Generalized Numeraire Portfolios

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Abstract

Given a set of assets, a numeraire portfolio (Long, 1990) is a self-financing portfolio with positive value and whose return process is a stochastic discount factors process. By relaxing the self-financing constraint, we define the generalized numeraire portfolios, and state necessary and sufficient conditions for their existence. We show that a set of assets admits generalized numeraire portfolios if and only if it is arbitrage free and at least one trading strategy has positive value. We also show that generalized numeraire portfolios are solutions to the optimal growth problem under the weaker constraint that the self-financing condition holds in conditional discounted expected value. Since the numeraire portfolio is unique (up to a scale factor), it generates only one admissible stochastic discount factor process. Generalized numeraire portfolios generate instead an infinite subset of, and, under some conditions, all the admissible one-period stochastic discount factors. Finally, we propose some interesting tests that exploit the notion of generalized numeraire portfolios and provide preliminary empirical evidence.

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1 Introduction

Since the work of Harrison and Kreps (1979), a considerable amount of research in asset pricing has been devoted to the characterization of stochastic discount factors that can be used to price securities in arbitrage-free markets.

In one line of work, researchers have attempted to impose as little structure as possible to the analysis in order to identify the set of *admissible* stochastic discount factors for a given set of assets. For example, in Hansen and Jagannathan (1991) admissibility is defined with respect to the first two moments of the discount factors. In Gallant, Hansen, and Tauchen (1990) and Bekaert and Liu (1999) the same concept is applied in a conditional framework.

As an alternative, researchers have imposed more assumptions on the economic model in an attempt to identify specific stochastic discount factors. The work of Long (1990) on the numeraire portfolio is one interesting example in this area. Long calls numeraire portfolio a self-financing trading strategy with positive value such that, when prices and cash flows are expressed in its units, the current net-of-cash-flow prices are the best predictors of the next period cum-cashflow prices. Long also shows that the numeraire portfolio corresponds to the self-financing strategy with positive value and highest continuously compounded return over a given time interval. For this reason, the numeraire portfolio is also referred to as the optimal-growth portfolio. Due to its appealing economic interpretation, the numeraire portfolio has been further analyzed by Artzner(1995), Johnson (1996), Bajeux-Besnainou and Portait (1997), Dijkstra (1998), Hentschel, Kang and Long (1999), Bansal (1997), and Bansal and Lehman (1998).

In this paper, we propose an extension of Long's results by introducing what we call generalized numeraire portfolios. In particular, we show that a given set of assets admits generalized numeraire portfolios if and only if: (a) it is free of arbitrage opportunities, and (b) it allows to manage at least one trading strategy with strictly positive value. Compared to the numeraire portfolio, our portfolios represent an important generalization for a variety of reasons. First, the numeraire portfolio is only a special case of generalized numeraire portfolio, because the strategy that it generates is required to be self-financing. In addition, the conditions for its existence are stronger then those for the existence of generalized numeraire portfolios: it is possible to produce robust examples of arbitrage-free securities markets for which the generalized numeraire portfolios are well defined but the numeraire portfolio is not. Finally, the numeraire portfolio is unique up to a scale factor, so that the reciprocals of its per-period gross returns generate a unique one-period stochastic discount factor process. The generalized numeraire portfolios, instead, generate all the one-period stochastic discount factors that characterize no-arbitrage prior to the terminal period. This fact is particularly important when the securities market is incomplete over some period prior to the last one, so that, under no-arbitrage, the set of one-period stochastic discount factor processes is infinite. In this case, the generalized numeraire portfolios track either a non trivial subset, or under some further conditions, all elements of that set. By contrast, the numeraire portfolio tracks only one element of the set of one-period stochastic discount factor processes. Hence generalized numeraire portfolios provide a far richer characterization of the information in asset prices.

For empirical purposes, the (generalized) numeraire portfolio approach provides a very intuitive characterization of stochastic discount factors as the inverse of the gross price appreciation rate of the numeraire portfolios. We derive several empirical implications of the theory and describe how to estimate generalized numeraire portfolios. We implement our approach on US decile portfolios and show that while both the usual market index proxies and fixed weight portfolios estimates of Long's numeraire portfolio fail to price all decile portfolios, the generalized numeraire extension of these portfolios do so. Although preliminary in nature, these results suggests that the generalized numeraire portfolio approach may provide an attractive alternative to empirically characterize admissible stochastic discount factors.

The paper is structured as follows. In Section 2, we model a multiperiod, finite horizon securities market in which assets trade at each date. We also review the notion of one-period stochastic discount factor processes, and relate them to the notion of no-arbitrage. Section 3 contains the core of the theoretical contribution of the paper. We define the generalized numeraire portfolios, analyze the links between generalized numeraire portfolios and one-period stochastic discount factors processes, and present necessary and sufficient conditions for their existence. Finally, we explain their relation with an extended version of the optimal growth problem. In Section 4, we discuss the main testable implications of our theoretical results. Section 5 presents some preliminary empirical evidence. Section 6 concludes the paper.

2 Assumptions and notation

We consider a frictionless securities market in which J assets are traded over the investment horizon $\mathcal{T} = \{0, 1, ..., T\}$. We denote by $d_j(t)$ the cashflow distributed by asset j at date t, by $S_j(t)$ the date t price of asset j *net of the current cashflow*, and regard $S_j(t)$ and $d_j(t)$ as random variables with a finite, but otherwise arbitrary, number of possible outcomes. Without loss of generality, we impose that the assets distribute no cashflow at date 0, and a liquidating one at date T, i.e., $d_j(0) = S_j(T) = 0$. At each date t, the investors share a common information set, $\mathcal{P}_t$, constituted by all the histories of prices and cash flows' outcomes up to t, and agree on the probability measure, P, that governs the occurrence of the events in $\mathcal{P}_t$.

Our definition of asset is quite general, and can be specialized to fit the basic features of most securities commonly traded in actual financial markets. For example, if asset j is a *share of common stock*, we interpret $S_j(t)$ as the ex-dividend price, $d_j(t)$ as the dividend, and impose $d_j(t) \geq 0$ for all t. If asset j is instead a *unit zero-coupon bond* with maturity t', we interpret $S_j(t)$ as the bond price and let $d_j(t) = 0$ for all $t \neq t'$, and $d_j(t') = 1$. As a third example, suppose that asset j is a *marked-to-market futures position*; in this case, we interpret $S_j(t)$ as the value of the position, $d_j(t)$ as the margin generated by marking the position to market, and impose $S_j(t) = 0$ for all t,[1] and $d_j(t) = f(t) - f(t-1)$, with $f(t)$ the date t futures price. Finally, if asset j is a *European call option* with maturity T and strike K on asset i, we interpret $S_j(t)$ as the call premium and let $d_j(t) = 0$ for all $t \leq T - 1$, and $d_j(T) = \max(d_i(T) - K, 0)$.

We describe intertemporal trading by means of sequences $\theta = \{\theta(t)\}_{t=0}^{T-1}$ of J-dimensional random variables, that is $\theta(t) = \{\theta_1(t), \theta_2(t), ..., \theta_J(t)\}$, where $\theta_j(t)$ represents the position (in number of units) taken in assets j at date t, and liquidated at date $t+1$. We call such $\theta's$ *dynamic trading strategies,* and restrict $\theta_j(t)$ to depend on information in $\mathcal{P}_t$ only.

We denote by $V_\theta(t)$ the date t value of a dynamic trading strategy, defined as the cost of establishing the positions in the J assets at their net-of-cashflow prices, if t precedes the last

[1] We disregard initial and maintenance margins since we assume frictionless trading.

trading date, and, at T, as the payoff from the final liquidation of θ. Therefore:

$$V_\theta(t) = \begin{cases} \theta(t) \cdot S(t) & t < T \\ \theta(T-1) \cdot d(T) & t = T \end{cases} \tag{1}$$

In what follows, we refer to the sequence $V_\theta = \{V_\theta(t)\}_{t=0}^{T}$ as to the *value process* of θ.

At date t, a dynamic trading strategy produces a cashflow $x_\theta(t)$, equal to the difference between the resources obtained from liquidating the positions taken at $t-1$ at the cum-cashflow prices $S(t)+d(t)$, and the cost to establish the new positions at the net-of-cashflow prices $S(t)$. The cashflow $x_\theta(t)$ is therefore related to the value $V_\theta(t)$ as follows:

$$x_\theta(t) = \begin{cases} -V_\theta(0) & t = 0 \\ \theta(t-1) \cdot [S(t) + d(t)] - V_\theta(t) & 0 < t < T \\ V_\theta(T) & t = T \end{cases} \tag{2}$$

Henceforth, we call the sequence $x_\theta = \{x_\theta(t)\}_{t=0}^{T}$ the *cashflow process* of θ.

The basic economic assumption underlying our results is the absence of arbitrage opportunities, a minimal requirement for the existence of equilibrium in a frictionless securities market populated by non-satiated investors. In our framework, a dynamic trading strategy θ gives rise to an arbitrage opportunity when it generates a cashflow process non-negative at all dates, and positive with positive probability at some date, that is

$$P(x_\theta(t) \geq 0) = 1 \qquad \textit{for all } t$$
$$P(x_\theta(t) > 0) > 0 \qquad \textit{for some } t$$

Absence of arbitrage opportunities at the given level of prices and cash flows can be characterized in terms of *stochastic discount factors processes,* defined as follows.

Definition 1 *A stochastic discount factors (SDFs) process for a set of assets is a sequence* $m = \{m(t)\}_{t=1}^{T}$ *of random variables that are strictly positive, depend only on information in* $\mathcal{P}_t$*, and satisfy*

$$S_j(t-1) = E\{m(t)[S_j(t) + d_j(t)] \mid \mathcal{P}_{t-1}\}, \quad \textit{for all } j,t \tag{3}$$

The sense in which the SDFs processes characterize no-arbitrage is that a set of assets is arbitrage-free if, and only if, it admits SDFs processes (see, e.g., Duffie, 1996).

The SDFs processes, moreover, constitute the tool for pricing under no-arbitrage the intertemporal cash flows generated by trading in a set of assets. Given a dynamic trading strategy θ, if we multiply both sides of (3) by $\theta(t-1)$ and exploit the definitions of value and cashflow processes, we obtain

$$V_\theta(t-1) = \begin{cases} E\{m(t)[V_\theta(t) + x_\theta(t)] \mid \mathcal{P}_{t-1}\} & t < T \\ E[m(T)\, x_\theta(T) \mid \mathcal{P}_{T-1}] & t = T \end{cases} \tag{4}$$

Next, we apply (4) in an iterative fashion, make use of the law of iterated expectation, and let $M(t-1,\tau) \equiv \prod_{l=t}^{\tau} m(l)$ represent the multiperiod stochastic discount factors derived from the SDFs process m, to obtain

$$V_\theta(t-1) = E\left[\sum_{\tau=t}^{T} M(t-1,\tau)\, x_\theta(\tau) \;\middle|\; \mathcal{P}_{t-1}\right], \quad \textit{for all } t \tag{5}$$

Under no-arbitrage, therefore, the date $t-1$ value of a dynamic trading strategy θ is equal to the conditional expected present value of the cash flows generated by θ after $t-1$, cash flows discounted back to $t-1$ with the multiperiod stochastic discount factors $M(t-1,\tau)$.

3 Generalized numeraire portfolios

To introduce the pivotal concept of this paper, we denote by Θ_+ the set collecting all the dynamic trading strategies whose value process is certainly strictly positive, that is

$$\Theta_+ = \{\theta \mid P(V_\theta(t) > 0) = 1, \text{ for all } t\}$$

We define the generalized numeraire portfolios as follows

Definition 2 *A generalized numeraire portfolio for a set of assets is a dynamic trading strategy* $\theta_{GN} \in \Theta_+$ *such that*

$$\frac{S_j(t-1)}{V_{\theta_{GN}}(t-1)} = E\left[\frac{S_j(t) + d_j(t)}{V_{\theta_{GN}}(t)} \;\middle|\; \mathcal{P}_{t-1}\right], \quad \textit{for all } j, t \tag{6}$$

The value of a generalized numeraire portfolio is a unit of account under which the net-of-cashflow prices at date $t-1$ are the best predictors of the cum-cashflow prices at t. Applying the law of iterated expectations to (6), we see that

$$\frac{S_j(t-1)}{V_{\theta_{GN}}(t-1)} = E\left[\sum_{\tau=t}^{T} \frac{d_j(\tau)}{V_{\theta_{GN}}(\tau)} \;\middle|\; \mathcal{P}_{t-1}\right], \quad \textit{for all } j, t$$

When prices and cash flows are expressed in units of the value of a generalized numeraire portfolio, therefore, the ex-dividend price of each asset is equal to the sum of its expected future cash flows.

The following result states necessary and sufficient conditions for a set of assets to admit generalized numeraire portfolios. All proofs are in the appendix.

Theorem 1 *A set of assets admits generalized numeraire portfolios if, and only if, it is arbitrage-free and satisfies* $\Theta_+ \neq \emptyset$.

The existence of generalized numeraire portfolios, therefore, is completely characterized by only two conditions. The first is the no arbitrage condition, that we have discussed in the previous section. The second is the existence of at least one trading strategy whose value process is strictly positive with probability one. This condition guarantees that the value of at least one portfolio can serve as a bona fide numeraire.

3.1 Generalized numeraire portfolios as SDFs processes

The generalized numeraire portfolios are closely related to the SDFs processes. Observe indeed that (6) is equivalent to

$$S_j(t-1) = E\left\{ \left[\frac{V_{\theta_{GN}}(t)}{V_{\theta_{GN}}(t-1)} \right]^{-1} [S_j(t) + d_j(t)] \,\middle|\, \mathcal{P}_{t-1} \right\}, \quad \textit{for all } j, t \tag{6'}$$

Compare then (6') with Definition 1 in the previous section, to realize that a generalized numeraire portfolio θ_{GN} generates the SDFs process $m_{\theta_{GN}}$ defined as follows:

$$m_{\theta_{GN}} \equiv \left\{ \left[\frac{V_{\theta_{GN}}(t)}{V_{\theta_{GN}}(t-1)} \right]^{-1} \right\}_{t=1}^{T} \tag{7}$$

To give a financial interpretation to $m_{\theta_{GN}}$, we exploit the definition of value in (1), and the relation between value and cashflow in (2), to see that

$$\frac{V_{\theta_{GN}}(t)}{V_{\theta_{GN}}(t-1)} = \begin{cases} \dfrac{\theta_{GN}(t-1)\cdot[S(t)+d(t)]}{\theta_{GN}(t-1)\cdot S(t-1)} - \dfrac{x_{\theta_{GN}}(t)}{\theta_{GN}(t-1)\cdot S(t-1)} & t<T \\[2ex] \dfrac{\theta_{GN}(T-1)\cdot d(T)}{\theta_{GN}(T-1)\cdot S(T-1)} & t=T \end{cases} \tag{8}$$

The components of the SDFs process $m_{\theta_{GN}}$ generated by θ_{GN} via (7) are then the reciprocals of the gross returns on θ_{GN} from $t-1$ to t, net of the *cashflow yield* $\frac{x_{\theta_{GN}}(t)}{\theta_{GN}(t-1)\cdot S(t-1)}$ for the periods before the last.

All marketed cash flows can therefore be priced using the gross returns on a generalized numeraire portfolio. Recall indeed from (5) that the no-arbitrage value of a trading strategy can be computed from its generated cashflow using the multiperiod stochastic discount factors $M(t-1,\tau)$ derived from any SDFs process m. Given a generalized numeraire portfolios θ_{GN}, we can then plug in (5) the multiperiod stochastic discount factors $M_{\theta_{GN}}(t-1,\tau) \equiv \prod_{l=t}^{\tau}\left[\frac{V_{\theta_{GN}}(l)}{V_{\theta_{GN}}(l-1)}\right]^{-1}$ derived from $m_{\theta_{GN}}$. From this standpoint, the date $t-1$ value of a dynamic trading strategy is the conditional expected present value of its future cash flows, with each date τ cashflow discounted back to $t-1$ at the $(\tau-t+1)-$periods rate obtained compounding the one-period gross returns on θ_{GN} (net of the cashflow yields, before the last period).

At this point, the following questions arise: do the generalized numeraire portfolios generate *all* the SDFs processes, that is, do we obtain *all* the SDFs processes by letting θ_{GN} vary in (7)? Or, instead, the SDFs processes that can be represented as the reciprocals of the gross returns, net of the cashflow yield before the last period, on a generalized numeraire portfolio are just a subset of the entire set of SDFs processes? And, if so, what are the properties that characterize this subset?

To answer, we let $\mathcal{M}$ denote the entire set of SDFs processes, and $\mathcal{M}_{GN}$ the set of all SDFs processes generated via (7), that is

$$\mathcal{M}_{GN} = \left\{ m \in \mathcal{M} \;\middle|\; m(t) = m_{\theta_{GN}}(t) \equiv \left[\frac{V_{\theta_{GN}}(t)}{V_{\theta_{GN}}(t-1)}\right]^{-1} \;\forall t, \textit{for some } \theta_{GN} \right\} \tag{9}$$

We define then the following subset $\mathcal{M}_T$ of $\mathcal{M}$:

$$\mathcal{M}_T = \left\{ m \in \mathcal{M} \;\middle|\; m(T) = \left[\frac{V_\theta(T)}{V_\theta(T-1)}\right]^{-1} \textit{for some } \theta \right\} \tag{10}$$

The only restriction imposed on the elements of $\mathcal{M}_T$ is that their last component be the reciprocal of the gross return on some dynamic trading strategy. This restriction, however, is quite tight, since it implies that *all the SDFs processes in $\mathcal{M}_T$ have the same last component.* The reason is that all dynamic trading strategies such that the reciprocal of their last period gross return is a date T stochastic discount factor (hence, in particular, all generalized numeraire portfolios) have the same gross return from $T-1$ to T.[2]

We characterize now the relationships among the sets $\mathcal{M}$, $\mathcal{M}_T$ and $\mathcal{M}_{GN}$. The comparison of $\mathcal{M}$ and $\mathcal{M}_{GN}$ is based on the notion of *complete markets at date* T. Specifically, we say that markets are complete at date T if any payoff x depending only on information in $\mathcal{P}_T$ is marketed at date $T-1$, i.e. $x = \theta(T-1) \cdot d(T)$ for some asset allocation $\theta(T-1)$ depending only on information in $\mathcal{P}_{T-1}$.

Theorem 2 *If a set of assets satisfies $\Theta_+ \neq \emptyset$, then $\mathcal{M}_{GN} = \mathcal{M}_T$, that is, for any SDFs process $m \in \mathcal{M}_T$ there exists a generalized numeraire portfolio θ_{GN} such that*

$$m(t) = m_{\theta_{GN}}(t) \equiv \left[\frac{V_{\theta_{GN}}(t)}{V_{\theta_{GN}}(t-1)}\right]^{-1}, \quad \textit{for all } t$$

If $\Theta_+ \neq \emptyset$ and markets are complete at date T, moreover, then $\mathcal{M}_{GN} = \mathcal{M}$, that is all the SDFs processes can be recovered by letting θ_{GN} vary in (7).

Theorem 2 supplies a detailed answer to the questions concerning the relationships among $\mathcal{M}$, $\mathcal{M}_T$ and $\mathcal{M}_{GN}$. First, it shows that when the generalized numeraire portfolios are well defined, they span via (7) a set of SDFs processes that need to satisfy only one condition: their last component must be the reciprocal of the gross return on some dynamic trading strategy. This can be rephrased as follows: if the last component of an SDFs process is the reciprocal of the gross return on some dynamic trading strategy, then *all of its components* are reciprocals of the one-period returns, net of the cashflow yield before the last period, on some generalized numeraire portfolio. When markets are complete at date T, moreover, the condition on the last component is not binding. In that case, there exists a unique stochastic discount factor that prices, at $t-1$, the liquidating cash flows, so that all SDFs processes have the same terminal component. Theorem 2 shows that, in this case, the generalized numeraire portfolios generate via (7) *the entire set* of SDFs processes that characterize no arbitrage for a given set of assets.

[2] This is proved as Lemma 1 in the Appendix.

3.2 Generalized numeraire portfolios as extended growth-optimal portfolios

The generalized numeraire portfolios are also related to the *growth-optimal portfolio* extensively discussed in the finance literature.[3] To review this concept, recall that a dynamic trading strategy θ is *self-financing* when at all the intermediate trading dates the resources obtained from liquidating $\theta(t-1)$ *exactly cover* the cost to establish $\theta(t)$, that is

$$x_\theta(t) = \theta(t-1) \cdot [S(t) + d(t)] - V_\theta(t) = 0, \quad t = 1, ..., T-1$$

The *growth-optimal portfolio* is the dynamic trading strategy which, *among the self-financing ones,* has the highest continuously compounded return from 0 to T, i.e. it is the solution to the *optimal growth problem*

$$\begin{aligned} &\max_{\theta \in \Theta_+} E\left\{\ln\left[\frac{V_\theta(T)}{V_\theta(0)}\right]\right\} \\ &s.t.\ x_\theta(t) = 0,\ t = 1, ..., T-1 \end{aligned} \tag{11}$$

Equivalently, the growth-optimal portfolio is the dynamic trading strategy that maximizes the expected logarithmic utility from terminal wealth.

Hereafter, we characterize the generalized numeraire portfolios as solutions to an extended version of the optimal growth problem, in which the dynamic trading strategies with the highest continuously compounded return from 0 to T are searched for in a set that includes, but is not exhausted by, the self-financing ones. Precisely, given a SDF process $m = \{m(t)\}_{t=1}^T$, we consider the *extended optimal growth problem* $\mathbb{P}(m)$, defined as follows:

$$\mathbb{P}(m): \quad \begin{cases} \max_{\theta \in \Theta_+} E\left\{\ln\left[\frac{V_\theta(T)}{V_\theta(0)}\right]\right\} \\ s.t.\ E\left[m(t)x_\theta(t) \mid \mathcal{P}_{t-1}\right] = 0, \quad t = 1, ..., T-1 \end{cases}$$

While the objective is the same as in the standard optimal growth problem, the feasible set of $\mathbb{P}(m)$ includes also non self-financing dynamic trading strategies, as long as they satisfy the following property: their intermediate cash flows multiplied by $m(t)$ have zero expected value conditional on information in $\mathcal{P}_{t-1}$. Recall now that $m(t)$ prices the cash flows marketed at

[3] See Latane' (1959), Markowitz (1959) and Breiman (1960) for early contributions, and Hakansson and Ziemba (1995) for a recent survey.

date t, conditional on the date $t-1$ information. Therefore, the feasible set of $\mathbb{P}(m)$ can be interpreted as the collection of dynamic trading strategies θ with strictly positive value and whose intermediate cash flows $x_\theta(t)$ would be assigned zero date $t-1$ price by $m(t)$, if they could be stripped from the payoff process $\{x_\theta(t)\}_{t=0}^{T}$ and marketed by themselves.[4]

We now characterize the generalized numeraire portfolio in terms of solutions to $\mathbb{P}(m)$.

Theorem 3 *A generalized numeraire portfolio θ_{GN} is a solution to $\mathbb{P}(m_{\theta_{GN}})$, i.e. to the extended optimal growth problem parameterized by the SDFs process $m_{\theta_{GN}} \equiv \left\{ \left[\frac{V_{\theta_{GN}}(t)}{V_{\theta_{GN}}(t-1)} \right]^{-1} \right\}_{t=1}^{T}$ generated by θ_{GN} via (7).*

To interpret this result, recall that $\left[\frac{V_{\theta_{GN}}(t)}{V_{\theta_{GN}}(t-1)} \right]^{-1}$ is the reciprocal of the net-of-cashflow-yield return on θ_{GN}. A generalized numeraire portfolio, therefore, has the highest continuously compounded return from 0 to T among all θ in Θ_+ whose intermediate cash flows $x_\theta(t)$, discounted back to $t-1$ at the gross return on θ_{GN} net-of-cashflow-yield, have zero conditional expected value. In other words, θ_{GN} maximizes the continuously compounded return among the strategies in Θ_+ whose intermediate cash flows $x_\theta(t)$, if stripped away from $\{x_\theta(t)\}_{t=0}^{T}$ and marketed by themselves, would be assigned zero date $t-1$ price by the stochastic discount factor generated by θ_{GN}.

3.3 Generalized numeraire portfolios versus Long's numeraire portfolio

Our definition of generalized numeraire portfolios is closely related to Long's (1990) *numeraire portfolio*. Long calls *numeraire portfolio* a *self-financing* dynamic trading strategy with strictly positive value process such that, when prices and cash flows are expressed in its units, the current net-of-cashflow prices are the best predictors of the next-period cum-cashflow ones. In our notation, therefore, the numeraire portfolio is a dynamic trading strategy $\theta_N \in \Theta_+$ that

[4] The formal sense in which an intermediate payoff $x_\theta(t)$ is marketable by itself when stripped away from $\{x_\theta(t)\}_{t=0}^{T}$ is the following: *there exists θ' such that* $x_{\theta'}(t) = x_\theta(t)$ and $x_{\theta'}(\tau) = 0$ if $\tau \neq t$. Thus, $x_\theta(t)$ is marketed by itself if some dynamic trading strategy generates a payoffs process equal to $x_\theta(t)$ at date t, and null at all other times. We use the conditional in the interpretation of the strategies that are feasible for $\mathbb{P}(m)$ because *we do not require* that the intermediate payoffs $x_\theta(t)$ be actually marketable by themselves (although we obviously allow for it).

satisfies

$$\begin{cases} \dfrac{S_j(t-1)}{V_{\theta_N}(t-1)} = E\left[\dfrac{S_j(t)+d_j(t)}{V_{\theta_N}(t)} \middle| \mathcal{P}_{t-1}\right] & \textit{for all } j,t \\ x_{\theta_N}(t) = 0 & t = 1,...,T-1 \end{cases}$$

As pointed out by Long, moreover, *the numeraire portfolio* θ_N *solves the optimal growth problem* (11), i.e. it is the self-financing dynamic trading strategy in Θ_+ with the highest continuously compounded return from 0 to T.[5]

Clearly, the generalized numeraire portfolios extend Long's numeraire portfolio by relaxing the self-financing constraint. The property with financial significance, indeed, is the existence of a dynamic trading strategy whose value is a numeraire under which the current net-of-cashflow prices are the best predictors of the next-period cum-cashflow ones, while the self-financing requirement seems dispensable. It is now natural to question the actual gains from relaxing the self-financing constraint. We address the theoretical side of this issue below, while we discuss its empirical implications in Section 4.

A first measure of the gains from removing the self-financing requirement comes from comparing the generalized numeraire portfolios to the numeraire portfolio in terms of existence conditions. As shown by Long (1990, Theorem 1), a set of assets admits a numeraire portfolio if, and only if, it is arbitrage-free and, among the dynamic trading strategies with strictly positive value, at least one is self-financing. It is however possible (see Girotto and Ortu, 1996) to supply robust examples of sets of assets which satisfy the following three conditions: 1. no-arbitrage holds, 2. at least one trading strategy has strictly positive value, and 3. no self-financing trading strategy has strictly positive value.[6] By Theorem 1 in this paper, the generalized numeraire portfolios are well defined in these cases, while the numeraire portfolio is not. The generalized numeraire portfolios, therefore, exists for a larger class of assets than the numeraire portfolio, and hence have a wider spectrum of applicability.

A second way to gauge the relevance of our extension is based on comparing the generalized numeraire portfolios to the numeraire portfolio in terms of SDFs processes. Recall to this end that, by Theorem 2, the generalized numeraire portfolios span via (7) the set $\mathcal{M}_T$ of all

[5] This follows directly from Theorem 3 by observing that the numeraire portfolio is a particular generalized numeraire portfolio.

[6] In this framework, for *robust example* we mean a situation in which any sufficiently small, but otherwise arbitrary, change in the realizations of prices and cashflows does not perturb properties 1. to 3.

SDFs processes whose last component is the reciprocal of the gross return on some dynamic trading strategy. The numeraire portfolio, instead, generates via (7) a unique element of $\mathcal{M}_T$. Indeed, all numeraire portfolios have the same one-period gross returns even if they may call for different asset allocations,[7] that is

$$\frac{V_{\theta'_N}(t)}{V_{\theta'_N}(t-1)} = \frac{V_{\theta''_N}(t)}{V_{\theta''_N}(t-1)}, \quad \textit{for any } \theta'_N, \theta''_N \textit{ and for all } t$$

Therefore, the SDFs process

$$m_N \equiv \left\{ \left[\frac{V_{\theta_N}(t)}{V_{\theta_N}(t-1)} \right]^{-1} \right\}_{t=1}^{T}$$

is independent from the composition of θ_N, and hence it constitutes a unique element of $\mathcal{M}_T$. Consider now the case in which no arbitrage holds but *markets are incomplete at some date* $t < T$, in the sense that some payoff $\overline{x}$ that depends only on information in $\mathcal{P}_t$ is not marketed at date $t-1$.[8] In this case, the set of SDFs processes that differ (at least) by their $t-th$ component is infinite,[9] which implies that $\mathcal{M}_T$ is infinite as well. When markets are incomplete at some date before the last one, therefore, the generalized numeraire portfolios span an infinite set of SDFs processes, and hence convey more information about the properties of the prices and cash flows than the numeraire portfolios, which generates a single SDFs process. This distinction, moreover, is maximized in the particular case in which markets are incomplete before the last trading date, but complete at T. By Theorem 2, in this case *the generalized numeraire portfolios span the entire, infinite set of SDFs processes, only one of which is generated by the numeraire portfolio.*

Summing up, the generalized numeraire portfolios constitute a robust extension of the numeraire portfolio when, under no arbitrage, at least one of these two conditions is satisfied:

[7] The composition of the numeraire portfolio is not unique whenever there exist redundant assets, i.e. assets whose cashflow can be replicated by the cashflow process of a dynamic trading strategy involving the other assets. The equality of gross returns across numeraire portfolios with different compositions is established in Long, 1990, Theorem 1.

[8] Formally, this means that there exists $\overline{x}$ in $\mathcal{P}_t$ such that $\overline{x} \neq \theta(t-1) \cdot [S(t) + d(t)]$ for any asset allocation $\theta(t-1)$ that depends only on information in $\mathcal{P}_{t-1}$.

[9] In our framework, a component $m(t)$ of a SDFs process is a random variable with a finite number of strictly positive realizations that satisfy the linear system of equations (3). When markets are incomplete at t, this system has more unknowns than linearly independent equations, and hence and infinite number of (strictly positive) solutions. A formal proof of this fact can be obtained following the proof of the second part of Theorem 2.

1. there exist trading strategies with strictly positive value process, but none of them is self-financing; 2. markets are incomplete at some date before the last. In the first case, the generalized numeraire portfolios are well defined although the numeraire portfolio is not. In the second case, the generalized numeraire portfolios convey more information on the prices and cash flows even if the numeraire portfolio is well defined, and this is so because the numeraire portfolio generates a unique SDFs process, while the generalized numeraire portfolios map an infinite set of SDFs process (*all the* SDFs processes, if markets are complete at date T).

3.4 Generalized numeraire portfolios and risk-neutral valuation

In this section, we compare the generalized numeraire portfolios to the approach taken by Girotto and Ortu (1996, 1997) to extend the set of numeraires commonly employed in the *risk-neutral valuation literature.* In this literature, the standard paradigm consists in selecting a numeraire *a-priori*, to show then the equivalence between no arbitrage and the existence of a probability Q [10] under which the newly denominated current net-of-cashflow prices are the best predictors of the future cum-cashflow ones. The probability Q is usually referred to as an *equivalent martingale measure,* or *risk-neutral probability, associated with the fixed numeraire.* Typical examples of fixed numeraires are the cost of rolling over single-period pure discount bonds (e.g., Duffie, 1996), the price of a zero-coupon bond with maturity the last trading date (e.g., Jamshidian, 1989, Geman, El Karoui and Rochet, 1995), and, more generally, the strictly positive value process of any self-financing dynamic trading strategy (e.g., Schroeder, 1999).

Girotto and Ortu (1996) show that the set of numeraires employable in the risk-neutral valuation framework can be extended by incorporating dynamic trading strategies that are not self-financing. Specifically, they call numeraire any dynamic trading strategy θ with strictly positive value process and that satisfies, for some equivalent martingale measure Q, the relation

$$\frac{S_j(t-1)}{V_\theta(t-1)} = E_Q\left[\frac{S_j(t)+d_j(t)}{V_\theta(t)}\,\middle|\,\mathcal{P}_{t-1}\right], \quad \textit{for all } j,t \tag{12}$$

Comparing (12) with (6) in Definition 2, we see that, from a risk-neutral valuation perspective, the generalized numeraire portfolios can be interpreted as the numeraires for which some associated *equivalent martingale measure Q coincides with the actual probability P.*

[10] The probability Q is required to be equivalent to P, in the sense that P and Q assign positive probability to the same events.

It is interesting to note that the existence of numeraires as defined in (12) is characterized by the same conditions that, in Theorem 1, characterize the existence of generalized numeraire portfolios, that is no arbitrage and $\Theta_+ \neq \emptyset$ (Girotto and Ortu, 1996, Theorem 3.1). If the value of some dynamic trading strategy is a unit of account such that, under a probability Q equivalent to P, the current net-of-cashflow prices are the best predictors of the future cum-cashflow one, then there is also some dynamic trading strategy whose value is a unit of account under which the current net-of-cashflow prices optimally predict the future cum-cashflow ones *under the natural probability* P.

4 Testable implications

In this section, we discuss how to test the empirical content of the extension achieved by the generalized numeraire portfolios. To this end, we characterize the existence of generalized numeraire portfolios in terms of rates of returns and portfolio weights, rather than prices, cash flows and dynamic trading strategies. For simplicity, we concentrate our attention on sets of assets whose net-of-cashflow prices are certainly non null at all times, so that the gross return from $t-1$ to t on each asset j,

$$R_j(t) = \frac{S_j(t) + d_j(t)}{S_j(t-1)}$$

is well-defined at all dates. We denote by $R(t)$ the vector of returns on all the J assets, and interpret any $J-$dimensional random variable $\omega(t) = \{\omega_1(t), ..., \omega_J(t)\}$ that depends only on information in $\mathcal{P}_t$ and satisfies $\sum_{j=1}^{J} \omega_j(t) = 1$ as a set of portfolio weights. We call portfolio weights process any sequence $\omega = \{\omega(t)\}_{t=0}^{T-1}$ of sets of portfolio weights. The following proposition restates Theorem 1 in terms of returns and portfolio weights.

Proposition 1 *A set of assets admits a generalized numeraire portfolio if, and only if, there exist a portfolio weights process ω_{GN}, and a sequence $\{\alpha(t)\}_{t=1}^{T}$ of random variables, with each $\alpha(t)$ depending only on information in $\mathcal{P}_t$ and $\alpha(T) \equiv 0$, such that*

$$P[\alpha(t) + \omega_{GN}(t-1) \cdot R(t) > 0] = 1 \qquad \textit{for all } t \tag{13}$$

$$E\left[\frac{R_j(t)}{\alpha(t) + \omega_{GN}(t-1) \cdot R(t)} - 1 \middle| \mathcal{P}_{t-1}\right] = 0 \qquad \textit{for all } j, t \tag{14}$$

Next, we use conditions (13) and (14) to discuss some of the testable implications of our results on generalized numeraire portfolios.

Implication 1. For any set of asset with well defined returns, by combining Theorem 1 with Proposition 1, we see that satisfying conditions (13) and (14) is equivalent to the absence of arbitrage plus the existence of at least one dynamic trading strategy θ with terminal value $V_\theta(T)$ certainly strictly positive. Therefore, the orthogonality conditions (13) and (14) lend themselves naturally to perform a direct test of the no-arbitrage condition for any set of limited liability assets that allow to generate strictly positive wealth at the end of a given finite horizon.

Implication 2. The empirical relevance of the extension proposed in this paper can be also addressed as follows. One can use the Generalized Method of Moments (Hansen, 1982) to estimate the portfolio weights process $\{\omega_{GN}(t)\}_{t=0}^{T-1}$ and the sequence $\{\alpha(t)\}_{t=1}^{T}$, $\alpha(T) = 0$ from the set of orthogonality conditions (14). Note that from Theorem 3, the sequence $\{\alpha(t)\}_{t=1}^{T}$ must also satisfy the following condition

$$E\left[\left.\frac{\alpha(t)}{\alpha(t)+\omega_{GN}(t-1)\cdot R(t)}\right|\mathcal{P}_{t-1}\right]=0 \qquad \text{for all } t \tag{15}$$

To test if the solution is indeed an estimate of (the per-period simple return on) a generalized numeraire portfolio, one can employ Hansen's test of overidentifying restrictions to determine whether the orthogonality conditions in (14) and (15) are satisfied. Additional orthogonality conditions can be generated by observing that the inverse of the τ period compound capital gain return on the generalized portfolio is also an admissible stochastic discount factor for the τ period compound gross return on any asset, for $1 \leq \tau \leq T$. It can readily be shown that

$$E\left[\left.\frac{\prod_{s=t}^{s=t+\tau} R_j(s)}{\prod_{s=t}^{s=t+\tau}\left(\alpha(s)+\omega_{GN}(s-1)\cdot R(s)\right)}-1\right|\mathcal{P}_{t-1}\right]=0 \qquad \text{for all } j, t \leq T-\tau \tag{16}$$

Implication 3 The existence of Long's numeraire portfolios is equivalent to the existence of a portfolio weights process $\omega_N = \{\omega_N(t)\}_{t=0}^{T-1}$ such that

$$E\left[\left.\frac{R_j(t)}{\omega_N(t)\cdot R(t)}-1\right|\mathcal{P}_{t-1}\right]=0 \qquad \textit{for all } j, t \tag{17}$$

The existence of Long's numeraire portfolios, therefore, can be tested along the same lines outlined for the generalized numeraire portfolios, with the exception that now one imposes the additional restriction $\alpha(t) = 0$ *for all* $t = 1, .., T-1$. The generalized numeraire portfolios can then be compared to the numeraire portfolio in terms of the average pricing errors for each asset computed in the two cases: the intuition is that the average pricing errors are likely to deteriorate in the case of the numeraire portfolio, since the estimation is performed under the further restriction $\alpha(t) = 0$ *for all* $t = 1, .., T-1$. By the same token, it could also be the case that the set of orthogonality conditions (17) is rejected while that in (14) and (15) are not.

Implication 4. Various portfolios of assets are commonly used in the empirical literature to test the pricing restrictions of a given model. A typical example is the value-weighted equity index computed by the Center for Research on Security Prices (CRSP) at the University of Chicago. Long (1990) suggests the following procedure to evaluate the performance of those portfolios as proxies of the numeraire portfolio. If $R_P = \{R_P(t)\}_{t=1}^{T}$ denotes the sequence of one-period returns on a candidate proxy, one substitutes $\omega_N(t) \cdot R(t)$ with $R_P(t)$ in (17) and then measure the pricing errors $E\left[\left.\frac{R_j(t)}{R_P(t)} - 1\right| \mathcal{P}_{t-1}\right] = 0$ for each asset at each trading date t. The theoretical results on the generalized numeraire portfolios allow us to improve over Long's approach in the following way. Given the return process R_P on a candidate proxy, compute the average pricing errors $E\left[\left.\frac{R_j(1)}{\alpha(t)+R_P(t)} - 1\right| \mathcal{P}_{t-1}\right] = 0$ using in this orthogonality condition the GMM estimates for $\{\alpha(1), \alpha(2), ..., \alpha(T-1)\}$. Obviously, when using the generalized numeraire portfolios their existence can be tested so that the average pricing errors can only improve.

The generalized numeraire portfolios can also be used to measure abnormal returns. For example, let R_f be the risk-free rate of return, β_j the beta of some security and R_M the return on the market portfolio, and consider the CAPM-based abnormal returns $\varepsilon_j = R_j - R_f - \beta_j(R_M - R_f)$. Although a testable implication of the CAPM is $E(\varepsilon_j) = 0$, one typically runs into (at least) three types of problems: first, one needs a sensible proxy for the market portfolio, second one has to cope with CAPM parameters that can be cursed by estimation errors, and eventually, if the no-abnormal-returns hypothesis is rejected, one has to decide if this is a rejection of the CAPM or, more profoundly, of market efficiency/no-arbitrage. The generalized numeraire portfolios supply the following alternative: given the return R_{GN} on a generalized numeraire portfolio, define the abnormal returns on any asset j as $\eta_j = \frac{R_j}{R_{GN}} - 1$. A direct testable implication of our theoretical results is $E\left(\eta_j\right) = 0$. In performing this test, one encounters only two types of problems: the identification of a proxy for a generalized numeraire portfolio, and the fact that rejecting $E\left(\eta_j\right) = 0$ implies rejecting the joint hypothesis of no-arbitrage and $\Theta_+ \neq \emptyset$, i.e. the existence of at least one trading strategy with strictly positive value. However, since the condition $\Theta_+ \neq \emptyset$ is typically satisfied, in all the interesting cases the rejection of $E\left(\eta_j\right) = 0$ is to be attributed to either the choice of the proxy or to the existence of arbitrage opportunities.

5 Data and tests

In this section, we investigate the empirical relevance of the generalized numeraire portfolio. In particular, we will start by investigating how well the commonly used empirical proxies for the numeraire portfolio suggested by Long (1990) or the fixed weight portfolios suggested by Hentschel, Kang and Long (1998) perform in pricing size-based portfolios and whether the simple extension proposed above yields significant improvement in the pricing of the test portfolios.

We use monthly data for the period January 1962 to December 1997, for a total of 432 observations. To measure the return on the market portfolio we use end-of-month total returns on the NYSE-AMEX-NASDAQ composite stock market index computed by the Center for Research in Security Prices (CRSP) at the University of Chicago. We use both the market value-weighted and equally-weighted indices as our market portfolio proxies. The test assets are the 10 size decile portfolios computed by CRSP. The stocks in portfolio j for month t are those of the firms in the j^{th} size decile of NYSE-AMEX-NASDAQ firms at the end of month $t-1$.

We use a number of instruments to model the dynamics of the numeraire portfolio trading strategy. Specifically, we use the lagged dividend price ratio (denoted XDP) on the CRSP market indices in excess of the 1 month T-Bill rate, the lagged default premium (DFP), and the first difference in the monthly returns on the three-month T-Bill (Δr_{3m}.) The T-Bill rates are from the U.S. Government Bond Files developed by CRSP. The default premium is computed as the difference between the yield on Moody's BAA and AAA corporate bond indices.

Table 1 reports the results of the GMM tests of condition (17) for the two market portfolio indices as prespecified proxies of Long's numeraire portfolio. We perform the tests using both gross returns and excess returns as it can readily be shown that condition (14) can be restated as

$$E\left[\frac{R_j(t)}{\alpha(t)+\omega_{GN}(t-1)\cdot R(t)} - \frac{R_1(t)}{\alpha(t)+\omega_{GN}(t-1)\cdot R(t)}\middle|\mathcal{P}_{t-1}\right]=0 \qquad \text{for all } j>1,t \tag{18}$$

Excess returns are computed in excess of the one month return on the 3 month Treasury bill[11]. The results show that for both versions of the test, the market indices are unable to price small stocks correctly. The abnormal returns for decile portfolio 1 are significantly different from zero when denominated in either numeraire proxies. Further, using the equally weighted market index as numeraire is also unsatisfactory for decile portfolios 5, 6 and 7. The tests of the overidentifying restrictions are soundly rejected in all cases. Hence neither the equally weighted nor the value weighted market indices are satisfactory deflators of returns for empirical purposes.

Next we investigate the performance of fixed weight portfolios including two, three or four decile portfolios as proxies for the numeraire portfolio. We use GMM to estimate the weights of the decile portfolios included in the numeraire and impose the restriction that the weight of decile 10 (the highest market capitalization stocks is equal to $1-\omega' i$, where ω denotes the vector of portfolio weights of the other decile portfolios included in the numeraire and i is a conformable vector on ones[12]. Table 2 presents the estimated weights invested in each of the decile portfolios as well as their robust standard errors and the test of the overidentifying restrictions. In all but one cases the portfolios include a large long position in the smallest decile portfolio and a large short position in the next size decile and positive positions of smaller magnitude in the other decile portfolios. As a result, all these portfolios, while having

[11] The excess return specification of the orthogonality condition tends to be slightly easier to estimate. Note though that the raw return and excess return tests are not exactly identical. Implicitly, in the excess return test the numeraire portfolio has to price the ten decile portfolios plus the 3 month bill, while in the raw return test, it prices only the ten decile portfolios.

[12] We duplicated all the estimations and test for fixed weight numeraire and generalized numeraire portfolios that included the decile portfolios and the 3 month Treasury bill. The results are essentially the same.

always a positive value, exhibit extremely large monthly return volatility. In all experiments, at most 2 weights are statistically significantly different from zero, and in all cases, the test of the overidentifying restrictions are rejected.

Next, as a first step to investigate whether generalized numeraire portfolios may have empirical relevance, we estimate a generalized numeraire portfolio of the form

$$GNP(t) = \alpha(t) + R_{NP}(t), \qquad \alpha(t) = -\lambda(t)[R_{NP}(t) - 1], \qquad \lambda(t) = f(\gamma, Z_{t-1}) \tag{19}$$

where R_{NP} is either one of the prespecified market index proxies investigated earlier or a fixed weight portfolio of the decile portfolios, Z_{t-1} a set of conditioning variables known at the beginning of period t, and γ a vector of parameters. We specify the λ process to be linear in the information variables, and choose Z_t to include the lagged dividend price ratio on the market portfolio and the default premium, variables which have been shown to be useful in predicting returns. This particular specification is chosen because it generates (empirically) strictly positive stochastic discount factor, and has an intuitive interpretation has in terms of the cash flow process it implies: α is the fraction of portfolio gains that the investor withdraws from the numeraire when it experiences gains and the fraction he reinvest in his portfolio when the numeraire experiences losses. It is estimated under the constraint that expected numeraire denominated value of the α process is zero, which is imposed through orthogonality condition (15). Parameter estimates and test results are reported in Table 3 when we use the market portfolio proxies and in Table 4 when we use the fixed weight portfolios. First, consider Table 3. For both market index proxies and both returns specifications, the coefficients of the conditioning variables are highly significant. This suggests that the α process varies significantly with market conditions and that such a variation is necessary to account for the time variation in the returns on decile portfolios. This could be due for example, to changes in the prices of risk or to time variation in the investment opportunity set. Next consider the test results for the decile portfolios. The abnormal returns of the decile portfolios, denominated in units of the generalized numeraire portfolios are not significantly different from zero, for the small stock portfolio as well as the other decile portfolios. Moreover, in all cases the test of the overidentifying restrictions is no longer significant. This suggests that the generalized numeraire portfolios can also account for the cross sectional differences in decile portfolio returns.

Consider next the generalized numeraire portfolios constructed from a fixed proportion investment in decile portfolios and a cash flow process. Column two to three report the coefficients of the conditioning variables determining the cash flow process and the next four columns the proportions invested in the decile portfolios. The evidence in panel a is consistent with the evidence reported in Table 3. The coefficients on the conditioning variables are of similar magnitude and sign, and the test of the overidentifying restrictions is not significantly different from zero. Interestingly, the estimated decile portfolio weights, while still large in absolute value, are four to five times smaller that those reported Table 2. This suggest that sufficient volatility of the stochastic discount factor is achieved not through extreme stock positions but through the combination of significant stock positions and a large cash flow process. The results in panel b are less sanguine. The evidence for the estimated GNP including decile 1 and 10 only is very similar to that reported in panel a. However when either decile 5 or decile 4 and 7 are included to the GNP, the parameters of the cash flow process are significantly different and the investment proportions of the included assets are much larger. For the last portfolio, the overidentifying restrictions are also rejected[13].

These results are promising but still preliminary. We need to investigate further specifications of the numeraire and generalized numeraire portfolios along the lines of Long (1990). In the current version of the paper we examine one specification of the generalized numeraire portfolio cash flow process. Other specifications need to be investigated. Second, we may want to add other assets, for example corporate and treasury bond portfolios, to the universe of assets on which the tests are implemented. This would allow for more general forms of the (generalized) numeraire portfolios as well as impose a more severe test on the pricing performance of the implied stochastic discount factors. Lastly, it would be of interest to empirically characterize the whole set of admissible generalized numeraire portfolios. This requires further methodological advances that we leave for future research.

[13] To investigate these results, we estimates a GNP including five decile portfolios (deciles 1, 3, 5, 7 and 10) for both raw returns and excess returns. In this case, the results are consistent for both raw returns and excess returns and the both the cash flow coefficients and the estimated weights are of similar magnitude and sign as in panel a. This suggests that the results for the last two portfolios in Table 4 may be due to some estimation problems

6 Conclusion

In this paper, we generalize the concept of numeraire portfolio proposed by Long, 1990, to the case in which the self-financing restriction is relaxed. For a given set of assets, a numeraire portfolio is a unique (up to a scale factor) self-financing portfolio with positive value and whose return process is a one-period stochastic discount factor process. By relaxing the self-financing constraint, we define the generalized numeraire portfolios and show that they are solutions to an extended optimal growth problem under the weaker constraint that the self-financing condition hold in conditional expected value. In contrast to the numeraire portfolio, in markets incomplete prior to the terminal period, the generalized numeraire portfolios generate an infinite subset of the set of all the one-period stochastic discount factor processes compatible with no-arbitrage .

We use GMM to provide preliminary empirical evidence of the relevance of the generalized numeraire portfolios for decile portfolios. We find that while the numeraire portfolio proposed by Long is rejected by the data due to its failure to price small stocks, the generalized numeraire portfolios deliver consistently small pricing errors for all portfolios and is not rejected by the test of the overidentifying restrictions. However further work is necessary to examine the robustness and general applicability of our approach.

Appendix

To prove our results we need to introduce some further notation. First, we denote by f_k^t the *generic history of prices and cash flows' outcomes up to date t*. Since we assume that prices and cash flows are random variables with a finite number of outcomes, at each date t there is a finite number, s_t, of such histories, so that the information set common to all investors is $\mathcal{P}_t = \{f_k^t, k = 1, ..., s_t\}$. We assume, without loss of generality, that $P(f_k^t) > 0$ for all t, k, i.e. all histories of prices and cash flows have strictly positive probability to occur.

We denote by $S_j(f_k^t)$, $d_j(f_k^t)$, the date t net-of-cashflow price, respectively cashflow of asset j under history f_k^t. We translate the requirement that the components $\theta_j(t)$ of a dynamic trading strategy θ depend only on information in $\mathcal{P}_t$ by denoting their realizations by $\theta_j(f_k^t)$. In other words, $\theta_j(f_k^t)$ represents the position in asset j taken at date t if, up to that date, the history of prices and cash flows' outcomes has been f_k^t. We denote then by $V_\theta(f_k^t)$, respectively $x_\theta(f_k^t)$ the date t value, respectively cashflow, of strategy θ under history f_k^t. Observe that, in this notation, $\Theta_+ = \{\theta \mid V_\theta(f_k^t) > 0 \; \forall t, k\}$. Finally, we translate the requirement that each component $m(t)$ of an SDFs process $m = \{m(t)\}_{t=1}^T$ depend only on information in $\mathcal{P}_t$ by denoting its realizations by $m(f_k^t)$.

Given $f_l^{t-1} \in \mathcal{P}_{t-1}$, we denote by $\mathcal{P}_l^t$ the set of prices and cash flows' histories up to t that are compatible with the history up to $t-1$ having been f_l^{t-1}. In other words, $\mathcal{P}_l^t$ collects the information available at t conditional on having observed the history f_l^{t-1} up to $t-1$. A sequence of random variables $m = \{m(t)\}_{t=1}^T$ is then an SDFs process if $m(f_k^t) > 0 \; \forall t, k$, and

$$S_j\left(f_l^{t-1}\right) = \sum_{f_k^t \in \mathcal{P}_l^t} \frac{P(f_k^t)}{P(f_l^{t-1})} \left\{ m\left(f_k^t\right) \left[S_j\left(f_k^t\right) + d_j\left(f_k^t\right) \right] \right\}, \quad \forall\, j, t, l \tag{A1}$$

A dynamic trading strategy $\theta_{GN} \in \Theta_+$ is a generalized numeraire portfolio if

$$S_j\left(f_l^{t-1}\right) = \sum_{f_k^t \in \mathcal{P}_l^t} \frac{P(f_k^t)}{P(f_l^{t-1})} \left\{ \left[\frac{V_{\theta_{GN}}\left(f_k^t\right)}{V_{\theta_{GN}}\left(f_l^{t-1}\right)} \right]^{-1} \left[S_j\left(f_k^t\right) + d_j\left(f_k^t\right) \right] \right\}, \quad \forall\, j, t, l \tag{A2}$$

It is more efficient to prove Theorem 2 before Theorem 1. However, we first need to establish

Lemma 1 *All SDFs processes in the set $\mathcal{M}_T$ defined in (10) have the same last component, that is $m'(T) = m''(T)$ for any m', $m'' \in \mathcal{M}_T$.*

Proof. Given m', $m'' \in \mathcal{M}_T$, let θ' be such that $m'(T) = \frac{V_{\theta'}(T)}{V_{\theta'}(T-1)}$, and θ'' be such that $m''(T) = \frac{V_{\theta''}(T)}{V_{\theta''}(T-1)}$. By (3) in the main text we have

$$S_j\left(T-1\right) = E\left\{\left[\frac{V_{\theta'}(T)}{V_{\theta'}(T-1)}\right]^{-1} d_j\left(T\right) \mid \mathcal{P}_{t-1}\right\}, \quad \forall\, j$$

since, by assumption, $S_j\left(T\right) = 0\ \forall j$. Multiplying both sides of this relation by $\theta''_j(T-1)$, summing up over j, and exploiting the definition of value process we obtain

$$E\left\{\left[\frac{V_{\theta'}(T)}{V_{\theta'}(T-1)}\right]^{-1} \frac{V_{\theta''}(T)}{V_{\theta''}(T-1)} \mid \mathcal{P}_{t-1}\right\} = 1$$

Inverting the order of θ' and θ'' in this procedure, we get instead

$$E\left\{\left[\frac{V_{\theta''}(T)}{V_{\theta''}(T-1)}\right]^{-1} \frac{V_{\theta'}(T)}{V_{\theta'}(T-1)} \mid \mathcal{P}_{t-1}\right\} = 1$$

By Jensen's inequality, these two conditions imply $\frac{V_{\theta'}(T)}{V_{\theta'}(T-1)} = \frac{V_{\theta''}(T)}{V_{\theta''}(T-1)}$, i.e. $m'(T) = m''(T)$ ∎

Proof of Theorem 2. We first show that $\mathcal{M}_{GN} = \mathcal{M}_T$ if $\Theta_+ \neq \emptyset$. To this end, it is enough to show that for any $m \in \mathcal{M}_T$ there exists $\theta \in \Theta_+$ such that

$$m\left(f_k^t\right) = \left[\frac{V_\theta\left(f_k^t\right)}{V_\theta\left(f_l^{t-1}\right)}\right]^{-1}, \quad \forall\, k : f_k^t \in \mathcal{P}_l^t,\ l = 1, ..., s_{t-1},\ t = 1, ..., T \tag{A3}$$

Comparing with (A1) and (A2), it is clear that any such θ is a generalized numeraire portfolio.

Given then $m \in \mathcal{M}_T$, we first observe that, by the definition of $\mathcal{M}_T$, there exists $\widetilde{\theta}$ that satisfies (A3) for $t = T$, that is

$$m\left(f_k^T\right) = \left[\frac{V_{\widetilde{\theta}}\left(f_k^T\right)}{V_{\widetilde{\theta}}\left(f_l^{T-1}\right)}\right]^{-1}, \quad \forall\, k : f_k^T \in \mathcal{P}_l^T,\ l = 1, ..., s_{T-1} \tag{A4}$$

Since $\Theta_+ \neq \emptyset$, there is no loss of generality in assuming that $\widetilde{\theta} \in \Theta_+$. We supply now a recursive procedure that, in $T-1$ steps, transforms $\widetilde{\theta}$ into $\theta \in \Theta_+$ that satisfies (A3). To initialize the procedure, we let $\theta^{(0)} \equiv \widetilde{\theta}$, and define the sequence $\left\{\theta^{(\tau)}\right\}_{\tau=1}^{T-1}$ of dynamic trading strategies according to the following recursive mechanism: $\theta^{(\tau)}(t) = \theta^{(\tau-1)}(t)$ if $t \neq \tau$, while the realizations of $\theta^{(\tau)}(\tau)$ across the histories f_k^τ are given by

$$\theta^{(\tau)}(f_k^\tau) = \left[\frac{V_{\theta^{(\tau-1)}}\left(f_l^{\tau-1}\right)}{m(f_k^\tau)V_{\theta^{(\tau-1)}}\left(f_k^\tau\right)}\right]\theta^{(\tau-1)}(f_k^\tau), \quad \forall\, k : f_k^\tau \in \mathcal{P}_l^\tau,\ l = 1, ..., s_{\tau-1} \tag{A5}$$

We show by induction that, for $\tau = 1, ..., T-1$, $\theta^{(\tau)} \in \Theta_+$ and satisfies (A3) for all $t \leq \tau$. The first step is to establish this for $\tau = 1$. Indeed, since $\theta^{(1)}(t) = \theta^{(0)}(t) \equiv \widetilde{\theta}(t)$ for all $t \neq 1$ and $\widetilde{\theta} \in \Theta_+$, $V_{\theta^{(1)}}(t)$ is strictly positive for all $t \neq 1$, so that we need only to establish $V_{\theta^{(1)}}(f_k^1) > 0$ $\forall k$, and that (A3) holds with $\theta^{(1)}$ for $t = 1$. Let then $\tau = 1$ in (A5), multiply both sides by $S(f_k^1) \equiv (S_1(f_k^1), ..., S_J(f_k^1))$, and exploit the definition of value process, to obtain

$$\begin{aligned} V_{\theta^{(1)}}(f_k^1) &= \theta^{(1)}(f_k^1) \cdot S(f_k^1) \\ &= \left[\frac{V_{\theta^{(0)}}(0)}{m(f_k^1) V_{\theta^{(0)}}(f_k^1)} \right] \theta^{(0)}(f_k^1) \cdot S(f_k^1) \\ &= \left[\frac{V_{\theta^{(0)}}(0)}{m(f_k^1) V_{\theta^{(0)}}(f_k^1)} \right] V_{\theta^{(0)}}(f_k^1) \\ &= \frac{V_{\theta^{(0)}}(0)}{m(f_k^1)} > 0, \quad \forall k \end{aligned} \tag{A6}$$

where the sign follows from $m(f_k^1) > 0$ $\forall k$, and $V_{\theta^{(0)}}(0) > 0$ since $\theta^{(0)}(0) \equiv \widetilde{\theta}(0)$ and $\widetilde{\theta} \in \Theta_+$. Moreover, $V_{\theta^{(0)}}(0) = V_{\theta^{(1)}}(0)$ since by construction $\theta^{(1)}(0) = \theta^{(0)}(0)$, so that (A6) implies

$$m\left(f_k^1\right) = \left[\frac{V_{\theta^{(1)}}\left(f_k^1\right)}{V_{\theta^{(1)}}(0)} \right]^{-1}, \quad \forall\ k = 1, ..., s_1$$

Therefore, $\theta^{(1)} \in \Theta_+$ and satisfies (A3) for $t = 1$. Given now any $\tau \geq 2$, we show that $\theta^{(\tau)} \in \Theta_+$ and that $\{V_{\theta^{(\tau)}}(t)\}_{t=0}^{T}$ satisfies (A3) for all $t \leq \tau$ if $\theta^{(\tau-1)} \in \Theta_+$ and satisfies (A3) for all $t \leq \tau - 1$. Since by construction $\theta^{(\tau)}(t) = \theta^{(\tau-1)}(t)$ for all $t \neq \tau$ and $\theta^{(\tau-1)} \in \Theta_+$, clearly $V_{\theta^{(\tau)}}(t)$ is strictly positive for all $t \neq \tau$ and satisfies (A3) for $t \leq \tau - 1$. Therefore, we only need to show that $V_{\theta^{(\tau)}}(f_k^\tau) > 0$ $\forall k$, and that it satisfies (A3) for $t = \tau$. Multiply then both sides of (A5) by $S(f_k^\tau) \equiv (S_1(f_k^\tau), ..., S_J(f_k^\tau))$, $f_k^\tau \in \mathcal{P}_l^\tau$ and exploit the definition of value process, to obtain

$$\begin{aligned} V_{\theta^{(\tau)}}(f_k^\tau) &= \theta^{(\tau)}(f_k^\tau) \cdot S(f_k^\tau) \\ &= \left[\frac{V_{\theta^{(\tau-1)}}\left(f_l^{\tau-1}\right)}{m(f_k^\tau) V_{\theta^{(\tau-1)}}(f_k^\tau)} \right] \theta^{(\tau-1)}(f_k^\tau) \cdot S(f_k^\tau) \\ &= \left[\frac{V_{\theta^{(\tau-1)}}\left(f_l^{\tau-1}\right)}{m(f_k^\tau) V_{\theta^{(\tau-1)}}(f_k^\tau)} \right] V_{\theta^{(\tau-1)}}(f_k^\tau) \\ &= \frac{V_{\theta^{(\tau-1)}}\left(f_l^{\tau-1}\right)}{m(f_k^\tau)} > 0, \quad \forall\ k : f_k^\tau \in \mathcal{P}_l^\tau,\ l = 1, ..., s_{\tau-1} \end{aligned} \tag{A7}$$

where the sign follows from $m(f_k^\tau) > 0 \; \forall k$ and $V_{\theta^{(\tau-1)}}\left(f_l^{\tau-1}\right) > 0$ for all l since by construction $\theta^{(\tau)}(\tau-1) = \theta^{(\tau-1)}(\tau-1)$ and $\theta^{(\tau-1)} \in \Theta_+$. Moreover, since $\theta^{(\tau)}(\tau-1) = \theta^{(\tau-1)}(\tau-1)$ then $V_{\theta^{(\tau-1)}}\left(f_l^{\tau-1}\right) = V_{\theta^{(\tau)}}\left(f_l^{\tau-1}\right)$, so that (A7) implies

$$m\left(f_k^\tau\right) = \left[\frac{V_{\theta^{(\tau)}}(f_k^\tau)}{V_{\theta^{(\tau)}}\left(f_l^{\tau-1}\right)}\right]^{-1}, \quad \forall\, k : f_k^\tau \in \mathcal{P}_l^\tau,\, l = 1, ..., s_{\tau-1}$$

which proves our claim on $\theta^{(\tau)}$ for all $\tau = 1, ..., T-1$. For $\tau = T-1$, in particular, $\theta^{(T-1)} \in \Theta_+$ and satisfies (A3) for all $t < T$. To conclude the proof, therefore, we need only to show that $\theta^{(T-1)}$ satisfies (A3) also for $t = T$. To do so, we observe that

$$\begin{aligned} \theta^{(T-1)}(f_l^{T-1}) &= \left[\frac{V_{\theta^{(T-2)}}\left(f_n^{T-2}\right)}{m(f_l^{T-1})V_{\theta^{(T-2)}}\left(f_l^{T-1}\right)}\right]\theta^{(T-2)}(f_l^{T-1}) \\ &= \left[\frac{V_{\theta^{(T-2)}}\left(f_n^{T-2}\right)}{m(f_l^{T-1})V_{\theta^{(T-2)}}\left(f_l^{T-1}\right)}\right]\widetilde{\theta}(f_l^{T-1}), \quad \forall l : f_l^{T-1} \in \mathcal{P}_n^{T-1},\, n = 1, ..., s_{T-2} \end{aligned} \tag{A8}$$

where the first equality comes from (A5), and the second from the fact that, by construction, $\theta^{(\tau)}(T-1) = \theta^{(\tau-1)}(T-1)$ $\tau = 1, ..., T-2$, and $\theta^{(0)}(T-1) \equiv \widetilde{\theta}(T-1)$. Letting then $d(f_k^T) \equiv \left(d_1(f_k^T), ..., d_1(f_k^T)\right)$, from (A8) and the definition of value we have

$$\begin{aligned} V_{\theta^{(T-1)}}(f_k^T) &= \theta^{(T-1)}(f_l^{T-1}) \cdot d(f_k^T) \\ &= \left[\frac{V_{\theta^{(T-2)}}\left(f_n^{T-2}\right)}{m(f_l^{T-1})V_{\theta^{(T-2)}}\left(f_l^{T-1}\right)}\right]\widetilde{\theta}(f_l^{T-1}) \cdot d(f_k^T) \\ &= \left[\frac{V_{\theta^{(T-2)}}\left(f_l^{T-1}\right)}{m(f_l^{T-1})V_{\theta^{(T-2)}}\left(f_l^{T-1}\right)}\right]V_{\widetilde{\theta}}(f_k^T), \quad \forall k : f_k^T \in \mathcal{P}_l^T,\, l : f_l^{T-1} \in \mathcal{P}_n^{T-1},\, n \leq s_{T-2} \end{aligned}$$

and

$$\begin{aligned} V_{\theta^{(T-1)}}(f_l^{T-1}) &= \theta^{(T-1)}(f_l^{T-1}) \cdot S(f_l^{T-1}) \\ &= \left[\frac{V_{\theta^{(T-2)}}\left(f_n^{T-2}\right)}{m(f_l^{T-1})V_{\theta^{(T-2)}}\left(f_l^{T-1}\right)}\right]\widetilde{\theta}(f_l^{T-1}) \cdot S(f_l^{T-1}) \\ &= \left[\frac{V_{\theta^{(T-2)}}\left(f_n^{T-2}\right)}{m(f_l^{T-1})V_{\theta^{(T-2)}}\left(f_l^{T-1}\right)}\right]V_{\widetilde{\theta}}(f_l^{T-1}), \quad \forall k : f_k^T \in \mathcal{P}_l^T,\, l : f_l^{T-1} \in \mathcal{P}_n^{T-1},\, n \leq s_{T-2} \end{aligned}$$

Putting the last two expressions together with (A4), we have

$$\left[\frac{V_{\theta^{(T-1)}}(f_k^T)}{V_{\theta^{(T-1)}}(f_l^{T-1})}\right]^{-1} = \left[\frac{V_{\tilde{\theta}}\left(f_k^T\right)}{V_{\tilde{\theta}}\left(f_l^{T-1}\right)}\right]^{-1} = m\left(f_k^T\right), \quad \forall\, k : f_k^T \in \mathcal{P}_l^T, l = 1, ..., s_{T-1}$$

hence $\theta^{(T-1)}$ satisfies (A3) for $t = T$ as well. Therefore, $\theta \equiv \theta^{(T-1)} \in \Theta_+$ and satisfies (A3), i.e. $\mathcal{M}_{GN} = \mathcal{M}_T$ when $\Theta_+ \neq \emptyset$.

We show now that $\mathcal{M}_{GN} = \mathcal{M}$ when $\Theta_+ \neq \emptyset$ and markets are complete at date T. We need only to prove this statement when $\mathcal{M} \neq \emptyset$. Moreover, since $\mathcal{M}_{GN} = \mathcal{M}_T$ when $\Theta_+ \neq \emptyset$, by Lemma 1 it is enough to show that all SDFs processes have the same last component when markets are complete at date T, i.e. that (A1) has a unique strictly positive solution in the variables $m(f_k^T)$ when $t = T$. Markets completeness at date T means that, given any payoff x in $\mathcal{P}_T$, the following linear system in the variables $\theta_j(f_l^{T-1})$ is feasible for all $l = 1, ..., s_{T-1}$:

$$x(f_k^T) = \sum\nolimits_j \theta_j(f_l^{T-1}) d_j(f_k^T), \quad \forall\, k : f_k^T \in \mathcal{P}_l^T$$

Therefore, the matrix $\left[d_j(f_k^T)\right]_{j=1,..,J;\; k: f_k^T \in \mathcal{P}_l^T}$ has rank equal to the cardinality of $\mathcal{P}_l^T$, $l = 1, ..., s_{T-1}$, which implies that if the linear system

$$S_j\left(f_l^{T-1}\right) = \sum_{f_k^t \in \mathcal{P}_l^t} y(f_k^T) d_j(f_k^T), \quad \forall\, k : f_k^T \in \mathcal{P}_l^T$$

is feasible, it has a unique solution. That this system is feasible when $\mathcal{M} \neq \emptyset$ is readily acknowledged by letting $y(f_k^T) = \dfrac{P(f_k^t)}{P(f_l^{t-1})} m\left(f_k^t\right)$, comparing with (A1), and recalling the assumption $S_j(f_k^T) = 0\ \forall f_k^T$. For $t = T$, therefore, (A1) has indeed a unique solution in the variables $m(f_k^T)$. This concludes the proof ■

We prove now Theorem 1, that establishes necessary and sufficient conditions for a set of assets to admit generalized numeraire portfolios.

Proof of Theorem 1. Since the only if part is obvious, we concentrate on the if part. To establish the existence of generalized numeraire portfolios when no arbitrage holds and $\Theta_+ \neq \emptyset$, it is enough by Theorem 2 to show that $\mathcal{M}_T \neq \emptyset$ under these assumptions. To do so, we establish the existence of θ such that, for all $l = 1, ..., s_{T-1}$, $V_\theta(f_l^{T-1}) = 1$ and

$$S_j\left(f_l^{T-1}\right) = \sum_{f_k^T \in \mathcal{P}_l^T} \frac{P(f_k^T)}{P(f_l^{T-1})} \frac{d_j\left(f_k^T\right)}{V_\theta\left(f_k^T\right)}, \quad \forall\, j,\; k : f_k^T \in \mathcal{P}_l^T$$

Since, by the definition of value process, $V_\theta(f_l^{T-1}) = \sum_j \theta_j(f_l^{T-1})S_j\left(f_l^{T-1}\right)$ and $V_\theta(f_k^T) = \sum_j \theta_j(f_l^{T-1})d_j(f_k^T)$ for all $k : f_k^T \in \mathcal{P}_l^T$, it is readily seen that the above expression constitutes the set of first order condition for an interior solution to the following optimization problem:

$$\max_{\theta(f_l^{T-1})} \sum_{f_k^T \in \mathcal{P}_l^T} \frac{P(f_k^T)}{P(f_l^{T-1})} \ln V_\theta\left(f_k^T\right)$$

$$s.t.\ V_\theta(f_l^{T-1}) = 1$$

The proof is concluded by employing the no arbitrage assumption to observe that, for all $l = 1, ..., s_{T-1}$, this optimization problem has indeed optimal interior solutions ■

Next, we prove Theorem 3, that characterizes the generalized numeraire portfolios as solutions to the extended optimal growth problems $\mathbb{P}(m)$ defined in Subsection 3.2.

Proof of Theorem 3. For simplicity, we prove the result for the case $T = 2$. In this case, problem $\mathbb{P}(m)$ can be explicitly written as follows:

$$\max_{\left\{\left(\theta_j(0), \theta_j(f_l^1)\right) > 0\right\}} \sum_{l=1}^{s_1} \sum_{f_k^2 \in \mathcal{P}_l^2} P(f_k^2) \ln\left[\sum_{j=1}^{J} \theta_j(f_l^1) d_j(f_k^2)\right]$$

$$s.t. \begin{cases} \sum_{l=1}^{s_1} P(f_l^1) m(f_l^1) \left\{ \sum_{j=1}^{J} \theta_j(0) \left[S_j\left(f_l^1\right) + d_j\left(f_l^1\right)\right] - \sum_{j=1}^{J} \theta_j(f_l^1) S_j\left(f_l^1\right) \right\} = 0 \\ \sum_{j=1}^{J} \theta_j(0) S_j\left(0\right) = 1 \end{cases}$$

where we have exploited the fact that there is no loss of generality in normalizing the solutions so that the initial value is 1. Since m is an SDFs process, arbitrage opportunities are ruled out so that the above problem has optimal solutions. Moreover, all solutions can be characterized by the first-order conditions of the Lagrangian associated to our problem. Precisely, denoting by λ_1 the multiplier for the first constraint, and by λ_2 that for the second, the solutions to our problems are the solutions to the following set of first-order conditions:

$$\begin{cases} \sum_{f_k^2 \in \mathcal{P}_l^2} \frac{P(f_k^2) d_j(f_k^2)}{\sum_{i=1}^J \theta_i(f_l^1) d_i(f_k^2)} - \lambda_1 P(f_l^1) m(f_l^1) S_j(f_l^1) = 0 & \begin{matrix} j = 1, ..., J \\ l = 1, ..., s_1 \end{matrix} \\ \lambda_1 \sum_{l=1}^{s_1} P(f_l^1) m(f_l^1) \left[S_j(f_l^1) + d_j(f_l^1) \right] - \lambda_2 S_j(0) = 0 & j = 1, ..., J \\ \sum_{l=1}^{s_1} P(f_l^1) m(f_l^1) \left\{ \sum_{j=1}^J \theta_j(0) \left[S_j(f_l^1) + d_j(f_l^1) \right] - \sum_{j=1}^J \theta_j(f_l^1) S_j(f_l^1) \right\} = 0 & \\ \sum_{j=1}^J \theta_j(0) S_j(0) = 1 & \end{cases} \tag{FOCs}$$

Let now θ_{GN} be a generalized numeraire portfolio and assume, without loss of generality, that $V_{\theta_{GN}}(0) = \sum_j \theta_{j,GN}(0) S_j(0) = 1$. Our proof is concluded by showing that if $m(f_l^1) = \frac{1}{V_{\theta_{GN}}(f_l^1)}$, $l = 1, ..., s_1$, then there exist λ_1, λ_2, such that θ_{GN} solves (FOCs). This is readily done by letting $\lambda_1 = \lambda_2 = 1$, observing that $\sum_{i=1}^J \theta_{i,GN}(f_l^1) d_i(f_k^2) = V_{\theta_{GN}}(f_k^2)$, using the definition of SDFs process in (A2), and observing that

$$\sum_{l=1}^{s_1} \frac{P(f_l^1)}{V_{\theta_{GN}}(f_l^1)} \left\{ \sum_{j=1}^J \theta_{j,GN}(0) \left[S_j(f_l^1) + d_j(f_l^1) \right] - \sum_{j=1}^J \theta_{j,GN}(f_l^1) S_j(f_l^1) \right\} =$$

$$= \sum_{j=1}^J \theta_{j,GN}(0) \sum_{l=1}^{s_1} P(f_l^1) \frac{\left[S_j(f_l^1) + d_j(f_l^1) \right]}{V_{\theta_{GN}}(f_l^1)} - 1$$

$$= \sum_j \theta_{j,GN}(0) S_j(0) - 1 = 0$$

which concludes the proof ■

Finally, we prove the characterization in terms of portfolio weights of the existence of generalized numeraire stated in Proposition 1.

Proof of Proposition 1. Given a generalized numeraire portfolio θ_{GN}, define the portfolio weights process ω_{GN} component-wise as follows:

$$\omega_{j,GN}(t) = \frac{\theta_{j,GN}(t) S_j(t)}{\theta_{GN}(t) \cdot S(t)}, \quad \textit{for all } t < T$$

and let

$$\alpha(t) = \begin{cases} -\dfrac{x_{\theta_{GN}}(t)}{\theta_{GN}(t-1)\cdot S(t-1)} & t = 1, ..., T-1 \\ 0 & t = T \end{cases}$$

Some simple algebraic manipulations shows that

$$\alpha(t) + \omega_{GN}(t-1)\cdot R(t) = \frac{V_{\theta_{GN}}(t)}{V_{\theta_{GN}}(t-1)}, \quad \textit{for all } t \tag{A9}$$

Since $\theta_{GN} \in \Theta_+$, then

$$P[\alpha(t) + \omega_{GN}(t-1)\cdot R(t) > 0] = 1, \quad \textit{for all } t$$

which establishes (13) in Proposition 1. Moreover, substituting (A9) in (6') in the main text, dividing both sides by $S_j(t-1)$, and recalling that $R_j(t) = \frac{S_j(t)+d_j(t)}{S_j(t-1)}$, we have

$$E\left\{\frac{R_j(t)}{\alpha(t) + \omega_{GN}(t-1)\cdot R(t)} - 1 \,\middle|\, \mathcal{P}_{t-1}\right\} = 0, \quad \textit{for all } j, t$$

which establishes (14) in Proposition 1, and concludes the proof of the only if part.

To prove the if part, let the portfolio weights process ω_{GN} and the sequence $\{\alpha(t)\}_{t=1}^{T}$, with $\alpha(t)$ in $\mathcal{P}_t$ and $\alpha(T) \equiv 0$, satisfy (13) and (14) in Proposition 1. Define then θ component-wise according to the following recursive procedure:

$$\theta_j(t) = \begin{cases} \dfrac{\omega_{j,GN}(t)}{S_j(t)} & t = 0 \\ \dfrac{\omega_{j,GN}(t)}{S_j(t)}\left[\theta(t-1)\cdot(S(t)+d(t)) + V_\theta(t-1)\alpha(t)\right] & t = 1, ..., T-1 \end{cases}$$

Observe that $V_\theta(0) = 1$ and, by (1) and (2) in the main text, $x_\theta(t) = -V_\theta(t-1)\alpha(t)$ for $t = 1, ..., T-1$, so that θ satisfies

$$\frac{\theta_j(t)S_j(t)}{V_\theta(t)} = \omega_{j,GN}(t), \quad \textit{for all } t$$

Multiplying both sides of this expression by $R_j(t+1) = \frac{S_j(t+1)+d_j(t+1)}{S_j(t)}$ and summing up over j we obtain

$$\frac{\theta(t)\cdot(S(t+1)+d(t+1))}{V_\theta(t)} = \omega_{GN}(t)\cdot R(t+1), \quad t = 0, ..., T-1$$

Summing then $\alpha(t+1)$ to both sides of this expression, and recalling that $x_\theta(t+1) = -V_\theta(t)\alpha(t+1)$ so that $\theta(t) \cdot (S(t+1) + d(t+1)) + V_\theta(t)\alpha(t+1) = V_\theta(t+1)$, we have

$$\frac{V_\theta(t+1)}{V_\theta(t)} = \alpha(t+1) + \omega_{GN}(t) \cdot R(t+1), \quad t = 0, ..., T-1 \tag{A10}$$

which by (13) implies that $\theta \in \Theta_+$. Moreover, substituting (A10) into (14), and multiplying by $S_j(t)$, after some elementary algebra we have

$$\frac{S_j(t)}{V_\theta(t)} = E\left\{ \frac{S_j(t+1) + d_j(t+1)}{V_\theta(t+1)} \middle| \mathcal{P}_t \right\}, \quad \textit{for all } j, t < T$$

which shows that θ is indeed a generalized numeraire portfolio, and concludes the proof ■

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Table 1: Pricing decile portfolios with Long's NP

The table reports tests of whether decile portfolio returns deflated by the return on the numeraire portfolio have mean zero (orthogonality condition (17)). The tests are performed using raw returns or returns in excess of the 3 month Treasury bill return and two proxies for the numeraire portfolio, the value weighted and the equally weighted market index. We report the test for individual decile portfolios (χ_1) as well as a joint test of all the orthogonality conditions (χ_{10}). All estimations are performed using GMM. The sample covers the period January 1962 to December 1997 (432 observations) and is extracted from the CRSP NYSE-AMEX-NASDAQ monthly database.

	Numeraire Portfolio Proxy			
	Value-Weighted Index		Equally-Weighted Index	
Panel a: raw returns				
Dec.	Test	p-value	Test	p-value
1	5.046	.025	6.058	.014
2	1.541	.214	0.335	.563
3	0.576	.448	0.731	.393
4	0.682	.409	1.152	.283
5	0.399	.528	4.647	.031
6	0.356	.551	5.139	.023
7	0.322	.571	3.500	.061
8	1.030	.310	0.797	.372
9	0.987	.321	0.619	.431
10	0.981	.322	0.641	.423
χ_{10}	22.20	.014	22.12	.015
Panel b: excess returns				
Dec	Test	p-value	Test	p-value
1	5.136	.023	3.778	.049
2	2.305	.129	1.324	.249
3	1.443	.229	0.701	.403
4	1.544	.214	0.801	.371
5	1.281	.256	0.643	.423
6	1.253	.262	0.656	.418
7	1.241	.265	0.692	.407
8	1.709	.191	1.109	.292
9	1.689	.194	1.193	.275
10	1.279	.258	1.176	.278
χ_{10}	23.04	.011	23.30	.010

Table 2: Pricing Decile portfolios with fixed weight NP's

The table shows estimates of numeraire portfolio weights and tests of whether decile portfolio returns deflated by the return on the numeraire portfolio have mean zero (orthogonality condition (17)). The numeraire portfolio consists of a set of decile portfolios with returns R_{it} such that the return on the numeraire portfolio is $R_{NPt} = \sum \omega_i R_{it} + (1 - \sum \omega_i) R_{10t}$. J-stat is the test of the overidentifying restrictions and is distributed as a $\chi_{10-(n-1)}$, where n is the number of decile portfolios included in the numeraire portfolio. The tests are performed using raw returns or returns in excess of the 3 month Treasury bill return. All estimations are performed using GMM. The sample covers the period January 1962 to December 1997 (432 observations) and is extracted from the CRSP NYSE-AMEX-NASDAQ monthly database.

Component Decile	ω_1	ω_i	ω_j	ω_{10}	J-stat
Panel a: raw returns					
Dec. 1 & 10	0.999	-	-	0.001	21.46
	(0.001)	-	-	-	0.006
Dec. 1, 5 & 10	5.619	-6.409	-	1.716	15.106
	(1.515)	(2.391)	-	-	0.035
Dec. 1, 4, 7 & 10	6.833	-9.572	3.013	0.795	14.805
	(1.631)	(3.755)	(3.549)	-	0.022
Panel b: excess returns					
Dec. 1 & 10	1.928	-	-	-0.928	21.266
	(0.916)	-	-	-	0.012
Dec. 1, 5 & 10	5.038	-6.153	-	2.115	15.751
	(1.482)	(2.316)	-	-	0.046
Dec. 1, 4, 7 & 10	5.910	-8.070	1.879	1.282	14.730
	(1.704)	(4.034)	(3.849)	-	0.039

Table 3: Pricing decile portfolios with GNP's

The table shows the estimates of the coefficients of the cash flow process and reports tests of whether decile portfolio returns deflated by the return on the generalized numeraire portfolio have mean zero (orthogonality conditions (14) and (15).) The generalized numeraire portfolio is estimated as follows

$$GNP(t) = \alpha(t) + R_{NP}(t), \qquad \alpha(t) = -\lambda(t)[R_{NP}(t) - 1], \qquad \lambda(t) = \gamma Z_t$$

where the elements of Z_t are the default premium and the market index dividend price ratio in excess of the risk free rate. The tests are performed using raw returns or returns in excess of the 3 month Treasury bill return and two proxies for the numeraire portfolio, the value weighted and the equally weighted market index. We report the test for individual decile portfolios (χ_1) as well as the test of the overidentifying restriction (χ_9). All estimations are performed using GMM. The sample covers the period January 1962 to December 1997 (432 observations) and is extracted from the CRSP NYSE-AMEX-NASDAQ monthly database.

Panel a: raw returns

	R_{NP} : Value-Weighted Index		R_{NP} : Equally-Weighted Index	
Z_i	γ_i	t-stat	γ_i	t-stat
DFP	-0.598	-1.79	-0.695	-3.923
XDP	-2.226	-3.118	-1.937	-4.775
Dec.	Test	p-value	Test	p-value
1	0.249	.617	0.283	.595
2	0.568	.451	0.438	.508
3	0.718	.397	0.479	.489
4	0.753	.386	0.447	.504
5	0.820	.365	0.444	.505
6	0.873	.350	0.457	.499
7	0.892	.345	0.427	.514
8	0.830	.362	0.365	.546
9	0.895	.344	0.361	.548
10	1.059	.303	0.359	.549
Overall test:				
χ_9	14.389	.109	12.704	.176

Table 3: Pricing decile portfolios with GNP's

Panel b: excess returns

	R_{NP}: Value-Weighted Index		R_{NP}: Equally-Weighted Index	
Z_i	γ_i	t-stat	γ_i	t-stat
DFP	-0.580	-2.893	-0.691	-4.118
XDP	-2.227	-3.380	-1.942	-4.730
Dec.	Test	p-value	Test	p-value
1	2.187	.139	0.230	.632
2	0.776	.378	0.029	.861
3	0.362	.547	0.057	.811
4	0.344	.558	0.021	.884
5	0.224	.636	0.024	.876
6	0.160	.690	0.032	.859
7	0.146	.702	0.009	.929
8	0.282	.596	0.018	.894
9	0.175	.676	0.023	.879
10	0.002	.966	0.008	.931
Overall test:				
χ_9	14.392	.109	12.295	.197

Table 4: Pricing Decile portfolios with fixed weight GNP's

The table reports the estimates of the coefficients of the cash flow process, of generalized numeraire portfolio weights, and tests of whether decile portfolio returns deflated by the return on the generalized numeraire portfolio have mean zero (orthogonality condition (14) & (15)). The generalized numeraire portfolio is estimated as follows

$$
\begin{aligned}
GNP(t) &= \alpha(t) + R_{NP}(t), \qquad \alpha(t) = -\lambda(t)[R_{NP}(t) - 1], \qquad \lambda(t) = \gamma Z_t \\
R_{NPt} &= \sum \omega_i R_{it} + (1 - \sum \omega_i) R_{10t}.
\end{aligned}
$$

The elements of Z_t are the default premium and the market index dividend price ratio in excess of the risk free rate. J-stat is the test of the overidentifying restrictions and is distributed as a $\chi_{11-(n+1)}$, where n is the number of decile portfolios included in the numeraire portfolio. The tests are performed using raw returns or returns in excess of the 3 month Treasury bill return. All estimations are performed using GMM. The sample covers the period January 1962 to December 1997 (432 observations) and is extracted from the CRSP NYSE-AMEX-NASDAQ monthly database.

Component Decile	γ_{DFP}	γ_{XDP}	ω_1	ω_i	ω_j	ω_{10}	J-stat
Panel a: raw returns							
Dec. 1 & 10	-0.668	-2.440	0.560	-	-	0.440	10.11
	(0.215)	(0.455)	(0.938)	-	-	-	0.183
Dec. 1, 5 & 10	-0.550	-1.648	1.200	-1.484	-	1.284	9.619
	(0.324)	(0.752)	(1.144)	(1.711)	-	-	0.142
Dec. 1, 4, 7 & 10	-0.571	-1.585	1.307	-1.070	-0.553	1.317	9.688
	(0.238)	(0.994)	(0.876)	(1.161)	(0.992)	-	0.085
Panel b: excess returns							
Dec. 1 & 10	-0.607	-2.478	0.667	-	-	0.333	10.31
	(0.160)	(0.589)	(0.927)	-	-	-	0.245
Dec. 1, 5 & 10	-0.028	0.836	1.955	-2.281	-	1.326	12.37
	(0.047)	(0.325)	(0.460)	(0.502)	-	-	0.089
Dec. 1, 4, 7 & 10	-0.005	0.669	2.729	-3.323	0.516	1.079	13.511
	(0.038)	(0.362)	(0.994)	(2.155)	(1.635)	-	0.036