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A Formula for K-Theory Truncation Schubert Calculus

Allen Knutson and Alexander Yong

1 Introduction

Let Flags($\mathbb{C}^n$) denote the variety of complete flags in $\mathbb{C}^n$. To each permutation π in the symmetric group S_n, there is an associated Schubert variety $X_\pi \subseteq \text{Flags}(\mathbb{C}^n)$. The classes of the Schubert structure sheaves $[\mathcal{O}_{X_\pi}]$ form an additive $\mathbb{Z}$-linear basis of the K-theory (Grothendieck) ring K(Flags($\mathbb{C}^n$)) of algebraic vector bundles over Flags($\mathbb{C}^n$). The *Schubert structure constants* are the integers defined by

$$[\mathcal{O}_{X_\sigma}] \cdot [\mathcal{O}_{X_\rho}] = \sum_{\pi \in S_n} C^{\pi}_{\sigma,\rho} [\mathcal{O}_{X_\pi}]. \tag{1.1}$$

It is known [4] that $(-1)^{\ell(\sigma)+\ell(\rho)-\ell(\pi)} C^{\pi}_{\sigma,\rho} \geq 0$, where $\ell(\alpha)$ is the minimum ℓ such that α is expressible as a product of ℓ simple transpositions $s_i = t_{i \leftrightarrow i+1}$. In the *cohomology case*, that is, when $\ell(\sigma) + \ell(\rho) = \ell(\pi)$, these are the structure constants for the analogous expansion of the product of Schubert classes $[X_\sigma] \cdot [X_\rho]$ in the cohomology ring $H^*(\text{Flags}(\mathbb{C}^n))$. These structure constants count the number of points in the intersection of general triple translates of $X_\sigma, X_\rho, X_{w_0\pi}$ (where w_0 denotes the longest permutation in S_n). The expansion (1.1) behaves well with respect to the inclusion $S_n \hookrightarrow S_{n+1}$. In particular, for any two permutations σ, ρ, and n sufficiently large, (1.1) stabilizes. Therefore, it will be unambiguous (and convenient) to call $(\sigma, \rho, \pi) \in S^3_\infty$ a *Schubert problem*, where $S_\infty = \bigcup_{n \geq 1} S_n$.

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It is a famous open problem to give a general subtraction-free combinatorial formula applicable to any Schubert problem. The analogous problem for Grassmannians is solved in the cohomology case by the Littlewood-Richardson rule, and more recently in the K-theory by Buch [6]. A solution for the flag variety would provide an important generalization of the Littlewood-Richardson rule. However, the known generalized Littlewood-Richardson rules [6, 21] handle only limited cases of the Schubert problems.

Our main result is a subtraction-free combinatorial formula for the family we call *truncation Schubert problems* (defined below). This formula specializes to compute the K-theory generalizations of the numbers considered by Kogan [21] and the K-theory Littlewood-Richardson coefficients of [6]. Actually, our main result gives formulas for many other combinatorial numbers studied in connection to the Schubert calculus [2, 6, 22], formulas for Schubert and Grothendieck polynomials [3, 13, 14, 23], degeneracy loci [5, 7, 8, 9, 16], and quantum Schubert polynomials [8, 11, 12]; see, for example, [10] and the references therein. We find it interesting that our formula also applies to new cases of Schubert problems where *neither* class is a pullback from a Grassmannian.

The fact that these numbers all arise from the single framework of problems isolated here suggests that their common combinatorial and geometric features ought to be better understood.

On the combinatorial side, in Section 3, we present our formula in terms of simple "marching" moves of the diagram of a permutation. We would like to understand how, for example, various combinatorial aspects of the classical Littlewood-Richardson coefficients might extend to this family of numbers. It would be interesting to understand the relations between the formula given here and formulas for the aforementioned special cases, and other related formulas, for example, see [17, 25, 27, 29, 30].

One feature of our proof is that it is both short and completely combinatorial. It is based on "truncation" techniques concerning Grothendieck polynomials [23] and in particular, the "transition" formula of Lascoux [22]. These methods (at least in cohomology) can be considered classical in the subject. Indeed, in previous work [22] (see also [24]), similar techniques were applied to give new formulas for the K-theory Littlewood-Richardson coefficients (after [6]). However, it is perhaps surprising that such methods are in fact applicable to more general Schubert calculus problems, and in particular, Kogan's Schubert problems. Our principal novelty of three simultaneous observations is reflected respectively in the three equalities found in (3.8).

Thus, since Kogan's Schubert problems form a special case of the truncation Schubert problems, our formula covers new cases of the Schubert problem in the K-theory (and, moreover, our proof makes transparent the role of Kogan's conditions).

However, we emphasize that our formula handles new cases beyond that in [21], even in cohomology.

A further goal of this paper is to present the diagram marching moves. One reason to use such (recursive) combinatorics is that the moves have a natural *geometric* interpretation. In a sequel [20] to this paper, we interpret the moves in terms of Gröbner degeneration of matrix Schubert varieties [15] via *diagonal* term orders (in an important contrast to the *anti*diagonal term orders used in [19]). For example, in the cohomology case, our formula can be interpreted as counting certain components of a partially degenerated matrix Schubert variety. It would be interesting to understand what relations exist between the formula presented here and the geometric Littlewood-Richardson rule of Vakil [31], which is also based on degeneration.

Finally, one other advantage of the approach presented here is the possible extensions to other Schubert calculus settings, for example, the cohomology/K-theory ring of flag varieties corresponding to the other classical Lie-type BCD (work in progress with F. Sottile).

2 Diagram moves and the main result

Let $G(\pi)$ denote the permutation matrix associated to $\pi \in S_n$, and call the nonzero entries of $G(\pi)$ its *dots*. The *diagram* of a permutation π is the following subset of $[n] \times [n]$:

$$D(\pi) := \{(p, q),\ 1 \le p, q \le n,\ \pi(p) > q,\ \pi^{-1}(q) > p\}. \tag{2.1}$$

Equivalently, $(p, q) \in D(\pi)$ if $(p, \pi^{-1}(q))$ is an inversion of π; thus $\#D(\pi) = \ell(\pi)$. Graphically, $D(\pi)$ is obtained from $G(\pi)$ by drawing a "hook" consisting of lines going east and south from each dot. The diagram appears as a collection of "connected components" of squares not in the hook of any dot (see Example 2.1).

Call the southernmost, then eastmost, box $(l, m) \in D(\pi)$ the *maximal corner*. Note that the maximal corner of π is in row l if and only if the last *descent* of π is in row l, that is, the largest index l such that $\pi(l) > \pi(l + 1)$. Call any dot that is maximally southeast with respect to the condition that it is northwest of (l, m) a *pivot*. There are no pivots if and only if the maximal corner is in the connected component of $D(\pi)$ attached to the top left corner of $[n] \times [n]$. See Example 2.1.

In the following definitions, it is convenient to describe the cohomology versions first before explaining their K-theory analogs. In Section 3, we will connect what follows to the Grothendieck transition formula of [22].

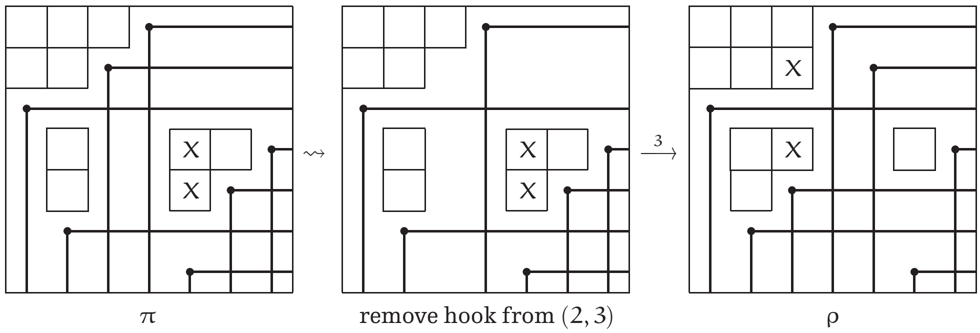


Figure 2.1

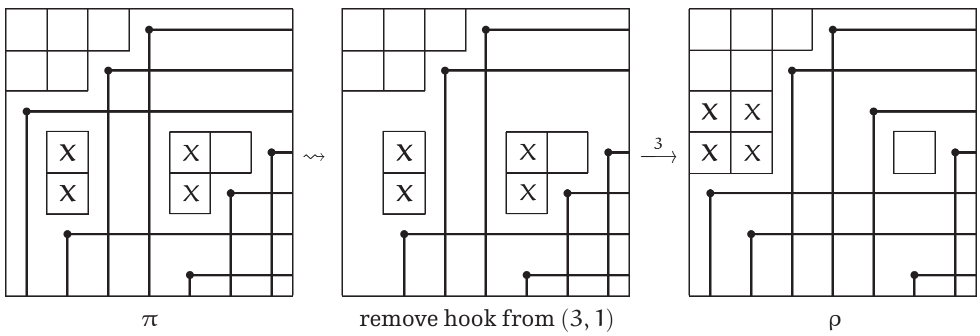


Figure 2.2

First, we describe the *marching* operation on $D(\pi)$. Suppose the maximal corner is at (l, m). If the input permutation has no pivots, declare the output of the marching to be null "$\varnothing$," and write $\pi \to \varnothing$. Otherwise, consider a pivot $(i, j) \in G(\pi)$. Remove the hook emanating from (i, j), and move *strictly* to the northwest every diagram box in the rectangle with the corners $(i, j), (l, m)$ into the only spaces available (i.e., by "hopping" over any hooks in the way). Do this by starting with the unique northwest box in the rectangle and continue left to right along the rows, and from top to bottom. It is easy to check that the resulting collection of boxes is necessarily the diagram of a permutation ρ. Let $\pi \xrightarrow{i} \rho$ denote that ρ is obtained from marching on $D(\pi)$ *towards the pivot* in row i.

Example 2.1. Let $\pi = 4317625$. We have $D(\pi) = \{(1,1), (1,2), (1,3), (2,1), (2,2), (4,2), (4,5), (4,6), (5,2), (5,5)\}$. The maximal corner is $(5,5)$ and its pivots are the dots at $(1,4), (2,3)$, and $(3,1)$. The boxes in the $(3,1), (5,5)$ rectangle of π are marked with Xs. Marching towards the pivot $(2,3)$, we get $\rho = 4517326$ as in Figure 2.1. Note that two adjacent X's at $(4,5)$ and $(5,5)$ can become separated (to $(2,3)$ and $(4,3)$, respectively) after marching. Marching instead towards the pivot $(3,1)$, we get $\rho = 4357126$ as shown in Figure 2.2. This time, some nonadjacent boxes in π become adjacent in ρ.

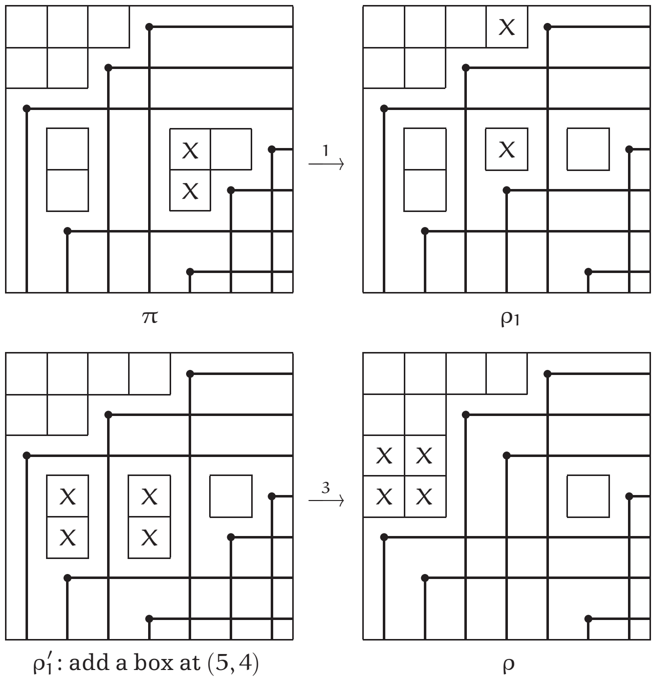


Figure 2.3

More generally, suppose that $1 \leq i_1 < i_2 < \cdots < i_k < l$ represent a subset of the rows that contain pivots of (l, m). Consider $\pi \xrightarrow{i_1} \rho_1$. Add a box to the diagram of ρ_1 at $(l, \rho_1(l))$. This is the diagram of a new permutation ρ'_1, where the added box is the maximal corner for ρ'_1, and the dot in row i_2 is a pivot for this box. Now march $\rho'_1 \xrightarrow{i_2} \rho_2$ and add a maximal corner similarly in row l to give ρ'_2. Repeat this process of marching and adding a box in row l until we obtain $\rho = \rho_k$. We write $\pi \xrightarrow{i_1, i_2, \ldots, i_k} \rho$ to denote this more general K-*marching* operation; a total of $k - 1$ boxes are added.

Example 2.2. Let π be as in the above example and suppose we march to the pivots $(1, 4)$ and $(3, 1)$ of the maximal corner $(5, 5)$, in succession (see Figure 2.3).

For any $\beta \in S_n$ and any positive integer t, we define a rooted, labeled tree $\mathcal{T}_t(\beta)$ whose vertices are labeled either by $\varnothing$ or by a permutation (repetitions allowed). The root is labeled by β. If a vertex is labeled by a permutation that has its last descent weakly smaller than t, or is labeled by $\varnothing$, then declare that vertex to be a leaf. Otherwise, the children of a vertex are indexed by the output of all ways of marching from that vertex. One can check easily that in finitely many steps, this growth process terminates, giving $\mathcal{T}_t(\beta)$. Note that $\mathcal{T}_t(\beta)$ is a pruning of $\mathcal{T}_s(\beta)$ for $t \leq s$. Define $K\mathcal{T}_t(\beta)$ similarly, using instead the K-marching operation (and, similarly, $K\mathcal{T}_t(\beta) \subseteq K\mathcal{T}_s(\beta)$). Finally, if a leaf vertex v is labeled π, call it a π-*leaf*.

We will be particularly interested in the cases that $K\mathcal{T}_s(\beta)$ or $\mathcal{T}_s(\beta)$ has exactly one labeled leaf (i.e., not by $\varnothing$). The best-behaved cases are when β is "2143-avoiding," also known as "vexillary," in which case, $K\mathcal{T}_s(\beta)$ has at most one leaf for any s. In the other direction, if π has a unique descent $\pi(i) > \pi(i+1)$, at $i = s$ (called a *Grassmannian permutation*), $N \in \mathbb{N}$, and β is the N-*stabilization* of π, meaning

$$\beta(i) = \begin{cases} i & \text{for } i \leq N, \\ N + \pi(i-N) & \text{for } i > N, \end{cases} \tag{2.2}$$

then $K\mathcal{T}_s(\beta)$ will have only one labeled leaf, and it will be labeled π.

We are now ready to introduce the family of Schubert problems covered by our main theorem. There is a standard operation $\star_n$ on two permutations $\sigma, \alpha \in S_n$. Let $\sigma \star_n \alpha$ be the permutation in S_{2n} whose matrix is the direct sum of $G(\sigma)$ and $G(\alpha)$. For example, $\mathrm{id} \star_n \alpha$ is just the n-stabilization of α. Let $\sigma \in S_n$ be a permutation whose last descent is l, and let $l \leq t \leq 2n$ be an integer. Suppose that $\alpha \in S_n$ is such that $K\mathcal{T}_t(\mathrm{id} \star_n \alpha)$ contains a single leaf ν with $\mathrm{label}(\nu) \neq \varnothing$; let $\mathrm{label}(\nu) = \rho$. Under these circumstances, call $(\sigma, \rho, \pi) \in S_n^2 \times S_\infty$ a *truncation Schubert problem subjugate to* (t, α).

Theorem 2.3. If $(\sigma, \rho, \pi) \in S_n^2 \times S_\infty$ is a truncation Schubert problem subjugate to (t, α), then $(-1)^{\ell(\sigma)+\ell(\rho)-\ell(\pi)} C^{\pi}_{\sigma,\rho}$ is the number of π-leaves of $K\mathcal{T}_t(\sigma \star_n \alpha)$. A simpler formula is available in the cohomology case: $C^{\pi}_{\sigma,\rho}$ is the number of π-leaves of $\mathcal{T}_t(\sigma \star_n \alpha)$. □

Example 2.4. Let $\sigma = 3412$ and $\alpha = 3214$ be permutations in S_4; so $\sigma \star_4 \alpha = 34127658 \in S_8$. One can check that $K\mathcal{T}_4(\mathrm{id} \star_4 \alpha)$ has a single labeled leaf, labeled by the permutation 12463578. Now $K\mathcal{T}_4(\sigma \star_4 \alpha)$ is given in Figure 2.4, and so by Theorem 2.3,

$$\begin{aligned} [\mathcal{O}_{X_{3412}}] \cdot [\mathcal{O}_{X_{12463578}}] = {} & [\mathcal{O}_{X_{46123578}}] + [\mathcal{O}_{X_{36142578}}] + [\mathcal{O}_{X_{35162478}}] + [\mathcal{O}_{X_{34261578}}] \\ & - [\mathcal{O}_{X_{46132578}}] - [\mathcal{O}_{X_{36152478}}] - [\mathcal{O}_{X_{36241578}}] \\ & - [\mathcal{O}_{X_{35261478}}] + [\mathcal{O}_{X_{36251478}}], \end{aligned} \tag{2.3}$$

where the expansion (1.1) has been done in the case $\mathrm{Flags}(\mathbb{C}^8)$.

As mentioned before the theorem, and spelled out in the corollary below, one family of truncation Schubert problems comes from Grassmannian permutations. In the cohomology case, these were given a (different) positive combinatorial formula by Kogan [21].

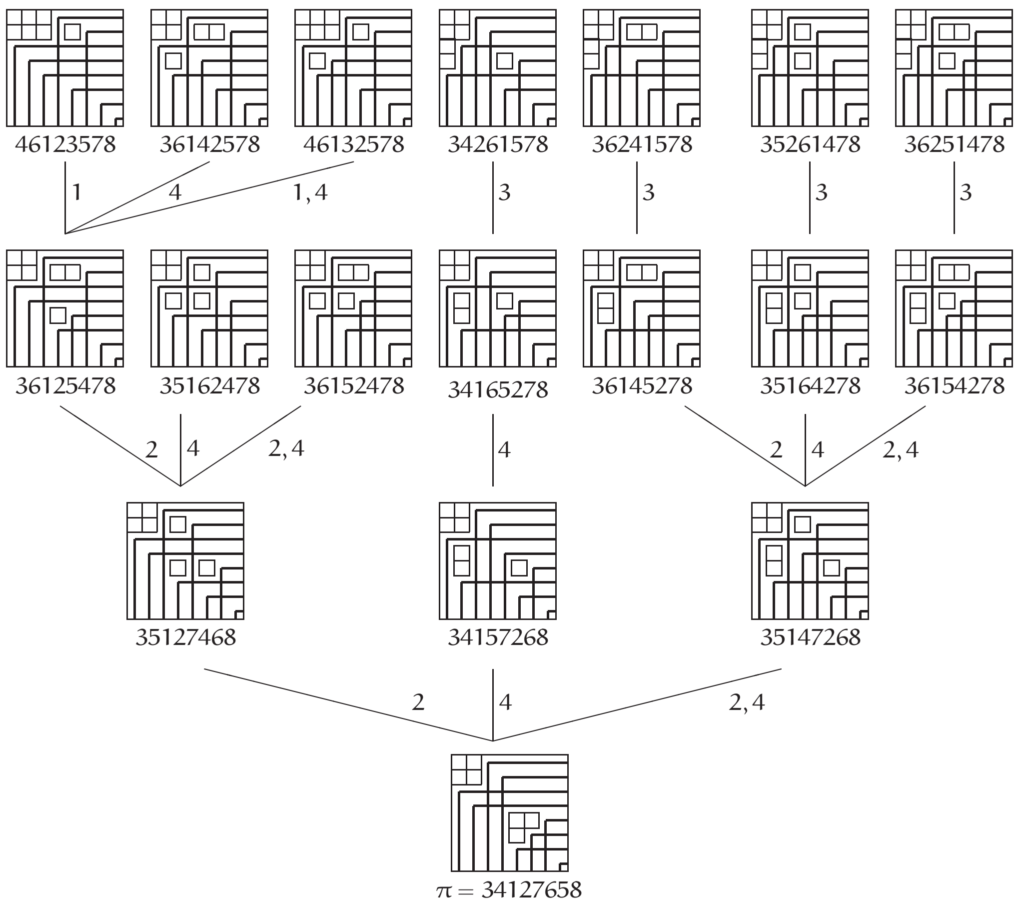


Figure 2.4 The tree $K\mathcal{T}_4(34127657)$.

Corollary 2.5. Let $\sigma \in S_n$ have the last descent at l and let ρ be a Grassmannian permutation with unique descent at t, where $l \le t \le n$. Then for any $\pi \in S_\infty$, $(-1)^{\ell(\sigma)+\ell(\rho)-\ell(\pi)} C^{\pi}_{\sigma,\rho}$ is the number of π-leaves of $K\mathcal{T}_t(\sigma \star_n \rho)$. In the cohomology case (treated in [21]), $C^{\pi}_{\sigma,\rho}$ is the number of π-leaves of $\mathcal{T}_t(\sigma \star_n \rho)$. □

If we also assume that σ is Grassmannian and moreover $t = l$, the first conclusion of Corollary 2.5 computes the K-theory Littlewood-Richardson coefficients of [6], while the second conclusion computes the classical Littlewood-Richardson coefficients. See Section 4.1 for more details.

Example 2.6. Let $\sigma = 321 \in S_3$, $\rho = 132$, and $\sigma \star_3 \rho = 321465$. The tree $K\mathcal{T}_2(321465)$ is given in Figure 2.5. Using this, the expansion (1.1) for $\text{Flags}(\mathbb{C}^6)$ is

$$[\mathcal{O}_{X_{321}}] \cdot [\mathcal{O}_{X_{132}}] = [\mathcal{O}_{X_{421356}}] + [\mathcal{O}_{X_{341256}}] - [\mathcal{O}_{X_{431256}}]. \tag{2.4}$$

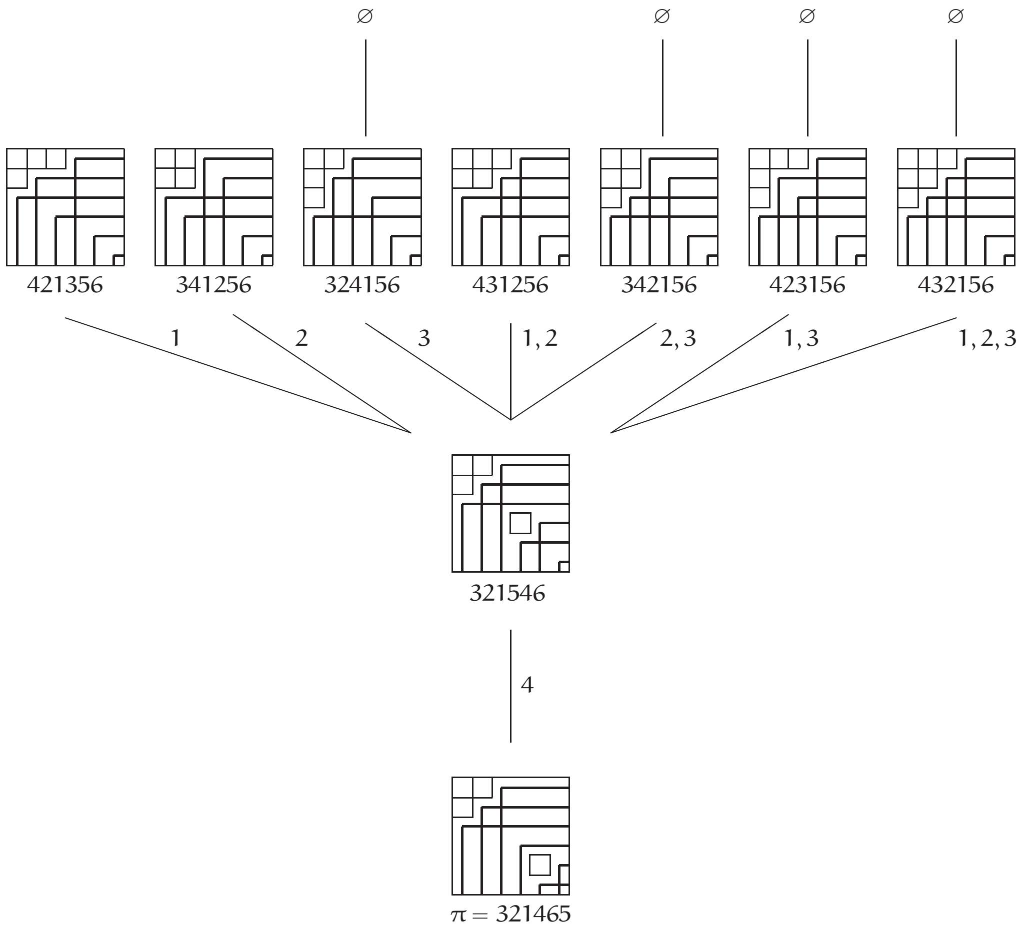


Figure 2.5 The tree $K\mathcal{T}_2(321465)$.

There is an isomorphism of $\mathrm{Flags}(\mathbb{C}^n)$ to itself induced by sending each vector subspace V to its orthogonal complement $V^\perp$ (with respect to an arbitrarily chosen bilinear form). The induced automorphism of $K(\mathrm{Flags}(\mathbb{C}^n))$ gives the symmetry $C^{\pi}_{\sigma,\rho} = C^{w_0\pi w_0}_{w_0\sigma w_0, w_0\rho w_0}$. This observation, combined with the corollary (or the theorem), gives, for example, a subtraction-free formula also for the Schubert numbers $C^{\pi}_{\sigma,\rho}$, where ρ is Grassmannian of descent t which is weakly *smaller* than the *first* descent of σ.

Theorem 2.3 also handles some new (but apparently limited) cases of Schubert problems (σ, ρ), where neither σ nor ρ are Grassmannian permutations. This differs from other formulas; see, for example, [2, 6, 10, 21, 22, 25, 27, 28, 30, 31].

Example 2.7. The tree $K\mathcal{T}_7(123459876\underline{10})$ has a single leaf indexed by a permutation, and that permutation is $12346985 7\underline{10}$. Hence a product of $[\mathcal{O}_{X_{12346985 7\underline{10}}}]$ with any $[\mathcal{O}_{X_\rho}]$ where

$\rho \in S_5$ is covered by Theorem 2.3, and in particular in $K(\mathrm{Flags}(\mathbb{C}^{10}))$,

$$[\mathcal{O}_{X_{12346985 7\underline{10}}}] \cdot [\mathcal{O}_{X_{41352}}] = [\mathcal{O}_{X_{41362985 7\underline{10}}}] + [\mathcal{O}_{X_{41356982 7\underline{10}}}] - [\mathcal{O}_{X_{41365982 7\underline{10}}}] \tag{2.5}$$

is a nontrivial expansion which is not computed by any previously known (subtraction-free) multiplication formula. We remark that Theorem 2.3 is the first to give a positive formula for even the cohomology expansion in $H^*(\mathrm{Flags}(\mathbb{C}^{10}))$:

$$[X_{12346985 7\underline{10}}] \cdot [X_{41352}] = [X_{41362985 7\underline{10}}] + [X_{41356982 7\underline{10}}]. \tag{2.6}$$

3 Proof of Theorem 2.3 and Corollary 2.5

We begin by recalling Lascoux and Schützenberger's *Grothendieck polynomials* [23], albeit via a rather unconventional definition. Let $X = \{x_1, x_2, \ldots\}$ be a collection of commuting independent variables. To each $\pi \in S_\infty$, there is an associated Grothendieck polynomial in the $\{x_i\}$, and these polynomials satisfy the following crucial recursion.

Theorem 3.1 (cf. [22, 25]). For any permutation $\gamma \in S_\infty$ with the last descent g, let $m > g$ be the largest integer such that $\gamma(m) < \gamma(g)$ and set $\gamma' = \gamma t_{g \leftrightarrow m}$. Suppose that $1 \le i_1 < i_2 < \cdots < i_s < g$ are the positions such that $\ell(\gamma' t_{i_j \leftrightarrow g}) = \ell(\gamma') + 1$. Then the (K-theory) transition formula of Lascoux [22] (as formulated in [25, Corollary 3.10]) holds:

$$\mathfrak{G}_\gamma(X) = \mathfrak{G}_{\gamma'}(X) + (x_g - 1)\left[\mathfrak{G}_{\gamma'}(X) \cdot (I - t_{i_1 \leftrightarrow g}) \cdots (I - t_{i_s \leftrightarrow g})\right], \tag{3.1}$$

where $t_{j \leftrightarrow l}$ acts on the $\{\mathfrak{G}_\xi(X)\}$ by $\mathfrak{G}_\xi(X) \cdot t_{j \leftrightarrow l} = \mathfrak{G}_{\xi t_{j \leftrightarrow l}}(X)$ and I acts as the identity operator. □

This and the base case $\mathfrak{G}_{id}(X) = 1$ uniquely determine the Grothendieck polynomials (the usual definition is via isobaric divided difference operators). Together, these polynomials form a $\mathbb{Z}$-linear basis of $\mathbb{Z}[X]$ and satisfy

$$\mathfrak{G}_\sigma(X)\mathfrak{G}_\rho(X) = \sum_{\pi \in S_\infty} C_{\sigma,\rho}^{\pi} \mathfrak{G}_\pi(X). \tag{3.2}$$

For any positive integer t, define the *truncation homomorphism* $r_t : \mathbb{Z}[X] \to \mathbb{Z}[X]$ by $r_t(f(X)) = f(x_1, \ldots, x_t, 0, 0, \ldots)$.

Theorem 2.3 is immediate from the second formula of the following result.

Theorem 3.2. For any $\gamma \in S_\infty$,

$$r_t\big(\mathfrak{G}_\gamma(X)\big) = \sum_{\nu} (-1)^{\ell(\gamma)-\ell(\text{label}(\nu))} \mathfrak{G}_{\text{label}(\nu)}(X), \tag{3.3}$$

where the sum is over all leaves ν of $K\mathcal{T}_t(\gamma)$ such that $\text{label}(\nu) \neq \varnothing$.

If $\sigma \in S_n$ has its last descent weakly smaller than t, and $\alpha \in S_n$ is arbitrary, then

$$\mathfrak{G}_\sigma(X) r_t\big(\mathfrak{G}_{\text{id} \star_n \alpha}(X)\big) = \sum_{\nu} (-1)^{\ell(\sigma)+\ell(\alpha)-\ell(\text{label}(\nu))} \mathfrak{G}_{\text{label}(\nu)}(X), \tag{3.4}$$

where the sum is over all leaves ν of $K\mathcal{T}_t(\sigma \star_n \alpha)$ such that $\text{label}(\nu) \neq \varnothing$. □

Proof. To expand $r_t(\mathfrak{G}_\gamma(X))$, we will need the following lemma, which connects the diagram moves of Section 2 to the (K-theory) transition formula in Theorem 3.1. It also gives an alternative form of a substitution formula of [26].

Lemma 3.3 (cf. [26])**.** Under the assumptions of Theorem 3.1,

(i) $(i_1, \gamma(i_1)), \ldots, (i_s, \gamma(i_s))$ are the pivots of the maximal corner $(g, \gamma^{-1}(m)) \in D(\gamma)$;

(ii) the following formula holds:

$$r_{g-1}\big(\mathfrak{G}_\gamma(X)\big) = \sum_{\gamma \xrightarrow{\mathcal{I}} \tau} (-1)^{\ell(\gamma)-\ell(\tau)} r_{g-1}\big(\mathfrak{G}_\tau(X)\big), \tag{3.5}$$

where the summation ranges over all subsets $\mathcal{I}$ of $\{i_1, \ldots, i_s\}$. □

Proof. Observe that the diagram of γ' differs from the diagram of γ only in that the maximal accessible box of γ has been removed (and thus there is a dot of $G(\gamma')$ in that position instead).[1] Now, for any index $1 \leq a < g$, $\ell(\gamma' t_{a \leftrightarrow g}) = \ell(\gamma') + 1$ holds if and only if $\gamma'(a) < \gamma'(g)$ and the rectangle defined by $(a, \gamma'(a))$ and $(g, \gamma'(g))$ contains no other dots of $G(\gamma')$; that is, if and only if $(a, \gamma'(a))$ is a pivot of the maximal accessible box of γ. Hence (i) holds.

Thus in view of (i), conclusion (ii) follows easily by expanding the K-transition formula from Theorem 3.1, observing that $(-1)^{\#\mathcal{I}} = (-1)^{\ell(\gamma)-\ell(\tau)}$, and setting $x_g = 0$. ■

[1]This seems to be the main reason to work with diagrams of permutations rather than inversion sets (which more easily generalize to other root systems).

From the above lemma, we have

$$\begin{aligned} r_{g-1}\big(\mathfrak{G}_\gamma(X)\big) &= \sum_{\gamma \xrightarrow{\mathcal{I}} \tau} (-1)^{\ell(\gamma)-\ell(\tau)} r_{g-1}\big(\mathfrak{G}_\tau(X)\big) \\ &= \sum_{\mathcal{A}} (-1)^{\ell(\gamma)-\ell(\tau)} \mathfrak{G}_\tau + \sum_{\mathcal{B}} (-1)^{\ell(\gamma)-\ell(\tau)} r_{g-1}\big(\mathfrak{G}_\tau(X)\big), \end{aligned} \tag{3.6}$$

where $\mathcal{A}$ consists of those τ appearing from marching from γ that have the last descent at $g-1$ or smaller, and $\mathcal{B}$ consists of those that still have their last descent at g. It is not hard to see that a finite number of iterations of marches from γ results in $\varnothing$ or a permutation with the last descent weakly smaller than $g-1$. Thus after repeated application of Lemma 3.3 on (3.6), we expand $r_{g-1}(\mathfrak{G}_\gamma(X))$ into the sum of Grothendieck polynomials indexed by such permutations (in particular, we have just used the fact that if a permutation τ has the last descent t and has no pivots, then $r_{t-1}(\mathfrak{G}_\tau(X)) = 0$). Therefore, the first conclusion follows by iterating the operation of setting the variables $x_{g-1}, x_{g-2}, \ldots, x_{t+1}$ to zero in succession.

Only a little more is necessary for the second conclusion of the theorem. Using Theorem 3.1 and induction, it is easy to check that

$$\mathfrak{G}_\sigma(X)\mathfrak{G}_{\mathrm{id} \star_n \rho}(X) = \mathfrak{G}_{\sigma \star_n \rho}(X). \tag{3.7}$$

Since the last descent of σ is l, then only the variables $x_1, \ldots, x_l$ appear in $\mathfrak{G}_\sigma(X)$. Hence because $l \leq t$, we have $r_t(\mathfrak{G}_\sigma(X)) = \mathfrak{G}_\sigma(X)$ and so

$$\begin{aligned} \mathfrak{G}_\sigma(X) r_t\big(\mathfrak{G}_{\mathrm{id} \star_n \alpha}(X)\big) &= r_t\big(\mathfrak{G}_\sigma(X)\big) r_t\big(\mathfrak{G}_{\mathrm{id} \star_n \alpha}(X)\big) \\ &= r_t\big(\mathfrak{G}_\sigma(X)\mathfrak{G}_{\mathrm{id} \star_n \alpha}(X)\big) = r_t\big(\mathfrak{G}_{\sigma \star_n \alpha}(X)\big). \end{aligned} \tag{3.8}$$

We conclude by applying the first conclusion to $\gamma = \sigma \star_n \alpha$ and observing that $\ell(\sigma \star_n \alpha) = \ell(\sigma) + \ell(\alpha)$. ■

Proof of Corollary 2.5. Since $r_t(\mathfrak{G}_{\mathrm{id} \star_n \rho}(X)) = \mathfrak{G}_\rho(X)$ [13], it follows from our above discussions that the hypotheses of Theorem 2.3 hold. ■

4 Remarks

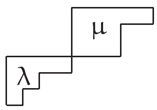
4.1 Comparisons with the Buch and Kogan formulas

We make a few remarks comparing the formula given by Theorem 2.3 with Buch's rule for the K-theory of Grassmannians, and Kogan's generalized Littlewood-Richardson rule. For this purpose, we give a simple example where all these results apply.

Let $\mathrm{Gr}(k, n)$ denote the Grassmannian of k-dimensional subspaces of $\mathbb{C}^n$. The Schubert varieties $\{X_\lambda\}$ of $\mathrm{Gr}(k, n)$ are indexed by partitions λ whose shape fits inside the $k \times (n-k)$ rectangle. Buch's rule is for expanding the product of classes of Schubert structure sheaves

$$[\mathcal{O}_{X_\lambda}] \cdot [\mathcal{O}_{X_\mu}] = \sum_{\nu \subseteq k \times (n-k)} C_{\lambda,\mu}^{\nu} [\mathcal{O}_{X_\nu}] \in K\big(\mathrm{Gr}(k, n)\big). \tag{4.1}$$

Let $\lambda \star \mu$ denote the shape obtained by placing the shapes of λ and μ corner to corner, as shown below:

$\lambda \star \mu =$ 

(4.2)

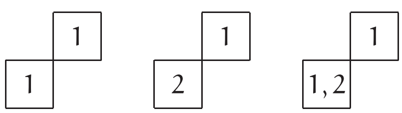
Define a *set-valued tableau* T *of shape* $\lambda \star \mu$ to be a filling of the "boxes" of $\lambda \star \mu$ by subsets of $\{1, 2, \ldots, k\}$ such that the entries are weakly increasing along rows as we read from left to right, and they are strictly increasing along columns as we read from top to bottom. We read the entries of any single box in increasing order.

Define the *column word of* T to be the sequence of integers obtained by reading the boxes of the tableaux from bottom to top and from left to right. Suppose $(w_1, w_2, \ldots, w_t)$ is this column word. We say that this is a *reverse lattice word* if each subsequence $(w_s, w_{s+1}, \ldots, w_t)$ for $s \geq 1$ has more i's than $(i+1)$'s for $i \geq 1$. Let the *content of* T be the vector $(c_1, c_2, \ldots, c_k)$, where c_i records the number of i's appearing in T. Finally, let $|\lambda|$ denote the number of boxes of a partition λ.

Then Buch's rule states that $(-1)^{|\lambda|+|\mu|-|\nu|} C_{\lambda,\mu}^{\nu}$ is the number of set-valued tableaux of shape $\lambda \star \mu$ of content μ whose column word is a reverse lattice word. Thus for example,

$$[\mathcal{O}_{X_{(1)}}]^2 = [\mathcal{O}_{X_{(2)}}] + [\mathcal{O}_{X_{(1,1)}}] - [\mathcal{O}_{X_{(2,1)}}] \in K\big(\mathrm{Gr}(2, 4)\big). \tag{4.3}$$

The terms on the right-hand side of (4.3) correspond respectively to the following set-valued tableaux from Buch's rule:



(4.4)

It is well known that the natural "forgetting subspaces" projection from $\mathrm{Flags}(\mathbb{C}^n)$ to $\mathrm{Gr}(k,n)$ induces an injective ring homomorphism in the other direction between their respective K-theory rings. This sends $[\mathcal{O}_{X_\lambda}] \in K(\mathrm{Gr}(k,n))$ to $[\mathcal{O}_{X_\sigma}] \in K(\mathrm{Flags}(\mathbb{C}^n))$, where σ is the Grassmannian permutation with unique descent at k uniquely determined by setting

$$\sigma(i) = i + \lambda_{k-i+1} \quad \text{for } 1 \le i \le k. \tag{4.5}$$

Thus, the calculation corresponding to (4.3) is $[\mathcal{O}_{X_{1324}}]^2 \in K(\mathrm{Flags}(\mathbb{C}^4))$. In the notation of Theorem 2.3, we have $\sigma = \rho = 1324$ and hence $\sigma \star_4 \rho = 13245768$. We invite the reader to draw out $K\mathcal{T}_2(13245768)$ in order to conclude

$$\left[\mathcal{O}_{X_{1324}}\right]^2 = \left[\mathcal{O}_{X_{1423}}\right] + \left[\mathcal{O}_{X_{2314}}\right] - \left[\mathcal{O}_{X_{2413}}\right], \tag{4.6}$$

in agreement with (4.3). In general, it is easy to read off the partitions from the diagrams in the leaves of the tree: simply turn the diagram upside down and remove empty columns.

Now we turn to Kogan's generalized Littlewood-Richardson rule. The original formulation [21, Theorem 4.1] is in terms of an insertion algorithm on RC-graphs [1, 14]. We will instead use the mild reformulation given in [10, Section 3], since this seems to us to be easier to state and use.

A *saturated chain* γ in the t-Bruhat order is a sequence of permutations

$$\gamma : \sigma = \sigma_0 \longrightarrow \sigma_1 \longrightarrow \sigma_2 \longrightarrow \cdots \longrightarrow \sigma_{|\lambda|} = \pi, \tag{4.7}$$

where $\ell(\sigma_i) = \ell(\sigma) + i$ and $\sigma_{i-1}^{-1}\sigma_i$ is a transposition $t_{a_i \leftrightarrow b_i}$ with $a_i \le t < b_i$ for each $i = 1, \ldots, |\lambda|$. The *word* of such a chain γ is the sequence of integers

$$\sigma_1(b_1), \sigma_2(b_2), \ldots, \sigma_{|\lambda|}(b_{|\lambda|}). \tag{4.8}$$

Under the assumptions of Corollary 2.5 (in the cohomology case), Kogan's formula asserts that $C^{\pi}_{\sigma,\rho}$ is the number of saturated chains in the t-Bruhat order from σ to π whose word is the column word of a semistandard tableau of shape λ. Then to recover the cohomology part of (4.6), we set $t = 2$ to get

$$[X_{1324}]^2 = [X_{1423}] + [X_{2413}] \in H^*(\text{Flags}(\mathbb{C}^4)), \tag{4.9}$$

again, in agreement with the above calculations. The terms in the expansion correspond to the saturated chains starting from 1324 determined by the transpositions $t_{2\leftrightarrow 4}$ and $t_{1\leftrightarrow 3}$, respectively. These in turn give, respectively, the tableaux

$$\boxed{3} \qquad \boxed{1} \tag{4.10}$$

In comparing Theorem 2.3 with the Buch and Kogan rules, we find our formula to be more handy for calculations of (1.1). For example, with even modest size examples, it is difficult in practice to know that one has found all of the set-valued tableaux (or saturated chains) that contribute to a coefficient. Exhaustively checking all possible cases usually takes much more effort than computing the tree needed in our formula. For this reason, we suspect that Theorem 2.3 should be more computationally efficient than the above alternatives.

Many aspects of the combinatorics of Theorem 2.3 remain mysterious to us, and thus there are a number of natural open questions. For example, it would be interesting to find a bijection between Theorem 2.3 and Buch or Kogan's rule in the common cases of applicability. We mention that it is not difficult to establish a bijection between Theorem 2.3 and the classical Littlewood-Richardson rule in the relevant cases. We plan to report on this bijection in [20].

4.2 Extensions to equivariant cohomology

The structure constants in the equivariant cohomology ring $H_T^*(\text{Flags}(\mathbb{C}^n))$ are polynomials in a second set of variables $\{y_1, y_2, \ldots\}$, and are known to have a positive expansion in $\{y_{i+1} - y_i\}$ (as proven in [18]). It does not seem easy to extend our techniques to apply to this richer problem. While the transition formula does have an equivariant extension, and one can state an equivariant truncation formula, this formula involves the $\{y_i\}$ individually rather than as differences. In trying to group them into differences, one leaves the realm of subtraction-free formulas.

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References

[1] N. Bergeron and S. Billey, *RC-graphs and Schubert polynomials*, Experiment. Math. **2** (1993), no. 4, 257–269.

[2] N. Bergeron and F. Sottile, *Schubert polynomials, the Bruhat order, and the geometry of flag manifolds*, Duke Math. J. **95** (1998), no. 2, 373–423.

[3] S. C. Billey, W. Jockusch, and R. P. Stanley, *Some combinatorial properties of Schubert polynomials*, J. Algebraic Combin. **2** (1993), no. 4, 345–374.

[4] M. Brion, *Positivity in the Grothendieck group of complex flag varieties*, J. Algebra **258** (2002), no. 1, 137–159.

[5] A. S. Buch, *Grothendieck classes of quiver varieties*, Duke Math. J. **115** (2002), no. 1, 75–103.

[6] ———, *A Littlewood-Richardson rule for the* K*-theory of Grassmannians*, Acta Math. **189** (2002), no. 1, 37–78.

[7] A. S. Buch and W. Fulton, *Chern class formulas for quiver varieties*, Invent. Math. **135** (1999), no. 3, 665–687.

[8] A. S. Buch, A. Kresch, H. Tamvakis, and A. Yong, *Schubert polynomials and quiver formulas*, Duke Math. J. **122** (2004), no. 1, 125–143.

[9] ———, *Grothendieck polynomials and quiver formulas*, to appear in Amer. J. Math.

[10] A. S. Buch, F. Sottile, and A. Yong, *Quiver coefficients are Schubert structure constants*, preprint, 2003.

[11] I. Ciocan-Fontanine, *On quantum cohomology rings of partial flag varieties*, Duke Math. J. **98** (1999), no. 3, 485–524.

[12] S. Fomin, S. Gelfand, and A. Postnikov, *Quantum Schubert polynomials*, J. Amer. Math. Soc. **10** (1997), no. 3, 565–596.

[13] S. Fomin and A. N. Kirillov, *Grothendieck polynomials and the Yang-Baxter equation*, Proceedings of the 6th Intern. Conf. on Formal Power Series and Algebraic Combinatorics, DIMACS, New Jersey, 1994, pp. 183–190.

[14] ———, *The Yang-Baxter equation, symmetric functions, and Schubert polynomials*, Discrete Math. **153** (1996), no. 1–3, 123–143.

[15] W. Fulton, *Flags, Schubert polynomials, degeneracy loci, and determinantal formulas*, Duke Math. J. **65** (1992), no. 3, 381–420.

[16] ———, *Universal Schubert polynomials*, Duke Math. J. **96** (1999), no. 3, 575–594.

[17] W. Fulton and A. Lascoux, *A Pieri formula in the Grothendieck ring of a flag bundle*, Duke Math. J. **76** (1994), no. 3, 711–729.

[18] W. Graham, *Positivity in equivariant Schubert calculus*, Duke Math. J. **109** (2001), no. 3, 599–614.

[19] A. Knutson and E. Miller, *Gröbner geometry of Schubert polynomials*, to appear in Ann. of Math.

[20] A. Knutson and A. Yong, *Gröbner geometry of Schubert transition and Littlewood-Richardson rules*, in preparation.

[21] M. Kogan, *RC-graphs and a generalized Littlewood-Richardson rule*, Int. Math. Res. Not. **2001** (2001), no. 15, 765–782.

[22] A. Lascoux, *Transition on Grothendieck polynomials*, Physics and Combinatorics (Nagoya, 2000), World Scientific Publishing, New Jersey, 2001, pp. 164–179.

[23] A. Lascoux and M.-P. Schützenberger, *Structure de Hopf de l'anneau de cohomologie et de l'anneau de Grothendieck d'une variété de drapeaux*, C. R. Acad. Sci. Paris Sér. I Math. **295** (1982), no. 11, 629–633 (French).

[24] ———, *Schubert polynomials and the Littlewood-Richardson rule*, Lett. Math. Phys. **10** (1985), no. 2-3, 111–124.

[25] C. Lenart, *A* K*-theory version of Monk's formula and some related multiplication formulas*, J. Pure Appl. Algebra **179** (2003), no. 1-2, 137–158.

[26] C. Lenart, S. Robinson, and F. Sottile, *Grothendieck polynomials via permutation patterns and chains in the Bruhat order*, preprint, 2004.

[27] C. Lenart and F. Sottile, *A Pieri-type formula for the* K*-theory of a flag manifold*, preprint, 2004.

[28] L. Manivel, *Symmetric Functions*, *Schubert Polynomials and Degeneracy Loci*, SMF/AMS Texts and Monographs, vol. 6, American Mathematical Society, Rhode Island, 2001, translated by J. R. Swallow.

[29] H. Pittie and A. Ram, *A Pieri-Chevalley formula in the* K*-theory of a* G/B*-bundle*, Electron. Res. Announc. Amer. Math. Soc. **5** (1999), 102–107.

[30] F. Sottile, *Pieri's formula for flag manifolds and Schubert polynomials*, Ann. Inst. Fourier (Grenoble) **46** (1996), no. 1, 89–110.

[31] R. Vakil, *A geometric Littlewood-Richardson rule*, preprint, 2003.

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