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Finite Regularity of VI^n-module

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ABSTRACT OF THE DISSERTATION

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Let R be a commutative Noetherian ring. The category VI^n is the product category $\underbrace{VI \times \cdots \times VI}_{n \text{ copies}}$ and a VI^n-module is a functor from the category VI^n to the category of R-modules. In this thesis, we will prove any finitely generated VI^n-module has finite regularity along with many interesting results, among them is the analogue of the Shift Theorem.

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Chapter 1

Introduction

1.1 Basic Notion

Let R be a commutative Noetherian ring and fix a finite field $\mathbb{F} = \mathbb{F}_q$. We assume that q is invertible in R and we call this the "non-describing characteristic" case.

Let VI be the category of finite dimensional vector spaces over $\mathbb{F}$ with morphisms being the linear injections. Then VI^n is the product category, $\underbrace{VI \times \cdots \times VI}_{n}$ whose objects are n-tuples of finite dimensional vector spaces over $\mathbb{F}$ and morphisms are n-tuples of linear injections.

For $\overline{W} = (W_1, \cdots, W_n) \in \mathrm{Ob}(VI^n)$, define the *norm* $|\overline{W}|$ by

$$|\overline{W}| := \dim(W_1) + \cdots \dim(W_n).$$

For $\overline{W}, \overline{Y} \in \mathrm{Ob}(VI^n)$, we say that

$$\overline{W} \leq \overline{Y} \quad \text{if} \quad \dim(W_i) \leq \dim(Y_i) \qquad \text{for all } i.$$

Let R-$\mathcal{M}od$ denoted the category of R-modules with morphisms R-module maps.

Definition 1.1.1. *A* **$\mathbf{VI^n}$*-module*** M *is a functor* $M : VI^n \to R\text{-}\mathcal{M}od$. *In particular,*

- *For* $\overline{W} = (W_1, \cdots, W_n) \in \mathrm{Ob}(VI^n)$, $M(\overline{W})$ *is an* R-module.

- *For each* n*-tuple of linear injections* $\overline{f} = (f_1, \cdots, f_n) : \overline{W} \to \overline{Y}$, *the* transition map $\overline{f}_* = M(\overline{f}) : M(\overline{W}) \to M(\overline{Y})$ *is an* R-module *map.*

Notation 1.1.2. *Since vector spaces of the same dimension are isomorphic,*

$$\overline{W} \cong (\mathbb{F}^{d_1}, \cdots, \mathbb{F}^{d_n})$$

where $d_i = \dim(W_i)$. *For ease of notation, sometimes we write* $M(d_1, \cdots, d_n)$ *for* $M(\mathbb{F}^{d_1}, \cdots, \mathbb{F}^{d_n})$. *If there is no risk of confusion, we also write* $\overline{f}$ *for the transition map* $\overline{f}_*$.

As with any algebraic object, we can also talk about its subobject.

Definition 1.1.3. *Given a* VI^n-module M, *a* ***submodule*** N *of* M *is a functor from the category* VI^n *to category* $R\text{-}\mathcal{M}od$ *such that*

- $N(\overline{W})$ *is an* R*-submodule of* $M(\overline{W})$ *for all* $\overline{W} \in \mathrm{Ob}(VI^n)$.

- N *is closed under the action of* VI^n *morphisms i.e for any transition map* $\overline{f} : \overline{W} \to \overline{Y}$, $\overline{f}_*(N(\overline{W})) \subseteq N(\overline{Y})$.

A map of VI^n-modules is a natural transformation of functors. More formally:

Definition 1.1.4. *If* M, N *are* VI^n*-modules, then a* **$\mathbf{VI^n}$*-module map*** $F : M \to N$ *is a natural transformation of functors from* M *to* N. *In particular, for any transition map*

$\overline{f} : \overline{W} \to \overline{Y}$, the diagram below is commutative

$$
\begin{array}{ccc}
M(\overline{W}) & \xrightarrow{M(\overline{f})} & M(\overline{Y}) \\
{\scriptstyle F_{\overline{W}}} \downarrow & & \downarrow {\scriptstyle F_{\overline{Y}}} \\
N(\overline{W}) & \xrightarrow[N(\overline{f})]{} & N(\overline{Y})
\end{array}
$$

We also have the concepts of kernel, image and cokernel for a VI^n-module map.

Definition 1.1.5. *Given a VI^n-module map $F : M \to N$, define*

- $\ker(F)(\overline{W}) := \ker F_{\overline{W}}$ *for any* $\overline{W} \in Ob(VI^n)$.
- $\ker(F)(\overline{f}) := M(\overline{f})|_{\ker F_{\overline{W}}}$ *for any transition map* $\overline{f} : \overline{W} \to \overline{Y}$.

Then, $\ker(F)$ *is a VI^n-submodule of M.*

The image and the cokernel of a VI^n-module map are defined similarly and both are VI^n-modules.

Definition 1.1.6. *For a VI^n-module M, we define the* ***degree*** *of M to be*

$$deg(M) = sup\{\ |\overline{W}| : M(\overline{W}) \neq 0\}.$$

By convention, if M is the zero module, we define $\deg(M) = -\infty$.

Recall that for an R-module A and a subset $S \subseteq A$, the R-submodule generated by S is defined to be the smallest R-submodule of A containing S. In a way, the presence of transition maps of a VI^n-module allows us to "generate" elements of higher degree.

Definition 1.1.7. *A VI^n-module M has* ***generation degree*** $\leq d$ *if there exists a set $S \subseteq \bigsqcup_{\overline{W}} M(\overline{W})$ with $|\overline{W}| \leq d$ such that the smallest VI^n-submodule of M containing S is M itself. We say that M is* ***finitely generated*** *if the set S is finite.*

Remark 1.1.8. *This implies that for each $v \in M(\overline{Y})$ where $|\overline{Y}| > d$, we can find a finite set $\{v_1, ..., v_n\} \subseteq \bigsqcup_{\overline{W}} M(\overline{W})$ with $|\overline{W}| \leq d$ and morphisms $\overline{f}_1, ..., \overline{f}_n$ such that v is in the R-submodule of $M(\overline{Y})$ generated by $\{\overline{f}_1(v_1), ..., \overline{f}_1(v_n)\}$. When M is finitely generated, the set $\{v_1, ..., v_n\}$ can be chosen independently of v.*

It is a useful fact in module theory that for any R-module V, there exists a surjection $\oplus_{i\in I} F_i \twoheadrightarrow V$ where F_i is a free/projective R-module. We have a similar result but first, we need to talk about *free* VI^n*- modules.*

1.2 Free VI$^{\mathbf{n}}$-mod

For each $\overline{d} = (d_1, \cdots, d_n) \in \mathbb{N}^n$, we define a VI^n-module $I(\overline{d})$ as follows:

For any $\overline{W}$,

$$I(\overline{d})(\overline{W}) \coloneqq \text{free } R\text{-module with basis } \mathrm{Hom}_{VI^n}\left((\mathbb{F}^{d_1}, \cdots, \mathbb{F}^{d_n}), \overline{W}\right)$$

and for any $\overline{f} : \overline{W} \to \overline{Y}$, define

$$\overline{f}_* = I(\overline{d})(\overline{f}) : I(\overline{d})(\overline{W}) \to I(\overline{d})(\overline{Y}) \text{ by } \overline{f}_*(\overline{g}) = \overline{f} \circ \overline{g},$$

then it's not hard to see that $I(\overline{d})$ is a VI^n-module. We call $I(\overline{d})$ a *free* VI^n -module.

Remark 1.2.9. *Note that for $v \in V(\overline{d})$ where V is a VI^n* -module*, there exists a VI^n* -module *map from $I(\overline{d})$ to V where for each $\overline{W}$, it is given by defining*

$$I(\overline{d})(\overline{W}) \to V(\overline{W}), \quad \overline{f} = (f_1, \cdots, f_n) \mapsto \overline{f}_*(v).$$

It's not hard to see from the remark that $I(\overline{d})$ is a projective VI^n-module. Furthermore, this also gives us the following familiar fact from module theory.

Proposition 1.2.10. *Let M be a VI^n-module, then there exists a VI^n-surjection $P \twoheadrightarrow M$ where P is projective. Furthermore, if M is generated in degree $\leq d$, then P can be chosen to be generated in degree $\leq d$ as well.*

Proof. From the remark, if S is a generating set of M, then for each $s \in S$, we can find a corresponding map $I(\overline{d}_s) \to V$ where $s \in M(\overline{d}_s)$. Combining all these maps give a surjection $\oplus_{s\in S} I(\overline{d}_s) \longrightarrow M$. Let $P = \oplus_{s\in S} I(\overline{d}_s)$, then P is projective. This proves the first part. For the second part, if M is generated in degree $\leq d$, then $|\overline{d}_s| \leq d$ for all $s \in S$. Since each $I(\overline{d}_s)$ is generated at position $\overline{d}_s$, it's clear that generation degree of P is $\leq d$. ∎

From the previous proposition, for any VI^n-module M generated in degree $\leq d$, there exists a short exact sequence (SES) of VI^n-modules

$$0 \to K \to P \to M \to 0$$

where P is a projective VI^n-module generated in degree $\leq d$.

Definition 1.2.11. *For a VI^n-module M with a SES as above, we say that M has* ***relation degree*** *$\leq r$ if K is generated in degree $\leq r$ and we denoted the relation degree of M by* ***rel deg***(M).

1.3 Induced/Semi-Induced $\mathbf{VI^n}$-modules

The free VI^n-module above is a special case of what we called an *induced* VI^n-module. To introduce Induced VI^n-module, one needs to know the concept of a VB^n-module. Similar to the category of VI^n, the category VB^n is the product category $\underbrace{VB \times \cdots \times VB}_{n}$, whose objects are n-tuples of finite dimensional vector space over $\mathbb{F}$ and morphisms are n-tuples of isomorphisms. Then, a VB^n-module is a functor from the category of VB^n

to the category of R-modules. Note that any VB^n-module V can be viewed as a VI^n-module simply by letting non-isomorphism of VI^n acts by zero map on V i.e $\overline{f}_* = 0$ for non-isomorphism $\overline{f}$.

Definition 1.3.12. *Given a VB^n-module V, an* ***induced** VI^n**-mod*** $I(V)$ *is of the form*

$$I(V) = \bigoplus_{\overline{d}\in\mathbb{N}^n} I(\overline{d}) \otimes_{R[GL_{d_1}\times\cdots\times GL_{d_n}]} V(\overline{d}).$$

We also say the module $I(V)$ is induced from the VB^n-module V.

Remark 1.3.13. *To see that a free VI^n*-module *module $I(\overline{d})$ is an induced module, let $V(\_) := R\,\mathrm{Hom}_{VB^n}(\overline{d},\_)$, then V is a VB^n-module and it's not hard to see that $I(\overline{d}) = I(V)$.*

Similar to how the concept of semi-simple module is related to simple module, a related concept to Induced VI^n-Modules is the Semi-Induced VI^n-modules.

Definition 1.3.14. *A VI^n-modules M is* ***semi-induced*** *if there exists a finite filtration of M by VI^n-modules such that corresponding quotients are induced i.e we can find a finite sequence of VI^n-submodules of M, $0 \subseteq M_1 \subseteq M_2 \subseteq \cdots \subseteq M_n = M$ such that M_i/M_{i-1} is an induced VI^n* -module.

1.4 The Functor H_0 and H_i

As stated in the previous section, any VB^n-module can be upgraded to be a VI^n-module simply by letting non-isomorphisms acts by zero maps. It's not hard to see that this give us a functor from the category of VB^n-modules denoted VB^n-$\mathcal{M}od$ to the category of VI^n-modules denoted VI^n-$\mathcal{M}od$, say $\psi : VB^n\text{-}\mathcal{M}od \to VI^n\text{-}\mathcal{M}od$.

The functor ψ has a left adjoint functor and we call this functor $H_0 : VI^n\text{-}\mathcal{M}od \to VB^n$-$\mathcal{M}od$. More explicitly, given a VI^n-module M, define

$$M_{<\overline{W}} := \text{the smallest } VI^n\text{-submodule of } M \text{ that contains } M(\overline{Y}) \text{ with } \overline{Y} < \overline{W}.$$

Then,

$$H_0(M)(\overline{W}) := \left(\frac{M}{M_{<\overline{W}}}\right)(\overline{W})$$

or

$$H_0(M)(\overline{W}) = \frac{M(\overline{W})}{\left\langle \overline{f}_*(M(\overline{Y})) \text{ for any } \overline{f} \in \mathrm{Hom}_{VI^n}(\overline{Y}, \overline{W}),\ \overline{Y} < \overline{W} \right\rangle}$$

Note that for $\overline{f} : \overline{W} \to \overline{Z}$, the induced map $\overline{f}_* : H_0(M)(\overline{W}) \to H_0(M)(\overline{Z})$ is the zero map because of how $H_0(M)$ is defined. Hence, we can think of $H_0(M)$ as a VB^n-module or a VI^n-modules with zero transition maps.

This functor H_0 can be defined not just for VI^n-modules but also for a $\mathcal{C}$-modules in general (here, a $\mathcal{C}$-modules is a functor from the category $\mathcal{C}$ to the category of R-modules) and it plays an important role in the theory of VI-, OI- and FI-modules or more generally, any combinatorial categories. For example, it can "detect" the generation degree of a VI^n-module as shown below.

Proposition 1.4.15. *M is a VI^n-module generated in degree $\leq d$ iff* $\deg(H_0(M)) \leq d$.

Proof. Let $\tilde{v} \in H_0(M)(\overline{Y})$ with $|\overline{Y}| > d$ and $v \in M(\overline{Y})$ such that $v \mapsto \tilde{v}$ via the quotient map $M(\overline{Y}) \to H_0(M)(\overline{Y})$.

As explained in Remark 1.1.8, for $v \in M(\overline{Y})$ with $|\overline{Y}| > d$, we can find a finite set $\{v_1, v_2, ..., v_n\} \in \bigsqcup_{\overline{W}} M(\overline{W})$ with $|\overline{W}| \leq d$ such that v is in the R-submodule of $M(\overline{Y})$ generated by $\{\overline{f}_1(v_1), \cdots, \overline{f}_n(v_n)\}$. But this implies $\{\overline{f}_1(v_1), \cdots, \overline{f}_n(v_n)\} \subseteq M_{<\overline{Y}}$ and by definition, $v \mapsto \tilde{v} = 0 \in H_0(M)(\overline{Y})$ so $\deg(H_0(M)) \leq d$.

Conversely, suppose $\deg(H_0(M) \leq d$. Let

$$M_{\leq a} \coloneqq \text{smallest } VI^n\text{-submodule of } M \text{ that contains } M(\overline{W}) \text{ for } |\overline{W}| \leq a.$$

We claim $M_{\leq d} = M_{\leq d+i}$ for $i \in \mathbb{N}$ and prove this by induction. The base case is clear so assume $M_{\leq d} = M_{\leq d+i}$ for $i < j$. Then,

$$H_0 M(d+j) = (M/M_{<d+j})(d+j) = 0$$

implies

$$M(d+j) = M_{<d+j}(d+j) = M_{\leq d+(j-1)}(d+j) = M_{\leq d}(d+j).$$

Combining this fact with the induction hypothesis, we have $M_{\leq d+j} = M_{\leq d}$.

Therefore, $M_{\leq d} = M_{\leq d+i}$ for $i \in \mathbb{N}$. It's not hard to see that this result yields $M_{\leq d} = M$.

Hence, $M_{\leq d} = M$ which implies that M is generated in degree $\leq d$ as claimed. ■

It can be checked directy that H_0 is right exact (or using the fact that H_0 is a left adjoint) so we can talk about its ith left derived functors $L_i H_0 : VI^n\text{-}\mathcal{M}od \to VI^n\text{-}\mathcal{M}od$ which we denote by H_i. For a VI^n-module M, $H_i(M)$ is also a VI^n-module so we can talk about the degree of $H_i(M)$ which is defined by

$$t_i(M) \coloneqq \deg(H_i(M))$$

and is referred to as *homological degrees* of M.

Due to Prop 1.4.15 above, we call $t_0(M)$ the *generating degree* of M.

The homological degrees $t_i(M)$ allow us to define an important homological invariant called the *Castelnuovo-Mumford regularity* or *regularity* for short.

Definition 1.4.16. *The **Castelnuovo-Mumford regularity** or **regularity** of a VI^n-module M denoted by reg(M) is defined by reg$(M) \coloneqq \sup\{\deg(H_i(M)) - i \mid i \geq 0\}$*

Chapter 2

Main Results

As noted in previous chapter, the regularity of a VI^n-module is a significant homological invariant and a common question in the literature is to investigate whether the regularity is finite or not; better yet, if it is finite, is there an explicit formula for the bound?

When $n = 1$ i.e for a VI-module, in [4], Nagpal was able to conclude that a finitely generated VI-module has finite regularity and even gave the formula for the bounds in term of local cohomology degrees.

In the same paper, Nagpal proved many interesting properties of VI-module, among them is the Shift Theorem which states, "a finitely generated VI-module, when shifted sufficiently many times, is semi-induced". In [1], using the Shift Theorem in a nontrivial way, Gan and Li proved a better bound of regularity for finitely generated VI-module in terms of generation and relation degrees. They also give the upper bound for the number of times that one must shift a finitely generated VI-module in order for it to be semi-induced as well.

A natural question arises from Nagpal's result, for $n \in \mathbb{N}$, do finitely generated VI^n-module also have finite regularity? Furthermore, can we find an explicit formula in term of n for the bound of regularity?

In this paper, we will give an affirmative answer to the first question, that is we will prove the following theorem:

Theorem 2.0.17. *(Finiteness of Regularity) Let M be a finitely generated VI^n-module. Then M has finite regularity.*

As for the second question, we suspect there might be a recursive formula with respect to n for the bound but we are unsure whether a explicit, closed-form formula for the bound exists.

Besides proving the finiteness of regularity, we will also establish many interesting and useful properties of VI^n-modules, similar to those of VI-modules in Nagpal's paper [4]; among these is the analogue of the Shift Theorem for VI^n-module. More precisely, we will prove the following theorem:

Theorem 2.0.18. *(Shift Theorem for VI^n-modules) Let M be a finitely generated VI^n-module with generation degree $d = t_0(M)$ and relation degree r. Then, for $x > d + r$,*

$$\overline{\Sigma}_1^x \overline{\Sigma}_2^x \cdots \overline{\Sigma}_n^x M \textit{ and } \Sigma_1^x \Sigma_2^x \cdots \Sigma_n^x M \textit{ are semi-induced.}$$

We will give the proofs of these two theorems and discuss in details about the shift functors and its variant, Σ_i and $\overline{\Sigma}_i$ in later chapters; more specifically, we will prove finiteness of regularity in chapter 3 and the Shift Theorem in chapter 4.

Chapter 3

Proof of Finiteness of Regularity

3.1 Shift Functor I

The main ingredient to prove Finiteness of Regularity is the Shift Theorem for VI-module of Nagpal. In order to state the Shift Theorem, we need to talk about the Shift Functor. First, we discuss the Shift Functor and its variant for VI-modules and then later, we'll see how this functor can be applied to a VI^n-module.

Let $X \in \mathrm{Ob}(VI)$, we have a functor $\tau^X : VI \to VI$ such that $\tau^X(V) \coloneqq V \oplus X$ and for any linear injection $f : V \to W$, $\tau^X(f) : \tau^X(V) \to \tau^X(W)$ is defined as $f \oplus \mathrm{Id}_X$. Since a VI-module is a functor $M : VI \to R\text{-}\mathcal{M}od$, this gives us a functor of VI-modules $\Sigma^X : VI\text{-}\mathcal{M}od \to VI\text{-}\mathcal{M}od$ defined as

$$\Sigma^X M(V) \coloneqq M(\tau^X(V)) = M(V \oplus X)$$

with transition map $\Sigma^X M(f) \coloneqq M(\tau^X(f)) : \Sigma^X M(V) \to \Sigma^X M(W)$.

For a VI-module map $F : M \to N$, $\Sigma^X F : \Sigma^X M \to \Sigma^X N$ is defined as

$$\Sigma^X F_V \coloneqq F_{V \oplus X}$$

This is a VI-module map because F is. Hence Σ^X is indeed a functor and we call it the *Shift Functor*. It is so-called the Shift Functor because applying Σ^X to a VI-module M at the vector space V essentially "shift" V by the vector space X.

Remark 3.1.19. *When* $\dim(X) = r$*, we use the notation* Σ^r *for* Σ^X *interchangeably and when* $\dim(X) = 1$*, we write* Σ^r *as* Σ *for short.*

The Shift Functor has some useful properties, among them is Prop 4.3 of [4].

Proposition 3.1.20. *(Prop 4.3 of* [4]*) Shift of an induced* VI*-module generated in degree* $\leq d$ *is an induced* VI*-module generated in degree* $\leq d$ *and shift of a projective* VI*-module is also projective.*

Along with the shift functor, we also have a natural transformation of $\mathrm{Id}_{VI\text{-}\mathcal{M}od} \to \Sigma^X$ induced by the inclusion $\iota_V : V \hookrightarrow V \oplus X$ for each vector space V. This gives us a canonical VI-module map $M \to \Sigma^X M$ defined by $M(\iota_V) : M(V) \to \Sigma^X M(V)$.

We denote the *kernel* of the map $M \to \Sigma^X M$ to be $\kappa^X M$ and its *cokernel* to be $\Delta^X M$.

It's not hard to see that Σ^X is an exact functor and hence, does not increase the generation degree and relation degree.

Proposition 3.1.21. *Let* M *be a* VI*-module with generation degree* $\leq d$ *and relation degree* $\leq r$*, then* $\Sigma^X M$ *also has generation degree* $\leq d$ *and relation degree* $\leq r$*.*

Proof. Since M be a VI-module with generation degree $\leq d$ and relation degree $\leq r$, we have a SES of VI-modules

$$0 \to R \to F \to M \to 0$$

where F is free with generation degree $\leq d$ and R with generation degree $\leq r$.

Since Σ^X is exact, we have another SES

$$0 \to \Sigma^X R \to \Sigma^X F \to \Sigma^X M \to 0.$$

By Prop 3.1.20, $\Sigma^X F$ is also free in generation degree $\leq d$ and $\Sigma^X R$ is generated in degree $\leq r$. But this implies that $\Sigma^X M$ is generated in degree $\leq d$ and related in degree $\leq r$. ■

As mentioned before, there are two versions of the Shift Theorem, one in Nagpal's paper [4] and the other in Gan-Li's paper [1]. In the former paper, Nagpal did not specify the number of times one must shift a VI-module to make it semi-induced but only state this holds for a sufficiently large number of shifts and in the latter, Gan-Li gives the precise number of shifts in terms of the generation and relation degree which is much more useful in practice. Therefore, we state the Shift Theorem from Gan-Li's [1].

Theorem 3.1.22. *(Shift Theorem, Corollary 4.4 of* [1]*) Let M be a finitely generated VI-module. If $X \in Ob(VI)$ such that* $\dim(X) \geq t_0(M) + t_1(M)$*, then $\Sigma^X M$ is semi-induced.*

One reason the Shift Theorem is so useful because shifting a VI-module sufficiently many times can make the module simpler to work with.

Proposition 3.1.23. *(Proposition 3.10 of* [4]*) Let M be a VI-module generated in finite degrees. Then M is homology acyclic i.e $H_i(M) = 0$ for $i \geq 1$ if and only if M is semi-induced.*

3.2 Shift Functor II

In this section, we seek another variant of the Shift Functor Σ that has better formal properties in addition to those of Σ.

Let $\mathcal{F}$ be a flag of the vector space X so $0 = X_0 \subseteq X_1 \subseteq \cdots \subseteq X_n = X$. The *parabolic subgroup corresponding to* $\mathcal{F}$ denoted by $\mathbf{P}(\mathcal{F})$ is the subgroup consisting of stabilizers of $\mathcal{F}$ in $GL(X)$, the general linear group of X.

We then have a canonical map $\mathbf{P}(\mathcal{F}) \to \bigoplus_{i=1}^{n} GL(X_i/X_{i-1})$ and the kernel of this map is called the *unipotent radical* of $\mathbf{P}(\mathcal{F})$, denoted by $\mathbf{U}(\mathcal{F})$.

We define $\overline{\Sigma}^X$, a variant of Σ^X as follows:

Fix a maximal flag of X so $0 = X_0 \subseteq X_1 \subseteq \cdots \subseteq X_n = X$ and $n = \dim(X)$. Now for a vector space V, let $V_0 = 0$ and $V_{i+1} = V \oplus X_i$ for $i \geq 0$, then

$$0 = V_0 \subseteq V_1 \subseteq \cdots \subseteq V_{n+1} = V \oplus X$$

is a flag of $V \oplus X$ and we denote the unipotent radical corresponding to this flag by $\mathbf{U}_X(V)$.

Let M be a VI-module, we define $\overline{\Sigma}^X M$ by

$$\overline{\Sigma}^X M(V) \coloneqq (\Sigma^X M(V))_{\mathbf{U}_X(V)} = M(V \oplus X)_{\mathbf{U}_X(V)}$$

and for $f : V \to W$, the transition map $\overline{\Sigma}^X M(f) : \overline{\Sigma}^X M(V) \to \overline{\Sigma}^X M(W)$ is defined by

$$\overline{a} \in M(V \oplus X)_{\mathbf{U}_X(V)} \mapsto \overline{M(\tau^X(f))(a)} \in M(W \oplus X)_{\mathbf{U}_X(W)}.$$

This map can be readily checked to be well-defined so we have another Shift Functor $\overline{\Sigma}^X : VI\text{-}\mathcal{M}od \to VI\text{-}\mathcal{M}od$. When $\dim(X) = 1$, we write $\overline{\Sigma}$ for $\overline{\Sigma}^X$ for short.

Similar to Σ, we also have a canonical map $M \to \overline{\Sigma}^X M$ from the composition of $M(V) \to \Sigma^X M(V) \to \overline{\Sigma}^X M(V)$. Kernel of this map is denoted $\overline{\kappa}^X M$ and cokernel $\overline{\Delta}^X M$.

Remark 3.2.24. *In the non-describing characteristic case, the order of $\mathbf{U}_X(V)$ for $V \in Ob(VI)$ is invertible in R so taking coinvariant is exact; therefore, $\overline{\Sigma}^X$ is also an exact functor like Σ^X.*

Because of the remark above, it turns out that $\overline{\kappa}^X M$ is actually kernel of the canonical map, $M \to \Sigma^X M$.

Proposition 3.2.25. *Let M be a VI-module, the kernel of the canonical map $M \to \overline{\Sigma}^X M$ is $\kappa^X M$.*

Proof. For any $V \in \mathrm{Ob}(VI)$, we have an exact sequence

$$0 \to \kappa^X M(V) \to M(V) \to \Sigma^X M(V).$$

Since the map $M(V) \to \Sigma^X M(V)$ is $\mathbf{U}_X(V)$-equivariant and $\mathbf{U}_X(V)$ acts trivially on $M(V)$, we have the sequence

$$0 \to (\kappa^X M(V))_{\mathbf{U}_X(V)} \to M(V)_{\mathbf{U}_X(V)} \to (\Sigma^X M(V))_{\mathbf{U}_X(V)}$$

which is

$$0 \to \kappa^X M(V) \to M(V) \to \overline{\Sigma}^X M(V).$$

The exactness of this sequence follows from remark 3.2.24 which gives the result. ■

While $\overline{\Sigma}^X$ shares many similar properties with Σ^X, the following proposition from Nagpal's [4] is significant and is the reason why we need to construct $\overline{\Sigma}^X$.

Proposition 3.2.26. *(Prop 4.12 of [4]) Let X be a vector space of dimension 1 and $I(W)$ be an induced VI-module. Then,*

$$\overline{\Sigma} I(W) = I(W) \oplus I(\overline{\Sigma} W) \quad \textit{and} \quad \overline{\Delta} I(W) = I(\overline{\Sigma} W).$$

Since $\overline{\Sigma}^X$ is an exact functor and by previous proposition 3.2.26, the Shift Functor $\overline{\Sigma}^X$ takes an induced module generated in degree $\leq d$ to direct sum of induced module also generated in degree $\leq d$, we see that $\overline{\Sigma}^X$ does not increase generation and relation degree.

Corollary 3.2.27. *Let M be a VI-module generated in degree $\leq t_0(M)$ and related in degree $\leq r$ i.e $\exists$ a SES*

$$0 \to K \to P \to M \to 0$$

where P is an induced VI-module generated in degree $\leq t_0(M)$ and K is generated in degree $\leq r$. Then,

$$t_0(\overline{\Sigma}^X M) \leq t_0(M) \text{ and rel deg}(\overline{\Sigma}^X M) \leq r.$$

One might wonder whether there is an analogue of the Shift Theorem (Theorem 3.1.22) for the Shift Functor $\overline{\Sigma}^X$, the answer is affirmative and the proof is a minor modification of the proof of the Shift Theorem for Σ^X in [1].

Theorem 3.2.28. *(Shift Theorem for $\overline{\Sigma}^X$) Let M be a finitely generated VI-module. If $X \in Ob(VI)$ such that $\dim(X) \geq t_0(M) + t_1(M)$, then $\overline{\Sigma}^X M$ is semi-induced.*

Proof. Suppose $\dim(X) \geq t_0(M) + t_1(M)$. First, recall that we have a complex of VI-modules $I^\bullet$ associated with M as constructed in Theorem 4.34 of [4]. By construction, $I^\bullet$ is a finite complex, starting with $I^0 = M$, such that each $I^i, (i \geq 1)$ is a semi-induced module. By theorem 4.1 of [1], the complex $\Sigma^X I^\bullet$ is exact. Now, for any $i \geq 0$, the ith homology of $\overline{\Sigma}^X I^\bullet$ is

$$H^i(\overline{\Sigma}^X I^\bullet) \underbrace{\cong}_{\text{b/c } \overline{\Sigma}^X \text{ is exact}} \overline{\Sigma}^X H^i(I^\bullet) = (\Sigma^X H^i(I^\bullet))_{U_X}$$

$$\underbrace{\cong}_{\text{b/c } \Sigma^X \text{ is exact}} (H^i(\Sigma^X I^\bullet))_{U_X} \underbrace{=}_{\Sigma^X I^\bullet \text{ is exact}} (0)_{U_X} = 0.$$

Hence, $\overline{\Sigma}^X I^\bullet$ is also an exact complex.

For $i \geq 1$, I^i is semi-induced by construction so by Proposition 4.12 of [4], $\overline{\Sigma}^X I^i$ is also semi-induced. Since $\overline{\Sigma}^X I^\bullet$ is a finite complex because $I^\bullet$ is, Corollary 4.23 of [4] implies $\overline{\Sigma}^X I^0 = \overline{\Sigma}^X M$ is also semi-induced. ∎

3.3 Application of Shift Functor to VI^n-Modules

Even though the Shift Functor and its variant are defined for a VI-module, we can still apply them to a VI^n-module as follows.

Given a VI^n-module M, if we fixed $n-1$ component of M i.e $M(d_1, d_2, \cdots, d_{i-1}, \_\_, d_{i+1}, \cdots, d_n)$, then $\widetilde{M_i} \coloneqq M(d_1, d_2, \cdots, d_{i-1}, \_\_, d_{i+1}, \cdots, d_n)$ is a VI-module. This simple observation is the key for this and for our proof of finite regularity. Since $\widetilde{M_i}$ is a VI-module, we can apply the Shift Functor Σ^X and its variant $\overline{\Sigma}^X$ to $\widetilde{M_i}$.

Definition 3.3.29. *Given a VI^n-module M and for any $1 \leq i \leq n$, we define the Shift Functor $\overline{\Sigma}_i^X : VI^n\text{-}\mathcal{M}od \to VI^n\text{-}\mathcal{M}od$ to be*

$$\overline{\Sigma}_i^X M(\overline{d}) \coloneqq \overline{\Sigma}^X \widetilde{M_i}(d_i).$$

Similar definition can be made to extend the application of the Shift Functor Σ^X to a VI^n-module.

As in the case of VI-module, we also have a canonical map $M \to \overline{\Sigma}_i^X M$ along with its *kernel* and *cokernel*, denoted $\overline{\kappa}_i^X M$ and $\overline{\Delta}_i^X M$ respectively.

Remark 3.3.30. *Many properties of the Shift Functor for VI-module still holds for VI^n-modules so we list a few results below:*

1. *For any $1 \leq i \leq n$, $\overline{\Sigma}_i^X$ is an exact functor.*

2. *$\overline{\Sigma}_i^X$ does not increase generation and relation degrees i.e $t_0(\overline{\Sigma}_i^X M) \leq t_0(M)$ and $rel\ deg(\overline{\Sigma}_i^X M) \leq rel\ deg(M)$.*

3. *$\overline{\kappa}_i^X M$ coincides with kernel, $\kappa_i^X M$, of the canonical map $M \to \Sigma_i^X M$.*

4. *$\overline{\Sigma}_i I(W) = I(W) \oplus I(\overline{\Sigma}_i W)$ and $\overline{\Delta}_i I(W) = I(\overline{\Sigma}_i W)$.*

3.4 The Functors H_0^{ver} and H_0^{hor}

In this section, we introduce two functors, related to H_0 of Sect 1.4, from the paper (in preparation) of Gan-T [3] (see Appendix 5) that are fundamental in proving finite regularity and the analogue of the Shift Theorem for VI^n-modules.

Using notation of the paper [3], let $\mathcal{A} = VI$ and $\mathcal{B} = VI^{n-1}$. As defined there, H_0^{hor} : $VI^n\text{-}\mathcal{M}od \to VI^n\text{-}\mathcal{M}od$ is a functor defined by

$$H_0^{\text{hor}} M(a, \overline{d}) := (H_0^{VI} M(\_\_, \overline{d}))|_a \text{ where } \overline{d} \in \mathbb{N}^{n-1} \text{ and } a \in \mathbb{N}$$

and similarly, $H_0^{\text{ver}} : VI^n\text{-}\mathcal{M}od \to VI^n\text{-}\mathcal{M}od$ is a functor given by

$$H_0^{\text{ver}} M(a, \overline{d}) := (H_0^{VI^{n-1}} M(a, \_\_))|_{\overline{d}}$$

Since H_0^{ver} and H_0^{hor} are right exact, we can define their left derived functors $L_i H_0^{\text{ver}}$, $L_i H_0^{\text{hor}}$ and denoted them as H_i^{ver} and H_i^{hor} respectively.

Theorem 2.4 of [3] implies, for any VI^n-module M, we have two convergent "horizontal-vertical" spectral sequences

$$H_p^{\text{ver}} H_q^{\text{hor}}(M) \Rightarrow H_{p+q}(M) \quad \text{and} \quad H_p^{\text{hor}} H_q^{\text{ver}}(M) \Rightarrow H_{p+q}(M).$$

We also need this technical Lemma that plays a key role in our proof of finite regularity and Shift Theorem.

Lemma 3.4.31. *Let M be VI^n-module, for $a \in \mathbb{N}, \overline{d} \in \mathbb{N}^{n-1}$ and for $p \geq 0$ we have:*

1. $H_p^{\text{ver}} M(a, \overline{d}) \cong H_p^{VI^{n-1}} M(a, \_\_)|_{\overline{d}}$

2. $H_p^{\text{hor}} M(a, \overline{d}) \cong H_p^{VI} M(\_\_, \overline{d})|_a$

Proof. We give the proof for part 1 since the proof of the second part follows similarly.

Suppose we have a free VI^{n-1}-module resolution

$$\cdots \to F_1 \to F_0 \to M \to 0.$$

For any $a \in \mathbb{N}$, $F_i(a, \_\_)$ is a free VI^{n-1}-module because

$$I(d_1, \overline{d})_{(a,\_\_)} \cong \bigoplus_{|\operatorname{Hom}_{VI}(d_1,a)|} I(\overline{d})$$

which is a free VI^{n-1}-module. Thus we have a free VI^{n-1}-resolution of $M(a, \_\_)$

$$\cdots \to F_1(a, \_\_) \to F_0(a, \_\_) \to M(a, \_\_) \to 0.$$

For any $\overline{d} \in \mathbb{N}^{n-1}$, we have a commutative diagram

$$\begin{array}{ccccccccc}
\cdots \longrightarrow & H_0^{\text{ver}} F_1(a, \overline{d}) & \longrightarrow & H_0^{\text{ver}} F_0(a, \overline{d}) & \longrightarrow & H_0^{\text{ver}} M(a, \overline{d}) & \longrightarrow & 0 \\
& \| & & \| & & \| & & \\
\cdots \longrightarrow & H_0^{VI^{n-1}} F_1(a, __)|_{\overline{d}} & \longrightarrow & H_0^{VI^{n-1}} F_0(a, __)|_{\overline{d}} & \longrightarrow & H_0^{VI^{n-1}} M(a, __)|_{\overline{d}} & \longrightarrow & 0
\end{array}$$

The pth homology of the top row gives $H_p^{\text{ver}} M(a, \overline{d})$ while the pth homology of the bottom row gives $H_p^{VI^{n-1}} M(a, \_\_)|_{\overline{d}}$.

Hence,

$$H_p^{\text{ver}} M(a, \overline{d}) \cong H_p^{VI^{n-1}} M(a, \_\_)|_{\overline{d}}.$$

■

3.5 Proof of Finiteness of Regularity

To prove the result, we first induct on n i.e assume any finitely generated VI^{n-1}-module has finite regularity. This is true for VI-module by Theorem 1.7 of Nagpal's [4] so the base case is established. We now prove a preliminary lemma:

Lemma 3.5.32. *If V is a finitely generated VI^n-module, then $\kappa_1 V = \ker(V \to \overline{\Sigma}_1 V)$ has finite regularity.*

Proof. By theorem 8.3.1 of [5], the category VI is quasi-Grobner and by proposition 4.3.5 of [5] which states "The cartesian product of finitely many quasi-Grobner categories is quasi-Grobner" so VI^n is also a quasi-Grobner category. Finally, theorem 1.1.3 of [5] implies any finitely generated VI^n-module over a commutative Noetherian ring R is also Noetherian. Therefore, V is a finitely generated Noetherian VI^n-module and since $\kappa_1 V \subseteq V$, $\kappa_1 V$ is also finitely generated and let $\alpha = t_0(\kappa_1 V)$.

Since "horizontal" maps act by the zero map on $\kappa_1 V$ i.e $\kappa_1 V(f, \mathrm{Id}, \mathrm{Id}, \cdots, \mathrm{Id}) = 0$, we have the diagram

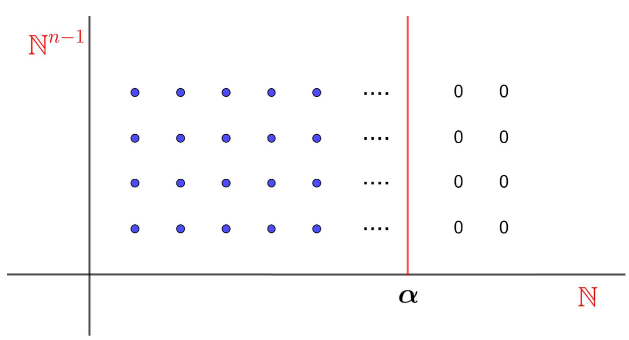

Figure 3.1: $\kappa_1 V$

where each dot at position $(a, \overline{d}) \in \mathbb{N} \times \mathbb{N}^{n-1}$ represents $\kappa_1 V_{(a,\overline{d})}$ and $\kappa_1 V_{(a,\overline{d})} = 0$ for $a > \alpha$. From Theorem 2.4 of [3] regarding VI^n as $VI \times VI^{n-1}$, we have a first quadrant spectral sequence $H_p^{\text{hor}}(H_q^{\text{ver}}(\kappa_1 V))$ with $H_p^{\text{hor}}(H_q^{\text{ver}}(\kappa_1 V)) \Rightarrow H_{p+q}(\kappa_1 V)$.

For $a \in \mathbb{N}$, each "column" of $\kappa_1 V$, $\kappa_1 V(a, \_\_)$, is a VI^{n-1}-module that is also finitely generated so induction hypothesis implies each "column" has finite regularity. By Lemma 3.4.31, this means we can find a $N'_a \in \mathbb{N}$ such that $\deg H_q^{\text{ver}}(\kappa_1 V)(a, \_\_) \leq q + N'_a$.

Since $\kappa_1 V(a, \_\_) = 0$ for $a > \alpha$, let

$$\tilde{N}' = \max\{N'_a \mid 0 \leq a \leq \alpha\},$$

then

$$H_q^{\text{ver}}(\kappa_1 V)(a, \overline{d}) = 0 \text{ for } a > \alpha \text{ or } |\overline{d}| > q + \tilde{N}' \tag{3.1}$$

Since H_q^{ver} is right exact, $H_q^{\text{ver}}(\kappa_1 V)$ is also finitely generated with generation degree $\leq \alpha$ so each "row" of $H_q^{\text{ver}}(\kappa_1 V)$, $H_q^{\text{ver}}(\kappa_1 V)(\_\_, \overline{d})$, is also a finitely generated VI-module with $\deg(H_q^{\text{ver}}(\kappa_1 V)(\_\_, \overline{d})) \leq \alpha$. Lemma 5.10 of [4] then tells us degree of each "row" of $H_p^{\text{hor}}(H_q^{\text{ver}}(\kappa_1 V))$ is bounded by $\alpha + p$, that is, by part 2 of Lemma 3.4.31

$$H_p^{\text{hor}}(H_q^{\text{ver}}(\kappa_1 V))(a, \overline{d}) = 0 \text{ if } a > \alpha + p.$$

From Equation 3.1, we also have $H_p^{\text{hor}}(H_q^{\text{ver}}(\kappa_1 V))(a, \overline{d}) = 0$ for $|\overline{d}| > q + \tilde{N}'$ and this gives

$$H_p^{\text{hor}}(H_q^{\text{ver}}(\kappa_1 V))(a, \overline{d}) = 0$$

for $|(a, \overline{d})| = a + d_1 + \cdots d_{n-1} > \alpha + p + q + \tilde{N}'$.

Since the spectral sequence $H_p^{\text{hor}}(H_q^{\text{ver}} \kappa_1 V) \Longrightarrow H_{p+q}(\kappa_1 V)$, letting $i = p + q$, we see that

$$H_i(\kappa_1 V)(a, \overline{d}) = 0 \text{ for } |(a, \overline{d})| > \alpha + i + \tilde{N}'.$$

Therefore,

$$\deg H_i(\kappa_1 V) \leq \alpha + i + \tilde{N}' \text{ or } \operatorname{reg}(\kappa_1 V) \leq \alpha + \tilde{N}',$$

so $\kappa_1 V$ has finite regularity as claimed. ■

Now let M be a finitely generated VI^n-module with generation degree $\leq t_0(M)$ and relation degree $\leq r$. We also induct on the generating degree of a VI^n-module i.e we assume that any finitely generated VI^n-module with generating degree $< t_0(M)$ also has finite regularity. Since a VI^n-module generated in degree < 0 is the zero module, it obviously has finite regularity so the base case holds as well.

Step 1: We show if $\mathbf{dim(X) \geq t_0(M) + r}$, then $\mathbf{\overline{\Sigma}_1^X M}$ has finite regularity.

Fixing the last $n-1$ components of M gives a VI-module i.e for $\overline{d} \in \mathbb{N}^{n-1}$, $M(\_\_, \overline{d})$ is a VI-module. Note that $M(\_\_, \overline{d})$ is also a finitely generated VI-module with generation degree $\leq t_0(M)$ and $t_1(M(\_\_, \overline{d})) \leq r$.

Hence, if we choose $X \in \mathrm{Ob}(VI)$ such that $\dim(X) \geq t_0(M) + r$, then the Shift Theorem for $\overline{\Sigma}^X$ of VI-module implies that each "row" of $\overline{\Sigma}_1^X M$ i.e $\overline{\Sigma}_1^X M(\_\_, \overline{d})$ is a semi-induced VI-module.

From Theorem 2.4 of [3] regarding VI^n as $VI \times VI^{n-1}$, we have a "horizontal-vertical" spectral sequence

$$H_p^{\mathrm{ver}} H_q^{\mathrm{hor}}(\overline{\Sigma}_1^X M) \Rightarrow H_{p+q}(\overline{\Sigma}_1^X M).$$

Since for each $\overline{d} \in \mathbb{N}^{n-1}$, $\overline{\Sigma} M(\_\_, \overline{d})$ is a semi-induced VI-module, Prop 3.1.23 and Lemma 3.4.31 implies

$$H_q^{\mathrm{hor}}(\overline{\Sigma}_1^X M) = \begin{cases} H_0^{\mathrm{hor}}(\overline{\Sigma}_1^X M) & \text{if } q = 0 \\ 0 & \text{if } q > 0 \end{cases}$$

Therefore, the E^2-level of our spectral sequence $H_\bullet^{\mathrm{ver}} H_\bullet^{\mathrm{hor}}(\overline{\Sigma}_1^X M)$ has the form

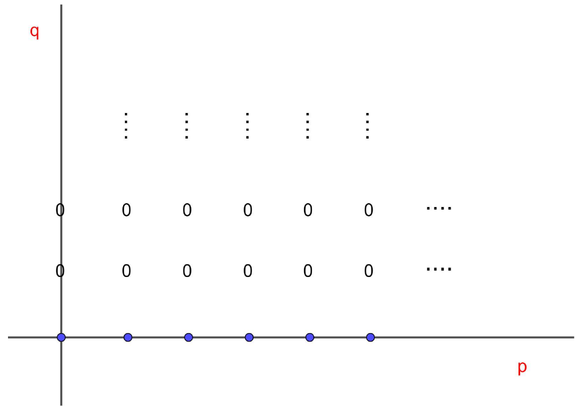

Figure 3.2: $H^{\text{ver}}_{\bullet} H^{\text{hor}}_{\bullet}(\overline{\Sigma}_1^X M)$

This implies for $p \geq 0$, we have

$$H_p(\overline{\Sigma}_1^X M) \cong H_p^{\text{ver}}(H_0^{\text{hor}}(\overline{\Sigma}_1^X M)). \tag{3.2}$$

Now, M is finitely generated with generating degree $\leq t_0(M)$ as a VI^n-module which implies $\overline{\Sigma}_1^X M$ is also a finitely generated VI^n-module with generating degree $\leq t_0(M)$. This yields, for $\overline{d} \in \mathbb{N}^{n-1}$,

$$H_0^{\text{hor}}(\overline{\Sigma}_1^X M)(a, \overline{d}) = 0 \quad \text{if} \quad a > t_0(M). \tag{3.3}$$

We can visualize this on the $\mathbb{N}$- and $\mathbb{N}^{n-1}$-axes below, where each dot at position $(a, \overline{d}) \in \mathbb{N} \times \mathbb{N}^{n-1}$ represents $H_0^{\text{hor}}(\overline{\Sigma}_1^X M)(a, \overline{d})$.

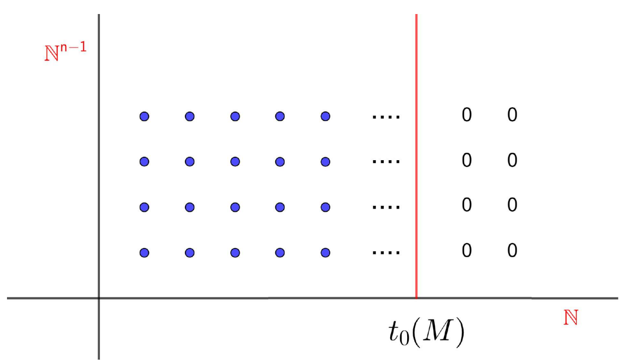

Figure 3.3: $H_0^{\text{hor}}(\overline{\Sigma}_1^X M)$

So the equation above says that $H_0^{\text{hor}}(\overline{\Sigma}_1^X M)$ is zero to the right of the vertical line $t_0(M)$.

Since H_0^{hor} is right exact, $H_0^{\text{hor}}(\overline{\Sigma}_1^X M)$ is also a finitely generated VI^n-module and so for each $a \in \mathbb{N}$, $H_0^{\text{hor}}(\overline{\Sigma}_1^X M)(a, \_\_)$ is a finitely generated VI^{n-1}-module so by induction hypothesis, $H_0^{\text{hor}}(\overline{\Sigma}_1^X M)(a, \_\_)$ has finite regularity i.e there exists $N_a > 0$ such that, by Lemma 3.4.31, the "vertical degree"

$$\deg(H_p^{\text{ver}}(H_0^{\text{hor}}(\overline{\Sigma}_1^X M))(a, \_\_)) \leq p + N_a \text{ for } p \geq 1.$$

Since $H_0^{\text{hor}}(\overline{\Sigma}_1^X M)$ is zero to the right of the vertical line $t_0(M)$, let

$$\tilde{N} = \max\{N_a | 0 \leq a \leq t_0(M)\},$$

then for all $a \in \mathbb{N}$,

$$\deg(H_p^{\text{ver}}(H_0^{\text{hor}}(\overline{\Sigma}_1^X M))(a, \_\_)) \leq p + \tilde{N} \text{ for } p \geq 1.$$

Also, Equation 3.3 yields

$$H_p^{\text{ver}}(H_0^{\text{hor}}(\overline{\Sigma}_1^X M))(a, \_\_) = 0 \text{ for } a \geq t_0(M).$$

Hence, as a VI^n-module,

$$\deg(H_p^{\text{ver}}(H_0^{\text{hor}}(\overline{\Sigma}_1^X M))) \leq p + \tilde{N} + t_0(M),$$

which Equation 3.2 implies

$$\deg(H_p(\overline{\Sigma}_1^X M)) \leq p + \tilde{N} + t_0(M)$$

for $p \geq 1$ as well i.e $\overline{\Sigma}_1^X M$ has finite regularity.

Step 2: For each $s \in \mathbb{N}$, we show if $\overline{\mathbf{\Sigma}}_1^{\mathbf{s+1}}\mathbf{M}$ has finite regularity, then $\overline{\mathbf{\Sigma}}_1^{\mathbf{s}}\mathbf{M}$ has finite regularity.

Now, for each $s \in \mathbb{N}$, we have two SES's of VI^n-modules

$$0 \longrightarrow \kappa_1(\overline{\Sigma}_1^s M) \longrightarrow \overline{\Sigma}_1^s M \longrightarrow \overline{\Sigma}_1^s M / \kappa_1(\overline{\Sigma}_1^s M) \longrightarrow 0 \tag{3.4}$$

$$0 \longrightarrow \overline{\Sigma}_1^s M/\kappa_1(\overline{\Sigma}_1^s M) \longrightarrow \overline{\Sigma}_1^{s+1} M \longrightarrow \overline{\Delta}_1(\overline{\Sigma}_1^s M) \longrightarrow 0 \tag{3.5}$$

where $\kappa_1(\overline{\Sigma}_1^s M) = \ker(\overline{\Sigma}_1^s M \to \overline{\Sigma}_1^{s+1} M)$ by part 3 of Remark 3.3.30 and $\overline{\Delta}_1(\overline{\Sigma}_1^s M) = \operatorname{coker}(\overline{\Sigma}_1^s M \to \overline{\Sigma}_1^{s+1} M)$.

Since M is finitely generated, $\overline{\Sigma}_1^s M$ is also finitely generated by right exactness of $\overline{\Sigma}_1$ so by Lemma 3.5.32, $\kappa_1(\overline{\Sigma}_1^s M)$ has finite regularity for any $s \in \mathbb{N}$.

We also have $t_0(\overline{\Delta}_1(\overline{\Sigma}_1^s M)) < t_0(M)$ because by part 2 of Remark 3.3.30, $t_0(\overline{\Sigma}_1^s M) \le t_0(M)$ and we have an exact sequence

$$\bigoplus_{\overline{d}} I(V_{\overline{d}}) \longrightarrow \overline{\Sigma}_1^s M \longrightarrow 0$$

where $V_{\overline{d}}$ are VB^n-module supported in degree $\overline{d}$ such that $|\overline{d}| = d_1 + \cdots + d_n \le t_0(M)$.

Applying the right-exact functor $\overline{\Delta}_1$ gives another exact sequence

$$\bigoplus_{\overline{d}} \overline{\Delta}_1 I(V_{\overline{d}}) \longrightarrow \overline{\Delta}_1(\overline{\Sigma}_1^s M) \longrightarrow 0\ .$$

By part 4 of Remark 3.3.30,

$$\overline{\Delta}_1 I(V_{\overline{d}}) = I(\overline{\Sigma}_1 V_{\overline{d}}),$$

so we have

$$\bigoplus_{\overline{d}} I(\overline{\Sigma}_1 V_{\overline{d}}) \longrightarrow \overline{\Delta}_1(\overline{\Sigma}_1^s M) \longrightarrow 0$$

But $\overline{\Sigma}_1 V_{\overline{d}}$ is supported in degree $d_1 - 1, d_2, \cdots, d_n$ so $I(\overline{\Sigma}_1 V_{\overline{d}}) = \overline{\Delta}_1 I(V_{\overline{d}})$ is generated in degree $d_1 - 1 + d_2 + \cdots + d_n < t_0(M)$. Therefore, $t_0(\overline{\Delta}_1(\overline{\Sigma}_1^s M)) < t_0(M)$.

By induction hypothesis on generating degree, $\overline{\Delta}_1(\overline{\Sigma}_1^s M)$ has finite regularity.

Hence, if $\overline{\Sigma}_1^{s+1} M$ has finite regularity, then the SES 3.5 along with finiteness of regularity of $\overline{\Delta}_1(\overline{\Sigma}_1^s M)$ implies $\overline{\Sigma}_1^s M/\kappa_1(\overline{\Sigma}_1^s M)$ also has finite regularity.

But then, the SES (3.4) along with finiteness of regularity of $\kappa_1(\overline{\Sigma}_1^s M)$ implies $\mathbf{\overline{\Sigma}_1^s M}$ also has finite regularity.

Step 3: We show M has finite regularity.

Notice from step 1 above, we already showed that $\overline{\Sigma}_1^X M$ has finite regularity if $s = \dim(X) \geq t_0(M) + r$ so $\overline{\Sigma}_1^{s-1} M$ also has finite regularity. This then implies $\overline{\Sigma}_1^{s-2} M$ also has finite regularity and so on. Eventually we can conclude that M also has finite regularity as desired.

Chapter 4

Shift Theorem and Miscellaneous Results

Our goal is this chapter is to prove the Shift Theorem 2.0.18 for VI^n-modules as stated in Chapter 2 and we build up to the proof of that by establishing interesting results along the way.

We will need to use the two functors H_0^{hor} and H_0^{ver} defined in Chapter 3 in the proof of various results below so we recall them here for convenience.

Using notation of v1 notes, let $\mathcal{A} = VI$ and $\mathcal{B} = VI^{n-1}$. As defined in the notes, $H_0^{\text{hor}} : VI^n\text{-}\mathcal{M}od \to VI^n\text{-}\mathcal{M}od$ is a functor defined by

$$H_0^{\text{hor}} M(a, \overline{d}) := (H_0^{VI} M(\_\_, \overline{d}))|_a \text{ where } \overline{d} \in \mathbb{N}^{n-1} \text{ and } a \in \mathbb{N}$$

and similarly, $H_0^{\text{ver}} : VI^n\text{-}\mathcal{M}od \to VI^n\text{-}\mathcal{M}od$ is a functor given by

$$H_0^{\text{ver}} M(a, \overline{d}) := (H_0^{VI^{n-1}} M(a, \_\_))|_{\overline{d}}$$

Proposition 4.0.33. *Let M be a VI^n-module, then $\overline{\Sigma}_1 H_0(M) \cong H_0\overline{\Delta}_1(M)$*

Proof. By proposition 4.15b of [4], we have for VI-module, $H_0^{VI}\overline{\Delta} \cong \overline{\Sigma} H_0^{VI}$ which implies $H_0^{\text{hor}}\overline{\Delta}_1 \cong \overline{\Sigma}_1 H_0^{\text{hor}}$. Recall that we have $H_0 = H_0^{\text{ver}} H_0^{\text{hor}}$ and this yields

$$H_0\overline{\Delta}_1 = H_0^{\text{ver}} H_0^{\text{hor}}\overline{\Delta}_1 \cong H_0^{\text{ver}}\overline{\Sigma}_1 H_0^{\text{hor}}.$$

We claim $H_0^{\text{ver}}\overline{\Sigma}_1 \cong \overline{\Sigma}_1 H_0^{\text{ver}}$ which would give us our result because

$$H_0^{\text{ver}}\overline{\Sigma}_1 H_0^{\text{hor}} \cong \overline{\Sigma}_1 H_0^{\text{ver}} H_0^{\text{hor}} \cong \overline{\Sigma}_1 H_0.$$

We now prove the claim that $H_0^{\text{ver}}\overline{\Sigma}_1 \cong \overline{\Sigma}_1 H_0^{\text{ver}}$.

Let X be a vector space of dimension 1 so $\Sigma_1^X = \Sigma_1$. By definition,

$$\overline{\Sigma}_1 H_0^{\text{ver}} M(V, \overline{W}) := [H_0^{\text{ver}} M(V \oplus X, \overline{W})]_{(\mathbf{U}_X(V), \overline{Id})}$$

where $V \in \text{Ob}(VI)$, $\overline{W} \in \text{Ob}(VI^{n-1})$.

By definition of H_0^{ver}, this is equal to

$$= \left[\frac{M(V \oplus X, \overline{W})}{\text{Image of } (Id_{V\oplus X}, \overline{f})_*}\right]_{(\mathbf{U}_X(V), \overline{Id})}$$

where $\overline{f} : \overline{Y} \to \overline{W}$ with $\overline{W} > \overline{Y}$.

In non-describing characteristic, taking coinvariant by $(\mathbf{U}_X(V), \overline{Id})$ is exact so

$$\left[\frac{M(V \oplus X, \overline{W})}{\text{Image of } (Id_{V\oplus X}, \overline{f})_*}\right]_{(\mathbf{U}_X(V), \overline{Id})} \cong \frac{[M(V \oplus X, \overline{W})]_{(\mathbf{U}_X(V), \overline{Id})}}{[\text{Image of } (Id_{V\oplus X}, \overline{f})_*]_{(\mathbf{U}_X(V), \overline{Id})}}$$

Now,

$$H_0^{\text{ver}}\overline{\Sigma}_1 M(V, \overline{W}) := \frac{[M(V \oplus X, \overline{W})]_{(\mathbf{U}_X(V), \overline{Id})}}{\text{Image of } \overline{\Sigma}_1(Id_V, \overline{f})_*}$$

But $\overline{\Sigma}_1(Id_V, \overline{f})_* = M(Id_{V\oplus X}, \overline{f})_{(\mathbf{U}_X(V), \overline{Id})}$ and since taking coinvariant preserves surjectivity, we have

$$\frac{[M(V \oplus X, \overline{W})]_{(\mathbf{U}_X(V), \overline{Id})}}{\text{Image of } \overline{\Sigma}_1(Id_V, \overline{f})_*} = \frac{[M(V \oplus X, \overline{W})]_{(\mathbf{U}_X(V), \overline{Id})}}{[\text{Image of } (Id_{V\oplus X}, \overline{f})_*]_{(\mathbf{U}_X(V), \overline{Id})}}.$$

Hence,

$$\overline{\Sigma}_1 H_0^{\text{ver}} M(V, \overline{W}) \cong H_0^{\text{ver}}\overline{\Sigma}_1 M(V, \overline{W})$$

which proves the claim. ∎

Remark 4.0.34. *In the proof of the previous proposition, we established the property*

$$H_0^{ver}\,\Sigma_1 \cong \Sigma_1\, H_0^{ver} \quad \textit{and} \quad H_0^{ver}\,\overline{\Sigma}_1 \cong \overline{\Sigma}_1\, H_0^{ver}.$$

Similar reasoning with minor modifications also yield the following property:

$$H_0^{hor}\,\Sigma_i \cong \Sigma_i\, H_0^{hor} \quad \textit{and} \quad H_0^{hor}\,\overline{\Sigma}_i \cong \overline{\Sigma}_i\, H_0^{hor} \quad \textit{with } 1 < i \leq n.$$

The property mentioned in the Remark above turns out to be fundamental in proving the Shift Theorem. Below, we state and prove some results that are interesting in its own right but can be combined to yield a special long exact sequence of VI^n-modules.

Proposition 4.0.35. *If F is an induced VI^n-module, then $L_i\overline{\Delta}_1(F) = 0$ for $i > 0$.*

Proof. F is induced so $F = I(V)$ for some VB^n-module V.

Let

$$\cdots \longrightarrow P_2 \longrightarrow P_1 \longrightarrow P_0 \longrightarrow V \longrightarrow 0$$

be a projective resolution of VB^n-modules.

Since $I(\_)$ is exact and take projective VB^n-module to projective,

$$\cdots \longrightarrow I(P_2) \longrightarrow I(P_1) \longrightarrow I(P_0) \longrightarrow I(V) \longrightarrow 0$$

is also a projective resolution of $I(V) = F$.

Applying $\overline{\Delta}_1$ yields

$$\cdots \longrightarrow \overline{\Delta}_1 I(P_2) \longrightarrow \overline{\Delta}_1 I(P_1) \longrightarrow \overline{\Delta}_1 I(P_0) \longrightarrow \overline{\Delta}_1 F \longrightarrow 0\,.$$

By 4 of Remark 3.3.30, this is isomorphic to

$$\cdots \longrightarrow I(\overline{\Sigma}_1 P_2) \longrightarrow I(\overline{\Sigma}_1 P_1) \longrightarrow I(\overline{\Sigma}_1 P_0) \longrightarrow I(\overline{\Sigma}_1 V) \longrightarrow 0\,.$$

But this is exact since $P_\bullet \to V$ is a projective resolution and $I(\_)$, $\overline{\Sigma}_1(\_)$ are exact functors so $L_i\overline{\Delta}_1(F) = 0$ for $i > 0$. ■

Proposition 4.0.36. *If M is a VI^n-module, $L_p\overline{\Delta}_1 M = 0$ for $p \geq 2$.*

Proof. Let M be a VI^n-module, then we have an exact sequence

$$0 \to K \to F \to M \to 0$$

where F is a free VI^n-module. This yields a LES

$$\cdots \to L_p\overline{\Delta}_1 K \to L_p\overline{\Delta}_1 F \to L_p\overline{\Delta}_1 M \to L_{p-1}\overline{\Delta}_1 K \to \cdots$$

By Proposition 4.0.35 above, $\overline{\Delta}_1(F) = 0$ for all $i > 0$ and if we have $L_{p-1}\overline{\Delta}_1(A) = 0$ for any VI^n-module A, then the LES above would imply $L_p\overline{\Delta}_1(M) = 0$, therefore we can use induction to show $L_p\overline{\Delta}_1(M) = 0$ for $p \geq 2$.

For the base case of showing $L_2\overline{\Delta}_1(M) = 0$, we only need to establish that $L_1\overline{\Delta}_1(K) = 0$. The tail of the LES, $0 \to L_1\overline{\Delta}_1(M) \to \overline{\Delta}_1 K \to \overline{\Delta}_1 F$, implies

$$L_1\overline{\Delta}_1(M) = \ker(\overline{\Delta}_1 K \to \overline{\Delta}_1 F).$$

We claim $\kappa_1 M \cong \ker(\overline{\Delta}_1 K \to \overline{\Delta}_1 F)$. To see this, note that we have a commutative diagram

$$\begin{array}{ccccccccc}
 & & \kappa_1 K & & \kappa_1 F & & \kappa_1 M & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & K & \longrightarrow & F & \longrightarrow & M & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \overline{\Sigma}_1 K & \longrightarrow & \overline{\Sigma}_1 F & \longrightarrow & \overline{\Sigma}_1 M & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & \overline{\Delta}_1 K & & \overline{\Delta}_1 F & & \overline{\Delta}_1 M & &
\end{array}$$

Snakes Lemma says we have an exact sequence

$$0 \to \kappa_1 K \to \kappa_1 F \to \kappa_1 M \to \overline{\Delta}_1 K \to \overline{\Delta}_1 F \to \overline{\Delta}_1 M \to 0. \tag{4.1}$$

Since F is free, $\kappa_1 F = 0$ and this gives $0 \to \kappa_1 M \to \overline{\Delta}_1 K \to \overline{\Delta}_1 F$ which implies

$$\kappa_1 M \cong \ker(\overline{\Delta}_1 K \to \overline{\Delta}_1 F).$$

Hence, $L_1\overline{\Delta}_1(M) \cong \kappa_1 M$.

Since M is arbitrary, $\kappa_1 K \cong L_1\overline{\Delta}_1(K)$ but from $(*)$ above, $\kappa_1 K = 0$ so $L_1\overline{\Delta}_1(K) = 0$ as needed. It follows by induction that $L_p\overline{\Delta}_1 M = 0$ for $p \geq 2$. ■

Proposition 4.0.37. *If M is an induced VI^n-module, then $H_i(M) = 0$ for $i > 0$.*

Proof. Since M is an induced VI^n-module, we have $M = I(V)$ for some VB^n-module V. Suppose

$$\cdots \longrightarrow P_2 \longrightarrow P_1 \longrightarrow P_0 \longrightarrow V \longrightarrow 0$$

is a projective resolution of V where P_i is a projective VB^n-module.

Since $I(P_i)$ is a projective VI^n-module and $I(\_)$ is an exact functor,

$$\cdots \longrightarrow I(P_2) \longrightarrow I(P_1) \longrightarrow I(P_0) \longrightarrow I(V) \longrightarrow 0$$

is also a projective resolution of $I(V) = M$.

Applying $H_0(\_)$ and note that for any VB^n-module W, $H_0(I(W)) \cong W$ where W is considered as a VI^n-module with $\overline{f}_* = 0$ for any non-invertible linear map f, we have a commutative diagram

$$\begin{array}{ccccccccccc}
\cdots & \longrightarrow & H_0^{VI^2} I(P_2) & \longrightarrow & H_0^{VI^2} I(P_1) & \longrightarrow & H_0^{VI^2} I(P_0) & \longrightarrow & H_0^{VI^2} I(V) & \longrightarrow & 0 \\
 & & \cong \downarrow & & \cong \downarrow & & \cong \downarrow & & \cong \downarrow & & \\
\cdots & \longrightarrow & P_2 & \longrightarrow & P_1 & \longrightarrow & P_0 & \longrightarrow & V & \longrightarrow & 0
\end{array}$$

Since $P_\bullet \to V$ is a projective resolution, bottom row is exact so top row is also exact and

$$H_i(M) = H_i(I(V)) = 0 \text{ for } i > 0.$$

■

Corollary 4.0.38. *If M is an induced VI^n-module, then $H_i(\overline{\Delta}_1 M) = 0$ for $i \geq 1$.*

Proof. We have $M = I(V)$ for some VB^n-module V and so each fixed row of M is also an induced VI-module so by 4 of Remark 3.3.30,

$$\overline{\Delta}_1 M = \overline{\Delta}_1 I(V) = I(\overline{\Sigma}_1 V).$$

By Prop 4.0.37 above, $H_i(\overline{\Delta}_1 M) = H_i(I(\overline{\Sigma}_1 V)) = 0$ for $i \geq 1$. ■

By combining previous results, we can come up with a special long exact sequence for any VI^n-module.

Proposition 4.0.39. *If M is a VI^n-module, then we have a LES*

$$\cdots \longrightarrow \overline{\Sigma}_1 H_p(M) \longrightarrow H_p\overline{\Delta}_1 M \longrightarrow H_{p-2}(\kappa_1 M) \longrightarrow \overline{\Sigma}_1 H_{p-1}(M) \longrightarrow \cdots$$

where $\kappa_1 M = \ker(M \to \overline{\Sigma}_1 M) \cong L_1\overline{\Delta}_1(M)$.

Proof. Since both H_0, $\overline{\Delta}_1$ are right exact and by Corollary 4.0.38, $\overline{\Delta}_1$ sends projective module to H_0-acyclic module, we have a first-quadrant Grothendieck spectral sequence $(L_p H_0)(L_q\overline{\Delta}_1)(M)$ converging to $L_n(H_0\overline{\Delta}_1)(M)$.

Furthermore, this spectral sequence only has two rows because $L_q\overline{\Delta}_1(M) = 0$ for $q \geq 2$ by Proposition 4.0.36.

Since $\overline{\Sigma}_1$ is exact, we also have

$$L_n(H_0\overline{\Delta}_1)(M) \underbrace{=}_{\text{by Prop 4.0.33}} L_n(\overline{\Sigma}_1 H_0)(M) = \overline{\Sigma}_1 L_n(H_0)(M) = \overline{\Sigma}_1 H_n(M).$$

Thus, we have a two-row spectral sequence converging to $\overline{\Sigma}_1 H_n(M)$. By ex 5.2.2 of Weibel's standard text [6], this two-row spectral sequence yields the desired LES

$$\cdots \longrightarrow \overline{\Sigma}_1 H_p(M) \longrightarrow H_p\overline{\Delta}_1 M \longrightarrow H_{p-2}(\kappa_1 M) \longrightarrow \overline{\Sigma}_1 H_{p-1}(M) \longrightarrow \cdots$$

and we already showed $\kappa_1 M \cong L_1\overline{\Delta}_1(M)$ the proof of Prop 4.0.36. ■

Theorem 4.0.40. (Shift Theorem for VI^n-modules) *Let M be a finitely generated VI^n-module with generation degree $\leq t_0(M)$ and relation degree $\leq r$. Then, for $x > t_0(M) + r$,*

$$\overline{\Sigma}_1^x \overline{\Sigma}_2^x \cdots \overline{\Sigma}_n^x M \text{ and } \Sigma_1^x \Sigma_2^x \cdots \Sigma_n^x M \text{ are semi-induced.}$$

To prove the Shift Theorem, we need a couple results from [4] whose proofs for VI case can be similarly adapted to the VI^n case. We state these results below for completeness.

For a VI^n-module M, let $M_{\prec d}$ and $M_{\preceq d}$ be the smallest VI^n-submodules of M containing $M(\overline{W})$ for any $\overline{W} \in \mathrm{Ob}(VI^n)$ with $|\overline{W}| < d$ and $|\overline{W}| \leq d$ respectively.

The lemma below follows from definition of H_0 immediately.

Lemma 4.0.41. *We have $H_0(M_{\prec d}) = H_0(M)_{<d}$ where $(\_\_)_{<d} : VI^n\text{-}\mathcal{M}od \to VI^n\text{-}\mathcal{M}od$ is a functor defined as, given a VI^n-module A, then $A_{<d}(\overline{W}) = 0$ if $|\overline{W}| \geq d$ and $A_{<d}(\overline{W}) = A(\overline{W})$ if $|\overline{W}| < d$ with obvious transition maps. Similarly, $H_0(M_{\preceq d}) = H_0(M)_{\leq d}$. Furthermore, if $m < n$, then the natural map $H_0(M_{\prec m}) \to H_0(M_{\prec n})$ is the canonical inclusion $H_0(M)_{<m} \to H_0(M)_{<n}$.*

Lemma 4.0.42. *Suppose $H_1(Q) = 0$ and $H_0(Q)$ is concentrated in degree d. Then Q is induced from d.*

Proof. $H_0(Q)$ is concentrated in degree d implies $H_0(Q)(\overline{W}) = Q(\overline{W})$ with $|\overline{W}| = d$ and $H_0(Q)(\overline{W}) = 0$ otherwise. This gives us a surjective map $\phi : I(H_0(Q)) \to Q$ with kernel K. The LES induced by applying H_0 to the SES $\quad 0 \to K \to I(H_0(Q)) \to Q \to 0 \quad$ is

$$\cdots \longrightarrow \underbrace{H_1(Q)}_{0} \longrightarrow H_0(K) \longrightarrow H_0(I(H_0(Q))) \longrightarrow H_0(Q) \longrightarrow \cdots.$$

Notice that the map $\underbrace{H_0(I(H_0(Q)))}_{\cong H_0(Q)} \to H_0(Q)$ is an isomorphism so $H_0(K) = 0$ which implies $K = 0$. Therefore, $Q \cong I(H_0(Q))$ is induced from d. ■

Proposition 4.0.43. *Let M be a VI^n-module with finite generation degree. Then M is homology acyclic iff M is semi-induced. More generally, if $H_1(M) = 0$, then M is semi-induced.*

Proof. The backward direction is clear. To prove the forward direction, we prove the second statement. So suppose $H_1(M) = 0$, we show M is semi-induced.

Let d be the generation degree of M, then we have a natural filtration

$$0 \subseteq M_{\preceq 1} \subseteq \cdots \subseteq M_{\preceq d} = M.$$

We claim that graded quotients $Q_i = M_{\preceq i}/M_{\prec i}$ are induced. We prove this by induction on the generating degree.

Note that $H_0(Q_i)$ is concentrated in degree i and we have a SES

$$0 \to M_{\prec d} \to M \to Q_d \to 0$$

which yields a LES

$$\cdots \longrightarrow H_2(Q_d) \longrightarrow H_1(M_{\prec d}) \longrightarrow H_1(M) \longrightarrow H_1(Q_d) \longrightarrow H_0(M_{\prec d}) \longrightarrow H_0(M) \longrightarrow \cdots$$

Since $H_1(M) = 0$, we have $H_1(Q_d) = \ker(H_0(M_{\prec d}) \to H_0(M))$ from the LES. By Lemma 4.0.41, $H_0(M_{\prec d}) \to H_0(M)$ is just the inclusion $H_0(M)_{<d} \to H_0(M)$.

Therefore, $H_1(Q_d) = 0$ and Lemma 4.0.42 yields Q_d is induced from d. Furthermore, the LES gives Q_d is also homology acyclic so $H_2(Q_d) = 0$.

As a result, $H_1(M_{\prec d}) = 0$ and the induction hypothesis applies to give Q_i with $1 \leq i < d$ are all induced. Hence, M is semi-induced as needed. ■

We need this last technical Lemma for the Proof of Shift Theorem.

Lemma 4.0.44. *If I is a free VI^n-module, then for any $a \in \mathbb{N}$, $(H_0^{hor} I)(a, \_\_)$ is a free VI^{n-1}-module.*

Proof. It suffices to assume $I = I(d_1, \overline{d})$ where $a \in \mathbb{N}$ and $\overline{d} \in \mathbb{N}^{n-1}$.

Now for $a > d_1$,

$$(H_0^{\text{hor}} I)(a, \_\_) = 0$$

and for $a = d_1$,

$$(H_0^{\text{hor}} I)(a, \_\_) = R[\mathbb{GL}_{d_1}] \otimes I(\overline{d}) \cong \bigoplus_{|\mathbb{GL}_{d_1}|} I(\overline{d}).$$

But this is an induced VI^{n-1}-module. ■

Proof of Shift Theorem, 4.0.40: We prove the result for $\overline{\Sigma}_1^X \overline{\Sigma}_2^X \cdots \overline{\Sigma}_n^X M$, an almost-identical proof shows the result for $\Sigma_1^X \Sigma_2^X \cdots \Sigma_n^X M$.

Let $t_0(M)$ and r be generation and relation degrees of M.

We prove the theorem by inducting on n so assume the result holds for VI^{n-1}-module i.e for any finitely generated VI^{n-1}-module V, if X is a vector space with $\dim(X) > t_0(M) + r$, then $\overline{\Sigma}_1^X \overline{\Sigma}_2^X \cdots \overline{\Sigma}_{n-1}^X V$ is semi-induced. Since $t_1(M) \leq r$, the base case is just the Shift Theorem for VI-module (Theorem 3.2.28).

In the proof of finite regularity in chapter 3, we showed that for $p \geq 0$ and if each row of $\overline{\Sigma}_1^X M$, that is $\overline{\Sigma}_1^X M(\_\_, \overline{d})$, is a semi-induced VI-module, then

$$H_p(\overline{\Sigma}_1^X M) \cong H_p^{\text{ver}}(H_0^{\text{hor}}(\overline{\Sigma}_1^X M)). \tag{4.2}$$

Let's choose X such that $\dim(X) > t_0(M) + r$. Since for any vector space X, each $\overline{\Sigma}_i^X$ preserves generation and relation degree by part 2 of Remark 3.3.30, $\overline{\Sigma}_2^X \overline{\Sigma}_3^X \cdots \overline{\Sigma}_n^X M$ also has generation degree $\leq t_0(M)$ and relation degree $\leq r$ as well. Hence, $\overline{\Sigma}_2^X \overline{\Sigma}_3^X \cdots \overline{\Sigma}_n^X M(\_\_, \overline{d})$

is a VI-module with generation degree $\leq t_0(M)$ and relation degree $\leq r$.

Since $\dim(X) \geq t_0(M)+r$, the Shift Theorem for VI-module (Theorem 3.2.28) implies that each row, $\overline{\Sigma}_1^X(\overline{\Sigma}_2^X\overline{\Sigma}_3^X \cdots \overline{\Sigma}_n^X)M(\_\_,\overline{d})$, is a semi-induced VI-module so we can apply 4.2 to get for $p \geq 0$

$$H_p(\overline{\Sigma}_1^X\overline{\Sigma}_2^X\overline{\Sigma}_3^X \cdots \overline{\Sigma}_n^X M) \cong H_p^{\text{ver}}(H_0^{\text{hor}}(\overline{\Sigma}_1^X\overline{\Sigma}_2^X\overline{\Sigma}_3^X \cdots \overline{\Sigma}_n^X M)).$$

Since $\overline{\Sigma}_i\overline{\Sigma}_j = \overline{\Sigma}_j\overline{\Sigma}_i$ and by the property in Remark 4.0.34, we have

$$\begin{aligned} H_p^{\text{ver}}(H_0^{\text{hor}}(\overline{\Sigma}_1^X\overline{\Sigma}_2^X\overline{\Sigma}_3^X \cdots \overline{\Sigma}_n^X M)) &\cong H_p^{\text{ver}}(H_0^{\text{hor}}(\overline{\Sigma}_2^X\overline{\Sigma}_3^X \cdots \overline{\Sigma}_n^X\overline{\Sigma}_1^X M)) \\ \underbrace{\cong}_{H_0^{\text{hor}}\overline{\Sigma}_i \cong \overline{\Sigma}_i H_0^{\text{hor}} \text{ for } i \neq 1} & \quad H_p^{\text{ver}}(\overline{\Sigma}_2^X\overline{\Sigma}_3^X \cdots \overline{\Sigma}_n^X H_0^{\text{hor}}(\overline{\Sigma}_1^X M)). \end{aligned}$$

Note that generation degree and relation degree of $\overline{\Sigma}_1^X M$ are $\leq t_0(M)$ and $\leq r$ respectively, which implies $t_0(H_0^{\text{hor}}(\overline{\Sigma}_1^X M)) \leq t_0(M)$. For any $a \in \mathbb{N}$, we claim that that the relation degree of $H_0^{\text{hor}}(\overline{\Sigma}_1^X M)(a,\_\_)$ as a VI^{n-1}-module is also less than or equal to r.

To see this, since generation degree and relation degree of M is $\leq t_0(M)$ and $\leq r$ respectively, we can find a SES

$$0 \to K \to F \to M \to 0$$

with F, a free VI^n-modules generated in degree $\leq d$ and $t_0(K) \leq r$.

Applying the exact functor $\overline{\Sigma}_1^X$ yields another SES

$$0 \to \overline{\Sigma}_1^X K \to \overline{\Sigma}_1^X F \to \overline{\Sigma}_1^X M \to 0$$

with $\overline{\Sigma}_1^X F$ still being a free VI^n-module by part 4 of Remark 3.3.30 and $t_0(\overline{\Sigma}_1^X K) \leq r$.

Applying the right exact functor H_0^{hor} to this SES, we can find another SES

$$0 \to K' \to H_0^{\text{hor}}(\overline{\Sigma}_1^X F) \to H_0^{\text{hor}}(\overline{\Sigma}_1^X M) \to 0 \tag{4.3}$$

where $K' = \text{Im}(H_0^{\text{hor}}(\overline{\Sigma}_1^X K) \to H_0^{\text{hor}}(\overline{\Sigma}_1^X F))$. Note that since K' is the image of a VI^n-

module generated in degree $\leq r$, we have $t_0(K') \leq r$.

For any $a \in \mathbb{N}$, the SES (4.3) induces a SES of VI^{n-1}-modules

$$0 \to K'(a, \_\_) \to H_0^{\text{hor}}(\overline{\Sigma}_1^X F)(a, \_\_) \to H_0^{\text{hor}}(\overline{\Sigma}_1^X M)(a, \_\_) \to 0$$

Since $\overline{\Sigma}_1^X F$ is a free VI^n-module, Lemma 4.0.44 implies $H_0^{\text{hor}}(\overline{\Sigma}_1^X F)(a, \_\_)$ is a free VI^{n-1}-module generated in degree $\leq t_0(M)$. Since $t_0(K') \leq r$ as a VI^n-module, $K'(a, \_\_)$ is also generated $\leq r$ as a VI^{n-1}-modules.

Hence, each "column" of $H_0^{\text{hor}}(\overline{\Sigma}_1^X M)$ i.e $H_0^{\text{hor}}(\overline{\Sigma}_1^X M)(a, \_\_)$ also has generation degree $\leq t_0(M)$ and relation degree $\leq r$ as a VI^{n-1}-module.

Induction hypothesis now tells us that $\overline{\Sigma}_2^X \overline{\Sigma}_3^X \cdots \overline{\Sigma}_n^X H_0^{\text{hor}}(\overline{\Sigma}_1^X M)(a, \_\_)$ is a semi-induced VI^{n-1}-module which means for $p > 0$, by Lemma 3.4.31

$$H_p^{\text{ver}}(\overline{\Sigma}_2^X \overline{\Sigma}_3^X \cdots \overline{\Sigma}_n^X H_0^{\text{hor}}(\overline{\Sigma}_1^X M)) = 0$$

We can then conclude, for $p > 0$, $H_p(\overline{\Sigma}_1^X \overline{\Sigma}_2^X \overline{\Sigma}_3^X \cdots \overline{\Sigma}_n^X M) = 0$ and so $\overline{\Sigma}_1^X \overline{\Sigma}_2^X \overline{\Sigma}_3^X \cdots \overline{\Sigma}_n^X M$ is homology acyclic. Proposition 4.0.43 above then concludes what we need to show.

Chapter 5

Appendix

Results of this appendix are from the paper, in preparation, "Bounding Regularity of FI^m-Modules" of Dr. Wee Liang Gan with the author.

5.1 Spectral Sequences

Let $\mathcal{C}$ is a skeletal small category. Define a relation $\preceq$ on $\mathrm{Ob}(\mathcal{C})$ by $X \preceq Y$ if $\mathcal{C}(X,Y) \neq \emptyset$. We write $X \prec Y$ if $X \preceq Y$ but not $Y \preceq X$. We say that $\mathcal{C}$ is *directed* if the relation $\preceq$ on $\mathrm{Ob}(\mathcal{C})$ is a partial order.

Now, let $\mathcal{C}$ be the product category $\mathcal{A} \times \mathcal{B}$, where $\mathcal{A}$ and $\mathcal{B}$ are directed skeletal small categories; in particular, $\mathcal{C}$ is a directed skeletal small category.

Let V be a $\mathcal{C}$-module. For any $(X,Y) \in \mathrm{Ob}(\mathcal{C})$, define k-submodules $V^{\mathrm{hor}}_{(X,Y)}$ and $V^{\mathrm{ver}}_{(X,Y)}$ of $V_{(X,Y)}$ by

$$V^{\mathrm{hor}}_{(X,Y)} = \textstyle\sum_{Z \prec X} \left(\sum_{f \in \mathcal{A}(Z,X)} (f, \mathrm{id}_Y)_* \left(V_{(Z,Y)}\right) \right),$$
$$V^{\mathrm{ver}}_{(X,Y)} = \textstyle\sum_{Z \prec Y} \left(\sum_{f \in \mathcal{B}(Z,Y)} (\mathrm{id}_X, f)_* \left(V_{(X,Z)}\right) \right).$$

Lemma 5.1.45. *(i) The assignment* $(X,Y) \mapsto V^{\mathrm{hor}}_{(X,Y)}$ *defines a* $\mathcal{C}$*-submodule* V^{hor} *of* V*.*

(ii) The assignment $(X,Y) \mapsto V^{\mathrm{ver}}_{(X,Y)}$ *defines a* $\mathcal{C}$*-submodule* V^{ver} *of* V*.*

(iii) One has: $\widetilde{V} = V^{\mathrm{hor}} + V^{\mathrm{ver}}$*.*

Proof. (i) Let $f \in \mathcal{A}(Z,X)$ and $(g,h) \in \mathcal{C}((X,Y),(X',Y'))$. Then

$$(g,h)(f,\mathrm{id}_Y) = (gf,h) = (gf,\mathrm{id}_{Y'})(\mathrm{id}_Z,h),$$

which implies

$$(g,h)_*\left((f,\mathrm{id}_Y)_*\left(V_{(Z,Y)}\right)\right) \subseteq (gf,\mathrm{id}_{Y'})_*\left(V_{(Z,Y')}\right).$$

Moreover, $Z \prec X$ implies $Z \prec X'$. Therefore $(g,h)_*\left(V^{\mathrm{hor}}_{(X,Y)}\right) \subseteq V^{\mathrm{hor}}_{(X',Y')}$.

(ii) Similar to (i).

(iii) It is clear that $V^{\mathrm{hor}} + V^{\mathrm{ver}} \subseteq \widetilde{V}$.

Now suppose $(f,g) \in \mathcal{C}((Z,W),(X,Y))$ where $(Z,W) \prec (X,Y)$. If $Z \prec X$, then

$$(f,g)_*\left(V_{(Z,W)}\right) \subseteq (f,\mathrm{id}_Y)_*\left(V_{(Z,Y)}\right) \subseteq V^{\mathrm{hor}}_{(X,Y)}.$$

If $W \prec Y$, then

$$(f,g)_*\left(V_{(Z,W)}\right) \subseteq (\mathrm{id}_X,g)_*\left(V_{(X,W)}\right) \subseteq V^{\mathrm{ver}}_{(X,Y)}.$$

Hence $\widetilde{V} \subseteq V^{\mathrm{hor}} + V^{\mathrm{ver}}$. ■

By the preceding lemma, we may define functors

$$\mathrm{H}^{\mathrm{hor}}_0 : \mathcal{C}\text{-Mod} \to \mathcal{C}\text{-Mod}, \qquad V \mapsto V/V^{\mathrm{hor}};$$

$$\mathrm{H}^{\mathrm{ver}}_0 : \mathcal{C}\text{-Mod} \to \mathcal{C}\text{-Mod}, \qquad V \mapsto V/V^{\mathrm{ver}};$$

moreover, there are canonical isomorphisms

$$\mathrm{H}^{\mathrm{ver}}_0(\mathrm{H}^{\mathrm{hor}}_0(V)) \cong \mathrm{H}^{\mathcal{C}}_0(V) \cong \mathrm{H}^{\mathrm{hor}}_0(\mathrm{H}^{\mathrm{ver}}_0(V)). \tag{5.1}$$

The functors $\mathrm{H}^{\mathrm{hor}}_0$ and $\mathrm{H}^{\mathrm{ver}}_0$ are right exact and we can define their left derived functors.

For each integer $i \geqslant 1$, let $\mathrm{H}_i^{\mathrm{hor}} : \mathcal{C}\text{-Mod} \to \mathcal{C}\text{-Mod}$ be the i-th left derived functor of $\mathrm{H}_0^{\mathrm{hor}}$, and let $\mathrm{H}_i^{\mathrm{ver}} : \mathcal{C}\text{-Mod} \to \mathcal{C}\text{-Mod}$ be the i-th left derived functor of $\mathrm{H}_0^{\mathrm{ver}}$. We call $\mathrm{H}_i^{\mathrm{hor}}(V)$ the i-th *horizontal homology* of V, and $\mathrm{H}_i^{\mathrm{ver}}(V)$ the i-th *vertical homology* of V.

5.1.1

Let $\mathcal{C}^{\mathrm{hor}}$ be the subcategory of $\mathcal{C}$ such that:

- every object of $\mathcal{C}$ is in $\mathcal{C}^{\mathrm{hor}}$;
- a morphism (f, g) in $\mathcal{C}$ is in $\mathcal{C}^{\mathrm{hor}}$ if and only if $g = \mathrm{id}_Y$ for some $Y \in \mathrm{Ob}(\mathcal{B})$.

We call $\mathcal{C}^{\mathrm{hor}}$ the *horizontal subcategory* of $\mathcal{C}$. Similarly, let $\mathcal{C}^{\mathrm{ver}}$ be the subcategory of $\mathcal{C}$ such that:

- every object of $\mathcal{C}$ is in $\mathcal{C}^{\mathrm{ver}}$;
- a morphism (f, g) in $\mathcal{C}$ is in $\mathcal{C}^{\mathrm{ver}}$ if and only if $f = \mathrm{id}_X$ for some $X \in \mathrm{Ob}(\mathcal{A})$.

We call $\mathcal{C}^{\mathrm{ver}}$ the *vertical subcategory* of $\mathcal{C}$. There are equivalences of categories:

$$\mathcal{C}^{\mathrm{hor}} \simeq \bigsqcup_{Y \in \mathrm{Ob}(\mathcal{B})} \mathcal{A}, \qquad \mathcal{C}^{\mathrm{ver}} \simeq \bigsqcup_{X \in \mathrm{Ob}(\mathcal{A})} \mathcal{B}.$$

Let $\mathrm{T}^{\mathrm{hor}}$ (respectively $\mathrm{T}^{\mathrm{ver}}$) be the restriction functor from the category $\mathcal{C}\text{-Mod}$ to the category $\mathcal{C}^{\mathrm{hor}}\text{-Mod}$ (respectively $\mathcal{C}^{\mathrm{ver}}\text{-Mod}$).

Lemma 5.1.46. *(i) The functor* $\mathrm{T}^{\mathrm{hor}}$ *(respectively* $\mathrm{T}^{\mathrm{ver}}$*) sends projective* $\mathcal{C}$*-modules to projective* $\mathcal{C}^{\mathrm{hor}}$*-modules (respectively projective* $\mathcal{C}^{\mathrm{ver}}$*-modules).*

(ii) Let V *be a* $\mathcal{C}$*-module. For each* $i \geqslant 0$*, one has:*

$$\mathrm{T}^{\mathrm{hor}}(\mathrm{H}_i^{\mathrm{hor}}(V)) \cong \mathrm{H}_i^{\mathcal{C}^{\mathrm{hor}}}(\mathrm{T}^{\mathrm{hor}}(V)),$$

$$\mathrm{T}^{\mathrm{ver}}(\mathrm{H}_i^{\mathrm{ver}}(V)) \cong \mathrm{H}_i^{\mathcal{C}^{\mathrm{ver}}}(\mathrm{T}^{\mathrm{ver}}(V)).$$

Proof. (i) It suffices to see that $\mathrm{T}^{\mathrm{hor}}$ (respectively $\mathrm{T}^{\mathrm{ver}}$) sends principal projective $\mathcal{C}$-modules to projective $\mathcal{C}^{\mathrm{hor}}$-modules (respectively projective $\mathcal{C}^{\mathrm{ver}}$-modules). For any $(X, Y) \in \mathrm{Ob}(\mathcal{C})$, one has:

$$\mathrm{T}^{\mathrm{hor}}\, M(\mathcal{C}, (X, Y)) \cong \bigoplus_{W \in \mathrm{Ob}(\mathcal{B})} \bigoplus_{g \in \mathcal{B}(Y,W)} M(\mathcal{C}^{\mathrm{hor}}, (X, W)),$$

$$\mathrm{T}^{\mathrm{ver}}\, M(\mathcal{C}, (X, Y)) \cong \bigoplus_{Z \in \mathrm{Ob}(\mathcal{A})} \bigoplus_{f \in \mathcal{A}(X,Z)} M(\mathcal{C}^{\mathrm{ver}}, (Z, Y)).$$

(ii) The case $i = 0$ is trivial; the case $i > 0$ follows by (i) and exactness of the functors $\mathrm{T}^{\mathrm{hor}}$ and $\mathrm{T}^{\mathrm{ver}}$. ■

5.1.2

The spectral sequences in the following theorem are special cases of the Grothendieck spectral sequence associated to the composition of two functors.

Theorem 5.1.47. *Let V be a $\mathcal{C}$-module. Then there are two convergent first-quadrant spectral sequences:*

$${}^{I}E^2_{pq} = \mathrm{H}^{\mathrm{ver}}_p(\mathrm{H}^{\mathrm{hor}}_q(V)) \Rightarrow \mathrm{H}^{\mathcal{C}}_{p+q}(V),$$

$${}^{II}E^2_{pq} = \mathrm{H}^{\mathrm{hor}}_p(\mathrm{H}^{\mathrm{ver}}_q(V)) \Rightarrow \mathrm{H}^{\mathcal{C}}_{p+q}(V).$$

Proof. We claim that $\mathrm{H}^{\mathrm{hor}}_0$ (respectively $\mathrm{H}^{\mathrm{ver}}_0$) sends projective $\mathcal{C}$-modules to $\mathrm{H}^{\mathrm{ver}}_0$-acyclic (respectively $\mathrm{H}^{\mathrm{hor}}_0$-acyclic) $\mathcal{C}$-modules. It suffices to verify the claim for principal projective $\mathcal{C}$-modules. Set

$$P = \mathrm{H}^{\mathrm{hor}}_0(M(\mathcal{C}, (X, Y))), \quad \text{where } (X, Y) \in \mathrm{Ob}(\mathcal{C}).$$

One has:

$$\mathrm{T}^{\mathrm{ver}}(P) \cong \bigoplus_{f \in \mathcal{A}(X,X)} M(\mathcal{C}^{\mathrm{ver}}, (X, Y)).$$

Hence, by Lemma 5.1.46, for each $i > 0$, one has:

$$\mathrm{T}^{\mathrm{ver}}(\mathrm{H}_i^{\mathrm{ver}}(P)) \cong \mathrm{H}_i^{\mathcal{C}^{\mathrm{ver}}}(\mathrm{T}^{\mathrm{ver}}(P)) \cong \bigoplus_{f\in\mathcal{A}(X,X)} \mathrm{H}_i^{\mathcal{C}^{\mathrm{ver}}}(M(\mathcal{C}^{\mathrm{ver}},(X,Y))) = 0,$$

which implies $\mathrm{H}_i^{\mathrm{ver}}(P) = 0$. Hence, the claim holds for $\mathrm{H}_0^{\mathrm{hor}}$, and similarly for $\mathrm{H}_0^{\mathrm{ver}}$. By (5.1), we may now apply the Grothedieck spectral sequence to the two compositions $H_0^{\mathrm{ver}}H_0^{\mathrm{hor}}$ and $H_0^{\mathrm{hor}}H_0^{\mathrm{ver}}$ to obtain the spectral sequences stated in the theorem. ■

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