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Author
Lee, Chul-hee

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by

Chul-hee Lee

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Committee in charge:

Professor Richard E. Borcherds, Chair
Professor Lauren K. Williams
Professor Petr Hořava

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Abstract

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We will discuss several questions related to q-hypergeometric series and modular functions. Nahm's conjecture is an attempt to answer the question of when a certain q-hypergeometric series can have modularity. With this conjecture as motivation, we will consider a few closely related algebraic objects such as Bloch groups, Y-systems, T-systems, Q-systems, the dilogarithm function and its variants.

Dedicated to my parents, Sangok Lee and Younghee Oh

who gave me life and education

Contents

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Chapter 1

Introduction

The theory of q-hypergeometric series and theory of modular functions are both old and have been studied for a long time. However, understanding the overlap between the two theories is far from being complete. As Slater's list [34] shows, there are many concrete examples found without a general principle to organize them.

Nahm's conjecture is an attempt to provide a guiding principle in this area of study. The conjecture is about the overlap of q-hypergeometric series and modular functions and it provides a good meeting place of various kinds of mathematics like number theory, conformal field theory and its integrable perturbations, algebraic K-theory, representation theory of Kac-Moody algebras, the Virasoro algebra and so on. The development of the theory of cluster algebras now seems to provide a fresh viewpoint on old partition identities and Nahm's conjecture.

1.1 Old identities of Ramanujan

Let us begin with a familiar example. We want to recall some old results of Ramanujan, which are still sources of motivation and inspiration to many number theorists. This example is a very representative one which captures many important ingredients of our study.

In a letter of Ramanujan to Hardy, Ramanujan presents an identity which amazed Hardy [13, p. 9]:

$$\cfrac{1}{1+\cfrac{e^{-2\pi}}{1+\cfrac{e^{-4\pi}}{1+\dots}}} = \left(\sqrt{\frac{5+\sqrt{5}}{2}} - \frac{\sqrt{5}+1}{2}\right) e^{2\pi/5}. \tag{1.1}$$

This identity represents a beautiful interaction between the theory of q-series and the theory of complex multiplication i.e. theory of modular functions. It can be derived from the Rogers-Ramanujan identities :

$$G(q) = \sum_{n=0}^{\infty} \frac{q^{n^2}}{(q;q)_n} = \frac{1}{(q;q^5)_\infty (q^4;q^5)_\infty} \tag{1.2}$$
$$H(q) = \sum_{n=0}^{\infty} \frac{q^{n^2+n}}{(q;q)_n} = \frac{1}{(q^2;q^5)_\infty (q^3;q^5)_\infty}$$

where $(a;q)_n$ denotes the Pochhammer symbol defined in Section 2.1.

The continued fraction in the left-hand side of (1.1) can be explained by a recursion formula (3.1) that the Rogers-Ramanujan identities satisfy, which is due to them being q-hypergeometric series. The right-hand side can be obtained as singular moduli, the values of modular functions at quadratic imaginary integers. See Section 3.1 for more about this.

There is another interesting identity involving the dilogarithm function which is again due to this interaction of the two theories. The definition of the dilogarithm function is given in Section 2.3. One virtue of modular functions is that the behavior at the points where q is a root of unity can be nicely described. For example, a modular function $f(q)$ behaves asymptotically like $\exp(\frac{r\pi^2}{t})$ as $t \searrow 0$ for some rational number r where $q = e^{-t}$. More precise statements and relevant results will be given in Section 2.6.

For a q-hypergeometric series to be modular, its asymptotic behavior as $t \searrow 0$ must match that of modular functions. For rational numbers $a > 0$ and b, we have the following asymptotics

$$\sum_{n=0}^{\infty} \frac{q^{\frac{a}{2}n^2+bn}}{(q)_n} \sim \exp\left(\frac{L(x)}{t}\right) \tag{1.3}$$

where x is the unique positive solution of the equation $x = (1-x)^a$ such that $0 < x < 1$ and L is the Rogers dilogarithm function. See (2.6.2) for this kind of asymptotics. One can see that modularity puts a strong constraint on the possible values of a since $L(x)$ must be a rational multiple of $L(1) = \frac{\pi^2}{6}$.

In the case of the Rogers-Ramanujan identities, the above asymptotics (1.3) gives

$$G(q), H(q) \sim \exp\left(\frac{\pi^2}{15t}\right).$$

The number $\frac{\pi^2}{15}$ inside the exponential term is obtained from the identity

$$L\left(\frac{3-\sqrt{5}}{2}\right) = \frac{\pi^2}{15} = \frac{2}{5}L(1). \tag{1.4}$$

Note that here the number $(3-\sqrt{5})/2$ is a solution of $x = (1-x)^2$. Interestingly, the number $2/5$ on the right-hand side of (1.4) predicts the appearance of 2 congruence classes modulo 5 on the infinite product side of the Rogers-Ramanujan identities (1.2).

There is a way to understand (1.2) as characters of certain modules of the Virasoro algebra as we can see in Section 3.2. In this context, the number 2/5 which has a purely arithmetic meaning (counting congruence classes appearing in partition identities) becomes the effective central charge of a corresponding minimal model in conformal field theory. The fact that this number can be obtained from the evaluation of the Rogers dilogarithm function is related to the integrable perturbation of conformal field theory which is regarded as a physical background for Rogers-Ramanujan-like identities.

1.2 Outline of this thesis

In Chapter 2, Nahm's conjecture will be introduced after defining a few necessary notions. In particular, several results about the asymptotics of modular functions and q-hypergeometric series will be given. In Chapter 3, a few examples of r-fold q-hypergeometric series associated with a pair of Dynkin diagrams will be presented. These examples are mainly intended to give motivation to study the topics of the following chapters. A connection between Nahm's conjecture and quantum dilogarithm identities is also explored.

In Chapter 4, we introduce Y-systems, T-systems and Q-systems which are very closely related to the asymptotics of an r-fold q-hypergeometric series. In particular, we will prove that solving the system of equations associated with a pair of Dynkin diagrams gives torsion elements of Bloch groups in support of Nahm's conjecture. In the following chapter, we focus on positive solutions of various equations considered in the previous chapter and prove a few of their properties.

In Chapter 6, we look for Lie algebraic solutions of equations considered in the previous chapters. We will introduce the concept of quantum dimensions and its generalization to understand some conjectured properties of level restricted Q-systems given in Conjecture 6.3.2. The affine Weyl group will also be introduced to attack this problem. The conjecture, if it is true, allows us to express positive solutions in terms of quantum dimensions quite concretely. Within the setting suggested, several additional conjectures will be proposed.

Chapter 2

Nahm's conjecture

The main goal of this chapter is to state Nahm's conjecture. To achieve it, we define necessary notions and review relevant results. This conjecture is discussed in detail in Nahm's paper [27] from the viewpoint of conformal field theory and in Zagier's [38] from that of number theory.

2.1 q-hypergeometric series

Let us begin with a few standard terminology. Set $(a;q)_0 = 1$ and

$$(a;q)_n = \prod_{k=0}^{n-1}(1-aq^k) = (1-a)(1-aq)\cdots(1-aq^{n-1})$$

for a positive integer n. This expression is called the Pochhammer symbol. The special case $(q;q)_n$ will be denoted by $(q)_n$. We also set

$$(a;q)_\infty = \prod_{k=0}^{\infty}(1-aq^k) \tag{2.1}$$

$$(q)_\infty = \prod_{k=1}^{\infty}(1-q^k). \tag{2.2}$$

The left-hand side of the following classical identity of Euler

$$\sum_{n=0}^{\infty}\frac{q^{n(n-1)/2}}{(q)_n}z^n = (-z;q)_\infty \tag{2.3}$$

is an example of q-hypergeometric series. Another identity of Euler is

$$\sum_{n=0}^{\infty}\frac{z^n}{(q)_n} = \frac{1}{(z;q)_\infty}. \tag{2.4}$$

The q-binomial theorem states that

$$\sum_{n=0}^{\infty} \frac{(a;q)_n}{(q)_n} z^n = \frac{(az;q)_\infty}{(z;q)_\infty}. \tag{2.5}$$

See Section 3.6 for an application of these identities.

Now we introduce the main object of this chapter.

Definition 2.1.1. Let A be a positive definite symmetric $r \times r$ matrix, B be a vector of length r, and C a scalar, all three with rational entries. $f_{A,B,C}$ denotes an r-fold q-hypergeometric series defined by

$$f_{A,B,C}(q) = \sum_{n=(n_1,\ldots,n_r)\in(\mathbb{Z}_{\geq 0})^r} \frac{q^{\frac{1}{2}n^t An + B^t n + C}}{(q)_{n_1} \ldots (q)_{n_r}}$$

where $q \in \mathbb{C}$ with $|q| < 1$.

2.2 Modular forms and functions

Let $\mathrm{SL}(2,\mathbb{Z})$ be group of 2×2 matrices

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

with coefficients in $\mathbb{Z}$ satisfying $ad - bc = 1$. This group acts on the upper half-plane defined by $\mathbb{H} = \{z \in \mathbb{C} | z = x + iy, y > 0\}$ as linear fractional transformations. If $\alpha = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2,\mathbb{Z})$, we set

$$\alpha z = \frac{az+b}{cz+d}.$$

This defines a group action. Note that the element $\pm I$ act on $\mathbb{H}$ trivially.

Definition 2.2.1. We call the group $\mathrm{SL}(2,\mathbb{Z})$ the modular group .

This group is generated by two elements $S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ and $T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$. There are important subgroups of finite index of the modular group called congruence subgroups. Let

$$\Gamma(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2,\mathbb{Z}) | \begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \pmod{N} \right\}.$$

We call a subgroup of $\mathrm{SL}(2,\mathbb{Z})$ a congruence subgroup of level N if it contains $\Gamma(N)$. Let Γ be a congruence subgroup of level N. Since $\Gamma(N)$ is a normal subgroup, $\alpha^{-1}\Gamma\alpha$ is also a congruence subgroup of level N for any $\alpha \in \mathrm{SL}(2,\mathbb{Z})$.

Definition 2.2.2. Let k be an integer. If a meromorphic function f on $\mathbb{H}$ satisfies

$$f\left(\frac{a\tau+b}{c\tau+d}\right) = (c\tau+d)^k f(\tau)$$

for all $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$, we call f weakly modular of weight k with respect to Γ.

Let us put $j(\gamma,\tau) = cz+d$ for $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2,\mathbb{Z})$. If we adopt the notation $f[\alpha]_k = j(\alpha,z)^{-k} f(\alpha z)$, f is weakly modular of weight k with respect to Γ if $f[\alpha]_k = f$ for all $\alpha \in \Gamma$.

Let $g_1 = \begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix}$ and $g_2 = \begin{pmatrix} a_2 & b_2 \\ c_2 & d_2 \end{pmatrix} \in \mathrm{SL}(2,\mathbb{Z})$. Then

$$g_3 = g_2 g_1 = \begin{pmatrix} a_3 & b_3 \\ c_3 & d_3 \end{pmatrix} = \begin{pmatrix} * & * \\ c_2 a_1 + d_2 c_1 & c_2 b_1 + d_2 d_1 \end{pmatrix}.$$

From $j(g_1,\tau) = c_1\tau + d_1$ and

$$j(g_2, g_1\tau) = c_2 g_1(\tau) + d_2 = c_2 \frac{a_1\tau + b_1}{c_1\tau + d_1} + d_2 = \frac{c_3\tau + d_3}{c_1\tau + d_1},$$

we can see that $j(g_2 g_1, \tau) = j(g_2, g_1\tau) j(g_1, \tau)$.

Lemma 2.2.3. *Let f be a weakly modular function of weight k with respect to Γ. $f[\alpha]_k$ is also weakly modular of weight k with respect to $\alpha^{-1}\Gamma\alpha$.*

Proof. Define $F(\tau) = (f[\alpha]_k)(\tau) = j(\alpha,\tau)^{-k} f(\alpha\tau)$. We want to prove that $F[g]_k = F$ for any $g \in \alpha^{-1}\Gamma\alpha$. Let $g_0 \in \Gamma$ and $g = \alpha^{-1} g_0 \alpha$.

$$\begin{aligned}
F[g]_k(\tau) &= j(g,\tau)^{-k} F(g\tau) && (2.6)\\
&= j(g,\tau)^{-k} j(\alpha, g\tau)^{-k} f(\alpha g\tau) = j(g_0\alpha,\tau)^{-k} f(g_0\alpha\tau) && (2.7)\\
&= j(g_0\alpha,\tau)^{-k} j(g_0,\alpha\tau)^{k} f(\alpha\tau) = j(\alpha,\tau)^{-k} f(\alpha\tau) && (2.8)\\
&= F(\tau) && (2.9)
\end{aligned}$$

So $F[g]_k = F$. □

Definition 2.2.4. Let k be an integer. A holomorphic function $f : \mathbb{H} \to \mathbb{C}$ is a modular form of weight k with respect to Γ if

1. f is weakly modular of weight k with respect to Γ,

2. $f[\alpha]_k$ is meromorphic at ∞ for all $\alpha \in \mathrm{SL}(2, \mathbb{Z})$ i.e. it has a Fourier expansion of the form
$$f[\alpha]_k = \sum_{n=m}^{\infty} a_n q_N^n$$
for some integer $m \in \mathbb{Z}$ where $q_N = e^{2\pi i\tau/N}$.

Example 2.2.5. The Dedekind eta function is defined by

$$\eta(\tau) = q^{1/24} \prod_{n=1}^{\infty} (1 - q^n) = q^{1/24} (q)_\infty$$

where $q = e^{2\pi i\tau}$. Its behavior under the action of the modular group is given by

$$\eta(\tau + 1) = e^{\frac{\pi i}{12}} \eta(\tau)$$
$$\eta(-\frac{1}{\tau}) = \sqrt{\frac{\tau}{i}} \eta(\tau) = \sqrt{-i\tau}\,\eta(\tau).$$

So $\Delta = \eta^{24}$ is an example of a holomorphic modular form of weight 12 with respect to the full modular group $\mathrm{SL}(2, \mathbb{Z})$.

Theta functions

Many examples of modular q-hypergeometric series in our study will be expressed using theta functions. See [30] for a systematic exposition of the theory.

Definition 2.2.6. Let L be a positive definite integral lattice of rank n and $v \in L \otimes_{\mathbb{Z}} \mathbb{R}$. Define
$$\theta_{L+v}(\tau) = \sum_{x \in L+v} q^{\frac{x^2}{2}}$$
where $q = e^{2\pi i\tau}$ as usual.

An important property of theta functions is their modularity. Instead of stating general theorem, let us look at an example since this will be enough for our purpose.

Example 2.2.7. Let N be an even positive integer. Let L be the lattice $(\mathbb{Z}, Nx^2)$ of rank 1 and $L^* = (\frac{1}{N}\mathbb{Z}, Nx^2)$ its dual lattice. Define theta functions

$$\Theta_{N,r}(\tau) = \sum_{n \in \mathbb{Z}} q^{\frac{(Nn+r)^2}{2N}}$$

for $r = 0, 1, \cdots, N - 1$. Let us consider functions obtained by dividing theta functions by the Dedekind eta function:
$$K_{N,r}(\tau) = \frac{\Theta_{N,r}(\tau)}{\eta(\tau)}.$$

Their modular transformation properties are given by

$$K_{N,r}(\tau+1) = e^{2\pi i[(r^2/2N)-1/24]}K_{N,r}(\tau) \tag{2.10}$$
$$K_{N,r}(-1/\tau) = \sum_{s=0}^{N-1}\frac{1}{\sqrt{N}}e^{2\pi i rs/N}K_{N,s}(\tau).$$

2.3 The dilogarithm function and its variants

Definition 2.3.1. The dilogarithm function is defined by

$$\mathrm{Li}_2(z) = -\int_0^z \frac{\log(1-t)}{t}dt$$

for $z \in \mathbb{C} - [1,\infty)$.

For $|z| < 1$, we have the following power series for the dilogarithm function :

$$\mathrm{Li}_2(z) = \sum_{n=1}^{\infty}\frac{z^n}{n^2}.$$

It turns up in the study of q-hypergeometric series quite naturally. Consider the Pochhammer symbol $(q)_n = (1-q)(1-q^2)\cdots(1-q^n)$. For $0 < q < 1$, taking logarithm in both sides, we get

$$\log(q)_n = \sum_{t=1}^{n}\log(1-q^t).$$

If n is big enough, we can approximate this sum by the integral

$$\int_1^n \log(1-q^t)\,dt = \frac{1}{\log q}\int_q^{q^n}\log(1-x)\,\frac{dx}{x} = \frac{1}{\log q}(\mathrm{Li}_2(q) - \mathrm{Li}_2(q^n)).$$

Now we define a variant of the dilogarithm function.

Definition 2.3.2. The Rogers dilogarithm function is defined by

$$L(x) = \mathrm{Li}_2(x) + \frac{1}{2}\log x\log(1-x) = -\frac{1}{2}\int_0^x \frac{\log(1-y)}{y} + \frac{\log(y)}{1-y}dy$$

for $x \in (0,1)$. We set $L(0) = 0$ and $L(1) = \pi^2/6$ so that L is continuous on $[0,1]$.

There are functional identities satisfied by the Rogers dilogarithm function. It satisfies the reflection property

$$L(x) + L(1-x) = L(1)$$

for $0 \leq x \leq 1$. The five-term relation is as follows :

$$L(x) + L(1-xy) + L(y) + L(\frac{1-y}{1-xy}) + L\left(\frac{1-x}{1-xy}\right) = \frac{\pi^2}{2} = 3L(1) \tag{2.11}$$

for $0 \leq x, y \leq 1$.

Let us define another variant of the dilogarithm function.

Definition 2.3.3. The Bloch-Wigner dilogarithm function is given by

$$D(z) = \text{Im}(\text{Li}_2(z)) + \log|z| \arg(1-z).$$

$D(z)$ is a real analytic function on $\mathbb{C}$ except at 0 and 1, where it is continuous but not differentiable. Since $D(\bar{z}) = -D(\bar{z})$, it vanishes on $\mathbb{R}$. It satisfies following functional equations :

$$D(x) + D(1-xy) + D(y) + D(\frac{1-y}{1-xy}) + D(\frac{1-x}{1-xy}) = 0 \tag{2.12}$$

$$D(x) + D(1-x) = D(x) + D(\frac{1}{x}) = 0. \tag{2.13}$$

This will be important in Section 2.4 where we discuss the Bloch group.

Remark 2.3.4. Note that the five numbers

$$x_0 = x, x_1 = 1-xy, x_2 = y, x_3 = \frac{1-y}{1-xy}, x_4 = \frac{1-x}{1-xy}$$

appearing in the functional identities (2.11) and (2.12) satisfy the relations

$$\begin{cases} x_0 & = & x \\ x_2 & = & y \\ x_{m-1}x_{m+1} & = & 1-x_m \end{cases}$$

with $x_m = x_{m+5}$. So the recurrence relation $x_{m-1}x_{m+1} = 1 - x_m$ defines a periodic sequence which gives rise to a functional dilogarithm identity. In fact, there is a nice generalization of this observation. Later in Chapter 4, we will see that there exists a family of recurrence equations called Y-systems defined for each pair of Dynkin diagrams, having periodicities and associated dilogarithm identities.

2.4 Bloch groups

In this section, we give a definition of the Bloch group for a field and explain the role of the Bloch-Wigner dilogarithm function in its study. Nahm's conjecture is formulated in terms of Bloch groups. The Bloch-Wigner dilogarithm is a useful tool to study the Bloch group. See [38] and [37] for a more thorough treatment.

Definition 2.4.1. Let F be a field. $\Lambda^2 F^*$ denotes the abelian group of formal sums of $x \wedge y, x, y \in F^*$ modulo the relations $x \wedge x = 0, (x_1 x_2) \wedge y = x_1 \wedge y + x_2 \wedge y$ and $x \wedge (y_1 y_2) = x \wedge y_1 + x \wedge y_2$.

Let $\partial : \mathbb{Z}[F^* \backslash \{1\}] \to \Lambda^2(F^*) \otimes_{\mathbb{Z}} \mathbb{Q}$ be a $\mathbb{Z}$-linear map defined by $\partial([x]) = x \wedge (1-x)$. Let $A(F) = \ker \partial$ and $C(F)$ the subgroup of $A(F)$ generated by the elements

$$[x] + [1-xy] + [y] + [\frac{1-y}{1-xy}] + [\frac{1-x}{1-xy}] \tag{2.14}$$

$$[x] + [1-x] \tag{2.15}$$

$$[x] + [\frac{1}{x}]. \tag{2.16}$$

It is convenient to set $[0] = [1] = [\infty] = 0$ in $A(F)$. We call (2.14) the five-term relation. The Bloch group $\mathcal{B}(F)$ of F is defined by $\mathcal{B}(F) = A(F)/C(F)$.

The Bloch-Wigner dilogarithm $D(z)$ can be used to define a map from $\mathcal{B}(\mathbb{C})$ to $\mathbb{R}$. For $\xi = \sum_i n_i [x_i] \in \mathcal{B}(\mathbb{C})$, let $D(\xi) = \sum_i n_i D(x_i)$. By (2.12) and (2.13), it is well-defined. Let F be a number field of degree $r_1 + 2r_2$ over $\mathbb{Q}$ where r_1 denotes the number of real embeddings and r_2 denotes the number of pairs of complex conjugate non-real embeddings. For an embedding $\sigma : F \hookrightarrow \mathbb{C}$ and $\xi \in \mathcal{B}(F)$, we may consider $D(\sigma(\xi))$. If $D(\sigma(\xi)) = 0$ for all such embeddings σ, then $\xi \in \mathcal{B}(F)$ is a torsion element in $\mathcal{B}(F)$. This is because the Bloch-Wigner dilogarithm function can be used to construct the regulator map from K_3 to $\mathbb{R}^{r_2}$. See [37, section 4].

Let $\mathbb{C}(y)$ be the field of rational functions in $(y_i)_{i \in I}$ where I is a finite set. For $f \in \mathbb{C}(y)$, $f|_{\mathbf{a}}$ denotes the evaluation of f at $(y_i) = \mathbf{a} = (a_i) \in \mathbb{C}^n$.

Proposition 2.4.2. *Suppose that we have a set S of rational functions $f_i \in \mathbb{C}(y)$ such that $\sum_{f_i \in S} (1-f_i) \wedge f_i = 0$ in $\Lambda^2 \mathbb{C}(y)^*$. $\sum_{f_i \in S} D(f_i|_{\mathbf{x}})$ is independent of $\mathbf{x} \in \mathbb{C}^n$.*

See [38, Chapter II. Section 2.A.] and references therein for a proof. If one wants to obtain such a set of rational functions satisfying the condition of the above statement, one should look at Y-systems. Y-systems are good suppliers for such rational functions. See Proposition 4.1.12.

2.5 Properties of the equation $\mathbf{x} = (1-\mathbf{x})^A$

When the asymptotic behavior of $f_{A,B,C}$ as z approaches to 0 is considered, one is led to a system of equations associated to the matrix $A = (a_{ij})$ given by

$$x_i = \prod_{j=1}^{r} (1-x_j)^{a_{ij}}, \qquad (i = 1, ..., r) \tag{2.17}$$

This is a system of r equations of r variables $x_1, \cdots, x_r$. When there is no confusion, this system of equations (2.17) will be denoted by $\mathbf{x} = (1-\mathbf{x})^A$.

Let us first consider solutions of $\mathbf{x} = (1-\mathbf{x})^A$ in the interval $[0,1]$.

Theorem 2.5.1. *Let A be a positive definite symmetric $r \times r$ matrix. Then the equation $\mathbf{x} = (1-\mathbf{x})^A$ has a unique real solution $\mathbf{x} = (x_i)$ such that $0 < x_i < 1$ for all $i = 1, \cdots r$.*

See [28] and [36] for proofs. This fact will play a very important role in Chapter 5.

Lemma 2.5.2. *Let A be a positive definite symmetric $r \times r$ matrix. Then $\mathbf{x} = (x_i)$ cannot be a solution of $\mathbf{x} = (1-\mathbf{x})^A$ if x_i is 0 or 1 for some $i = 1, \cdots r$.*

Proof. Since A is a positive definite matrix, all diagonal entries are positive. So from the equation (2.17) one can see that x_i is neither 0 nor 1. □

For a complex solution $\mathbf{x} = (x_i)$ of this equation, we will assume the existence of complex r-tuples $\mathbf{u} = (u_i) \in \mathbb{C}^n$ and $\mathbf{v} = (v_i) \in \mathbb{C}^n$ satisfying the following system of equations

$$\begin{cases} v_i = \sum_{j=1}^r a_{ij} u_j & i = 1, \cdots, r \\ e^{u_i} + e^{v_i} = 1 & i = 1, \cdots, r \end{cases}$$

and then we can put $x_i = e^{v_i}$.

Proposition 2.5.3. *Let A be a positive definite symmetric $r \times r$ matrix with integer coefficients. Any solution of the equation $\mathbf{x} = (1-\mathbf{x})^A$ in a number field $F \subseteq \mathbb{C}$ is an element of $\mathcal{B}(F)$.*

Proof. Let $\mathbf{x} = (x_i) \in F^n$ be a solution of

$$x_i = \prod_{j=1}^r (1-x_j)^{a_{ij}}.$$

If $\xi_x = \sum_i [x_i] \in \mathbb{Z}[F^* \backslash \{1\}]$, $\partial(\xi_x) = \sum_{i=1}^r a_{ij} x_i \wedge (1-x_i) = \sum_{i=1}^r \sum_{j=1}^r a_{ij} x_i \wedge x_j$. Since a_{ij} is symmetric, $\partial(\xi_x) = 0$ and thus ξ_x is an element of $\mathcal{B}(F)$. □

2.6 Asymptotic behavior of modular forms and q-hypergeometric series

In Chapter 1, we mentioned that the equation $x = (1-x)^2$ is important in the description of the asymptotic behavior of the Rogers-Ramanujan identities. A similar kind of equation also appears when the asymptotic behavior of $f_{A,B,C}(z)$ is considered. To see this, we review a few results from [38, 36] about the asymptotics of modular forms and q-hypergeometric series in this section.

Asymptotic behavior of modular forms

Let us begin with the asymptotics of modular forms.

Theorem 2.6.1. *Let f be a modular form of weight k with respect to a congruence subgroup Γ. For any positive integer N,*

$$f(e^{-t}) \sim bt^{-k} \exp(\frac{r\pi^2}{t}) \left(1 + o(t^N)\right)$$

for some $r \in \mathbb{Q}$ and $b \in \mathbb{C}$ as $t \searrow 0$.

Proof. Let $g(\tau) = \tau^{-k} f(-1/\tau)$. Then g is also a modular form of weight k with respect to $S\Gamma S$ by Lemma 2.2.3 where $S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$. We have

$$g(\tilde{\tau}) = \sum_{n=m}^{\infty} b_n \tilde{q}_l^n$$

where $\tilde{\tau} = -1/\tau$ and $\tilde{q}_l = e^{2\pi i \tilde{\tau}/l}$ for some integer l.

If we set $\tau = \frac{it}{2\pi}$, we have $q = e^{2\pi i \tau} = e^{-t}$, $\tilde{\tau} = -1/\tau = \frac{2\pi i}{t}$ and $\tilde{q} = e^{-\frac{4\pi^2}{t}}$. This implies

$$f(\tau) = (\tilde{\tau})^k g(\tilde{\tau}) \sim (2\pi i)^k b_m t^{-k} \exp(\frac{r\pi^2}{t}) \left(1 + o(t^N)\right).$$

□

Asymptotic behavior of r-fold q-hypergeometric series

Now we consider the asymptotics of an r-fold q-hypergeometric series

$$f_{A,B,C}(q) = \sum_{n \in (\mathbb{Z}_{\geq 0})^r} \frac{q^{\frac{1}{2}n^t A n + B^t n + C}}{(q)_{n_1} \cdots (q)_{n_r}}.$$

Theorem 2.6.2. *[36, Theorem 2.3.] Let $\mathbf{x} = (x_i)$ be the unique positive solution of $\mathbf{x} = (1 - \mathbf{x})^A$ with $0 < x_i < 1$ for $i = 1, \cdots, r$. Let $\tilde{A}$ be the matrix*

$$A + \operatorname{diag}(\frac{1 - x_1}{x_1}, \cdots, \frac{1 - x_r}{x_r})$$

and let

$$\alpha = \sum_{i=1}^{r} L(x_i), \tag{2.18}$$

$$\beta = \frac{1}{\sqrt{\det \tilde{A}}} \prod_{i=1}^{r} \frac{(1 - x_i)^{B_i}}{\sqrt{x_i}}, \tag{2.19}$$

$$\gamma = C + \frac{1}{24} \sum_{i=1}^{r} \frac{2 - x_i}{x_i}. \tag{2.20}$$

As $t \searrow 0$*,*

$$f_{A,B,C}(e^{-t}) \sim \beta \exp(\frac{\alpha}{t} - \gamma t)\left(1 + \sum_{m=1}^{\infty} c_m t^m\right) \tag{2.21}$$

where c_m*'s are certain numbers depending on* A, B, C*.*

See [36] for a proof. For $f_{A,B,C}$ to be modular, the asymptotic behavior as $t \searrow 0$ must be consistent with that of modular functions. So by Theorem 2.6.1 and Theorem 2.6.2, we obtain several constraints on $f_{A,B,C}$ as follows :

Corollary 2.6.3. *If* $f_{A,B,C}$ *is a modular form of weight* k *with respect to a congruence subgroup, then the followings must be satisfied :*

- *the weight* k *of* f *is 0.*
- $\sum_{i=1}^{r} L(x_i) \in \pi^2\mathbb{Q}$*.*
- $c_m = \frac{\gamma^m}{m!}$ *for all* $m \geq 1$*.*

Theorem 2.6.2 is closely related to the asymptotic growth of coefficients of

$$f_{A,B,C}(q) = \sum_{n \in (\mathbb{Z}_{\geq 0})^r} \frac{q^{\frac{1}{2}n^t An + B^t n + C}}{(q)_{n_1} \dots (q)_{n_r}} = \sum_{n=0}^{\infty} r(n) q^n$$

which is given by

$$\log^2 r(n) \sim 4n \sum_{i=1}^{r} L(x_i) + o(n).$$

Remark 2.6.4. Before we close this section, we consider the asymptotic behavior of a function given by infinite products since these types of functions show up quite frequently in this study. A theorem of Meinardus is about the asymptotics of an infinite product of the form

$$f(\tau) = q^c \prod_{j=1}^{\infty} (1 - q^j)^{-a_n} = 1 + \sum_{n=1}^{\infty} r(n) q^n$$

for a non-negative integer sequence $(a_n)_{n \in \mathbb{Z}^{>0}}$. See [2] for a detailed exposition. Let us consider a special case where $a_1, a_2, \cdots$ defines a periodic sequence with period m such that $a_j = a_{m-j}$ for $j = 1, \cdots, m-1$. Consider a Dirichlet series defined by

$$L(s) = \sum_{n=1}^{\infty} \frac{a_n}{n^s}.$$

If κ denotes the residue of L at $s = 1$, then

$$\log^2 r(n) \sim 4n \frac{\kappa \pi^2}{6} + o(n).$$

From this, one can see that if $f_{A,B,C}$ can be written as an infinite product of this particular form, $\alpha = \sum_{i=1}^{k} L(x_i) = \kappa \frac{\pi^2}{6}$ must hold.

Theorem 2.6.5. *[16, Lemma 13.11.2.] Let $(a_n)_{n\in\mathbb{Z}^{>0}}$ be a periodic sequence of integers with period m such that $a_j = a_{m-j}$ for $j = 1, \cdots, m-1$. Let $b = \sum_{j=1}^{m} a_j$ and $f_c(\tau) = q^c \prod_{j=1}^{\infty}(1-q^j)^{a_j}$. Then $f_c(\tau)$ is a modular form if and only if*

$$c = \frac{bm}{24} - \frac{1}{4m}\sum_{j=1}^{m-1} j(m-j)a_j.$$

This result can be useful to many examples discussed in Chapter 3.

2.7 Nahm's conjecture

In [27], Nahm considered a question of when an r-fold q-hypergeometric series $f_{A,B,C}$ is modular and made a conjecture relating this question to algebraic K-theory, motivated by integrable perturbations of rational conformal field theories.

Definition 2.7.1. Consider an r-fold q-hypergeometric series

$$f_{A,B,C}(\tau) = \sum_{n=(n_1,\ldots,n_r)\in(\mathbb{Z}_{\geq 0})^r} \frac{q^{\frac{1}{2}n^tAn+B^tn+C}}{(q)_{n_1}\cdots(q)_{n_r}}$$

where $q = e^{2\pi i\tau}$. If $f_{A,B,C}$ is a modular form, then we call (A, B, C) a modular triple and A the matrix part of it.

Example 2.7.2. Simple examples of modular triples are provided by the Rogers-Ramanujan identities:

$$G(q) = \sum_{n=0}^{\infty} \frac{q^{n^2}}{(q;q)_n} = \frac{1}{(q;q^5)_\infty(q^4;q^5)_\infty}$$

$$H(q) = \sum_{n=0}^{\infty} \frac{q^{n^2+n}}{(q;q)_n} = \frac{1}{(q^2;q^5)_\infty(q^3;q^5)_\infty}.$$

In terms of theta functions,

$$q^{-1/60}G(q) = \frac{\theta_{5,1}}{\eta}$$

and

$$q^{11/60}H(q) = \frac{\theta_{5,3}}{\eta}$$

where

$$\theta_{5,1} = \sum_{k\in\mathbb{Z}} (-1)^k q^{\frac{(10k+1)^2}{40}}, \tag{2.22}$$

$$\theta_{5,3} = \sum_{k\in\mathbb{Z}} (-1)^k q^{\frac{(10k+3)^2}{40}}. \tag{2.23}$$

In this case, we have modular triples $((2),(0),-1/60)$ and $((2),(1),11/60)$.

Let $\mathbf{x} = (x_1, ..., x_r) \in \mathbb{C}^n$ be a solution of $\mathbf{x} = (1-\mathbf{x})^A$. Let $F \subseteq \mathbb{C}$ be a number field. If $\mathbf{x} \in F^n$, we consider $\xi_{\mathbf{x}} = [x_1] + ... + [x_r]$ in $\mathbb{Z}[F^*\backslash\{1\}]$. Nahm's conjecture is as follows :

Conjecture 2.7.3. *Let A be a positive definite symmetric $r \times r$ matrix with rational entries. The followings are equivalent:*

(i) For any solution $x = (x_1, ..., x_r) \in F^r$ of $\mathbf{x} = (1-\mathbf{x})^A$, the element $\xi_x \in \mathcal{B}(F)$ is a torsion element of $\mathcal{B}(F)$.

(ii) There exists a modular triple (A, B, C).

In addition to that we would expect that for such matrices A, there exist a finite number of modular triples $(A, B_i, C_i)_{i\in I}$ indexed by a finite set I and $(f_{A,B_i,C_i})_{i\in I}$ spans a vector space which is invariant under the modular transformations.

Several counterexamples to the above conjecture have been found in [36]. This result is discussed in Section 2.10. Thus the conjecture must be modified and reformulated.

2.8 Nahm's conjecture in rank 1 case

The classification of rank 1 case is completed. See [35, 38]. Only three matrices are allowed to be a matrix part of a modular triple : $(\frac{1}{2}), (1)$ and (2). One interesting observation is that

A	B	C
2	0	-1/60
2	1	11/60
1	0	-1/48
1	1/2	1/24
1	-1/2	1/24
1/2	0	-1/40
1/2	1/2	1/40

Table 2.1: Classification of modular triples of rank 1

these are all of the form $\mathcal{C}(X) \otimes \mathcal{C}(X')^{-1}$ where $\mathcal{C}(X)$ and $\mathcal{C}(X')$ are $A_1 = (2)$ or $T_1 = (1)$ Cartan matrices.

2.9 Nahm's conjecture in rank 2 case

In the rank 2 case, many examples are also given in [38] and more examples are found in [36]. In [14], Huang and Lee considered a matrix of rank 2 with integer entries as an initial attempt to classify modular triples with rank 2 matrices and obtained the following result :

Theorem 2.9.1. *If* $\begin{pmatrix} a & b \\ b & d \end{pmatrix}$ *is a positive definite symmetric matrix with integer entries such that all complex solutions to the system of equations*

$$1 - x_1 = x_1^a x_2^b$$
$$1 - x_2 = x_1^b x_2^d$$

are real, then $\begin{pmatrix} a & b \\ b & d \end{pmatrix}$ *equals one of* $\begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$, $\begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}$, $\begin{pmatrix} 4 & 2 \\ 2 & 2 \end{pmatrix}$, $\begin{pmatrix} 2 & 2 \\ 2 & 4 \end{pmatrix}$, $\begin{pmatrix} 1 & -1 \\ -1 & 2 \end{pmatrix}$, $\begin{pmatrix} 2 & -1 \\ -1 & 1 \end{pmatrix}$, $\begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}$, $\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$, $\begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$, $\begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}$, $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$.

Since a totally real field has its Bloch group consisting of only torsion elements, proving all solutions are real is enough to conclude these solutions are all torsion in the Bloch group. As in the rank 1 classification result, one can again observe a mysterious appearance of the Cartan matrices. To be precise, all these matrices are of the form $\mathcal{C}(X) \otimes \mathcal{C}(X')^{-1}$ where (X, X') is a pair of Dynkin diagram of type A_1, T_1, A_2, T_2 or just direct sum of two rank 1 Cartan matrices $\mathcal{C}(A_1)$ and $\mathcal{C}(T_1)$.

Since it is known that the matrix $\begin{pmatrix} 4 & 1 \\ 1 & 1 \end{pmatrix}$ produces a modular triple and allows non-real complex solutions, this result is not a complete one but it is believed that this list contains a large part of all rank 2 integral matrices producing modular triples. This suggests that instead of considering general matrices A without knowing where to look for, one may consider matrices given by a tensor product of two Cartan matrices and see what is going in more detail. This will be the topic of the following chapters.

2.10 Counterexamples

In [36], Zwegers and Vlasenko found counterexamples to Nahm's conjecture. Here we briefly describe one of them. Let $A = \begin{pmatrix} 3/4 & -1/4 \\ -1/4 & 3/4 \end{pmatrix}$, $B = (1/4, -1/4)$ and $C = -1/80$. Consider the function

$$f_{A,B,C}(z) = \frac{\theta_{5,1}(\frac{z}{8})\eta(z)}{\eta(\frac{z}{2})\eta(2z)}$$

which is modular. One can see that

$$\mathbf{x} = (x_1, x_2) = (\frac{1-\sqrt{-3}}{2}, \frac{1-\sqrt{-3}}{2})$$

is a solution of $\mathbf{x} = (1 - \mathbf{x})^A$ and this is not a torsion element of the Bloch group $\mathcal{B}(\mathbb{C})$.

Chapter 3

q-hypergeometric series associated with a pair of Dynkin diagrams

In this chapter, we will see several examples of r-fold q-hypergeometric series $f_{A,B,C}$ with its matrix part A obtained from a pair of Cartan matrices. They have been studied in the theory of partitions and continued fraction representations of related series. We will also see how they are related to the representation theory of the Virasoro algebra. In Section 3.6, we will define another variant of the dilogarithm, the quantum dilogarithm and see an application of it to Nahm's conjecture. This will show how one can use quantum dilogarithm identities to construct a new modular triple from a known modular triple.

3.1 The Rogers-Ramanujan continued fraction

Consider a q-hypergeometric series

$$R(z) = \sum_{n=0}^{\infty} \frac{z^n q^{n^2}}{(q)_n}.$$

The Rogers-Ramanujan identities are obtained as specializations $H(q) = R(q)$ and $G(q) = R(1)$. An important result about the Rogers-Ramanujan identities is the following recursive identity satisfied by $R(z)$.

Theorem 3.1.1. *$R(z)$ satisfies the following :*

$$R(z) = R(zq) + zqR(zq^2). \tag{3.1}$$

See [2]. This recursive relation can be used to get the Rogers-Ramanujan continued fraction. Since

$$R(q^n) = R(q^{n+1}) + q^{n+1}R(q^{n+2})$$

we get

$$\frac{R(q^{n+1})}{R(q^n)} = \frac{1}{1+q^{n+1}\dfrac{R(q^{n+2})}{R(q^{n+1})}}.$$

By making use of this, one can get the Rogers-Ramanujan continued fraction

$$\frac{H(q)}{G(q)} = \frac{R(q)}{R(1)} = \frac{1}{1+q\dfrac{R(q^2)}{R(q)}} = \frac{1}{1+\dfrac{q}{1+q^2\dfrac{R(q^3)}{R(q^2)}}} = \cdots .$$

From this identity, a curious looking identity

$$\frac{1}{1+\dfrac{e^{-2\pi}}{1+\dfrac{e^{-4\pi}}{1+\dots}}} = \left(\sqrt{\frac{5+\sqrt{5}}{2}} - \frac{\sqrt{5}+1}{2}\right) e^{2\pi/5} \tag{3.2}$$

of Ramanujan mentioned in Chapter 1 can be deduced if one knows the value of the modular function

$$r(\tau) = q^{\frac{1}{5}}\frac{H(q)}{G(q)} = \frac{q^{\frac{1}{5}}}{1+\dfrac{q}{1+\dfrac{q^2}{1+\dfrac{q^3}{1+\cdots}}}}$$

at $\tau = i$.

3.2 Representations of the Virasoro algebra and their characters

In physics, our objects have been studied in the context of conformal field theory and the representation theory of the Virasoro algebra.

Definition 3.2.1. The Virasoro algebra Vir is a complex Lie algebra spanned by L_n, $n \in \mathbb{Z}$ and the central element C satisfying the following commutation relations :

$$[L_m, L_n] = (m-n)L_{m+n} + \frac{C}{12}(m^3-m)\delta_{m+n,0}$$

where $\delta_{k,0}$ denotes the Kronecker delta.

Let $c, h \in \mathbb{C}$. For a complex vector space V, we call a representation $\rho : \mathrm{Vir} \to \mathrm{End}\, V$ a highest weight representation with highest weight (c, h) if there exists a non-zero vector $v_0 \in V$ and the followings are satisfied :

- $\rho(L_0)v_0 = hv_0$
- $\rho(C)v_0 = cv_0$
- $\rho(L_n)v_0 = 0$ for $n > 0$
- V is spanned by v_0 and the elements of the form $\rho(L_{-n_1})\cdots\rho(L_{-n_k})v_0$ with $n_1 \geq \cdots \geq n_k > 0$ and $n_i \in \mathbb{Z}$.

In this case, we have the direct sum decomposition

$$V = \bigoplus_{n \geq 0} V_n$$

with $V_0 = \mathbb{C}v_0$ and V_n for $n > 0$ is generated by $\rho(L_{-n_1})\cdots\rho(L_{-n_k})v_0$ satisfying $n_1 + \cdots + n_k = n$, $n_i \in \mathbb{Z}$. For $v \in V_n$, we have $\rho(L_0)v = (n + h)v$. The character of a highest weight module V is defined by

$$\mathrm{tr}_V\, q^{L_0 - c/24} = \sum_{n=0}^{\infty} (\dim V_n) q^{h - c/24 + n}.$$

For each $c, h \in \mathbb{C}$, we have an irreducible highest weight representation $M(c, h)$ of the Virasoro algebra. These two distinguished numbers c and h play important roles in the classification of representations. c is called the central charge and h is called the conformal dimension or conformal weight.

A conformal field theory with a finite number of primary fields is called rational conformal field theory. Here we simply assume that a finite number of closely related Virasoro modules are given by a rational conformal field theory. It is a source of modular functions since the characters of their modules may have modular invariant properties. Minimal models are good examples of rational conformal field theory. See [8, Chapter 7-8].

Minimal models and their characters

Minimal models are parametrized by a pair of positive integers (p, p') with $p \geq 2$ and $p' \geq p$. The minimal model $c(p, p')$ consists of the irreducible highest weight Virasoro modules $M(c, h_{r,s}^{(p,p')})$ where the central charge is given by $c = 1 - 6\frac{(p'-p)^2}{p'p}$ and their conformal dimensions are

$$h_{r,s}^{(p,p')} = \frac{(p'r - ps)^2 - (p' - p)^2}{4pp'}$$

for $1 \leq r \leq p - 1$ and $1 \leq s \leq p' - 1$. Note that $h_{r,s}^{(p,p')} = h_{p-r,p'-s}^{(p,p')}$. The number $c - 24h_{\min}$ is called the effective central charge of the minimal model where $h_{\min}$ is the minimum value among conformal dimensions $h_{r,s}^{(p,p')}$. The characters are given by the Rocha-Caridi formula as follows :

Theorem 3.2.2. *[33]*

$$\chi_{r,s}^{(p,p')} = \frac{q^{h_{r,s}^{(p,p')}-\frac{c}{24}}}{(q)_\infty} \sum_{n=-\infty}^{\infty} (q^{pp'n^2+(rp'-sp)n} - q^{(pn+r)(p'n+s)})$$

If we use the notations in Example 2.2.7,

$$\chi_{r,s}^{(p,p')}(\tau) = K_{2pp',p'r-ps}(\tau) - K_{2pp',p'r+ps}(\tau).$$

Their modular transformation properties can be obtained using (2.10).

In particular, the $c(2,2k+1)$ minimal model has characters which can be neatly written as

$$\chi_{1,j}^{(2,2k+1)}(\tau) = q^{h_j - c/24} \prod_{n \neq 0, \pm(j+1) \pmod{2k+1}} (1-q^n)^{-1} \tag{3.3}$$

where $1 \leq j \leq k$, $h_j = h_{1,j}^{(2,2k+1)} = \frac{(2k+1-2j)^2-(2k+1-2)^2}{4\cdot 2\cdot(2k+1)}$ and $c = 1 - 6\frac{(2k+1-2)^2}{(2k+1)\cdot 2}$. The effective central charge is $c - 24h_k = (2k-2)/(2k+1)$. One may apply the Jacobi triple product identity to get this infinite product (3.3) from Theorem 3.2.2.

Example 3.2.3. For example, consider the $c(2,5)$ minimal model. Theorem 3.2.2 gives characters

$$\chi_{1,1}^{(2,5)} = \frac{q^{11/60}}{(q)_\infty} \sum_{n=-\infty}^{\infty} (q^{10n^2+3n} - q^{(2n+1)(5n+4)}) \tag{3.4}$$

$$\chi_{1,2}^{(2,5)} = \frac{q^{-1/60}}{(q)_\infty} \sum_{n=-\infty}^{\infty} (q^{10n^2+3n} - q^{(2n+1)(5n+3)}). \tag{3.5}$$

One can notice that these are

$$\chi_{1,1}^{(2,5)} = q^{11/60} H(q) = \frac{q^{11/60}}{(q^2;q^5)_\infty (q^3;q^5)_\infty} \tag{3.6}$$

$$\chi_{1,2}^{(2,5)} = q^{-1/60} G(q) = \frac{q^{-1/60}}{(q;q^5)_\infty (q^4;q^5)_\infty}. \tag{3.7}$$

Example 3.2.4. For another concrete example, let us consider the $c(3,4)$ minimal model. It is called the Ising minimal model because of its close relation to the Ising model in statistical mechanics. Its central charge is $c = 1/2$. There are three irreducible representations with conformal dimensions $0, 1/16, 1/2$ and their characters are

$$\chi_{1,1}^{(3,4)} = \frac{q^{-1/48}}{(q;q)_\infty}\left(\sum_{n=-\infty}^{\infty} q^{12n^2+n} - q^{(3n+1)(4n+1)}\right) \tag{3.8}$$

$$\chi_{1,2}^{(3,4)} = \frac{q^{1/24}}{(q;q)_\infty}\left(\sum_{n=-\infty}^{\infty} q^{12n^2-2n} - q^{(3n+1)(4n+2)}\right) \tag{3.9}$$

$$\chi_{1,3}^{(3,4)} = \frac{q^{23/48}}{(q;q)_\infty}\left(\sum_{n=-\infty}^{\infty} q^{12n^2-5n} - q^{(3n+1)(4n+3)}\right). \tag{3.10}$$

See Example 3.5.1 for modular triples related to the above identities.

3.3 Pairs of Cartan matrices

It has been long known (and conjectured) that there is an interesting class of modular triples whose matrix part is given by the Kronecker product of a pair of Cartan matrices of $ADET$ type.

If M, N are positive definite matrices, then $M \otimes N$ is also positive definite, where $\otimes$ denotes the Kronecker product of matrices. Let us consider a matrix $A = \mathcal{C}(X) \otimes \mathcal{C}(X')^{-1}$ where X and X' are one of $ADET$ Dynkin diagrams and $\mathcal{C}(X)$ and $\mathcal{C}(X')$ their Cartan matrices. See Appendix A for definitions. Note that $A = \mathcal{C}(X) \otimes \mathcal{C}(X')^{-1}$ is positive definite since both Cartan matrices are positive definite. For such matrices A, many modular triples have been found.

3.4 $\mathcal{C}(A_1) \otimes \mathcal{C}(T_n)^{-1}$ case

Now we will see some examples of modular q-hypergeometric series of the form $f_{A,B,C}$ with $A = \mathcal{C}(A_1) \otimes \mathcal{C}(T_n)^{-1}$. In the case of the Rogers-Ramanujan identities, the matrix part of the modular triple is (2) which can be written as $\mathcal{C}(A_1) \otimes \mathcal{C}(T_1)^{-1}$. The Andrews-Gordon identities are generalizations of the Rogers-Ramanujan identities and they give modular triples with matrix parts of the form $A = \mathcal{C}(A_1) \otimes \mathcal{C}(T_n)^{-1}$. Here, we state the Andrews-Gordon identities.

Theorem 3.4.1. *For two integers k, i such that $k \geq 2$ and $1 \leq i \leq k$,*

$$\sum_{n_1,\cdots,n_{k-1}\geq 0} \frac{q^{N_1^2+\cdots+N_{k-1}^2+N_i+\cdots+N_{k-1}}}{(q)_{n_1}\cdots(q)_{n_{k-1}}} = \prod_{\substack{r\neq 0,\pm i \\ (\mathrm{mod}\ 2k+1)}} \frac{1}{1-q^r}$$

where if $j \leq k-1$, $N_j = n_j + \cdots + n_{k-1}$ and if $j = k$, we set $N_j = 0$.

One can see that the right-hand side is nothing but a character of the $c(2,2k+1)$ minimal model, up to a fractional power of q. The left-hand side can be written in the form of $f_{A,B,C}$. So these identities can be a source of many modular triples. If we set $n = k-1$, their matrix part is $A = \mathcal{C}(A_1) \otimes \mathcal{C}(T_n)^{-1}$, or more concretely,

$$\mathcal{C}(A_1) \otimes \mathcal{C}(T_n)^{-1} = \begin{pmatrix} 2 & 2 & 2 & 2 & 2 \\ 2 & 4 & 4 & 4 & 4 \\ 2 & 4 & 6 & 6 & 6 \\ 2 & 4 & 6 & \ddots & \vdots \\ 2 & 4 & 6 & \cdots & 2n \end{pmatrix}.$$

Example 3.4.2. When $n = 1$, we recover the Rogers-Ramanujan identities. For $i = 1$, one gets

$$H(q) = \sum_{n=0}^{\infty} \frac{q^{n^2+n}}{(q;q)_n} = \frac{1}{(q^2;q^5)_\infty (q^3;q^5)_\infty}.$$

For $i = 2$,

$$G(q) = \sum_{n=0}^{\infty} \frac{q^{n^2}}{(q;q)_n} = \frac{1}{(q;q^5)_\infty (q^4;q^5)_\infty}.$$

Example 3.4.3. Let us look at more examples with $n = 2$. If $i = 1$,

$$\sum_{n_1,n_2 \geq 0} \frac{q^{n_1^2+2n_1n_2+2n_2^2+n_1+2n_2}}{(q)_{n_1}(q)_{n_2}} = \prod_{r \neq 0,\pm 1 \pmod 7} \frac{1}{1-q^r} = \frac{(q;q^7)_\infty (q^6;q^7)_\infty (q^7;q^7)_\infty}{(q)_\infty}.$$

For $i = 2$,

$$\sum_{n_1,n_2 \geq 0} \frac{q^{n_1^2+2n_1n_2+2n_2^2+n_2}}{(q)_{n_1}(q)_{n_2}} = \prod_{r \neq 0,\pm 2 \pmod 7} \frac{1}{1-q^r} = \frac{(q^2;q^7)_\infty (q^5;q^7)_\infty (q^7;q^7)_\infty}{(q)_\infty}.$$

For $i = 3$,

$$\sum_{n_1,n_2 \geq 0} \frac{q^{n_1^2+2n_1n_2+2n_2^2}}{(q)_{n_1}(q)_{n_2}} = \prod_{r \neq 0,\pm 3 \pmod 7} \frac{1}{1-q^r} = \frac{(q^3;q^7)_\infty (q^4;q^7)_\infty (q^7;q^7)_\infty}{(q)_\infty}.$$

So from these identities we obtain three modular triples

$$(A, (1,2), 17/42) \tag{3.11}$$
$$(A, (0,1), 5/42) \tag{3.12}$$
$$(A, (0,0), -1/42) \tag{3.13}$$
$$\tag{3.14}$$

where

$$A = \begin{pmatrix} 2 & 2 \\ 2 & 4 \end{pmatrix}.$$

In [32], Richmond and Szekeres studied the asymptotic behavior of Andrews-Gordon identity and proved the following dilogarithm identity

$$\sum_{i=1}^{n} L(\delta_i) = \frac{2n}{2n+3} L(1)$$

where $\mathbf{x} = (\delta_i)$ is the unique solution of $\mathbf{x} = (1-\mathbf{x})^{\mathcal{C}(A_1)\otimes\mathcal{C}(T_n)^{-1}}$ such that $0 < \delta_i < 1$. We can obtain this identity also from the formula in Theorem 4.2.5 of the next chapter by computing

$$\sum_{i=1}^{n} L(\delta_i) = \frac{h(A_1)r(A_1)r(T_n)}{h(A_1)+h(T_n)} L(1) = \frac{2n}{2+2n+1} L(1) = \frac{2n}{2n+3} L(1).$$

This dilogarithm identity gives the effective central charge $c - 24h_k = (2k-2)/(2k+1)$ of the non-unitary $c(2, 2k+1)$ minimal models.

3.5 $\mathcal{C}(T_1) \otimes \mathcal{C}(T_n)^{-1}$ case

We also have a few modular triples whose matrix part is of the form $A = \mathcal{C}(T_1) \otimes \mathcal{C}(T_n)^{-1}$.

Example 3.5.1. The matrix $\mathcal{C}(T_1) \otimes \mathcal{C}(T_1)^{-1} = (1)$ of rank 1 is related to the identity in (2.3) of Euler,

$$\prod_{n=0}^{\infty} (1 + zq^n) = \sum_{n=0}^{\infty} \frac{q^{n(n-1)/2}}{(q)_n} z^n. \tag{3.15}$$

When $z = 1$, it becomes

$$\prod_{n=0}^{\infty} (1 + q^n) = \sum_{n=0}^{\infty} \frac{q^{n(n-1)/2}}{(q)_n}. \tag{3.16}$$

If we specialize $z = q^{1/2}$, we get

$$\prod_{n=1}^{\infty} (1 + q^{n-\frac{1}{2}}) = \sum_{n=0}^{\infty} \frac{q^{n^2/2}}{(q)_n}. \tag{3.17}$$

Another specialization $z = q$ gives

$$\prod_{n=1}^{\infty} (1 + q^n) = \sum_{n=0}^{\infty} \frac{q^{n(n+1)/2}}{(q)_n}. \tag{3.18}$$

Note that (3.16) and (3.18) are just scalar multiples of each other. In terms of minimal model characters, we have

$$\chi_{1,2}^{(3,4)} = \frac{\eta(2\tau)}{\eta(\tau)} = q^{1/24} \sum_{m=-\infty}^{\infty} (-1)^m q^{3m^2-m} = q^{1/24} \sum_{n=0}^{\infty} \frac{q^{n(n+1)/2}}{(q)_n}.$$

These are in fact, with a minor modification, known as Weber's modular functions

$$\mathfrak{f}(\tau) = \frac{e^{-\frac{\pi i}{24}}\eta(\frac{\tau+1}{2})}{\eta(\tau)} = q^{-1/48}\prod_{n=1}^{\infty}(1+q^{n-\frac{1}{2}}) = q^{-1/48}\sum_{n=0}^{\infty}\frac{q^{n^2/2}}{(q)_n}, \tag{3.19}$$

$$\mathfrak{f}_1(\tau) = \frac{\eta(\frac{\tau}{2})}{\eta(\tau)} = q^{-1/48}\prod_{n=1}^{\infty}(1-q^{n-\frac{1}{2}}), \tag{3.20}$$

$$\mathfrak{f}_2(\tau) = \sqrt{2}\frac{\eta(2\tau)}{\eta(\tau)} = \sqrt{2}q^{1/24}\prod_{n=1}^{\infty}(1+q^{n}) = \sqrt{2}q^{1/24}\sum_{n=0}^{\infty}\frac{q^{n(n+1)/2}}{(q)_n}. \tag{3.21}$$

From these considerations, we can obtain three modular triples

$$((1),(0),-1/48) \tag{3.22}$$

$$((1),(1/2),1/24) \tag{3.23}$$

$$((1),(-1/2),1/24)\,. \tag{3.24}$$

The equation $x = 1 - x$ has the unique solution $x = 1/2$. From (3.21), one can derive the identity

$$L(\frac{1}{2}) = \frac{1}{2}L(1) = \frac{\pi^2}{12}.$$

Theorem 4.2.5 also gives the same dilogarithm identity

$$L(\frac{1}{2}) = \frac{h(T_1)r(T_1)r(T_1)}{h(T_1)+h(T_1)}L(1) = \frac{3}{6}L(1) = \frac{1}{2}L(1).$$

Example 3.5.2. For the matrix $\mathcal{C}(T_1)\otimes\mathcal{C}(T_2)^{-1}$, let us take a look at the q-series identity

$$f(q,z) = \sum_{k\geq 0}\frac{q^{k(k+1)/2}(-zq;q)_k}{(q)_k} = (-zq^2;q^2)_\infty(-q;q)_\infty = \prod_{m=1}^{\infty}(1+zq^{2m})(1+q^m).$$

This is called the Lebesgue's identity and the left-hand side of it can be rewritten as

$$\sum_{k\geq 0}\frac{q^{k(k+1)/2}(-zq;q)_k}{(q)_k} = \sum_{i,j\geq 0}\frac{z^j q^{\frac{i^2}{2}+ij+j^2+\frac{i}{2}+j}}{(q)_i(q)_j}. \tag{3.25}$$

In [1, p.278-279], this identity was studied from the viewpoint of partition identities and continued fractions. To prove (3.25), one can use the q-binomial identity

$$(-zq;q)_k = \sum_{r=0}^{k}\begin{bmatrix}k\\r\end{bmatrix}_q q^{r(r+1)/2}z^r$$

where

$$\begin{bmatrix}k\\r\end{bmatrix}_q = \frac{(q)_k}{(q)_r(q)_{k-r}}.$$

Thus from this, one can produce modular triples with its matrix part given by $\begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix} = \mathcal{C}(T_1) \otimes \mathcal{C}(T_2)^{-1}$ by specializing z appropriately. For example, $z = 1$ gives

$$f(q,1) = \sum_{i,j\geq 0} \frac{q^{\frac{i^2+2ij+2j^2}{2}+\frac{i+2j}{2}}}{(q)_i(q)_j} = (-q^2;q^2)_\infty(-q;q)_\infty = \frac{(q^4;q^4)_\infty}{(q;q)_\infty} = \frac{1}{(q^1;q^4)_\infty(q^2;q^4)_\infty(q^3;q^4)_\infty}.$$

So

$$q^{1/8} \sum_{i,j\geq 0} \frac{q^{\frac{i^2+2ij+2j^2}{2}+\frac{i+2j}{2}}}{(q)_i(q)_j} = \frac{\eta(4\tau)}{\eta(\tau)}$$

gives a modular triple $(\mathcal{C}(T_1) \otimes \mathcal{C}(T_2)^{-1}, (1/2, 1), 1/8)$. By solving the system of equations

$$\begin{cases} x_1 &= (1-x_1)(1-x_2) \\ x_2 &= (1-x_1)(1-x_2)^2 \end{cases},$$

we can get the corresponding dilogarithm identity

$$L(\sqrt{2}-1) + L\left(\frac{1}{2}(2-\sqrt{2})\right) = \frac{3}{4}L(1).$$

Note also that

$$\frac{h(T_1)r(T_1)r(T_2)}{h(T_1)+h(T_2)} = \frac{6}{8} = \frac{3}{4}.$$

3.6 The quantum dilogarithm and its application to Nahm's conjecture

Recall the identity (2.3)

$$\prod_{n=0}^{\infty}(1+zq^n) = \sum_{n=0}^{\infty} \frac{q^{n(n-1)/2}}{(q)_n} z^n.$$

This can be used to define the quantum dilogarithm function as follows :

Definition 3.6.1. The quantum dilogarithm function is defined by

$$(z;q)_\infty = \prod_{n=0}^{\infty}(1-zq^n) = \sum_{n=0}^{\infty} \frac{(-1)^n q^{n(n-1)/2}}{(q)_n} z^n.$$

Like other variants of the dilogarithm function, the quantum dilogarithm function also satisfies the five-term relation whose semiclassical limit recovers the five-term relation of the dilogarithm function.

Theorem 3.6.2. *[5] (quantum five-term relation) Let u and v be noncommutative variables satisfying the relation $uv = qvu$ where q is a central element. Then the following identity holds :*

$$(v;q)_\infty(u;q)_\infty = (u;q)_\infty(-vu;q)_\infty(v;q)_\infty$$

In [38, Proposition 3], Zagier shows the following identity is equivalent to the quantum five-term relation.

Theorem 3.6.3.

$$\sum_{\substack{r,s,t\geq 0\\ r+s=m,s+t=n}} \frac{q^{rt}}{(q)_r(q)_s(q)_t} = \frac{1}{(q)_m(q)_n} \tag{3.26}$$

Proof. We will use the identity (2.4)

$$\sum_{n=0}^{\infty} \frac{x^n}{(q)_n} = \frac{1}{(x;q)_\infty}$$

and (2.5)

$$\sum_{n=0}^{\infty} \frac{(a;q)_n}{(q)_n} x^n = \frac{(ax;q)_\infty}{(x;q)_\infty}$$

several times. We will prove (3.26) by showing their generating functions are equal. For the right-hand side, we get the generating function

$$\sum_{m,n\geq 0} \frac{x^m y^n}{(q)_m(q)_n} = \frac{1}{(x;q)_\infty(y;q)_\infty}.$$

Now for the left-hand side,

$$\sum_{m,n\geq 0}\left(\sum_{\substack{r,s,t\geq 0\\ r+s=m,s+t=n}} \frac{q^{rt}}{(q)_r(q)_s(q)_t}\right) x^m y^n \tag{3.27}$$

$$= \sum_{r,s,t\geq 0} \frac{q^{rt}x^{r+s}y^{s+t}}{(q)_r(q)_s(q)_t} = \sum_{r,s,t\geq 0} \frac{(xq^t)^r(xy)^s y^t}{(q)_r(q)_s(q)_t} \tag{3.28}$$

$$= \sum_{s,t\geq 0} \frac{(xy)^s y^t}{(q)_s(q)_t} \frac{1}{(xq^t;q)_\infty} = \sum_{s,t\geq 0} \frac{(xy)^s y^t}{(q)_s(q)_t} \frac{(x;q)_t}{(x;q)_\infty} \tag{3.29}$$

$$= \frac{1}{(xy;q)_\infty} \frac{1}{(x;q)_\infty} \sum_{t\geq 0} \frac{(x;q)_t y^t}{(q)_t} = \frac{1}{(xy;q)_\infty} \frac{1}{(x;q)_\infty} \frac{(xy;q)_\infty}{(y;q)_\infty} \tag{3.30}$$

$$= \frac{1}{(x;q)_\infty(y;q)_\infty} \tag{3.31}$$

□

An application of (3.26) to Nahm's conjecture is the following identity between two q-hypergeometric series.

Theorem 3.6.4. *[40]* [1] *Let $f_{A,B,C}$ be an r-fold q-hypergeometric series with*

$$A = \begin{pmatrix} a & b \\ b & c \end{pmatrix}, B = (k, l).$$

If we set

$$A' = \begin{pmatrix} a & b+1 & a+b \\ b+1 & c & b+c \\ a+b & b+c & a+2b+c \end{pmatrix}, B' = (k, l, k+l),$$

the equality $f_{A,B,C} = f_{A',B',C}$ holds.

Proof.

$$f_{A,B,C} = q^C \sum_{m,n\geq 0} \frac{q^{(m,n)A(m,n)^t + B\cdot(m,n)}}{(q)_m(q)_n} \tag{3.32}$$

$$= q^C \sum_{m,n\geq 0} \sum_{\substack{s,r,t\geq 0 \\ r+s=m, s+t=n}} \frac{q^{\frac{1}{2}(am^2+2bmn+cn^2)+B\cdot(m,n)} q^{rt}}{(q)_r(q)_s(q)_t} \tag{3.33}$$

$$= q^C \sum_{s,r,t\geq 0} \frac{q^{\frac{1}{2}(a(r+s)^2+2b(r+s)(s+t)+c(s+t)^2)} q^{rt} q^{k(r+s)+l(s+t)}}{(q)_r(q)_s(q)_t} \tag{3.34}$$

Let us compute the exponent in the numerator :

$$a(r+s)^2 + 2b(r+s)(s+t) + c(s+t)^2 + 2rt \tag{3.35}$$
$$= (a)r^2 + (b+1)rt + (a+b)rs + (b+1)tr + (c)t^2 \tag{3.36}$$
$$+ (b+c)ts + (a+b)rs + (b+c)ts + (a+2b+c)s^2. \tag{3.37}$$

Therefore, setting

$$A' = \begin{pmatrix} a & b+1 & a+b \\ b+1 & c & b+c \\ a+b & b+c & a+2b+c \end{pmatrix}, B' = (k, l, k+l),$$

we obtain $f_{A,B,C} = f_{A',B',C}$. □

[1] This is an identity presented by S. Zwegers during a talk in the AIM workshop and may not have been published.

This shows that the quantum five-term relation allows us to construct new modular triples from known ones. One can also write down a solution of $\mathbf{x} = (1-\mathbf{x})^{A'}$ in terms of a solution of $\mathbf{x} = (1-\mathbf{x})^{A}$. If $\mathbf{x} = (x_1, x_2)$ is a solution of $\mathbf{x} = (1-\mathbf{x})^{A}$, $\mathbf{z} = (z_1, z_2, z_3)$ is a solution of $\mathbf{z} = (1-\mathbf{z})^{A'}$ where

$$z_1 = \frac{x_1\,(1-x_2)}{1-x_1x_2} \tag{3.38}$$

$$z_2 = \frac{(1-x_1)\,x_2}{1-x_1x_2} \tag{3.39}$$

$$z_3 = x_1x_2. \tag{3.40}$$

Recall Theorem 2.5.1. One can see that if $0 < x_i < 1$ for $i = 1, 2$, we have $0 < z_i < 1$ for $i = 1, 2, 3$. This transformation preserves positive solutions in $(0, 1)$. Moreover, we get a dilogarithm identity

$$L(x_1) + L(x_2) = L\left(\frac{x_1(1-x_2)}{1-x_1x_2}\right) + L\left(\frac{(1-x_1)x_2}{1-x_1x_2}\right) + L(x_1x_2)$$

which is nothing but the five-term relation (2.11).

Example 3.6.5. For example, let us apply this to the modular triples (3.11) considered before. The construction gives

$$(A, (1, 2), 17/42) \mapsto (A', (1, 2, 3), 17/42) \tag{3.41}$$

$$(A, (0, 1), 5/42) \mapsto (A', (0, 1, 1), 5/42) \tag{3.42}$$

$$(A, (0, 0), -1/42) \mapsto (A', (0, 0, 0), -1/42) \tag{3.43}$$

where

$$A = \begin{pmatrix} 2 & 2 \\ 2 & 4 \end{pmatrix}, A' = \begin{pmatrix} 2 & 3 & 4 \\ 3 & 4 & 6 \\ 4 & 6 & 10 \end{pmatrix}.$$

3.7 q-hypergeometric series and functional dilogarithm identities

In [12], Gordon and McIntosh derive a few functional dilogarithm identities from well-known identities involving q-hypergeometric series. Let us briefly review their results.

Let us look at the Euler's identity (2.3)

$$(-z; q)_\infty = \sum_{n=0}^{\infty} \frac{q^{n(n-1)/2}}{(q)_n} z^n$$

again. The asymptotic analysis of this identity leads to the functional dilogarithm identity

$$L(z) + L(1/z) = L(1).$$

If we consider Y-systems and their associated functional dilogarithm identities as in Theorem 4.1.7, this corresponds to the case of (A_1, A_1)

Recall the q-binomial theorem (2.5)

$$\sum_{n=0}^{\infty} \frac{(a;q)_n}{(q)_n} x^n = \frac{(ax;q)_\infty}{(x;q)_\infty}.$$

The asymptotic analysis of this allows us to prove the functional dilogarithm identity

$$L(a) + L(b) = L(ab) + L\left(\frac{a(1-b)}{1-ab}\right) + L\left(\frac{b(1-a)}{1-ab}\right).$$

The q-Gauss sum is another well-known q-series identity, which is given by

$$\sum_{n=0}^{\infty} \frac{(a;q)_n (b;q)_n}{(c;q)_n (q;q)_n} (\frac{c}{ab})^n = \frac{(c/a;q)_\infty (c/b;q)_\infty}{(c;q)_\infty (c/(ab);q)_\infty}.$$

It corresponds to the functional dilogarithm identity involving 9-terms :

$$\begin{aligned} &L(a) + L(b) + L(c) + L\left(\frac{1-c}{d}\right) \\ =&L(ac) + L(bc) + L\left(\frac{a(1-c)}{d}\right) + L\left(\frac{b(1-c)}{d}\right) + L\left(\frac{(1-ac)(1-bc)}{d}\right) \end{aligned} \tag{3.44}$$

Remark 3.7.1. Recently, in [18] and [21], many new quantum dilogarithm identities associated with pairs of Dynkin diagrams were obtained by exploiting quantum cluster algebras. In this context, the quantum five-term relation is a special case corresponding to the Dynkin pair (A_2, A_1).

From the study of wall-crossing invariants in [3], Cecotti, Neitzke and Vafa considered the trace of quantum monodromy operators associated with a pair of Dynkin diagrams and explained how one can produce q-hypergeometric series involving a tensor product of two Cartan matrices. Based on the idea of wall-crossing invariance, they obtained new q-series identities.

It is plausible that one might organize the huge list of partition identities appearing in Slater's list [34] in the context of more general mathematical theory based on a pair of Cartan matrices along with the same line of quantum dilogarithm identities and trace of monodromy operators.

Chapter 4

Y-systems, T-systems, Q-systems and torsion elements of the Bloch group

In this chapter, we study close relatives of the equation $\mathbf{x} = (1 - \mathbf{x})^A$. The definitions of Y-systems, T-systems and Q-systems will be given in this chapter. It is quite remarkable that this simple looking equation $\mathbf{x} = (1 - \mathbf{x})^A$, in fact, has a rich theory behind it. An extensive and authoritative survey of this huge subject is given in [26]. It was mentioned that this equation has a close relation to some topics of mathematical physics. From mathematical sides, a good algebraic framework to deal with these objects is the theory of cluster algebras.

In Section 4.1, we will study Y-systems. Many properties of Y-systems will be used in the following section. Recall that Nahm's conjecture is formulated in terms of the Bloch group. In Section 4.2, we will prove that solutions of $\mathbf{x} = (1-\mathbf{x})^A$ are indeed torsion elements of the Bloch group. In Section 4.3 and Section 4.4, T-systems and Q-systems will be introduced. T-systems and Q-systems are especially good when one wants to get solutions of $\mathbf{x} = (1-\mathbf{x})^A$ explicitly. Q-systems will be central in Chapter 6.

4.1 Y-systems

The Y-system for a pair of ADE Dynkin diagrams

In this section, we closely follow the notations of [29]. Let X be a Dynkin diagram of ADE type with the index set I. The rank and the Coxeter number of X will be denoted by r and h. We write Cartan matrix of X as $\mathcal{C}(X)$ and the adjacency matrix $\mathcal{I}(X) = 2I_r - \mathcal{C}(X)$ where I_r denotes the $r \times r$ identity matrix. We call a decomposition $I = I_+ \cup I_-$ bipartite if $\mathcal{I}(X)_{ij} = 1$ implies $(i, j) \in I_+ \times I_-$ or $(i, j) \in I_- \times I_+$.

Now consider an ordered pair of Dynkin diagrams (X, X'). For another Dynkin diagram X', $I' = I'_+ \cup I'_-$, r', h', $\mathcal{C}(X')$, and $\mathcal{I}(X')$ will be defined analogously.

We give an alternate bicoloring on the pair of Dynkin diagrams. Let us fix bipartite decompositions of I and I'. Let $\mathbf{I} = I \times I'$ and $\mathbf{I} = \mathbf{I}_+ \sqcup \mathbf{I}_-$ where $\mathbf{I}_+ = (I_+ \times I'_+) \sqcup (I_- \times I'_-)$

and $\mathbf{I}_- = (I_+ \times I'_-) \sqcup (I_- \times I'_+)$. Let $\epsilon : \mathbf{I} \to \{1, -1\}$ be the function defined by $\epsilon(\mathbf{i}) = \pm 1$ for $\mathbf{i} \in \mathbf{I}_\pm$ and $P_\pm = \{(\mathbf{i}, u) \in \mathbf{I} \times \mathbb{Z} | \epsilon(\mathbf{i})(-1)^u = \pm 1\}$. Roughly speaking, we want our alternate bicoloring interchanges their colors as $u \in \mathbb{Z}$ changes by 1.

Definition 4.1.1. For a family of variables, $\{Y_{ii'}(u) | i \in I, i' \in I', u \in \mathbb{Z}\}$, the Y-system $\mathbb{Y}(X, X')$ associated with a pair (X, X') of ADE Dynkin diagrams is defined as a system of recurrence relations as follows :

$$Y_{ii'}(u-1)Y_{ii'}(u+1) = \frac{\prod_{j:j\sim i}(1 + Y_{ji'}(u))}{\prod_{j':j'\sim i'}(1 + Y_{ij'}(u)^{-1})} \tag{4.1}$$

where $a \sim b$ means a is adjacent to b.

Note that $\mathbb{Y}(X, X')$ consists of two decoupled copies, $\{Y_\mathbf{i}(u) | (\mathbf{i}, u) \in P_+\}$ and $\{Y_\mathbf{i}(u) | (\mathbf{i}, u) \in P_-\}$. If $(\mathbf{i}, u) \in P_+$, $Y_\mathbf{i}(u)$ can be written as a rational function of variables $\{Y_\mathbf{i}(0) | \mathbf{i} \in \mathbf{I}_+\}$ and $\{Y_\mathbf{i}(-1) | \mathbf{i} \in \mathbf{I}_-\}$ whereas if $(\mathbf{i}, u) \in P_-$, $Y_\mathbf{i}(u)$ only depends on $\{Y_\mathbf{i}(0) | \mathbf{i} \in \mathbf{I}_-\}$ and $\{Y_\mathbf{i}(-1) | \mathbf{i} \in \mathbf{I}_+\}$.

Example 4.1.2. Let us consider the example of $\mathbb{Y}(A_2, A_1)$. Since the index set I' of the A_1 Dynkin diagram consists of the single element 1, we just set $Y_{i,1} = Y_i$ for $i = 1, 2$. The recurrence relation of the Y-system is

$$Y_i(u-1)Y_i(u+1) = \prod_{j:j\sim i}(1 + Y_j(u)).$$

If we write the sequence explicitly, we get the following :

u	$Y_1(u)$	$Y_2(u)$
$\vdots$	$\vdots$	$\vdots$
-1	$\frac{1}{\alpha}$	x
0	y	β
1	$\alpha(\beta+1)$	$\frac{y+1}{x}$
2	$\frac{x+y+1}{xy}$	$\alpha + \frac{\alpha+1}{\beta}$
3	$\frac{\alpha+1}{\alpha\beta}$	$\frac{x+1}{y}$
4	x	$\frac{1}{\alpha}$
5	β	y
6	$\frac{y+1}{x}$	$\alpha(\beta+1)$
7	$\alpha + \frac{\alpha+1}{\beta}$	$\frac{x+y+1}{xy}$
8	$\frac{x+1}{y}$	$\frac{\alpha+1}{\alpha\beta}$
9	$\frac{1}{\alpha}$	x
10	y	β
11	$\alpha(\beta+1)$	$\frac{y+1}{x}$
$\vdots$	$\vdots$	$\vdots$

One can clearly observe the decoupling of the Y-system and that this sequence is periodic with period 10. Another important thing to note is that all terms are Laurent polynomials of initial conditions.

Definition 4.1.3. If a solution $\{Y_{\mathbf{i}}(u)|\mathbf{i} \in \mathbf{I}, u \in \mathbb{Z}\}$ of $\mathbb{Y}(X, X')$ does not have any dependence on u so that $Y_{\mathbf{i}}(u) = y_{\mathbf{i}}$ for each $\mathbf{i}$ in a field, it must satisfy the following system of rr' equations of rr' variables :

$$y_{ii'}^2 = \frac{\prod_{j:j\sim i}(1 + y_{ji'})}{\prod_{j':j'\sim i'}(1 + y_{ij'}^{-1})}. \tag{4.2}$$

We call it the constant Y-system and denote it by $\mathbb{Y}_c(X, X')$.

One can see the importance of the constant Y-system in Proposition 4.2.4.

Properties of Y-systems

In fact, many conjectures about Y-systems such as periodicities and functional dilogarithm identities, originated from the thermodynamic Bethe ansatz approach of conformal field theory [39, 31, 11], had remained open for years but now have been proved rigorously due to recent development of the theory of cluster algebras. (See [7] and [22]). One may hope that a correct reformulation of Nahm's conjecture incorporates this and it would help us find new directions toward understanding modular triples and modular q-hypergeometric series.

Now we state several important results about Y-systems. They will be used in our proof of theorem 4.2.1.

Theorem 4.1.4. *[22]A solution $\{Y_{\mathbf{i}}(u)|\mathbf{i} \in \mathbf{I}, u \in \mathbb{Z}\}$ of $\mathbb{Y}(X, X')$ is periodic and*

$$Y_{\mathbf{i}}(u + 2(h + h')) = Y_{\mathbf{i}}(u). \tag{4.3}$$

For all these theorems below, we assume that $\{Y_{\mathbf{i}}(u)|\mathbf{i} \in \mathbf{I}, u \in \mathbb{Z}\}$ satisfies the Y-system $\mathbb{Y}(X, X')$ associated to a pair of ADE Dynkin diagrams. Let $(y_{\mathbf{i}})_{\mathbf{i}\in\mathbf{I}}$ be indeterminates and set

$$Y_{\mathbf{i}}(0) = y_{\mathbf{i}}, \mathbf{i} \in \mathbf{I}_+ \tag{4.4}$$

$$Y_{\mathbf{i}}(-1) = y_{\mathbf{i}}^{-1}, \mathbf{i} \in \mathbf{I}_-. \tag{4.5}$$

Then each $Y_{\mathbf{i}}(u)$ with $(\mathbf{i}, u) \in P_+$ can be regarded as a rational function in $y_{\mathbf{i}}$'s. Let $\mathbb{Q}(y)$ be the field of rational functions in $y_{\mathbf{i}}$'s.

Theorem 4.1.5. *[29] Let $(\mathbf{i}, u) \in P_+$. Then*

$$Y_{\mathbf{i}}(u) = G_{\mathbf{i}}(u)T_{\mathbf{i}}(u) \in \mathbb{Q}(y) \tag{4.6}$$

where $G_{\mathbf{i}}(u) \in \mathbb{Q}(y)$ satisfies $G_{\mathbf{i}}(u)|_{(0,\cdots,0)} = 1$ and $T_{\mathbf{i}}(u) \neq 1$ is a positive or negative monomial in $y_{\mathbf{i}}$'s i.e. $T_{\mathbf{i}}(u)$ can be written as a product of $y_{\mathbf{i}}$'s or as a product of $y_{\mathbf{i}}^{-1}$'s.

Let $S_+ = \{(\mathbf{i}, u) | 0 \leq u \leq 2(h+h') - 1, (\mathbf{i}, u) \in P_+\}$. We state a property of the Y-system which Nakanishi called the constancy condition in [29].

Theorem 4.1.6. *[29, Proposition 3.2 (i)]*

$$\sum_{(\mathbf{i},u)\in S_+} Y_{\mathbf{i}}(u) \wedge (1 + Y_{\mathbf{i}}(u)) = 0 \in \Lambda^2\mathbb{Q}(y)^*. \tag{4.7}$$

This theorem enables us to prove a class of functional dilogarithm identities associated with Y-systems.

Theorem 4.1.7. *[29, Theorem 2.8] The following functional dilogarithm identity holds :*

$$\sum_{(\mathbf{i},u)\in S_+} L\left(\frac{Y_{\mathbf{i}}(u)}{1 + Y_{\mathbf{i}}(u)}\right) = hrr'L(1). \tag{4.8}$$

To get the right-hand side of (4.8), one needs to count how many of $f_{\mathbf{i}}(u)|_{(0,\cdots,0)}$ becomes 1 for each $\mathbf{i} \in S_+$ where $f_{\mathbf{i}}(u) = \frac{Y_{\mathbf{i}}(u)}{1+Y_{\mathbf{i}}(u)}$ and this part is crucial in [29].

Example 4.1.8. In Example 4.1.2, $\mathbb{Y}(A_2, A_1)$ was explicitly worked out and we saw that

$$S = \left\{x, y, \frac{y+1}{x}, \frac{x+y+1}{xy}, \frac{x+1}{y}\right\}$$

forms a half-period of $\mathbb{Y}(A_2, A_1)$. So we have $r = 2, h = 3$ and $r' = 1, h' = 2$. They are all Laurent polynomials in x and y. From this, one can get functional dilogarithm identities

$$\begin{aligned}
&\sum_{a\in S} L\left(\frac{a}{1+a}\right) \\
=&L\left(\frac{x}{1+x}\right) + L\left(\frac{y}{1+y}\right) + L\left(\frac{1+y}{x(1+\frac{1+y}{x})}\right) + L\left(\frac{1+x+y}{xy(1+\frac{1+x+y}{xy})}\right) + L\left(\frac{1+x}{(1+\frac{1+x}{y})y}\right) \\
=&L\left(\frac{x}{x+1}\right) + L\left(\frac{y}{y+1}\right) + L\left(\frac{y+1}{x+y+1}\right) + L\left(\frac{x+y+1}{xy+x+y+1}\right) + L\left(\frac{x+1}{x+y+1}\right) \\
=&3L(1) = \frac{\pi^2}{2}
\end{aligned}$$

and

$$\begin{aligned}
&\sum_{a\in S} L\left(\frac{1}{1+a}\right) \\
=&L\left(\frac{1}{x+1}\right) + L\left(\frac{1}{y+1}\right) + L\left(\frac{1}{\frac{y+1}{x}+1}\right) + L\left(\frac{1}{\frac{x+y+1}{xy}+1}\right) + L\left(\frac{1}{\frac{x+1}{y}+1}\right) \\
=&L\left(\frac{1}{x+1}\right) + L\left(\frac{1}{y+1}\right) + L\left(\frac{x}{x+y+1}\right) + L\left(\frac{xy}{xy+x+y+1}\right) + L\left(\frac{y}{x+y+1}\right) \\
=&2L(1) = \frac{\pi^2}{3}.
\end{aligned}$$

The Y-system for a pair of foldings of ADE Dynkin diagrams

The results in the previous section can be extended to include all foldings of ADE diagrams without much effort.

First note that for a pair (X, X') of directed graphs with the index sets I and I', we can redefine the Y-system in terms of their adjacency matrices of graphs as follows :

$$Y_{ii'}(u-1)Y_{ii'}(u+1) = \frac{\prod_{j\in I}(1+Y_{ji'}(u))^{\mathcal{I}(X)_{ij}}}{\prod_{j'\in I'}\left(1+Y_{ij'}(u)^{-1}\right)^{\mathcal{I}(X')_{i'j'}}}. \tag{4.9}$$

Let X be a Dynkin diagram of ADE type. We can regard it as a directed graph with the adjacency matrix $\mathcal{I}(X)$. For a group G of diagram automorphisms of X, we can define a quotient diagram $\bar{X} = X/G$ as follows : $\bar{X}$ has the vertex set $\bar{I}$, the orbit of I under G. $(\bar{i}, \bar{j})$ is an edge of $\bar{X}$ if (i, j) is an edge of G and the multiplicity $\mathcal{I}(\bar{X})_{\bar{i}\bar{j}}$ is defined as the number of preimages of $\bar{j}$ for a fixed representative i of $\bar{i}$. Let us call $\bar{X}$ the folding of X by G.

Note that $\bar{X}$ is generally a directed graph and its adjacency matrix $\mathcal{I}(\bar{X})$ may not be symmetric. Let us call the matrix $\mathcal{C}(\bar{X}) = 2I_r - \mathcal{I}(\bar{X})$ the Cartan matrix of $\bar{X}$. If G is a trivial group, we just get $\bar{X} = X$. The Coxeter number of $\bar{X}$ is the same as the Coxeter number of X.

The tadpole graph T_r is obtained as the folding of $X = A_{2r}$ by the diagram automorphism group of order 2. The Cartan matrix $\mathcal{C}(T_r)$ is the same as $\mathcal{C}(A_r)$ except that a diagonal entry corresponding to the vertex with a loop is 1 instead of 2. See Appendix A.

$\bar{X}$ inherits the bipartite decomposition $\bar{I} = \bar{I}_+ \cup \bar{I}_-$ of X except when $\bar{X} = T_n$ because T_n diagram has a loop and cannot be bipartite, in which case, we just set $\bar{I} = \bar{I}_+ = \bar{I}_-$.

Let $(\bar{X}, \bar{X}')$ be a pair of foldings of ADE Dynkin diagrams. Let $\bar{\mathbf{I}} = \bar{I} \times \bar{I}'$ and define a decomposition $\bar{\mathbf{I}} = \bar{\mathbf{I}}_+ \cup \bar{\mathbf{I}}_-$ where $\bar{\mathbf{I}}_+ = (\bar{I}_+ \times \bar{I}'_+) \cup (\bar{I}_- \times \bar{I}'_-)$ and $\bar{\mathbf{I}}_- = (\bar{I}_+ \times \bar{I}'_-) \cup (\bar{I}_- \times \bar{I}'_+)$. When $\bar{X} = T_r$ or $\bar{X}' = T_{r'}$, we just get $\bar{\mathbf{I}} = \bar{\mathbf{I}}_+ = \bar{\mathbf{I}}_-$.

When $\bar{X} = T_r$ or $\bar{X}' = T_r$, we also set $\bar{P}_+ = \bar{P}_- = \bar{\mathbf{I}} \times \mathbb{Z}$. Otherwise, we can define $\bar{P}_\pm$ similarly as in the previous section.

From a solution $(Y_{\mathbf{i}}(u))_{(\mathbf{i},u)\in\mathbf{I}\times\mathbb{Z}}$ of $\mathbb{Y}(X, X')$, one can obtain a solution $(Y_{\bar{\mathbf{i}}}(u))_{(\bar{\mathbf{i}},u)\in\bar{\mathbf{I}}\times\mathbb{Z}}$ of the Y-system $\mathbb{Y}(\bar{X}, \bar{X}')$ by just setting $Y_{\bar{\mathbf{i}}}(u) = Y_{\mathbf{i}}(u)$.

Now we restate Theorems in the previous section for $\mathbb{Y}(\bar{X}, \bar{X}')$ associated to a pair of foldings of ADE Dynkin diagrams.

Theorem 4.1.9. *Theorem 4.1.4 and Theorem 4.1.5 hold true for a solution of* $\mathbb{Y}(\bar{X}, \bar{X}')$.

Let $(y_{\bar{\mathbf{i}}})_{\bar{\mathbf{i}}} \in \bar{\mathbf{I}}$ be indeterminates and set

$$Y_{\bar{\mathbf{i}}}(0) = y_{\bar{\mathbf{i}}}, \bar{\mathbf{i}} \in \bar{\mathbf{I}}_+ \tag{4.10}$$
$$Y_{\bar{\mathbf{i}}}(-1) = y_{\bar{\mathbf{i}}}^{-1}, \bar{\mathbf{i}} \in \bar{\mathbf{I}}_-. \tag{4.11}$$

Then again each $Y_{\bar{\mathbf{i}}}(u)$ with $(\bar{\mathbf{i}}, u) \in \bar{P}_+$ can be regarded as an element of $\mathbb{Q}(y)$, the field of rational functions in $y_{\bar{\mathbf{i}}}$'s.

Let us define $d_{\bar{\mathbf{i}}}(u)$ as the number of the preimages of $(\bar{\mathbf{i}}, u)$ under the quotient map $\mathbf{I} \times \mathbb{Z} \to \bar{\mathbf{I}} \times \mathbb{Z}$ and set $\bar{S}_+ = \{(\bar{\mathbf{i}}, u) | 0 \le u \le 2(h + h') - 1, (\bar{\mathbf{i}}, u) \in \bar{P}_+\}$. Then Theorem 4.1.6 can be restated as follows :

Theorem 4.1.10.

$$\sum_{(\bar{\mathbf{i}},u)\in\bar{S}_+} d_{\bar{\mathbf{i}}}(u)(Y_{\bar{\mathbf{i}}}(u) \wedge (1 + Y_{\bar{\mathbf{i}}}(u))) = 0 \in \Lambda^2\mathbb{Q}(y)^*. \tag{4.12}$$

In other words, the element

$$\sum_{(\bar{\mathbf{i}},u)\in\bar{S}_+} d_{\bar{\mathbf{i}}}(u)[\frac{Y_{\bar{\mathbf{i}}}(u)}{1 + Y_{\bar{\mathbf{i}}}(u)}] \tag{4.13}$$

of the group ring of $\mathbb{Q}(y)$ *is an element of the Bloch group* $\mathcal{B}(\mathbb{Q}(y))$.

Finally, Theorem 4.1.7 can be written as follows :

Theorem 4.1.11. *The following functional dilogarithm identity holds :*

$$\sum_{(\bar{\mathbf{i}},u)\in\bar{S}_+} d_{\bar{\mathbf{i}}}(u)L\left(\frac{Y_{\bar{\mathbf{i}}}(u)}{1 + Y_{\bar{\mathbf{i}}}(u)}\right) = hrr'L(1).$$

Now we prove that Y-system produces a torsion element of the Bloch group.

Proposition 4.1.12. *Let* $f_{\bar{\mathbf{i}}}(u) = \frac{Y_{\bar{\mathbf{i}}}(u)}{1+Y_{\bar{\mathbf{i}}}(u)} \in \mathbb{Q}(y)$. *Then*

$$\sum_{(\bar{\mathbf{i}},u)\in\bar{S}_+} d_{\bar{\mathbf{i}}}(u)D(f_{\bar{\mathbf{i}}}(u)|_{\mathbf{x}}) = 0 \tag{4.14}$$

for any $\mathbf{x} = (x_{\bar{\mathbf{i}}}) \in \mathbb{C}^n$ *where* $n = rr'$.

Proof. To employ Proposition 2.4.2, we check the following condition

$$\sum_{(\bar{\mathbf{i}},u)\in\bar{S}_+} d_{\bar{\mathbf{i}}}(u)\left(f_{\bar{\mathbf{i}}}(u) \wedge (1 - f_{\bar{\mathbf{i}}}(u))\right) = 0. \tag{4.15}$$

This is equivalent to

$$\sum_{(\bar{\mathbf{i}},u)\in\bar{S}_+} d_{\bar{\mathbf{i}}}(u)\left((\frac{Y_{\bar{\mathbf{i}}}(u)}{1 + Y_{\bar{\mathbf{i}}}(u)}) \wedge (\frac{1}{1 + Y_{\bar{\mathbf{i}}}(u)})\right) = 0,$$

which reduces to the constancy condition of the Y-system,

$$\sum_{(\bar{\mathbf{i}},u)\in\bar{S}_+} d_{\bar{\mathbf{i}}}(u)\left(Y_{\bar{\mathbf{i}}}(u) \wedge (1 + Y_{\bar{\mathbf{i}}}(u))\right) = 0.$$

Thus the condition (4.15) is satisfied by Theorem 4.1.10.

Now all we have to check is that there is a point $\mathbf{a} \in \mathbb{C}^n$ such that $\sum_{(\bar{\mathbf{i}},u)\in\bar{S}_+} D(f_{\bar{\mathbf{i}}}(u)|_{\mathbf{a}}) = 0$. By Theorem 4.1.9, $f_{\bar{\mathbf{i}}}(u) = \frac{G_{\bar{\mathbf{i}}}(u)T_{\bar{\mathbf{i}}}(u)}{1+G_{\bar{\mathbf{i}}}(u)T_{\bar{\mathbf{i}}}(u)}$. Since $G_{\bar{\mathbf{i}}}(u)|_{(0,\cdots,0)} = 1$ and $T_{\bar{\mathbf{i}}}(u) \neq 1$ is a positive or negative monomial in $y_{\bar{\mathbf{i}}}$'s, $f_{\bar{\mathbf{i}}}(u)|_{(0,\cdots,0)}$ is always 0 or 1 depending on whether $T_{\bar{\mathbf{i}}}(u)$ is positive or negative. So we can simply choose $\mathbf{a} = (0,\cdots,0)$ and then $\sum_{(\bar{\mathbf{i}},u)\in\bar{S}_+} D(f_{\bar{\mathbf{i}}}(u)|_{\mathbf{a}}) = 0$ and therefore $\sum_{(\bar{\mathbf{i}},u)\in\bar{S}_+} D(f_{\bar{\mathbf{i}}}(u)|_{\mathbf{x}}) = 0$ for any $\mathbf{x} = (x_{\bar{\mathbf{i}}}) \in \mathbb{C}^n$ by Proposition 2.4.2. □

Remark 4.1.13. This proposition generalizes [10, Theorem 2] and [6, Corollary 6.14].

Corollary 4.1.14. *Let $(\bar{X}, \bar{X}')$ be a pair of ADET diagrams. If $(y_{\bar{\mathbf{i}}}) \in F^n$ is a solution of the constant Y-system $\mathbb{Y}_c(\bar{X}, \bar{X}')$ for a number field F,*

$$\sum_{\bar{\mathbf{i}}\in\bar{\mathbf{I}}}[\frac{y_{\bar{\mathbf{i}}}}{1+y_{\bar{\mathbf{i}}}}] \in \mathcal{B}(F) \tag{4.16}$$

is a torsion element of the Bloch group $\mathcal{B}(F)$.

Proof. Let $\sigma : F \hookrightarrow \mathbb{C}$ be an embedding. By Proposition 4.1.12, we know

$$\sum_{(\mathbf{i},u)\in\bar{S}_+} d_{\bar{\mathbf{i}}}(u) D\left(\sigma(\frac{y_{\bar{\mathbf{i}}}}{1+y_{\bar{\mathbf{i}}}})\right) = 0 \tag{4.17}$$

Note that when $(\bar{X}, \bar{X}')$ is given by a pair of $ADET$ diagrams, $d_{\bar{\mathbf{i}}}(u)$ is the same for all $(\bar{i}, u)$. If $(\bar{X}, \bar{X}') = (T_r, T_{r'})$, $d_{\bar{\mathbf{i}}}(u) = 2$ and $d_{\bar{\mathbf{i}}}(u) = 1$ otherwise.

Thus

$$\sum_{\bar{\mathbf{i}}\in\bar{\mathbf{I}}} D\left(\sigma(\frac{y_{\bar{\mathbf{i}}}}{1+y_{\bar{\mathbf{i}}}})\right) = 0. \tag{4.18}$$

Since this is true for any $\sigma : F \hookrightarrow \mathbb{C}$, $\sum_{\bar{\mathbf{i}}\in\bar{\mathbf{I}}}[\frac{y_{\bar{\mathbf{i}}}}{1+y_{\bar{\mathbf{i}}}}]$ is a torsion element of the Bloch group. □

The Y-system, which can be defined for a pair of Dynkin diagrams, turns out to be very useful to study the equation $\mathbf{x} = (1-\mathbf{x})^A$ as one can see in Proposition 4.2.4. We can relate the equation $\mathbf{x} = (1-\mathbf{x})^A$ to the Y-system and then using properties of Y-systems, we can show that all solutions give torsion elements of the Bloch group.

4.2 Torsion elements of the Bloch group

We are now ready to prove the following theorem.

Theorem 4.2.1. *Let $A = \mathcal{C}(X) \otimes \mathcal{C}(X')^{-1}$ where (X, X') is a pair of ADET Dynkin diagrams. For every solution $\mathbf{x} = (x_{\mathbf{i}})_{\mathbf{i}\in\mathbf{I}}$ of the equation $\mathbf{x} = (1-\mathbf{x})^A$ in a number field F, $\xi_{\mathbf{x}} = \sum_{\mathbf{i}\in\mathbf{I}}[x_{\mathbf{i}}]$ is a torsion element of the Bloch group $\mathcal{B}(F)$.*

The proof is obtained using properties of Y-systems. Frenkel and Szenes studied dilogarithm identities and their relation to torsion elements in algebraic K-theory [9] and Y-systems [10]. In [27, section 4], Nahm briefly explains how one can obtain a proof of the above statement assuming the periodicity of the Y-system. It seems that, however, more structural properties of Y-systems need to be used to complete the proof. Nakanishi's paper [29] contains most of results used here except relating results to the Bloch group.

We give the following definition for a notational unity.

Definition 4.2.2. Let (X, X') be a pair of $ADET$ Dynkin diagrams. The system of equations

$$\prod_{j' \in I'} x_{ij'}^{\mathcal{C}(X')_{i'j'}} = \prod_{j \in I} (1 - x_{ji'})^{\mathcal{C}(X)_{ij}} \tag{4.19}$$

with $(i, i') \in I \times I' = \mathbf{I}$ will be denoted by $\mathbb{X}(X, X')$.

Proposition 4.2.3. *Two systems of equations $\mathbf{x} = (1 - \mathbf{x})^A$ and $\mathbb{X}(X, X')$ are equivalent. In other words, they have the same set of complex solutions.*

Proof. Let $(x_\mathbf{i})$ be a solution of $\mathbf{x} = (1 - \mathbf{x})^A$. Recall that from Lemma 2.5.2, we know that $x_\mathbf{i} \neq 0, 1$ for all $\mathbf{i} \in \mathbf{I}$. We can see that

$$x_\mathbf{i} = \prod_{\mathbf{j} \in \mathbf{I}} (1 - x_\mathbf{j})^{(\mathcal{C}(X) \otimes \mathcal{C}(X')^{-1})_{\mathbf{ij}}}, \tag{4.20}$$

which is the same as

$$x_{ii'} = \prod_{(j,j') \in I \times I'} (1 - x_{jj'})^{(\mathcal{C}(X) \otimes \mathcal{C}(X')^{-1})_{\mathbf{ij}}} \tag{4.21}$$

implies

$$\prod_{j' \in I'} x_{ij'}^{\mathcal{C}(X')_{i'j'}} = \prod_{j \in I} (1 - x_{ji'})^{\mathcal{C}(X)_{ij}}. \tag{4.22}$$

Conversely, we can see that $\mathbb{X}(X, X')$ cannot have any solution with $x_\mathbf{i} = 0$ or $x_\mathbf{i} = 1$ for some $\mathbf{i} \in \mathbf{I}$. It is so because X and X' are connected graphs, which implies that for all $\mathbf{i} \in \mathbf{I}$, both $x_\mathbf{i}$ and $1 - x_\mathbf{i}$ show up as denominators in some equations of $\mathbb{X}(X, X')$. Then (4.22) implies (4.20). $\square$

Now we relate a solution of the equation $\mathbb{X}(X, X')$ to the constant Y-system $\mathbb{Y}_c(X, X')$. This will show that the constant Y-system $\mathbb{Y}_c(X, X')$ is just a disguised form of the equation $\mathbb{X}(X, X')$.

Proposition 4.2.4. *If $\mathbf{x} = (x_\mathbf{i})$ is a solution to $\mathbb{X}(X, X')$ in a number field F, $y_\mathbf{i} = \frac{x_\mathbf{i}}{1 - x_\mathbf{i}}$ is a solution to the constant Y-system $\mathbb{Y}_c(X, X')$. Conversely, if $\mathbf{y} = (y_\mathbf{i})$ with $y_\mathbf{i} \neq -1, 0$ for all $\mathbf{i} \in \mathbf{I}$ satisfies $\mathbb{Y}_c(X, X')$, $x_\mathbf{i} = \frac{y_\mathbf{i}}{1 + y_\mathbf{i}}$ satisfies $\mathbb{X}(X, X')$.*

Proof. Use the change of variables $y_{\mathbf{i}} = \frac{x_{\mathbf{i}}}{1-x_{\mathbf{i}}}$ or $x_{\mathbf{i}} = \frac{y_{\mathbf{i}}}{1+y_{\mathbf{i}}} = \frac{1}{1+y_{\mathbf{i}}^{-1}}$.

From (4.22), we obtain

$$\prod_{j'\in I'}(\frac{1}{1+y_{ij'}^{-1}})^{\mathcal{C}(X')_{i'j'}} = \prod_{j\in I}(\frac{1}{1+y_{ji'}})^{\mathcal{C}(X)_{ij}} \tag{4.23}$$

$$1 = \frac{\prod_{j\in I}(1+y_{ji'})^{-\mathcal{C}(X)_{ij}}}{\prod_{j'\in I'}(1+y_{ij'}^{-1})^{-\mathcal{C}(X')_{i'j'}}}. \tag{4.24}$$

This can be written as

$$\left(\frac{1}{1+y_{ii'}^{-1}}\right)^2 (1+y_{ii'})^2 = \frac{\prod_{j\in I}(1+y_{ji'})^{\mathcal{I}(X)_{ij}}}{\prod_{j'\in I'}\left(1+y_{ij'}^{-1}\right)^{\mathcal{I}(X')_{i'j'}}}. \tag{4.25}$$

Thus we finally get the constant Y-system

$$y_{ii'}^2 = \frac{\prod_{j\in I}(1+y_{ji'})^{\mathcal{I}(X)_{ij}}}{\prod_{j'\in I'}\left(1+y_{ij'}^{-1}\right)^{\mathcal{I}(X')_{i'j'}}}. \tag{4.26}$$

These steps can be reversed to prove the converse statement. □

Now we can finish the proof of Theorem 4.2.1.

Proof. Let $\mathbf{x} = (x_{\mathbf{i}})$ be a solution to $\mathbf{x} = (1-\mathbf{x})^A$. By Proposition 4.2.4, $y_{\mathbf{i}} = \frac{x_{\mathbf{i}}}{1-x_{\mathbf{i}}}$ is a solution to the constant Y-system $\mathbb{Y}_c(X, X')$. Then the theorem follows from Corollary 4.1.14. □

Theorem 2.5.1 says that we have a special positive solution of $\mathbf{x} = (1-\mathbf{x})^A$ satisfying $0 < x_{\mathbf{i}} < 1$ for all $\mathbf{i} \in \mathbf{I}$. If we apply Proposition 4.2.4 to this solution, Theorem 4.1.11 implies the following :

Theorem 4.2.5. *Let X and X' be one of ADET Dynkin diagrams. For $A = \mathcal{C}(X)\otimes\mathcal{C}(X')^{-1}$ and its unique positive solution $\mathbf{x} = (1-\mathbf{x})^A$ satisfying $0 < x_{\mathbf{i}} < 1$ for all $\mathbf{i} \in \mathbf{I}$, we get a dilogarithm identity*

$$\sum_{\mathbf{i}\in\mathbf{I}} L(x_{\mathbf{i}}) = \frac{hrr'}{h+h'}L(1).$$

Remark 4.2.6. The proof shows that knowing general properties of Y-systems is enough to prove that solutions are torsion elements in the Bloch group. The Y-system is partly a built-in structure of cluster algebras. If we use Y-systems, solving equation $\mathbf{x} = (1-\mathbf{x})^A$ becomes not really an essential part although finding concrete solutions is still an interesting problem.

Although counterexamples to Nahm's conjecture have been found as explained in Section 2.10, it is reasonable to believe that a fairly general theory about q-series associated to pairs of the Dynkin diagrams exists. Since being a torsion element in the Bloch group is a simple consequence implied by properties of the Y-system associated a pair of Cartan matrices, the Y-system itself looks more natural object to look at in relation to Nahm's conjecture than the Bloch group. This may give a hint on how to reformulate Nahm's conjecture.

There are a few immediate questions along this line. The matrix $\begin{pmatrix} 4 & 1 \\ 1 & 1 \end{pmatrix}$ is among the list of rank 2 modular triples and it cannot be written as a tensor product of two Cartan matrices of $ADET$ type. One can ask the following :

Question 4.2.7. Can one understand matrices like $\begin{pmatrix} 4 & 1 \\ 1 & 1 \end{pmatrix}$, which are not tensor product of two Cartan matrices, in the context of Y-systems?

In view of Theorem 2.9.1, one may initially guess that all solutions of $\mathbf{x} = (1-\mathbf{x})^A$ for $A = \mathcal{C}(X) \otimes \mathcal{C}(X')^{-1}$ are totally real, but Nahm's calculation in [27] shows that it may have non-real complex solutions (for example, (A_2, A_3) case). We can also see this in Section 6.7.

4.3 T-systems

Now we define another system of recurrence relations which has similar properties as Y-systems.

Definition 4.3.1. For a pair (X, X') of $ADET$ Dynkin diagrams with the index sets I and I', consider the system of recurrence relations

$$T_{ii'}(u-1)T_{ii'}(u+1) = \prod_{j \in I} T_{ji'}(u)^{\mathcal{I}(X)_{ij}} + \prod_{j' \in I'} T_{ij'}(u)^{\mathcal{I}(X')_{i'j'}} \tag{4.27}$$

with $(i, i') \in \mathbf{I}$ and $u \in \mathbb{Z}$. We call it the T-system and denote it by $\mathbb{T}(X, X')$.

Quite analogously to 4.1.4, we have a periodicity result.

Theorem 4.3.2. *[15, Corollary 4.28.]A solution of the T-system is periodic and*

$$T_{\mathbf{i}}(u + 2(h+h')) = T_{\mathbf{i}}(u), \tag{4.28}$$
$$T_{i,i'}(u + (h+h')) = T_{\omega(i),\omega'(i')}(u) \tag{4.29}$$

where ω and ω' are graph automorphisms defined in Appendix A.2.

As in the case of Y-systems, if (4.27) does not depend on $u \in \mathbb{Z}$, we call it the constant T-system. Since it has a very close relationship with the Q-system which will be defined in the next section, we introduce the following notation.

Definition 4.3.3. For a pair (X, X') of $ADET$ Dynkin diagrams, we denote the system of equations

$$Q_{ii'}^2 = \prod_{j\in I} Q_{ji'}^{\mathcal{I}(X)_{ij}} + \prod_{j'\in I'} Q_{ij'}^{\mathcal{I}(X')_{i'j'}}. \tag{4.30}$$

with $(i, i') \in \mathbf{I}$ by $\mathbb{Q}(X, X')$

Now we prove that $\mathbb{Q}(X, X')$ is obtained from the equation $\mathbb{X}(X, X')$ by a suitable change of variables.

Proposition 4.3.4. *If $\mathbf{x} = (x_{\mathbf{i}})$ is a solution to $\mathbb{X}(X, X')$ in a number field F, $z_{ii'} = \prod_{j\in I} x_{ji'}^{-\mathcal{C}(X)^{-1}_{ij}}$ is a solution to the constant T-system $\mathbb{Q}(X, X')$ and $z_{\mathbf{i}} \neq 0$ for $\mathbf{i} \in \mathbf{I}$.*

Proof. Let us rewrite the equation $\mathbf{x} = (1 - \mathbf{x})^A$:

$$x_{\mathbf{i}} = \prod_{\mathbf{j}\in\mathbf{I}} (1 - x_{\mathbf{j}})^{(\mathcal{C}(X)\otimes\mathcal{C}(X')^{-1})_{\mathbf{ij}}} \tag{4.31}$$

$$\prod_{\mathbf{j}\in\mathbf{I}} x_{\mathbf{j}}^{(\mathcal{C}(X)^{-1}\otimes\mathcal{C}(X'))_{\mathbf{ij}}} = 1 - x_{\mathbf{i}}. \tag{4.32}$$

Since $x_{ii'} \neq 0$, $z_{ii'} = \prod_{j\in I} x_{ji'}^{-\mathcal{C}(X)^{-1}_{ij}} \neq 0$. Now apply the change of variables

$$x_{ii'} = \prod_{j\in I} z_{ji'}^{-\mathcal{C}(X)_{ij}}$$

to (4.32) so that we get

$$\prod_{j'\in I'} z_{ij'}^{-\mathcal{C}(X')_{i'j'}} = 1 - \prod_{j\in I} z_{ji'}^{-\mathcal{C}(X)_{ij}}, \tag{4.33}$$

$$\prod_{j\in I} z_{ji'}^{-\mathcal{C}(X)_{ij}} + \prod_{j'\in I'} z_{ij'}^{-\mathcal{C}(X')_{i'j'}} = 1. \tag{4.34}$$

By multiplying $z_{ii'}^2$ both sides, we have

$$\prod_{j\in I} z_{ji'}^{\mathcal{I}(X)_{ij}} + \prod_{j'\in I'} z_{ij'}^{\mathcal{I}(X')_{i'j'}} = z_{ii'}^2. \tag{4.35}$$

□

Before we move on to the next section, let us put the relations among solutions of various systems considered so far together.

Theorem 4.3.5. *Let* $(Q_{\mathbf{i}})$ *be a solution of* $\mathbb{Q}(X, X')$ *such that* $Q_{\mathbf{i}} \neq 0$ *for all* $\mathbf{i} \in \mathbf{I}$. *Let*

$$x_{i,i'} = \frac{\prod_{j \in I} Q_{j,i'}^{\mathcal{I}(X)_{i,j}}}{Q_{i,i'}^2}$$

and

$$y_{i,i'} = \frac{\prod_{j \in I} Q_{j,i'}^{\mathcal{I}(X)_{i,j}}}{\prod_{j' \in I'} Q_{i,j'}^{\mathcal{I}(X')_{i'j'}}}.$$

Then $\mathbf{x} = (x_{\mathbf{i}})$ *and* $\mathbf{y} = (y_{\mathbf{i}})$ *are solutions of the systems* $\mathbb{X}(X, X')$ *and* $\mathbb{Y}_c(X, X')$, *respectively.*

Proof. Let $(Q_{\mathbf{i}})$ be a solution of $\mathbb{Q}(X, X')$ so that

$$\prod_{j \in I} Q_{j,i'}^{\mathcal{I}(X)_{i,j}} + \prod_{j' \in I'} Q_{i,j'}^{\mathcal{I}(X')_{i',j'}} = Q_{i,i'}^2. \tag{4.36}$$

Dividing both sides by $Q_{i,i'}^2$, we get

$$\prod_{j \in I} Q_{j,i'}^{-\mathcal{C}(X)_{i,j}} + \prod_{j' \in I'} Q_{i,j'}^{-\mathcal{C}(X')_{i',j'}} = 1 \tag{4.37}$$

$$\prod_{j' \in I'} Q_{i,j'}^{-\mathcal{C}(X')_{i',j'}} = 1 - \prod_{j \in I} Q_{j,i'}^{-\mathcal{C}(X)_{i,j}}. \tag{4.38}$$

Let $x_{i,i'} = \prod_{j \in I} Q_{j,i'}^{-\mathcal{C}(X)_{i,j}}$. We have

$$1 - x_{i,i'} = \prod_{j' \in I'} Q_{i,j'}^{-\mathcal{C}(X')_{i',j'}}.$$

We want to prove

$$\prod_{j' \in I'} x_{ij'}^{\mathcal{C}(X')_{i'j'}} = \prod_{j \in I} (1 - x_{ji'})^{\mathcal{C}(X)_{ij}}. \tag{4.39}$$

This can be proved by a straightforward computation as follows :

$$\prod_{j' \in I'} x_{ij'}^{\mathcal{C}(X')_{i'j'}} \tag{4.40}$$

$$= \prod_{j' \in I'} \prod_{j \in I} (Q_{j,j'}^{-\mathcal{C}(X)_{i,j}})^{\mathcal{C}(X')_{i'j'}} \tag{4.41}$$

$$= \prod_{j' \in I'} \prod_{j \in I} Q_{j,j'}^{-\mathcal{C}(X)_{i,j}\mathcal{C}(X')_{i'j'}} \tag{4.42}$$

$$= \prod_{j \in I} \prod_{j' \in I'} (Q_{j,j'}^{-\mathcal{C}(X')_{i'j'}})^{\mathcal{C}(X)_{i,j}} \tag{4.43}$$

$$= \prod_{j \in I} (1 - x_{ji'})^{\mathcal{C}(X)_{ij}}. \tag{4.44}$$

So $x = (x_{\mathbf{i}})$ is a solution of the equation $\mathbb{X}(X, X')$.

Note that $x_{i,i'} \neq 1$ is impossible because

$$x_{i,i'} = \frac{\prod_{j\in I} Q_{j,i'}^{\mathcal{I}(X)_{i,j}}}{Q_{i,i'}^2} = 1$$

implies

$$\prod_{j'\in I'} Q_{i,j'}^{\mathcal{I}(X')_{i',j'}} = Q_{i,i'}^2 - \prod_{j\in I} Q_{j,i'}^{\mathcal{I}(X)_{i,j}} = 0$$

which contradicts to the assumption. Thus, by Proposition 4.2.4

$$y_{i,i'} = \frac{x_{i,i'}}{1 - x_{i,i'}} = \frac{\frac{\prod_{j\in I} Q_{j,i'}^{\mathcal{I}(X)_{i,j}}}{Q_{i,i'}^2}}{1 - \frac{\prod_{j\in I} Q_{j,i'}^{\mathcal{I}(X)_{i,j}}}{Q_{i,i'}^2}} = \frac{\prod_{j\in I} Q_{j,i'}^{\mathcal{I}(X)_{i,j}}}{\prod_{j'\in I'} Q_{i,j'}^{\mathcal{I}(X')_{i'j'}}}$$

is a solution of $\mathbb{Y}_c(X, X')$. □

4.4 Q-systems

Definition 4.4.1. Let X be a Dynkin diagram of ADE type with the index set I. For a family of variables, $\{Q_m^{(a)} | a \in I, m \in \mathbb{Z}^{\geq 0}\}$, consider the recurrence equation given by

$$(Q_m^{(a)})^2 = \prod_{b\in I} (Q_m^{(b)})^{\mathcal{I}(X)_{ab}} + Q_{m-1}^{(a)} Q_{m+1}^{(a)}. \tag{4.45}$$

We call it the Q-system of type X. It is usual to put boundary conditions $Q_0^{(a)} = 1$ for all $a \in I$.

Theorem 4.4.2. *(4.45) defines a sequence of polynomials in* $\{Q_1^{(a)} | a \in I\}$.

See [19, 4] for a proof. This is related to a general phenomenon called the Laurent phenomenon in cluster algebras.

Let us begin with a simple example of the Q-system when $X = A_1$. In fact, the Chebyshev polynomials of the second kind satisfy the above recurrence relations. So one can say that Q-systems are generalizations of the Chebyshev polynomials.

Chebyshev polynomials and generalized Chebyshev polynomials

The Chebyshev polynomials of the second kind are defined by the recurrence relations

$$\begin{cases} U_0(x) &= 1 \\ U_1(x) &= 2x \\ U_{n+1}(x) &= 2xU_n(x) - U_{n-1}(x) \end{cases}.$$

One can easily prove the following :

Proposition 4.4.3. *$n \geq 1$,*

$$U_n(x)^2 = 1 + U_{n+1}(x)U_{n-1}(x) \tag{4.46}$$

Proof. This can be done easily using induction. For $n = 1$, we have $U_0(x) = 1, U_1(x) = 2x, U_2(x) = 4x^2 - 1$. For general $n \in \mathbb{N}$,

$$\begin{aligned}
&U_{n+1}(x)U_{n-1}(x) = (2xU_n(x) - U_{n-1}(x))(U_{n-1}(x)) && (4.47)\\
=&2xU_n(x)U_{n-1}(x) - U_{n-1}^2(x) = 2xU_n(x)U_{n-1}(x) - (1 + U_n(x)U_{n-2}(x)) && (4.48)\\
=&U_n(x)(2xU_{n-1}(x) - U_{n-2}(x)) - 1 && (4.49)\\
=&U_n(x)^2 - 1 && (4.50)
\end{aligned}$$

□

With $U_0(x) = 1$ and $U_1(x) = 2x$, the recursion (4.46) can be used to define the Chebyshev polynomials. If we adopt this as a starting point, the fact that we always get polynomials from this recursion may not be very obvious because it involves divisions of polynomials.

An interesting observation about $U_n(x)$ is that it can be written as the determinant of $\mathcal{C}(A_n)$ slightly modified. For example,

$$U_3(x) = \begin{vmatrix} 2x & -1 & 0 \\ -1 & 2x & -1 \\ 0 & -1 & 2x \end{vmatrix}$$

Here the matrix is the same as $\mathcal{C}(A_n)$ with its diagonal entries replaced by $2x$.

The Chebyshev polynomials of the first kind are defined by the recurrence equations :

$$\begin{cases} T_0(x) & = & 1 \\ T_1(x) & = & 2x \\ T_{n+1}(x) & = & 2xT_n(x) - T_{n-1}(x) \end{cases}.$$

Similar properties hold in this case and $T_n(x)$ can be written as the determinant of the matrix $\mathcal{C}(T_n)$ with a similar modification as above. For example, we have

$$T_3(x) = \begin{vmatrix} 2x & -1 & 0 \\ -1 & 2x & -1 \\ 0 & -1 & x \end{vmatrix}.$$

Example 4.4.4. If $X = A_2$, we get a generalization of Chebyshev polynomials in two variables. When we impose the condition $Q_1^{(1)} = Q_1^{(2)} = x$, we get a sequence of polynomials in x. One may think this as generalized Chebyshev polynomials associated with the Dynkin diagram of type $X = T_1$. They can also be defined by the following recursion relations :

$$\begin{cases} Q_0(x) & = & 1 \\ Q_1(x) & = & x \\ Q_n(x)^2 - Q_n(x) & = & Q_{n-1}(x)Q_{n+1}(x) \end{cases}.$$

The few first terms are

$$Q_0(x) = 1 \tag{4.51}$$
$$Q_1(x) = x \tag{4.52}$$
$$Q_2(x) = x^2 - x \tag{4.53}$$
$$Q_3(x) = x^3 - 2x^2 + 1 \tag{4.54}$$
$$Q_4(x) = x^4 - 3x^3 + x^2 + 2x \tag{4.55}$$
$$Q_5(x) = x^5 - 4x^4 + 3x^3 + 3x^2 - 2x \tag{4.56}$$
$$\vdots \qquad .$$

Remark 4.4.5. One way to realize the Q-system comes from the representation theory of quantum groups and Lie algebras. The subject actually originated from this study. This enables us to handle them with many tools in Lie theory. This will be presented in Section 6.2.

Before we close the chapter, let us state a question which seems not having been answered.

Question 4.4.6. Are all solutions of $\mathbb{X}(X, X')$, $\mathbb{Y}_c(X, X')$ and $\mathbb{Q}(X, X')$ contained in cyclotomic fields?

See Section 5.5 and Conjecture 6.3.2 for related discussions.

Chapter 5

On positive solutions of $\mathbb{X}(X, X')$, $\mathbb{Y}_c(X, X')$ and $\mathbb{Q}(X, X')$

Let (X, X') be an ordered pair of Dynkin diagrams of $ADET$ type. In this chapter, we focus on positive solutions of $\mathbb{X}(X, X')$, $\mathbb{Y}_c(X, X')$ and $\mathbb{Q}(X, X')$. The index sets of X and X' will be denoted by I and I' respectively. We will use the notations $\mathbf{I} = I \times I'$ and $\mathbf{i} = (i, i') \in \mathbf{I}$ as in the previous chapters.

5.1 The principal positive solution of $\mathbb{X}(X, X')$

Consider the system of equations $\mathbf{x} = (1 - \mathbf{x})^A$ associated to a positive definite symmetric matrix $A = (a_{ij})$. Recall Theorem 2.5.1 which says that there is a unique real solution such that $0 < x_i < 1$ for all $i = 1, \cdots r$. By applying this to the matrix $A = \mathcal{C}(X) \otimes \mathcal{C}(X')^{-1}$, we get the following definition.

Definition 5.1.1. We call the unique solution $\mathbf{x} = (x_{\mathbf{i}})$ of $\mathbb{X}(X, X')$ with $0 < x_{\mathbf{i}} < 1$ the principal positive solution of $\mathbb{X}(X, X')$.

Remark 5.1.2. One cannot say there exists a unique positive solution of $\mathbb{X}(X, X')$. For example, $x = (1 - x)^2$ has two positive solutions $x = \frac{1}{2}\left(3 - \sqrt{5}\right)$ or $x = \frac{1}{2}\left(3 + \sqrt{5}\right)$.

Now we consider how the symmetry of Dynkin diagrams is reflected on the solutions. One main reason to establish the following is to prove Theorem 5.3.6 which is closely related to Conjecture 6.3.2.

Theorem 5.1.3. *The principal positive solution of $\mathbb{X}(X, X')$ inherits the symmetry of the Dynkin diagrams.*

Proof. Essentially the uniqueness of the principal positive solution implies the symmetry. More concretely, the principal positive solution $\mathbf{x} = (x_{i,i'})$ satisfies

$$\prod_{j' \in I'} x_{ij'}^{\mathcal{C}(X')_{i'j'}} = \prod_{j \in I} (1 - x_{ji'})^{\mathcal{C}(X)_{ij}} \tag{5.1}$$

for each $(i, i') \in \mathbf{I}$.

Let σ and σ' be graph automorphisms of X and X' respectively. Then we get

$$\prod_{j' \in I'} x_{\sigma(i),j'}^{\mathcal{C}(X')_{\sigma(i'),j'}} = \prod_{j \in I} (1 - x_{j,\sigma'(i')})^{\mathcal{C}(X)_{\sigma(i),j}} \tag{5.2}$$

$$\prod_{j' \in I'} x_{\sigma(i),\sigma'(j')}^{\mathcal{C}(X')_{\sigma(i'),\sigma(j')}} = \prod_{j \in I} (1 - x_{\sigma(j),\sigma'(i')})^{\mathcal{C}(X)_{\sigma(i),\sigma(j)}} \tag{5.3}$$

$$\prod_{j' \in I'} x_{\sigma(i),\sigma'(j')}^{\mathcal{C}(X')_{i',j'}} = \prod_{j \in I} (1 - x_{\sigma(j),\sigma'(i')})^{\mathcal{C}(X)_{i,j}}. \tag{5.4}$$

This implies that $\mathbf{x} = (x_{\sigma(i),\sigma'(i')})$ must be the principal positive solution. So we can conclude that $x_{\sigma(i),\sigma'(i')} = x_{i,i'}$ from its uniqueness. □

5.2 The positive solution of $\mathbb{Y}_c(X, X')$

Theorem 5.2.1. *There exists a unique positive solution of the constant Y-system $\mathbb{Y}_c(X, X')$ and it inherits the symmetry of the Dynkin diagrams.*

Proof. This follows from Proposition 4.2.4 and Theorem 5.1.3. □

Definition 5.2.2. We call the solution $\mathbf{y} = (y_{\mathbf{i}})$ of $\mathbb{Y}_c(X, X')$ characterized in Theorem 5.2.1 the positive solution of $\mathbb{Y}_c(X, X')$.

5.3 The positive solution of $\mathbb{Q}(X, X')$

Let us consider the system of equations $\mathbb{Q}(X, X')$ associated to a pair of Dynkin diagrams of $ADET$ type. Recall Definition 4.3.3. The system is given by

$$Q_{ii'}^2 = \prod_{j \in I} Q_{ji'}^{\mathcal{I}(X)_{ij}} + \prod_{j' \in I'} Q_{ij'}^{\mathcal{I}(X')_{i'j'}}. \tag{5.5}$$

Lemma 5.3.1. *If $\mathbf{z} = (z_{\mathbf{i}})$ is a positive solution of the equation $\mathbb{Q}(X, X')$, $x_{i,i'} = \prod_{j \in I} z_{j,i'}^{-\mathcal{C}(X)_{i,j}}$ is the principal positive solution of $\mathbb{X}(X, X')$. Conversely, if $\mathbf{x} = (x_{\mathbf{i}})$ is the principal positive solution of $\mathbb{X}(X, X')$, $z_{i,i'} = \prod_{j \in I} x_{j,i'}^{-(\mathcal{C}(X)^{-1})_{ij}}$ is a positive solution of $\mathbb{Q}(X, X')$.*

Proof. Let $\mathbf{z} = (z_{\mathbf{i}})$ be a solution of the equation $\mathbb{Q}(X,X')$ with $z_{\mathbf{i}} > 0$ for all $i \in \mathbf{I}$. Theorem 4.3.5 implies that $x_{i,i'} = \prod_{j\in I} z_{j,i'}^{-\mathcal{C}(X)_{i,j}} > 0$ is a solution of $\mathbb{X}(X,X')$. Moreover, one can see that

$$1 - x_{i,i'} = \prod_{j'\in I'} z_{i,j'}^{-\mathcal{C}(X')_{i',j'}} > 0,$$

which implies $x_{i,i'} < 1$. So $\mathbf{x} = (x_{\mathbf{i}})$ must be the principal positive solution of the equation $\mathbb{X}(X,X')$. The second statement follows from Proposition 4.3.4. □

Theorem 5.3.2. *There exist a unique positive solution* $\mathbf{z} = (z_{\mathbf{i}})$ *of* $\mathbb{Q}(X,X')$. *Moreover, the inequality* $z_{\mathbf{i}} > 1$ *holds for all* $\mathbf{i} \in \mathbf{I}$.

Proof. The existence is guaranteed by Lemma 5.3.1. Let $\mathbf{z} = (z_{i,i'})$ be a positive solution of $\mathbb{Q}(X,X')$. By Lemma 5.3.1,

$$x_{i,i'} = \prod_{j\in I} z_{j,i'}^{-\mathcal{C}(X)_{i,j}}$$

must be the principal positive solution of $\mathbb{X}(X,X')$. Since $\mathcal{C}(X)$ is invertible,

$$z_{i,i'} = \prod_{j\in I} x_{j,i'}^{-(\mathcal{C}(X)^{-1})_{ij}} \tag{5.6}$$

holds. Thus the uniqueness of positive solutions of $\mathbb{Q}(X,X')$ follows from the uniqueness of the principal positive solution of $\mathbb{X}(X,X')$. Taking the logarithm on both sides of (5.6), we get

$$\log z_{i,i'} = \sum_{j\in I} (\mathcal{C}(X)^{-1})_{ij} (-\log x_{j,i'}). \tag{5.7}$$

Since the inverse Cartan matrix $\mathcal{C}(X)^{-1}$ has only positive entries and $-\log x_{\mathbf{i}} > 0$ for all $\mathbf{i}$, $\log z_{\mathbf{i}}$ must be strictly positive and therefore $z_{\mathbf{i}} > 1$ for all $\mathbf{i}$. □

Definition 5.3.3. We call the solution of $\mathbb{Q}(X,X')$ characterized in Theorem 5.3.2 the positive solution of $\mathbb{Q}(X,X')$.

Theorem 5.3.4. *The positive solution of the* $\mathbb{Q}(X,X')$ *inherits the symmetry of the Dynkin diagrams.*

Proof. This follows from the uniqueness of the positive solution of $\mathbb{Q}(X,X')$ similarly as in Theorem 5.1.3 or one can see this concretely using (5.7) with Theorem 5.1.3. Let σ and σ' be graph automorphisms of X and X' respectively. Then

$$\log z_{\sigma(i),\sigma'(i')} = \sum_{j\in I} (\mathcal{C}(X)^{-1})_{\sigma(i)j} (-\log x_{j,\sigma'(i')}) \tag{5.8}$$

$$= \sum_{j\in I} (\mathcal{C}(X)^{-1})_{ij} (-\log x_{j,i'}) \tag{5.9}$$

$$= \log z_{i,i'}. \tag{5.10}$$

□

Example 5.3.5. Let us consider the example of $\mathbb{Q}(D_4, D_4)$. Because of the symmetry, the number of equations for the positive solution can be significantly reduced. Since $X = X' = D_4$ the positive solution enjoys an additional symmetry arising from the interchange of X and X'. Thus after applying the symmetry, the system of equations reduces to

$$\begin{cases} (Q_{1,1})^2 &= 2Q_{1,2} \\ (Q_{1,2})^2 &= (Q_{1,1})^3 + Q_{2,2} \\ (Q_{2,2})^2 &= 2(Q_{1,2})^3 \end{cases}$$

together with

$$\begin{cases} Q_{1,1} &= Q_{1,2} = Q_{1,3} = Q_{1,4} = Q_{3,1} = Q_{3,3} = Q_{3,4} = Q_{4,1} = Q_{4,3} = Q_{4,4} > 0 \\ Q_{1,2} &= Q_{2,1} = Q_{2,3} = Q_{2,4} = Q_{3,2} = Q_{4,2} > 0 \\ Q_{2,2} &> 0. \end{cases}$$

By solving these equations, one can find the positive solution

$$\begin{bmatrix} Q_{1,1} & Q_{1,2} & Q_{1,3} & Q_{1,4} \\ Q_{2,1} & Q_{2,2} & Q_{2,3} & Q_{2,4} \\ Q_{3,1} & Q_{3,2} & Q_{3,3} & Q_{3,4} \\ Q_{4,1} & Q_{4,2} & Q_{4,3} & Q_{4,4} \end{bmatrix} = \begin{bmatrix} 6 & 18 & 6 & 6 \\ 18 & 108 & 18 & 18 \\ 6 & 18 & 6 & 6 \\ 6 & 18 & 6 & 6 \end{bmatrix}$$

of $\mathbb{Q}(D_4, D_4)$. Interestingly, all the entries of this solution are in $\mathbb{Z}$.

Now we prove a theorem which will be important in the following section and in the next chapter.

Theorem 5.3.6. *Let X' be a Dynkin diagram of ADE type of rank r' and $\mathbf{z} = (z_{\mathbf{i}})$ be the unique positive solution of $\mathbb{Q}(A_n, X')$. Then it satisfies the following properties :*

1. *(symmetry) $z_{i,i'} = z_{n+1-i,i'}$ for $1 \le i \le n$ and $1 \le i' \le r'$.*
2. *(unimodality) $z_{i-1,i'} < z_{i,i'}$ for $i = 1, \cdots, m = \lfloor \frac{n+1}{2} \rfloor$ and $1 \le i' \le r'$ where $\lfloor x \rfloor$ is the floor function.*

Proof. For the symmetry property, we may use Theorem 5.3.2. The symmetry of the positive solution implies the symmetry condition $z_{i,i'} = z_{n+1-i,i'}$.

Now we prove the unimodality of the positive solution $\mathbf{z} = (z_{i,i'})$. Lemma 5.3.1 implies that

$$x_{i,i'} = \prod_{j \in I} z_{j,i'}^{-\mathcal{C}(A_n)_{ij}} = \frac{z_{i-1,i'} z_{i+1,i'}}{z_{i,i'}^2}$$

is the principal positive solution of $\mathbf{x} = (1 - \mathbf{x})^A$ so that $0 < x_{i,i'} < 1$ for all (i, i').

If n is even, by the symmetry of solutions, we have $z_{m+1,i'} = z_{m,i'}$. Then

$$x_{m,i'} = \frac{z_{m-1,i'} z_{m+1,i'}}{z_{m,i'} z_{m,i'}} = \frac{z_{m-1,i'}}{z_{m,i'}} < 1$$

because $x_{m,i'} < 1$.

If n is odd, again the symmetry condition implies $z_{m+1,i'} = z_{m-1,i'}$. Using this, we get

$$x_{m,i'} = \frac{z_{m-1,i'}z_{m+1,i'}}{z_{m,i'}z_{m,i'}} = \frac{z_{m-1,i'}^2}{z_{m,i'}^2} = (\frac{z_{m-1,i'}}{z_{m,i'}})^2 < 1.$$

In both cases, we can obtain the inequality $z_{m-1,i'} < z_{m,i'}$. Since

$$x_{i,i'} = \frac{z_{i-1,i'}z_{i+1,i'}}{z_{i,i'}^2} < 1,$$

we have

$$z_{i-1,i'} < z_{i,i'}(\frac{z_{i,i'}}{z_{i+1,i'}}) < z_{i,i'}$$

for $1 \le i \le m-1$. Thus we have proved the unimodality. □

5.4 Positive solutions of level restricted Q-systems

Definition 5.4.1. Let X be a Dynkin diagram of ADE type. Consider the Q-system (4.45)

$$(Q_m^{(a)})^2 = \prod_{b\in I}(Q_m^{(b)})^{\mathcal{I}(X)_{ab}} + Q_{m-1}^{(a)}Q_{m+1}^{(a)}$$

with boundary conditions $Q_0^{(a)} = 1$ for all $a \in I$. Let $k \ge 2$ be an integer. We are interested in solutions of the Q-system satisfying the unit boundary conditions

$$Q_k^{(a)} = 1$$

for all $a \in I$. We call the resulting system of equations for $Q_m^{(a)}$ with $0 \le m \le k$ and $1 \le a \le r$ the level k restricted Q-system of type X.

Note that the level k restricted Q-system is nothing but the constant T-system $\mathbb{Q}(X, A_n)$ given by

$$Q_{i,i'}^2 = \prod_{j\in I} Q_{j,i'}^{\mathcal{I}(X)_{ij}} + \prod_{j'\in I'} Q_{i,j'}^{\mathcal{I}(A_n)_{i'j'}}$$

where $n = k-1$ by the identification $Q_{i,i'} = Q_{i'}^{(i)}$. Thus from Theorem 5.3.6, we can deduce the following.

Corollary 5.4.2. *Let X be a Dynkin diagram of ADE type of rank r. There exists a unique solution $\mathbf{z} = (z_i^{(a)})$ of the level k restricted Q-system satisfying $z_i^{(a)} > 0$ for $0 \le i \le k$ and $1 \le a \le r$. Moreover, it satisfies the following additional properties:*

1. *(symmetry) $z_i^{(a)} = z_{k-i}^{(a)}$ for $0 \le i \le k$ and $1 \le a \le r$.*

2. *(unimodality)* $z_{i-1}^{(a)} < z_i^{(a)}$ *for* $i = 1, \cdots, m = \lfloor \frac{k}{2} \rfloor$ *and* $1 \leq a \leq r$ *where* $\lfloor x \rfloor$ *is the floor function.*

Definition 5.4.3. We call the solution $\mathbf{z} = (z_i^{(a)})$ with $0 \leq i \leq k$ and $1 \leq a \leq r$ characterized in Corollary 5.4.2 the positive solution of the level k restricted Q-system.

Example 5.4.4. Let us take a look at the example of the level $n+1$ restricted Q-system of type A_1. There is a nice realization of this system of equations in elementary geometry. Let us consider a regular $(n+3)$-gon with the sides of length 1. Fix a vertex v and let v_i, $i = 0, 1, \cdots, n+1$ be the vertices such that the polygon is given by $vv_0v_1 \cdots v_{n+1}$. Let us denote the length of the edge $\overline{v_0v_i}$ as d_i. Then the lengths of diagonals satisfy the following recurrence relations :

$$\begin{cases} d_0 &= 1 \\ d_1^2 &= 1 + d_0 d_2 \\ &\vdots \\ d_i^2 &= 1 + d_{i-1} d_{i+1} \\ &\vdots \\ d_n^2 &= 1 + d_{n-1} d_{n+1} \\ d_{n+1} &= 1 \end{cases}.$$

If one considers this recurrence from a geometric viewpoint, the properties stated in Corollary 5.4.2 are quite easy to see. The lengths of diagonals are given by the formula

$$d_i = \frac{\sin\left(\frac{\pi(i+1)}{n+3}\right)}{\sin\left(\frac{\pi}{n+3}\right)}$$

for $i = 0, 1, \cdots, n+1$. One can show this using the law of sines in elementary geometry. See Section 5.5 for more general explicit formula.

5.5 Explicit expressions for positive solutions

There are a few cases we can write down positive solutions very concretely. Recall Theorem 4.3.5. If $Q = (Q_{i,i'})$ is a solution of $\mathbb{Q}(X, X')$ with $Q_{i,i'} \neq 0$,

$$x_{i,i'} = \frac{\prod_{j \in I} Q_{j,i'}^{\mathcal{I}(X)_{i,j}}}{Q_{i,i'}^2} \tag{5.11}$$

$$y_{i,i'} = \frac{\prod_{j \in I} Q_{j,i'}^{\mathcal{I}(X)_{i,j}}}{\prod_{j' \in I'} Q_{i,j'}^{\mathcal{I}(X')_{i'j'}}} \tag{5.12}$$

are solutions of $\mathbb{X}(X, X')$ and $\mathbb{Y}_c(X, X')$ respectively.

(A_r, A_n) case

The positive solution of $\mathbb{Q}(A_r, A_n)$ is given by

$$Q_{a,m} = \prod_{i=1}^{a} \prod_{j=1}^{r+1-a} \frac{\sin\left(\frac{\pi(i+j+m-1)}{n+r+2}\right)}{\sin\left(\frac{\pi(i+j-1)}{n+r+2}\right)}.$$

This expression first appeared in [23]. See also Theorem 6.7.2. By (5.11), the principal positive solution of $\mathbb{X}(A_r, A_n)$ is

$$x_{a,m} = \frac{\sin\left(\frac{\pi a}{n+r+2}\right)\sin\left(\frac{\pi(r+1-a)}{n+r+2}\right)}{\sin\left(\frac{\pi(a+m)}{n+r+2}\right)\sin\left(\frac{\pi(r+1-a+m)}{n+r+2}\right)}.$$

Also by (5.12), the positive solution of $\mathbb{Y}_c(A_r, A_n)$ is

$$y_{a,m} = \frac{\sin\left(\frac{\pi a}{n+r+2}\right)\sin\left(\frac{\pi(r+1-a)}{n+r+2}\right)}{\sin\left(\frac{\pi m}{n+r+2}\right)\sin\left(\frac{\pi(n+1-m)}{n+r+2}\right)}.$$

The dilogarithm identity (4.2.5) associated with this is

$$\sum_{i=1}^{n}\sum_{a=1}^{r} L\left(\frac{\sin\left(\frac{\pi a}{n+r+2}\right)\sin\left(\frac{\pi(r+1-a)}{n+r+2}\right)}{\sin\left(\frac{\pi(a+m)}{n+r+2}\right)\sin\left(\frac{\pi(r+1-a+m)}{n+r+2}\right)}\right) = \frac{nr(r+1)}{n+r+2}L(1).$$

(D_r, A_1) case

In this case, one can show that the positive solution of $\mathbb{Q}(D_r, A_1)$ is given by

$$Q_{a,1} = \begin{cases} a+1 & 1 \le a \le r-2 \\ \sqrt{r} & a = r-1, r \end{cases}.$$

Then the principal positive solution of $\mathbb{X}(A_1, D_r)$ is

$$x_{1,a} = \begin{cases} 1/(a+1)^2 & 1 \le a \le r-2 \\ 1/r & a = r-1, r \end{cases}.$$

And the positive solution of $\mathbb{Y}_c(A_1, D_r)$ is

$$y_{1,a} = \begin{cases} 1/(a^2+2a) & 1 \le a \le r-2 \\ 1/(r-1) & a = r-1, r \end{cases}.$$

Lastly, the corresponding dilogarithm identity is

$$\sum_{j=2}^{r-1} L\left(\frac{1}{j^2}\right) + 2L\left(\frac{1}{r}\right) = \frac{1\cdot r\cdot 2}{(2r-2)+2}L(1) = L(1)$$

Chapter 6

Level restricted Q-systems and generalized quantum dimensions

In this chapter, we focus on the problem of finding explicit solutions of various equations. As we saw in Chapter 4 and Chapter 5, many systems of equations such as $\mathbb{X}(X, X')$, $\mathbb{Y}_c(X, X')$ and $\mathbb{Q}(X, X')$ are closely related to each other. Among them, the Q-system allows an approach using characters of irreducible representations of a simple Lie algebra. We will look for Lie algebraic solutions of the level restricted Q-system defined in Section 5.4 and investigate their properties stated in Conjecture 6.3.2.

This topic has been discussed in several works : see [24], [25]. [27, Section 5] and [20, Chapter 3]. It seems that the affine Weyl group has not been used much in this context. The main goal of this chapter is to illustrate the use of generalized quantum dimensions and their affine Weyl group symmetry in the study of Lie algebraic solutions of level restricted Q-systems.

6.1 Review of Lie theory

In this section, let us briefly review a few notions from Lie theory to fix notations. Let $\mathfrak{g}$ be a simple Lie algebra of rank r and $\mathfrak{h}$ a Cartan subalgebra. $\mathfrak{h}^*$ denotes the dual space of $\mathfrak{h}$ and we use the symbol $\langle \cdot, \cdot \rangle$ to denote the natural pairing between $\mathfrak{h}$ and $\mathfrak{h}^*$.

Let $\Phi \subset \mathfrak{h}^*$ be the root system with its Dynkin diagram of type X. We will assume X is one of ADE type. Let Q be the root lattice and P be the weight lattice. We fix a set Φ^+ of positive roots with the set $\Pi = \{\alpha_i | i = 1, \cdots, r\}$ of simple roots. Let $\theta = \sum_{i=1}^{r} a_i \alpha_i$ be the highest root where a_i denotes the Dynkin labels as given in Appendix A. Let $(\cdot|\cdot)$ be the non-degenerate invariant symmetric bilinear form on $\mathfrak{h}$ and on $\mathfrak{h}^*$ normalize $(\cdot|\cdot)$ so that $(\theta|\theta) = 2$.

$Q^\vee$ denotes the coroot lattice which is $\mathbb{Z}$-dual of the weight lattice P. We will choose the basis $\Pi^\vee = \{h_i \in \mathfrak{h} | i = 1, \cdots, r\}$ of the coroot lattice so that $\langle \alpha_i, h_j \rangle = a_{ji}$ where $(a_{ij}) = \mathcal{C}(X)$ denotes the Cartan matrix of X. Let $\{\omega_i \in P | i = 1, \cdots, r\}$ be the dual basis of P for $\Pi^\vee$ so that $\langle \omega_i, h_j \rangle = \delta_{ij}$ and we call the elements of this set fundamental weights. Let $P^\vee$ be the coweight lattice $\mathbb{Z}$-dual to Q. Note that $Q \subset P$ and $Q^\vee \subset P^\vee$.

For a root $\alpha \in \Phi$, we can define the Weyl reflection s_α on P by

$$s_\alpha \lambda = \lambda - 2\frac{(\lambda|\alpha)}{(\alpha|\alpha)}\alpha.$$

Note that if X is one of ADE type,

$$s_\alpha \lambda = \lambda - (\lambda|\alpha)\alpha.$$

We call the group generated by all these reflections the Weyl group W^0 and it is generated by the elements $s_1, \cdots, s_r$ where $s_i = s_{\alpha_i}$. The signature of $w \in W^0$ will be denoted by $(-1)^{\ell(w)}$.

We have the group algebra $\mathbb{Z}[P]$ with $\mathbb{Z}$-basis of elements of the form e^λ, $\lambda \in P$. The multiplication is given by

$$e^\lambda e^\mu = e^{\lambda+\mu}.$$

We can regard e^λ as a function defined on $\mathfrak{h}$ by $x \mapsto e^{2\pi i \langle \lambda, x\rangle}$ or as a function defined on $\mathfrak{h}^*$ by $\mu \mapsto e^{2\pi i (\lambda|\mu)}$.

Let $\rho = \sum_{i=1}^r \omega_i \in P$ be the Weyl vector. For a dominant weight $\lambda \in P^+$, χ_λ denotes the character of an irreducible highest weight module with highest weight λ. The Weyl character formula says

$$\chi_\lambda = \frac{A_{\lambda+\rho}}{A_\rho} \tag{6.1}$$

where

$$A_\mu = \sum_{w \in W^0} (-1)^{\ell(w)} e^{w\mu} \in \mathbb{C}[P]. \tag{6.2}$$

The Weyl denominator formula says

$$A_\rho = \sum_{w \in W^0} (-1)^{\ell(w)} e^{w\rho} = e^\rho \prod_{\alpha>0} (1 - e^{-\alpha}). \tag{6.3}$$

For $\mu \in \mathfrak{h}^*$, we can express $A_\rho(\mu)$ as

$$A_\rho(\mu) = \prod_{\alpha>0} (2i) \sin \pi(\mu|\alpha). \tag{6.4}$$

Now we turn our attention to Q-system considered in Section 4.4.

6.2 Realizations of Q-systems as characters of Lie algebras

We are looking for Lie algebraic solutions of level k restricted Q-systems in this chapter. An important link is provided by the fact that there is a way to realize the Q-system in terms of characters of irreducible representations of a simple Lie algebra.

When $X = A_r$,

$$Q_m^{(a)} = \chi_{m\omega_a} \tag{6.5}$$

with $a = 1, \cdots, r$ and $m = 0, 1, \cdots$ satisfies the Q-system of type A_r.

For $X = D_r$, the expression is more complicated and it is given by

$$Q_m^{(a)} = \begin{cases} \sum \chi_{k_a\omega_a + k_{a-2}\omega_{a-2} + \cdots + k_1\omega_1} & 1 \le a < r-1, a \equiv 1 \pmod 2, \\ \sum \chi_{k_a\omega_a + k_{a-2}\omega_{a-2} + \cdots + k_0\omega_0} & 1 \le a < r-1, a \equiv 0 \pmod 2, \\ \chi_{m\omega_a} & a = r-1, r \end{cases} \tag{6.6}$$

where $\omega_0 = 0$ and the summation is over all nonnegative integers satisfying $k_a + k_{a-2} + \cdots + k_1 = m$ for a odd and $k_a + k_{a-2} + \cdots + k_0 = m$ for a even.

The fact that they satisfy Q-systems has a long story involving the Kirillov-Reshetikhin conjecture. For more information, see [26, Section 13]. The above decomposition will be used much in the following sections, especially in Section 6.7.

6.3 Conjectures on positive solutions of level restricted Q-systems

In this section, we regard $Q_m^{(a)}$ as a sum of irreducible characters of a simple Lie algebra of type X using the decomposition given in Section 6.2. If we evaluate them at some element $\mu \in \mathfrak{h}^*$, $Q_m^{(a)}(\mu)$ still satisfies the Q-system. An interesting question is as follows :

Question 6.3.1. For which elements $\mu \in \mathfrak{h}^*$ is $\mathbf{z} = \left(Q_m^{(a)}(\mu)\right)$ a solution of the level k restricted Q-system?

Recall that there exist a unique positive solution $\mathbf{z} = (z_\mathbf{i})$ of the level k restricted Q-system. In the previous chapter, we obtained Corollary 5.4.2 which is about the positive solution of the level k restricted Q-system. We saw symmetries of the positive solution and its unimodality. Although its existence is guaranteed, one serious problem is that we still do not know how one can express such solutions explicitly in general.

In [26], a conjecture about the form of the positive solution of the level k restricted Q-system and its several properties is proposed. It suggests that a distinguished element

$$\frac{\rho}{h+k} \in \mathfrak{h}^*$$

plays a very important role. The conjecture is as follows :

Conjecture 6.3.2. *[26, Conjecture 14.2.] Let* $z_m^{(a)} = Q_m^{(a)}(\frac{\rho}{h+k})$ *for* $a = 1, \cdots, r$ *and* $m = 0, 1, \cdots$. *It satisfies the following properties :*

1. *(positivity)* $z_m^{(a)} > 0$ *for* $0 \leq m \leq k$ *and* $1 \leq a \leq r$.
2. *(symmetry)* $z_m^{(a)} = z_{k-m}^{(a)}$ *for* $1 \leq m \leq k-1$ *and* $1 \leq a \leq r$.
3. *(unimodality)* $z_{m-1}^{(a)} < z_m^{(a)}$ *holds true for* $m = 1, \cdots, \lfloor \frac{k}{2} \rfloor$ *and* $1 \leq a \leq r$ *where* $\lfloor x \rfloor$ *is the floor function.*
4. *(unit boundary condition)* $z_k^{(a)} = 1$ *for* $1 \leq a \leq r$.
5. *(occurrence of 0)* $z_{k+1}^{(a)} = z_{k+2}^{(a)} = \cdots = z_{k+h-1}^{(a)} = 0$ *for* $1 \leq a \leq r$.

So if the conjecture is true, $\mathbf{z} = \left(Q_m^{(a)}\left(\frac{\rho}{h+k}\right)\right)$ with $1 \leq m \leq n$ and $1 \leq a \leq r$ must be the positive solution of $\mathbb{Q}(A_n, X)$ where $n = k-1$ and we can also see that the positive solution is contained in cyclotomic fields.

Remark 6.3.3. This conjecture is formulated for simple Lie algebras including non-simply laced types. We already know that this is true for $X = A_r$.

Remark 6.3.4. For $X = D_r$, the unit boundary condition implies all the other properties except the occurrences of 0's. The reasoning is as follows : one can see the positivity of $Q_m^{(a)}(\frac{\rho}{h+k})$ for $m = 1, \cdots, \lfloor \frac{k}{2} \rfloor$ directly from the decomposition (6.6) since the contribution from each character in the summand can be shown to be positive. Since the unit boundary condition is satisfied, it is a solution of the constant T-system. Then we can apply the half-periodicity (4.29) of the T-system which implies $z_m^{(a)} = z_{k-m}^{(\omega(a))}$. Thus the positivity condition follows. Therefore it must be the unique positive solution characterized in Corollary 5.4.2. The symmetry condition and the unimodality condition now follow from it.

In the rest of this chapter, a few useful concepts and tools to deal with this conjecture will be introduced. A fruitful observation is that the character evaluated at some elements of $\mathfrak{h}^*$ also shows up in the theory of affine Kac-Moody algebras as entries of modular S-matrix. Since we began with Nahm's conjecture about modular q-hypergeometric series, this connection should not be too surprising. See Section 6.6 for a further remark. The affine Weyl group will be shown to be a crucial ingredient to understand Conjecture 6.3.2.

6.4 Affinizations of weights and affine Weyl group actions

Affinizations of weights

Let U be the free $\mathbb{Z}$-module generated by two elements $\hat{\omega}_0$ and δ with the symmetric bilinear form $(\cdot|\cdot)$ defined by

$$(x_1\hat{\omega}_0 + x_2\delta|y_1\hat{\omega}_0 + y_2\delta) = x_1y_2 + x_2y_1.$$

Let $\hat{P} = P \oplus U$ be the orthogonal sum of two lattices P and U. It is a lattice with the basis ω_i $(i = 1, \cdots, r)$, $\hat{\omega}_0$ and δ with the non-degenerate symmetric bilinear form $(\cdot|\cdot)$ obtained as the orthogonal sum of symmetric bilinear forms on P and U. So an element of $\hat{\lambda} \in \hat{P}$ can be written as $\hat{\lambda} = \lambda + k\hat{\omega}_0 + n\delta \in \hat{P}$ where λ is an element of P and $k, n \in \mathbb{Z}$. Let us denote it by $\hat{\lambda} = (\lambda; k; n)$. Thus $\hat{\omega}_0 = (0; 1; 0)$ and $\delta = (0; 0; 1)$. Let us set $\omega_0 = 0$. We call $\hat{\omega}_i = (\omega_i; a_i; 0)$, $i = 0, 1 \cdots, r$ the affine fundamental weights and $\hat{\rho} = (\rho; h; 0)$ the affine Weyl vector .

Definition 6.4.1. Let $\lambda \in P$. Any element of the form

$$\hat{\lambda} = \lambda + k\hat{\omega}_0 + n\delta = (\lambda; k; n) \in \hat{P}$$

for some $n \in \mathbb{Z}$ is called a level k affinization of λ. For an element $\hat{\mu} = (\mu; k; n) \in \hat{P}$, we call μ the finite part of $\hat{\mu}$ and sometimes denote it by $\overline{\hat{\mu}}$.

The affine Weyl group

Now we define the notion of the affine Weyl group. Since the root lattice Q is contained in P, we can regard Q as contained in $\hat{P}$. We call elements of the form $\hat{\alpha} = (\alpha; 0; n) \in \hat{P}$ as affine roots. Note that $(\hat{\alpha}|\hat{\alpha}) = (\alpha|\alpha) = 2$. Let us put $\alpha_0 = (-\theta; 0; 1)$ where θ is the highest root and $\alpha_i = (\alpha_i; 0; 0)$ for $i = 1, \cdots, r$ by abusing the notation. For an affine root $\hat{\alpha}$, we can define the Weyl reflection on $\hat{P}$ as

$$s_{\hat{\alpha}}\hat{\lambda} = \hat{\lambda} - (\hat{\lambda}|\hat{\alpha})\hat{\alpha}.$$

More concretely, for $\hat{\alpha} = (\alpha; 0; m)$ and $\hat{\lambda} = (\lambda; k; n)$, we have

$$s_{\hat{\alpha}}\hat{\lambda} = (s_\alpha(\lambda + km\alpha); k; n - m(\lambda|\alpha) - km^2). \tag{6.7}$$

We call the group W generated by all these reflections the affine Weyl group and it is generated by the elements $s_0, s_1, \cdots, s_r$ where $s_i = s_{\alpha_i}$.

Proposition 6.4.2. *Let $\lambda \in P$ be a weight and $\hat{\lambda} \in \hat{P}$ be its level k affinization. The finite part of $s_i\hat{\lambda}$ is $s_i\lambda$ for $i = 1, \cdots, r$.*

Proof. Let $\hat{\lambda} = (\lambda; k; n)$. We get

$$s_i\hat{\lambda} = (s_i\lambda; k; n). \tag{6.8}$$

□

Let us take a look at the action of s_0 on $\hat{P}$. This result will play an important role later.

Proposition 6.4.3. *Let $\lambda \in P$ be a weight and $\hat{\lambda} \in \hat{P}$ be its level k affinization. The finite part of $s_0\hat{\lambda}$ is $s_\theta\lambda + k\theta$.*

Proof. Recall that $\alpha_0 = (-\theta; 0; 1)$ and $\hat{\lambda} = (\lambda; k; n)$. From (6.7), we can observe that

$$\begin{aligned} & s_0\hat{\lambda} && (6.9)\\ =&(s_\theta(\lambda - k\theta); k; n + (\lambda|\theta) - k) && (6.10)\\ =&(\lambda - k\theta - (\lambda - k\theta, \theta)\theta; k; n + (\lambda|\theta) - k) && (6.11)\\ =&(\lambda - (\lambda, \theta)\theta + k\theta; k; n + (\lambda|\theta) - k) && (6.12)\\ =&(s_\theta\lambda + k\theta; k; n + (\lambda|\theta) - k). && (6.13) \end{aligned}$$

Therefore we get $\overline{s_0\hat{\lambda}} = s_\theta\lambda + k\theta$. □

Classifications of elements of Γ_k

Let k be a positive integer. Let $\hat{P}_k$ be the subset of $\hat{P}$ defined by

$$\{(\lambda; k; n) \in \hat{P} | \lambda \in P, n \in \mathbb{Z}\}.$$

(6.7) shows that the affine Weyl group acts on $\hat{P}_k$. Let Γ_k be the set obtained by quotienting out δ from $\hat{P}_k$. In other words, Γ_k is the subset of the group $\hat{P}/\mathbb{Z}\delta$ obtained as the image of $\hat{P}_k$. This can also be regarded as the set of level k affinizations of all elements of P. So there is a bijection between P and Γ_k. One reason to introduce Γ_k is to make the role of the affine Weyl group transparent.

Any element of Γ_k can be represented by

$$\hat{\lambda} = \sum_{i=0}^{r} \lambda_i\hat{\omega}_i + n\delta \in \hat{P}$$

where λ_i's satisfy $\sum_{i=0}^{r} a_i\lambda_i = k$. We denote it by $(\lambda_0, \cdots, \lambda_r)$.

Let us define $\Gamma_k^+ = \{(\lambda_0, \cdots, \lambda_r) \in \Gamma_k | \lambda_i \in \mathbb{Z}^{\geq 0}\}$ and $\Gamma_k^{++} = \{(\lambda_0, \cdots, \lambda_r) \in \Gamma_k | \lambda_i \in \mathbb{Z}^{>0}\}$. The elements of Γ_k are classified by the affine Weyl group action as follows :

Theorem 6.4.4. *Any W-orbit in Γ_k contains a unique element of Γ_k^+.*

Proof. See [17] for a proof. □

Definition 6.4.5. One can define the action of the affine Weyl group on the set $\hat{P}$ as

$$w \cdot \hat{\lambda} = w(\hat{\lambda} + \hat{\rho}) - \hat{\rho}$$

for each $w \in W$. We will call this action the shifted affine Weyl group action.

The shifted affine Weyl group action also preserves Γ_k. Note that the classification problem of Γ_k under the shifted affine Weyl group action is essentially reduced to the classification problem of Γ_{h+k} under the usual affine Weyl group action.

6.5 Quantum dimensions

Before we introduce the concept of quantum dimensions, let us review the Weyl character and dimension formula. The Weyl denominator formula (6.4) implies

$$A_\mu\left(\frac{\rho}{h+k}\right) = A_\rho\left(\frac{\mu}{h+k}\right) = \prod_{\alpha>0}(2i)\sin\frac{\pi(\mu|\alpha)}{h+k} \tag{6.14}$$

for any $\mu \in P$. Let $\lambda \in P$ be a dominant weight. Using the formula (6.14), one can see

$$A_\lambda\left(\frac{\rho}{h+k}\right) = \prod_{\alpha>0}(2i)\sin\frac{\pi(\lambda|\alpha)}{h+k}$$

and

$$A_{\lambda+\rho}\left(\frac{\rho}{h+k}\right) = \prod_{\alpha>0}(2i)\sin\frac{\pi(\lambda+\rho|\alpha)}{h+k}.$$

Then the Weyl character formula implies

$$\chi_\lambda\left(\frac{\rho}{h+k}\right) = \frac{A_{\lambda+\rho}\left(\frac{\rho}{h+k}\right)}{A_\rho\left(\frac{\rho}{h+k}\right)} = \frac{\prod_{\alpha>0}\sin\frac{\pi(\lambda+\rho|\alpha)}{h+k}}{\prod_{\alpha>0}\sin\frac{\pi(\rho|\alpha)}{h+k}}.$$

If we take the limit as $k \to \infty$, the Weyl dimension formula

$$\dim(V_\lambda) = \frac{\prod_{\alpha>0}(\lambda+\rho|\alpha)}{\prod_{\alpha>0}(\rho|\alpha)}$$

can be obtained.

Definition 6.5.1. Let $\lambda \in P$ and $\hat{\lambda} \in \Gamma_k$ be its level k affinization. The quantum dimension or q-dimension of $\hat{\lambda}$ is defined by

$$\mathcal{D}_{\hat{\lambda}} = \chi_\lambda\left(\frac{\rho}{h+k}\right) = \frac{\prod_{\alpha>0}\sin\frac{\pi(\lambda+\rho|\alpha)}{h+k}}{\prod_{\alpha>0}\sin\frac{\pi(\rho|\alpha)}{h+k}}. \tag{6.15}$$

See [8, Section 16.3] for a reference. Now $Q_m^{(a)}\left(\frac{\rho}{h+k}\right)$ in Conjecture 6.3.2 can be written as a linear combination of quantum dimensions with positive integer coefficients. Note that it does not mean that these are positive numbers.

Theorem 6.5.2. *Let $\lambda = \sum_{i=1}^{l} \lambda_i \omega_i \in P^+$ be a dominant weight such that $\sum_{i=1}^{l} a_i \lambda_i \le k$. For its level k affinization $\hat{\lambda} \in \Gamma_k^+$, $\mathcal{D}_{\hat{\lambda}} > 0$.*

Proof. Let us use the product formula (6.15) for the quantum dimension

$$\mathcal{D}_{\hat{\lambda}} = \frac{\prod_{\alpha>0} \sin \frac{\pi(\lambda+\rho|\alpha)}{h+k}}{\prod_{\alpha>0} \sin \frac{\pi(\rho|\alpha)}{h+k}}.$$

The maximum value of $\{(\lambda+\rho|\alpha)|\alpha > 0\}$ is obtained when $\alpha = \theta$, the highest root of the root system. Since $(\lambda+\rho|\theta) = (\lambda|\theta) + (\rho|\theta) = \sum_{i=1}^{l} a_i\lambda_i + (h-1) \le k+h-1$, $\mathcal{D}_{\hat{\lambda}} > 0$. □

Now we look at the role of the symmetry of the extended Dynkin diagrams.

Theorem 6.5.3. *Suppose that $\hat{\lambda}_1, \hat{\lambda}_2 \in \Gamma_k$ are conjugate by an automorphism of the extended Dynkin diagram. Then $\mathcal{D}_{\hat{\lambda}_1} = \mathcal{D}_{\hat{\lambda}_2}$.*

Proof. See [8, Section 16.3] for a proof.

□

Corollary 6.5.4. *Let $\lambda = k\omega_i \in P^+$ be a dominant weight where ω_i is a fundamental weight such that $\hat{\omega}_i$ is conjugate to $\hat{\omega}_0$ by an automorphism of the extended Dynkin diagram. For its level k affinization $\hat{\lambda} \in \Gamma_k$, $\mathcal{D}_{\hat{\lambda}} = 1$.*

Proof. Note that $\mathcal{D}_{\hat{0}} = 1$. Then the desired result follows from Theorem 6.5.3. □

6.6 Generalized quantum dimensions

For a pair of weights $\lambda, \mu \in P$ and their level k affinizations $\hat{\lambda}, \hat{\mu} \in \Gamma_k$, we consider the quantity

$$S_{\hat{\lambda}\hat{\mu}} = A_{\lambda+\rho}\left(\frac{\mu+\rho}{h+k}\right) = \sum_{w \in W^0} (-1)^{\ell(w)} \exp \frac{2\pi i (w(\lambda+\rho)|\mu+\rho)}{h+k}.$$

Note that $S_{\hat{\lambda}\hat{\mu}} = S_{\hat{\mu}\hat{\lambda}}$. The quantum dimension can be written in terms of S :

$$\mathcal{D}_{\hat{\lambda}} = \frac{S_{\hat{\lambda}\hat{0}}}{S_{\hat{0}\hat{0}}} = \frac{\prod_{\alpha>0} \sin \frac{\pi(\lambda+\rho|\alpha)}{h+k}}{\prod_{\alpha>0} \sin \frac{\pi(\rho|\alpha)}{h+k}}.$$

For $\hat{\mu} \in \Gamma_k^+$,

$$S_{\hat{0}\hat{\mu}} = \mathcal{D}_{\hat{\mu}} S_{\hat{0}\hat{0}} \neq 0 \tag{6.16}$$

by Theorem 6.5.2.

Remark 6.6.1. The quantities $S_{\hat{\lambda}\hat{\mu}}$, up a minor correction factor, are elements of modular S-matrix in the theory of affine Kac-Moody algebras and play important roles there. See [17]. They are very similar to the Gauss sum in number theory and the Gauss sum is useful to describe the modular transformation property of theta functions. $S_{\hat{\lambda}\hat{\mu}}$ shows up in the theory of affine Kac-Moody algebras for essentially the same reason.

Definition 6.6.2. Let $\hat{\lambda} \in \Gamma_k$ and $\hat{\mu} \in \Gamma_k^+$. We can assume that $S_{\hat{0}\hat{\mu}} \neq 0$ by (6.16). The generalized quantum dimension of $\hat{\lambda}$ twisted by $\hat{\mu}$ is defined by

$$\mathcal{D}_{\hat{\lambda}}[\hat{\mu}] = \frac{S_{\hat{\lambda}\hat{\mu}}}{S_{\hat{0}\hat{\mu}}}.$$

Then the quantum dimension is a special case of generalized quantum dimensions since $\mathcal{D}_{\hat{\lambda}} = \mathcal{D}_{\hat{\lambda}}[\hat{0}]$.

Definition 6.6.3. For $\hat{\lambda} \in \Gamma_k$, let us define the function

$$S_{\hat{\lambda}} : \Gamma_{h+k} \to \mathbb{C}$$

by

$$S_{\hat{\lambda}}(\hat{\nu}) = \sum_{w \in W^0} (-1)^{\ell(w)} \exp \frac{2\pi i (w(\lambda+\rho)|\nu)}{h+k}$$

where $\hat{\nu} \in \Gamma_{h+k}$.

Note that $S_{\hat{\lambda}\hat{\mu}} = S_{\hat{\lambda}}(\hat{\mu}+\hat{\rho})$ and $\mathcal{D}_{\hat{\lambda}}[\hat{\mu}] = \frac{S_{\hat{\lambda}}(\hat{\mu}+\hat{\rho})}{S_{\hat{0}}(\hat{\mu}+\hat{\rho})}$. We prove that the $S_{\hat{\lambda}}$ has the affine Weyl group symmetry.

Theorem 6.6.4. *Let $\hat{\lambda} \in \Gamma_k$, $\hat{\nu} \in \Gamma_{h+k}$ and w be an element of the affine Weyl group W. Then*

$$S_{w\cdot\hat{\lambda}}(\hat{\nu}) = (-1)^{\ell(w)} S_{\hat{\lambda}}(\hat{\nu}).$$

Proof. It is enough to check that this holds for $w = s_0, s_1, \cdots s_r \in W$ since they generate W.

For $s_i \in W$, $i \geq 1$, it can be easily seen as the rearrangement of the sum over the finite Weyl group. Note that $\overline{s_i(\hat{\lambda}+\hat{\rho})} = s_i(\lambda+\rho)$ holds for $i = 1, \cdots, r$ by Proposition 6.4.2.

$$S_{s_i\cdot\hat{\lambda}}(\hat{\nu}) \tag{6.17}$$
$$= \sum_{w\in W^0}(-1)^{\ell(w)}\exp\frac{2\pi i(w(\overline{s_i\cdot\hat{\lambda}})|\nu)}{h+k} \tag{6.18}$$
$$\tag{6.19}$$
$$= \sum_{w\in W^0}(-1)^{\ell(w)}\exp\frac{2\pi i(w(s_i(\lambda+\rho))|\nu)}{h+k} \tag{6.20}$$
$$= \sum_{w\in W^0}(-1)^{\ell(w)}\exp\frac{2\pi i(ws_i(\lambda+\rho)|\nu)}{h+k} \tag{6.21}$$
$$= (-1)\sum_{w\in W^0}(-1)^{\ell(w)}\exp\frac{2\pi i(w(\lambda+\rho)|\nu)}{h+k} \tag{6.22}$$
$$= -S_{\hat{\lambda}}(\hat{\nu}) \tag{6.23}$$

For $s_0 \in W$, let us use Proposition 6.4.3. We first check

$$(\overline{s_0\cdot\hat{\lambda}}|\nu) \tag{6.24}$$
$$= (\overline{s_0(\hat{\lambda}+\hat{\rho})}|\nu) \tag{6.25}$$
$$= (s_\theta(\lambda+\rho)+(h+k)\theta|\nu) \tag{6.26}$$
$$\equiv (s_\theta(\lambda+\rho)|\nu) \pmod{(h+k)\mathbb{Z}}. \tag{6.27}$$

Note that we have used the fact that $(\theta|\nu)\in\mathbb{Z}$. From this, we have

$$S_{s_0\cdot\hat{\lambda}}(\hat{\nu}) \tag{6.28}$$
$$= \sum_{w\in W^0}(-1)^{\ell(w)}\exp\frac{2\pi i(w(\overline{s_0\cdot\hat{\lambda}})|\nu)}{h+k} \tag{6.29}$$
$$= \sum_{w\in W^0}(-1)^{\ell(w)}\exp\frac{2\pi i(\overline{s_0\cdot\hat{\lambda}}|w^{-1}\nu)}{h+k} \tag{6.30}$$
$$= \sum_{w\in W^0}(-1)^{\ell(w)}\exp\frac{2\pi i(s_\theta(\lambda+\rho)|w^{-1}\nu)}{h+k} \tag{6.31}$$
$$= \sum_{w\in W^0}(-1)^{\ell(w)}\exp\frac{2\pi i(ws_\theta(\lambda+\rho)|\nu)}{h+k} \tag{6.32}$$
$$= (-1)\sum_{w\in W^0}(-1)^{\ell(w)}\exp\frac{2\pi i(w(\lambda+\rho)|\nu)}{h+k} \tag{6.33}$$
$$= -S_{s_0\cdot\hat{\lambda}}(\hat{\nu}). \tag{6.34}$$

Therefore, $S_{w\cdot\hat{\lambda}}(\hat{\nu}) = (-1)^{\ell(w)}S_{\hat{\lambda}}(\hat{\nu})$ holds for any $w\in W$. □

Corollary 6.6.5. *With the same assumptions as Theorem 6.6.4, the following holds :*

$$S_{\hat{\lambda}}(w\hat{\nu}) = (-1)^{\ell(w)} S_{\hat{\lambda}}(\hat{\nu}).$$

Proof. There exists $\hat{\mu} \in \Gamma_k$ such that $\hat{\nu} = \hat{\mu} + \hat{\rho}$. Note that $S_{\hat{\lambda}}(\hat{\nu}) = S_{\hat{\lambda}}(\hat{\mu} + \hat{\rho}) = S_{\hat{\mu}}(\hat{\lambda} + \hat{\rho})$. Therefore we get

$$S_{\hat{\lambda}}(w\hat{\nu}) \tag{6.35}$$
$$= S_{\hat{\lambda}}(w(\hat{\mu} + \hat{\rho})) \tag{6.36}$$
$$= S_{\hat{\lambda}}(w \cdot \hat{\mu} + \hat{\rho}) \tag{6.37}$$
$$= S_{w \cdot \hat{\mu}}(\hat{\lambda} + \hat{\rho}) = (-1)^{\ell(w)} S_{\hat{\lambda}}(\hat{\nu}) \tag{6.38}$$

□

We can apply Theorem 6.6.4 to generalized quantum dimensions.

Corollary 6.6.6. *Let $\hat{\lambda} \in \Gamma_k$, $\hat{\mu} \in \Gamma_k^+$ and $w \in W$. $\mathcal{D}_{w \cdot \hat{\lambda}}[\hat{\mu}] = (-1)^{\ell(w)} \mathcal{D}_{\hat{\lambda}}[\hat{\mu}]$. Especially, $\mathcal{D}_{w \cdot \hat{\lambda}} = (-1)^{\ell(w)} \mathcal{D}_{\hat{\lambda}}$.*

Proof.
$$\mathcal{D}_{w \cdot \hat{\lambda}}[\hat{\mu}] = \frac{S_{w \cdot \hat{\lambda}}(\hat{\mu} + \hat{\rho})}{S_{\hat{0}}(\hat{\mu} + \hat{\rho})} = \frac{(-1)^{\ell(w)} S_{\hat{\lambda}}(\hat{\mu} + \hat{\rho})}{S_{\hat{0}}(\hat{\mu} + \hat{\rho})} = (-1)^{\ell(w)} \mathcal{D}_{\hat{\lambda}}[\hat{\mu}].$$

□

Another useful consequence of Theorem 6.6.4 is the following :

Corollary 6.6.7. *Let $w \in W$ be an element of odd signature. If $w \cdot \hat{\lambda} = \hat{\lambda}$, $S_{\hat{\lambda}} = 0$ on Γ_{h+k}.*

Proof. $S_{\hat{\lambda}}(\hat{\nu}) = S_{w \cdot \lambda}(\hat{\nu}) = (-1) S_{\hat{\lambda}}(\hat{\nu})$. □

The following result will be very useful in the next section when we discuss the conjectured properties of level k restricted Q-systems.

Theorem 6.6.8. *Let $\lambda = \sum_{i=1}^r \lambda_i \omega_i \in P^+$ be a dominant weight such that $\sum_{i=1}^r a_i \lambda_i = k+1$. For its level k affinization $\hat{\lambda} \in \Gamma_k$, $S_{\hat{\lambda}} = 0$ on Γ_{h+k}. In particular, $\mathcal{D}_{\hat{\lambda}}[\hat{\mu}] = 0$ for any $\hat{\mu} \in \Gamma_k^+$.*

Proof. Let us prove that $s_0 \cdot \hat{\lambda} = \hat{\lambda}$. From

$$\hat{\lambda} + \hat{\rho} = (-\hat{\omega}_0 + \sum_{i=1}^r \lambda_i \hat{\omega}_i) + \sum_{i=0}^r \hat{\omega}_i = \sum_{i=1}^r (\lambda_i + 1) \hat{\omega}_i,$$

we can see that $s_0 \cdot \hat{\lambda} = s_0(\hat{\lambda} + \hat{\rho}) - \hat{\rho} = \hat{\lambda} + \hat{\rho} - \hat{\rho} = \hat{\lambda}$. Therefore, $S_{\hat{\lambda}} = 0$ by Corollary 6.6.7 and this implies

$$\mathcal{D}_{\hat{\lambda}}[\hat{\mu}] = \frac{S_{\hat{\lambda}}(\hat{\mu} + \hat{\rho})}{S_{\hat{0}}(\hat{\mu} + \hat{\rho})} = 0.$$

□

6.7 Applications to level restricted Q-systems

Let us see how the affine Weyl group symmetry can be used to understand Conjecture 6.3.2. We want to regard $Q_m^{(a)}$ as a function defined on Γ_{h+k}^{++}. Let us write the decomposition of $Q_m^{(a)}$ into classical characters as

$$Q_m^{(a)} = \sum_{\omega \in \Omega_m^{(a)}} \chi_\omega.$$

For the rest of this chapter, we will regard

$$Q_m^{(a)} = \sum_{\omega \in \Omega_m^{(a)}} \frac{S_{\hat{\omega}}}{S_{\hat{0}}} \tag{6.39}$$

as a function defined on Γ_{h+k}^{++}. Then for any $\hat{\mu} \in \Gamma_k^+$ and $\hat{\nu} = \hat{\mu} + \hat{\rho} \in \Gamma_{h+k}^{++}$, $Q_m^{(a)}(\hat{\nu})$ is a linear combination of generalized quantum dimensions with integer coefficients,

$$Q_m^{(a)}(\hat{\nu}) = Q_m^{(a)}(\hat{\mu} + \hat{\rho}) = \sum_{\omega \in \Omega_m^{(a)}} \mathcal{D}_{\hat{\omega}}[\hat{\mu}].$$

A_r case

If $X = A_r$, we know $Q_m^{(a)} = S_{(k-m)\hat{\omega}_0 + m\hat{\omega}_a}/S_{\hat{0}}$. Let us first prove the unit boundary condition of Conjecture 6.3.2 and its implication.

Theorem 6.7.1. *For $X = A_r$, $Q_k^{(a)}\left(\hat{\rho} + \hat{0}\right) = \mathcal{D}_{k\hat{\omega}_a} = 1$ for all $a = 1, \cdots, r$.*

Proof. We have

$$Q_k^{(a)} = \frac{S_{k\hat{\omega}_a}}{S_{\hat{0}}}.$$

So we can apply Corollary 6.5.4. □

Now we have found the positive solution of $\mathbb{Q}(A_r, A_{k-1})$ in terms of quantum dimensions.

Theorem 6.7.2. *Let $k \geq 2$. Put $z_{a,m} = \mathcal{D}_{(k-m)\hat{\omega}_0 + m\hat{\omega}_a}$ for $1 \leq a \leq r$ and $1 \leq m \leq k-1$. Then $\mathbf{z} = (z_{a,m})$ is the positive solution of $\mathbb{Q}(A_r, A_{k-1})$.*

Proof. It follows from Theorem 6.7.1. □

To see the vanishing of $Q_{k+1}^{(a)}$, it is enough to see that the level k affinizations of $(k+1)\omega_a$ are fixed by an element of the affine Weyl group whose signature is odd.

Theorem 6.7.3. *For $X = A_r$, $Q_{k+1}^{(a)} = 0$ identically on Γ_{h+k}^{++} for $a = 1, \cdots, r$.*

Proof. Since

$$Q_{k+1}^{(a)} = \frac{S_{-\hat{\omega}_0+(k+1)\hat{\omega}_a}}{S_{\hat{0}}},$$

we can apply Theorem 6.6.8 which implies that $S_{-\hat{\omega}_0+(k+1)\hat{\omega}_a} = 0$. □

This implies that whenever $Q_k^{(1)}(\hat{\nu}) = 1$ for a $\hat{\nu} \in \Gamma_{h+k}^{++}$, the unit boundary condition $Q_k^{(a)}(\hat{\nu}) = 1$ for all $a = 1, \cdots, r$ is satisfied by the recurrence relation of the Q-system (4.45). Then one gets a solution of $\mathbb{Q}(A_r, A_{k-1})$ which is not necessarily positive. Let us look at an example.

Example 6.7.4. Let $X = A_3$ and $k = 3$. Here the Coxeter number is $h = 4$. From Theorem 6.7.3, we know

$$S_{\hat{0}} \begin{bmatrix} Q_1^{(1)} & Q_1^{(2)} & Q_1^{(3)} \\ Q_2^{(1)} & Q_2^{(2)} & Q_2^{(3)} \\ Q_3^{(1)} & Q_3^{(2)} & Q_3^{(3)} \\ Q_4^{(1)} & Q_4^{(2)} & Q_4^{(3)} \\ \vdots & \vdots & \vdots \end{bmatrix} \tag{6.40}$$

$$= \begin{bmatrix} S_{2\hat{\omega}_0+\hat{\omega}_1} & S_{2\hat{\omega}_0+\hat{\omega}_2} & S_{2\hat{\omega}_0+\hat{\omega}_3} \\ S_{\hat{\omega}_0+2\hat{\omega}_1} & S_{\hat{\omega}_0+2\hat{\omega}_2} & S_{\hat{\omega}_0+2\hat{\omega}_3} \\ S_{3\hat{\omega}_1} & S_{3\hat{\omega}_2} & S_{3\hat{\omega}_3} \\ S_{4\hat{\omega}_1-\hat{\omega}_0} & S_{4\hat{\omega}_2-\hat{\omega}_0} & S_{4\hat{\omega}_3-\hat{\omega}_0} \\ \vdots & \vdots & \vdots \end{bmatrix} \tag{6.41}$$

$$= \begin{bmatrix} S_{2\hat{\omega}_0+\hat{\omega}_1} & S_{2\hat{\omega}_0+\hat{\omega}_2} & S_{2\hat{\omega}_0+\hat{\omega}_3} \\ S_{\hat{\omega}_0+2\hat{\omega}_1} & S_{\hat{\omega}_0+2\hat{\omega}_2} & S_{\hat{\omega}_0+2\hat{\omega}_3} \\ S_{3\hat{\omega}_1} & S_{3\hat{\omega}_2} & S_{3\hat{\omega}_3} \\ 0 & 0 & 0 \\ \vdots & \vdots & \vdots \end{bmatrix}. \tag{6.42}$$

Now having $Q_4^{(a)} = 0$ for $a = 1, 2, 3$, the Q-system equations for $m = 3$

$$\begin{cases} \left(Q_3^{(1)}\right)^2 = Q_2^{(1)} Q_4^{(1)} + Q_3^{(2)} \\ \left(Q_3^{(2)}\right)^2 = Q_3^{(1)} Q_3^{(3)} + Q_2^{(2)} Q_4^{(2)} \\ \left(Q_3^{(3)}\right)^2 = Q_3^{(2)} + Q_2^{(3)} Q_4^{(3)} \end{cases}$$

imply

$$\begin{cases} \left(Q_3^{(1)}\right)^2 = Q_3^{(2)} \\ \left(Q_3^{(2)}\right)^2 = Q_3^{(1)}Q_3^{(3)} \\ \left(Q_3^{(3)}\right)^2 = Q_3^{(2)} \end{cases}.$$

We already know that

$$Q_3^{(1)}\left(\hat{\rho}+\hat{0}\right) = Q_3^{(2)}\left(\hat{\rho}+\hat{0}\right) = Q_3^{(3)}\left(\hat{\rho}+\hat{0}\right) = 1$$

from Theorem 6.7.1. Therefore,

$$\begin{bmatrix} Q_1^{(1)}\left(\hat{\rho}+\hat{0}\right) & Q_1^{(2)}\left(\hat{\rho}+\hat{0}\right) & Q_1^{(3)}\left(\hat{\rho}+\hat{0}\right) \\ Q_2^{(1)}\left(\hat{\rho}+\hat{0}\right) & Q_2^{(2)}\left(\hat{\rho}+\hat{0}\right) & Q_2^{(3)}\left(\hat{\rho}+\hat{0}\right) \\ Q_3^{(1)}\left(\hat{\rho}+\hat{0}\right) & Q_3^{(2)}\left(\hat{\rho}+\hat{0}\right) & Q_3^{(3)}\left(\hat{\rho}+\hat{0}\right) \\ Q_4^{(1)}\left(\hat{\rho}+\hat{0}\right) & Q_4^{(2)}\left(\hat{\rho}+\hat{0}\right) & Q_4^{(3)}\left(\hat{\rho}+\hat{0}\right) \\ \vdots & \vdots & \vdots \end{bmatrix} \tag{6.43}$$

$$= \begin{bmatrix} \mathcal{D}_{2\hat{\omega}_0+\hat{\omega}_1} & \mathcal{D}_{2\hat{\omega}_0+\hat{\omega}_2} & \mathcal{D}_{2\hat{\omega}_0+\hat{\omega}_3} \\ \mathcal{D}_{\hat{\omega}_0+2\hat{\omega}_1} & \mathcal{D}_{\hat{\omega}_0+2\hat{\omega}_2} & \mathcal{D}_{\hat{\omega}_0+2\hat{\omega}_3} \\ 1 & 1 & 1 \\ 0 & 0 & 0 \\ \vdots & \vdots & \vdots \end{bmatrix}. \tag{6.44}$$

So we get the positive solution of $\mathbb{Q}(A_3, A_2)$ in terms of quantum dimensions. One may ask for which element $\hat{\nu} \in \Gamma_{h+k}^{++} = \Gamma_7^{++}$, the unit boundary conditions $Q_4^{(a)}\left(\hat{\nu}\right) = 1$, $a = 1, 2, 3$ are satisfied. By a straightforward calculation, one can come up with the following 5 possibilities

$$\hat{\nu} = \begin{cases} 4\hat{\omega}_0 + \hat{\omega}_1 + \hat{\omega}_2 + \hat{\omega}_3 \\ 2\hat{\omega}_0 + 2\hat{\omega}_1 + \hat{\omega}_2 + 2\hat{\omega}_3 \\ 2\hat{\omega}_0 + \hat{\omega}_1 + 3\hat{\omega}_2 + \hat{\omega}_3 \\ \hat{\omega}_0 + 3\hat{\omega}_1 + 2\hat{\omega}_2 + \hat{\omega}_3 \\ \hat{\omega}_0 + \hat{\omega}_1 + 2\hat{\omega}_2 + 3\hat{\omega}_3 \end{cases}.$$

The first three give distinct real solutions and the last two are complex conjugates of each other.

D_r case

For $X = D_r$, the situation is a little different from the $X = A_r$ case because the decomposition of $Q_m^{(a)}$ into characters consists of several summands. It is given in (6.6). Let us begin with $Q_m^{(a)}$'s given by a single summand.

Theorem 6.7.5. *For $X = D_r$, $Q_k^{(a)}\left(\hat{\rho}+\hat{0}\right) = 1$ for $a = 1, r-1$ and r.*

Proof. One can prove this as in Theorem 6.7.1. Note that

$$Q_k^{(a)} = \frac{S_{k\hat{\omega}_a}}{S_{\hat{0}}}$$

for $a = 1, r-1$ and r. Then we get the result as a consequence of Corollary 6.5.4. □

Theorem 6.7.6. *For $X = D_r$, $Q_{k+1}^{(a)} = 0$ identically on Γ_{h+k}^{++} for $a = 1, r-1$ and r.*

Proof. Again we may apply Theorem 6.6.8 to

$$Q_{k+1}^{(a)} = \frac{S_{-\hat{\omega}_0+(k+1)\hat{\omega}_a}}{S_{\hat{0}}}$$

for $a = 1, r-1$ and r. □

Example 6.7.7. Let $X = D_4$ and $k = 3$. Here the Coxeter number is $h = 6$. By (6.6),

$$S_{\hat{0}}\begin{bmatrix} Q_1^{(1)} & Q_1^{(2)} & Q_1^{(3)} & Q_1^{(4)} \\ Q_2^{(1)} & Q_2^{(2)} & Q_2^{(3)} & Q_2^{(4)} \\ Q_3^{(1)} & Q_3^{(2)} & Q_3^{(3)} & Q_3^{(4)} \\ Q_4^{(1)} & Q_4^{(2)} & Q_4^{(3)} & Q_4^{(4)} \\ \vdots & \vdots & \vdots & \vdots \end{bmatrix} \tag{6.45}$$

$$= \begin{bmatrix} S_{2\hat{\omega}_0+\hat{\omega}_1} & S_{3\hat{\omega}_0} + S_{\hat{\omega}_0+\hat{\omega}_2} & S_{2\hat{\omega}_0+\hat{\omega}_3} & S_{2\hat{\omega}_0+\hat{\omega}_4} \\ S_{\hat{\omega}_0+2\hat{\omega}_1} & S_{3\hat{\omega}_0} + S_{\hat{\omega}_0+\hat{\omega}_2} + S_{2\hat{\omega}_2-\hat{\omega}_0} & S_{\hat{\omega}_0+2\hat{\omega}_3} & S_{\hat{\omega}_0+2\hat{\omega}_4} \\ S_{3\hat{\omega}_1} & S_{3\hat{\omega}_0} + S_{\hat{\omega}_0+\hat{\omega}_2} + S_{2\hat{\omega}_2-\hat{\omega}_0} + S_{3\hat{\omega}_2-3\hat{\omega}_0} & S_{3\hat{\omega}_3} & S_{3\hat{\omega}_4} \\ S_{4\hat{\omega}_1-\hat{\omega}_0} & S_{3\hat{\omega}_0} + S_{\hat{\omega}_0+\hat{\omega}_2} + S_{2\hat{\omega}_2-\hat{\omega}_0} + S_{3\hat{\omega}_2-3\hat{\omega}_0} + S_{4\hat{\omega}_2-5\hat{\omega}_0} & S_{4\hat{\omega}_3-\hat{\omega}_0} & S_{4\hat{\omega}_4-\hat{\omega}_0} \\ \vdots & \vdots & \vdots & \vdots \end{bmatrix}. \tag{6.46}$$

By applying Theorem 6.6.8, we can remove many S's so that

$$S_{\hat{0}}\begin{bmatrix} Q_1^{(1)} & Q_1^{(2)} & Q_1^{(3)} & Q_1^{(4)} \\ Q_2^{(1)} & Q_2^{(2)} & Q_2^{(3)} & Q_2^{(4)} \\ Q_3^{(1)} & Q_3^{(2)} & Q_3^{(3)} & Q_3^{(4)} \\ Q_4^{(1)} & Q_4^{(2)} & Q_4^{(3)} & Q_4^{(4)} \\ \vdots & \vdots & \vdots & \vdots \end{bmatrix} \tag{6.47}$$

$$= \begin{bmatrix} S_{2\hat{\omega}_0+\hat{\omega}_1} & S_{3\hat{\omega}_0} + S_{\hat{\omega}_0+\hat{\omega}_2} & S_{2\hat{\omega}_0+\hat{\omega}_3} & S_{2\hat{\omega}_0+\hat{\omega}_4} \\ S_{\hat{\omega}_0+2\hat{\omega}_1} & S_{3\hat{\omega}_0} + S_{\hat{\omega}_0+\hat{\omega}_2} & S_{\hat{\omega}_0+2\hat{\omega}_3} & S_{\hat{\omega}_0+2\hat{\omega}_4} \\ S_{3\hat{\omega}_1} & S_{3\hat{\omega}_0} + S_{\hat{\omega}_0+\hat{\omega}_2} + S_{3\hat{\omega}_2-3\hat{\omega}_0} & S_{3\hat{\omega}_3} & S_{3\hat{\omega}_4} \\ 0 & S_{3\hat{\omega}_0} + S_{\hat{\omega}_0+\hat{\omega}_2} + S_{3\hat{\omega}_2-3\hat{\omega}_0} + S_{4\hat{\omega}_2-5\hat{\omega}_0} & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots \end{bmatrix}. \tag{6.48}$$

We want to verify the equality

$$S_{3\hat{\omega}_0} + S_{\hat{\omega}_0+\hat{\omega}_2} + S_{3\hat{\omega}_2-3\hat{\omega}_0} + S_{4\hat{\omega}_2-5\hat{\omega}_0} = 0.$$

One can use the shifted affine Weyl group action to show it. Note that

$$s_0 \cdot (4\hat{\omega}_2 - 5\hat{\omega}_0) = 3\hat{\omega}_0$$

and

$$s_0 \cdot (3\hat{\omega}_2 - 3\hat{\omega}_0) = \hat{\omega}_0 + \hat{\omega}_2.$$

Thus $S_{3\hat{\omega}_0} + S_{4\hat{\omega}_2-5\hat{\omega}_0} = 0$ and $S_{3\hat{\omega}_2-3\hat{\omega}_0} + S_{\hat{\omega}_0+\hat{\omega}_2} = 0$ by Theorem 6.6.4 and therefore,

$$S_{3\hat{\omega}_0} + S_{\hat{\omega}_0+\hat{\omega}_2} + S_{3\hat{\omega}_2-3\hat{\omega}_0} + S_{4\hat{\omega}_2-5\hat{\omega}_0} = 0. \tag{6.49}$$

From these calculations, we get

$$S_{\hat{0}} \begin{bmatrix} Q_1^{(1)} & Q_1^{(2)} & Q_1^{(3)} & Q_1^{(4)} \\ Q_2^{(1)} & Q_2^{(2)} & Q_2^{(3)} & Q_2^{(4)} \\ Q_3^{(1)} & Q_3^{(2)} & Q_3^{(3)} & Q_3^{(4)} \\ Q_4^{(1)} & Q_4^{(2)} & Q_4^{(3)} & Q_4^{(4)} \\ \vdots & \vdots & \vdots & \vdots \end{bmatrix} \tag{6.50}$$

$$= \begin{bmatrix} S_{2\hat{\omega}_0+\hat{\omega}_1} & S_{3\hat{\omega}_0} + S_{\hat{\omega}_0+\hat{\omega}_2} & S_{2\hat{\omega}_0+\hat{\omega}_3} & S_{2\hat{\omega}_0+\hat{\omega}_4} \\ S_{\hat{\omega}_0+2\hat{\omega}_1} & S_{3\hat{\omega}_0} + S_{\hat{\omega}_0+\hat{\omega}_2} & S_{\hat{\omega}_0+2\hat{\omega}_3} & S_{\hat{\omega}_0+2\hat{\omega}_4} \\ S_{3\hat{\omega}_1} & S_{3\hat{\omega}_0} & S_{3\hat{\omega}_3} & S_{3\hat{\omega}_4} \\ 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots \end{bmatrix}. \tag{6.51}$$

Now we know that $Q_4^{(a)} = 0$ for $a = 1, 2, 3, 4$. Let us look at the parts of the Q-system equations with $m = 3$,

$$\begin{cases} \left(Q_3^{(1)}\right)^2 = Q_2^{(1)}Q_4^{(1)} + Q_3^{(2)} \\ \left(Q_3^{(2)}\right)^2 = Q_3^{(1)}Q_3^{(3)}Q_3^{(4)} + Q_2^{(2)}Q_4^{(2)} \\ \left(Q_3^{(3)}\right)^2 = Q_3^{(2)} + Q_2^{(3)}Q_4^{(3)} \\ \left(Q_3^{(4)}\right)^2 = Q_3^{(2)} + Q_2^{(4)}Q_4^{(4)} \end{cases}.$$

This becomes

$$\begin{cases} \left(Q_3^{(1)}\right)^2 = Q_3^{(2)} \\ \left(Q_3^{(2)}\right)^2 = Q_3^{(1)}Q_3^{(3)}Q_3^{(4)} \\ \left(Q_3^{(3)}\right)^2 = Q_3^{(2)} \\ \left(Q_3^{(4)}\right)^2 = Q_3^{(2)} \end{cases}. \tag{6.52}$$

Since $S_{3\hat{\omega}_0} = S_{\hat{0}}$, $Q_3^{(2)}(\hat{\nu}) = 1$ for any $\hat{\nu} \in \Gamma_{h+k}^{++}$. Thus we know that $Q_3^{(2)} = 1$ identically on Γ_{h+k}^{++} and we can see that there are only four possible solutions of (6.52) :

$$(Q_3^{(1)}, Q_3^{(2)}, Q_3^{(3)}, Q_3^{(4)}) = \begin{cases} (1,1,1,1) \\ (1,1,-1,-1) \\ (-1,1,-1,1) \\ (-1,1,1,-1) \end{cases}.$$

If we evaluate them at $\hat{\rho} + \hat{0} \in \Gamma_{h+k}^{++}$, we get

$$\left(Q_3^{(1)}(\hat{\rho}+\hat{0}), Q_3^{(2)}(\hat{\rho}+\hat{0}), Q_3^{(3)}(\hat{\rho}+\hat{0}), Q_3^{(4)}(\hat{\rho}+\hat{0})\right) = (1,1,1,1)$$

by Theorem 6.7.5. Therefore we finally get

$$\begin{bmatrix} Q_1^{(1)}\left(\hat{\rho}+\hat{0}\right) & Q_1^{(2)}\left(\hat{\rho}+\hat{0}\right) & Q_1^{(3)}\left(\hat{\rho}+\hat{0}\right) & Q_1^{(4)}\left(\hat{\rho}+\hat{0}\right) \\ Q_2^{(1)}\left(\hat{\rho}+\hat{0}\right) & Q_2^{(2)}\left(\hat{\rho}+\hat{0}\right) & Q_2^{(3)}\left(\hat{\rho}+\hat{0}\right) & Q_2^{(4)}\left(\hat{\rho}+\hat{0}\right) \\ Q_3^{(1)}\left(\hat{\rho}+\hat{0}\right) & Q_3^{(2)}\left(\hat{\rho}+\hat{0}\right) & Q_3^{(3)}\left(\hat{\rho}+\hat{0}\right) & Q_3^{(4)}\left(\hat{\rho}+\hat{0}\right) \\ Q_4^{(1)}\left(\hat{\rho}+\hat{0}\right) & Q_4^{(2)}\left(\hat{\rho}+\hat{0}\right) & Q_4^{(3)}\left(\hat{\rho}+\hat{0}\right) & Q_4^{(4)}\left(\hat{\rho}+\hat{0}\right) \\ \vdots & \vdots & \vdots & \vdots \end{bmatrix} \tag{6.53}$$

$$= \begin{bmatrix} \mathcal{D}_{2\hat{\omega}_0+\hat{\omega}_1} & 1+\mathcal{D}_{\hat{\omega}_0+\hat{\omega}_2} & \mathcal{D}_{2\hat{\omega}_0+\hat{\omega}_3} & \mathcal{D}_{2\hat{\omega}_0+\hat{\omega}_4} \\ \mathcal{D}_{\hat{\omega}_0+2\hat{\omega}_1} & 1+\mathcal{D}_{\hat{\omega}_0+\hat{\omega}_2} & \mathcal{D}_{\hat{\omega}_0+2\hat{\omega}_3} & \mathcal{D}_{\hat{\omega}_0+2\hat{\omega}_4} \\ 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots \end{bmatrix}. \tag{6.54}$$

So we obtained the positive solution of the system $\mathbb{Q}(D_4, A_2)$ in terms of quantum dimensions explicitly. The positivity is clearly seen from Theorem 6.5.2. The symmetry of the positive solution can now be understood in terms of the symmetry of the extended Dynkin diagram.

Example 6.7.8. Let us work out another example with $X = D_4$ and the level $k = 4$. Again, we want to show that $Q_5^{(a)} = 0$ for $a = 1, \cdots, 4$. Let us consider $Q_5^{(2)}0$. We have

$$S_{\hat{0}} Q_5^{(2)} = S_{4\hat{\omega}_0} + S_{2\hat{\omega}_2} + S_{2\hat{\omega}_0+\hat{\omega}_2} + S_{3\hat{\omega}_2-2\hat{\omega}_0} + S_{4\hat{\omega}_2-4\hat{\omega}_0} + S_{5\hat{\omega}_2-6\hat{\omega}_0}.$$

Using the shifted affine Weyl group action, we can see that

$$s_0 \cdot (5\hat{\omega}_2 - 6\hat{\omega}_0) = 4\hat{\omega}_0 \tag{6.55}$$

$$s_0 \cdot (4\hat{\omega}_2 - 4\hat{\omega}_0) = 2\hat{\omega}_0 + \hat{\omega}_2 \tag{6.56}$$

$$s_0 \cdot (3\hat{\omega}_2 - 2\hat{\omega}_0) = 2\hat{\omega}_2. \tag{6.57}$$

Thus $Q_5^{(2)} = 0$. This result together with Theorem 6.6.8 implies

$$S_{\hat{0}} \begin{bmatrix} Q_1^{(1)} & Q_1^{(2)} & Q_1^{(3)} & Q_1^{(4)} \\ Q_2^{(1)} & Q_2^{(2)} & Q_2^{(3)} & Q_2^{(4)} \\ Q_3^{(1)} & Q_3^{(2)} & Q_3^{(3)} & Q_3^{(4)} \\ Q_4^{(1)} & Q_4^{(2)} & Q_4^{(3)} & Q_4^{(4)} \\ Q_5^{(1)} & Q_5^{(2)} & Q_5^{(3)} & Q_5^{(4)} \\ \vdots & \vdots & \vdots & \vdots \end{bmatrix} \tag{6.58}$$

$$= \begin{bmatrix} S_{3\hat{\omega}_0+\hat{\omega}_1} & S_{4\hat{\omega}_0} + S_{2\hat{\omega}_0+\hat{\omega}_2} & S_{3\hat{\omega}_0+\hat{\omega}_3} & S_{3\hat{\omega}_0+\hat{\omega}_4} \\ S_{2\hat{\omega}_0+2\hat{\omega}_1} & S_{4\hat{\omega}_0} + S_{2\hat{\omega}_2} + S_{2\hat{\omega}_0+\hat{\omega}_2} & S_{2\hat{\omega}_0+2\hat{\omega}_3} & S_{2\hat{\omega}_0+2\hat{\omega}_4} \\ S_{\hat{\omega}_0+3\hat{\omega}_1} & S_{4\hat{\omega}_0} + S_{2\hat{\omega}_0+\hat{\omega}_2} & S_{\hat{\omega}_0+3\hat{\omega}_3} & S_{\hat{\omega}_0+3\hat{\omega}_4} \\ S_{4\hat{\omega}_1} & S_{4\hat{\omega}_0} & S_{4\hat{\omega}_3} & S_{4\hat{\omega}_4} \\ 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots \end{bmatrix}. \tag{6.59}$$

If we evaluate them at $\hat{\rho} + \hat{0} \in \Gamma_{h+k}^{++}$, we get

$$\begin{bmatrix} Q_1^{(1)}\left(\hat{\rho}+\hat{0}\right) & Q_1^{(2)}\left(\hat{\rho}+\hat{0}\right) & Q_1^{(3)}\left(\hat{\rho}+\hat{0}\right) & Q_1^{(4)}\left(\hat{\rho}+\hat{0}\right) \\ Q_2^{(1)}\left(\hat{\rho}+\hat{0}\right) & Q_2^{(2)}\left(\hat{\rho}+\hat{0}\right) & Q_2^{(3)}\left(\hat{\rho}+\hat{0}\right) & Q_2^{(4)}\left(\hat{\rho}+\hat{0}\right) \\ Q_3^{(1)}\left(\hat{\rho}+\hat{0}\right) & Q_3^{(2)}\left(\hat{\rho}+\hat{0}\right) & Q_3^{(3)}\left(\hat{\rho}+\hat{0}\right) & Q_3^{(4)}\left(\hat{\rho}+\hat{0}\right) \\ Q_4^{(1)}\left(\hat{\rho}+\hat{0}\right) & Q_4^{(2)}\left(\hat{\rho}+\hat{0}\right) & Q_4^{(3)}\left(\hat{\rho}+\hat{0}\right) & Q_4^{(4)}\left(\hat{\rho}+\hat{0}\right) \\ Q_5^{(1)}\left(\hat{\rho}+\hat{0}\right) & Q_5^{(2)}\left(\hat{\rho}+\hat{0}\right) & Q_5^{(3)}\left(\hat{\rho}+\hat{0}\right) & Q_5^{(4)}\left(\hat{\rho}+\hat{0}\right) \\ \vdots & \vdots & \vdots & \vdots \end{bmatrix} \tag{6.60}$$

$$= \begin{bmatrix} \mathcal{D}_{3\hat{\omega}_0+\hat{\omega}_1} & \mathcal{D}_{2\hat{\omega}_0+\hat{\omega}_2} + 1 & \mathcal{D}_{3\hat{\omega}_0+\hat{\omega}_3} & \mathcal{D}_{3\hat{\omega}_0+\hat{\omega}_4} \\ \mathcal{D}_{2\hat{\omega}_0+2\hat{\omega}_1} & \mathcal{D}_{2\hat{\omega}_2} + \mathcal{D}_{2\hat{\omega}_0+\hat{\omega}_2} + 1 & \mathcal{D}_{2\hat{\omega}_0+2\hat{\omega}_3} & \mathcal{D}_{2\hat{\omega}_0+2\hat{\omega}_4} \\ \mathcal{D}_{\hat{\omega}_0+3\hat{\omega}_1} & \mathcal{D}_{2\hat{\omega}_0+\hat{\omega}_2} + 1 & \mathcal{D}_{\hat{\omega}_0+3\hat{\omega}_3} & \mathcal{D}_{\hat{\omega}_0+3\hat{\omega}_4} \\ 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots \end{bmatrix}. \tag{6.61}$$

Thus again we have found the positive solution of the system $\mathbb{Q}(D_4, A_3)$ in terms of quantum dimensions. The positivity is clear from Theorem 6.5.2 and the symmetry of the extended Dynkin diagram explains the symmetry of the positive solution.

6.8 Periodicitiicies of Q-systems

As we saw before, we have periodicity results for both Y-systems (Theorem 4.1.4) and T-systems (Theorem 4.3.2). However, it does not make sense to talk about the periodicity of Q-systems if one only considers Q-systems as polynomials of initial conditions as stated in

Theorem 4.45. In the previous section, we saw that $Q_m^{(a)}$ can be thought of as a function defined on Γ_{h+k}^{++}. This setting allows us to formulate a conjecture about the periodicity of the Q-system.

Recall also that Conjecture 6.3.2 says

$$Q_{k+1}^{(a)}(\hat{\rho}+\hat{0}) = Q_{k+2}^{(a)}(\hat{\rho}+\hat{0}) = \cdots = Q_{k+h-1}^{(a)}(\hat{\rho}+\hat{0}) = 0.$$

In fact, this seems to be true for any $\hat{\nu} \in \Gamma_{h+k}^{++}$. We state it here as a conjecture and it should be considered together with Conjecture 6.3.2.

Conjecture 6.8.1. *Regard the elements of* $\{Q_m^{(a)} | 1 \leq a \leq r, m \in \mathbb{Z}^{\geq 0}\}$ *as functions on the set* Γ_{h+k}^{++} *as given in (6.39). Then the following properties are satisfied :*

1. *(occurrence of 0)* $Q_{k+1}^{(a)} = Q_{k+2}^{(a)} = \cdots = Q_{k+h-1}^{(a)} = 0$ *for* $1 \leq a \leq r$.
2. *(periodicity)* $Q_m^{(a)} = Q_{m+2M(k+h)}^{(a)}$ *for* $1 \leq a \leq r$ *and* $m = 0, 1, \cdots$ *where* M *is some positive integer depending on* X.

Example 6.8.2. Let $X = A_3$, $k = 3$ as in Example 6.7.4 and $\hat{\mu} \in \Gamma_k^+$.

$$\begin{bmatrix} Q_1^{(1)}(\hat{\rho}+\hat{\mu}) & Q_1^{(2)}(\hat{\rho}+\hat{\mu}) & Q_1^{(3)}(\hat{\rho}+\hat{\mu}) \\ Q_2^{(1)}(\hat{\rho}+\hat{\mu}) & Q_2^{(2)}(\hat{\rho}+\hat{\mu}) & Q_2^{(3)}(\hat{\rho}+\hat{\mu}) \\ \vdots & \vdots & \vdots \end{bmatrix} \tag{6.62}$$

$$= \begin{bmatrix} \mathcal{D}_{2\hat{\omega}_0+\hat{\omega}_1}[\hat{\mu}] & \mathcal{D}_{2\hat{\omega}_0+\hat{\omega}_2}[\hat{\mu}] & \mathcal{D}_{2\hat{\omega}_0+\hat{\omega}_3}[\hat{\mu}] \\ \mathcal{D}_{\hat{\omega}_0+2\hat{\omega}_1}[\hat{\mu}] & \mathcal{D}_{\hat{\omega}_0+2\hat{\omega}_2}[\hat{\mu}] & \mathcal{D}_{\hat{\omega}_0+2\hat{\omega}_3}[\hat{\mu}] \\ \mathcal{D}_{3\hat{\omega}_1}[\hat{\mu}] & \mathcal{D}_{3\hat{\omega}_2}[\hat{\mu}] & \mathcal{D}_{3\hat{\omega}_3}[\hat{\mu}] \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ -\mathcal{D}_{3\hat{\omega}_1}[\hat{\mu}] & \mathcal{D}_{3\hat{\omega}_2}[\hat{\mu}] & -\mathcal{D}_{3\hat{\omega}_3}[\hat{\mu}] \\ -\mathcal{D}_{2\hat{\omega}_1+\hat{\omega}_2}[\hat{\mu}] & \mathcal{D}_{\hat{\omega}_0+2\hat{\omega}_2}[\hat{\mu}] & -\mathcal{D}_{\hat{\omega}_2+2\hat{\omega}_3}[\hat{\mu}] \\ -\mathcal{D}_{\hat{\omega}_1+2\hat{\omega}_2}[\hat{\mu}] & \mathcal{D}_{2\hat{\omega}_0+\hat{\omega}_2}[\hat{\mu}] & -\mathcal{D}_{2\hat{\omega}_2+\hat{\omega}_3}[\hat{\mu}] \\ -\mathcal{D}_{3\hat{\omega}_2}[\hat{\mu}] & \mathcal{D}_{3\hat{\omega}_0}[\hat{\mu}] & -\mathcal{D}_{3\hat{\omega}_2}[\hat{\mu}] \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ \mathcal{D}_{3\hat{\omega}_2}[\hat{\mu}] & \mathcal{D}_{3\hat{\omega}_0}[\hat{\mu}] & \mathcal{D}_{3\hat{\omega}_2}[\hat{\mu}] \\ \mathcal{D}_{2\hat{\omega}_2+\hat{\omega}_3}[\hat{\mu}] & \mathcal{D}_{2\hat{\omega}_0+\hat{\omega}_2}[\hat{\mu}] & \mathcal{D}_{\hat{\omega}_1+2\hat{\omega}_2}[\hat{\mu}] \\ \mathcal{D}_{\hat{\omega}_2+2\hat{\omega}_3}[\hat{\mu}] & \mathcal{D}_{\hat{\omega}_0+2\hat{\omega}_2}[\hat{\mu}] & \mathcal{D}_{2\hat{\omega}_1+\hat{\omega}_2}[\hat{\mu}] \\ \mathcal{D}_{3\hat{\omega}_3}[\hat{\mu}] & \mathcal{D}_{3\hat{\omega}_2}[\hat{\mu}] & \mathcal{D}_{3\hat{\omega}_1}[\hat{\mu}] \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ -\mathcal{D}_{3\hat{\omega}_3}[\hat{\mu}] & \mathcal{D}_{3\hat{\omega}_2}[\hat{\mu}] & -\mathcal{D}_{3\hat{\omega}_1}[\hat{\mu}] \\ -\mathcal{D}_{\hat{\omega}_0+2\hat{\omega}_3}[\hat{\mu}] & \mathcal{D}_{\hat{\omega}_0+2\hat{\omega}_2}[\hat{\mu}] & -\mathcal{D}_{\hat{\omega}_0+2\hat{\omega}_1}[\hat{\mu}] \\ -\mathcal{D}_{2\hat{\omega}_0+\hat{\omega}_3}[\hat{\mu}] & \mathcal{D}_{2\hat{\omega}_0+\hat{\omega}_2}[\hat{\mu}] & -\mathcal{D}_{2\hat{\omega}_0+\hat{\omega}_1}[\hat{\mu}] \\ -\mathcal{D}_{3\hat{\omega}_0}[\hat{\mu}] & \mathcal{D}_{3\hat{\omega}_0}[\hat{\mu}] & -\mathcal{D}_{3\hat{\omega}_0}[\hat{\mu}] \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ \mathcal{D}_{3\hat{\omega}_0}[\hat{\mu}] & \mathcal{D}_{3\hat{\omega}_0}[\hat{\mu}] & \mathcal{D}_{3\hat{\omega}_0}[\hat{\mu}] \\ \mathcal{D}_{2\hat{\omega}_0+\hat{\omega}_1}[\hat{\mu}] & \mathcal{D}_{2\hat{\omega}_0+\hat{\omega}_2}[\hat{\mu}] & \mathcal{D}_{2\hat{\omega}_0+\hat{\omega}_3}[\hat{\mu}] \\ \mathcal{D}_{\hat{\omega}_0+2\hat{\omega}_1}[\hat{\mu}] & \mathcal{D}_{\hat{\omega}_0+2\hat{\omega}_2}[\hat{\mu}] & \mathcal{D}_{\hat{\omega}_0+2\hat{\omega}_3}[\hat{\mu}] \\ \mathcal{D}_{3\hat{\omega}_1}[\hat{\mu}] & \mathcal{D}_{3\hat{\omega}_2}[\hat{\mu}] & \mathcal{D}_{3\hat{\omega}_3}[\hat{\mu}] \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ \vdots & \vdots & \vdots \end{bmatrix} \tag{6.63}$$

Example 6.8.3. Let $X = D_4$, $k = 4$ as in Example 6.7.8 and $\hat{\mu} \in \Gamma_k^+$.

$$\begin{bmatrix} Q_1^{(1)}(\hat{\rho}+\hat{\mu}) & Q_1^{(2)}(\hat{\rho}+\hat{\mu}) & Q_1^{(3)}(\hat{\rho}+\hat{\mu}) & Q_1^{(4)}(\hat{\rho}+\hat{\mu}) \\ Q_2^{(1)}(\hat{\rho}+\hat{\mu}) & Q_2^{(2)}(\hat{\rho}+\hat{\mu}) & Q_2^{(3)}(\hat{\rho}+\hat{\mu}) & Q_2^{(4)}(\hat{\rho}+\hat{\mu}) \\ \vdots & \vdots & \vdots & \vdots \end{bmatrix} \tag{6.64}$$

$$= \begin{bmatrix}
\mathcal{D}_{3\hat{\omega}_0+\hat{\omega}_1}[\hat{\mu}] & \mathcal{D}_{4\hat{\omega}_0}[\hat{\mu}] + \mathcal{D}_{2\hat{\omega}_0+\hat{\omega}_2}[\hat{\mu}] & \mathcal{D}_{3\hat{\omega}_0+\hat{\omega}_3}[\hat{\mu}] & \mathcal{D}_{3\hat{\omega}_0+\hat{\omega}_4}[\hat{\mu}] \\
\mathcal{D}_{2\hat{\omega}_0+2\hat{\omega}_1}[\hat{\mu}] & \mathcal{D}_{4\hat{\omega}_0}[\hat{\mu}] + \mathcal{D}_{2\hat{\omega}_2}[\hat{\mu}] + \mathcal{D}_{2\hat{\omega}_0+\hat{\omega}_2}[\hat{\mu}] & \mathcal{D}_{2\hat{\omega}_0+2\hat{\omega}_3}[\hat{\mu}] & \mathcal{D}_{2\hat{\omega}_0+2\hat{\omega}_4}[\hat{\mu}] \\
\mathcal{D}_{\hat{\omega}_0+3\hat{\omega}_1}[\hat{\mu}] & \mathcal{D}_{4\hat{\omega}_0}[\hat{\mu}] + \mathcal{D}_{2\hat{\omega}_0+\hat{\omega}_2}[\hat{\mu}] & \mathcal{D}_{\hat{\omega}_0+3\hat{\omega}_3}[\hat{\mu}] & \mathcal{D}_{\hat{\omega}_0+3\hat{\omega}_4}[\hat{\mu}] \\
\mathcal{D}_{4\hat{\omega}_1}[\hat{\mu}] & \mathcal{D}_{4\hat{\omega}_0}[\hat{\mu}] & \mathcal{D}_{4\hat{\omega}_3}[\hat{\mu}] & \mathcal{D}_{4\hat{\omega}_4}[\hat{\mu}] \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
\mathcal{D}_{4\hat{\omega}_1}[\hat{\mu}] & \mathcal{D}_{4\hat{\omega}_0}[\hat{\mu}] & \mathcal{D}_{4\hat{\omega}_3}[\hat{\mu}] & \mathcal{D}_{4\hat{\omega}_4}[\hat{\mu}] \\
\mathcal{D}_{\hat{\omega}_0+3\hat{\omega}_1}[\hat{\mu}] & \mathcal{D}_{4\hat{\omega}_0}[\hat{\mu}] + \mathcal{D}_{2\hat{\omega}_0+\hat{\omega}_2}[\hat{\mu}] & \mathcal{D}_{\hat{\omega}_0+3\hat{\omega}_3}[\hat{\mu}] & \mathcal{D}_{\hat{\omega}_0+3\hat{\omega}_4}[\hat{\mu}] \\
\mathcal{D}_{2\hat{\omega}_0+2\hat{\omega}_1}[\hat{\mu}] & \mathcal{D}_{4\hat{\omega}_0}[\hat{\mu}] + \mathcal{D}_{2\hat{\omega}_2}[\hat{\mu}] + \mathcal{D}_{2\hat{\omega}_0+\hat{\omega}_2}[\hat{\mu}] & \mathcal{D}_{2\hat{\omega}_0+2\hat{\omega}_3}[\hat{\mu}] & \mathcal{D}_{2\hat{\omega}_0+2\hat{\omega}_4}[\hat{\mu}] \\
\mathcal{D}_{3\hat{\omega}_0+\hat{\omega}_1}[\hat{\mu}] & \mathcal{D}_{4\hat{\omega}_0}[\hat{\mu}] + \mathcal{D}_{2\hat{\omega}_0+\hat{\omega}_2}[\hat{\mu}] & \mathcal{D}_{3\hat{\omega}_0+\hat{\omega}_3}[\hat{\mu}] & \mathcal{D}_{3\hat{\omega}_0+\hat{\omega}_4}[\hat{\mu}] \\
\mathcal{D}_{4\hat{\omega}_0}[\hat{\mu}] & \mathcal{D}_{4\hat{\omega}_0}[\hat{\mu}] & \mathcal{D}_{4\hat{\omega}_0}[\hat{\mu}] & \mathcal{D}_{4\hat{\omega}_0}[\hat{\mu}] \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
\mathcal{D}_{4\hat{\omega}_0}[\hat{\mu}] & \mathcal{D}_{4\hat{\omega}_0}[\hat{\mu}] & \mathcal{D}_{4\hat{\omega}_0}[\hat{\mu}] & \mathcal{D}_{4\hat{\omega}_0}[\hat{\mu}] \\
\mathcal{D}_{3\hat{\omega}_0+\hat{\omega}_1}[\hat{\mu}] & \mathcal{D}_{4\hat{\omega}_0}[\hat{\mu}] + \mathcal{D}_{2\hat{\omega}_0+\hat{\omega}_2}[\hat{\mu}] & \mathcal{D}_{3\hat{\omega}_0+\hat{\omega}_3}[\hat{\mu}] & \mathcal{D}_{3\hat{\omega}_0+\hat{\omega}_4}[\hat{\mu}] \\
\mathcal{D}_{2\hat{\omega}_0+2\hat{\omega}_1}[\hat{\mu}] & \mathcal{D}_{4\hat{\omega}_0}[\hat{\mu}] + \mathcal{D}_{2\hat{\omega}_2}[\hat{\mu}] + \mathcal{D}_{2\hat{\omega}_0+\hat{\omega}_2}[\hat{\mu}] & \mathcal{D}_{2\hat{\omega}_0+2\hat{\omega}_3}[\hat{\mu}] & \mathcal{D}_{2\hat{\omega}_0+2\hat{\omega}_4}[\hat{\mu}] \\
\mathcal{D}_{\hat{\omega}_0+3\hat{\omega}_1}[\hat{\mu}] & \mathcal{D}_{4\hat{\omega}_0}[\hat{\mu}] + \mathcal{D}_{2\hat{\omega}_0+\hat{\omega}_2}[\hat{\mu}] & \mathcal{D}_{\hat{\omega}_0+3\hat{\omega}_3}[\hat{\mu}] & \mathcal{D}_{\hat{\omega}_0+3\hat{\omega}_4}[\hat{\mu}] \\
\mathcal{D}_{4\hat{\omega}_1}[\hat{\mu}] & \mathcal{D}_{4\hat{\omega}_0}[\hat{\mu}] & \mathcal{D}_{4\hat{\omega}_3}[\hat{\mu}] & \mathcal{D}_{4\hat{\omega}_4}[\hat{\mu}] \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots
\end{bmatrix} \tag{6.65}$$

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Appendix A

Dynkin diagrams and Cartan matrices

A.1 Dynkin diagrams of $ADET$ type and their Cartan matrices

The Dynkin diagrams of $ADET$ type are given in Figure A.1. Their Cartan matrices are as follows :

$$\mathcal{C}(A_n) = \begin{pmatrix} 2 & -1 & 0 & 0 & 0 \\ -1 & 2 & -1 & \ddots & 0 \\ 0 & -1 & \ddots & \ddots & 0 \\ 0 & \ddots & \ddots & 2 & -1 \\ 0 & 0 & 0 & -1 & 2 \end{pmatrix}$$

$$\mathcal{C}(D_n) = \begin{pmatrix} 2 & -1 & 0 & 0 & 0 \\ -1 & 2 & \ddots & 0 & 0 \\ 0 & \ddots & 2 & -1 & -1 \\ 0 & 0 & -1 & 2 & 0 \\ 0 & 0 & -1 & 0 & 2 \end{pmatrix}$$

$$\mathcal{C}(T_n) = \begin{pmatrix} 2 & -1 & 0 & 0 & 0 \\ -1 & 2 & -1 & \ddots & 0 \\ 0 & -1 & \ddots & \ddots & 0 \\ 0 & \ddots & \ddots & 2 & -1 \\ 0 & 0 & 0 & -1 & 1 \end{pmatrix}$$

$$\mathcal{C}(E_6) = \begin{pmatrix} 2 & -1 & 0 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 & -1 \\ 0 & 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & 0 & -1 & 2 & 0 \\ 0 & 0 & -1 & 0 & 0 & 2 \end{pmatrix}$$

$$\mathcal{C}(E_7) = \begin{pmatrix} 2 & -1 & 0 & 0 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 & 0 & -1 \\ 0 & 0 & -1 & 2 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & 0 & 0 & -1 & 2 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 2 \end{pmatrix}$$

$$\mathcal{C}(E_8) = \begin{pmatrix} 2 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 2 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 2 & -1 & 0 & -1 \\ 0 & 0 & 0 & 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 2 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 2 \end{pmatrix}.$$

A.2 Graph automorphisms of Dynkin diagrams

Let I be the set of vertices as in Figure A.1. We define a diagram automorphism $\omega : I \to I$ for each Dynkin diagram of $ADET$ type as follows :

1. For A_r, $\omega(v_i) = v_{r+1-i}$.
2. For D_{2n+1}, $\omega = \begin{cases} \omega(v_i) = v_i & 1 \leq i < 2n \\ \omega(v_{2n}) = v_{2n+1} & \\ \omega(v_{2n+1}) = v_{2n} & \end{cases}$
3. For E_6, $\omega(v_i) = v_{6-i}$ for $i \neq 6$ and $\omega(v_6) = v_6$.
4. For others, $\omega(v_i) = v_i$

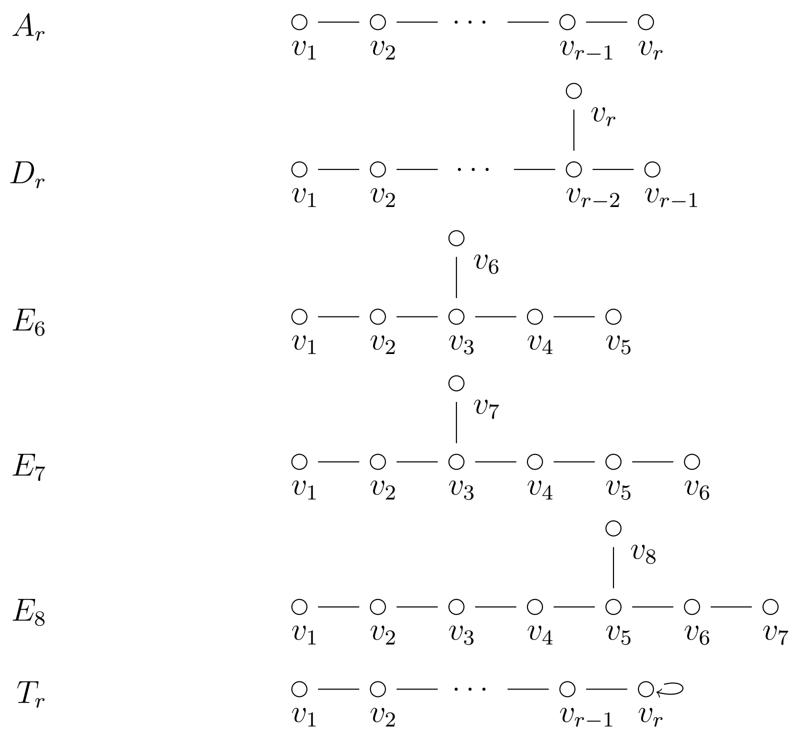

Figure A.1: *ADET* Dynkin diagrams with indexed vertices.

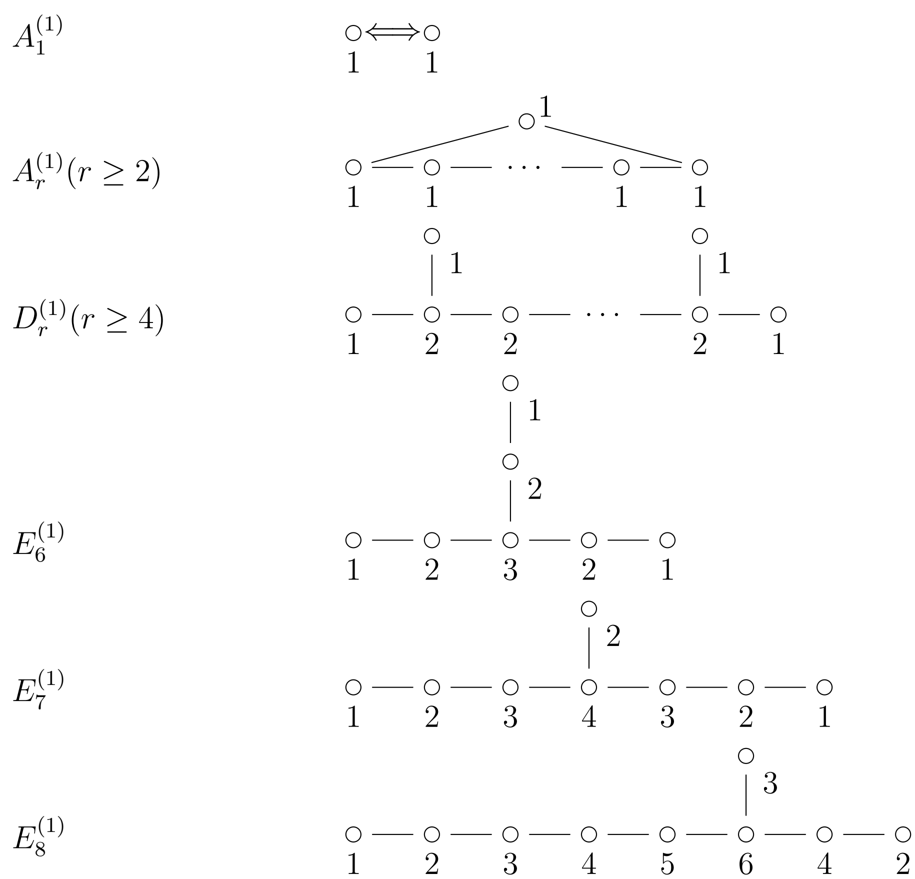

Figure A.2: Extended Dynkin diagrams of ADE type and Dynkin labels