of integers and on Euclidean space, both of which support a large literature within Ramsey theory. Finally, we will consider a problem in extremal combinatorics, a field that has a large intersection with Ramsey theory.

Chapter 1

Introduction

Ramsey theory is named after Frank Ramsey, who in 1930 proved what is now known as Ramsey's theorem [32]. Though not Ramsey's original formulation, one common special case of Ramsey's theorem states that given an arbitrary n, there exists a least $R(n)$ such that if the edges of the complete graph $K_{R(n)}$ are partitioned into two sets A and B, one of the parts must contain a copy of K_n. In the more typical "chromatic" terminology, we say that whenever the edges of $K_{R(n)}$ are 2-colored, there exists a monochromatic K_n.

This theorem is especially easy to illustrate in the case $n = 3$. Figure 1.1 shows a 2-coloring of the edges of K_5 with no monochromatic K_3, demonstrating that $R(3) > 5$. However, it is not possible to 2-color the edges of K_6 red and blue without forming a triangle, as is easily shown: let x be some vertex in K_6; x has degree 5, so by the pigeonhole principle, some 3 of the edges connected to x must be colored alike. Without loss of generality, suppose the edges $\{x, y_1\}$, $\{x, y_2\}$, and $\{x, y_3\}$ are all red. If any of the edges $\{y_1, y_2\}$, $\{y_1, y_3\}$, or $\{y_2, y_3\}$ are red, then we have a monochromatic red triangle. If all three of these edges are blue, then we have a monochromatic blue triangle. In either case, there must exist a monochromatic K_3, so $R(3) = 6$.

Ramsey's theorem is archetypical of Ramsey theory as a whole, and it demonstrates the central tenet of Ramsey theory: in a large enough system, there is always structure. A few other results in Ramsey theory, apart from those in the following chapters, are given here to show the massive scope of Ramsey theory:

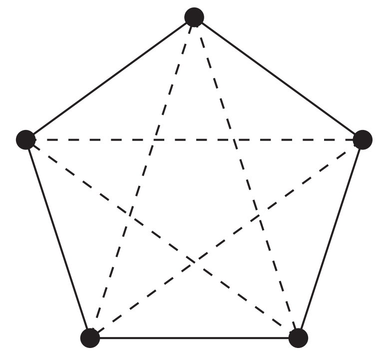

Figure 1.1: A 2-coloring of K_5 with no monochromatic triangle.

Theorem 1 (Pigeonhole Principle)**.** *If $k + 1$ balls are placed into k bins, then some bin contains at least 2 balls.*

Theorem 2 (Hindman [25])**.** *For any $k, r > 0$, if the subsets of $\mathbb{N}$ of size k are r-colored, then there exists an infinite $A \subset \mathbb{N}$ all of whose subsets of size k are the same color.*

Theorem 3 (Erdős-Szekeres [12])**.** *For any k, there exists n such that any n points in general position in the plane must contain a convex k-gon.*

Theorem 4 (Kneser)**.** *If G is a nontrivial abelian group, and A and B are nonempty finite subsets of G such that $|A| + |B| \leq |G|$, then there exists a proper subgroup H of G such that*

$$|H| \geq |A| + |B| - |A + B|.$$

Theorem 5 (Mantel's Theorem)**.** *A graph on n vertices with at least $\lfloor \frac{n^2}{4} \rfloor$ contains a triangle.*

The notion of a Ramsey-type theorem is very general, and Ramsey theory touches on almost any conceivable mathematical object. Here we will be particularly interested in two specific topics: first, we will discuss Ramsey theory on the integers, and then in Chapter 3 we will look at Euclidean Ramsey theory. Finally, in Chapter 4 we look at a problem in extremal graph theory, a subject many of whose theorems (including Mantel's Theorem above) might also be considered Ramsey theorems.

Chapter 2

Ramsey Theory on the Integers

As indicated in Chapter 1, when the integers are finitely colored in any way, there are highly structured monochromatic subsets. Several classical theorems deal with these structures; the first we introduce is Schur's theorem.

Theorem 6 (Schur [33])**.** *For any r, there is a least integer N such that if $[N]$ is r-colored, then there exists a monochromatic solution to the equation $x + y = z$ with $x > y > 0$.*

A generalization of Schur's theorem is Folkman's theorem, published 54 years later; below, for $S \subset \mathbb{N}$, let

$$\sum(S) := \left\{ \sum_{s \in A} s : A \subseteq S, A \neq \emptyset \right\}.$$

Theorem 7 (Folkman [13])**.** *For any r, if $\mathbb{N}$ is r-colored there exist arbitrarily large finite $S \subset \mathbb{N}$ with $\sum(S)$ monochromatic.*

A further extension of this idea comes from Neil Hindman, in a 1974 paper:

Theorem 8 (Hindman [24])**.** *For any r, if $\mathbb{N}$ is r-colored, then there exists $S \subseteq \mathbb{N}$ infinite such that $\sum(S)$ is monochromatic.*

The three theorems above do not tell the complete story of this problem. Schur's theorem has other well-known extensions – see chapters 8 and 9 of *Ramsey Theory on the Integers* [30] and chapter 3 of *Ramsey Theory (Second Edition)* [20] for more on these. There are still many related open questions, though, that have resisted attack.

For instance, Schur's theorem guarantees that if the integers are finitely colored, there is a monochromatic set of the form $\{x, y, x + y\}$. It is also known (following quickly from Schur's theorem) that the same is true of sets of the form $\{x, y, xy\}$. It is still an open question, though, whether for every r, if the integers are r-colored, there must be a monochromatic set of the form $\{x, y, x + y, xy\}$.

Now we turn to the celebrated theorem of van der Waerden on arithmetic progressions:

Theorem 9 (van der Waerden [41])**.** *For any $k, r \in \mathbb{N}$, there is a least integer $w(k, r)$ such that if $[w(k, r)]$ is r-colored there must exist a monochromatic k-term arithmetic progression (that is, a set of the form $\{a + bd : 0 \leq b \leq k - 1, d > 0\}$).*

Van der Waerden's theorem is probably the most widely-known Ramsey theorem on the integers, and there is a wide literature surrounding it. Shelah showed that the upper bounds on $w(k, r)$ are primitive recursive [34], and in 2001 W.T. Gowers showed [19] that

$$w(k, 2) \leq 2^{2^{2^{2^{2^{k+9}}}}},$$

a result of work that led to his receiving the Fields medal.

There are other interesting and highly nontrivial facts about arithmetic progressions – in 1975, Endre Szemerédi proved a much strengthened generalization of van der Waerden's theorem, now known as Szemerédi's Theorem:

Theorem 10 (Szemerédi [39])**.** *If $A \subset \mathbb{N}$ has positive upper density – that is, if*

$$\limsup_{n \to \infty} \frac{|A \cap [n]|}{n} > 0,$$

then A contains arbitrarily long arithmetic progressions.

In 2004, Ben Green and Terence Tao proved that the primes contain arbitrarily long arithmetic progressions [21]; of course, the primes do not have positive upper density. As $\sum_{p \text{ prime}} \frac{1}{p} = \infty$, this work is a special case of a famous conjecture of Erdős:

Conjecture 1. *If $A \subset \mathbb{N}$ satisfies*

$$\sum_{a \in A} \frac{1}{a} = \infty,$$

then A contains arbitrarily long arithmetic progressions.

To date, it has not been shown that a set A satisfying the conditions of the conjecture must contain even a 3-term arithmetic progression – the field of Ramsey theory on the integers is alive and well.

In Landman and Robertson's book *Ramsey Theory on the Integers*, there are discussions of several different variants of the van der Waerden numbers (i.e., replacing the arithmetic progressions in van der Waerden's theorem with other related structures), many of which have interesting and surprising properties. We examine some of these below.

2.1 Variants of van der Waerden numbers

2.1.1 Multi-Arithmetic Progressions

A *multi-arithmetic progression* (or *MAP*) of length k and *gap size* m is an increasing sequence $\{x_j\}_{j=1}^{k} \subset \mathbb{N}$ such that $\{x_{j+1} - x_j : j > 1\}$ has cardinality m. Given an increasing sequence $x_1, \ldots, x_k$, call the $x_{j+1} - x_j$ the *lengths* of the gaps. Note that multi-arithmetic progressions with gap size 1 are simply arithmetic progressions.

Define $B_m(k, r)$ to be the least integer N such that for any r-coloring of $[N]$, there exists a monochromatic MAP of length k and gap size m. $B_m(k, r)$ exists for all $m, k, r > 0$ because $B_1(k, r)$ exists by van der Waerden's theorem, and clearly $B_m(k, r) \leq B_n(k, r)$ for $m > n$. Given m, k, r below, we will call an r-coloring of $[N]$ *good* if it contains no monochromatic MAP of length k and gap size m.

Proposition 1. *If* $k + \frac{m(m+1)}{2} > (k-1)r + 1$*, then* $B_m(k, r) = (k-1)r + 1$*.*

Proof. To see that $B_m(k, r) \geq (k-1)r + 1$, note that in the interval $[(k-1)r]$, letting each color appear $k - 1$ times gives a good coloring. To see the other direction, observe that for $B_m(k, r)$ to exceed $(k-1)r + 1$, there must exist a good coloring χ of $[(k-1)r + 1]$, which implies by the pigeonhole principle that some color (say red) must show up at least k times. Moreover, these k red elements must have at least $m + 1$ distinct consecutive differences, or else they form a MAP. There are $k - 1$ total consecutive differences between our red elements, of which $(k-1) - m$ can be taken to be 1. The m remaining consecutive differences must be distinct from 1, so of course the minimal case is that we take these to be $2, 3, \ldots, m + 1$. Thus, our interval must be of length at least k (the number of our red elements) plus the sum of the differences:

$$\begin{aligned} N &:= k + [(k-1) - m] + \sum_{i=2}^{m+1} i \\ &= k + [(k-1) - m] + \frac{(m+1)(m+2)}{2} - 1 \\ &= 2k - m - 2 + \frac{(m+1)(m+2)}{2} \end{aligned}$$

to accommodate our red elements. If $N > (k-1)r + 1$, then there is no good coloring of $[(k-1)r + 1]$, so $B_m(k, r) = (k-1)r + 1$. □

In Table 2.1 we list some nontrivial values of $B_m(k, r)$, obtained with a computer and an optimized (but essentially brute force) algorithm. Note that though these numbers appear to grow much more slowly than the van der Waerden numbers, there are very many more multi-arithmetic progressions than arithmetic progressions. For any k, there are $O(n^2)$ arithmetic progressions in $[n]$; for multi-arithmetic progressions with gap size 2, there are $\binom{n}{2}$ choices of gap lengths. Given a pair $a < b$ of gap lengths, and a number of times $1 \leq i < k - 1$ a appears as a gap length, there are $O(n)$ multi-arithmetic progressions, and hence $O(n^3)$ total MAPs of gap size 2. This makes exhausting over the space of possible r-colorings computationally very expensive, and of course the exponent grows with the gap size. The following proposition, used in some of the computer calculations, shows that this estimate is essentially correct.

Proposition 2. *There are fewer than*

$$\frac{m^{k-1} n^{m+1}}{2(k-1)}$$

MAPs of length k and gap size m in $[n]$.

Proof. Fix n and let $f(x)$ be the number of MAPs in $[n]$ of length k and gap size m whose minimal gap length is x. Since there are $k - 1$ gaps, $f(x) = 0$ for $x \geq n/(k-1)$. Since a MAP of length k and minimal gap length x spans at least $k + x(k-1)$ integers, there are only $(n - k - x(k-1) + 1) \leq (n - x(k-1))$ valid starting positions for such a MAP. There are fewer than $\binom{n-x}{m-1}$ choices for the remaining gap lengths and m^{k-1} MAPs associated to each starting position and set of gap lengths, so

$$f(x) \leq \binom{n-x}{m-1}(n - x(k-1))m^{k-1} \leq n^{m-1}(n - x(k-1))m^{k-1}.$$

Therefore the total number of MAPs of length k and gap size m is at most

$$\begin{aligned}
\sum_{x=0}^{\lfloor n/(k-1)\rfloor} f(x) &\leq \sum_{x=0}^{\lfloor n/(k-1)\rfloor} n^{m-1}(n - x(k-1))m^{k-1} \\
&\leq n^{m-1}m^{k-1}\left[\left(\frac{n^2}{k-1}\right) - (k-1)\sum_{x=0}^{n/(k-1)} x\right] \\
&= n^{m-1}m^{k-1}\left[\left(\frac{n^2}{k-1}\right) - \left(\frac{k-1}{2}\right)\left(\frac{n}{k-1}\right)\left(\frac{n+k-1}{k-1}\right)\right] \\
&< n^{m-1}m^{k-1}\left[\left(\frac{n^2}{k-1}\right) - \left(\frac{n}{2}\right)\left(\frac{n}{k-1}\right)\right] \\
&= n^{m-1}m^{k-1}\left(\frac{n^2}{2(k-1)}\right) \\
&= \frac{m^{k-1}n^{m+1}}{2(k-1)}.
\end{aligned}$$

□

$B_2(4,2) = 9$	$B_3(5,3) = 17$	$B_4(6,3) = 19$
$B_2(4,3) = 16$	$B_3(5,4) = 27$	$B_4(6,4) = 29$
$B_2(4,4) = 25$	$B_3(6,3) = 25$	$B_4(7,3) = 24$
$B_2(4,5) = 37$	$B_3(7,2) = 15$	$B_4(8,3) = 31$
$B_2(5,2) = 14$	$B_3(7,3) = 35$	$B_4(11,2) = 23$
$B_2(5,3) = 35$	$B_3(8,2) = 19$	$B_4(12,2) = 25$
$B_2(6,2) = 21$	$B_3(9,2) = 23$	$B_4(13,2) = 29$
$B_2(7,2) = 28$	$B_3(10,2) = 27$	$B_4(14,2) = 31$
$B_2(8,2) = 41$	$B_3(11,2) = 32$	$B_4(15,2) = 35$
$B_2(9,2) = 53$	$B_3(12,2) = 37$	$B_4(16,2) = 38$
		$B_4(17,2) = 41$

Table 2.1: Some nontrivial values of $B_m(k,r)$.

Proposition 3. *For $m+2 \le k \le 2m$ and $r \ge 2$,*

$$B_m(k,r) < \binom{k-m}{2}(B_m(k,r-1)^2 + B_m(k,r-1)).$$

Proof. Fix $m \ge 1$, $m+2 \le k \le 2m$, and $r \ge 2$. Now let $N \in \mathbb{N}$ and consider a coloring χ of the interval $[N]$. Without loss of generality, call $\chi(1)$ the color red. Observe that since $k \le 2m$, if:

1. there are at least k red elements in $[N]$, and
2. among the gaps between consecutive red elements, some gap length ℓ appears at least $k-m$ times,

then χ cannot be a good coloring of $[N]$. Indeed, suppose that both these conditions hold, and let $\{x_j\}$ be the red elements straddling the first $k-m$ gaps of length ℓ. Since the gap of length ℓ appears $k-m$ times, there are at least $k-m+1$ elements among the x_j (this will happen when the gaps of length ℓ are all consecutive). If the set $\{x_j\}$ does not comprise at least k elements, add red elements arbitrarily so that we have k elements.

In this case it is easy to see that $\{x_j\}$ is a red MAP of length k with gap size m: among the k red elements in this sequence, the gap length ℓ appears at least $k-m$ times, so of the $k-1$ total gaps, there can be at most m distinct gaps. That is, χ is a bad coloring.

Now suppose χ is a good coloring of $[N]$. Then the above two properties cannot hold for any color, in particular not for red. Observe now that the gap lengths between consecutive red elements can be at most $B_m(k,r-1)-1$, for by definiton there is no coloring of $[B_m(k,r-1)]$ with $r-1$ colors that avoids a MAP of length k and gap size m. Thus the only possible gap lengths between red colors are $\{0,1,\ldots,B_m(k,r-1)-1\}$, and since χ is a good coloring, each of these can appear at most $k-m-1$ times between red elements; in addition, there may be at most $B_m(k,r-1)-1$ elements of other colors after the last red element. Since $\{0,1,\ldots,B_m(k,r-1)-1\}$ has cardinality $B_m(k,r-1)$, there can be at most $(k-m-1)B_m(k,r-1)+1$ red elements.

Let $B := B_m(k,r-1)$. Summing the maximal possible red elements and the possible gap lengths, along with the elements of other colors that may appear after the last red element, we get

$$
\begin{aligned}
N &\leq [(k-m-1)B+1] + \left[(k-m-1)\sum_{i=0}^{B-1} i\right] + [B-1] \\
&= [(k-m-1)B+1] + \left[(k-m-1)\frac{B(B-1)}{2}\right] + [B-1] \\
&= \left(\frac{k-m-1}{2}\right)B^2 + \left(\frac{k-m+1}{2}\right)B \\
&= \left(\frac{k-m}{2}\right)B^2 + \left(\frac{k-m}{2}\right)B + \frac{B-B^2}{2} \\
&< \left(\frac{k-m}{2}\right)(B^2+B)) - 1,
\end{aligned}
$$

where

$$\frac{B_m(k,r-1) - B_m(k,r-1)^2}{2} < -1$$

since $k \geq m+2$ implies $k \geq 3$, so $B_m(k,r-1) \geq B_m(k,1) = k \geq 3$.

Now we have shown that assuming χ is a good coloring of $[N]$ implies that

$$N < \left(\frac{k-m}{2}\right)(B_m(k,r-1)^2 + B_m(k,r-1)) - 1,$$

so that even adding 1 to the righthand side will force the coloring to be bad:

$$B_m(k,r) < \left(\frac{k-m}{2}\right)(B_m(k,r-1)^2 + B_m(k,r-1)).$$

□

Corollary 1. *For $m+2 \leq k \leq 2m$,*

$$B_m(k,r) \leq k^{2^r-1},$$

with equality iff $r = 1$.

Proof. Fix $m \geq 1$ and $m+2 \leq k \leq 2m$. $B_m(k,1) = k$, so by the above

$$B_m(k,2) < (k-m)(k^2+k) = k^3 + k^2 - mk^2 - mk < k^3.$$

Now suppose $B_m(k,r) < k^{2^r-1}$ for some $r \geq 2$. Then by the above proposition,

$$\begin{aligned} B_m(k, r+1) &< (k-m)\left(\left(k^{2^r-1}\right)^2 + k^{2^r-1}\right) \\ &= (k-m)\left(k^{2^{r+1}-2} + k^{2^r-1}\right) \\ &= k^{2^{r+1}-1} + k^{2^r} - mk^{2^{r+1}-2} - mk^{2^r-1} \\ &< k^{2^{r+1}-1} + k^{2^r} - k^{2^{r+1}-2} \\ &< k^{2^{r+1}-1} + k^{2^r} - k^{2^r} \\ &= k^{2^{r+1}-1}, \end{aligned}$$

so the result holds by induction. □

Proposition 4. *For all r, we have the inequalities*

$$w(3,r) = B_1(3,r) \leq B_2(5,r),$$

$$w(3,r) = B_1(3,r) \leq B_3(9,r),$$

and

$$w(4,r) = B_1(4,r) \leq B_2(11,r).$$

Proof. We will prove these three inequalities in order, treating the consecutive differences among MAP elements as words to facilitate the proof. That is, the MAP $\{1,3,5,6,8\}$ would correspond to the word 2212. For the first inequality, consider an r-coloring of $[B_2(5,r)]$. By definition, this coloring contains a monochromatic 5-term MAP with gap size 2. Call the two gap lengths a and b. If any pair of these appears in succession, that forms a monochromatic 3-term AP. Otherwise, up to symmetry, the only possibility is that the sequence of gaps is *abab*, so that the first, third, and fifth terms in the MAP form a 3-term AP. In either case, $w(3,r) \leq B_2(5,r)$.

Now consider an r-coloring of $[B_3(9,r)]$. Again by definition, this coloring contains a monochromatic 9-term MAP with gap size 3; call the gap lengths a, b, and c. Without loss of generality let the first two gaps be a and b (as above, no two can appear in succession). Applying the rule that no two blocks of gaps with the same sum can appear in succession (as in *abccab*, for $a+b+c = c+a+b$, the tree in Figure 2.1 shows the only possible gap sequences that can occur. None has as many as 8 gaps, contradicting the

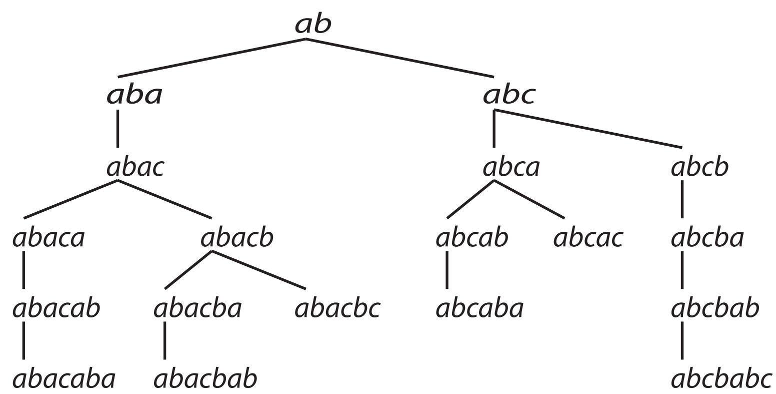

Figure 2.1: Proof of the inequality that $w(3, r) \leq B_3(9, r)$ for all r.

fact that there exists a monochromatic 9-term MAP. Thus, our r-coloring does indeed contain a monochromatic 3-term AP.

Finally, let χ be an r-coloring of $[B_2(11, r)]$; then χ necessarily contains a MAP of length 11 and gap size 2 in some color, say red. Call the gap lengths a and b. Now we have a gap sequence that looks like $x_1, x_2, \ldots, x_{10}$, where the x_j are valued either a or b. If, for any j, $x_j = x_{j+1} = x_{j+2}$, or $x_j + x_{j+1} = x_{j+2} + x_{j+3} = x_{j+4} + x_{j+5}$, or similarly for nine gaps, then clearly we will have discovered a red 4-term AP. However, it is easy to check (as in the second inequality) that there is no sequence of ten gaps that can avoid such a configuration; the unique maximal configurations with nine members are, lexicographically, *aabbabbaa*, *aabbabbab*, *abaabaabb*, *babbabbaa*, *bbaabaabb*, and *bbaabaaba*. Thus χ contains a monochromatic AP of length 4, completing the proof. □

Interestingly, after trying to extend the above ideas, I found paper by T. C. Brown entitled "Is there a sequence on four symbols in which no two adjacent segments are permutations of one another?" in the American Mathematical Monthly from 1971 [3].

Later, a paper appears by F. M. Dekking showing that there is a sequence on two symbols with no four adjacent segments that are permutations of each other, and also that there is a sequence on three symbols with no three adjacent segments that are permutations of one another [6].

Finally, in 1992 Veikko Keränen answered Brown's question in the affirmative [27], and so Proposition 4 above cannot be extended via the methods used in its proof. Now we present our final upper bound on values of $B_m(k, r)$, after a definition and a lemma.

Definition 1. *A k-**cube** H is a set of integers of the form* $a + \{0, d_1\} + \{0, d_2\} + \cdots + \{0, d_k\}$ *with* $d_i \neq d_j$ *for* $i \neq j$*. H is said to be **nondegenerate** if* $|H| = 2^k$ *(this would not occur if, for example,* $a + d_1 + d_2 = a + d_3$*). Finally, we will call H **proper** if*

$$d_j > \sum_{i<j} d_i.$$

Proposition 5 (Specialization of Szemerédi's Cube Lemma)**.** *Among any r-coloring of* $[n]$ *there exists a monochromatic proper K-cube, where*

$$K \geq \log\log n - \log\log r - 1.$$

Proof. Let $A_0 \subset [n]$ be the largest monochromatic subset under our coloring, so that $|A_0| \geq n/r$. We will define A_{k+1} recursively as follows: among all the subsets of A_k, choose the largest $S \subset A_k$ such that S is symmetric with respect to reflection about some a or $a + 1/2$, $a \in [n]$, with $a \notin S$ if S is symmetric about a. Define A_{k+1} to be the first half of S. Provided that $|A_k| \geq 2$, it will always be possible to define A_{k+1} nontrivially. This process must stop; eventually some A_K will have precisely one element, which corresponds to a 2-cube in A_{K-1}, further to a proper 3-cube in A_{K-2}, and so on. We would like to estimate K.

Observe that if $[m]$ is r-colored and there are $\ell \geq 2$ red elements, we can consider reflections about each element of m and also about the points between consecutive elements. Since we care only about pairings of red elements across points of reflection, there are only $2m - 3$ nontrivial reflections: those about $3/2, 2, \ldots, m - 1/2$. If we let $f(x)$ be the total number of pairings produced by reflection about x, then since each pair of red elements contributes exactly 1 pairing,

$$\sum_{i=1}^{2m-3} f(1 + i/2) = \binom{\ell}{2}.$$

In particular, there must be some reflection that results in at least

$$\frac{\binom{\ell}{2}}{2m-3} > \frac{\binom{\ell}{2}}{2m}$$

pairings.

Since A_k lies within the first half of A_{k-1}, A_k is contained in an interval of length at most $\lfloor n/2^k \rfloor$, so there are less than $n2^{1-k}$ nontrivial reflections of A_k, among which $\binom{|A_k|}{2}$ pairings must occur. I claim now that as long as $|A_{k-1}| \geq 2$,

$$|A_k| > \frac{n}{2^{2^k+k-1} r^{2^k}} - \frac{1}{r^k}.$$

Clearly this bound holds for A_0. Suppose it holds for A_k and that $|A_k| \geq 2$; then

$$\begin{aligned}
|A_{k+1}| &> \frac{\binom{|A_k|}{2}}{n2^{1-k}} \\
&> \left(\frac{1}{n2^{2-k}}\right)\left(\frac{n}{2^{2^k+k-1} r^{2^k}} - \frac{1}{r^k}\right)\left(\frac{n}{2^{2^k+k-1} r^{2^k}} - \frac{1+r^k}{r^k}\right) \\
&> \left(\frac{1}{n2^{2-k}}\right)\left(\frac{n^2}{(2^{2^k+k-1} r^{2^k})^2} - \frac{n(r^k+2)}{r^k 2^{2^k+k-1} r^{2^k}}\right) \\
&= \left(\frac{1}{n2^{2-k}}\right)\left(\frac{n^2}{2^{2^{k+1}+2k-2} r^{2^{k+1}}} - \frac{n(r^k+2)}{r^k 2^{2^k+k-1} r^{2^k}}\right) \\
&= \frac{n}{2^{2^{k+1}+k} r^{2^{k+1}}} - \frac{r^k+2}{2^{2^k+1} r^{2^k+k}} \\
&> \frac{n}{2^{2^{k+1}+k} r^{2^{k+1}}} - \frac{4r^k}{2^{2^k+1} r^{2^k+k}} \\
&\geq \frac{n}{2^{2^{k+1}+k} r^{2^{k+1}}} - \frac{1}{r^{2^k}} \\
&\geq \frac{n}{2^{2^{k+1}+k} r^{2^{k+1}}} - \frac{1}{r^{k+1}}.
\end{aligned}$$

Therefore, the result holds by induction, and to estimate K above we simply need to determine how large k can be while still ensuring that $|A_k| \geq 2$. By our bound above, we require only that

$$\frac{n}{2^{2^k+k-1} r^{2^k}} - \frac{1}{r^k} \geq 1,$$

which will hold if

$$\frac{n}{2^{2^k+k-1} r^{2^k}} \geq 2,$$

or, equivalently, if $n \geq 2^{2^k+k} r^{2^k}$. Since

$$n \geq 2^{2^k+k} r^{2^k} \Longleftarrow n \geq r^{2^{k+2}} \Longleftarrow \log\log_r n \geq k+2,$$

it is sufficient that $k \leq \log\log_r n - 2$. Since this inequality is sufficient to imply $|A_k| \geq 2$, and we can proceed one step in our construction beyond this point,

$$K \geq \log\log_r n - 1 = \log\log n - \log\log r - 1.$$

□

Corollary 2. *From the proof above we get the estimate*

$$B_{\lceil \log k \rceil}(k,r) < 2^{2^k+k} r^{2^k}.$$

Of course, the most interesting goal would be to find bounds on the function $B_2(k,r)$, as that is the most basic extension (in this direction) of the van der Waerden numbers.

2.1.2 Arithmetic Progressions in Arbitrary Sets

Another natural way to generalize van der Waerden's theorem is to change the set that is being partitioned, rather than changing the structure that is forbidden (as in the case of multi-arithmetic progressions). In this section we will restrict ourselves to the case of 2 colors, though the ideas here generalize to r colors. Define $w^*(k)$ to be the least integer such that there exists a set $A \subset \mathbb{N}$ with $|A| = w^*(k)$ and with the property that any 2-coloring of A contains a monochromatic k-term arithmetic progression (kAP for brevity). Clearly $w^*(k) \leq w(k,2)$ for all k; in fact, Ron Graham has asked whether $w(k,2) - w^*(k) \to \infty$.

Before we proceed, we need one more definition. Define the function $f(s,t)$ ($s < t$) to be the least integer such that there exists $A \subset \mathbb{N}$ with $|A| = f(s,t)$ and with the properties that any 2-coloring of A contains a monochromatic s-term arithmetic progression, and A contains no t-term arithmetic progression. The function $f(s,t)$ was shown to exist for all $s < t$ by Joel Spencer.

Proposition 6. $f(3,4) = w^*(3) = w(3,2) = 9$.

We have the trivial inequalities $w^*(3) \leq w(3,2)$ and $w^*(3) \leq f(3,4)$, and it is well-known that $w(3,2) = 9$. Throughout this section, let $S := \{1,3,5,9,10,15,17,19,29\}$; this set demonstrates that $f(3,4) \leq 9$. To prove Proposition 6, then, it will suffice to prove that $w^*(3) > 8$. The final steps of the proof require computer calculations, to which we will apply the following lemma:

Lemma 1. *For $A \subset \mathbb{N}$ with $|A| = n$, the number of 3-term arithmetic progressions in A is less than or equal to the number of 3-term APs in $[n]$.*

Proof. Let $A = \{x_1, \ldots, x_n\}$, $x_1 < \cdots < x_n$, and let $P = \{(x_i, x_j, x_k) \in A^3 : i < j < k, x_j - x_i = x_k - x_j\}$ be the set of all 3-term APs in A. Let $P_d = \{(x_i, x_j, x_k) \in P : \min(j - i, k - j) = d\}$. The P_d clearly partition P. If we define Q and Q_d analogously for the set $[n]$, it will suffice to show that $|P_d| \leq |Q_d|$. So fix d and define a function $f : P_d \to Q_d$ as follows. For $(x_i, x_j, x_k) \in P_d$,

$$f((x_i, x_j, x_k)) = \begin{cases} (i, j, j + d) & \text{if } (j - i) = d \leq (k - j), \\ (j - d, j, k) & \text{if } (k - j) = d < (j - i). \end{cases}$$

f is clearly well-defined. To see that f is injective, note that the only preimages of (i, j, k) under f are (x_i, x_j, x_ℓ) for $\ell \geq k$ and (x_h, x_j, x_k) for $h \leq i$. Clearly (x_i, x_j, x_ℓ) a 3-term AP precludes (x_i, x_j, x_m) from being a 3-term AP for all $m \neq \ell$. Thus the only difficulty is if (x_h, x_j, x_k) and (x_i, x_j, x_ℓ) are distinct 3-term APs (i.e., $x_h \neq x_i$). However this cannot happen; it would imply that (x_i, x_j, x_k) is a 3-term AP, a contradiction to $x_h \neq x_i$. Thus we have shown that $|P_d| \leq |Q_d|$ for arbitrary d; it follows that $|P| \leq |Q|$. □

We can give a short proof that $w^*(3) > 6$ by hand.

Proposition 7. $w^*(3) > 6$.

Proof. We will show that for any 6 natural numbers $x_1 < \cdots < x_6$, there is a 2-coloring that avoids a monochromatic 3-term AP. Let $x_1 < \cdots < x_6$ be arbitrary natural numbers, and consider the colorings (where r and b represent red and blue, respectively) *rrrbbb* and *rrbbbr* (that is, in the former coloring x_1 is red, x_2 is red, and so on). Suppose both of these colorings result in a monochromatic 3-term AP. Then among the two pairs $\{(x_1, x_2, x_3), (x_4, x_5, x_6)\}$ and $\{(x_1, x_2, x_6), (x_3, x_4, x_5)\}$, one member of each pair must be in arithmetic progression. Without loss of generality, we may assume that $\{x_1, x_2, x_3\}$ is in progression. Thus $\{x_1, x_2, x_6\}$ cannot be, so $\{x_3, x_4, x_5\}$ must be. Note that this precludes $\{x_3, x_4, x_6\}$ from being in progression, and likewise $\{x_1, x_2, x_5\}$ cannot be in progression, so that the coloring *rrbbrb* avoids monochromatic 3-term APs. □

It is not too hard to show that $w^*(3) > 7$ by hand, though the proof is a lengthy case analysis. We give only part of the proof here.

Proposition 8. $w^*(3) > 7$.

Proof. Let $x_1 < \cdots < x_7$ be arbitrary natural numbers, $A = \{x_1, \ldots, x_7\}$. Suppose by way of contradiction that A cannot be 2-colored to avoid monochromatic 3-term arithmetic progressions. Note that if some x_i is in at most one 3-term AP, then by Proposition 7 we can color the remaining six elements of A to avoid monochromatic 3-term APs, and then color x_i to achieve a coloring of A with no monochromatic 3-term APs. Now if $\{x_1, x_2, x_7\}$ are in progression, then x_1 cannot be involved in any other 3-term APs: if $x_j > x_2$, then $\{x_1, x_j, x_k\}$ being in progression would imply that $x_k > x_7$, a contradiction. Similarly, $\{x_1, x_6, x_7\}$ cannot be a 3-term AP. Therefore, without loss of generality we may assume that neither $\{x_1, x_2, x_7\}$ nor $\{x_1, x_6, x_7\}$ are in arithmetic progression.

Since A cannot be 2-colored to avoid a monochromatic 3-term AP, we give in Table 2.1.2 a list of some colorings, each of which is associated to a list of 3-term APs, one of which must occur among the x_i for that coloring to contain a monochromatic 3-term AP. The colorings are of the form used in the proof of Proposition 7, and each is numbered for later reference. Before we begin the case analysis, suppose that $\{x_i, x_j, x_k\}$ $(i < j < k)$ is in arithmetic progression and observe that certain other triples cannot also be in progression. None of $\{x_{i'}, x_j, x_k\}$ $(i \neq i')$, $\{x_i, x_{j'}, x_k\}$ $(j \neq j')$ or $\{x_i, x_j, x_{k'}\}$ $(k \neq k')$ can be in progression. Nor can $\{x_{i'}, x_{j'}, x_k\}$ for $i' < i$ and $j' > j$ or for $i' > i$ and $j' < j$, $\{x_{i'}, x_j, x_{k'}\}$ for $i' < i$ and $k' < k$ or for $i' > i$ and $k' > k$, or $\{x_i, x_{j'}, x_{k'}\}$ for $j' < j$ and $k' > k$ or for $j' > j$ and $k' < k$. These facts will be used below.

Now we assume, for a later contradiction, that none of the colorings above is without a monochromatic 3-term AP. Considering coloring 10, we have two cases by symmetry: either $\{x_1, x_2, x_6\}$ is an AP, or $\{x_3, x_4, x_5\}$ is. First suppose $\{x_1, x_2, x_6\}$ is an AP. This precludes $\{x_1, x_2, x_3\}$, $\{x_1, x_2, x_4\}$, $\{x_1, x_2, x_5\}$, $\{x_1, x_3, x_4\}$, $\{x_1, x_3, x_5\}$, $\{x_1, x_3, x_6\}$, $\{x_1, x_4, x_5\}$, $\{x_1, x_4, x_6\}$ and $\{x_1, x_5, x_6\}$ from being APs.

Now coloring 12, *rbrrbrb*, implies that either $\{x_3, x_4, x_6\}$ or $\{x_2, x_5, x_7\}$ are in arithmetic progression. If $\{x_3, x_4, x_6\}$ is in progression, then coloring 11 implies that $\{x_2, x_6, x_7\}$ is in progression; now colorings 1 and 2 imply that $\{x_2, x_3, x_4\}$ and $\{x_2, x_3, x_5\}$ are in progression, respectively, which is impossible. The other case (we are still assuming from coloring 10 that $\{x_1, x_2, x_6\}$ is an AP) from coloring 12 is that $\{x_2, x_5, x_7\}$ is an AP. In this case coloring 14 implies that $\{x_3, x_5, x_6\}$ is in progression, and coloring 17 implies that $\{x_4, x_5, x_6\}$ is in progression, again a contradiction.

Table 2.2: Colorings of A and associated progressions that must occur in A to force a monochromatic 3-term AP

Coloring		3-term APs				
1	*rrrrbbb*	$\{x_1, x_2, x_3\}$	$\{x_1, x_2, x_4\}$	$\{x_1, x_3, x_4\}$	$\{x_2, x_3, x_4\}$	$\{x_5, x_6, x_7\}$
2	*rrrbrbb*	$\{x_1, x_2, x_3\}$	$\{x_1, x_2, x_5\}$	$\{x_1, x_3, x_5\}$	$\{x_2, x_3, x_5\}$	$\{x_4, x_6, x_7\}$
3	*rrrbbrb*	$\{x_1, x_2, x_3\}$	$\{x_1, x_2, x_6\}$	$\{x_1, x_3, x_6\}$	$\{x_2, x_3, x_6\}$	$\{x_4, x_5, x_7\}$
4	*rrrbbbr*	$\{x_1, x_2, x_3\}$	$\{x_1, x_3, x_7\}$	$\{x_2, x_3, x_7\}$	$\{x_4, x_5, x_6\}$	
5	*rrbrrbb*	$\{x_1, x_2, x_4\}$	$\{x_1, x_2, x_5\}$	$\{x_1, x_4, x_5\}$	$\{x_2, x_4, x_5\}$	$\{x_3, x_6, x_7\}$
6	*rrbrbrb*	$\{x_1, x_2, x_4\}$	$\{x_1, x_2, x_6\}$	$\{x_1, x_4, x_6\}$	$\{x_2, x_4, x_6\}$	$\{x_3, x_5, x_7\}$
7	*rrbrbbr*	$\{x_1, x_2, x_4\}$	$\{x_1, x_4, x_7\}$	$\{x_2, x_4, x_7\}$	$\{x_3, x_5, x_6\}$	
8	*rrbbrrb*	$\{x_1, x_2, x_5\}$	$\{x_1, x_2, x_6\}$	$\{x_1, x_5, x_6\}$	$\{x_2, x_5, x_6\}$	$\{x_3, x_4, x_7\}$
9	*rrbbrbr*	$\{x_1, x_2, x_5\}$	$\{x_1, x_5, x_7\}$	$\{x_2, x_5, x_7\}$	$\{x_3, x_4, x_6\}$	
10	*rrbbbrr*	$\{x_1, x_2, x_6\}$	$\{x_2, x_6, x_7\}$	$\{x_3, x_4, x_5\}$		
11	*rbrrrbb*	$\{x_1, x_3, x_4\}$	$\{x_1, x_3, x_5\}$	$\{x_1, x_4, x_5\}$	$\{x_2, x_6, x_7\}$	$\{x_3, x_4, x_5\}$
12	*rbrrbrb*	$\{x_1, x_3, x_4\}$	$\{x_1, x_3, x_6\}$	$\{x_1, x_4, x_6\}$	$\{x_2, x_5, x_7\}$	$\{x_3, x_4, x_6\}$
13	*rbrrbbr*	$\{x_1, x_3, x_4\}$	$\{x_1, x_3, x_7\}$	$\{x_1, x_4, x_7\}$	$\{x_2, x_5, x_6\}$	$\{x_3, x_4, x_7\}$
14	*rbrbrrb*	$\{x_1, x_3, x_5\}$	$\{x_1, x_3, x_6\}$	$\{x_1, x_5, x_6\}$	$\{x_2, x_4, x_7\}$	$\{x_3, x_5, x_6\}$
15	*rbrbrbr*	$\{x_1, x_3, x_5\}$	$\{x_1, x_3, x_7\}$	$\{x_1, x_5, x_7\}$	$\{x_2, x_4, x_6\}$	$\{x_3, x_5, x_7\}$
16	*rbrbbrr*	$\{x_1, x_3, x_6\}$	$\{x_1, x_3, x_7\}$	$\{x_2, x_4, x_5\}$	$\{x_3, x_6, x_7\}$	
17	*rbbrrrb*	$\{x_1, x_4, x_5\}$	$\{x_1, x_4, x_6\}$	$\{x_1, x_5, x_6\}$	$\{x_2, x_3, x_7\}$	$\{x_4, x_5, x_6\}$

Above we handled the case (concerning coloring 10) of $\{x_1, x_2, x_6\}$ or $\{x_2, x_6, x_7\}$ being an AP; for the remaining case, we suppose neither of these is, and so $\{x_3, x_4, x_5\}$ is an AP (because we are assuming coloring 10 is not without a monochromatic 3-term AP. Since $\{x_3, x_4, x_5\}$ is an AP, none of $\{x_1, x_4, x_5\}$, $\{x_2, x_4, x_5\}$, $\{x_3, x_4, x_6\}$, or $\{x_3, x_4, x_7\}$ is. We omit the details of the case analysis, for they are longer but identical in kind to the previous case.

Since in all cases we reach a contradiction, it must be that some coloring avoids a monochromatic 3-term arithmetic progression, regardless of the original set A. □

In fact, by precisely the methods above (and a longer list of colorings), one can show that:

Proposition 9. $w^*(3) > 8$.

This last result was achieved with the help of computer calculations. Proposition 9 directly implies Proposition 6.

2.2 The Hales-Jewett Number HJ(3,2)

The Hales-Jewett Theorem [23] says that for any positive integers k and r there exists $n = HJ(k, r)$ such that whenever the set of length n words over a k-letter alphabet are r-colored, there must exist a monochromatic line. Here perhaps a bit of explanation is in order. A *word* over an *alphabet* (= set) A is a finite sequence in A, and the *length* of the word is the number of terms in the sequence. For our purposes, the informal view of a sequence as terms listed in order will do, so that, for example, 1323 is a length 4 word over the alphabet $\{1, 2, 3\}$ (and also over the alphabet $\{1, 2, 3, 4, 5\}$ for that matter). A *variable word* over A is a word over $A \cup \{v\}$ in which v occurs, where v is a "variable" which is not in A. If $w = w(v)$ is a variable word over A and $a \in A$, then $w(a)$ is the word in which all occurrences of v are replaced by a. Thus, for example, if $w(v) = 1v3v$, then $w(1) = 1131$ and $w(2) = 1232$. A *combinatorial line* over A is $\{w(a) : a \in A\}$ where $w(v)$ is a variable word over A. Again, if $A = \{1, 2, 3\}$, then $\{1131, 1232, 1333\}$ is the combinatorial line determined by $w(v) = 1v3v$.

A substantial amount of effort has been invested in finding the value of the smallest n which "works" for particular instances of Schur's Theorem, van der Waerden's Theorem, and Ramsey's Theorem. For example, the smallest n guaranteeing a monochromatic length k arithmetic progression when $[n]$ is 2-colored are respectively 9, 35, and 178 for $k = 3$, $k = 4$, and $k = 5$. See [20, Chapter 4] and [31] for substantial information about known specific values of van der Waerden numbers, Schur numbers, and Ramsey numbers.

The original proofs of these theorems produced exceedingly large upper bounds for n (except for Schur's Theorem, where the original proof shows that $n = \lfloor r!e \rfloor$ will do). The easiest way to prove Ramsey's Theorem and the Hales-Jewett theorem is to prove the infinite versions. One then deduces the finite versions, but this method yields no upper bounds at all. Twenty years ago there was a great deal of excitement when Shelah showed [34] that there are upper bounds for the van der Waerden and Hales-Jewett numbers that are primitive recursive. See [20] for a detailed discussion of the Hales-Jewett theorem and also of the proof by Shelah.

Uniquely among the classical theorems mentioned above, no nontrivial values of $HJ(k, r)$ had been known. It's clear that $HJ(k, 1) = 1$ for any k, and that $HJ(2, r) = r$

is not hard to prove. (If w is a word of length l over the alphabet $\{1,2\}$ and $\varphi(w)$ is the number of 1's occurring in w, then there is no monochromatic combinatorial line and so $HJ(2,r) \geq r$. If $w_i = a_{i,1}a_{i,2}\cdots a_{i,l}$ where $a_{i,t} = 2$ if $t < i$ and $a_{i,t} = 1$ if $t \geq i$, then whenever $i \neq j$, $\{w_i, w_j\}$ is a combinatorial line, and so $HJ(2,r) \leq r$.) The first nontrivial value of HJ, then, is $HJ(3,2)$, which we show here, in Section 2, to be 4. In Section 3 we present an algorithm which we used to determine that HJ(3,2)=4 before the detailed proof of Section 2 was found and present some lower bounds for other Hales-Jewett numbers obtained using that algorithm.

2.2.1 HJ(3,2)=4

This section is devoted entirely to a proof of the following theorem.

Theorem 11. *Let the length four words on the alphabet* $\{1,2,3\}$ *be two colored. Then there exists a monochromatic combinatorial line.*

Proof. Suppose instead that we have a 2-coloring of the 4-letter words over $\{1,2,3\}$ with respect to which there is no monochromatic combinatorial line. Let A be the set of words with the first color and let B be the set of words with the second color. Now $\{1111, 2222, 3333\}$ is a combinatorial line, so we may assume without loss of generality that $1111 \in A$, $2222 \in A$, and $3333 \in B$.

The proof now proceeds through four lemmas. In the proofs of these lemmas, we shall follow the customary abuse of notation wherein we substitute "$P \Rightarrow Q$" for the instance of modus ponens which should say "($P \Rightarrow Q$) and P, therefore Q".

Lemma 2. *If* $\{2111, 1211\} \subseteq A$*, then* $2211 \in B$.

Proof. Suppose instead that $\{2111, 1211, 2211\} \subseteq A$.

$$\begin{aligned} 1111 \in A \text{ and } 2211 \in A &\Rightarrow 3311 \in B. \\ 1211 \in A \text{ and } 2211 \in A &\Rightarrow 3211 \in B. \\ 1111 \in A \text{ and } 2111 \in A &\Rightarrow 3111 \in B. \end{aligned}$$

But $\{3311, 3211, 3111\}$ is a combinatorial line. □

Lemma 3. *It is not the case that* $\{1112, 1121, 1211, 2111\} \subseteq A$.

Proof. Suppose that $\{1112, 1121, 1211, 2111\} \subseteq A$.

$$\begin{array}{rcl}
1112 \in A \text{ and } 2222 \in A & \Rightarrow & 3332 \in B. \\
3332 \in B \text{ and } 3333 \in B & \Rightarrow & 3331 \in A. \\
3331 \in A \text{ and } 1111 \in A & \Rightarrow & 2221 \in B. \\
1111 \in A \text{ and } 1121 \in A & \Rightarrow & 1131 \in B. \\
1131 \in B \text{ and } 3333 \in B & \Rightarrow & 2232 \in A. \\
\text{Lemma 2} & \Rightarrow & 2211 \in B. \\
2221 \in B \text{ and } 2211 \in B & \Rightarrow & 2231 \in A. \\
2111 \in A \text{ and } 2222 \in A & \Rightarrow & 2333 \in B. \\
2232 \in A \text{ and } 2231 \in A & \Rightarrow & 2233 \in B. \\
1211 \in A \text{ and } 2222 \in A & \Rightarrow & 3233 \in B. \\
3233 \in B \text{ and } 3333 \in B & \Rightarrow & 3133 \in A. \\
2333 \in B \text{ and } 2233 \in B & \Rightarrow & 2133 \in A. \\
2233 \in B \text{ and } 3333 \in B & \Rightarrow & 1133 \in A.
\end{array}$$

But $\{1133, 2133, 3133\}$ is a combinatorial line. □

Lemma 4. *It is not the case that some two of* 1112*,* 1121*,* 1211*, and* 2111 *are in* A*.*

Proof. Suppose instead without loss of generality that $\{1211, 2111\} \subseteq A$. By Lemma 3 we can assume without loss of generality that $1112 \in B$.

$$\begin{array}{rcl}
2222 \in A \text{ and } 1211 \in A & \Rightarrow & 3233 \in B. \\
3333 \in B \text{ and } 3233 \in B & \Rightarrow & 3133 \in A. \\
1111 \in A \text{ and } 3133 \in A & \Rightarrow & 2122 \in B. \\
2122 \in B \text{ and } 1112 \in B & \Rightarrow & 3132 \in A. \\
3132 \in A \text{ and } 3133 \in A & \Rightarrow & 3131 \in B. \\
3131 \in B \text{ and } 3333 \in B & \Rightarrow & 3232 \in A. \\
1111 \in A \text{ and } 2111 \in A & \Rightarrow & 3111 \in B. \\
3232 \in A \text{ and } 2222 \in A & \Rightarrow & 1212 \in B. \\
3111 \in B \text{ and } 3333 \in B & \Rightarrow & 3222 \in A. \\
3232 \in A \text{ and } 3222 \in A & \Rightarrow & 3212 \in B. \\
3111 \in B \text{ and } 3212 \in B & \Rightarrow & 3313 \in A. \\
1212 \in B \text{ and } 3212 \in B & \Rightarrow & 2212 \in A.
\end{array}$$

But $\{1111, 2212, 3313\}$ is a combinatorial line. □

Lemma 5. $\{1112, 1121, 1211, 2111, 2221, 2212, 2122, 1222\} \subseteq B$*.*

Proof. Suppose not. We have not distinguished between 2 and 1 so we may assume without loss of generality that $2111 \in A$. We have that $\{1211, 1121, 1112\} \subseteq B$ by Lemma 4.

$$
\begin{array}{lcl}
1111 \in A \text{ and } 2111 \in A & \Rightarrow & 3111 \in B. \\
3111 \in B \text{ and } 3333 \in B & \Rightarrow & 3222 \in A. \\
2111 \in A \text{ and } 2222 \in A & \Rightarrow & 2333 \in B. \\
3222 \in A \text{ and } 2222 \in A & \Rightarrow & 1222 \in B. \\
2333 \in B \text{ and } 3333 \in B & \Rightarrow & 1333 \in A. \\
1222 \in B \text{ and } 1211 \in B & \Rightarrow & 1233 \in A. \\
1233 \in A \text{ and } 1333 \in A & \Rightarrow & 1133 \in B. \\
1133 \in B \text{ and } 3333 \in B & \Rightarrow & 2233 \in A. \\
1222 \in B \text{ and } 1112 \in B & \Rightarrow & 1332 \in A. \\
1332 \in A \text{ and } 1333 \in A & \Rightarrow & 1331 \in B. \\
1331 \in B \text{ and } 3333 \in B & \Rightarrow & 2332 \in A. \\
2332 \in A \text{ and } 1332 \in A & \Rightarrow & 3332 \in B. \\
2233 \in A \text{ and } 1233 \in A & \Rightarrow & 3233 \in B. \\
2332 \in A \text{ and } 2222 \in A & \Rightarrow & 2112 \in B. \\
3233 \in B \text{ and } 3333 \in B & \Rightarrow & 3133 \in A. \\
3133 \in A \text{ and } 1111 \in A & \Rightarrow & 2122 \in B. \\
2233 \in A \text{ and } 2222 \in A & \Rightarrow & 2211 \in B. \\
3332 \in B \text{ and } 3333 \in B & \Rightarrow & 3331 \in A. \\
3331 \in A \text{ and } 1111 \in A & \Rightarrow & 2221 \in B. \\
2122 \in B \text{ and } 2112 \in B & \Rightarrow & 2132 \in A. \\
2221 \in B \text{ and } 2211 \in B & \Rightarrow & 2231 \in A. \\
2221 \in B \text{ and } 1211 \in B & \Rightarrow & 3231 \in A. \\
2132 \in A \text{ and } 3133 \in A & \Rightarrow & 1131 \in B. \\
3231 \in A \text{ and } 2231 \in A & \Rightarrow & 1231 \in B.
\end{array}
$$

But $\{1131, 1231, 1331\}$ is a combinatorial line. □

We are now ready to conclude the proof of Theorem 11.

We have by Lemma 5 that $\{1112, 1121, 1211, 2111, 2221, 2212, 2122, 1222\} \subseteq B$ and we have not distinguished between 1 and 2. (We distinguished between 1 and 2 in the proof of Lemma 5, but that distinction has disappeared.) Since $\{3331, 3332, 3333\}$ is a combinatorial line, we may assume without loss of generality that $3331 \in A$.

We have that all words with three 1's and one 2 are in B and all words with three 2's and one 1 are in B, so all words with two 3's, one 1, and one 2 are in A. (To see for example that $3132 \in A$, use the fact that $2122 \in B$ and $1112 \in B$.)

$$\begin{aligned}
3331 \in A \text{ and } 3321 \in A &\Rightarrow 3311 \in B.\\
3331 \in A \text{ and } 2331 \in A &\Rightarrow 1331 \in B.\\
1331 \in B \text{ and } 3333 \in B &\Rightarrow 2332 \in A.\\
3311 \in B \text{ and } 3333 \in B &\Rightarrow 3322 \in A.\\
2332 \in A \text{ and } 2222 \in A &\Rightarrow 2112 \in B.\\
3322 \in A \text{ and } 2222 \in A &\Rightarrow 1122 \in B.\\
2112 \in B \text{ and } 2122 \in B &\Rightarrow 2132 \in A.\\
1112 \in B \text{ and } 1122 \in B &\Rightarrow 1132 \in A.
\end{aligned}$$

But $\{1132, 2132, 3132\}$ is a combinatorial line. □

2.2.2 An Algorithm

Another method of proving that $HJ(3,2) = 4$ requires a computer (or some months of free time), but is very elementary, and gives a reasonable idea for obtaining constructive lower bounds on other Hales-Jewett numbers. Owing to the extremely large upper bound, of course, it is possible that any constructive lower bound is still well short of the mark.

The algorithm is quite simple (and can easily be generalized, but we will use $k = 3$ and $r = 2$ here for clarity). First, one enumerates and stores the 2-colorings of the length 1 words (here and below, over the alphabet $\{1, 2, 3\}$) that avoid a monochromatic line (the "good" colorings); these are the 6 nonconstant colorings.

Now we make the simple observation that in any good 2-coloring of the length-2 words, each set of the form $\{1x, 2x, 3x\}$ with $x \in [3]$ must correspond to one of the 6 good colorings of $[3]^1$, or else that set comprises a monochromatic line. Using this fact, we can examine all of the possibly good colorings of the length 2 words by considering 6^3 possibilities instead of all $2^9 = 8^3$ colorings. The good colorings are stored – it turns out that there are 66 of them.

In any possible good 2-coloring of the words of length 3, each set of the form $\{11x$, $12x$, $13x$, $21x$, $22x$, $23x$, $31x$, $32x$, $33x\}$ with $x \in [3]$ must have one of the 66 colorings mentioned above. In searching the colorings of the length 3 words, this lets us examine just 66^3 possibilities instead of $2^{27} = 512^3$. Of the 66^3 we examine, we find 1644 good colorings, which are stored as before.

Repeating this process, in the 1644^3 possible good colorings of the length 4 words, we find in each case a monochromatic line. Thus, $HJ(3,2) = 4$. Note that in this last step, we have a search space of $1644^3 \approx 2^{32}$ instead of one with size 2^{81}.

This algorithm can be modified to produce lower bounds: for instance, though it's not practical to enumerate and store all of the good 2-colorings of $[4]^5$, a list of some known good colorings can still prove computationally useful. Using such a list, together with a simple simulated annealing algorithm (see [29] for a description of simulated annealing), we have easily obtained the bounds $HJ(4, 2) > 6$ and $HJ(3, 3) > 6$; the colorings proving these lower bounds are given in Appendix A. Note that even if, for example, $HJ(3, 3) = 7$, to prove this one would have to certify that each of the 3^{3^7} potential 3-colorings of $[3]^7$ contains a monochromatic line. This is a search space too large for the methods of this section to approach.

2.2.3 Acknowledgement

This chapter is based on the paper "The first nontrivial Hales-Jewett number is four," written by the author together with Neil Hindman.

Chapter 3

Euclidean Ramsey Theory

3.1 Nondegenerate triangles in the plane

In 1973, Erdős, Graham, Montgomery, Rothschild, Spencer, and Straus published the three seminal papers of Euclidean Ramsey theory ([8], [9], [10]). In the third of these, they consider this question: if the points in $\mathbb{E}^2$ are partitioned into two sets – say, red and blue – then which sets must occur monochromatically (that is, in one of the parts), and which can be avoided?

More formally, for a finite set $X \subset \mathbb{E}^2$, let *Cong(X)* be the set of all subsets of $\mathbb{E}^2$ which are congruent to X under some Euclidean motion (including reflection). Fixing a finite set $X \subset \mathbb{E}^2$, consider the set of all maps $\chi : \mathbb{E}^2 \to \{red, blue\}$. If in every case there is some X' in $\chi^{-1}(red)$ or in $\chi^{-1}(blue)$ with $X' \in Cong(X)$, we say that X cannot be avoided by two colors – there is always a monochromatic copy of X, regardless of the coloring χ. This notion extends in the obvious way to more than two colors.

It is easy to see that if X consists of two points, then we cannot avoid it with two colors: let d be the distance between the two points, and try to 2-color the vertices of any equilateral triangle of side d. In [10] the authors show that if X is an equilateral triangle of side d (by a triangle, we mean the set of its vertices), then it can be avoided, by coloring the plane with alternating horizontal red and blue strips of width $\sqrt{3}d/2$, each half-open at the top. There are various triangles that are known to be impossible to avoid with two colors; a list of some families of these is given in [10], and L. Shader has shown in [36] that all right triangles also belong on this list.

Conjecture 2. *[10] For any non-equilateral triangle T, every 2-coloring of $\mathbb{E}^2$ contains a monochromatic copy of T.*

This is still open, and we make no direct progress toward Conjecture 2 here. Instead, we note that in [36], as a lemma to the main result, we have:

Lemma 6. *For any real number a and 2-coloring of the plane, there is a monochromatic equilateral triangle of side ka, for some $k \in \{1, 3, 5, 7\}$.*

As a special case of Theorem 9 in [8], we have

Theorem 12. *If T is a set of three noncollinear points and χ is any 2-coloring of $\mathbb{E}^2$, then χ contains a monochromatic congruent copy of T, $2T$, or $\sqrt{3}T$ (where kT is just the triangle T scaled by a factor of k).*

Here we present a similar result.

3.1.1 The main result

Theorem 13. *If T is a set of three noncollinear points and χ is any 2-coloring of $\mathbb{E}^2$, then χ contains a monochromatic* **translate** *of T, $2T$, $3T$, or $4T$.*

Proof. Consider the triangle $4T$ built from copies of T, as in Figure 3.1. Note that this orientation was chosen to facilitate the proof, and below we will refer to the "top" vertex, etc., casually; of course T need not actually be oriented this way.

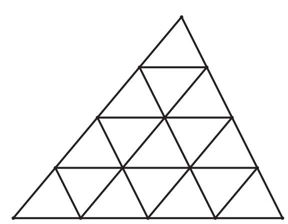

Figure 3.1: The triangle $4T$ formed from T.

Suppose by way of contradiction that we can color the 15 vertices of this diagram without producing a monochromatic T, $2T$, $3T$, or $4T$. Then the outermost vertices cannot be the same color (our two colors here will be black and white). Without loss of generality, color the top and leftmost vertices black, and the rightmost white. This leads us to Figure 3.2, which includes vertex labels that we will use below.

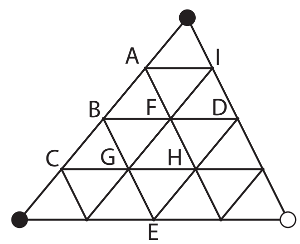

Figure 3.2: Coloring the outermost vertices.

Note that in Figure 3.2, vertex B must be white, otherwise vertices D and E would both be forced to be white, producing a white $2T$. Vertices A, B, and C cannot all be white, because then F and G would be forced to be black, and it would be impossible to color H. Note that this logic applies to any three consecutive vertices.

Thus, one of A and C is black; by symmetry, we may arbitrarily choose A. This forces I to be white, leaving us at Figure 3.3.

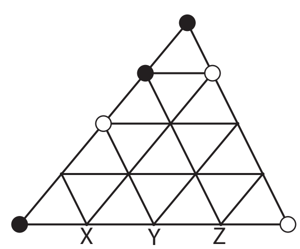

Figure 3.3: After coloring some more vertices.

Now to avoid a monochromatic $3T$, vertex X must be black and vertex Z must be white. It is now impossible to color vertex Y without producing three consecutive like-colored vertices, so the proof is complete. □

This leads to another result if we consider congruence instead of simply translation.

Corollary 3. *If T is a set of three noncollinear points and χ is any 2-coloring of $\mathbb{E}^2$, then χ contains a monochromatic congruent copy of T, $2T$, or $3T$.*

Proof. Fix χ and suppose there is no monochromatic congruent copy of T, $2T$, or $3T$. Then if the triangular lattice from Theorem 13 is placed onto the plane in any position, in any orientation, the outermost vertices will be monochromatic (otherwise that lattice would be colored to avoid any monochromatic T, $2T$, $3T$, or $4T$, a contradiction to Theorem 13). This easily implies that the whole plane is monochromatic, a contradiction. □

3.1.2 Conclusion

While the results above do not lead to any new forced monochromatic triangles among 2-colorings, observe this theorem in [10]:

Theorem 14. *Fix a 2-coloring of $\mathbb{E}^2$ and let T be a triangle with sides a, b, and c. Then T occurs monochromatically if and only if some equilateral triangle with side a, b, or c occurs monochromatically.*

Conjecture 2 is therefore equivalent to:

Conjecture 3. *Fix a 2-coloring of $\mathbb{E}^2$ and let T and T' be equilateral triangles with side lengths $d \neq d'$, respectively. Then at least one of T, T' occurs monochromatically.*

This is much stronger than any of the results above; in each of those conditional results, a list of three or more similar triangles is given, one of which must occur monochromatically. An intermediate problem would be to prove a conditional result like the ones above with a list of just two similar triangles; as far as we know, this has not been done even in the case of equilateral triangles.

3.2 Degenerate triangles in the plane

In Section 3.1 we discussed proper triangles in the plane; here we consider the case of degenerate triangles – that is, sets of three collinear points. In this section, an (a, b, c) triangle will refer to a triangle with side lengths a, b, and c (and as above, when we refer to a triangle in the plane, we really mean the set of its vertices).

For any collinear set S of 3 points, it is known that with 16 colors one can avoid a monochromatic copy of S in $\mathbb{E}^n$ for all n ([38]), but it is an open question if this is the best possible. Figure 3.4 shows that in the plane, it is possible to avoid the $(a, a, 2a)$ degenerate triangle with only 3 colors. This tiling extends to cover $\mathbb{E}^2$; each hexagon has diameter $2a$ and all of the hexagons are half-open as shown for the uppermost hexagon in Figure 3.4.

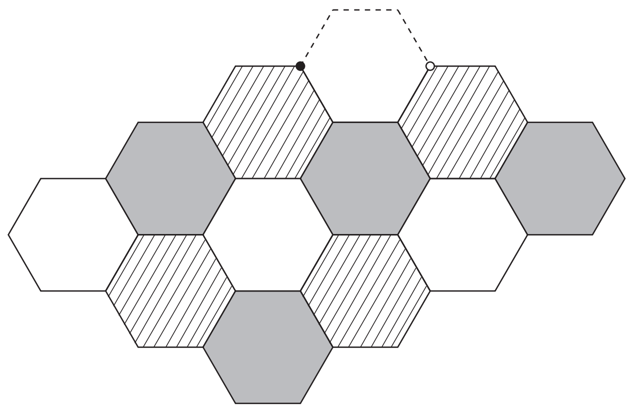

Figure 3.4: A sketch of the 3-coloring avoiding the $(a, a, 2a)$ triangle.

Proposition 10. *If χ is a 2-coloring of $\mathbb{E}^2$ that contains a monochromatic copy of the (a, a, a) triangle, then for any $b > 0$, χ also contains a monochromatic copy of the degenerate $(a, b, a + b)$ triangle.*

Proof. Let χ be a 2-coloring of $\mathbb{E}^2$ in the colors black and white, and suppose the three vertices of an (a, a, a) triangle in the plane are monochromatic, as in Figure 3.5 (all acute angles are $\pi/3$). Suppose by way of contradiction that we can avoid a monochromatic $(a, b, a+b)$ triangle. In the diagram, vertices A and B must then be colored white, forcing vertex C to be colored black. Then vertex E must be colored white. Since both E and B are white, it is impossible to color vertex D either black or white without producing a monochromatic $(a, b, a + b)$ triangle, thus completing the proof.

□

Proposition 11. *If χ is a 2-coloring of $\mathbb{E}^2$, and for some $a, b > 0$, χ contains a monochromatic copy of the $(a + b, a + b, a + b)$ triangle, χ also contains a monochromatic copy of the degenerate $(a, b, a + b)$ triangle.*

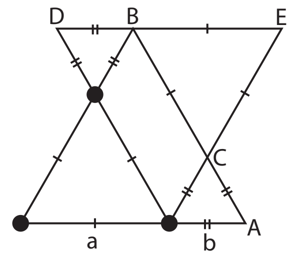

Figure 3.5: Sketch of the proof of Proposition 10.

Proof. Fix $a, b > 0$, and let χ be a 2-coloring of $\mathbb{E}^2$ in the colors black and white such that there is a monochromatic $(a+b, a+b, a+b)$ triangle, as in Figure 3.6 (again, all acute angles are $\pi/3$). Suppose by way of contradiction that we can avoid a monochromatic $(a, b, a+b)$ triangle. In the diagram, vertices A, B and C must be colored white, forcing vertex E to be colored black. Now, as in Proposition 10, it is impossible to color vertex D without producing a monochromatic $(a, b, a+b)$ triangle.

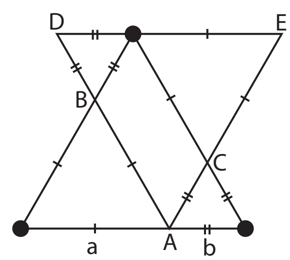

Figure 3.6: Sketch of the proof of Proposition 11.

□

3.3 Acknowledgement

Sections 3.1 and 3.2 of this chapter are based on the paper "Monochromatic triangles in $\mathbb{E}^2$," written by the author.

Chapter 4

Other Topics

4.1 An Intersection Theorem about Domino Tilings

A typical problem in extremal set theory is to give conditions that a family of sets must satisfy, and then ask what is the maximal size of a family of sets which can be formed satisfying these conditions. One simple example is to insist that every two pairs of sets in your family intersect at least ℓ times, or in other words the family of sets is ℓ-intersecting. One of the most celebrated results in extremal set theory looks at the maximal size of an ℓ-intersecting family.

Theorem 15 (Erdős-Ko-Rado [11])**.** *Let $\mathcal{F}$ be an ℓ-intersecting family of sets, with each element A_i a k-element subset of $\{1, \ldots, n\}$. Then for $n \geq (k - \ell + 1)(\ell + 1)$*

$$|\mathcal{F}| \leq \binom{n-\ell}{k-\ell}.$$

In the original statement of the proof this was shown to hold for $n \geq n_0(k, \ell)$. Frankl [15] established the above bound for $\ell \geq 15$ and then Wilson [42] established the bound in general. Taking all k element sets containing $\{1, \ldots, \ell\}$ forms an ℓ-intersecting family of size $\binom{n-k}{k-\ell}$. Theorem 15 then says that this is essentially best possible, in other words you cannot be more clever than doing the obvious thing.

This result has been generalized to other combinatorial objects which share a notion of intersection. The type of objects that have previously been studied include permutations [7], set partitions [28], colored sets [2], arithmetic progressions [14], strings [16], and vector spaces [17]. In this note we will consider a new type of intersection problem, namely the intersection of tilings.

A tiling consists of covering a board using tile pieces from a given set so that the board is completely covered and no two tiles overlap (for more about tilings we recommend the excellent survey paper by Ardila and Stanley [1]). We say that two tilings of the board intersect if there is a tile placed in the same position on both boards. For example, Figure 4.1 shows two tilings of a 4×5 board using dominoes. The shaded tile is placed the same way in both tilings so these intersect.

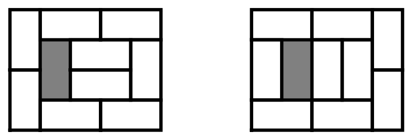

Figure 4.1: An example of intersecting tilings.

In this note we will find the maximal size of families of intersecting tilings for the cases of tiling the $2{\times}n$ strip (Section 4.1.1) and the $3{\times}(2n)$ strip (Section 4.1.2) by using dominoes.

4.1.1 Tilings of $2{\times}n$ using dominoes

It is well known that the number of tilings of the $2{\times}n$ strip using dominoes is $F(n+1)$ where $F(n)$ are the well known Fibonacci numbers, $F(1) = F(2) = 1$ and $F(n) = F(n-1) + F(n-2)$ (this is sequence `A000045` in the `OEIS` [35]).

Theorem 16. *Let $\mathcal{T}$ be an intersecting family of tilings of the $2{\times}n$ strip using dominoes. Then $|\mathcal{T}| \le F(n)$.*

Proof. We first note that by taking all the tilings of the $2{\times}n$ strip that begin with a vertical domino we have an intersecting family of size $F(n)$. So it remains to show that this cannot be improved upon.

Consider the graph which is formed by taking all possible tilings and putting an edge between two tilings it they do **not** intersect. The problem of finding a maximal intersecting family is equivalent to finding a maximal independent set in this graph. We can split the vertices into two sets $\mathcal{H}$ and $\mathcal{V}$. Where $\mathcal{H}$ is the $F(n-1)$ tilings that start with two horizontal tiles and $\mathcal{V}$ is the $F(n)$ tilings that start with a vertical tile. By definition, all edges in the graph are between $\mathcal{H}$ and $\mathcal{V}$ (i.e., the graph is bipartite).

We claim that there is a matching between $\mathcal{H}$ and a subset of $\mathcal{V}$. To see this, suppose that we have a tiling T in $\mathcal{H}$. Then we can decompose this tiling into a sequence of blocks where a block consists of two horizontal tiles followed by any number of vertical tiles. We now map $T \to S$ block by block using the rule shown in Figure 4.2. For any

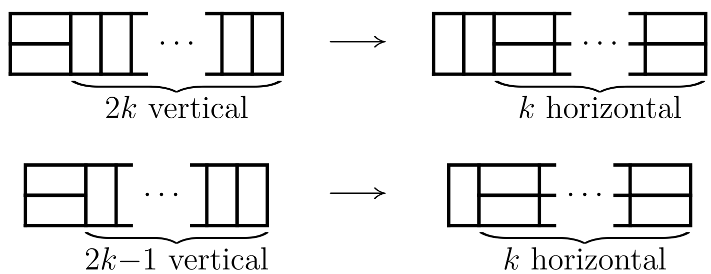

Figure 4.2: The rule for forming the matching between $\mathcal{H}$ and $\mathcal{V}$.

$T \in \mathcal{H}$ the resulting S will start with a vertical tile and so is in $\mathcal{V}$, further block by block it can be seen that S and T have no common tile, so there is an edge between S and T. Finally it is easy to check that this map is 1-to-1, so gives our desired matching.

Since there is a matching from every element of $\mathcal{H}$ to an element of $\mathcal{V}$ it follows that for any subset $\mathcal{Q}$ of $\mathcal{H}$ that the number of elements in $\mathcal{V}$ adjacent to $\mathcal{Q}$ has size at least $|\mathcal{Q}|$. (This is the rarely used direction of Hall's Marriage Theorem.) Now suppose that $\mathcal{T}$ is an intersecting family and let $\mathcal{Q} = \mathcal{T} \cap \mathcal{H}$ and $\mathcal{R} = \mathcal{T} \cap \mathcal{V}$. Since the elements of $\mathcal{R}$ cannot be adjacent to elements of $\mathcal{Q}$ the above comment implies that $|\mathcal{R}| \le |\mathcal{V}| - |\mathcal{Q}|$. So we have

$$|\mathcal{T}| = |\mathcal{Q}| + |\mathcal{R}| \le |\mathcal{Q}| + (|\mathcal{V}| - |\mathcal{Q}|) = |\mathcal{V}| = F(n). \qquad \square$$

4.1.2 Tilings of $3\times(2n)$ using dominoes

We now turn to tilings of the $3\times(2n)$ board. We first count the number of such tilings (this has been done previously and is `A001835` in the `OEIS` [35]). A commonly used approach is to set up a system of linear recurrences and then solve the system, we will do a variation where we count the number of weighted walks in a small graph.

The basic idea is to break the 3×(2n) strip into n small blocks of size 3×2, and consider how horizontal dominoes can intersect the break between consecutive blocks. Since the area of each block is even, it follows that in the breaks we must have an even number of horizontal dominoes. This gives the four possibilities shown in Figure 4.3, the fourth of which cannot happen in a tiling of 3×(2n), we will refer to the remaining possibilities, from left to right, as 工, 古 and 旦.

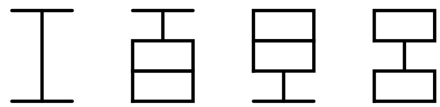

Figure 4.3: The different configuration of horizontal dominoes between blocks.

To count the total number of tilings we can take all possible configurations of horizontal dominoes in the breaks and then count the ways to fill in the remaining untiled portion of the strip. We can do this by using weighted walks in a small directed graph where the vertex set is {工, 古, 旦} and the weight of an edge is the number of ways to fill in the unused area of a block between the two column breaks indicated. For instance there are 3 ways to fill in a 3×2 strip so there is a loop of weight 3 for the edge 工工. Similarly, edges 古古, 旦旦, 工古, 古工, 工旦 and 旦工 have weight 1 since there is only one way to fill in the block, while 古旦 and 旦古 have weight 0 since there is no way to fill in the uncovered area using dominoes. This gives us the following adjacency matrix for the graph.

$$A = \begin{matrix} & \text{工} & \text{古} & \text{旦} \\ \text{工} & 3 & 1 & 1 \\ \text{古} & 1 & 1 & 0 \\ \text{旦} & 1 & 0 & 1 \end{matrix}$$

Since the left and right sides of the 3×(2n) board correspond to 工 we need to find the sum of the weight of walks of length n in the graph that start and end at 工. This is equivalent to finding the $(1,1)$ entry of A^n. The eigenvalues of A are $2+\sqrt{3}$, $2-\sqrt{3}$ and

1, using these along with their eigenvectors to form projection matrices we have

$$A^n = (2+\sqrt{3})^n \begin{pmatrix} \frac{3+\sqrt{3}}{6} & \frac{\sqrt{3}}{6} & \frac{\sqrt{3}}{6} \\ \frac{\sqrt{3}}{6} & \frac{3-\sqrt{3}}{12} & \frac{3-\sqrt{3}}{12} \\ \frac{\sqrt{3}}{6} & \frac{3-\sqrt{3}}{12} & \frac{3-\sqrt{3}}{12} \end{pmatrix} + (2-\sqrt{3})^n \begin{pmatrix} \frac{3-\sqrt{3}}{6} & -\frac{\sqrt{3}}{6} & -\frac{\sqrt{3}}{6} \\ -\frac{\sqrt{3}}{6} & \frac{3+\sqrt{3}}{12} & \frac{3+\sqrt{3}}{12} \\ -\frac{\sqrt{3}}{6} & \frac{3+\sqrt{3}}{12} & \frac{3+\sqrt{3}}{12} \end{pmatrix}$$
$$+ 1^n \begin{pmatrix} 0 & 0 & 0 \\ 0 & \frac{1}{2} & -\frac{1}{2} \\ 0 & -\frac{1}{2} & \frac{1}{2} \end{pmatrix}.$$

Taking the sum of the $(1,1)$ entries we have established the following.

Proposition 12. *It T_n is the number of tilings of $3\times(2n)$ by dominoes then*

$$T_n = \frac{3+\sqrt{3}}{6}(2+\sqrt{3})^n + \frac{3-\sqrt{3}}{6}(2-\sqrt{3})^n.$$

Looking at the possible forms of the 3×2 blocks we get nine possible shapes (note that nine is also the sum of the entries of A). These are shown in Figure 4.4. The tiles split into three groups, "blue" tiles with a single horizontal domino on the top, "red" tiles with a single horizontal domino on the bottom and a universal tile. Since *every* 3×2 block has at least one horizontal domino then any $3\times(2n)$ tiling which uses a universal tile will intersect every other tiling, i.e., it will be universally intersecting. It turns out that these are the only universally intersecting configurations.

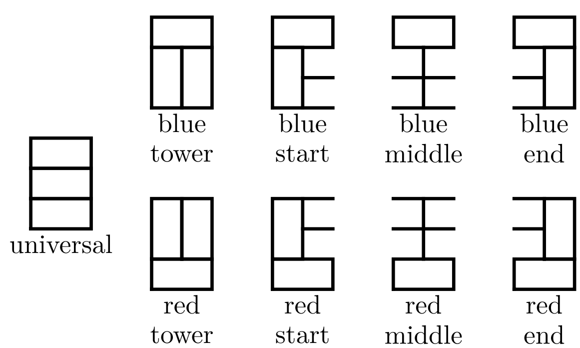

Figure 4.4: The possible 3×2 blocks.

We now count the number of tilings that do not have a universal tile. The previous approach is easily adopted and the only change is to remove a single possibility between

II, namely the one with three horizontal dominoes. This gives the following matrix:

$$B = \begin{pmatrix} 2 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}.$$

This matrix has eigenvalues 3, 1 and 0, so that for $n \geq 1$ we have, similarly to before,

$$B^n = 3^n \begin{pmatrix} \frac{2}{3} & \frac{1}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{1}{6} & \frac{1}{6} \\ \frac{1}{3} & \frac{1}{6} & \frac{1}{6} \end{pmatrix} + 1^n \begin{pmatrix} 0 & 0 & 0 \\ 0 & \frac{1}{2} & -\frac{1}{2} \\ 0 & -\frac{1}{2} & \frac{1}{2} \end{pmatrix}.$$

Taking the sum of the $(1, 1)$ entries we have the following.

Proposition 13. *If S_n is the number of tilings of $3\times(2n)$ by dominoes which does not have three horizontal dominoes in a column, then $S_n = 2{\cdot}3^{n-1}$.*

We are now ready to bound the size of a maximal intersecting family.

Theorem 17. *Let $\mathcal{T}$ be an intersecting family of tilings of the $3\times(2n)$ strip using dominoes. Then*

$$|\mathcal{T}| \leq \frac{3+\sqrt{3}}{6}(2+\sqrt{3})^n + \frac{3-\sqrt{3}}{6}(2-\sqrt{3})^n - 3^{n-1}.$$

Proof. As in the $2\times n$ case we form a graph where each vertex is a tile and two vertices are connected if they do not intersect. Any tiling which contains the universal tile will be an isolated vertex. The remaining tiles can be split into two groups, those that start with a red tile and those that start with a blue tile. As before this is a bipartition of our graph.

Claim. *There is a perfect matching in the set of tilings which do not contain the universal tile.*

Before we prove the claim let us show how this will give the statement of the theorem. In an intersecting family we can take any number of the isolated vertices and at most one of the tilings in each edge of the perfect matching. There are $T_n - S_n$ isolated vertices and $\frac{1}{2}S_n$ edges in the perfect matching; it follows that an intersecting family has

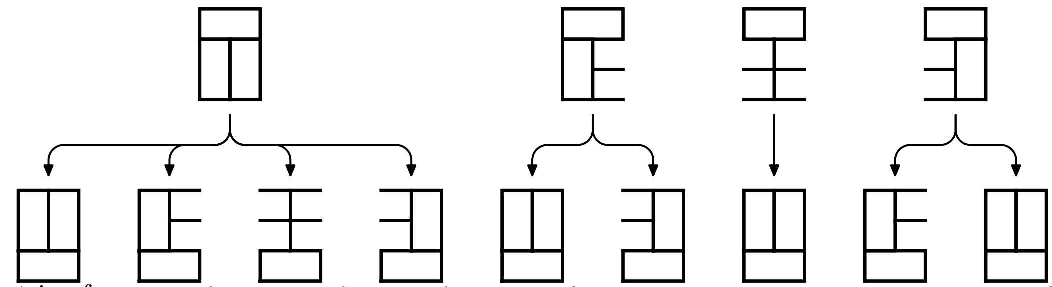

Figure 4.5: Rule for mapping in the 3×(2n) case.

at most $T_n - \frac{1}{2}S_n$ edges. Now using the results from Propositions 12 and 13 the result will follow.

To prove the claim we give a mapping between tilings that start with a blue tile to tilings that start with a red tile. So let T be a tiling. We break T into (maximal) blue and red strips. It suffices to give a mapping that takes a blue strip into a red strip of the same size (and vice-versa) which does not intersect. Such a mapping is given in Figure 4.5. It is easy to check that this mapping satisfies all the needed properties and concludes the proof of the theorem. □

4.1.3 Concluding remarks

Tiling problems have been very popular (both in looking at existence and enumeration of tilings). Looking for maximal intersecting family of tilings opens up an entirely new avenue of investigation of tilings. In this note we have restricted ourselves to domino tilings of the 2×n and 3×2n boards but one can more generally look at domino tilings of k×n boards.

Besides looking at domino tilings one can consider tilings with squares and dominoes, or squares and "L"s [5], or tetris pieces, or polyominoes (see Golomb's [18] excellent book on the subject which also deals extensively with tiling problems), or hexagonal animals, or three-dimensional tilings. For each problem one can also consider a variety of different board configurations. The possibilities of different problems are limited only by the imagination.

4.1.4 Acknowledgement

This chapter is based on the paper "An intersection theorem about domino tilings," written by the author together with Steve Butler and Paul Horn.

Appendix A

Lower Bounds on Hales-Jewett Numbers

A lower bound below will be in the form of an integer sequence, as in:

$HJ(3,2) > 2$:

```
001010100
```

This gives a 2-coloring (in the colors 0 and 1) of the elements of $\{0,1,2\}^2$ listed in lexicographic order. That is, 00 is colored 0, 01 is colored 0, 02 is colored 1, 10 is colored 0, and so on. The claim is that this coloring contains no monochromatic combinatorial lines (and of course it does not, as can be checked by hand).

Checking the examples below is best left to a computer.

$HJ(3,3) > 6$:

```
120012020022121212010122010202011212121211010200020021021220
002212122020012201102021212221202021121121200200022122210211
110022212022220212100112122201201110220101010220012221112210
012121202220212120112020200120200011002121201210020011100001
212121112212210201101202010022101010020011022100220211121212
102010002221221120221011202202120022121100010220221211020101
020200001021211010120101221100200211201012122122100200110200
221210022010022121201202201011001002122110220011221012122100
221100010121020122210212022102201211102020202021102122001 0
```

02222121211000110010211020200202020102121001011000111012221 0
112220101101211110002110211200201012012102121022212101212110
002100202112201110200001001022112100011112100110002101121002
211022221

$HJ(4, 2) > 6$:

0011001010001101011001010001111001011110100101011101010001 10
1000000100110010110100101110010101001110010100110011111001011
0001010101001100111000011011011010000110101010000111001000 01
0001011010111101101101101100010000101101100100011111011 01
1011010110100010001001101000010101011110001110001010100001 01
0011110110110100101001011101001011001110110110010010001010 01
1010110010100010010001111010000111010011010100100010110001 01
0011011111100111110011101001101011100001101000110010111000 01
0110110010011011100110010011010000100111111010001110001000 011
0100000111010101011001111011110010110100100110110101101010 11
1011010110110010011000010110010100011010100111000110110000 11
1110010010011100001111101011110010101001010001000101100001 10
1000011001010111110111000011101000111000011011001101100000 11
1110001110101110110110010001111010111000011110010100100111 01
0010110000001010001111000000100110110100111101101010110111 0001
0111101010111010100111010010111001001000010110101001010101 11
1110001011010010110010010011101010111111001000100010010110 100
1011011010100001000101101000001010111001000101101010101011 10
1000010101100101100100101101100110100101011010101000001110 00
0001011111011110000110000111100000110011010010010011111001 01
0011111001011000000010010011110000010010101011100110111000 001
0011011101001010111001011101001100101001100001101001011000 11
1000000101001001000101110101100111101110100011100101001000 10
0101100101010011001001111000100110100010110000100111001111 00
1010010010001001110001001101001001100001010110111101100010 00

001010011110011101100001101001001101001010011010011110100111
011111001100011111001000000101100010100010010010001011000110
010100011010011111001010110101101101110100101010011111011001
010000011000001100011100001110100101100100100111010010001010
101100011100101100010010100101100100111001100101100110000110
010010111000000111010010001010100011110101111010010101110011
001011100101001000101101101110011101010101100111001110001001
011110101011010010111100110101110111100100101010100000011110
100010100101110101010101101100100101010000101011101000010101
101001010101010011000010100110111001010000100101101100110111
110100101010110110111010011101000010001110101011110101110100
110010100101001100011000001111001000011001001001101100111110
100001110110100101101010110100010110011010000100100101010110
100000101001100101100101001010011011010001101000001101100111
110101001000110100100110100100100110011011011010110001010101
110100101001101110010111110000101001011001011011001100011010
110101111100101110101010110100010110110100100101100000010110
101000010010100000110101010101111010001010111010011010010010
011111011001110010000111101110001011101101000010110101010001
101100011000010101110100101010110001011110101110111001011011
001001011000001100111110101101000100001010000100110110011100
101001001101101100111110001010100101001101000111110100101001
010110000111111001011000011101011010001001011000000111011000
000101100101100000011110010100111111010110110110010111100010
010001100111110000011100000100010110010000100100110100011100
011110110010101000100101110101100110100010000101110110111110
010010111010110100110010000110111000011101110011110001101110
100000111101110010111010101101001010010111011110110001010100
101100110101101010100011011101111100011111000100100001011110
100011001100100110000011100000011100101000010110101100111010
100000110101110011100001011011001010011010011011000111010100

110010101011010100100101100100111010010101010011010100100110
101100010110010110000111110010001101010111001000001010111110
010100111010010110010010100100110110110000110100101001111110
101100011101110001110010100000010000111100101010011010011011 0
110110110100101100101110101110000101010010101101011100011100
101101001110010100011000010100010110100010100110000100111000
101101101100010100101100001010011100001110100111000110101110
101101111101101100101100100001110100110100101101001000111011
001000111101110101001110001101111000011111011100011111010010
001011000111100100111100100100101110001110001010111001101001
110101011100100001010111011100111110100000011110011100110101
011010101010100101110110111000110101100011011110100000111010
1000010110110110

Bibliography

[1] F. Ardila and R.P. Stanley, Tilings, `arXiv:math/0501170v3`.

[2] B. Bollobás and I. Leader, An Erdős-Ko-Rado theorem for signed sets, *Comput. Math. Appl.* **34** (1997), 9–13.

[3] T. C. Brown, Is there a sequence on four symbols in which no two adjacent segments are permutations of one another?, *Amer. Math. Monthly* **78** (1971)

[4] S. Butler, P. Horn, and E. Tressler, An intersection theorem about domino tilings, submitted.

[5] P. Chinn, R. Grimaldi and S. Heubach, Tilings with L's and squares, *J. Integer Sequences* **10** (2007), article 07.2.8, 17 pp.

[6] F. M. Dekking, Strongly non-repetitive sequences and progression-free sets, *J. Combinatorial Theory Ser. A* **27** (1979), 181–185

[7] M. Deza and P. Frankl, On the maximum number of permutations with given maximal or minimal distance, *J. Combinatorial Theory Ser. A* bf 22 (1977), 352–360.

[8] P. Erdős, R. L. Graham, P. Montgomery, B. L. Rothschild, J. Spencer, and E. G. Straus, Euclidean Ramsey theorems, I, *J. Combinatorial Theory Ser. A* **14** (1973), 341–363.

[9] P. Erdős, R. L. Graham, P. Montgomery, B. L. Rothschild, J. Spencer, and E. G. Straus, Euclidean Ramsey theorems, II, *Colloq. Math. Soc. Janos Bolyai* **10** (1973), 529–557.

[10] P. Erdös, R. Graham, P. Montgomery, B. Rothschild, J. Spencer, and E. G. Straus, Euclidean Ramsey theorems, III, *Colloq. Math. Soc. Janos Bolyai* **10** (1973), 559–583.

[11] P. Erdős, C. Ko and R. Rado, Intersection theorems for systems of finite sets, *Quart. J. Math. Oxford Ser. (2)* 12 (1961) 313–320.

[12] P. Erdős and G. Szekeres, A combinatorial problem in geometry, *Composito Math.* **2** (1935) 464–470.

[13] J. Folkman, Graphs with monochromatic complete subgraphs in every edge coloring, *SIAM J. Appl. Math.* **18** (1970), 19–24.

[14] K. Ford, Maximal collections of intersecting arithmetic progressions, *Combinatorica* **23** (2003), 263–281.

[15] P. Frankl, The Erdős-Ko-Rado theorem is true for $n = ckt$, *Combinatorics (Proc. Fifth Hungarian Colloq., Keszthely, 1976), Vol. I* 365–375.

[16] P. Frankl and Z. Füredi, The Erdős-Ko-Rado theorem for integer sequences, *SIAM J. Algebraic Discrete Methods* **1** (1980), 376–381.

[17] P. Frankl and R.M. Wilson, The Erdős-Ko-Rado theorem for vector spaces, *J. Combin. Theory Ser. A* **43** (1986), 228–236.

[18] S.W. Golomb, Polyominoes, 2nd ed., Princeton Univ. Press, Princeton, 1994, 184 pp.

[19] W. T. Gowers, A New Proof of Szemerdi's Theorem, *Geom. Func. Anal.* **11** (2001), 465–588.

[20] R. Graham, B. Rothschild, and J. Spencer, *Ramsey Theory (Second Edition)*, Wiley, (1990).

[21] B. Green and T. Tao, The primes contain arbitrarily long arithmetic progressions, `arXiv:math/0404188`.

[22] D. Hilbert, Uber die Irreducibilitat Ganzer Rationaler Functionen mit Ganzzahligen Coefficienten, *J. Reine Angew Math.* **110** (1892), 104–129.

[23] A. Hales and R. Jewett, Regularity and positional games, *Trans. Amer. Math. Soc.* **106** (1963), 222–229.

[24] N. Hindman, Finite sums from sequences within cells of a partition of $\mathbb{N}$, *J. Comb. Th. (A)* **17** (1974), 1–11.

[25] N. Hindman, Ultrafilters and combinatorial number theory, *Lecture Notes in Math* **751** (1979), Springer-Verlag, Berlin, 119–184.

[26] N. Hindman and E. Tressler, The first nontrivial Hales-Jewett number is four, to appear in *Ars Combinatoria*

[27] Veikko Keränen, Abelian squares are avoidable on 4 letters, *Automata, Languages, and Programming: Lecture notes in Computer Sciences* **623** (1992) Springer-Verlag, 41–52.

[28] C.Y. Ku and D. Renshaw, Erdős-Ko-Rado theorems for permutations and set partitions, *J. Combin. Theory Ser. A*, to appear.

[29] P. van Laarhoven and E. Aarts, *Simulated Annealing: Theory and Applications (Third Edition)*, Kluwer Academic Publishers, Dordrecht (1987).

[30] B. Landman and A. Robertson, *Ramsey Theory on the Integers*, Student Mathematical Library Vol. 24, AMS (2004).

[31] S. Radziszowski, Small Ramsey numbers, *Electronic J. Comb. (Dynamic Survey)*, `http://www.combinatorics.org`.

[32] F. Ramsey, On a problem of formal logic, *Proc. London Math. Soc.* **30** (1930), 264–286.

[33] I. Schur, Uber die Kongruenz $x^m + y^m = z^m \mod p$, *Jahresbericht der Deutschen Math.-Verein.* **25** (1916), 114–117.

[34] S. Shelah, Primitive recursive bounds for van der Waerden numbers, *J. Amer. Math. Soc.* **1** (1988), 683–697.

[35] N.J.A. Sloane, The on-line encyclopedia of integer sequences, `www.research.att.com/ njas/sequences/`.

[36] L. Shader, All Right Triangles Are Ramsey in $\mathbb{E}^2$, *J. Comb. Theory, Ser. A* **20(3)** (1976), 385–389.

[37] J. Spencer, Restricted Ramsey configurations, *J. Comb. Theory, Ser. A* **19** (1975), 278–286.

[38] E. G. Straus Jr., A combinatorial theorem in group theory, *Math. Comp.* **29** (1975), 303–309.

[39] E. Szemerédi, On sets of integers containing no k elements in arithmetic progression, *Acta Arithmetica* **27** (1975), 199-245.

[40] P. Turan, On the theory of graphs, *Colloq. Math.* **3** (1954), 19–30.

[41] B. van der Waerden, Beweis einer Baudetschen Vermutung, *Nieuw Arch. Wiskunde* **19** (1927), 212–216.

[42] R.M. Wilson, The exact bound in the Erdős-Ko-Rado theorem, *Combinatorica* **4** (1984), 247–257.