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FILTERING THE BACK-PROJECTION

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ABSTRACT

Expressions for the fan-beam projection and back-projection operators are given along with an evaluation of the impulse response for the composite of the projection and back-projection operations. The fan-beam geometries for both curved and flat detectors have back-projection operators such that the impulse response for the composite of the projection and back-projection operations is equal to $1/|r - r_0|$ if the data are taken over 360° and thus two-dimensional Fourier filter techniques can be used to reconstruct transverse sections from fan-beam projection data. It will be shown that the impulse response for the composite of the projection and back-projection operators for fan-beam geometry is not spatially invariant if the data are taken over 180° as is true for parallel-beam projection data. The convolution algorithm is approximately four times faster and requires 40% less memory than the filter of the back-projection algorithm. However, the filter of the back-projection algorithm has the advantage of easily implementing different frequency space filters which lends itself to easily changing the noise propagation vs. resolution properties of the convolution kernel.

1. INTRODUCTION

The reconstruction of transverse sections from fan-beam projection data has important applications in both transmission and emission tomographic systems. Fan-beam systems utilizing a transmission point source such as an x-ray tube, have the advantage over parallel-beam systems of collecting multiple angles simultaneously in a short time interval. Emission systems can utilize a fan-beam geometry by collecting data either using an array of converging collimators such as pin hole collimation or coincidence counters such as a positron ring detector. These systems are applicable to determining dynamic and physiological changes in the transverse section as a function of time.

Fan-beam and parallel-beam geometries have back-projection operators which give an image representing the true image convolved with a blurring function that is equal to approximately $1/r$ if sufficient views are available. The relationship between the true and back-projected image is given by the equation

$$\text{back-projection}(r, \phi) = f(r, \phi) * 1/r \quad (1)$$

where f is the true image and 'back-projection (r, ϕ) ' is the back-projection image. Using the two-dimensional Fourier filter techniques the true image is reconstructed from the back-projected image for parallel-beam and fan-beam geometries by de-convolving the $1/r$ blurring function using the relationship

$$f(r, \phi) = \mathcal{F}_2^{-1} \left\{ \tilde{c}(s) \cdot \mathcal{F}_2 [\text{back-projection}(r, \phi)] \right\} \quad (2)$$

where f is the transverse section, 'back-projection(r, ϕ)' is the back-projection image, $\tilde{c}$ is the filter function and s is the radial component of the frequency. Budinger and Gullberg [1] conjectured that this result was true for fan-beam geometries and for beams of arbitrary orientations such as curved rays and randomly oriented chords from positron annihilation events. This paper shows that this conjecture is true for fan-beam geometries which collect projection data using either a curved or flat detector. This result was obtained for parallel-beam geometry by Bates and Peters [2] and has been implemented for fan-beam geometries in comparison studies with other algorithms [3].

The methods of reconstructing fan-beam projection data can be divided into six categories:

1. Two-dimensional filtering of the back-projection [1,3]. This is denoted as the filter of the back-projection algorithm and is formally expressed in Eq. (2).
2. Re-ordering and rebinning the fan-beam projection data so that the data correspond to a parallel-beam geometry [4,5]. This allows the use of straight-forward iterative methods, convolution methods, or Fourier space methods which are presently developed for reconstructing parallel-beam geometry.
3. Special convolution methods [6-9]. First a convolution function is applied to the fan-beam projection data and the result is then back-projected.
4. Radon's integration method [10,11]. The projection data are first differentiated, then followed by a Hilbert transform or an approximation using the trapezoidal method of integration.

5. Back-projection of the filtered fan-beam projection data [12].
For each projection the data are first Fourier transformed, then filtered, next the inverse Fourier transform is taken and the result is back-projected to give an approximate solution.
6. Iterative least-squares method [3]. A chi-square function representing the difference between the projected transverse section and the measured projection data is minimized using steepest descent or conjugate gradient methods of optimization.

This paper first defines the projection and back-projection operators for the fan-beam geometries for both curved and flat detectors. Then the proof is given for the spatial invariance of the impulse response for the composite of the projection and back-projection operations only under the condition that 360° of data are taken. The numerical implementation of the filter of the back-projection algorithm is then described along with results for fan-beam projection data from phantoms.

2. FAN-BEAM PROJECTION OPERATORS

The two fan-beam geometries dealt with in this paper are shown in Figs. 1 and 2. In Fig. 1, the data for the rays are collected using a curved detector; whereas in Fig. 2 the fan-beam data are collected using a flat detector. Both geometries have rays which emanate from a point which is a distance R from the center of rotation. The circular detector lies on a circle with radius R and center at the vertex of the fan beam. It is assumed that the projection data are measured with coordinates (ξ, θ) for a curved detector and coordinates (ζ, θ) for a flat detector. Thus the detector can be thought of as being positioned at the center of rotation.

Using the equation of the line shown in Fig. 1, the fan-beam projection operator, P_C , for a curved detector is defined as $P_C: f \rightarrow p_C$ where

$$p_C(\xi, \theta) = \iint_{R^2} f(x, y) \delta(R \sin(\xi/R) + x \sin(\theta + \xi/R) - y \cos(\theta + \xi/R)) dx dy \quad (3)$$

The integral is taken over R^2 and it is assumed that $f(x, y) = 0$ outside the cone spanned by the fan beam. The limits of the cone are a function of the length of the curved detector and a function of R which is the distance between the vertex of the fan beam and the detector at the center of rotation.

The function $f(x, y) \delta(g(x, y, \xi, \theta))$ is interpreted as a line mass of density $f(x, y)$ on the ray defined by the equation $g(x, y, \xi, \theta) = 0$. Thus the function δ is a distribution, or generalized function which assigns to the function $f(x, y)$ the number $p_C(\xi, \theta)$ which is equal to the line integral of $f(x, y)$ along the ray defined by g [13].

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Fig. 1 Fan-beam geometry for data collected at projection angle θ using a curved detector. The projection data $p_c(\xi, \theta)$ represent line integrals for the lines $R \sin(\xi/R) + x \sin(\theta + \xi/R) - y \cos(\theta + \xi/R) = 0$ where R is the distance from the vertex of the fan to the center of rotation and ξ is the projected distance measured from the center of rotation along a curved detector positioned at the center of rotation.

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Fig. 2 Fan-beam geometry for data collected at projection angle θ using a flat detector. The projection data $p_f(\xi, \theta)$ represent line integrals for the lines $\zeta(x \cos \theta + y \sin \theta + R)/R + x \sin \theta - y \cos \theta = 0$ where R is the distance from the vertex of the fan to the center of rotation and ζ is the projected distance measured from the center of rotation along a flat detector positioned at the center of rotation.

Using the equation of the line shown in Fig. 2, the fan-beam projection operator, P_f , for a flat detector is defined as $P_f: f \rightarrow p_f$ where

$$p_f(\zeta, \theta) = \iint_{\mathbb{R}^2} f(x, y) \delta(\zeta(x \cos\theta + y \sin\theta + R)/R + x \sin\theta - y \cos\theta) dx dy \quad (4)$$

As in the case of the curved detector, the integral is taken over $\mathbb{R}^2$ and it is assumed that $f(x, y) = 0$ outside the cone spanned by the fan-beam.

The fan-beam projection operators for the curved and flat detectors are linear operators. For the curved detector the projections satisfy

$$p_c(\xi, \theta) = p_c(-\xi, \theta + \pi + 2\xi/R). \quad (5)$$

For the flat detector the projections satisfy

$$p_f(\zeta, \theta) = p_f(-\zeta, \theta + \pi + 2 \tan^{-1}(\zeta/R)). \quad (6)$$

Notice that both Eq. (5) and Eq. (6) do not satisfy a relationship similar to that for parallel-beam geometry (Fig. 3):

$$p_=(\rho, \psi) = p_=(-\rho, \psi + \pi) \quad (7)$$

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Fig. 3 Parallel-beam geometry for data collected at projection angle ψ .

The projection data $p_{\psi}(\rho, \psi)$ represent line integrals for the lines $\rho + x \sin \psi - y \cos \psi = 0$ where ρ is the projected distance measured from the center of rotation along a flat detector.

where

$$p_=(\rho, \psi) = \iint_{\tilde{R}^2} f(x,y) \delta(\rho + x \sin\psi - y \cos\psi) dx dy . \quad (8)$$

These relationships suggest that for fan-beam geometry the line integrals of the transverse section are not equally sampled if the data are taken over only π radians. It will be shown that the impulse response for the composite of the projection and back-projection operators for fan-beam geometry is not spatially invariant if the data are taken over π radians as is true for parallel-beam projection data. Data must be taken over 2π radians to ensure spatial invariance.

3. FAN-BEAM BACK-PROJECTION OPERATORS

The back-projection operation for parallel-beam geometry requires the summation of line integrals over the range of 180° and is given by the equation

$$b_{\parallel}(r, \phi) = \int_0^{\pi} p_{\parallel}(r \sin(\phi - \theta), \theta) d\theta. \quad (9)$$

Whereas, the fan-beam geometries require data samples around the full 360° . The back-projection operators for the fan-beam geometry are defined for a curved detector as $B_c: p_c \rightarrow b_c$ such that b_c is given by the equation

$$b_c(r, \phi) = 1/2 \int_0^{2\pi} p_c(\xi^*, \theta) d\theta \quad (10)$$

where

$$\xi^* = R \tan^{-1} \left[\frac{r \sin(\phi - \theta)}{R + r \cos(\phi - \theta)} \right] \quad (11)$$

and for a flat detector as $B_f: p_f \rightarrow b_f$ such that b_f is given by the equation

$$b_f(r, \phi) = 1/2 \int_0^{2\pi} p_f(\zeta^*, \theta) K(r, \phi, R, \theta) d\theta \quad (12)$$

where

$$\zeta^* = \frac{Rr \sin(\phi-\theta)}{R + r \cos(\phi-\theta)} \quad (13)$$

and

$$K(r, \phi, R, \theta) = \frac{[r^2 + R^2 + 2Rr \cos(\phi-\theta)]^{1/2}}{R + r \cos(\phi-\theta)} \quad (14)$$

Therefore, the fan-beam geometry requires a double sampling of projection data which is reflected in the factor of 1/2 in the definitions of the back-projection operators. Also notice in Eq. (12) that when using a flat detector a special weighting given by the function K in Eq. (14) is required for the back-projection operation.

Another operator that operates on the space of projection functions is the adjoint operator of the projection operator. A detailed analysis of this operator in the case of fan-beam geometry with a curved detector is given in reference [14]. The composite of this operator and the projection operator is shown not to be spatially invariant, and thus not applicable to reconstructing projection data using Fourier transform methods.

4. IMPULSE RESPONSE FOR THE COMPOSITE OF THE FAN-BEAM PROJECTION AND BACK-PROJECTION OPERATIONS USING A CURVED DETECTOR

The following theorem states that the composite of the projection and back-projection operations for a fan-beam geometry using a curved detector is space invariant, that is, the response to a point source $\delta(x-x_0) \delta(y-y_0)$ is given by $h_c(x-x_0, y-y_0)$ where h_c is the point spread function. In the proof below polar coordinates are used.

THEOREM 1. For the projection operator P_c defined in Eq. (3) and the back-projection operator B_c defined by Eq. (10), the composite of the projection and back-projection operators, $B_c \circ P_c$, has a point spread function

$$h_c(x,y) = \frac{1}{\sqrt{x^2 + y^2}} \quad (15)$$

such that $B_c \circ P_c [\delta(x-x_0) \delta(y-y_0)] = h_c(x-x_0, y-y_0)$.

PROOF: The impulse response of the fan-beam projection operator for a curved detector can be evaluated for a point source at the position (x_0, y_0) by substituting the two-dimensional delta function $f(x,y) = \delta(x-x_0) \delta(y-y_0)$ into Eq. (3). After transforming to polar coordinates $[(x,y) \rightarrow (r,\phi)]$ the projection operation gives

$$P_c [\delta(x-x_0) \delta(y-y_0)] = \delta(R \sin(\xi/R) + r_0 \sin(\theta + \xi/R - \phi_0)) \quad (16)$$

One can evaluate the composite operation of projecting and back-projecting for a point source by substituting Eq. (16) into Eq. (10), giving

$$h_c(r, \phi; r_o, \phi_o) = 1/2 \int_0^{2\pi} \delta(R \sin(\xi^*/R) + r_o \sin(\theta + \xi^*/R - \phi_o)) d\theta \quad (17)$$

where $h_c(r, \phi; r_o, \phi_o) = B_c \circ P_c [\delta(x-x_o) \delta(y-y_o)]$ and ξ^* is given by Eq. (11).

In order to evaluate Eq. (17) we use the relationship [15, p. 38]

$$1/2 \int_0^{2\pi} \delta(g_c(\theta)) d\theta = 1/2 \sum_i \int_0^{2\pi} \frac{\delta(\theta - \theta_i^*)}{\left| \frac{dg_c(\theta)}{d\theta} \right|_{\theta=\theta_i^*}} d\theta \quad (18)$$

where θ_i^* are the zeros of g_c ,

$$g_c(\theta) = R \sin(\xi^*/R) + r_o \sin(\theta + \xi^*/R - \phi_o). \quad (19)$$

The two relationships,

$$\sin(\xi^*/R) = \frac{r \sin(\phi - \theta)}{[r^2 + R^2 + 2rR \cos(\phi - \theta)]^{1/2}} \quad (20)$$

and

$$\cos(\xi^*/R) = \frac{R + r \cos(\phi - \theta)}{[r^2 + R^2 + 2rR \cos(\phi - \theta)]^{1/2}} \quad (21)$$

that follow from Eq. (11), allow us to rewrite $g_c(\theta)$ as

$$g_c(\theta) = \frac{a \cos \theta - b \sin \theta + c}{S^{1/2}} \quad (22)$$

where

$$a = R(r \sin \phi - r_o \sin \phi_o) \quad (23)$$

$$b = R(r \cos \phi - r_o \cos \phi_o) \quad (24)$$

$$c = r r_o \sin(\phi - \phi_o) \quad (25)$$

$$S = r^2 + R^2 + 2rR \cos(\phi - \theta). \quad (26)$$

Now $g_c(\theta)$ has two roots in the interval between 0 and 2π , namely θ_1^* and θ_2^* which satisfy the equations

$$\sin \theta_1^* = \frac{bc - a\sqrt{a^2 + b^2 - c^2}}{a^2 + b^2} \quad (27)$$

$$\cos \theta_1^* = \frac{-ac - b\sqrt{a^2 + b^2 - c^2}}{a^2 + b^2} \quad (28)$$

$$\sin \theta_2^* = \frac{bc + a\sqrt{a^2 + b^2 - c^2}}{a^2 + b^2} \quad (29)$$

$$\cos \theta_2^* = \frac{-ac + b\sqrt{a^2 + b^2 - c^2}}{a^2 + b^2} \quad (30)$$

The expressions for $a^2 + b^2$ and $a^2 + b^2 - c^2$ can be simplified using Eqs. (23)-(26), giving the vector equations

$$a^2 + b^2 = R^2(r^2 + r_o^2 - 2rr_o \cos(\phi - \phi_o)) = R^2|\underline{r} - \underline{r}_o|^2 \quad (31)$$

$$a^2 + b^2 - c^2 = R^2|\underline{r} - \underline{r}_o|^2 - r^2 r_o^2 \sin^2(\phi - \phi_o) = R^2|\underline{r} - \underline{r}_o|^2 - |\underline{r} \times \underline{r}_o|^2. \quad (32)$$

Next we evaluate $|dg_c(\theta)/d\theta|_{\theta=\theta_1^*}$ and $|dg_c(\theta)/d\theta|_{\theta=\theta_2^*}$ where the derivative of $g_c(\theta)$ [Eq. (22)] with respect to θ is

$$\frac{dg_c(\theta)}{d\theta} = \frac{(-a \sin\theta - b \cos\theta)S^{1/2} - (a \cos\theta - b \sin\theta + c)dS^{1/2}/d\theta}{S} \quad (33)$$

Substituting the expression for $\sin\theta_1^*$ and $\cos\theta_1^*$ given in Eq. (27) and Eq. (28) respectively and using the expression in Eq. (32), gives

$$\left| \frac{dg_c(\theta)}{d\theta} \right|_{\theta=\theta_1^*} = \frac{\sqrt{a^2+b^2-c^2}}{S^{1/2}(\theta_1^*)} = \frac{\sqrt{R^2|x-x_0|^2 - |x \times x_0|^2}}{S^{1/2}(\theta_1^*)} \quad (34)$$

(Notice that for the second term in the numerator of Eq. (33), $a \cos\theta_i^* - b \sin\theta_i^* + c = 0$ for $i = 1, 2$). Likewise for $\theta = \theta_2^*$ we have

$$\left| \frac{dg_c(\theta)}{d\theta} \right|_{\theta=\theta_2^*} = \frac{\sqrt{a^2+b^2-c^2}}{S^{1/2}(\theta_2^*)} = \frac{\sqrt{R^2|x-x_0|^2 - |x \times x_0|^2}}{S^{1/2}(\theta_2^*)} \quad (35)$$

The denominator in Eq. (34) and Eq. (35) is evaluated by substituting Eqs. (27)-(30) into the expression for S given in Eq. (26). First, for $S(\theta_1^*)$ we have

$$S(\theta_1^*) = \frac{(r^2+R^2)(a^2+b^2) - 2rR(ac \cos\phi - bc \sin\phi) - 2rR(a \sin\phi + b \cos\phi)\sqrt{a^2+b^2-c^2}}{a^2 + b^2}$$

(36)

where the expressions

$$2rR(ac \cos \phi - bc \sin \phi) = 2R^2 |\underline{r} \times \underline{r}_0|^2$$

and

$$2rR(a \sin \phi + b \cos \phi) \sqrt{a^2 + b^2 - c^2} = 2R^2 [\underline{r} \cdot (\underline{r} - \underline{r}_0)] \sqrt{R^2 |\underline{r} - \underline{r}_0|^2 - |\underline{r} \times \underline{r}_0|^2}.$$

Substituting these expressions along with Eqs. (31) and (32) into Eq. (36) gives

$$S(\theta_1^*) = \frac{R^2 |\underline{r} - \underline{r}_0|^2 - |\underline{r} \times \underline{r}_0|^2 - 2[\underline{r} \cdot (\underline{r} - \underline{r}_0)] \sqrt{R^2 |\underline{r} - \underline{r}_0|^2 - |\underline{r} \times \underline{r}_0|^2} + r^2 |\underline{r} - \underline{r}_0|^2 - |\underline{r} \times \underline{r}_0|^2}{|\underline{r} - \underline{r}_0|^2} \quad (37)$$

Since $[\underline{r} \cdot (\underline{r} - \underline{r}_0)]^2 = r^2 |\underline{r} - \underline{r}_0|^2 - |\underline{r} \times \underline{r}_0|^2$, the numerator in Eq. (37) is a perfect square and we can write $S(\theta_1^*)$ as

$$S(\theta_1^*) = \frac{\left\{ \sqrt{R^2 |\underline{r} - \underline{r}_0|^2 - |\underline{r} \times \underline{r}_0|^2} - \underline{r} \cdot (\underline{r} - \underline{r}_0) \right\}^2}{|\underline{r} - \underline{r}_0|^2} \quad (38)$$

Likewise for $S(\theta_2^*)$ we obtain

$$S(\theta_2^*) = \frac{\left\{ \sqrt{R^2 |\underline{r} - \underline{r}_0|^2 - |\underline{r} \times \underline{r}_0|^2} + \underline{r} \cdot (\underline{r} - \underline{r}_0) \right\}^2}{|\underline{r} - \underline{r}_0|^2} \quad (39)$$

Substituting the expression for $S(\theta_1^*)$ into Eq. (34) gives

$$\left| \frac{dg_c(\theta)}{d\theta} \right|_{\theta=\theta_1^*} = \frac{|\underline{r} - \underline{r}_0| \sqrt{R^2 |\underline{r} - \underline{r}_0|^2 - |\underline{r} \times \underline{r}_0|^2}}{\sqrt{R^2 |\underline{r} - \underline{r}_0|^2 - |\underline{r} \times \underline{r}_0|^2} - \underline{r} \cdot (\underline{r} - \underline{r}_0)} \quad (40)$$

and substituting the expression for $S(\theta_2^*)$ into Eq. (35) gives

$$\left| \frac{dg_c(\theta)}{d\theta} \right|_{\theta=\theta_2^*} = \frac{|\underline{r} - \underline{r}_0| \sqrt{R^2 |\underline{r} - \underline{r}_0|^2 - |\underline{r} \times \underline{r}_0|^2}}{\sqrt{R^2 |\underline{r} - \underline{r}_0|^2 - |\underline{r} \times \underline{r}_0|^2} + \underline{r} \cdot (\underline{r} - \underline{r}_0)} \quad (41)$$

Notice that the expressions for these derivatives differ only in the sign of the second term in the denominator. The denominator in Eq. (40) would normally be expressed as the absolute value since it comes from the square root of a perfect square. However, if we assume that R is greater than either r_0 or r then

$$\sqrt{R^2 |\underline{r} - \underline{r}_0|^2 - |\underline{r} \times \underline{r}_0|^2} > \underline{r} \cdot (\underline{r} - \underline{r}_0) = \sqrt{r^2 |\underline{r} - \underline{r}_0|^2 - |\underline{r} \times \underline{r}_0|^2}$$

and the denominator expressed in Eq. (40) is exactly equal to its absolute value.

Now we can evaluate the impulse response for the composite of the projection and back-projection operations for the fan-beam geometry with curved detector given in Eq. (17). Since $g_c(\theta)$ has only two roots in the interval $(0, 2\pi)$, we can use Eq. (18) and rewrite Eq. (17) as

$$h_c(r, \phi; r_o, \phi_o) = 1/2 \int_0^{2\pi} \frac{\delta(\theta - \theta_1^*)}{\left| \frac{dg(\theta)}{d\theta} \right|_{\theta=\theta_1^*}} + \frac{\delta(\theta - \theta_2^*)}{\left| \frac{dg(\theta)}{d\theta} \right|_{\theta=\theta_2^*}} d\theta \quad (42)$$

where θ_1^* and θ_2^* satisfy Eqs. (27)-(30) and $\left| \frac{dg_c(\theta)}{d\theta} \right|_{\theta=\theta_1^*}$ and $\left| \frac{dg_c(\theta)}{d\theta} \right|_{\theta=\theta_2^*}$ are given by Eqs. (40) and (41). Substituting these equations into Eq. (42) and integrating over θ gives

$$h_c(r, \phi; r_o, \phi_o) = \frac{\sqrt{R^2 |\tilde{r} - \tilde{r}_o|^2 - |\tilde{r} \times \tilde{r}_o|^2} - \tilde{r} \cdot (\tilde{r} - \tilde{r}_o)}{2 \sqrt{R^2 |\tilde{r} - \tilde{r}_o|^2 - |\tilde{r} \times \tilde{r}_o|^2} |\tilde{r} - \tilde{r}_o|} + \frac{\sqrt{R^2 |\tilde{r} - \tilde{r}_o|^2 - |\tilde{r} \times \tilde{r}_o|^2} + \tilde{r} \cdot (\tilde{r} - \tilde{r}_o)}{2 \sqrt{R^2 |\tilde{r} - \tilde{r}_o|^2 - |\tilde{r} \times \tilde{r}_o|^2} |\tilde{r} - \tilde{r}_o|} \quad (43)$$

$$= \frac{1}{|\tilde{r} - \tilde{r}_o|} \quad (44)$$

Expressing Eq. (44) in rectangular coordinates, we see that the response to the point source $\delta(x-x_0) \delta(y-y_0)$ is given by the expression

$$h_c(x-x_0, y-y_0) = \frac{1}{\sqrt{(x-x_0)^2 + (y-y_0)^2}}$$

where the point spread function h_c is given by

$$h_c(x, y) = \frac{1}{\sqrt{x^2 + y^2}}.$$

This proves the theorem. ||

Theorem 1 states that the linear operation of the composition of projecting and back-projecting for the fan-beam geometry with a curved detector is shift invariant; thus the operation can be characterized by the convolution equation

$$b_c(x, y) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x', y') h_c(x' - x, y' - y) dx' dy'. \quad (45)$$

Using the two-dimensional Fourier transform, the image $f(x, y)$ can be reconstructed from the back-projection image $b_c(x, y)$ using the expression

$$f(x, y) = \mathcal{F}_2^{-1} \left\{ \mathcal{F}_2[b_c(x, y)] / \mathcal{F}_2[h_c(x, y)] \right\} \quad (46)$$

As a corollary to the result given by Eq. (44), we see if the integration is taken from 0 to π in Eq. (42) then the result may not include both terms in Eq. (43). This is due to the fact that either the roots θ_1^* or θ_2^* or both may not be in the interval $(0, \pi)$. Therefore if we define the operator B_C^π as $B_C^\pi: p_C \rightarrow b_C^\pi$ such that b_C^π is given by the equation

$$b_C^\pi = 1/2 \int_0^\pi p_C(\xi^*, \theta) d\theta \quad (47)$$

and ξ^* is given by Eq. (11), then we have the following corollary to Theorem 1.

COROLLARY 1. For the operator P_C defined by Eq. (3) and the operator B_C^π defined by Eq. (47), the operator $B_C^\pi \circ P_C$ has an impulse response for a point source $\delta(x-x_0)\delta(y-y_0)$ given by

$$h_C^\pi(r, \phi; r_0, \phi_0) = \begin{cases} \frac{\sqrt{R^2 |\underline{r} - \underline{r}_0|^2 - |\underline{r} \times \underline{r}_0|^2} - \underline{r} \cdot (\underline{r} - \underline{r}_0)}{2 \sqrt{R^2 |\underline{r} - \underline{r}_0|^2 - |\underline{r} \times \underline{r}_0|^2} |\underline{r} - \underline{r}_0|} & \text{if } \theta_1^* \in (0, \pi) \\ \frac{\sqrt{R^2 |\underline{r} - \underline{r}_0|^2 - |\underline{r} \times \underline{r}_0|^2} + \underline{r} \cdot (\underline{r} - \underline{r}_0)}{2 \sqrt{R^2 |\underline{r} - \underline{r}_0|^2 - |\underline{r} \times \underline{r}_0|^2} |\underline{r} - \underline{r}_0|} & \text{if } \theta_2^* \in (0, \pi) \\ \frac{1}{|\underline{r} - \underline{r}_0|} & \text{if } \theta_1^*, \theta_2^* \in (0, \pi) \\ 0 & \text{otherwise} \end{cases} \quad (48)$$

where θ_1^* and θ_2^* satisfy Eqs. (27) - (30).

A similar result was given by Peters [12] for the point spread function of the fan-beam layergram. Therefore from Eq. (48) we see that the back-projection image cannot be characterized by a convolution equation if the projection data are integrated over 180° .

5. IMPULSE RESPONSE FOR THE COMPOSITE OF THE FAN-BEAM PROJECTION

AND

BACK-PROJECTION OPERATIONS USING A FLAT DETECTOR

The following theorem states that the composite of the projection and back-projection operations for a fan-beam geometry using a flat detector is also space invariant.

THEOREM 2. For the projection operator P_f defined by Eq. (4) and the back-projection operator B_f defined by Eq. (12), the composite of the projection and back-projection operators, $B_f \circ P_f$, has a point spread function

$$h_f(x,y) = \frac{1}{\sqrt{x^2 + y^2}} \quad (49)$$

such that $B_f \circ P_f [\delta(x-x_0) \delta(y-y_0)] = h_f(x-x_0, y-y_0)$.

PROOF: The impulse response of the fan-beam projection operator for a flat detector is given by

$$P_f[\delta(x-x_0) \delta(y-y_0)] = \delta[\zeta(r_0 \cos(\theta-\phi_0) + R)/R + r_0 \sin(\theta-\phi_0)] \quad (50)$$

Evaluating the composite operation of projecting and back-projecting for a point source, we substitute Eq. (50) into Eq. (12), and follow a similar proof as for the curved detector in Theorem 1 resulting in

$$h_f(r, \phi; r_o, \phi_o) = \frac{1}{|\underline{r} - \underline{r}_o|} \quad . \parallel$$

If we define the operator B_f^π as $B_f^\pi : P_f \rightarrow b_f^\pi$ such that b_f^π is given by the equation

$$b_f^\pi(r, \phi) = 1/2 \int_0^\pi p_f(\zeta^*, \theta) K(r, \phi, R, \theta) d\theta \quad (51)$$

and ζ^* is given by Eq. (13) and $K(r, \phi, R, \theta)$ is given by Eq. (14), then we have the following corollary to Theorem 2.

COROLLARY 2. For the operator P_f defined by Eq. (4) and the operator B_f^π defined by Eq. (51), the operator $B_f^\pi \circ P_f$ has an impulse response $h_f^\pi(r, \phi; r_o, \phi_o)$ for a point source $\delta(x-x_o) \delta(y-y_o)$ given by the same expressions as Eq. (48).

The results given by Theorem 2 and Corollary 2 are identical to the results obtained in Theorem 1 and Corollary 1 for the fan-beam geometry using instead a curved detector.

6. NUMERICAL IMPLEMENTATION

The RECLBL Library [3] is a package of computational subroutines that apply to the reconstruction of transverse sections from projection data. The library has 9 user-called reconstruction algorithms. The specific algorithm being investigated in this paper performs the following sequence of operations: back-project the projection data; Fourier transform the two-dimensional back-projection image; multiply the two-dimensionally distributed Fourier coefficients by one of the optional filter functions supplied by the RECLBL Library; and perform the two-dimensional inverse Fourier transform. These algorithm operations are symbolized in Eq. (2).

The derivation of the algorithm follows immediately from Eq. (46) and the fact that $\mathcal{F}_2(r^{-1}) = s^{-1}$ where s is the radial coordinate in frequency space. In order to obtain a desired roll off of $|s|$, this is replaced by the filter function $\tilde{c}$, which is equal to the product of a window function $w(s)$ and the absolute value of the radial component of the frequency:

$$\tilde{c}(s) = |s| w(s). \quad (52)$$

The purpose of this method of reconstruction is to effect a deconvolution of the true image f from the back-projection image.

The digital implementation of this algorithm by the RECLBL Library first back-projects the projection data p_{km} using the equation

$$b_{ij} = \sum_{km} F_{ij}^{km} p_{km} \quad (53)$$

where k is the projection bin index, m is the angle index, and F_{ij}^{km} are the weighting factors in the back-projection subroutine. For the fan-beam geometry using a flat detector, the factors F_{ij}^{km} also include the kernel K given in Eq. (14). The back-projection image is then discrete Fourier transformed using the equation

$$\tilde{b}_{k\ell} = \frac{1}{NDIM^2} \sum_{n=0}^{NDIM-1} \sum_{m=0}^{NDIM-1} b_{nm} \exp [-2\pi i (kn + \ell m) / NDIM]. \quad (54)$$

NDIM is equal to 2^{IPOW2} where $IPOW2 = 2 \times$ (the smallest power of two that is greater than or equal to NDIMU which is the linear dimension of the reconstruction array.) Next the discrete Fourier transformed values $\tilde{b}_{k\ell}$ are multiplied by a filter function $\tilde{c}$ giving

$$\tilde{f}_{k\ell} = \tilde{c} \left(\frac{\sqrt{k^2 + \ell^2}}{NDIM} \right) \tilde{b}_{k\ell} \quad (55)$$

Then the values $\tilde{f}_{k\ell}$ are inverse Fourier transformed and multiplied by a normalization factor to give the reconstruction

$$f_{nm} = \frac{\pi}{NANG * PWID} \sum_{k=0}^{NDIM-1} \sum_{\ell=0}^{NDIM-1} \tilde{f}_{k\ell} \exp [2\pi i (nk + m\ell) / NDIM] \quad (56)$$

where NANG is the total number of projection angles and PWID is the linear dimension of the pixels (digitized picture cells) in units of projection bin widths. The factor π/NANG is the step size in the numerical calculation of the back-projection integral. The factor $1/\text{PWID}$ is the result of scaling the reconstruction space.

Implementation of the algorithm involves a two-dimensional Fast Fourier Transform. A number of practical aspects would agree that this approach is less efficient than the commercially used convolution methods [7,8]. The reason for this is that a matrix which is four times the size of the reconstructed image is required so that the convolution result of one period does not overlap the convolution result of the succeeding period when implementing the Fast Fourier Transform [16, Chapter 7].

7. RESULTS

The results for the composite of the fan-beam projection and back-projection operations and the corresponding reconstructions are shown in Fig. (4) for various point source orientations with coordinates (r_0, ϕ_0) and rotations of 180° and 360° . The projection data represent a sampling at 1° increments using a curved detector; however the results are the same using a flat detector. The middle column of figures are images of the back-projection multiplied by $|r - r_0|$ where (r_0, ϕ_0) is the coordinate of the point source. This has uniform gray only if the back-projection is proportional to $1/|r - r_0|$. The back-projection operation for a rotation of 360° is approximately proportional to $1/|r - r_0|$, but due to the digitalization, the image of Back-Projection $\cdot |r - r_0|$ represents a 10% variation between the maximum and minimum values. The contrast for the case of 360° rotation was increased to illustrate this variation.

Notice that for point sources which lie along the central axis of the fan-beam (rows 1-3 of Fig. 4) the points (r, ϕ) in the transverse section have at least one root θ_1^* or θ_2^* which is in the interval $(0, \pi)$. Therefore Corollary 1 predicts that the result will be one of the top two expressions in Eq. (48). The expression in the first line of Eq. (48) is represented in the middle column of rows 1-3 of Fig. 4 as the lighter areas and the second expression is represented by brighter areas. Keep in mind that the value of $h_c^\pi(r, \phi)$ or $h_f^\pi(r, \phi)$ is determined by whether θ_1^* or θ_2^* is in the interval $(0, \pi)$. This in turn is dependent on the values of the coordinates (r, ϕ) .

If the point source lies along the x-axis, the root θ_2^* is in the interval $(0, \pi)$ for points (r, ϕ) in the upper half of the xy-plane and the root θ_1^* is in the interval $(0, \pi)$ for points (r, ϕ) in the lower half of the xy-plane.

Fig. 4 Comparison of the point source response as a function of position and rotation angle.

For example, we see from Eqs. (27) - (30) that for a centrally positioned source ($r_0 = 0$)

$$\sin \theta_1^* = \frac{-y}{|\tilde{r}|} \quad \cos \theta_1^* = \frac{-x}{|\tilde{r}|}$$

$$\sin \theta_2^* = \frac{y}{|\tilde{r}|} \quad \cos \theta_2^* = \frac{x}{|\tilde{r}|}$$

Therefore for points in the lower half of the plane $\theta_1^* \in (0, \pi)$ and Eq. (48) reduces to

$$|\tilde{r}| h_c^\pi(r, \phi) = \frac{1}{2} - \frac{|\tilde{r}|}{2R}$$

For points in the upper half of the plane $\theta_2^* \in (0, \pi)$ and Eq. (48) reduces to

$$|\tilde{r}| h_c^\pi(r, \phi) = \frac{1}{2} + \frac{|\tilde{r}|}{2R}$$

Therefore for a centrally positioned point source the upper and lower planes show a contrast (row 2 of Fig. 4) about the abscissa as the result of the discontinuity along the x-axis.

For point sources that are not on the central axis rows 4-7 of Fig. 4 show yet a different result. In rows 4 and 5 the bright areas represent where both θ_1^* and θ_2^* are in the interval $(0, \pi)$ and thus the back-projection is equal to $1/|r-r_0|$. The other areas are characterized by either one of the two top expressions in Eq. (48). There is no point where the back-projection equals to zero. However, for rows 6 and 7 of Fig. 4 the black areas are where the back-projection is equal to zero. One can see that this must be so since there are no fan-rays which will lie in the black area and intersect the point (r_0, ϕ_0) for any projection angle between 0 and π .

Fig. 5 gives a comparison of the results from filtering the parallel-beam and fan-beam back-projection for simulated phantom data. Artifacts are generated when the fan-beam back-projection operation is intergrated only over 180° . The Hann window was used instead of the rectangular window in order to eliminate the ringing which occurs with a sharp cut-off. Other windows such as the Hamming, Parzen, and Butterworth have been studied and the results are discussed in [3].

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Fig. 5 Comparison of results from filtering the back-projection for parallel-beam and fan-beam geometries.

8. CONCLUSION

Three key points are derived from the analysis presented:

1. There exists fan-beam back-projection operators for both curved and flat detectors for which the composite of the projection and back-projection operations give a result equal to $1/|\underline{r}-\underline{r}_0|$ for a point source.
2. These operators involve the integration of projection data taken over 360° and not just 180° which is sufficient for parallel-beam geometry.
3. The convolution algorithm [7] requires less memory and shorter computational time than the filter of the back-projection algorithm with existing hardware [3].

Point 1 implies that the back-projection operation gives an image which can be represented as a convolution of the original image with the point spread function $1/|\underline{r}|$, and thus applicable for the filter of the back-projection approach to transverse section reconstruction. A disadvantage of using this approach over other fan-beam reconstruction methods which first filter and then back-project, is that a matrix which is four times the size of the reconstructed image is required so that the convolution result of one period does not overlap the convolution result of the succeeding period when implementing the Fast Fourier Transform. Thus the filtered of the back-projection algorithm requires more computer memory and longer computational time. However the algorithm can be developed with various options for frequency space filters which lends itself to easily changing the noise propagation vs. resolution properties of the convolution kernel.

The results using the filter of the back-projection approach to reconstructing fan-beam data hints at a general approach to reconstructing projection data generated from nonparallel beams such as curved rays. It is required that a back-projection operator exists which gives a point source response which is either proportional to $1/|\underline{r}-\underline{r}_0|$ or is a function which depends only on the radial distance from the point source. It is also conceivable that there are back-projection operators for the attenuated Radon transform [17] and for the three-dimensional geometries such as the cone beam which give a back-projection image that can be deconvolved using the filter of the back-projection approach for reconstruction.

APPENDIX A

This appendix lists some of the properties of fan-beam geometry which are important for the implementation of other algorithms.

I. The coordinates (ρ, ψ) for the parallel-beam projection data are related to the coordinates (ξ, θ) of the fan-beam projection data for a curved detector through the transformations:

$$\begin{aligned}\rho &= R \sin(\xi/R) & \psi &= \theta + \xi/R \\ \xi &= R \sin^{-1}(\rho/R) & \theta &= \psi - \sin^{-1}(\rho/R)\end{aligned}\tag{A1}$$

and are related to the coordinates (ζ, θ) of the fan-beam projection data for a flat detector through the transformation

$$\begin{aligned}\rho &= R\zeta/\sqrt{\zeta^2 + R^2} & \psi &= \theta + \tan^{-1}(\zeta/R) \\ \zeta &= R\rho/\sqrt{R^2 - \rho^2} & \theta &= \psi - \sin^{-1}(\rho/R)\end{aligned}\tag{A2}$$

These transformations are used to reorganize the fan-beam data into parallel-beam data so that reconstruction algorithms applicable to parallel-beam geometry can be used.

II. Integration over ξ of the impulse response of the fan-beam projection operator for a curved detector (Eq. (16)) gives

$$\int_{-\infty}^{\infty} \delta(R \sin(\xi/R) + r_0 \sin(\theta + \xi/R - \phi_0)) d\xi = \frac{R}{\sqrt{r_0^2 + R^2 + 2r_0 R \cos(\phi_0 - \theta)}} \quad (A3)$$

Equation (A3) is equal to the ratio: (Distance from the fan vertex to the detector)/(Distance from the fan vertex to the point (r_0, ϕ_0) defined by the coordinate system rotated at the projection angle θ). Therefore the total density measured is not constant for each angle but varies as a function of the distance the point source is from the vertex of the fan beam. The iterative algorithms use Eq. (A3) as a model for the projection of data. This implies that pixels closer to the vertex of the fan-beam are given a larger weighting factor than those which are farther away.

Likewise for a flat detector, integration over ξ of Eq. (50) gives

$$\int_{-\infty}^{\infty} \delta \left[\zeta (r_0 \cos(\theta - \phi_0) + R)/R + r_0 \sin(\theta - \phi_0) \right] d\zeta = \frac{R}{R + r_0 \cos(\theta - \phi_0)} \quad (A4)$$

Eq. (A4) is equal to the ratio: (Distance from the fan source to the detector)/(Distance from the fan vertex to the point projected along the fan axis from the point (r_0, ϕ_0) defined by the coordinate system rotated a the projection angle θ).

III. The kernel K given in Eq. (14) can be expressed in the form

$$K(\zeta^*, R) = \sqrt{1 - (\zeta^*)^2/R} \quad . \quad (A5)$$

This allows us to rewrite the back-projection in Eq. (12) as

$$b_f(r, \phi) = 1/2 \int_0^{2\pi} \rho_f(\zeta^*, \theta) \sqrt{1 - (\zeta^*)^2/R} d\theta \quad . \quad (A6)$$

It was suggested by Dr. Ronald Huesman that this expression is computationally more efficient since the projection data $\rho_f(\zeta, \theta)$ can be premultiplied by the factor $\sqrt{1 - \zeta^2/R}$ before performing the back-projection operation.

APPENDIX B

Here a detailed proof of Theorem 2 is given. This proof is similar to the proof presented for Theorem 1.

PROOF: The projection of a point source is given by Eq. (50). Evaluating the composite operation of projecting and back-projecting for a point source, we substitute Eq. (50) into Eq. (12), giving

$$h_f(r, \phi; r_o, \phi_o) = \frac{1}{2} \int_0^{2\pi} \delta[\zeta^*(r_o \cos(\theta - \phi_o) + R)/R + r_o \sin(\theta + \phi_o)] \frac{[r^2 + R^2 + 2rR \cos(\phi - \theta)]^{\frac{1}{2}}}{R + r \cos(\phi - \theta)} d\theta \quad (B1)$$

where ζ^* is given by Eq. (13). Just as was done in the proof of Theorem 1, we will use the relationship

$$\frac{1}{2} \int_0^{2\pi} \delta(g_f(\theta)) \frac{S^{\frac{1}{2}}(\theta)}{R + r \cos(\phi - \theta)} d\theta = \frac{1}{2} \int_0^{2\pi} \sum_i \frac{\delta(\theta - \theta_i^*)}{\left| \frac{dg_f(\theta)}{d\theta} \right|_{\theta=\theta_i^*}} \frac{S^{\frac{1}{2}}(\theta)}{R + r \cos(\phi - \theta)} d\theta \quad (B2)$$

where $S(\theta)$ is given by Eq. (26) and θ_i^* are the zeros of g_f ,

$$g_f(\theta) = \zeta (r_o \cos(\theta - \phi_o) + R)/R + r_o \sin(\theta - \phi_o) \quad (B3)$$

The function $g_f(\theta)$ can be rewritten as

$$g_f(\theta) = \frac{a \cos\theta - b \sin\theta + c}{t} \quad (B4)$$

where a, b, and c are given by Eqs. (23)-(25) and

$$t = R + r \cos(\phi - \theta) \quad (B5)$$

It turns out that $g_f(\theta)$ has the same two roots in the interval 0 to 2π as did $g_c(\theta)$ in Eq. (22). These roots satisfy the Eqs. (27)-(30).

Next we need to evaluate $|dg_f(\theta)/d\theta|_{\theta=\theta_1^*}$ and $|dg_f(\theta)/d\theta|_{\theta=\theta_2^*}$ where the derivative of $g_f(\theta)$ with respect to θ is

$$\frac{dg_f(\theta)}{d\theta} = \frac{(-a \sin\theta - b \cos\theta)t - (a \cos\theta - b \sin\theta + c) dt/d\theta}{t^2} \quad (B6)$$

Substituting the expressions for $\sin \theta_1^*$ and $\cos \theta_1^*$ given in Eq. (27) and Eq. (28) respectively and using the expression in Eq. (32), gives

$$\left| \frac{dg_f(\theta)}{d\theta} \right|_{\theta=\theta_1^*} = \frac{\sqrt{a^2 + b^2 - c^2}}{t(\theta_1^*)} = \frac{\sqrt{R^2 |\underline{r} - \underline{r}_0|^2 - |\underline{r} \times \underline{r}_0|^2}}{t(\theta_1^*)} \quad (B7)$$

Likewise for $\theta = \theta_2^*$ we have

$$\left| \frac{dg_f(\theta)}{d\theta} \right|_{\theta = \theta_2^*} = \frac{\sqrt{a^2 + b^2 - c^2}}{t(\theta_2^*)} = \frac{\sqrt{R^2 |\underline{r} - \underline{r}_0|^2 - |\underline{r} \times \underline{r}_0|^2}}{t(\theta_2^*)} \quad (B8)$$

The denominators in Eq. (B7) and Eq. (B8) are evaluated by substituting the Eqs. (27)-(30) into the expression for t given in Eq. (B5). This gives the two expressions

$$t(\theta_1^*) = \frac{\sqrt{R^2 |\underline{r} - \underline{r}_0|^2 - |\underline{r} \times \underline{r}_0|^2} \left[\sqrt{R^2 |\underline{r} - \underline{r}_0|^2 - |\underline{r} \times \underline{r}_0|^2} - \underline{r} \cdot (\underline{r} - \underline{r}_0) \right]}{R |\underline{r} - \underline{r}_0|^2} \quad (B9)$$

$$t(\theta_2^*) = \frac{\sqrt{R^2 |\underline{r} - \underline{r}_0|^2 - |\underline{r} \times \underline{r}_0|^2} \left[\sqrt{R^2 |\underline{r} - \underline{r}_0|^2 - |\underline{r} \times \underline{r}_0|^2} + \underline{r} \cdot (\underline{r} - \underline{r}_0) \right]}{R |\underline{r} - \underline{r}_0|^2} \quad (B10)$$

We do not need to substitute the expressions given by Eq. (B9) and Eq. (B10) into Eqs. (B7) and (B8), since it will be shown below that these terms totally cancel. However, the explicit expressions for $t(\theta_1^*)$ and $t(\theta_2^*)$ are shown here to indicate that they are not singular as long as R is greater than r and r_0 .

Now we can evaluate the impulse response given in Eq. (B1). Since $g_f(\theta)$ has only two roots in the interval $(0, 2\pi)$ we can use Eq. (B2) and rewrite Eq. (B1) as

$$h_f(r, \phi; r_o, \phi_o) = \frac{1}{2} \int_0^{2\pi} \left[\frac{\delta(\theta - \theta_1^*)}{\left| \frac{dg_f(\theta)}{d\theta} \right|_{\theta=\theta_1^*}} + \frac{\delta(\theta - \theta_2^*)}{\left| \frac{dg_f(\theta)}{d\theta} \right|_{\theta=\theta_2^*}} \right] \frac{S^{\frac{1}{2}}(\theta)}{t(\theta)} d\theta \quad (B11)$$

where θ_1^* and θ_2^* satisfy Eqs. (27)-(30), $|dg_f(\theta)/d\theta|_{\theta=\theta_1^*}$ and $|dg_f(\theta)/d\theta|_{\theta=\theta_2^*}$ are given by Eqs. (B7) and (B8), S is given by Eq. (26), and t is given by Eq. (B5). Substituting these equations into Eq. (B11) and integrating over θ gives

$$h_f(r, \phi; r_o, \phi_o) = \frac{S^{\frac{1}{2}}(\theta_1^*)}{2\sqrt{R^2|\tilde{r}-\tilde{r}_o|^2 - |\tilde{r} \times \tilde{r}_o|^2}} + \frac{S^{\frac{1}{2}}(\theta_2^*)}{2\sqrt{R^2|\tilde{r}-\tilde{r}_o|^2 - |\tilde{r} \times \tilde{r}_o|^2}} \quad (B12)$$

Substituting Eqs. (38) and (39) for $S^{\frac{1}{2}}(\theta_1^*)$ and $S^{\frac{1}{2}}(\theta_2^*)$ gives

$$h_f(r, \phi; r_o, \phi_o) = \frac{\sqrt{R^2|\tilde{r}-\tilde{r}_o|^2 - |\tilde{r} \times \tilde{r}_o|^2} - \tilde{r} \cdot (\tilde{r}-\tilde{r}_o)}{2|\tilde{r}-\tilde{r}_o| \sqrt{R^2|\tilde{r}-\tilde{r}_o|^2 - |\tilde{r} \times \tilde{r}_o|^2}} + \frac{\sqrt{R^2|\tilde{r}-\tilde{r}_o|^2 - |\tilde{r} \times \tilde{r}_o|^2} + \tilde{r} \cdot (\tilde{r}-\tilde{r}_o)}{2|\tilde{r}-\tilde{r}_o| \sqrt{R^2|\tilde{r}-\tilde{r}_o|^2 - |\tilde{r} \times \tilde{r}_o|^2}} = \frac{1}{|\tilde{r} - \tilde{r}_o|} \quad (B13)$$

This proves the theorem. ||

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