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Pressure Loss of Turbulent Power Law Fluid Flow Between Two Parallel Rough Plates

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Pressure Loss of Turbulent Power Law Fluid Flow Between Two Parallel Rough Plates

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in

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by

Abdullah Sahal Shukri

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ABSTRACT OF THE THESIS

A Study of the Men Pressure Loss of Turbulent Power Law Fluid Flow Between Two Parallel Rough Plates

by

Abdullah Sahal Shukri

Master of Science, Graduate Program in Mechanical Engineering
University of California, Riverside, June 2016
Dr. Guanshu Xu, Chairperson

The pressure loss of turbulent power law fluid flow between parallel plates has been theoretically analyzed. Using Prandtl mixing length theory and Van Karman similarity hypotheses for turbulent flow, we have derived a new friction factor formula that resembles previous formula for turbulent power law fluid flow in rough pipes. The new formula is also valid for the transition region between smooth and wholly rough wall turbulence.

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List of Symbols

P: Pressure
F: Force
m: Mass
a: Acceleration
$\vec{V}$: Velocity vector
u: Velocity in x direction
v: Velocity in y direction
w: Velocity in z direction
t: Time
ρ: Density
μ: Dynamic Viscosity
ν: Kinematic Viscosity
$\forall$: Volume
τ: Shear stress
δ_{ij}: Kronecker delta
σ_{ij}: Stress
$\dot{q}$: Rate of volume of heat per unit mass
k: Thermal conductivity
K: Flow consistency index
$\dot{\gamma}$: Shear rate
n : Flow behavior index
τ_y: Yield stress
A_c: Cross section area
p: Wetted perimeter
Q: Flow Rate
f: Fanning friction Factor
f_D: Darcy friction Factor
Re: Reynolds Number
Re^*: Reynolds Number for non-Newtonian fluid
τ_w: Shear stress at the wall
u_{avg}: Average velocity
c: Average velocity
μ_w: Apparent viscosity at the wall
η: Kolmogorov length scale
ε: Dissipation rate
υ: Kolmogorov velocity scale
$\bar{u}$: Mean velocity in x direction
$\bar{v}$: Mean velocity in y direction
$\bar{w}$: Mean velocity in y direction
u': Fluctuating velocity in x direction
v': Fluctuating velocity in y direction
w': Fluctuating velocity in z direction
$\overline{u'}, \overline{v'}, \overline{w'}$: Averaged fluctuating values
κ: Von Karman constant
v_*: Frictional velocity
δ: Laminar (viscous) layer thickness
u_{max}: Maximum velocity at the center

1. Introduction

On 1959 Dodge and Metzner [16] carried out an investigation by applying dimensional analysis techniques for power law fluid. In the same year, Shaver and Merril worked on extending Blasius formula to power law fluid [17]. After 14 years, Clapp came up with dimensionless velocity distribution of turbulent power law fluid flow for smooth pipe [18]. The pressure drop for non-Newtonian turbulent flow through rough pipes has been studied by A. P. Szilas, E. Bobok, and L. Navratil on 1981 [1]. In this thesis, the same will be done for the flow between the parallel plates.

The first two chapters will summarize a basic fluid mechanic's theories and governing equations. Followed by the required principles and equations for a turbulent flow that are required to better understand the main work of this thesis. Finally, the last chapter will explain theoretical calculations for finding the friction factor for a turbulent non-Newtonian flow between two parallel plates.

2. Governing Equation

The basic of computational fluid dynamics is the fundamental governing equations which are continuity, momentum and energy. These equations explain the physics, and they are based on the mathematical statements of following fundamental:

a) conservation of mass
b) $F = ma$ Newton's second law
c) conservation energy

2.1 Materials Derivative

To be able to drive the governing equation for fluid dynamics, we need to understand an important physical theory called Materials derivative of substantial derivative [3].

Let's consider a fluid parcel is moving with the flow in a Cartesian coordinates as per figure 2.1. The unit vector for velocity is given as following:

$$\vec{V} = u\,\vec{i} + v\,\vec{j} + w\,\vec{k} \tag{2.1}$$

Where the velocity component for x, y and z, are given as following:

$u = u(x, y, z, t)$, $= v(x, y, z, t)$, $w = w(x, y, z, t)$ (2.2)

Also, the density is given by:

$f = f(x, y, z, t)$ (2.3)

f: is a physical quantity such as heat, density, temperature,e tc.

Since we are considering unsteady flow, where $u, v, w \,\&\, \rho$ are function of space and time.

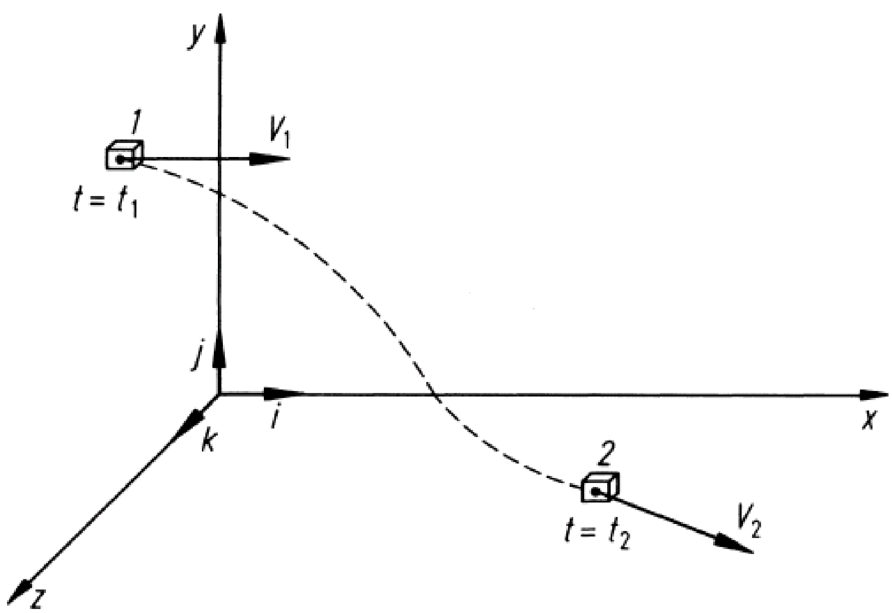

Figure 2.1: fluid parcel is moving with the flow from point 1 to 2 [3].

At point 1 the physical quantity f of the fluid parcel is:

$$f_1 = f(x_1, y_1, z_1, t_1) \tag{2.4}$$

And at point 2 the physical quantity f of the fluid parcel is:

$$f = f(x_2, y_2, z_2, t_2) \tag{2.5}$$

Now we can expand this function by using Tyler's series as following:

$$f_2 - f_1 = \left(\frac{\partial f_1}{\partial x}\right)(x_2 - x_1) + \left(\frac{\partial f_1}{\partial y}\right)(y_2 - y_1) + \left(\frac{\partial f_1}{\partial z}\right)(z_2 - z) + \left(\frac{\partial f_1}{\partial t}\right)(t_2 - t_1) \tag{2.6}$$

Divide both sides by $(t_2 - t_1)$:

$$\frac{f_2 - f_1}{(t_2 - t_1)} = \left(\frac{\partial f_1}{\partial x}\right)\frac{(x_2 - x_1)}{(t_2 - t_1)} + \left(\frac{\partial f_1}{\partial y}\right)\frac{(y_2 - y_1)}{(t_2 - t_1)} + \left(\frac{\partial f_1}{\partial z}\right)\frac{(z_2 - z)}{(t_2 - t_1)} + \left(\frac{\partial f_1}{\partial t}\right)\frac{(t_2 - t_1)}{(t_2 - t_1)} \tag{2.7}$$

Apply the limit for the LHS on equation (2.7):

$$\lim_{t_2 \to t_1}\left(\frac{f_2 - f_1}{t_2 - t_1}\right) = \frac{Df}{Dt} \tag{2.8}$$

Here, ${}^{Df}/_{Dt}$ stand for instantaneous time rate of change of f for a fluid parcel as it moving through point 1. In otherworld, we can say the time rate of change of f for a fluid parcel as it's moving through space. On other hand side, ${}^{\partial f}/_{\partial t}$ which is the time rate of change of f at point 1 only. So, ${}^{Df}/_{Dt}$ and ${}^{\partial f}/_{\partial t}$ have different physical meaning and numerical quantity. By taking the limit for the RHS of equation (2.1):

$$\lim_{t_2 \to t_1} \left(\frac{x_2 - x_1}{t_2 - t_1}\right) = u \ , \qquad \lim_{t_2 \to t_1} \left(\frac{y_2 - y_1}{t_2 - t_1}\right) = v \ , \quad \lim_{t_2 \to t_1} \left(\frac{z_2 - z_1}{t_2 - t_1}\right) = w \qquad (2.9)$$

After applying the limit for both hand sides of equation (2.7):

$$\frac{Df}{Dt} = \frac{\partial f}{\partial t} + u\ \frac{\partial f}{\partial x} + v\ \frac{\partial f}{\partial y} + w\ \frac{\partial f}{\partial z} \qquad (2.10)$$

The general form for material derivative is:

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + u\ \frac{\partial}{\partial x} + v\ \frac{\partial}{\partial y} + w\ \frac{\partial}{\partial z} \qquad (2.11)$$

Since for Cartesian coordinate$\nabla = \vec{\imath}\ \frac{\partial}{\partial x} + \vec{\jmath}\ \frac{\partial}{\partial y} + \ \vec{k}\ \frac{\partial}{\partial z}$.

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + (\vec{V}.\nabla \mathrm{f}) \qquad (2.12)$$

Equation (2.3) known as material derivative and it describe a rate of change of physical quantity in time and space. The first term on RHS is known as local rate of change and the second term is convective derivative which is the change rate of the physical quantity.

Another way of driving the material derivative. If $f = f(x, y, z, t)$, then by applying chain rule:

$$df = \frac{\partial f}{\partial t}\, dt + \frac{\partial f}{\partial x}\, dx + \frac{\partial f}{\partial y}\, dy + \frac{\partial f}{\partial z}\, dz \tag{2.13}$$

Divide both sides by dt:

$$\frac{df}{dt} = \frac{\partial f}{\partial t}\,\frac{dt}{dt} + \frac{\partial f}{\partial x}\,\frac{dx}{dt} + \frac{\partial f}{\partial y}\,\frac{dy}{dt} + \frac{\partial f}{\partial z}\,\frac{dz}{dt} \tag{2.14}$$

Substitute the following:

$$\frac{dx}{dt} = u\ , \quad \frac{dy}{dt} = v\ , \qquad \frac{dz}{dt} = w \tag{2.15}$$

We get:

$$\frac{df}{dt} = \frac{\partial f}{\partial t} + \frac{\partial f}{\partial x}\, u + \frac{\partial f}{\partial y}\, v + \frac{\partial f}{\partial z}\, w \tag{2.16}$$

Which is exactly same as (2.10).

2.2 Reynolds Transport Theorem

Reynolds transport method allows using both system and control volume concepts to describe fluid motion.

Control volume: known as a *Eulerian* approach which focuses on a fixed region of interest. Control volume can change the size and shape, and mass might cross the boundary of the control volume.

System: known as *Lagrangian,* which focuses on fixed packet of mass. The system can change size and shape, but have fixed mass packet which means no mass can cross the boundary.

Let's consider a control fluid volume in a fixed place and the system is a certain amount of fluid. At some time (t) both the system and the control volume take up the same space see figure 2.2. However at some time later $(t + dt)$ we noticed that control volume stays the same, but the system which is that amount of fluid we're looking for have moved and they are no longer identical. So, Reynolds transport theorem used to link both system and control volume.

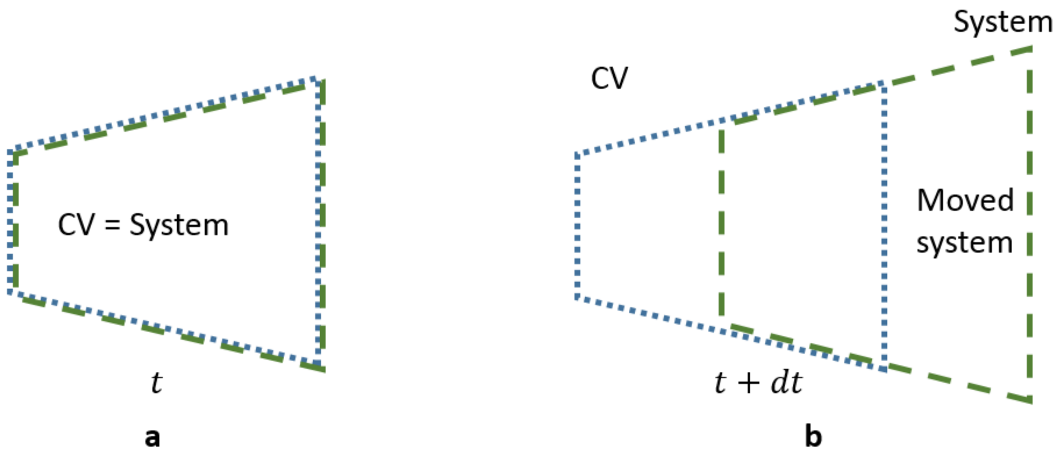

Figure 2.2: control volume and system. a. Control volume and system are identical. b. at a certain period of the system have moved

The challenge we face with the control volume is that the physics laws and expression applied to system only. For example newton's second law:

$$\sum F_{sys} = m_{sys}\, a_{sys} \tag{2.17}$$

$$\sum F_{CV} \neq m_{CV}\, a_{CV} \tag{2.18}$$

Let's start by defining B as any extensive property such as mass, momentum, volume, energy. In addition, b is the intensive property which is defined as following:

$$b = \frac{B}{m} \tag{2.19}$$

We can start by integral over a volume for intensive property as following:

$$B_{sys} = \int \rho b \, d\forall \tag{2.20}$$

At initial time t:

$$B_{sys}(t) = B_{CV}(t) \tag{2.21}$$

By looking at figure 2.3 at time $t + dt$:

I – mass that enters the CV to fill the space of the left system.

II- a portion of the system that is no longer in CV.

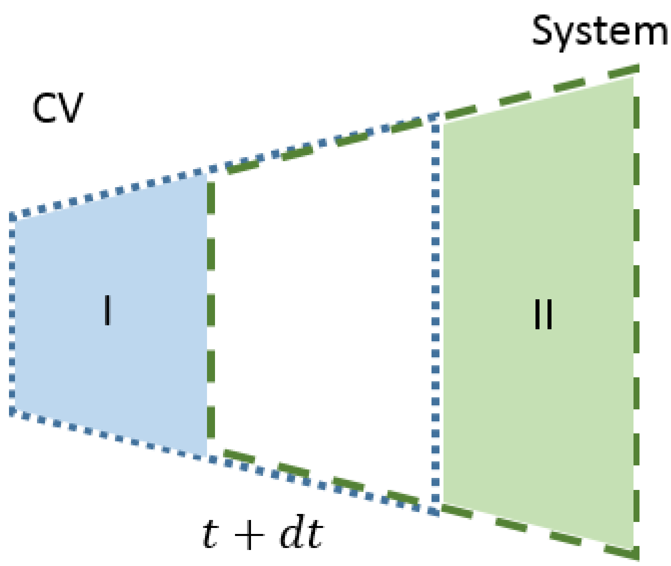

Figure 2.3: I- mass that enters the CV to fill the space of the left system. II- a portion of the system that is no longer in CV.

Which lead to the following:

$$B_{sys}(t + dt) = B_{CV}(t + dt) - B_I(t + dt) + B_{II}(t + dt) \tag{2.22}$$

The change of B_{sys} with respect to time can be expressed as following:

$$\frac{dB_{sys}}{dt} = \frac{B_{sys}(t+dt) - B_{sys}(t)}{dt} = \frac{B_{CV}(t+dt) - B_{CV}(t) - B_I(t+dt) + B_{II}(t+dt)}{dt} \tag{2.23}$$

By taking the limit of the system and control volume when $dt \to 0$:

$$\lim_{dt \to 0} \frac{B_{sys}(t+dt) - B_{sys}(t)}{dt} = \frac{DB_{sys}}{Dt} \quad (2.24)$$

$$\lim_{dt \to 0} \frac{B_{CV}(t+dt) - B_{CV}(t)}{dt} = \frac{\partial B_{CV}}{\partial t} \quad (2.25)$$

Now taking the limit for quantities that crossed the boundary:

$$\lim_{dt \to 0} \frac{-B_I(t+dt)}{dt} = -\rho A_1 V_1 b_1 = \dot{B_{in}} \quad (2.26)$$

$$\lim_{dt \to 0} \frac{B_{II}(t+dt)}{dt} = \rho A_2 V_2 b_2 = \dot{B_{out}} \quad (2.27)$$

Substitute back to (2.23):

$$\frac{DB_{sys}}{Dt} = \frac{\partial B_{CV}}{\partial t} + \dot{B}_{out} - \dot{B_{in}} \quad (2.28)$$

$$\{change\ within\ system\} = \{change\ with\ CV\} + \{stuff\ crossed\ boundary\}$$

Since this works only with uniform surfaces, areas, and velocity, we need to generalize (2.5). Let’s consider an arbitrary control surface which is a boundary for a control volume as per Figure 2.4.

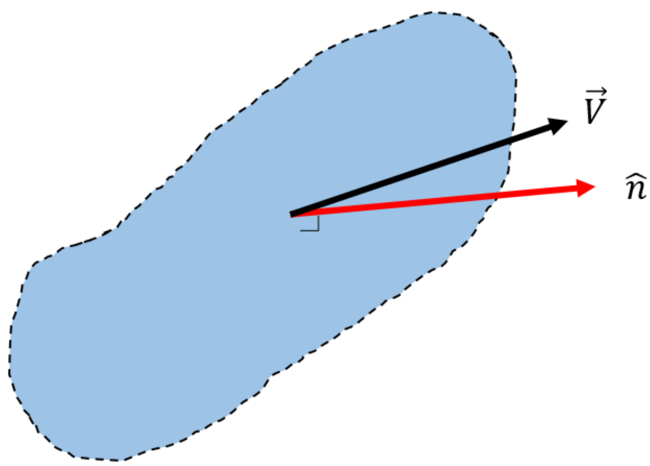

Figure 2.4: arbitrary control surface with unit vector ($\hat{n}$) normal to the surface.

$\hat{n}$: is unit vector and its normal to the surface.

Based on that we can have:

$$\dot{B} = \int_{CS} \rho \, b \, V.\hat{n} \, dA \tag{2.29}$$

$V.\hat{n}$: can cover the inlet and outlet, since the dot product will be negative if the velocity opposite to $\hat{n}$ direction.

Also, we may write $\frac{\partial B_{CV}}{\partial t}$ as following:

$$\frac{\partial B_{CV}}{\partial t} = \frac{\partial}{\partial t} \int_{CV} \rho \, b \, d\forall \tag{2.30}$$

So by substituting back to (2.5):

$$\frac{DB_{sys}}{Dt} = \frac{\partial}{\partial t} \int_{CV} \rho\, b\, d\forall + \int_{CS} \rho\, b\, V.\hat{n}\, dA \qquad (2.31)$$

$$\{change\ within\ system\} = \{change\ with\ CV\} + \{stuff\ crossed\ boundary\}$$

(2.32)

$$\{\text{Lagrangian}\} = \{\text{Eulerian}\} + \{\text{difference between Lagrangian \&Eulerian}\}$$

(2.33)

So (2.31) called Reynolds Transport theorem which describes the transport and conservation of b and relate Lagrangian and Eulerian perspective.

2.3 Conservation of Mass

Conservation of mass can be approached by both the finite control volume (*Eulerian*) model and fluid parcel models (*Lagrangian*) of the flow. This will help in understanding both models physical nature [3].

Eulerian model:

Let's start with Eulerian model and the fixed control volume is illustrated in figure 2.5. The flow velocity vector is $\vec{V}$, the vector element for the control volume is $d\forall$. In addition, the vector element for the surface area is dS. The physical principle for conservation of mass is applied to this control volume which mean the following: time rate of decrease of mass inside control volume

$$\begin{Bmatrix} \text{Net mass flow } out \text{ of control} \\ \text{volume through surface } S \end{Bmatrix} = \begin{Bmatrix} \text{time rate of decrease of mass} \\ \text{inside control volume} \end{Bmatrix} \qquad (2.34)$$

We can express the LHS of (2.34) as mass flow rate = (density) X (corss section area or dS) X (perpendicular velocity vector to dS) as shown below:

$$\dot{m} = \rho \, \vec{V} \, . \, \vec{d}S \qquad (2.35)$$

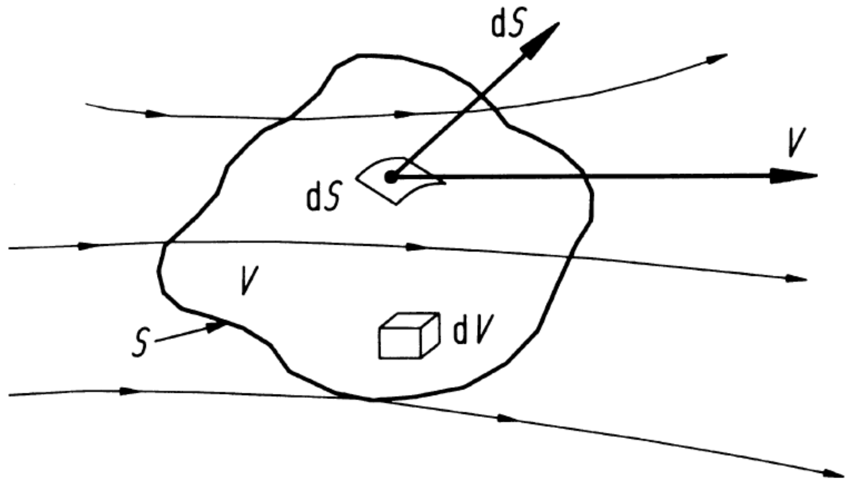

Figure 2.5: Eulerian model and the fixed control volume [3].

Since $\vec{dS}$ is always pointing outward of the control surface:

$$\vec{V}\, pointing\ outward \rightarrow \vec{V} \,.\, \vec{dS} > 0 \rightarrow \begin{pmatrix} flow\ is\ leaving\ the\ control \\ volume\ (outflow) \end{pmatrix}$$

$$\vec{V}\, pointing\ inward \rightarrow \vec{V} \,.\, \vec{dS} < 0 \rightarrow \begin{pmatrix} flow\ is\ entering\ the\ control \\ volume\ (inflow) \end{pmatrix}$$

Based on that the net mass flow rate outside a control volume which is the LHS of (2.7) can be expressed as integral over a control surface:

$$\oiint \rho\, \vec{V} \,.\, \vec{dS} \tag{2.36}$$

Now let's consider the RHS of (2.7), the mass within $d\forall$ is $\rho\, d\forall$. Therefore, the total mass inside control volume is:

$$\oiiint \rho \, d\forall \tag{2.37}$$

By considering the time rate of mass decrease:

$$-\frac{\partial}{\partial t}\oiiint \rho \, d\forall \tag{2.38}$$

Which is representing the RHS of (2.34). Substitute (2.36) and (2.38) back in (2.34):

$$\oiint \rho \, \vec{V} \, . \, \vec{d}S = -\frac{\partial}{\partial t}\oiiint \rho \, d\forall \tag{2.39}$$

Or

$$\frac{\partial}{\partial t}\oiiint \rho \, d\forall \, + \oiint \rho \, \vec{V} \, . \, \vec{d}S = 0 \tag{2.40}$$

Since we have fixed control volume, we can move $\partial/_{\partial t}$ inside the integration:

$$\oiiint \frac{\partial \rho}{\partial t} \, d\forall \, + \oiint \rho \, \vec{V} \, . \, \vec{d}S = 0 \tag{2.41}$$

Apply Gauss divergence theorem to the second term:

$$\oiint (\rho\,\vec{V}).\ \vec{d}S = \oiiint \nabla.(\rho\,\vec{V})\, d\forall \qquad (2.42)$$

Substitute (2.42) into (2.41):

$$\oiiint \frac{\partial\rho}{\partial t}\, d\forall \ + \ \oiiint \nabla.(\rho\,\vec{V})\, d\forall = 0 \qquad (2.43)$$

Which can be simplified as following:

$$\oiiint \left[\frac{\partial\rho}{\partial t} + \nabla.(\rho\,\vec{V})\ \right] d\forall \ = 0 \qquad (2.44)$$

Since we have arbitrary control volume as shown in figure 2.5, the only way of (2.44) to be zero is by having everything inside the brackets equal zero. This leads to the continuity equation which is:

$$\frac{\partial\rho}{\partial t} + \nabla.\left(\rho\,\vec{V}\right) = 0 \qquad (2.45)$$

The second term can be expanded as following:

$$\nabla.\left(\rho\,\vec{V}\right) = \rho\nabla.\left(\,\vec{V}\right) + \ \vec{V}\,\nabla.(\rho) \qquad (2.46)$$

Substitute back in (2.45)

$$\frac{\partial\rho}{\partial t} + \rho\nabla.\left(\,\vec{V}\right) + \ \vec{V}\,\nabla.(\rho) = 0 \qquad (2.47)$$

We noticed that the first and second terms were the material derivative for the density. So (2.47) can be written as following:

$$\frac{D\rho}{Dt} + \rho\nabla.\vec{V} = 0 \tag{2.48}$$

Lagrangian model:

Now we'll consider moving fluid parcel approach which illustrated by Figure 2.6. Since the mass is fixed for the fluid parcel δm and the parcel volume is $\delta\forall$, we can have the following:

$$\delta m = \rho\, \delta\forall \tag{2.49}$$

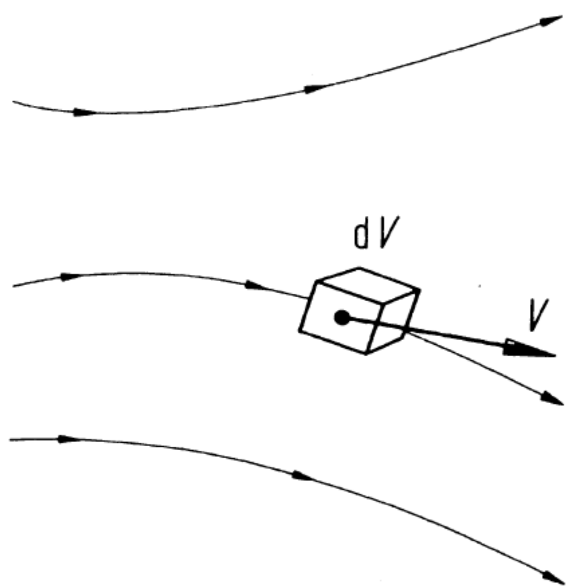

Figure 2.6: Lagrangian model with moving fluid parcel approach [3].

By considering a conserved mass for the fluid parcel, we can say the time rate of change of mass is equal to zero. Taking into account the material derivative we end up with the following:

$$\frac{D(\delta m)}{Dt} = 0 \tag{2.50}$$

Substitute (2.49) into (2.50):

$$\frac{D(\rho\ \delta\forall\ )}{Dt} = 0 \tag{2.51}$$

By applying the chain rule:

$$\frac{D(\rho\ \delta\forall\ )}{Dt} = \ \delta\forall\ \frac{D\rho}{Dt} + \rho\frac{D(\delta\forall\ )}{Dt} = 0 \tag{2.52}$$

Divide both terms by $\delta\forall$:

$$\frac{D\rho}{Dt} + \rho\left[\frac{1}{\delta\forall}\frac{D(\delta\forall\ )}{Dt}\right] = 0 \tag{2.53}$$

Replace $\frac{1}{\delta\forall}\frac{D(\delta\forall\ )}{Dt}$ with$\nabla.\vec{V}$, since they are equal. This leads us to the continuity equation:

$$\frac{D\rho}{Dt} + \rho\nabla.\vec{V} = 0 \tag{2.54}$$

Once again we were able to drive the continuity equation (2.148) and (2.54) by using fixed control volume (Eulerian) and moving fluid parcel (Lagrangian) models, and we end ups with the same conclusion.

2.4 Conservation of Momentum

The conservation of momentum based on a basic physical principle which is:

$\vec{F} = m\vec{a}$ Newton's second law

Based on that moving fluid parcel model will be used which is shown in figure 2.7 with all the forces in x-direction [3].

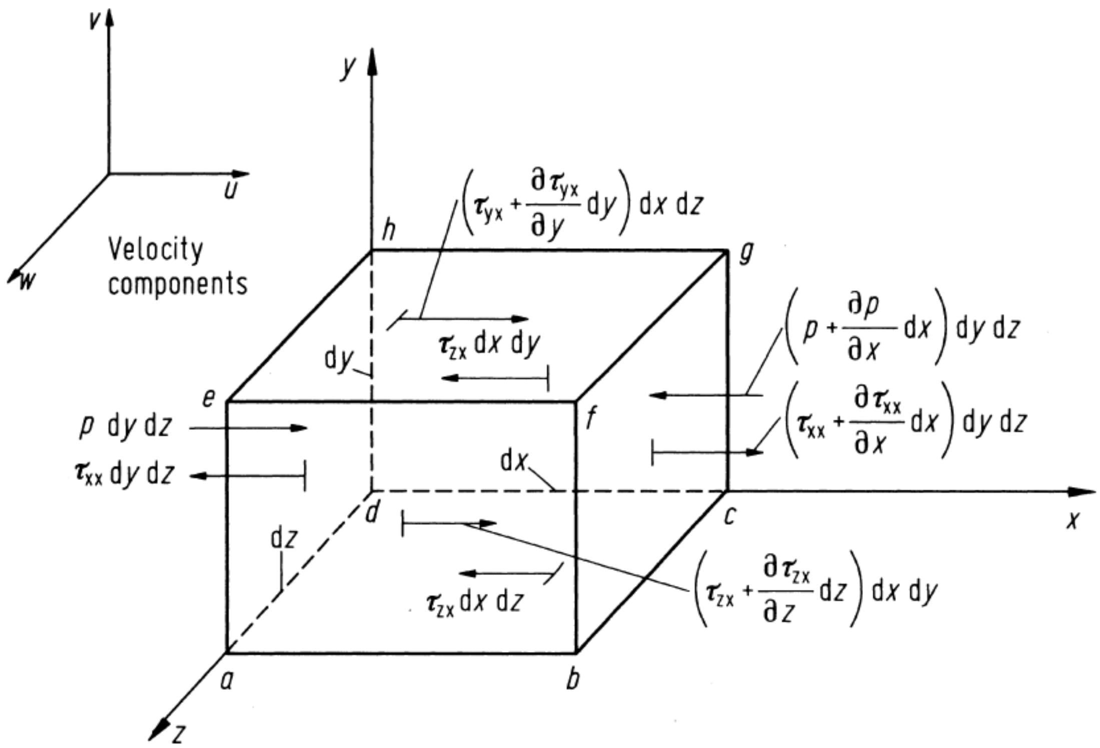

Figure 2.7: Moving fluid parcel showing all the forces in x-direction [3].

Newton's second law can be expressed for the moving fluid parcel as the net forces acting on the fluid parcel equal to fluid parcel mass multiplied by acceleration acting on the fluid parcel. The forces are acting on x,y and z directions, but we are considering x-component only as following:

$$F_x = ma_x \tag{2.55}$$

Let’s start by the LHS of (2.55) which the net force acting on x-direction. There are two types of forces. The first type is Body force; these forces are acting on the volumetric mass such as gravitational, magnetic and electrical forces. The second type is surface force; these forces are acting on the surface of the fluid parcel, and they can be caused by:

a) Pressure acting the surrounding area of the fluid parcel and acting like a force on the fluid parcel surface.

b) Normal and shear stress forces are acting on the fluid parcel surface. This mainly caused by fluid motion which leads to friction force on the fluid parcel.

For the body force, let’s consider the force per unit mass $\overrightarrow{f_x} = \vec{F_x}/_{m}$. The fluid parcel volume is $(dx\, dy\, dz)$. So we can get the following: Net body forces acting on fluid parcel on x-direction

$$\begin{Bmatrix} net\ body\ forces\ acting \\ on\ fluid\ parcel \\ for\ x-direction \end{Bmatrix} = \rho\, f_x\, (dx\, dy\, dz) \tag{2.56}$$

For the surface force, let’s start by the force caused by pressure distribution around the fluid parcel in x-direction only. As shown in figure 2.7, the pressure acting on the surface (*adhe)* and (*bcgf)* of the fluid parcel. Since force equal pressure times cross-section area perpendicular to that force which is $dy\,dz$ in this case, we end up with the following:

$$\begin{Bmatrix} net\ surface\ forces\ caused \\ by\ pressure\ distribution \\ acting\ in\ x-direction \end{Bmatrix} = \left[p - \left(p + \frac{\partial p}{\partial x}dx\right)\right] dy\ dz \tag{2.57}$$

For the normal and shear stress as surface force, will start by normal stress τ_{xx} which acting perpendicular on surface (*adhe)* and (*bcgf)* as shown in figure 2.7. For the force caused by normal stress equal normal stress times cross section area perpendicular to that force which is $dy\,dz$ in this case. We end up with the following:

$$\begin{Bmatrix} net\ surface\ forces\ caused \\ by\ normal\ stress \\ acting\ in\ x-direction \end{Bmatrix} = \left[\left(\tau_{xx} + \frac{\partial \tau_{xx}}{\partial x}dx\right) - \tau_{xx}\right] dy\ dz \tag{2.58}$$

Now for the shear stress, which acting parallel to surface (*abfe),* (*cghd), (efgh)* and *(abcd)* as shown in figure 2.7. For the force caused by shear stress, equal shear stress times cross section area parallel to that force which is $dx\,dy$ for (*abfe) and* (*cghd)* and $dx\,dz$ for *(efgh)* and *(abcd).* We end up with the following:

$$\begin{Bmatrix} net\ surface\ forces\ caused \\ by\ shear\ stress \\ acting\ in\ x-direction \end{Bmatrix} = \left[\left(\tau_{yx} + \frac{\partial \tau_{yx}}{\partial y} dy\right) - \tau_{yx}\right] dx\,dz$$

$$+ \left[\left(\tau_{zx} + \frac{\partial \tau_{zx}}{\partial z} dz\right) - \tau_{zx}\right] dx\,dy \qquad (2.59)$$

Now we can combine all the surface force under one equation from (2.57), (2.58) and (2.59):

$$\begin{Bmatrix} net\ surface\ forces \\ acting\ in\ x-direction \end{Bmatrix} = \left[p - \left(p + \frac{\partial p}{\partial x} dx\right)\right] dy\,dz$$

$$+ \left[\left(\tau_{xx} + \frac{\partial \tau_{xx}}{\partial x} dx\right) - \tau_{xx}\right] dy\,dz$$

$$+ \left[\left(\tau_{yx} + \frac{\partial \tau_{yx}}{\partial y} dy\right) - \tau_{yx}\right] dx\,dz$$

$$+ \left[\left(\tau_{zx} + \frac{\partial \tau_{zx}}{\partial z} dz\right) - \tau_{zx}\right] dx\,dy \qquad (2.60)$$

Which can be simplified as following:

$$\begin{Bmatrix} net\ surface\ forces \\ acting\ in\ x-direction \end{Bmatrix} = \left[-\frac{\partial p}{\partial x}dx\right]dy\,dz \quad + \left[\frac{\partial \tau_{xx}}{\partial x}dx\right]dy\,dz$$

$$+ \left[\frac{\partial \tau_{yx}}{\partial y}dy\right]\ dx\,dz + \left[\frac{\partial \tau_{zx}}{\partial z}dz\right]\ dx\,dy \qquad (2.61)$$

$$\begin{Bmatrix} net\ surface\ forces \\ acting\ in\ x-direction \end{Bmatrix} = \left(-\frac{\partial p}{\partial x} + \frac{\partial \tau_{xx}}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z}\right)dx\,dy\,dz \qquad (2.62)$$

The total force in x-direction is the summation of (2.22) and (2.26), which leads to the following:

$$F_x = \left(-\frac{\partial p}{\partial x} + \frac{\partial \tau_{xx}}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z}\right)dx\,dy\,dz + \rho\, f_x\,(dx\,dy\,dz) \qquad (2.63)$$

Now we have the LHS of (2.55) as shown in (2.63), let's look for the RHS of (2.55). Since we are considering a moving fluid parcel with fixed mass, we can calculate the mass as following:

$$m = \rho\, dx\, dy\, dz \qquad (2.64)$$

For the acceleration part, since the acceleration is time rate of change of its velocity which is u in x- direction. By taking into consideration the material derivative the acceleration can be found as following:

$$a_x = \frac{Du}{Dt} \qquad (2.65)$$

Now let's substitute (2.63), (2.28) and (2.65) back into (2.55):

$$\left(-\frac{\partial p}{\partial x}+\frac{\partial \tau_{xx}}{\partial x}+\frac{\partial \tau_{yx}}{\partial y}+\frac{\partial \tau_{zx}}{\partial z}\right) dx\, dy\, dz+\rho\, f_x\, dx\, dy\, dz=(\rho\, dx\, dy\, dz)\frac{Du}{Dt} \quad (2.66)$$

Divide both sides by $dx\, dy\, dz$:

$$\left(-\frac{\partial p}{\partial x}+\frac{\partial \tau_{xx}}{\partial x}+\frac{\partial \tau_{yx}}{\partial y}+\frac{\partial \tau_{zx}}{\partial z}\right)+\rho\, f_x \;=\rho\frac{Du}{Dt} \quad (2.67)$$

So the conservation of momentum equation for x- direction is:

$$\rho\frac{Du}{Dt}=-\frac{\partial p}{\partial x}+\frac{\partial \tau_{xx}}{\partial x}+\frac{\partial \tau_{yx}}{\partial y}+\frac{\partial \tau_{zx}}{\partial z}+\rho\, f_x \quad (2.68a)$$

For y- direction:

$$\rho\frac{Dv}{Dt}=-\frac{\partial p}{\partial y}+\frac{\partial \tau_{xy}}{\partial x}+\frac{\partial \tau_{yy}}{\partial y}+\frac{\partial \tau_{zy}}{\partial z}+\rho\, f_y \quad (2.68b)$$

For z- direction:

$$\rho\frac{Dw}{Dt}=-\frac{\partial p}{\partial z}+\frac{\partial \tau_{xz}}{\partial x}+\frac{\partial \tau_{yz}}{\partial y}+\frac{\partial \tau_{zz}}{\partial z}+\rho\, f_z \quad (2.68c)$$

Equation (2.68a, b and c) are the x, y and z component of the conservation of momentum equation. These equation were driven directly from a basic physical principle which is newton's second law. They can be summarized in one equation by using tensor notation, as following:

$$\rho \frac{D\,u_j}{Dt} = -P_{,j} + \tau_{ij,i} + \rho f_j \tag{2.69}$$

Some books consider the pressure gradient as part of normal and shear stress term, so (2.69) can be expressed as following:

$$\rho \frac{D\,u_j}{Dt} = \sigma_{ij,i} + \rho f_j \tag{2.70}$$

Newtonian Fluid:

The stress tensor for Newtonian fluid can be determent based on the following conditions:

a) When fluid at rest, the stress is hydrostatic pressure (thermodynamic pressure).

$$\sigma_{ij} = \text{thermodynamic pressure} + \text{stress due to fluid motion}$$

$$\sigma_{ij} = -P\,\delta_{ij} + \tau_{ij} \tag{2.71}$$

b) Stress is linearly related to strain

$$\tau_{ij} = \gamma_{ijkl}\,e_{kl} \tag{2.72}$$

$$e_{kl} = u_{k,l} = (\varepsilon_{kl} + \xi_{kl}) \tag{2.73}$$

c) No shear stress act during motion

$$\tau_{ij} = \epsilon_{ijkl}\varepsilon_{kl} = {}^{1}/_{2}\, \epsilon_{ijkl}\,(u_{k,l} + u_{l,k}) \tag{2.74}$$

d) Isotropy: independent on the orientation of coordinate system

ϵ_{ijkl} *has to be isotropic*

$$\tau_{ij} = \lambda\delta_{ij}u_{k,k} + \mu(u_{i,j}+u_{j,i}) \tag{2.75}$$

So, the stress tensor for Newtonian will be:

$$\sigma_{ij} = -P\,\delta_{ij} + \lambda\delta_{ij}u_{k,k} + \mu(u_{i,j}+u_{j,i}) \tag{2.76}$$

Based on (2.54), for incompressible fluid ${}^{D\rho}/_{Dt} = 0$ since the density is constant. Thus, $u_{k,k} = 0$ for incompressible fluid. λ is the second coefficient of viscosity (related to bulk viscosity). So (2.76) will be as following for incompressible Newtonian fluid:

$$\sigma_{ij} = -P\,\delta_{ij} + \mu(u_{i,j}+u_{j,i}) \tag{2.77}$$

Substitute back in (2.70):

$$\rho\frac{D\,u_j}{Dt} = [-P\,\delta_{ij} + \mu\big(u_{i,j}+u_{j,i}\big)\,]_{,i} + \rho f_j \tag{2.78}$$

Let's simplify the first term on RHS, $[-P\,\delta_{ij} + \mu\big(u_{k,l}+u_{l,k}\big)\,]_{,i}$:

$$[-P\,\delta_{ij} + \mu\big(u_{k,l}+u_{l,k}\big)\,]_{,i} = -P_{,i}\,\delta_{ij} + \mu\big(u_{i,ji}+u_{j,ii}\big) \tag{2.79}$$

$$u_{i,ji} = u_{i,ij} = zero\ for\ incomresable\ fluid$$

$$\rho \frac{D\,u_j}{Dt} = -P_{,j} + \mu\, u_{j,ii} + \rho f_j \tag{2.80}$$

This conservation of momentum for incompressible Newtonian fluid which is also called Navier stoke equation, and below are the expanded equation for x, y, and z-direction:

$$\rho\left(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z}\right) = -\frac{\partial P}{\partial x} + \mu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right) + \rho f_x \tag{2.81a}$$

$$\rho\left(\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z}\right) = -\frac{\partial P}{\partial y} + \mu\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial z^2}\right) + \rho f_y \tag{2.81b}$$

$$\rho\left(\frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z}\right) = -\frac{\partial P}{\partial z} + \mu\left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 w}{\partial z^2}\right) + \rho f_z \tag{2.81c}$$

Non-Newtonian Fluid:

In a Newtonian fluid, the relation is linear between the shear stress and the shear rate , passing through the origin, the coefficient of viscosity is the constant of proportionality. In a non-Newtonian fluid, the relation between the shear stress and the shear rate is different, and can even be time-dependent. The shear stress for non-Newtonian fluid:

$$\tau = \tau_0 + K\dot{\gamma}^n \qquad (2.82)$$

$\dot{\gamma}$: Share rate $\dot{\gamma} = \frac{du}{dy} = u_{j,i}$

K: Consistency index

n: Flow behavior index, $n = 1$ for Newtonian fluid

τ_0: yield stress

For power law fluid $\tau_0 = zero$, and the shear stress can be expressed as following:

$$\tau = K\dot{\gamma}^n \qquad (2.83)$$

Most of the hydraulic fracturing fluid considered as to obey power low which will be dealing with in this research. For Bingham fluid $n = 1$, and the shear stress can be expressed as following:

$$\tau = \tau_0 + K\dot{\gamma} \qquad (2.84)$$

2.5 Conservation of Energy

Energy equation will be driven by using moving fluid parcel model (Lagrangian). First law of thermodynamic is the basic principle for driving the energy equation for moving fluid parcel, which can be represented as following [3]:

$$\begin{Bmatrix} \text{energy rate of} \\ \text{change inside} \\ \text{fluid parcel} \end{Bmatrix} = \begin{Bmatrix} \text{net heat flux} \\ \text{into fluid parcel} \end{Bmatrix} + \begin{Bmatrix} \text{rate of work done} \\ \text{caused by body or} \\ \text{surface force} \end{Bmatrix} \qquad (2.85)$$

For the first term in (2.85), the total energy of moving fluid element per unit mass is the sum of its total internal energy per mass unit e, and the kinetic energy per unit mass $V^2/2$. So the total energy can be expresses as $e + V^2/2$. Since a moving fluid parcel is considered, time rate of change for energy per unit mass id given by material derivative. Fluid parcel mass is $\rho\, dx\, dy\, dz$, so the LHS of (2.31) can be expressed as following:

$$\begin{Bmatrix} \text{energy rate of} \\ \text{change inside} \\ \text{fluid parcel} \end{Bmatrix} = \rho \frac{D}{Dt}\left(e + \frac{V^2}{2}\right) dx\, dy\, dz \qquad (2.86)$$

For the first term in RHS of (2.85), which represent the net heat flux into a fluid parcel. This flux of heat can occur due to:

a) Volumetric heat like absorption or emission of radiation.

b) Transfer of heat across the surface due to temperature gradient.

For the volumetric heat consider $\dot{q}$ as rate of volume of heat added per unit mass which illustrated in figure 2.8. Fluid parcel mass is $\rho\, dx\, dy\, dz$, so Volumetric heat can be expressed as following:

$$\begin{Bmatrix} volumetric\ heat\ of \\ fluid\ parcel \end{Bmatrix} = \rho\, \dot{q}\ \ dx\, dy\, dz \qquad (2.87)$$

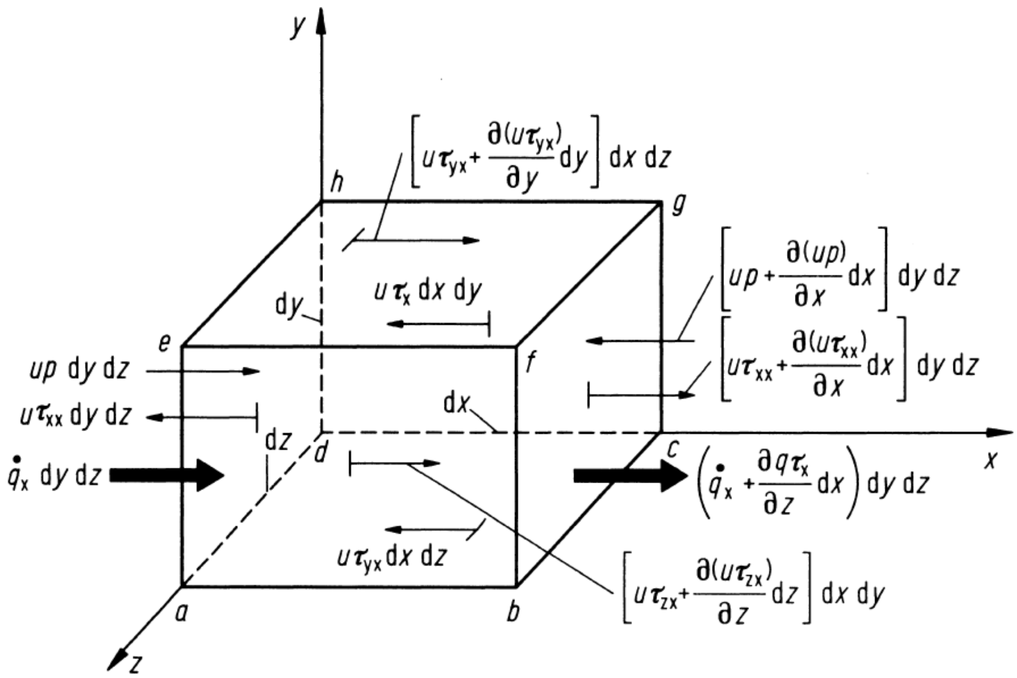

Figure 2.8: Energy flux across moving fluid parcel in x direction [3].

For the heat transfer due to temperature gradient which is referred to as thermal conductivity. $\dot{q}_x$ is the heat transfer by thermal conductivity per unit time per unit area in x- direction. Based on figure 2.8, the heat transfer into the fluid parcel through surface (*adhe) is* $\dot{q}_x\, dy\, dz.$ The heat transfer out of the fluid parcel through surface (*bcgf) is* $\left[\dot{q}_x + \frac{\partial\, \dot{q}_x}{\partial x}\, dx\right] dy\, dz.$ Therefore, the net heat transfer in x- direction through the fluid parcel by thermal conductivity is:

$$\left[\dot{q}_x - \left(\dot{q}_x + \frac{\partial\,\dot{q}_x}{\partial x}\;\,dx\right)\right] dy\,dz = -\frac{\partial\,\dot{q}_x}{\partial x}\;\,dx\,dy\,dz \qquad (2.88)$$

By taking into consideration y and z-direction, we end up with the following:

$$\begin{Bmatrix} heat\ transfer \\ for\ fluid\ parcel\ by \\ thermal\ conductivity \end{Bmatrix} = -\left(\frac{\partial\,\dot{q}_x}{\partial x} + \frac{\partial\,\dot{q}_y}{\partial y} + \frac{\partial\,\dot{q}_z}{\partial z}\right)\;dx\,dy\,dz \qquad (2.89)$$

Combining (2.33) and (2.34) represents the RHS first term of (2.31):

$$\begin{Bmatrix} net\ heat\ flux \\ into\ fluid\ parcel \end{Bmatrix} = \left[\rho\,\dot{q} - \left(\frac{\partial\,\dot{q}_x}{\partial x} + \frac{\partial\,\dot{q}_y}{\partial y} + \frac{\partial\,\dot{q}_z}{\partial z}\right)\right]\;dx\,dy\,dz \qquad (2.90)$$

Since heat transfer by thermal conductivity is related to temperature gradient as following:

$$\dot{q}_x = -k\frac{\partial T}{\partial x}\,, \qquad \dot{q}_y = -k\frac{\partial T}{\partial y}\,,\;\; \dot{q}_z = -k\frac{\partial T}{\partial z} \qquad (2.91)$$

So, (2.35) can be represented as:

$$\begin{Bmatrix} net\ heat\ flux \\ into\ fluid\ parcel \end{Bmatrix} = \left[\rho\, \dot{q} + \frac{\partial}{\partial x}\left(k\frac{\partial T}{\partial x}\right) + \frac{\partial}{\partial y}\left(k\frac{\partial T}{\partial y}\right) + \frac{\partial}{\partial z}\left(k\frac{\partial T}{\partial z}\right)\right]\ dx\, dy\, dz \tag{2.92}$$

For the second term in RHS of (2.85), which represent the rate of work done due to body or surface force on moving fluid parcel. Work is equal to force multiplied by distance, and the power is force multiplied by velocity, so power is the rate of work. Since f is force per unit mass we need to consider fluid parcel mass which is $\rho\, dx\, dy\, dz$. So the rate of work done by body force on moving fluid parcel can be expressed as following:

$$\begin{Bmatrix} rate\ of\ work\ done \\ by\ body\ force \end{Bmatrix} = \rho\, \vec{f}.\vec{V}\ \ dx\, dy\, dz \tag{2.93}$$

For the surface forces which are done by pressure normal and shear stress as shown in figure 2.7. For x- the direction the rate of work done by all the force equal force multiplied by u. So for force caused by pressure disruption around moving fluid parcel on surface (*adhe) is* $u\, P\, dy\, dz$*.* The net rate of work done by pressure on x- direction which are on surface (*adhe)* and *(bcfg) as shown on figure 2.8* can be represented as following:

$$\left[uP - \left(uP + \frac{\partial(uP)}{\partial x}dx\right)\right] dy\,dz = -\frac{\partial(uP)}{\partial x}\,dx\,dy\,dz \tag{2.94}$$

The same can be applied for the normal and shear stress force. The rate of work done by shear stress in x-direction which are applied on surface (abcd) and (efgh) as shown in figure 2.8 can be represented as following:

$$\left[\left(u\tau_{yx} + \frac{\partial(u\tau_{yx})}{\partial y}dy\right) - u\tau_{yx}\right] dx\,dz = \frac{\partial(u\tau_{yx})}{\partial y}\,dx\,dy\,dz \tag{2.95}$$

All the surface forces which represented on figure 2.8 can be expressed as following:

$$\begin{Bmatrix} rate\ of\ work\ for \\ all\ surface\ forces \\ in\ x-direction \end{Bmatrix} = \left[-\frac{\partial(uP)}{\partial x} + \frac{\partial(u\tau_{xx})}{\partial x}\frac{\partial(u\tau_{yx})}{\partial y}\frac{\partial(u\tau_{zx})}{\partial z}\right] dx\,dy\,dz \tag{2.96a}$$

Sam can be applied for y and z directions:

$$\begin{Bmatrix} rate\ of\ work\ for \\ all\ surface\ forces \\ in\ y-direction \end{Bmatrix} = \left[-\frac{\partial(vP)}{\partial y} + \frac{\partial(v\tau_{xy})}{\partial x}\frac{\partial(v\tau_{yy})}{\partial y}\frac{\partial(v\tau_{zy})}{\partial z}\right] dx\,dy\,dz \tag{2.96b}$$

$$\begin{Bmatrix} rate\ of\ work\ for \\ all\ surface\ forces \\ in\ z-direction \end{Bmatrix} = \left[-\frac{\partial(wP)}{\partial z} + \frac{\partial(w\tau_{xz})}{\partial x}\frac{\partial(w\tau_{yz})}{\partial y}\frac{\partial(w\tau_{zz})}{\partial z}\right] dx\,dy\,dz \tag{2.96c}$$

The rate of work done by body and surface forces can be represented by combining (2.93), (2.96a), (2.96b) and (2.96c):

$$\begin{Bmatrix} rate\ of\ work\ done \\ caused\ by\ body\ or \\ surface\ force \end{Bmatrix} = \Big[-\left(\frac{\partial(uP)}{\partial x}+\frac{\partial(vP)}{\partial y}+\frac{\partial(wP)}{\partial z}\right)$$

$$+\ \frac{\partial(u\tau_{xx})}{\partial x}\frac{\partial(u\tau_{yx})}{\partial y}\frac{\partial(u\tau_{zx})}{\partial z}$$

$$+\frac{\partial(v\tau_{xy})}{\partial x}\frac{\partial(v\tau_{yy})}{\partial y}\frac{\partial(v\tau_{zy})}{\partial z}$$

$$+\frac{\partial(w\tau_{xz})}{\partial x}\frac{\partial(w\tau_{yz})}{\partial y}\frac{\partial(w\tau_{zz})}{\partial z}\Big]\,dx\,dy\,dz$$

$$+\rho\,\vec{f}.\vec{V}\ \ dx\,dy\,dz \qquad (2.97)$$

The final form of energy equation can be represented by substituting (2.86), (2.92) and (2.97) into (2.85), so we end up with the following:

$$\rho\ \frac{D}{Dt}\left(e+\frac{V^2}{2}\right) = \ \rho\,\dot{q}+\frac{\partial}{\partial x}\left(k\frac{\partial T}{\partial x}\right)+\frac{\partial}{\partial y}\left(k\frac{\partial T}{\partial y}\right)+\frac{\partial}{\partial z}\left(k\frac{\partial T}{\partial z}\right)$$

$$+\Big[-\left(\frac{\partial(uP)}{\partial x}+\frac{\partial(vP)}{\partial y}+\frac{\partial(wP)}{\partial z}\right)$$

$$+\ \frac{\partial(u\tau_{xx})}{\partial x}\frac{\partial(u\tau_{yx})}{\partial y}\frac{\partial(u\tau_{zx})}{\partial z}$$

$$+\frac{\partial(v\tau_{xy})}{\partial x}\frac{\partial(v\tau_{yy})}{\partial y}\frac{\partial(v\tau_{zy})}{\partial z}$$

$$+\frac{\partial(w\tau_{xz})}{\partial x}\frac{\partial(w\tau_{yz})}{\partial y}\frac{\partial(w\tau_{zz})}{\partial z}\Big]\,dx\,dy\,dz$$

$$+\rho\,\vec{f}.\vec{V}\ \ dx\,dy\,dz \qquad (2.98)$$

3. Laminar Flow and Fluid Rheology

Matter responded to applied external forces to its surface by deformation. Based on that, liquid and solid act and response in a similar way while gasses have less resistance. Newtonian fluid can be considered as a special case of Non-Newtonian fluid. In order to better understand the Non-Newtonian fluid behavior, we will focus on the laminar flow between two parallel plates. We will start by driving the general governing equation for Non-Newtonian fluid. Then will simplify these equation to be used for Newtonian, power law and Bingham plastic fluids [6].

The general shear stress equation for Non-Newtonian fluid:

$$\tau = \tau_y + K\,\dot{\gamma}^n \qquad (3.1)$$

τ_y: Yield stress (used with Bingham fluid)

K : Consistency index

$\dot{\gamma}$: shear rate

n : Flow behavior index

The force balance for flow in a pipe or parallel plates (figure 3.1) is the relation between wall stress and pressure drop.

$$L\,p\ \tau = A_c\,\Delta P \tag{3.2}$$

A_c: *cross section area*

p: *wetted perimeter*

$$\tau = \frac{A_c}{p}\frac{\Delta P}{L} \tag{3.3}$$

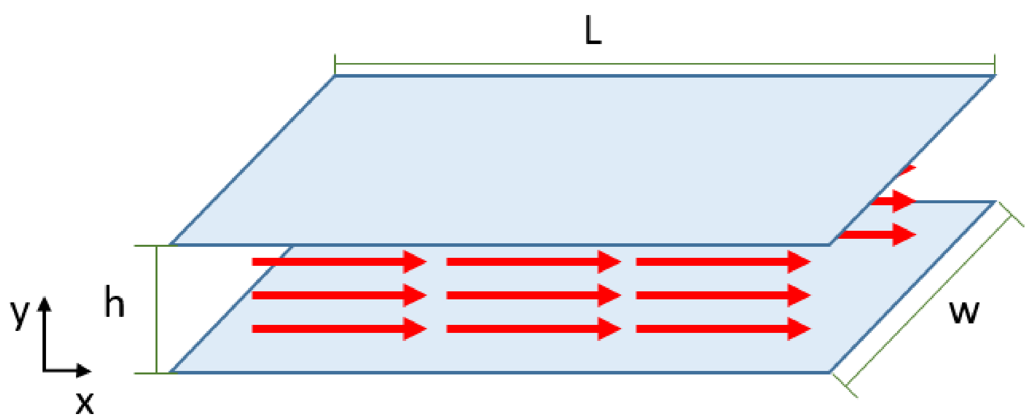

Figure 3.1: Schematic of two parallel plates flow geometry.

- For smaller cross section, since $\frac{A_c}{p}$ is linear and $\frac{\Delta P}{L}$ is constant, τ will change linearly with $\frac{A_c}{p}$.
- Stress ratio $\dot{\gamma} = \frac{du}{dy}$ can be obtained by velocity profile.
- Flow rate Q can be obtained by integrating velocity profile.

- Flow rate vs. pressure drop relation is the basic equation for flow geometry which can represent as the relation between friction factor and Reynolds number.
- Fanning friction factor defined as the proportional relation between wall stress and suitable approximation of kinetic energy [31].

$$\tau = f\ x\left(\tfrac{1}{2}\rho u_{avg}^2\right) \tag{3.4}$$

$$Re = \frac{\rho\ u_{avg}\ l}{\mu} = \frac{inertial\ force}{viscous\ force} \tag{3.5}$$

$$l = \frac{4\ A_c}{p} = \frac{cross\ section\ area}{wetted\ perimeter} \rightarrow hydrolic\ dimeter \tag{3.6}$$

For parallel plate flow with width w and height h, where $h/_{w} \rightarrow 0$. So,

$$\frac{A_c}{p} = \frac{hw}{2(h+w)} = \frac{h}{2} \tag{3.7}$$

The characteristic length $l = 2\,h$ (3.8)

Solution strategy will be as following:

a. Drive the velocity distribution.

b. Calculate the maximum velocity.

c. Calculate the rate.

d. Obtain the average velocity

e. Determine the quantities that are characterising the smoothness of the velocity profile.

For parallel plate flow, the force balance is

$$\tau_w = \frac{h\,\Delta P}{2\,L} \tag{3.9}$$

Since we have a linear relation for the shear stress, the shear stress at distance y from the wall is:

$$\tau = \frac{(\frac{h}{2} - y)}{(\frac{h}{2})}\,\tau_w \tag{3.10}$$

The increasing ratio of yield stress to the wall stress results in a wider plug for velocity profile.

$$\emptyset = \frac{\tau_y}{\tau_w} \tag{3.11}$$

At certain distance from the wall y_0, the fluid moving as fluid plug uniform velocity u_{max}.

$$y_0 = \frac{(1-\emptyset)h}{2} \tag{3.12}$$

The shear rate is the velocity derivative with respect to the distance from the wall

$$\dot{\gamma} = \frac{du}{dy} = \left(\frac{\tau - \tau_y}{K}\right)^{1/n} \tag{3.13}$$

Substitute shear stress from (3.10) to get a differential equation describing the velocity distribution.

$$\frac{du}{dy} = \left(\left(\frac{(\frac{h}{2} - y)}{(\frac{h}{2})}\,\tau_w - \tau_y\right)/K\right)^{1/n} \tag{3.14}$$

Integrate once from the wall to a given y and $u(0) = 0$ for no slip condition.

$$u(y) = \frac{n}{2(n+1)}\left\{h(1 - \emptyset)\left[\frac{\tau_w(1-\emptyset)}{K}\right]^{1/n} - [h(1 - \emptyset) - 2y]\left[\frac{\tau_w(1-\emptyset-2y/h)}{K}\right]^{1/n}\right\} \tag{3.15}$$

Maximum velocity obtained by substituting $y = y_0$:

$$u_{max} = \frac{n}{2(1+n)} h(1 - \emptyset)\left[\frac{\tau_w(1-\emptyset)}{K}\right]^{1/n} \tag{3.16}$$

The total flow rate is the sum of the flow rate in the plug $q_1$and the outer part q_2.

$$q_1 = w(h - 2y_0)u_{max} \tag{3.17}$$

$$q_2 = 2w\int_0^{y_0} u(y)\,dy \tag{3.18}$$

Substitute (3.15) and (3.16) into q_1 and q_2. After the integration the total flow rate is:

$$q = \frac{wh^2 n(1-\emptyset)(1+n+n\emptyset)}{2(1+n)(1+2n)} \left[\frac{\tau_w(1-\emptyset)}{K}\right]^{1/n} \tag{3.19}$$

So the average velocity is

$$u_{avg} = \frac{q}{hw} = \frac{h(1-\emptyset)(1+n+n\emptyset)}{2(1+n)(1+2n)} \left[\frac{\tau_w(1-\emptyset)}{K}\right]^{1/n} \tag{3.20}$$

The usual measure of velocity profile flatness can be obtained from (3.16) and (3.20):

$$\frac{u_{max}}{u_{avg}} = \frac{1+2n}{1+n+n\emptyset} \tag{3.21}$$

Flow rate as a function of the pressure gradient.

$$q = \frac{wh^2 n(1-\emptyset)(1+n+n\emptyset)}{2(1+n)(1+2n)} \left[\frac{h(1-\emptyset)}{2K} \frac{\Delta P}{L}\right]^{1/n} \tag{3.22}$$

From equation (3.4) and (3.20):

$$f = \frac{4(1+n)(1+2n)\, \tau_w^{1-1/n}}{u_{avg}\rho h(1-\emptyset)(1+n+n\emptyset)\left[\frac{(1-\emptyset)}{K}\right]^{1/n}} \tag{3.23}$$

For Newtonian slot flow $Re \times f = 24$, to generalize the Reynolds number for Non-Newtonian fluid let's preserve this property, therefore $Re^* \times f = 24$. From (3.16):

$$Re^* = \frac{6\, u_{avg}\rho h(1-\emptyset)(1+n+n\emptyset)\left[\frac{(1-\emptyset)}{K}\right]^{1/n}}{(1+n)(1+2n)\, \tau_w^{1-1/n}} \tag{3.24}$$

Based on $\tau = \tau_y + K\,\dot{\gamma}^n$ and $\emptyset = \frac{\tau_y}{\tau_w}$, the shear rate at the wall:

$$\dot{\gamma}_w = \left[\frac{(1-\emptyset)\tau_w}{K}\right]^{1/n} \tag{3.25}$$

And

$$\mu_w = \frac{\tau_w}{\dot{\gamma}} = \tau_w^{1-1/n}\left(\frac{K}{1-\emptyset}\right)^{1/n} \tag{3.26}$$

From (3.15) and (3.18):

$$u_{avg} = \frac{q}{hw} = \frac{h(1-\emptyset)(1+n+n\emptyset)}{2(1+n)(1+2n)}\dot{\gamma}_w \tag{3.27}$$

And from (3.13) and (3.17):

$$Re' = \frac{6\,u_{avg}\rho h(1-\emptyset)(1+n+n\emptyset)}{(1+n)(1+2n)\,\mu_w} \tag{3.28}$$

The shear rate for Newtonian fluid $\dot{\gamma}_w = \frac{6u_{avg}}{h}$ by using $\tau = \mu\dot{\gamma}$ and (3.13), the shear for non-Newtonian fluid is:

$$\dot{\gamma}_w = \left[\frac{1+3n+2n^2}{3n\,(1-\emptyset)(1+n+n\emptyset)}\right]\frac{6u_{avg}}{h} \tag{3.29}$$

So, the shear stress:

$$\tau_w = \tau_y + K\,\left[\frac{1+3n+2n^2}{3n\,(1-\emptyset)(1+n+n\emptyset)}\right]^n\left(\frac{6u_{avg}}{h}\right)^n \tag{3.30}$$

Now easily we can obtain the corresponding relations for Bingham Plastic, power low, and Newtonian fluid. It worth mentioning that sometimes real life fluid could show more than one behavior like

Plastic fluid which has the characteristics of Bingham and power law fluid see figure 3.2.

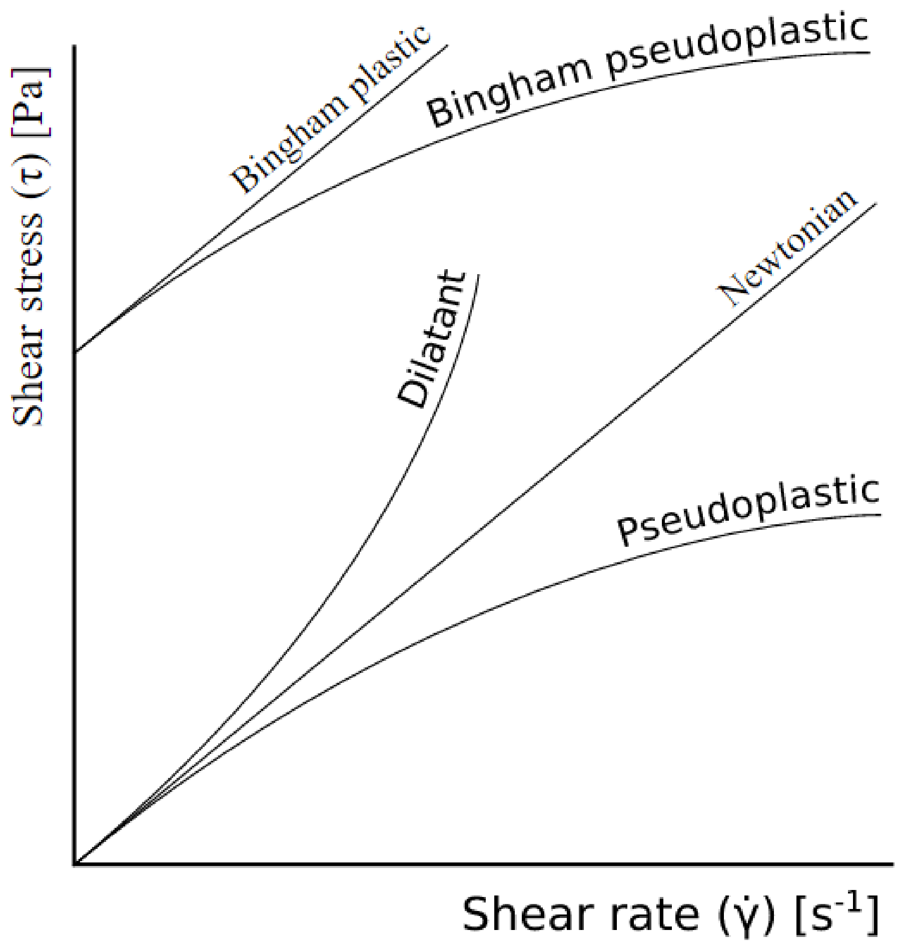

Figure 3.2: A stress-strain plot for Non-Newtonian fluids [32].

3.1 Newtonian Fluid

In order to solve for Newtonian fluid the yield stress $\tau_y = 0$, and fluid behavior index $n = 1$. By applying this to the previous equation, we can find the following:

$$\tau = \mu\dot{\gamma} \tag{3.31}$$

From (3.16):

$$u_{max} = \frac{h\tau_w}{4\mu}h \tag{3.32}$$

From (3.19) and (3.22):

$$q = \frac{wh^2\tau_w}{6\mu} = \frac{wh^3n}{12\mu}\frac{\Delta P}{L} \tag{3.33}$$

From (3.20):

$$u_{avg} = \frac{q}{hw} = \frac{h\tau_w}{6\mu} \tag{3.34}$$

From (3.28):

$$Re' = \frac{2h\,u_{avg}\,\rho}{\mu} \tag{3.35}$$

From (3.29):

$$\dot{\gamma}_w = \frac{6u_{avg}}{h} \tag{3.36}$$

From (3.30):

$$\tau_w = \mu \left(\frac{6u_{avg}}{h}\right) \tag{3.37}$$

From (3.9) and (3.22):

$$\frac{\Delta P}{L} = \frac{12\,\mu\,u_{avg}}{h^2} = \frac{12\,\mu\,q}{wh^3} \tag{3.38}$$

3.2 Power Law Fluid

In order to solve for Power-law fluid the yield stress $\tau_y = 0$. By applying this to the previous equation, we can find the following:

$$\tau = K\,\dot{\gamma}^n \tag{3.39}$$

From (3.16):

$$u_{max} = \frac{nh}{2(1+n)}\left(\frac{\tau_w}{K}\right)^{1/n} \tag{3.40}$$

From (3.19) and (3.22):

$$q = \frac{wh^2 n}{2(1+2n)}\left(\frac{\tau_w}{K}\right)^{1/n} = \frac{wh^2 n}{2(1+2n)}\left(\frac{h}{2K}\frac{\Delta P}{L}\right)^{1/n} \tag{3.41}$$

From (3.20):

$$u_{avg} = \frac{q}{hw} = \frac{hn}{2(1+2n)} \left(\frac{\tau_w}{K}\right)^{1/n} \tag{3.42}$$

From (3.28):

$$Re' = \frac{6n\, u_{avg} \rho h}{(1+2n)\, \tau_w^{1-1/n} K^{1/n}} \tag{3.43}$$

From (3.29):

$$\dot{\gamma}_w = \left(\frac{1+2n}{3n}\right) \frac{6u_{avg}}{h} \tag{3.44}$$

From (3.30):

$$\tau_w = K \left(\frac{1+2n}{3n}\right)^n \left(\frac{6u_{avg}}{h}\right)^n \tag{3.45}$$

From (3.9) and (3.22):

$$\frac{\Delta P}{L} = 2^{n+1} K \left(\frac{1+2n}{3n}\right)^n x\, h^{-(n+1)} {u_{avg}}^n = 2^{n+1} K \left(\frac{1+2n}{3n}\right)^n x\, w^{-n} h^{-(n+1)} q^n \tag{3.46}$$

Power-law fluid with $n < 1$ called shear thinning fluid and for $n > 1$ called shear thickening, see figure 4.2.

3.3 Bingham Plastic Fluid

For ideal Bingham fluid, flow behavior index $n = 1$. By applying this to the previous equation, we can find the following:

$$\tau = \tau_y + \mu_p \dot{\gamma} \tag{3.47}$$

From (3.16):

$$u_{max} = \frac{h(1-\emptyset)^2 \tau_w}{4\mu_p} \tag{3.48}$$

From (3.19) and (3.22):

$$q = \frac{wh^2(1-\emptyset)^2(2+\emptyset)\tau_w}{12\mu_p} = \frac{wh^3(1-\emptyset)^2(2+\emptyset)}{24\mu_p}\frac{\Delta P}{L} \tag{3.49}$$

From (3.20):

$$u_{avg} = \frac{q}{hw} = \frac{h(1-\emptyset)^2(2+\emptyset)\tau_w}{12\mu_p} \tag{3.50}$$

From (3.28):

$$Re' = \frac{12\ {\mu_{avg}}^2 \rho\ \emptyset}{\tau_y} \tag{3.51}$$

From (3.29):

$$\dot{\gamma}_w = \left[\frac{2}{(1-\emptyset)(2+\emptyset)}\right]\frac{6u_{avg}}{h} \tag{3.52}$$

From (3.30):

$$\tau_w = \tau_y + \mu_p \left[\frac{2}{(1-\emptyset)(2+\emptyset)}\right]^n \left(\frac{6u_{avg}}{h}\right) \tag{3.53}$$

From (3.9) and (3.22):

$$\frac{\Delta P}{L} = \frac{2\,\tau_y}{h\,\varphi} \tag{3.54}$$

$$\varphi = -2\sqrt{1 + \frac{4\mu_p u_{avg}}{h\tau_y}} \cos\frac{4\pi + \arccos\left(1 + \frac{4\mu_p u_{avg}}{h\tau_y}\right)^{-3}}{3} \tag{3.55}$$

Or

$$\varphi = -2\sqrt{1 + \frac{4\mu_p q}{h\,w^2\tau_y}} \cos\frac{4\pi + \arccos\left(1 + \frac{4\mu_p q}{h\,w^2\tau_y}\right)^{-3}}{3} \tag{3.56}$$

4 Turbulent Flow

This chapter will explain the basic principles for turbulent flow then shows the basic theories that are required for understanding this thesis.

4.1 Kolmogorov Scale

Turbulent start to form by introducing large eddies, these large eddies transfer the kinetic energy to newly formed smaller eddied and so on until it reached the smallest eddies that can't transfer energy any more. So the smallest eddies dissipate energy as heat due to viscosity as shown in figure 4.1.The smallest scale of turbulence was derived by Kolmogorov [20,21], with the assumption that the energy dissipation takes place at the smallest scale. The required physical parameter to describe this behavior dimensional wise is viscosity ν and dissipation rate ε of turbulent kinetic energy. The generalized unit for viscosity ν is viscosity $L^2/_T$ and for dissipation rate ε is $L^2/_{T^3}$. Based on that the Kolmogorov length scale expressed as following:

$$\eta = \left(\frac{\nu^3}{\varepsilon}\right)^{1/4} \tag{4.1}$$

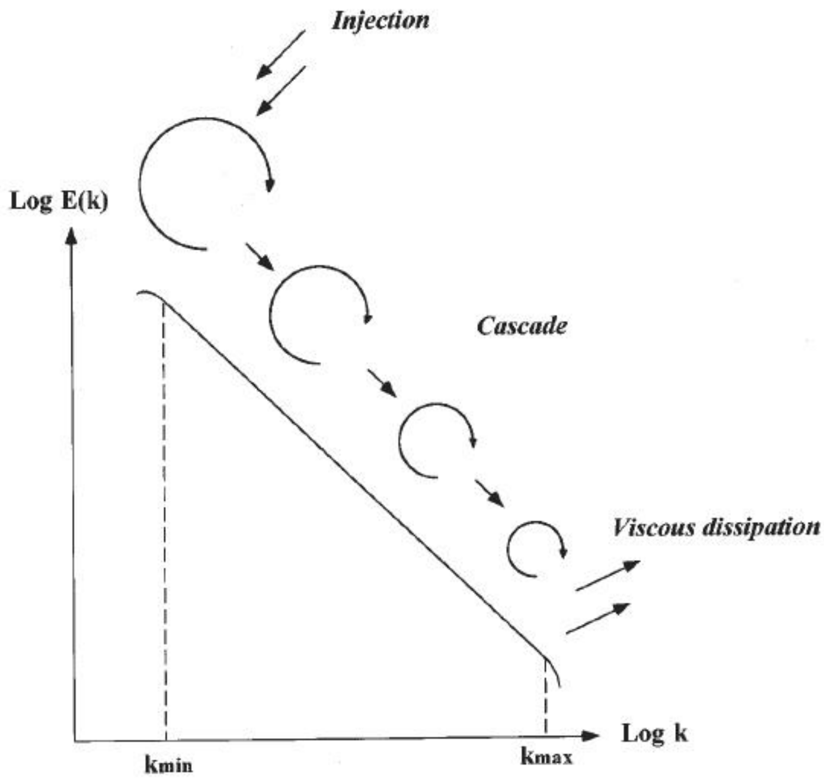

Figure 4.1: the large eddy transfer energy to smallest eddy which transfers energy as dissipated heat due to viscosity. E (k) is the spectral density and k is the wavenumber [33].

In the same way, the Kolmogorov time scale expressed as following:

$$\tau = \left(\frac{\nu}{\varepsilon}\right)^{1/2} \tag{4.2}$$

Lastly, for the Kolmogorov velocity scale:

$$\upsilon = (\nu\varepsilon)^{1/4} \tag{4.3}$$

Assume the following:

- Kinetic energy amount per unit mass in large eddy scale is proportional to u'^2.
- Rate of transfer of kinetic energy is proportional to $u'/_l$.
- Rate of energy supply to the smallest scale $u'^2 \quad u'/_l = \frac{|u'|^3}{l}$.

Based on these assumption the energy dissipation $\varepsilon = \frac{|u'|^3}{l}$. The integral length scale l can be compared with Kolmogorov length scale η by substituting $\varepsilon = \frac{|u'|^3}{l}$ back in (4.1):

$$\eta \sim \left(\frac{\nu^3}{|u'|^3/_l}\right)^{1/4} \tag{4.4}$$

Then we can introduce Reynolds number as following:

$$\frac{\eta}{l} \sim \left(\frac{\nu^3}{|u'|^3\, l^3}\right)^{1/4} \sim Re_l^{-3/4} \tag{4.5}$$

Or

$$\frac{l}{\eta} \sim Re_l^{3/4} \tag{4.6}$$

The ~ have been used to show that these relation based on dimensional analysis only.

4.2 Origen of Turbulence

Turbulent flow has fluctuating velocity in all the dimensions which make it difficult to solve by Naiver stoke equation. The fluctuating velocity can be introduced to Naiver Stoke equation by averaging.

As shown in figure 4.2, velocity component is actually an averaged mean velocity and fluctuating velocity which was explained by Klebanoff [23]. Velocity is decomposed into a mean value $\bar{u}$ and fluctuating value u'.

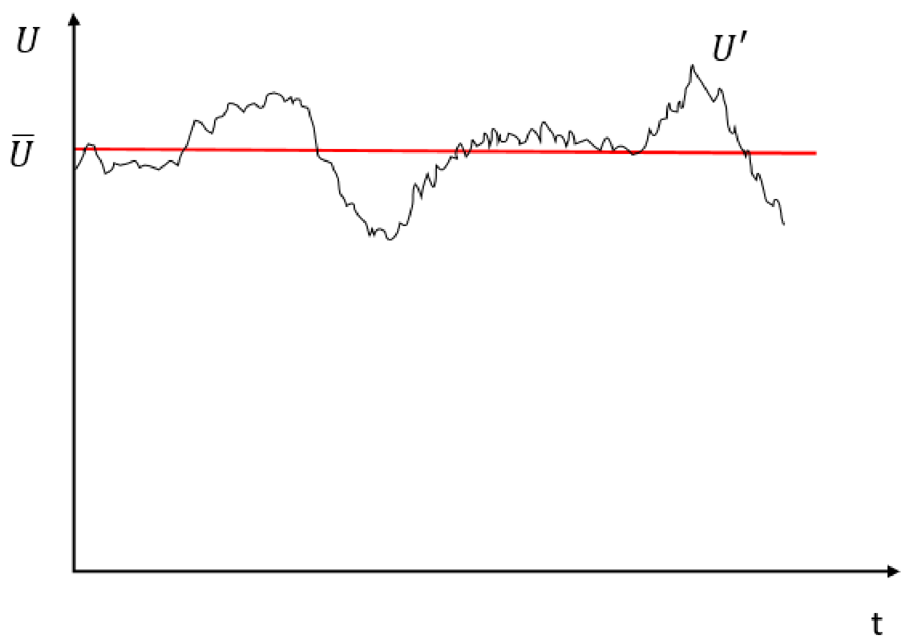

Figure 4.2: velocity components are mean velocity $\bar{u}$ and fluctuating velocity u'.

The velocity components are, time-averaged value $\bar{u}$ and fluctuating velocity u'. Similarly, we can apply this definition for the following values:

$$u = \bar{u} + u' , \quad v = \bar{v} + v', \quad w = \bar{w} + w', \tag{4.7}$$

$$P = \bar{P} + P' \quad , \quad \rho = \bar{\rho} + \rho' \quad , \quad T = \bar{T} + T'$$

The averaging method can be considered by taking mean values at a fixed place and averaged over a period of time that is large enough for the mean values to be independent of it.

$$\bar{u} = \lim_{T\to\infty} \frac{1}{T} \int_{t_0}^{t_0+\mathrm{T}} u \, dt \tag{4.8}$$

The time-averaged values for the fluctuating values are defined to be zero:

$$\overline{u'} = 0 \quad , \quad \overline{v'} = 0 \quad , \quad \overline{w'} = 0 \quad , \quad \overline{P'} = 0 \tag{4.9}$$

Summary of rules for time averaging:

$$\overline{\frac{\partial \overline{u}}{\partial x}} = \frac{\partial \overline{u}}{\partial x}\ ,\ \ \overline{\frac{\partial u'}{\partial x}} = 0$$

$$\bar{\bar{f}} = \bar{f}\ \ ,\quad \overline{f+g} = \bar{f} + \bar{g}, \qquad\qquad \overline{\bar{f}.g} = \bar{f}.\bar{g}, \tag{4.10}$$

$$\overline{\frac{\partial f}{\partial s}} = \frac{\partial \bar{f}}{\partial s}, \qquad\qquad \overline{\int f\, ds} = \int \bar{f}\, ds$$

But $\overline{f.g} \neq \bar{f}.\bar{g}$

First, start by averaging continuity equation by substituting (4.7) into (2.54) for incompressible flow ($\frac{D\rho}{Dt} = 0$):

$$\frac{\partial \overline{u}}{\partial x} + \frac{\partial u'}{\partial x} + \frac{\partial \overline{v}}{\partial y} + \frac{\partial v'}{\partial y} + \frac{\partial \overline{w}}{\partial z} + \frac{\partial w'}{\partial z} = 0 \tag{4.11}$$

Now, apply time averaging:

$$\overline{\frac{\partial \overline{u}}{\partial x} + \frac{\partial u'}{\partial x} + \frac{\partial \overline{v}}{\partial y} + \frac{\partial v'}{\partial y} + \frac{\partial \overline{w}}{\partial z} + \frac{\partial w'}{\partial z}} = 0 \tag{4.12}$$

So that the time averaged continuity equation is

$$\frac{\partial \overline{u}}{\partial x} + \frac{\partial \overline{v}}{\partial y} + \frac{\partial \overline{w}}{\partial z} = 0 \tag{4.13}$$

Now averaging NS equation, we will consider x-component only since the rest are identical, start by substituting (4.3) into (2.36a):

$$\rho\left(\frac{\partial(\bar{u}+u')}{\partial t}+(\bar{u}+u')\frac{\partial(\bar{u}+u')}{\partial x}+(\bar{v}+v')\frac{\partial(\bar{u}+u')}{\partial y}+(\bar{w}+w')\frac{\partial(\overline{u}+u')}{\partial z}\right)=-\frac{\partial(\bar{P}+P')}{\partial x}+$$

$$\mu\left(\frac{\partial^2(\bar{u}+u')}{\partial x^2}+\frac{\partial^2(\bar{u}+u')}{\partial y^2}+\frac{\partial^2(\bar{u}+u')}{\partial z^2}\right)+\rho f_x \qquad (4.14)$$

f_x: is not subjected to turbulence fluctuation.

Now, apply time averaging:

$$\rho\overline{\left(\frac{\partial(\bar{u}+u')}{\partial t}+(\bar{u}+u')\frac{\partial(\bar{u}+u')}{\partial x}+(\bar{v}+v')\frac{\partial(\bar{u}+u')}{\partial y}+(\bar{w}+w')\frac{\partial(\overline{u}+u')}{\partial z}\right)}=$$

$$-\overline{\frac{\partial(\bar{P}+P')}{\partial x}+\ \mu\left(\frac{\partial^2(\bar{u}+u')}{\partial x^2}+\frac{\partial^2(\bar{u}+u')}{\partial y^2}+\frac{\partial^2(\bar{u}+u')}{\partial z^2}\right)+\rho f_x} \qquad (4.15)$$

Simplify as following:

$$\rho\left(\frac{\partial\bar{u}}{\partial t}+\frac{\partial\overline{u}\overline{u}}{\partial x}+\frac{\partial\overline{u'u'}}{\partial x}+\frac{\partial\overline{u}\overline{v}}{\partial y}+\frac{\partial\overline{u'v'}}{\partial x}+\frac{\partial\overline{u}\overline{w}}{\partial z}+\frac{\partial\overline{u'w'}}{\partial z}+\right)=-\frac{\partial\bar{P}}{\partial x}+\ \mu\left(\frac{\partial^2\bar{u}}{\partial x^2}+\frac{\partial^2\bar{u}}{\partial y^2}+\right.$$

$$\left.\frac{\partial^2\bar{u}}{\partial z^2}\right)+\rho f_x \qquad (4.16)$$

$$\rho\left(\frac{\partial\bar{u}}{\partial t}+\bar{u}\frac{\partial\bar{u}}{\partial x}+\bar{v}\frac{\partial\bar{u}}{\partial y}+\bar{w}\frac{\partial\bar{u}}{\partial z}\right)=-\frac{\partial\bar{P}}{\partial x}+\ \mu\left(\frac{\partial^2\bar{u}}{\partial x^2}+\frac{\partial^2\bar{u}}{\partial y^2}+\frac{\partial^2\bar{u}}{\partial z^2}\right)+\rho f_x-$$

$$\rho\left(\frac{\partial\overline{u'u'}}{\partial x}+\frac{\partial\overline{u'v'}}{\partial x}+\frac{\partial\overline{u'w'}}{\partial z}\right) \qquad (4.17)$$

$$\rho\left(\frac{\partial\bar{u}}{\partial t}+\bar{u}\frac{\partial\bar{u}}{\partial x}+\bar{v}\frac{\partial\bar{u}}{\partial y}+\bar{w}\frac{\partial\bar{u}}{\partial z}\right)=-\frac{\partial\bar{P}}{\partial x}+\ \mu\,\Delta\bar{u}\ +\rho f_x-\rho\left(\frac{\partial\overline{u'u'}}{\partial x}+\frac{\partial\overline{u'v'}}{\partial y}+\frac{\partial\overline{u'w'}}{\partial z}\right)$$

(4.18a)

For y and z direction:

$$\rho\left(\frac{\partial\bar{v}}{\partial t}+\bar{u}\frac{\partial v}{\partial x}+\bar{v}\frac{\partial\bar{v}}{\partial y}+\bar{w}\frac{\partial\bar{v}}{\partial z}\right)=-\frac{\partial\bar{P}}{\partial y}+\ \mu\,\Delta\bar{v}\ +\rho f_y-\rho\left(\frac{\partial\overline{v'u'}}{\partial x}+\frac{\partial\overline{v'v'}}{\partial y}+\frac{\partial\overline{v'w'}}{\partial z}\right)$$

(4.18b)

$$\rho\left(\frac{\partial\bar{w}}{\partial t}+\bar{u}\frac{\partial\bar{u}}{\partial x}+\bar{v}\frac{\partial\bar{w}}{\partial y}+\bar{w}\frac{\partial\bar{w}}{\partial z}\right)=-\frac{\partial\bar{P}}{\partial z}+\ \mu\,\Delta\bar{w}\ +\rho f_z-\rho\left(\frac{\partial\overline{w'u'}}{\partial x}+\frac{\partial\overline{w'v'}}{\partial y}+\frac{\partial\overline{w'w'}}{\partial z}\right)$$

(4.18c)

For tensor notation:

$$\rho\left(\frac{D\bar{u}_i}{Dt}\right)=\rho f_i-\frac{\partial\bar{P}}{\partial x_i}+\ \mu\,\bar{u}_{i,jj}\ -\rho\left(\frac{\partial\overline{u_i'u_j'}}{\partial x_j}\right) \qquad (4.19)$$

The last term known as Reynolds stress [22] which represents the turbulence shear stress caused by fluctuating velocity components. We may simplify (4.19) as following:

$$\rho\left(\frac{D\bar{u}_i}{Dt}\right)=\rho f_i-\frac{\partial\bar{P}}{\partial x_i}+\frac{\partial}{\partial x_j}\left(\mu\frac{\partial\bar{u}_i}{\partial x_j}\ -\rho\overline{u_i'u_j'}\right) \qquad (4.20)$$

$$\rho\left(\frac{D\bar{u}_i}{Dt}\right)=\rho f_i-\frac{\partial\bar{P}}{\partial x_i}+\frac{\partial}{\partial x_j}\,\tau_{ij} \qquad (4.21)$$

Where $\tau_{ij}=\mu\frac{\partial\bar{u}_i}{\partial x_j}\ -\rho\overline{u_i'u_j'}$, which represent the laminar and turbulent shear stress.

Klebanoff preformed a detailed measurements for turbulence fluctuation on a flat wall [24, 26]. The same have been done for pipe flow by Laufer, J. [25]. Furthermore, the turbulent channel flow have been covered by Laufer, J. [27].

4.3 Prandtl's Mixing Length Theory

Prandtl's mixing length hypotheses [5,9] can be explained on parallel plate flow. Let's consider the simple case that the velocity is in x- direction and its function of y only.

$$\bar{u} = \bar{u}(y) \quad , \quad \bar{v} = 0 \, , \quad \bar{w} = 0 \tag{4.22}$$

The shearing stress for turbulent flow is:

$$\tau_{turbulant} = -\rho \, \overline{u'v'} \tag{4.23}$$

Let's consider a fluid parcel moved a distance l from y_1 to $y_1 - l$, as shown in figure 4.3. in this case the change in velocity can be expressed as following:

$$\Delta\bar{u}_1 = \bar{u}(y_1) - \bar{u}(y_1 - l) \approx l \left(\frac{\overline{\partial u}}{\partial y}\right)_1 \tag{4.24}$$

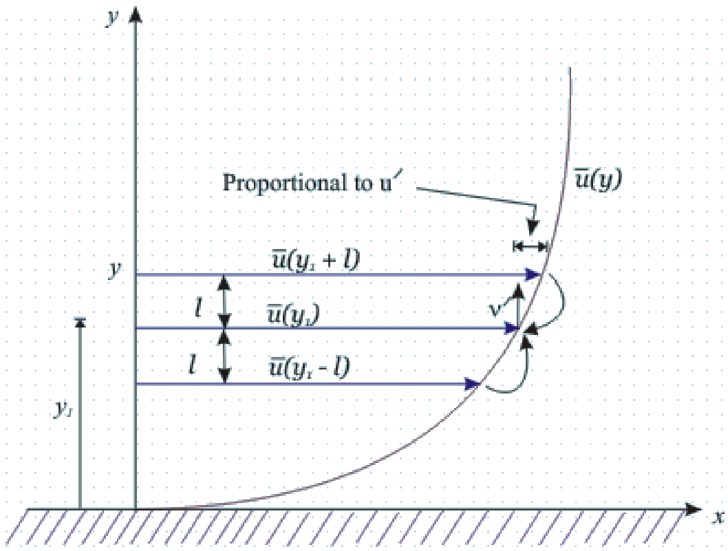

Figure 4.3: Prandtl's Mixing Length [5]

Similarly, a fluid parcel moves from $y_1 + l$ to y_1:

$$\Delta\bar{u}_2 = \bar{u}(y_1 + l) - \bar{u}(y_1) \approx l \left(\frac{\overline{\partial u}}{\partial y}\right)_1 \tag{4.25}$$

In order to drive an expression for the turbulent shear stress. We will take a closer look at the mean value $\overline{u'v'}$. When the fluid parcel arrive at layer y_1, it'll have positive v' and it'll reduce the momentum in the new layer so u' will be negative and vice versa. Thus, they can be summarized as following:

$v' < 0$ *Add momentum to the new layer* $\rightarrow u' > 0$

$v' > 0$ *Reduce momentum to the new layer* $\rightarrow u' < 0$

So Prandtl has assumed the following:

$$\overline{u'v'} = -l^2 \left(\frac{\partial \overline{u}}{\partial y}\right)^2 \quad (4.26)$$

So, the turbulent shear stress can be written as following:

$$\tau_{turbulant} = \rho\, l^2 \left(\frac{\partial \overline{u}}{\partial y}\right)^2 \quad (4.27)$$

Taking into account the sign of turbulent shear stress must change with $\frac{\partial \overline{u}}{\partial y}$, it's more correct to write it as following:

$$\tau_{turbulant} = \rho\, l^2 \left|\frac{\partial \overline{u}}{\partial y}\right| \frac{\partial \overline{u}}{\partial y} \quad (4.28)$$

This is Prandtl's mixing length hypotheses. Galbraith, R.M.A [10], have provided a good experimental support for mixing length theory.

4.4 Van Karman Similarity Hypothesis

Von Karman made an attempt to establish a rule that allows determining the dependence of mixing length on space coordinate [4]. Let's assume that the turbulence fluctuations are similar at all point of the flow field. So, they differ by time and length scale from point to point. For similarity rule, we consider a two dimensional mean flow in x direction, which mean $u = u(y)$ and $v = 0$. In this case it's possible to show the following:

$$l \sim \frac{d\bar{u}/dy}{d^2\bar{u}/dy^2} \tag{4.29}$$

Von Kaman has made the assumption for this relation to be equivalent by introducing κ which dimensionless constant.

$$l = \kappa \left| \frac{d\bar{u}/dy}{d^2\bar{u}/dy^2} \right| \tag{4.30}$$

Based on this hypothesis we may say that the mixing length l, is independent of the velocity magnitude and function of velocity distribution only. By substituting (4.30) into (4.28), we get the turbulent shear stress as following:

$$\tau_{turbulant} = \rho \left(\kappa \left| \frac{d\bar{u}/dy}{d^2\bar{u}/dy^2} \right| \right)^2 \left| \frac{\overline{\partial u}}{\partial y} \right| \frac{\overline{\partial u}}{\partial y} \tag{4.31}$$

$$\tau_{turbulant} = \rho \ \kappa^2 \frac{\left(d\bar{u}/_{dy}\right)^4}{\left(d^2\bar{u}/_{dy^2}\right)^2} \tag{4.32}$$

Prandtl then assumed the fluctuating component v proportional to the mixing length l and to the absolute value $\left|\frac{du}{dy}\right|$ of the velocity gradient. This assumption signifies that the correlation between u and v is supposedly unaffected by the location. Betz, A has shown a very detailed derivation for (4.30) [11]. After a while van Karman hypotheses has been extended to include compressible fluid by Lin, C.C. and Shen, S.F [12] and their work were observed by Bjorgum, O [13] and Hamel, G. [14]

4.5 Law of The Wall

The law of the wall proves that the average turbulent velocity at a certain point is proportional to the logarithmic distance to the wall, or the boundary of the fluid region. It could be referred to as universal velocity distribution which was first published by von

Kármán, in 1930. This law is valid for the flow region that is close to the wall which is less than 20% of the flow height. Thus, it provides a good approximation [5].

The law of the wall is valid for flows at high Reynolds numbers where the viscous layer overlap with main stream low with approximately constant shear stress near the wall.

$$\frac{u}{v_*} = \frac{1}{\kappa} \ln \frac{y\, v_*}{\upsilon} + C \tag{4.33}$$

$\frac{y\, v_*}{\upsilon}$	Distance y to the wall. The friction velocity v_* and kinematic viscosity υ used to have dimensionless term.
$\frac{u}{v_*}$	The velocity u is parallel to the wall and its function of y. The friction velocity v_* used to have dimensionless term
v_*	Frictional velocity, is a measure of turbulent eddies and correlation which exist between a fluctuating component in x and y directions.

Experiments results shows that von Karman constant to be $\kappa \approx 0.41$ and the constant $C \approx 5$ for a smooth wall.it can be written as following:

$$\frac{u}{v_*} = \frac{1}{\kappa} \ln \frac{y}{y_0} \qquad (4.34)$$

y_0 the distance from the boundary where the law of the wall velocity goes to zero. This important since the law of the wall velocity profile is turbulent and the law of the wall can't be applied for the laminar sublayer.

Viscous Sublayer

This region is below 5 wall unit which mean $\frac{y\,u_*}{v} < 5$, in this case we can use the following

$$\frac{y\,u_*}{v} = \frac{u}{v_*} \qquad (4.35)$$

It can be used for more than 5 wall unit but at 12 unit the error could reach up to 25%.

Buffer Layer

This region where main stream flow overlap with the viscous sublayer, and it's between 5 and 30 wall units which mean $5 < \frac{y\,u_*}{\upsilon} < 30$, in this region nether law are applicable as shown in figure 4.4.

$$\frac{y\,u_*}{\upsilon} \neq \frac{u}{v_*} \tag{4.36}$$

$$\frac{u}{v_*} \neq \frac{1}{\kappa} \ln \frac{y\,u_*}{\upsilon} + \mathrm{C} \tag{4.37}$$

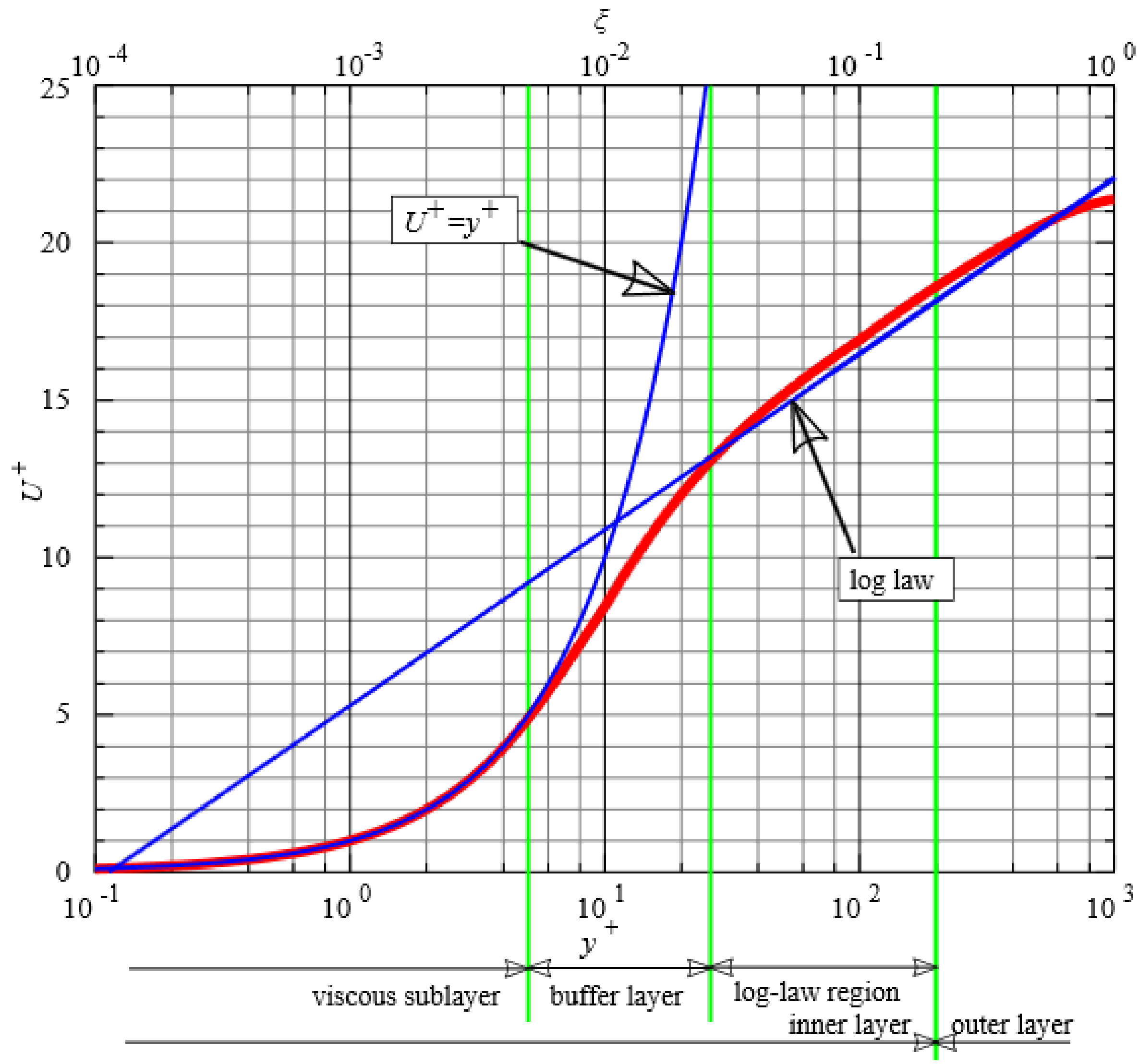

Figure 4.4: Law of the wall with the tree different layer solutions ($U^+ = \frac{u}{v_*}$ & $y^+ = \frac{y\,u_*}{v}$) [34].

5 Formulation of Problem

We are solving for a homogeneous and isotropic turbulent flow between two parallel plates. The fluid is Non-Newtonian and we are considering power law fluid in the solution. Moreover, the fluid is pressure driven fluid and it's incompressible.

5.1 General Equation

The coordinate system is chosen to be in the middle of the two parallel plates as shown in figure 5.1.

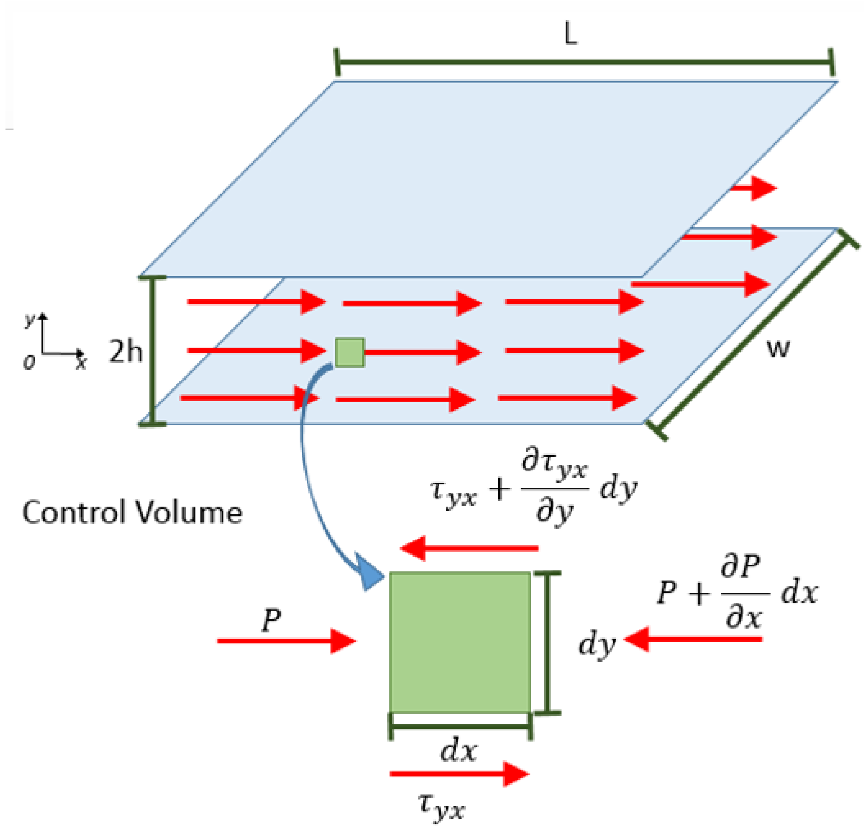

Figure 5.1: schematic of two parallel plate flow geometry and control volume

We can drive the governing equation for this case by using the force balance over the indicated control volume in figure 5.1 as following:

$$\sum F_x = 0 \;,\; F = PA_{Cross\ Section} \quad , \quad F = \tau\, A_{\text{Perpendicular}} \tag{5.1}$$

$$\left[P - \left(P + \frac{\partial P}{\partial x} dx\right)\right] dydz + \left[\tau_{yx} - \left(\tau_{yx} + \frac{\partial \tau_{yx}}{\partial y} dy\right)\right] dxdz = 0 \tag{5.2}$$

After simplifying the equation we end up with:

$$\frac{\partial P}{\partial x} = \frac{\partial \tau_{yx}}{\partial y} \tag{5.3}$$

Integrate once with respect to y:

$$y\frac{\partial P}{\partial x} = \tau_{yx} \tag{5.4}$$

For turbulent flow $\tau_{yx} = \tau_{Laminar} + \tau_{Turbulent}$, which can be expanded as following for power law fluid:

$$\tau_{yx} = K \left|\frac{\partial \bar{u}}{\partial y}\right|^n - \rho\, \overline{u'v'} \tag{5.5}$$

Substitute back τ_{yx} in (5.4):

$$y \frac{\partial \bar{P}}{\partial x} = K \left|\frac{\partial \bar{u}}{\partial y}\right|^n - \rho\, \overline{u'v'} \tag{5.6}$$

On the wall of the plate thin viscous sublayer (laminar sublayer) formed, which have negligible turbulent fluctuation. The thickness of the viscous sublayer refer to as δ, see figure 5.2.

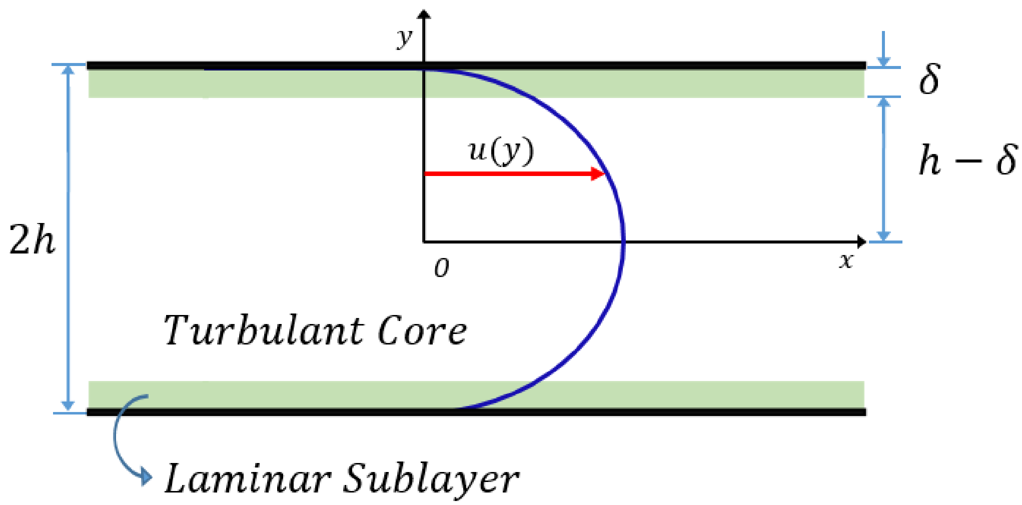

Figure 5.2: velocity profile between two parallel plates

On the other hand, the outside viscous sublayer (turbulent core) have negligible viscous shear stress effect. Next sept is to find the velocity distribution for each sublayer.

5.2 Inside Viscous Sublayer

Since turbulent fluctuation is negligible inside viscous sublayer $-\rho\,\overline{u'v'} = 0$, and we will end up with the following equation to solve:

$$y\,\frac{\partial \bar{P}}{\partial x} = K\left|\frac{\partial \bar{u}}{\partial y}\right|^{n} \tag{5.7}$$

Since the viscous sublayer is very thins, we may assume the shear stress to be constant throughout the whole δ range. So in this case $y = h$, which result in the following equation:

$$\tau = \frac{y}{h}\tau_w \quad (5.8)$$

$$\tau_w = h\,\frac{\partial \bar{P}}{\partial x} \quad (5.9)$$

τ_w is the shear stress at the wall $y = {}^{+}/{-}\, h$. Frictional velocity $v_* = \sqrt{\frac{\tau_w}{\rho}}$, so after dimensional consideration we get the following:

$$v_* = \sqrt{\frac{h}{\rho}\frac{\partial \bar{P}}{\partial x}} \quad (5.10)$$

After substituting v_* in (5.7) with $y = h$, we get the following:

$$h\,\frac{\partial \bar{P}}{\partial x} = \rho v_*^2 = K\,\left|\frac{\partial \bar{u}}{\partial y}\right|^n \quad (5.11)$$

Now solve for $\frac{\partial \bar{u}}{\partial y}$:

$$\frac{\partial \bar{u}}{\partial y} = \left(\frac{\rho}{K}v_*^2\right)^{1/n} \quad (5.12)$$

Integrate once with respect to y:

$$\bar{u} = \int_y^h \left(\frac{\rho}{K}v_*^2\right)^{1/n} dy \quad (5.13)$$

$$\bar{u} = \left(\frac{\rho}{K} v_*^2\right)^{1/n} (h - y) \tag{5.14}$$

In order to have dimensionless solution, write the velocity destruction as $\bar{u}/v_*$:

$$\boxed{\frac{\bar{u}}{v_*} = \left(\frac{\rho}{K} v_*^{2-n}\right)^{1/n} (h - y)} \tag{5.15}$$

(5.15) is the velocity distribution for inside viscous sublayer only, where $h - \delta \leq y \leq h$, see figure 5.4.

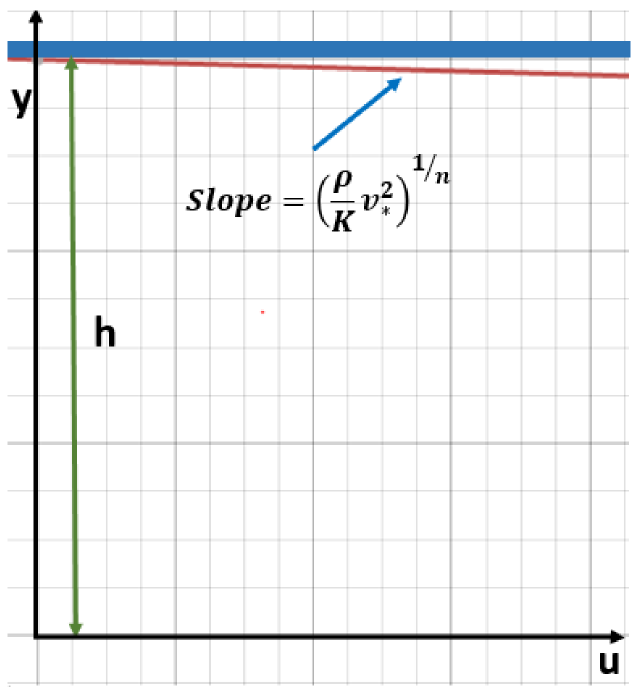

Figure 5.3: velocity distribution inside viscous sublayer from $y = \mathrm{h} - \delta$ to h.

5.3 Turbulent Core

For outside of viscous sublayer, shear stress effect is negligible compare to turbulent fluctuation. So viscous term can be omitted which lead to $K\left|\frac{\partial \overline{u}}{\partial y}\right|^{n}=0$, the equation of motion will be as following:

$$y\frac{\partial \bar{P}}{\partial x}=-\rho\,\overline{u'v'} \tag{5.16}$$

In order to solve the turbulent fluctuation term we need to have turbulent closure. In this case we will use Van Karman turbulent similarity hypotheses [4] which was explained before in section 4.4 [4].

$$\rho\overline{u'v'}=-\rho\kappa^{2}\left(\frac{\partial \overline{u}}{\partial y}\right)^{4}\left(\frac{\partial^{2}\overline{u}}{\partial y^{2}}\right)^{-2} \tag{5.17}$$

Substitute Van Karman closure in (5.16):

$$y\,\frac{\partial \bar{P}}{\partial x}=\rho\kappa^{2}\left(\frac{\partial \overline{u}}{\partial y}\right)^{4}\left(\frac{\partial^{2}\overline{u}}{\partial y^{2}}\right)^{-2} \tag{5.18}$$

Move ρ to LHS and y to RHS, then multiply both sides by h:

$$\frac{\partial \bar{P}}{\partial x}\frac{h}{\rho}=\frac{h}{y}\,\kappa^{2}\left(\frac{\partial \overline{u}}{\partial y}\right)^{4}\left(\frac{\partial^{2}\overline{u}}{\partial y^{2}}\right)^{-2} \tag{5.19}$$

Since $v_* = \sqrt{\frac{h}{\rho}\frac{\partial \bar{P}}{\partial x}}$, will replace $\frac{\partial \bar{P}}{\partial x}\frac{h}{\rho}$ with v_*^2:

$$v_*^2 = \frac{h}{y}\,\kappa^2\left(\frac{\partial \bar{u}}{\partial y}\right)^4\left(\frac{\partial^2 \bar{u}}{\partial y^2}\right)^{-2} \tag{5.20}$$

Move all terms to LHS and divide by $\frac{h}{y}\,\kappa^2$:

$$\frac{v_*^2}{h\,\kappa^2}\left(\frac{\partial^2 \bar{u}}{\partial y^2}\right)^2 y - \left(\frac{\partial \bar{u}}{\partial y}\right)^4 = 0 \tag{5.21}$$

So we end up with second order nonlinear ordinary differential equation, which has the following solution:

$$\bar{u} = \frac{v_*}{\sqrt{h}\,k}\left[\sqrt{y} - \frac{C_1}{2}\ln\left(2\sqrt{y} + C_1\right)\right] + C_2 \tag{5.22}$$

Apply the following boundary conditions:

$$\left.\frac{\partial \bar{u}}{\partial y}\right|_{y=h} = \infty \quad \& \quad \bar{u}|_{y=0} = u_{max} \tag{5.23}$$

We end up with the velocity distribution for turbulent flow between parallel plates see figure 5.4:

$$\boxed{\frac{\bar{u}}{v_*} = \frac{u_{max}}{v_*} + \frac{1}{k}\left[\sqrt{\frac{y}{h}} + \ln\left(1 - \sqrt{\frac{y}{h}}\right)\right]} \tag{5.24}$$

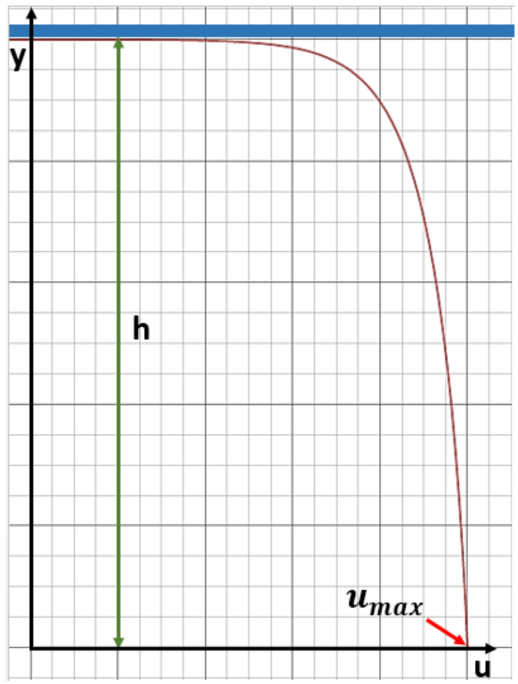

Figure 5.4: velocity distribution between parallel plates from $y = 0$ to h

5.4 Maximum Velocity

u_{max} still unknown in (5.24). The velocity distribution equation for inside and outside viscous sublayer are equal at $y = h - \delta$. Inside viscous sublayer velocity distribution (5.15) at $y = h - \delta$:

$$\frac{\bar{u}}{v_*} = \left(\frac{\rho}{K} v_*^{2-n}\right)^{1/n} \delta \tag{5.25}$$

Outside viscous sublayer velocity distribution (5.24) at $y = h - \delta$:

$$\frac{\bar{u}}{v_*} = \frac{u_{max}}{v_*} + \frac{1}{k}\left[\sqrt{\frac{(h-\delta)}{h}} + \ln\left(1 - \sqrt{1 - \frac{\delta}{h}}\right)\right] \tag{5.26}$$

After substituting $y = h - \delta$ for the inside (5.25) and outside (5.26) viscous sublayer equation, they became equal so we can solve for u_{max}:

$$\frac{u_{max}}{v_*} = -\frac{1}{k}\left[\sqrt{\frac{(h-\delta)}{h}} + \ln\left(1 - \sqrt{1 - \frac{\delta}{h}}\right)\right] + \left(\frac{\rho}{K} v_*^{2-n}\right)^{1/n} \delta \tag{5.27}$$

Since $\frac{\delta}{h} \ll 1$,

$$\frac{u_{max}}{v_*} = -\frac{1}{k}\left[1 + \ \ln\left(\frac{\delta}{2h}\right)\right] + \left(\frac{\rho}{K} v_*^{2-n}\right)^{1/n} \delta \qquad (5.28)$$

5.5 Reynolds Number for Power Law Fluid

Based on (5.28), δ is difficult to be determent numerically and by experiment, we can use prandtl's assumption [2,28] for Newtonian fluid:

$$\frac{v_* \delta \rho}{\mu} = constant \qquad (5.29)$$

Which is look like Reynolds number ($Re = \frac{c(2h)\rho}{\mu}$) after replacing average velocity c with frictional velocity v_* and replacing $2h$ by δ.The same can be assumed for Non-Newtonian fluid between two parallel plates. Use Reynolds number (3.43) for power law fluid. But we will replace h with $2h$:

$$Re^* = \frac{12\, n\, c\, \rho\, h}{(1+2n)\, \tau_w^{1-1/n} K^{1/n}} \tag{5.30}$$

τ_w Will be substituted from (4.45) with $2h$ instead of h:

$$\tau_w = K \left(\frac{1+2n}{3n}\right)^n \left(\frac{6\, c}{2h}\right)^n \tag{5.31}$$

$$\tau_w^{1-1/n} = K^{1-1/n} \left(\frac{1+2n}{3n}\right)^{n-1} \left(\frac{6\, c}{2h}\right)^{n-1} \tag{5.32}$$

Substitute $\tau_w^{1-1/n}$ back in Re^*:

$$Re^* = \frac{12\, n\, c\, \rho\, h}{(1+2n)\, K^{1/n}\, K^{1-1/n} \left(\frac{1+2n}{3n}\right)^{n-1} \left(\frac{6u_{avg}}{2h}\right)^{n-1}} \tag{5.33}$$

Simplify as following:

$$Re^* = \frac{12\, (2h)^n\, c^{2-n}\, \rho}{K \left(\frac{4n+2}{n}\right)^n} \tag{5.34}$$

Now we can use Prandlt's assumption for Re^* as:

$$\frac{12\, \delta^n\, {v_*}^{2-n}\, \rho}{K \left(\frac{4n+2}{n}\right)^n} = constant = \alpha \tag{5.35}$$

Solve for δ:

$$\delta = \left(\frac{\alpha\, K \left(\frac{4n+2}{n}\right)^n}{12\, {v_*}^{2-n}\, \rho}\right)^{1/n} \tag{5.36}$$

Substitute (5.36) into (5.28):

$$\frac{u_{max}}{v_*} = -\frac{1}{k}\left[1 + \ln\left(\frac{\left(\frac{\alpha\, K \left(\frac{4n+2}{n}\right)^n}{12\, {v_*}^{2-n}\, \rho}\right)^{1/n}}{2h}\right)\right] + \left(\frac{\rho}{K} v_*^{2-n}\right)^{1/n} \left(\frac{\alpha\, K \left(\frac{4n+2}{n}\right)^n}{12\, {v_*}^{2-n}\, \rho}\right)^{1/n} \quad (5.36a)$$

Simplify as following:

$$\frac{u_{max}}{v_*} = \frac{1}{n\,\kappa} \ln\left(Re^* \left(\frac{v_*}{c}\right)^{2-n}\right) - \frac{1}{\kappa}\left[1 + \frac{\ln(\alpha)}{n}\right] + \left(\frac{\alpha}{12}\right)^{1/n} \frac{4n+2}{n} \quad (5.37)$$

Since now we can deal with u_{max}, substitute (5.37) into outside viscous sublayer solution (5.21):

$$\frac{\bar{u}}{v_*} = \frac{1}{n\,\kappa} \ln\left(Re^* \left(\frac{v_*}{c}\right)^{2-n}\right) - \frac{1}{\kappa}\left[1 + \frac{\ln(\alpha)}{n}\right] + \left(\frac{\alpha}{12}\right)^{1/n} \frac{4n+2}{n} + \frac{1}{k}\left[\sqrt{\frac{y}{h}} + \ln\left(1 - \sqrt{\frac{y}{h}}\right)\right] \quad (5.38)$$

5.6 Average Velocity

In order to find the overall average velocity, we need to integrate (5.38). Which can be computed as following:

$$\frac{c}{v_*} = \frac{2}{2\,h}\int_0^h \frac{\overline{u}}{v_*}dy \tag{5.39}$$

Multiplied by 2, so we can integrate from $0\ to\ h$ instead of $-h\ to\ h$. Divided by $2h$, to keep the RHS dimensionless same as LHS. Substitute (5.38) into (5.39):

$$\frac{c}{v_*} = \frac{1}{h}\int_0^h \left\{\frac{1}{n\kappa}\ \ln\left(Re^*\left(\frac{v_*}{c}\right)^{2-n}\right) - \frac{1}{\kappa}\left[1 + \frac{\ln(\alpha)}{n}\right] + \left(\frac{\alpha}{12}\right)^{1/n}\frac{4n+2}{n} + \frac{1}{\kappa}\sqrt{\frac{y}{h}} + \frac{1}{\kappa}\ \ln\left(1 - \sqrt{\frac{y}{h}}\right)\right\}dy \tag{5.40}$$

The first three teams are not function of y, so assume the following for simplification:

$$G = \frac{1}{n\kappa}\ \ln\left(Re^*\left(\frac{v_*}{c}\right)^{2-n}\right) - \frac{1}{\kappa}\left[1 + \frac{\ln(\alpha)}{n}\right] + \left(\frac{\alpha}{12}\right)^{1/n}\frac{4n+2}{n} \tag{5.41}$$

Substitute G back to (5.40):

$$\frac{c}{v_*} = \frac{1}{h}\int_0^h \left\{G + \frac{1}{\kappa}\sqrt{\frac{y}{h}} + \frac{1}{\kappa}\ \ln\left(1 - \sqrt{\frac{y}{h}}\right)\right\}dy \tag{5.42}$$

Integrate each term alone:

First term

$$\int_0^h G\,dy = Gy|_0^h = G\,(h-0)$$

$$\int_0^h G\,dy = Gh \tag{5.43}$$

Second term

$$\int_0^h \frac{1}{\kappa}\sqrt{\frac{y}{h}}\,dy = \frac{1}{\kappa\sqrt{h}}\,\frac{2y^{3/2}}{3}\bigg|_0^h = \frac{1}{\kappa\sqrt{h}}\,\frac{2h^{3/2}}{3} - 0$$

$$\int_0^h \frac{1}{\kappa}\sqrt{\frac{y}{h}}\,dy = \frac{2h}{3\kappa} \tag{5.44}$$

Third term

$$\int_0^h \frac{1}{\kappa}\ \ln\left(\sqrt{\frac{h-y}{h}}\right)dy = \frac{1}{\kappa}\left[-\frac{y}{2} - \frac{(h-y)}{2}\ln\left(\frac{h-y}{h}\right)\right]\bigg|_0^h$$

$$\int_0^h \frac{1}{\kappa}\ \ln\left(\sqrt{\frac{h-y}{h}}\right)dy = -\frac{h}{2\ \kappa} \tag{5.45}$$

Substitute the three terms back together:

$$\frac{c}{v_*} = \frac{1}{h}\left\{Gh + \frac{2\,h}{3\kappa}\ - \frac{h}{2\ \kappa}\right\} \tag{5.46}$$

$$\frac{c}{v_*} = G + \frac{1}{6\,\kappa} \tag{5.47}$$

Substitute G back:

$$\frac{c}{v_*} = \frac{1}{n\kappa}\ln\left(Re^*\left(\frac{v_*}{c}\right)^{2-n}\right) - \frac{1}{\kappa}\left[1 + \frac{\ln(\alpha)}{n}\right] + \left(\frac{\alpha}{12}\right)^{1/n}\frac{4n+2}{n} + \frac{1}{6\,\kappa} \tag{5.48}$$

Simplify as following:

$$\frac{c}{v_*} = \frac{1}{\sqrt{2}\,\kappa}\left[\frac{1}{n}\ln\left(Re^*\left(\frac{v_*}{c}\right)^{2-n}\right) - \frac{\ln(\alpha)}{n} - \frac{5}{6}\right] + \left(\frac{\alpha}{12}\right)^{1/n}\frac{4n+2}{n} \tag{5.49}$$

Now we can solve for friction factor since $\frac{c}{v_*}$ is function of Re^* only.

5.7 Friction Factor

Friction factor can be driven by using force balance between two parallel plats as shown in figure 5.1:

$$\Delta P \; 2h \, W = \tau_w \; 2 \, L \, W \tag{5.50}$$

Since$\tau_w = \rho v_*^2$, we end up with the following:

$$\frac{\Delta P}{L} \; h = \rho v_*^2 \tag{5.51}$$

Darcy friction equation [30,31] for parallel plates, can be calculated by using the hydraulic diameter for parallel plates which is $4h$:

$$\frac{\Delta P}{L} = f_D \; \frac{\rho}{2} \; \frac{c^2}{4h} \tag{5.52}$$

Substitute Darcy friction equation into (5.51):

$$f_D \; \frac{\rho}{2} \; \frac{c^2}{4h} \; h = \rho v_*^2 \tag{5.53}$$

Solve for $c/_{v_*}$:

$$\frac{c}{v_*} = \sqrt{\frac{8}{f_D}} \tag{5.54}$$

Substitute (5.54) into (5.49): and solve for $1/\sqrt{f_D}$:

$$\frac{1}{\sqrt{f_D}} = \frac{1}{\sqrt{8}\,\kappa}\left[\frac{1}{n}\ln\left(Re^*\left(\frac{\sqrt{f_D}}{\sqrt{8}}\right)^{2-n}\right) - \frac{\ln(\alpha)}{n} - \frac{5}{6}\right] + \left(\frac{\alpha}{12}\right)^{1/n}\frac{2n+1}{\sqrt{2}n} \tag{5.55}$$

Simplify as following:

$$\frac{1}{\sqrt{f_D}} = \frac{1}{2\sqrt{2}\,\kappa}\left[\frac{1}{n}\ln\left(Re^*(f_D)^{1-n/2}\right) - \frac{\ln(8\alpha)}{n} - 1.246\right] + \left(\frac{\alpha}{12}\right)^{1/n}\frac{2n+1}{\sqrt{2}n} \tag{5.56}$$

Change $\ln\, to\, log$ for first term $(\log x = \ln x\ \log e)$:

$$\frac{1}{\sqrt{f_D}} = \frac{1}{2\sqrt{2}\,\kappa}\left[\frac{2.303}{n}\log\left(Re^*(f_D)^{1-n/2}\right) - \frac{\ln(8\alpha)}{n} - 1.246\right] + \left(\frac{\alpha}{12}\right)^{1/n}\frac{2n+1}{\sqrt{2}n} \tag{5.57}$$

Solve for Newtonian fluid by substituting $n = 1$:

$$\frac{1}{\sqrt{f_D}} = \frac{0.814}{\kappa}\, log\left(Re^*\sqrt{f_D}\right) - 0.354\frac{ln(8\alpha)}{\kappa} - \frac{0.441}{\kappa} + 0.177\,\alpha \tag{5.58}$$

Prandtl derived the skin friction factor for smooth pipe [19] and it was verified by Nikuradse [2,7], assume this to hold true for smooth plate:

$$\frac{1}{\sqrt{f_D}} = 2\log\left(Re^*\sqrt{f_D}\right) - 0.8 \qquad (5.59)$$

Equation (5.59) should be equal to equation (5.58), so we can solve for κ and α:

$$2 = \frac{0.814}{\kappa} \Longrightarrow \kappa = 0.407 \qquad (5.60)$$

Substitute κ in (5.58), to solve for α:

$$\frac{1}{\sqrt{f_D}} = 2\,log\left(Re^*\sqrt{f_D}\right) - 0.870\ \ ln(8\alpha) - 1.084 + 0.177\,\alpha \qquad (5.61)$$

Last three terms equal -0.8 :

$$-0.8 = -0.870\ \ ln(8\alpha) - 1.084 + 0.177\,\alpha$$

So, $\alpha = 28.247$ (5.62)

Substitute $\kappa \, and \, \alpha$ back in (5.57):

$$\frac{1}{\sqrt{f_D}} = 0.869\left[\frac{2.303}{n}\log\left(Re^*(f_D)^{1-n/2}\right) - \frac{\ln(193.976)}{n} - 1.246\right] + (2.354)^{1/n}\frac{2n+1}{\sqrt{2}n} \quad (5.63)$$

Simplify as following:

$$\frac{1}{\sqrt{f_D}} = \frac{2}{n}\log\left(Re^*(f_D)^{1-n/2}\right) + (2.354)^{1/n}\frac{1.414\,n+0.707}{n} - \frac{4.578}{n} - 1.083 \quad (5.64)$$

Or

$$\frac{1}{\sqrt{f_D}} = \frac{1}{n}\left[2\log\left(Re^*(f_D)^{1-n/2}\right) + (0.810)^{1/n}(1.414n+0.707) - 4.578\right] - 1.083 \quad (5.65)$$

For rough wall turbulence f_D can be calculated as following [5]:

$$\frac{1}{\sqrt{f_D}} = 2\log\left(14.82\,\frac{h}{\varepsilon}\right) \quad (5.66)$$

In order to solve for the transition region between smooth and rough wall and based on Colebrook, C. approach [15], we may propose the following:

$$\frac{1}{\sqrt{f_D}} = -2\log\left(\frac{10^{-\frac{\beta}{2}}}{Re^{*\,1/n}\,(f_D)^{(2-n)/2n}} + \frac{\varepsilon}{14.82\,h}\right) \tag{5.67}$$

$\beta = (\,2.354)^{1/n}\frac{1.414\,n+0.707}{n} \; - \frac{4.578}{n} - 1.083$ Last three terms of (5.64)

Figure 5.3 shows the plot of equation (5.23) with $n = 1$. On the rough region five roughness values (0.000001; 0.0001; 0.001; 0.01; 0.03) where used to show the difference. Figure 5.4 shows the plot of equation (5.23) with $n = 0.8, 1\;\&\;1.2$.

6 Findings

The turbulent friction factor diagram for flow between parallel plates looks almost like Moody diagram for the pipes [29] which was done by Moody, L. Figure 5.6 shows that for very smooth ($\varepsilon/h = 0.000001$) plates, the viscosity effect is dominant even at high Reynolds number($Re = 10^{10}$). For very rough plates ($\varepsilon/h = 0.03$), viscosity effect start to disappear at $Re = 10^6$ and turbulent shear force start to be dominant at higher Reynolds number.

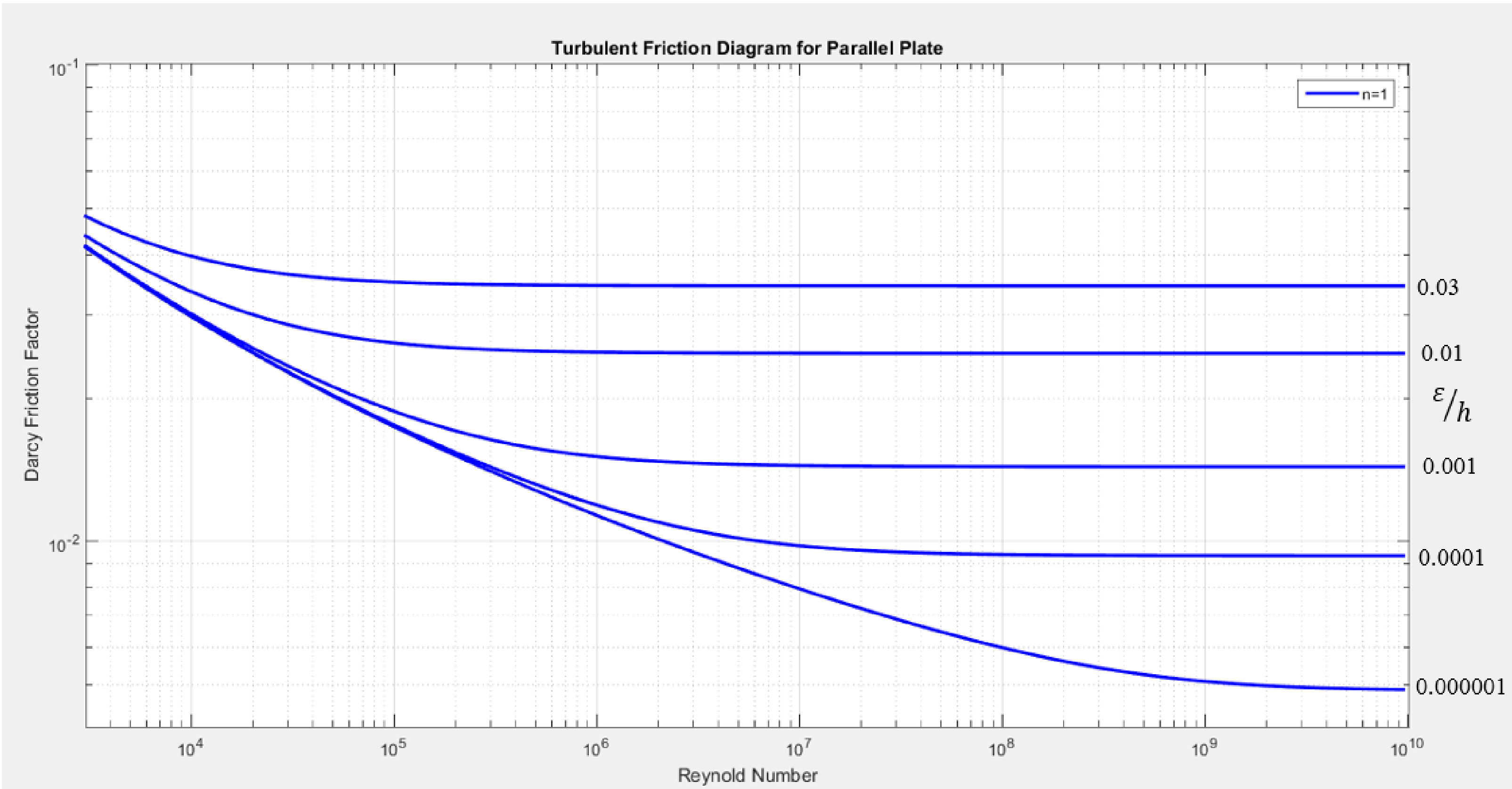

Figure 5.5: Darcy friction factor for flow between parallel plates

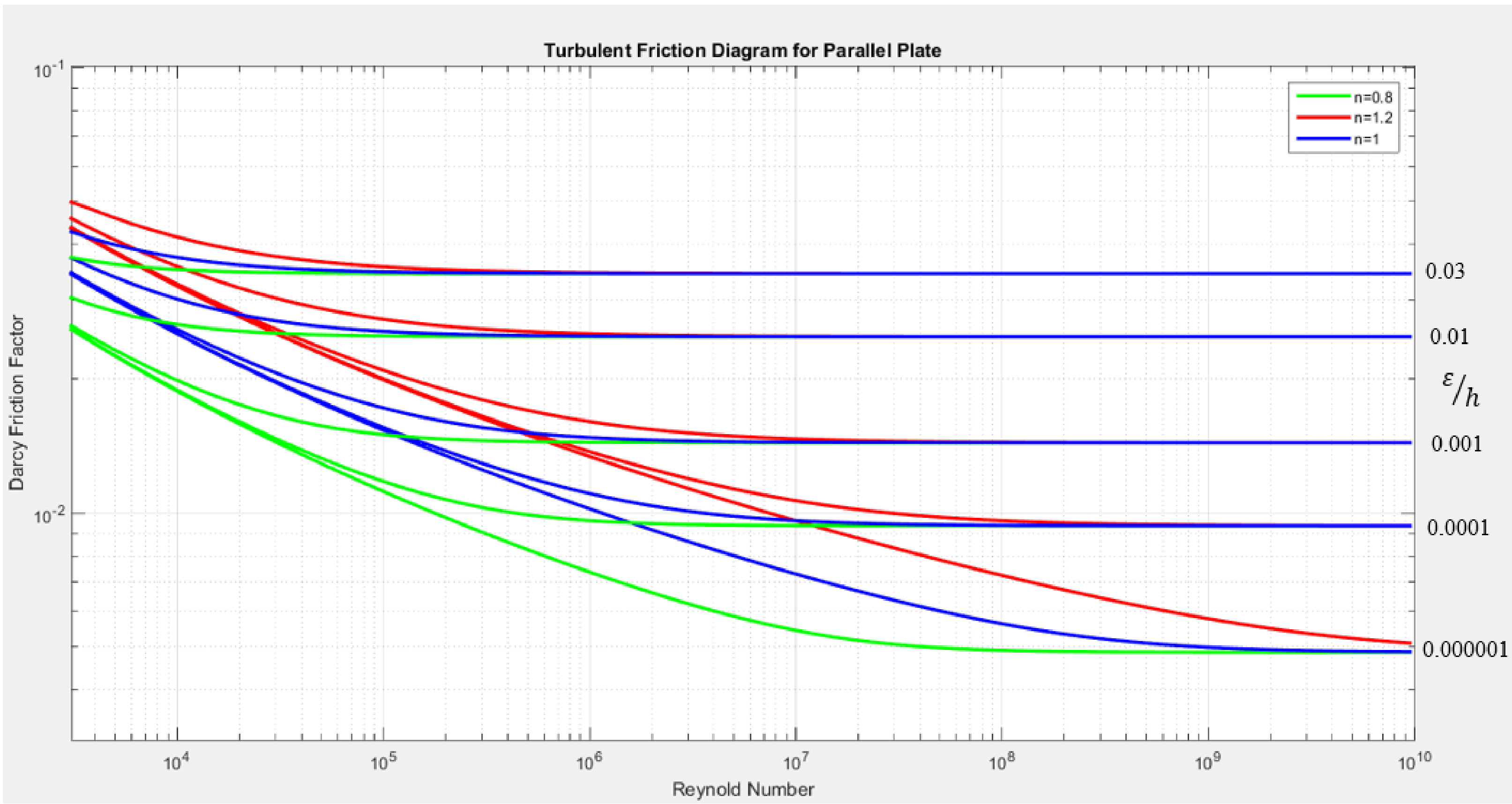

Figure 5.6: Darcy friction factor for flow between parallel plates with different values of flow behavior index

Appendix I

Second Order Nonlinear Ordinary Differential Equation Solution for Turbulent Core Velocity Distribution

$$\frac{v_*^2}{2h\,\kappa^2}\left(\frac{\partial^2\overline{u}}{\partial y^2}\right)^2 y-\left(\frac{\partial\overline{u}}{\partial y}\right)^4=0$$

In order to simplify the solution let $A=\frac{v_*^2}{2h\,\kappa^2}$, so equation (5.8) will be as following:

$$A\left(\frac{\partial^2\overline{u}}{\partial y^2}\right)^2 y-\left(\frac{\partial\overline{u}}{\partial y}\right)^4=0$$

Let $V(y)=\frac{\partial\overline{u}}{\partial y}$, which gives $\frac{\partial V}{\partial y}=\frac{\partial^2\overline{u}}{\partial y^2}$.

Substitute $V(y)$ and $\frac{\partial V}{\partial y}$ in :

$$A\left(\frac{\partial V}{\partial y}\right)^2 y-V^4=0$$

Solve forV'^2:

$$\left(\frac{\partial V}{\partial y}\right)^2=\frac{V^4}{Ay}$$

After taking the square root of both sides:

$\frac{\partial V}{\partial y} = \frac{V^2}{\sqrt{Ay}}$ or $\frac{\partial V}{\partial y} = -\frac{V^2}{\sqrt{Ay}}$

For First solution $\frac{\partial V}{\partial y} = \frac{V^2}{\sqrt{Ay}}$, divide both sides by $\frac{V^2}{\sqrt{A}}$ and:

$\frac{\sqrt{A}}{V^2}\frac{\partial V}{\partial y} = \frac{1}{\sqrt{y}}$

Integrate with respect to y:

$\sqrt{A} \int \frac{\frac{\partial V}{\partial y}}{V^2}\, dy = \int \frac{1}{\sqrt{y}}\, dy$

$-\frac{\sqrt{A}}{V} = 2\sqrt{y} + C_1$

Solve for V:

$V = -\frac{\sqrt{A}}{2\sqrt{y}+C_1}$

Since $V(y) = \frac{\partial \bar{u}}{\partial y}$:

$\frac{\partial \bar{u}}{\partial y} = -\frac{\sqrt{A}}{2\sqrt{y}+C_1}$

Integrate both sides with respect to y:

$\bar{u} = -\sqrt{A} \int \frac{1}{2\sqrt{y}+C_1} dy$

$\bar{u} = -\sqrt{A}\left[\sqrt{y} - \frac{C_1}{2}\ln\left(2\sqrt{y} + C_1\right)\right] + C_2$

Apply the following boundary conditions:

$\left.\frac{\partial \bar{u}}{\partial y}\right|_{y=h} = \infty$ & $\bar{u}|_{y=0} = u_{max}$

$$\frac{\partial \bar{u}}{\partial y} = -\frac{\sqrt{A}}{2\sqrt{y} + c_1}$$

At $y = h \Rightarrow \frac{\partial \bar{u}}{\partial y} = -\frac{\sqrt{A}}{2\sqrt{h} + c_1} = \infty \Rightarrow -\left(2\sqrt{h} + c_1\right) = 0 \Rightarrow \boxed{c_1 = -2\sqrt{h}}$

Substitute c_1:

$$\bar{u} = -\sqrt{A}\left[\sqrt{y} + h\ln\left(2\sqrt{y} - 2h\right)\right] + C_2$$

Apply the second boundary condition:

At $y = 0 \Rightarrow \bar{u} = -\sqrt{A}\sqrt{h}\ \ln\left(-2\sqrt{h}\right) + c_2 = u_{max} \Rightarrow$

$$\boxed{c_2 = u_{max} + \sqrt{A}\sqrt{h}\ \ln\left(-2\sqrt{h}\right)}$$

Substitute c_2 in (5.11):

$$\bar{u} = -\sqrt{A}\sqrt{y} - \sqrt{A}\sqrt{h}\ \ln\left(2\sqrt{y} - 2\sqrt{h}\right) + u_{max} + \sqrt{A}\sqrt{h}\ \ln\left(-2\sqrt{h}\right)$$

Simplify as per the following steps:

$$\bar{u} = \sqrt{A}\left[-\sqrt{y} - \sqrt{h}\ \ln\left(2\sqrt{y} - 2\sqrt{h}\right) + \sqrt{h}\ \ln\left(-2\sqrt{h}\right)\right] + u_{max}$$

$$\bar{u} = \sqrt{A}\left[-\sqrt{y} - \sqrt{h}\ \ln\left(2\sqrt{y} - 2\sqrt{h}\right) + \sqrt{h}\ \ln\left(-2\sqrt{h}\right)\right] + u_{max}$$

$$\bar{u} = \sqrt{A}\left[-\sqrt{y} + \sqrt{h}\ \ln\left(\frac{-2\sqrt{h}}{2\sqrt{y}-2\sqrt{h}}\right)\right] + u_{max}$$

$$\bar{u} = \sqrt{A}\left[-\sqrt{y} + \sqrt{h}\ \ln\left(\frac{\sqrt{h}}{\sqrt{h}-\sqrt{y}}\right)\right] + u_{max}$$

Substitute $A = \frac{v_*^2}{2h\,\kappa^2}$:

$$\bar{u} = \frac{v_*}{\sqrt{2h}\,k}\left[-\sqrt{y} + \sqrt{h}\ \ln\left(\frac{\sqrt{h}}{\sqrt{h}-\sqrt{y}}\right)\right] + u_{max}$$

$$\bar{u} = \frac{v_*}{\sqrt{2}\,k}\left[-\sqrt{\frac{y}{h}} + \ln\left(\frac{\sqrt{h}}{\sqrt{h}-\sqrt{y}}\right)\right] + u_{max}$$

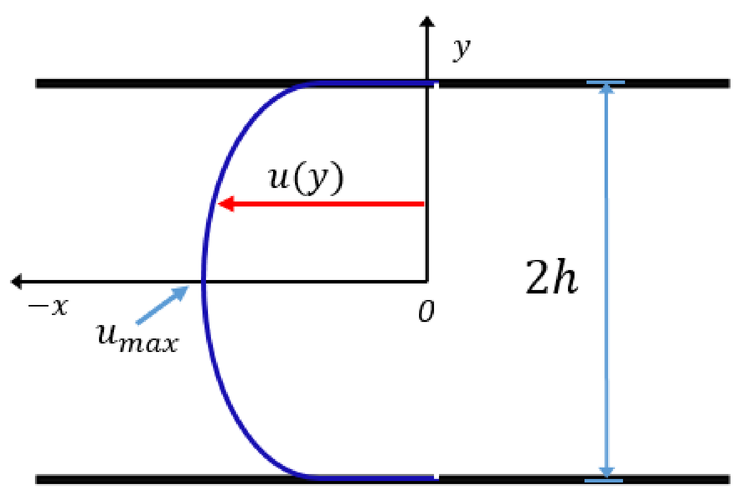

Figure A: First solution plot.

Based on the plot the curve is pointing to reverse direction of expected solution which does not make sense. Let's try the second solution.

For the second solution $\frac{\partial V}{\partial y} = -\frac{V^2}{\sqrt{Ay}}$, divide both sides by $\frac{V^2}{\sqrt{A}}$ and:

$$\frac{\sqrt{A}}{V^2}\frac{\partial V}{\partial y} = -\frac{1}{\sqrt{y}}$$

Integrate with respect to y:

$$\sqrt{A}\int \frac{\frac{\partial V}{\partial y}}{V^2}\, dy = -\int \frac{1}{\sqrt{y}}\, dy$$

$$-\frac{\sqrt{A}}{V} = -2\sqrt{y} + C_1$$

Solve for V:

$$V = \frac{\sqrt{A}}{2\sqrt{y}+C_1}$$

Since $V(y) = \frac{\partial \overline{u}}{\partial y}$:

$$\frac{\partial \overline{u}}{\partial y} = \frac{\sqrt{A}}{2\sqrt{y}+C_1}$$

Integrate both sides with respect to y:

$$\bar{u} = \sqrt{A}\int \frac{1}{2\sqrt{y}+C_1}dy$$

$$\bar{u} = \sqrt{A}\left[\sqrt{y} - \frac{C_1}{2}\ln\left(2\sqrt{y} + C_1\right)\right] + C_2$$

Apply the following boundary conditions:

$$\left.\frac{\partial \bar{u}}{\partial y}\right|_{y=h} = \infty \quad \& \quad \bar{u}|_{y=0} = u_{max}$$

$$\frac{\partial \bar{u}}{\partial y} = \frac{\sqrt{A}}{2\sqrt{y} + c_1}$$

At $y = h \Rightarrow \frac{\partial \bar{u}}{\partial y} = \frac{\sqrt{A}}{2\sqrt{h} + c_1} = \infty \Rightarrow \left(2\sqrt{h} + c_1\right) = 0 \Rightarrow \boxed{c_1 = -2\sqrt{h}}$

Substitute c_1:

$$\bar{u} = \sqrt{A}\sqrt{y} - \sqrt{A}\sqrt{h}\ \ln\left(2\sqrt{y} - 2\sqrt{h}\right) + c_2$$

At $y = 0 \Rightarrow \bar{u} = \sqrt{A}\sqrt{h}\ \ln\left(-2\sqrt{h}\right) + c_2 = u_{max} \Rightarrow$

$$\boxed{c_2 = u_{max} - \sqrt{A}\sqrt{h}\ \ln\left(-2\sqrt{h}\right)}$$

Substitute c_2:

$$\bar{u} = \sqrt{A}\sqrt{y} + \sqrt{A}\,\sqrt{h}\ \ln\!\left(2\sqrt{y} - 2\sqrt{h}\right) + u_{max} - \sqrt{A}\,\sqrt{h}\ \ln\!\left(-2\sqrt{h}\right)$$

$$\bar{u} = \sqrt{A}\sqrt{y} + \sqrt{A}\,\sqrt{h}\ (\ln\!\left(2\sqrt{y} - 2\sqrt{h}\right) - \ln\!\left(-2\sqrt{h}\right)) + u_{max}$$

$$\bar{u} = \sqrt{A}\sqrt{y} + \sqrt{A}\,\sqrt{h}\ (\ln\left(\frac{2\sqrt{y}-2\sqrt{h}}{-2\sqrt{h}}\right)) + u_{max}$$

$$\bar{u} = \sqrt{A}\sqrt{y} + \sqrt{A}\,\sqrt{h}\ (\ln\left(\frac{\sqrt{h}-\sqrt{y}}{\sqrt{h}}\right)) + u_{max}$$

$$\bar{u} = \sqrt{A}\left[\sqrt{y} + \sqrt{h}\ \ln\left(1 - \sqrt{\frac{y}{h}}\right)\right] + u_{max}$$

Substitute $A = \frac{v_*^2}{2h\,\kappa^2}$:

$$\bar{u} = \frac{v_*}{\sqrt{2h}\,k}\left[\sqrt{y} + \sqrt{h}\ \ln\left(1 - \sqrt{\frac{y}{h}}\right)\right] + u_{max}$$

$$\bar{u} = \frac{v_*}{\sqrt{2}\,k}\left[\sqrt{\frac{y}{h}} + \ln\left(1 - \sqrt{\frac{y}{h}}\right)\right] + u_{max}$$

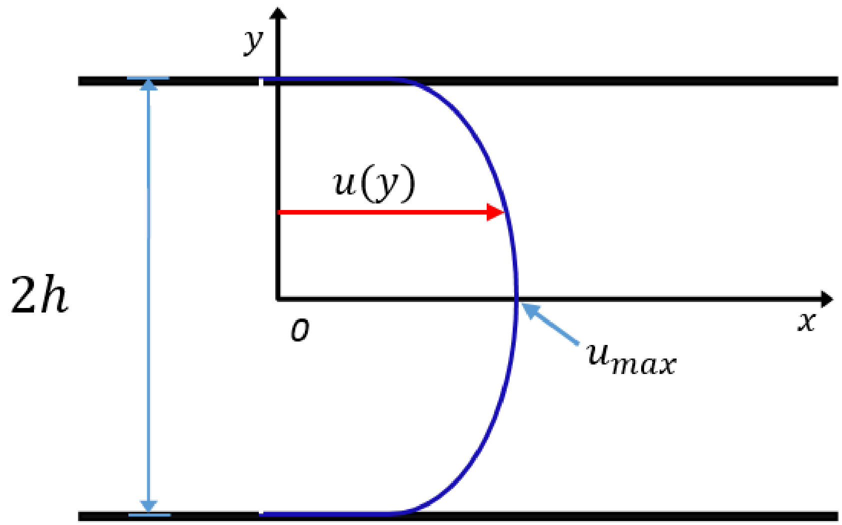

Figure B: second solution plot.

The second solution plot is pointing in the expected direction which makes sense.

Appendix II

Friction Factor Derivation for Smooth Pipe

Force balance in a pipe can be shown as following

$\sum F_x = 0$

$F = PA_{Cross\ Section}$, $F = \tau\ A_{\mathrm{Perpendicular}}$

$\Delta P\ \pi R^2 = \tau_w\ 2\pi R\ L$

$\frac{\Delta P}{L}\frac{R}{2} = \tau_w$

Substitute ${}^{\Delta P}/_{L}$ from Darcy–Weisbach friction equation $\frac{\Delta P}{L} =$

$\rho\frac{f}{D}\frac{\bar{u}^2}{2}$:

$\rho\frac{f}{2R}\frac{\bar{u}^2}{2}\frac{R}{2} = \tau_w$

$\rho f\ \frac{\bar{u}^2}{8} = \tau_w$

$f\ \frac{\bar{u}^2}{8} = {v_*}^2$

$\frac{\bar{u}}{v_*} = \sqrt{\frac{8}{f}}$

The general logarithmic formula for velocity distribution from the law of the wall:

$$\frac{u}{v_*} = \frac{1}{\kappa}\ln\left(\frac{yv_*}{\nu}\right) + C$$

The unknown $\frac{1}{\kappa}$ and C can be determent based on Nikurasce measurements, which are shown on figure C curve 3.

$$\frac{1}{\kappa} = 2.5 \quad C = 5.5$$

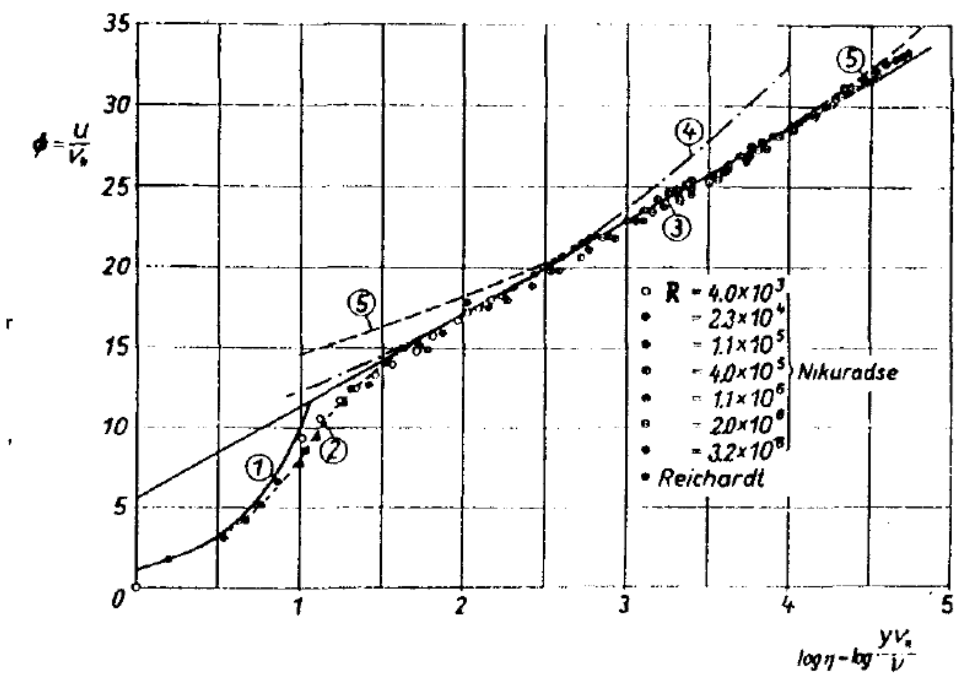

Figure C: universal velocity distribution for the smooth pipe. (Schlichting, H. - Boundary Layer Theory (Figure 20.4))

So the universal velocity distribution is:

$$\frac{u}{v_*} = 2.5 \ln\left(\frac{yv_*}{\nu}\right) + 5.5$$

or

$$\frac{u}{v_*} = 2.75 \log\left(\frac{yv_*}{\nu}\right) + 5.5$$

We need to have the previous equation LHS to be $\frac{\bar{u}}{v_*}$, T.E.

Stanton [8] deduced the following:

$$\frac{u_{max} - u}{v_*} = 5.75 \log\left(\frac{R}{y}\right)$$

Integrate one to get the average velocity distribution:

$$\bar{u} = u_{max} - 3.75 v_*$$

Substitute back to the universal velocity distribution:

$$\frac{\bar{u}}{v_*} = 2.75 \log\left(\frac{Rv_*}{\nu}\right) + 1.75$$

$\frac{Rv_*}{\nu}$ can be simplified as following:

$$\frac{Rv_*}{\nu} = \frac{1}{2}\frac{\bar{u}D}{\nu}\,\frac{v_*}{\bar{u}} = \frac{\bar{u}D}{\nu}\frac{1}{4}\sqrt{\frac{f}{2}}$$

And $\frac{\bar{u}}{v_*}$ can be substituted with $\sqrt{\frac{8}{f}}$, so we end up with universal law of friction for smooth pipe:

$$\frac{1}{\sqrt{f}} = 2.035 \log\left(\frac{\bar{u}D}{\nu}\sqrt{f}\right) - 0.91$$

The plot of $\frac{1}{\sqrt{f}}$ against $\log\left(\frac{\bar{u}D}{\nu}\sqrt{f}\right)$ shows a straight line as shown in figure D.

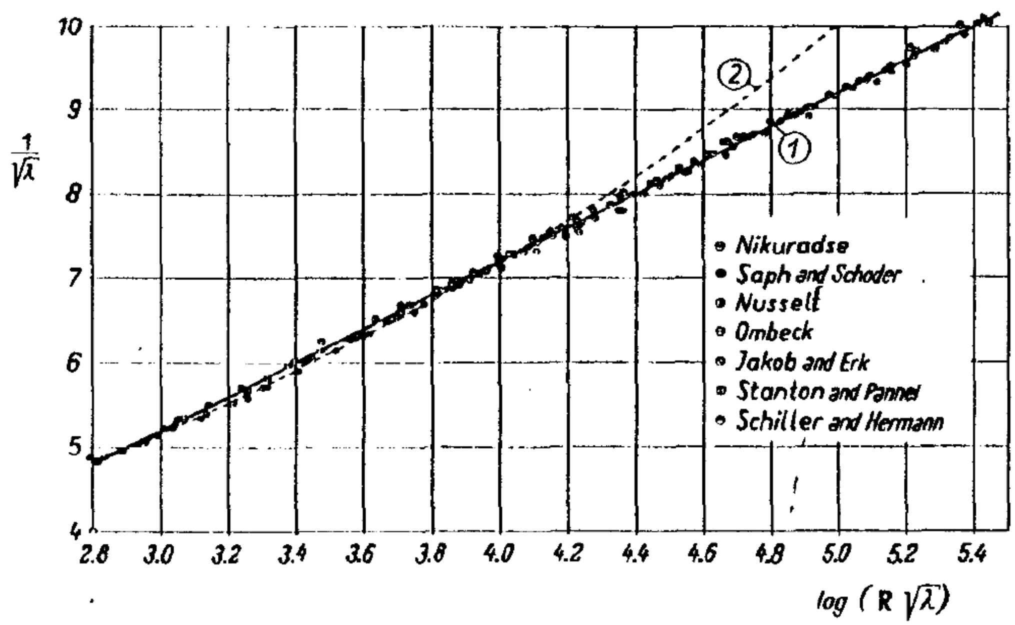

Figure D: universal law of friction for a smooth pipe. Curve.1 J. Nikuradse experiment result in line with $\frac{1}{\sqrt{f}} = 2 \log\left(\frac{\bar{u}D}{\nu}\sqrt{f}\right) - 0.8$.(Schlichting, H. - Boundary Layer Theory (Figure 20.9))

The experimental work was done by J. Nikuradse [7] and the result differ a little from the original equation as shown in figure B curve.1. The corrected formula is:

$$\frac{1}{\sqrt{f}} = 2\ \log\left(\frac{\bar{u}D}{\nu}\sqrt{f}\right) - 0.8$$

Which is known as Prandtl's universal friction factor for the smooth pipe.

Appendix III

Friction Factor Derivation for Rough Pipe

Starting from Prandtl's mixing length hypotheses:

$$\tau = \rho\, l^2 \left(\frac{\partial u}{\partial y}\right)^2$$

Next to the wall we may assume $l = \kappa\, y$, since the turbulent shear stress at the wall is zero. Based on that:

$$\tau = \rho\, \kappa^2\, y^2 \left(\frac{\partial u}{\partial y}\right)^2$$

$$v_* = \kappa^2\, y^2 \left(\frac{\partial u}{\partial y}\right)^2$$

$$\frac{\partial u}{\partial y} = \frac{v_*}{\kappa\, y}$$

Integrate once:

$$u = \frac{v_*}{\kappa} \ln(y) + C$$

At certain distance from the wall $u = 0$, let's consider this distance to be y_0:

$$u = \frac{v_*}{\kappa} (\ln y - \ln y_0)$$

It natural to make y_0 proportional to roughness height k_s, so

$$y_0 = \gamma\, k_s$$

$$u = \frac{v_*}{\kappa}(\ln y - \ln \gamma\, k_s)$$

$$u = \frac{v_*}{\kappa}\left(\ln \frac{y}{k_s} - \ln \gamma\right)$$

The constant γ depends on the nature of the particular roughness. Comparing this with Nikuradse's measurements for $\kappa = 0.4$, we can represent the equation as following:

$$\frac{u}{v_*} = 2.5\ \ln\left(\frac{y}{k_s}\right) + B$$

Where B have three different values for each roughness range. For completely rough pipe $B = 8.5$.

$$\frac{u}{v_*} = 2.5\ \ln\left(\frac{y}{k_s}\right) + 8.5$$

Or

$$\frac{u}{v_*} = 5.75\ \log\left(\frac{y}{k_s}\right) + 8.5$$

Since $\bar{u} = u - 3.75 v_*$,

$$\frac{\bar{u}}{v_*} = 5.75\ \log\left(\frac{R}{k_s}\right) + 4.75$$

Solve for friction factor as $\frac{\bar{u}}{v_*} = \sqrt{\frac{8}{f}}$:

$$\frac{1}{\sqrt{f}} = 2\ \log\left(\frac{R}{k_s}\right) + 1.68$$

This the formula for completely rough flow and it’s been corrected by Nikuradse’s experimental result as shown in figure E:

$$\frac{1}{\sqrt{f}} = 2\ \log\left(\frac{R}{k_s}\right) + 1.74$$

Or

$$\frac{1}{\sqrt{f}} = 2\ \log\left(3.7\,\frac{D}{k_s}\right)$$

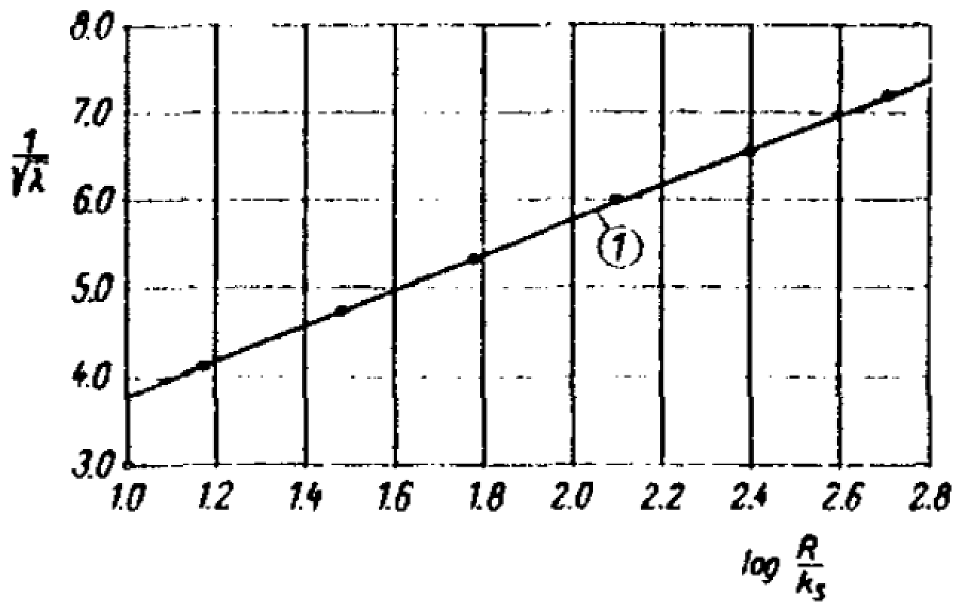

Figure E: resistance formula of sand roughened pipes in completely rough regime (Schlichting, H. - Boundary Layer Theory (Figure 20.23))

Appendix IV

Examples

Example 1:

Find the pressure drop and the hydraulic power for a flow rate between two parallel plates equal to $4\ {m^3}/{s}$. The plate have length of 2 m, width 2 m and the distance between the plates is 0.2 m. the relative roughness for the plates ${\varepsilon}/{h} = 0.001$ as shown in Figure F. The fluid properties are $n = 0.8,\quad \rho = 1450\ \frac{Kg}{m^3},\quad K = 0.8\ Pa.s^{0.8}$.

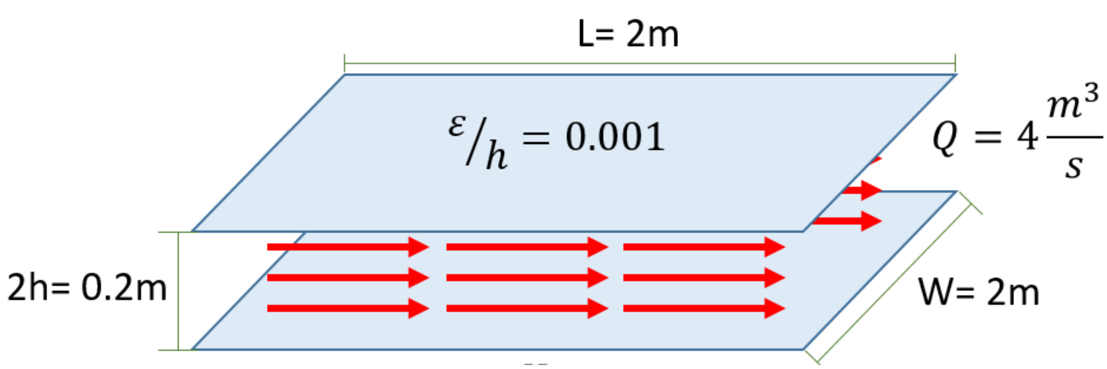

Figure F: Flow between Two Rough Parallel Plates

Average velocity $c = \frac{Q}{A} = \frac{4}{0.2\ (2)} = 10\ \frac{m}{s}$

$$Re^* = \frac{12\ \rho\ c^{2-n}\ h^n}{\left(\frac{1+2n}{n}\right)^n K} = 2.1X10^4 \rightarrow f\ = 0.017$$

$\frac{\Delta P}{L} = f\ \frac{\rho}{2}\ \ \frac{c^2}{4h} \rightarrow \Delta P = 6.16\ KPa$

$P_h = Q\ \Delta P = 61.6\ KW = 82.6\ hp$

Example 2:

Find the pressure drop and the hydraulic power for a flow rate between two parallel plates equal to 4 $m^3/_s$. The plate have length of 2 m, width 2 m and the distance between the plates is 0.2 m. the relative roughness for the plates $\varepsilon/_h = 0.001$ as shown in Figure F. The fluid properties are $n = 1,\quad \rho = 1600\ \frac{Kg}{m^3},\ \ K = 0.8\ Pa.s^{1.2}$.

Average velocity $c = \frac{Q}{A} = \frac{4}{0.2\ (2)} = 10\ \frac{m}{s}$

$Re^* = \frac{12\ \rho\ c^{2-n}\ h^n}{\left(\frac{1+2n}{n}\right)^n K} = 8X10^3 \rightarrow f = 0.049$

$\frac{\Delta P}{L} = f\ \frac{\rho}{2}\ \ \frac{c^2}{4h} \rightarrow \Delta P = 19.6\ KPa$

$P_h = Q\ \Delta P = 78.4\ KW = 105\ hp$

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