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Abstract

In this note, we develop in closed form the dynamical $SO(3)$ spin map determined by the Thomas-BMT equation for a charged particle with general gyromagnetic ratio in a uniform longitudinal magnetic and electric field. This field configuration provides an idealized model of uniform particle acceleration with transverse (solenoidal) confinement. The spin map is expressed in a factorized Lie-algebraic (axis-angle) form, making it straightforward to determine the corresponding $SU(2)$ or quaternion representation. To facilitate numerical implementation, the Lie generators are expressed as functions of the six phase space variables commonly used for charged particle transport in s-based tracking codes. The dynamical evolution of these phase space variables is also described. This allows exact transport of both phase space and spin variables over long distances in such a field region, without the need for numerical integration.

1 Hamiltonian and Charged-Particle Orbits

1.1 Electromagnetic Field and Potentials

We will treat the special case of a uniform longitudinal electric and magnetic field $\vec{E} = E_z\hat{z}$, $\vec{B} = B_z\hat{z}$, which we characterize using the following two parameters:

$$\lambda = \frac{qE_z}{mc^2}, \qquad k = \frac{qB_z}{mc}. \tag{1}$$

Here q denotes the particle charge, m denotes the particle mass, and c denotes the speed of light. Each of the quantities in (1) has units of inverse distance (1/m). The fields $\vec{E}$ and $\vec{B}$ may be expressed using the following potentials:

$$\frac{q\psi}{mc^2} = -\lambda z, \qquad \frac{qA_x}{mc} = -\frac{k}{2}y, \qquad \frac{qA_y}{mc} = \frac{k}{2}x, \qquad \frac{qA_z}{mc} = 0, \tag{2}$$

where $\vec{E} = -\nabla\psi$ and $\vec{B} = \nabla \times \vec{A}$. To describe the particle orbits, we will work in terms of the following set of six canonical variables:

$$X = x, \quad Y = y, \quad T = ct, \quad P_x = \frac{p_x + qA_x}{mc}, \quad P_y = \frac{p_y + qA_y}{mc}, \quad P_t = \frac{p_t - q\psi}{mc^2}. \tag{3}$$

Here $p_x = \gamma m v_x$, $p_y = \gamma m v_y$ and $p_t = -\gamma mc^2$ denote the relativistic "mechanical" momenta. Note that the (dimensionless) canonical momenta can be expressed in terms of the usual relativistic beta and gamma as:

$$P_x = \beta_x \gamma - \frac{k}{2} Y, \qquad P_y = \beta_y \gamma + \frac{k}{2} X, \qquad P_t = -\gamma + \lambda z. \tag{4}$$

The orbits are then determined using the following Hamiltonian, where the coordinate z is taken as the independent variable:

$$H(X, P_x, Y, P_y, T, P_t; z) = \\ -\sqrt{(P_t - \lambda z)^2 - 1 - \left(P_x + \frac{kY}{2}\right)^2 - \left(P_y - \frac{kX}{2}\right)^2}. \tag{5}$$

The reader may verify that Hamilton's equations of motion using (5) are consistent with the usual Lorentz force equations for the fields defined by (1).

1.2 Exact solution

Since (5) is independent of T, the canonical momentum P_t is an invariant of motion. Another convenient invariant of motion is given by the magnitude of the transverse mechanical momentum:

$$P_\perp^{\mathrm{mech}} = \sqrt{(P_x + \alpha Y)^2 + (P_y - \alpha X)^2}, \qquad \alpha = \frac{k}{2}. \tag{6}$$

It is straightforward to check that $\{H, P_\perp^{\mathrm{mech}}\} = 0$.

The complete solution for motion $z \mapsto z^f$, which is obtained in [1], is given by the map $(X, P_x, Y, P_y)^T \mapsto R_{4D}(X, P_x, Y, P_y)^T$ where:

$$R_{4D} = R_{\mathrm{foc}} R_{\mathrm{rot}}, \tag{7a}$$

$$R_{\mathrm{foc}} = \begin{pmatrix} \cos(\theta) & \sin(\theta)/\alpha & 0 & 0 \\ -\alpha\sin(\theta) & \cos(\theta) & 0 & 0 \\ 0 & 0 & \cos(\theta) & \sin(\theta)/\alpha \\ 0 & 0 & -\alpha\sin(\theta) & \cos(\theta) \end{pmatrix}, \tag{7b}$$

$$R_{\mathrm{rot}} = \begin{pmatrix} \cos(\theta) & 0 & \sin(\theta) & 0 \\ 0 & \cos(\theta) & 0 & \sin(\theta) \\ -\sin(\theta) & 0 & \cos(\theta) & 0 \\ 0 & -\sin(\theta) & 0 & \cos(\theta) \end{pmatrix}. \tag{7c}$$

Here the angle parameter θ is itself a function of the initial phase space variables. It is given by:

$$\theta = \frac{\alpha}{\lambda} \log\left[\frac{-P_t + \lambda z^f + P_z^f}{-P_t + \lambda z + P_z}\right], \qquad P_z = \sqrt{(P_t - \lambda z)^2 - 1 - (P_\perp^{\mathrm{mech}})^2}. \tag{8}$$

The solution for the longitudinal coordinates is given by:

$$T^f = T + \frac{1}{\lambda}\left(P_z^f - P_z\right), \qquad P_t^f = P_t. \tag{9}$$

Direct evaluation shows that these orbits satisfy Hamilton's equations of motion with the Hamiltonian (5).

1.3 On-axis design orbit

A nominal design orbit is given by taking $X = P_x = Y = P_y = 0$, in which case we have simply:

$$T^f = T + \frac{1}{\lambda}\left(P_z^f - P_z\right), \qquad P_t^f = P_t, \qquad P_z = \sqrt{(P_t - \lambda z)^2 - 1}. \tag{10}$$

It is convenient to re-express this result in terms of the relativistic factors β and γ on this orbit. Using (4) we obtain:

$$T^f = T + \frac{1}{\lambda}\left((\beta\gamma)^f - (\beta\gamma)\right), \qquad \gamma^f = \gamma + \lambda\Delta z. \tag{11}$$

When a design orbit with a specific value of γ is selected, the corresponding quantities will be denoted using the subscript d, defining γ_d, T_d, etc.

1.4 Paraxial solution

The paraxial (chromatic) solution is obtained by expanding the Hamiltonian (5) to second order in the transverse phase space variables (X, P_x, Y, P_y). This yields the Hamiltonian:

$$H_{\text{parax}}(X, P_x, Y, P_y, T, P_t; z) = \\ \frac{1}{2\sqrt{(P_t - \lambda z)^2 - 1}}\left\{P_x^2 + P_y^2 + k(P_x Y - P_y X) + \frac{k^2}{4}(X^2 + Y^2)\right\} - \sqrt{(P_t - \lambda z)^2 - 1}. \tag{12}$$

The orbits can be obtained by expanding the exact orbits to linear order in the transverse phase space variables. The solution is again given by $(X, P_x, Y, P_y)^T \mapsto R_{4D}(X, P_x, Y, P_y)^T$ where R_{4D} takes the same form as (7), except that the angle parameter θ is evaluated at $X = P_x = Y = P_y = 0$:

$$\theta = \frac{\alpha}{\lambda}\log\left[\frac{-P_t + \lambda z^f + P_z^f}{-P_t + \lambda z + P_z}\right], \qquad P_z = \sqrt{(P_t - \lambda z)^2 - 1}. \tag{13}$$

Expanding the solution for T to second order in (X, P_x, Y, P_y) yields the paraxial solution:

$$T^f = T + \frac{1}{\lambda}\left(P_z^f - P_z\right) + \frac{1}{2\lambda}\left(\frac{1}{P_z} - \frac{1}{P_z^f}\right)(P_\perp^{\text{mech}})^2. \tag{14}$$

The result is a solution of Hamilton's equations with the Hamiltonian (12). In this solution, T and P_t are absolute. To obtain T and P_t relative to the design orbit, one must subtract the design values thus:

$$\Delta T = T - T_d, \qquad \Delta P_t = P_t - P_{t,d}. \tag{15}$$

2 Thomas-BMT equation

The Thomas-BMT equation for the rest-frame spin 3-vector $\vec{\mathcal{S}}$ of a particle of mass m and charge q moving in the electromagnetic fields specified in (1) can be expressed in the following form [3, 4, 5], where the longitudinal coordinate z is taken as the independent variable:

$$\frac{d\vec{\mathcal{S}}}{dz} = \vec{\Omega} \times \vec{\mathcal{S}}, \qquad \vec{\Omega} \;\; = -\frac{1}{\gamma\beta_z}\left[(1+G\gamma)\,k\hat{z} - G(\gamma-1)ku_z\hat{u} - (G+1/(1+\gamma))\beta\gamma\lambda(\hat{u}\times\hat{z})\right]. \quad (16)$$

Here G denotes the gyromagnetic anomaly of the particle, given in terms of the gyromagnetic ratio g by $G = (g-2)/2$, γ and $\vec{\beta}$ are the relativistic factors evaluated on the particle orbit, and $\hat{u}$ is a unit vector pointing the direction of the instantaneous velocity (in the lab frame). Namely:

$$\hat{u} = \frac{\vec{\beta}}{||\vec{\beta}||}. \quad (17)$$

For a particle on the design orbit, where $\hat{u} = \hat{z}$ we have simply:

$$\vec{\Omega}_d(z) = -\frac{1}{(\gamma_d\beta_d)(z)}(1+G)k\hat{z}. \quad (18)$$

The spin map M_d for a particle on the design orbit must satisfy the 3×3 matrix equation:

$$\frac{dM_d}{dz} = (\vec{\Omega}_d(z)\cdot\vec{L})M_d, \qquad M_d(z_i) = I. \quad (19)$$

Here L_j, $j = 1,2,3$ denote standard 3×3 matrices forming a basis for the Lie algebra of $SO(3)$ [6], and I denotes the 3×3 identity matrix. Since the vectors $\vec{\Omega}_d(z)$ are parallel for all values of z, it follows that the solution of (19) is given by:

$$M_d = e^{\vec{\lambda}_d\cdot\vec{L}}, \qquad \vec{\lambda}_d = \int_{z_i}^{z_f}\vec{\Omega}_d(z)dz = -(1+G)k\hat{z}\int_{z_i}^{z_f}\frac{1}{(\gamma_d\beta_d)(z)}dz. \quad (20)$$

The latter integral is:

$$\int_{z_i}^{z_f}\frac{1}{(\gamma_d\beta_d)(z)}dz = \theta/\alpha, \quad (21)$$

where θ is just the expression in (13). As a result, it follows that:

$$\vec{\lambda}_d = -2\theta(1+G)\hat{z}. \quad (22)$$

We have now determined the spin map M_d on the design orbit. It is a rotation about the z-axis, whose rotation vector is given by (22).

Next, we will consider particles that do not lie on the design orbit. In terms of our canonical variables we have:

$$\gamma = -P_t + \lambda z, \qquad \beta_x\gamma = P_x + \alpha Y, \qquad \beta_y\gamma = P_y - \alpha X, \quad (23)$$

$$\beta_z\gamma = \sqrt{(P_t-\lambda z)^2 - 1 - (P_\perp^{\text{mech}})^2}, \quad (24)$$

so that the unit vector $\hat{u}$ appearing in (17) is given by:

$$u_x = \frac{P_x+\alpha Y}{\sqrt{(P_t-\lambda z)^2-1}}, \quad u_y = \frac{P_y-\alpha X}{\sqrt{(P_t-\lambda z)^2-1}}, \quad u_z = \frac{\sqrt{(P_t-\lambda z)^2-1-(P_\perp^{\text{mech}})^2}}{\sqrt{(P_t-\lambda z)^2-1}}. \quad (25)$$

Expanding the precession vector $\vec{\Omega}$ from (16) to linear order in $\Delta\zeta = (X, P_x, Y, P_y)^T$ yields the following result:

$$\vec{\Omega} = \vec{\Omega}_d + (D\Omega)\Delta\zeta + O(||\Delta\zeta||^2), \tag{26}$$

where $D\Omega$ is the 3×4 Jacobian matrix with components $D\Omega_{ij} = \partial\Omega_i/\partial\zeta_j|_{\vec{\zeta}=0}$. This is obtained as:

$$D\Omega = \begin{pmatrix} -\alpha g & f & \alpha f & g \\ -\alpha f & -g & -\alpha g & f \\ 0 & 0 & 0 & 0 \end{pmatrix}, \tag{27}$$

where the quantities f and g are functions of z given by:

$$f(z) = \frac{Gk}{1 - P_t + \lambda z}, \qquad g(z) = \frac{\lambda}{\sqrt{(P_t - \lambda z)^2 - 1}}\left(G + \frac{1}{1 - P_t + \lambda z}\right). \tag{28}$$

The spin map M is then the 3×3 matrix given in factorized Lie-algebraic form [6] by:

$$M = e^{\vec{\lambda}_d\cdot\vec{L}} e^{A\Delta\zeta\cdot\vec{L}} + O(||\Delta\zeta||^2), \qquad \Delta\zeta = (X, P_x, Y, P_y)^T. \tag{29}$$

Here the 3×4 matrix A is given by:

$$A = \int_{z_i}^{z_f} M_d(z)^T D\Omega(z) R_{4D}(z) dz, \tag{30}$$

where R_{4D} is the 4×4 paraxial transport matrix defined in (7). We make use of the following simplification of the integrand:

$$M_d(z)^T D\Omega(z) R_{4D}(z) = e^{2G\theta\hat{z}\cdot\vec{L}} D\Omega(z). \tag{31}$$

The problem is then reduced to evaluating the integral:

$$A = \int_{z_i}^{z_f} e^{2G\theta\hat{z}\cdot\vec{L}} D\Omega(z) dz. \tag{32}$$

The result is given by:

$$A = \begin{pmatrix} \alpha G & F & \alpha F & -G \\ -\alpha F & G & \alpha G & F \\ 0 & 0 & 0 & 0 \end{pmatrix}, \tag{33a}$$

where

$$F = \nu(z_i)\sin(2G\theta) + \frac{\lambda(1 - \cos(2G\theta))}{2\alpha} + \{\nu(z_f) - \nu(z_i)\}\sin(2G\theta), \tag{33b}$$

$$G = \nu(z_i)(1 - \cos(2G\theta)) - \frac{\lambda\sin(2G\theta)}{2\alpha} - \{\nu(z_f) - \nu(z_i)\}\cos(2G\theta), \tag{33c}$$

and we have introduced the function ν given by:

$$\nu(z) = \frac{\sqrt{(P_t - \lambda z)^2 - 1}}{1 - P_t + \lambda z} = \frac{\beta\gamma}{1 + \gamma}. \tag{33d}$$

To see that (32) holds, note first that $z_i = z_f \Rightarrow \theta = 0$, so that $F = G = 0$, and hence $A = 0$.

Calculating the z-derivatives of F and G in (33) shows that:

$$F' = 2G\theta'\nu_f \cos(2G\theta) + \big(\lambda G\theta'/\alpha + \nu'\big)\sin(2G\theta), \tag{34a}$$
$$G' = 2G\theta'\nu_f \sin(2G\theta) - \big(\lambda G\theta'/\alpha + \nu'\big)\cos(2G\theta). \tag{34b}$$

However, evaluating the derivatives θ' and ν' directly we have that:

$$f(z) = 2G\theta'\nu(z), \qquad g(z) = \frac{G\lambda}{\alpha}\theta' + \nu'. \tag{35}$$

Using (33-35) to evaluate dA/dz, and comparing the result with (27), we see that:

$$\frac{dA}{dz} = e^{2G\theta\hat{z}\cdot\vec{L}} D\Omega(z). \tag{36}$$

This shows that (32) holds, as desired.

The result (33) will be revisited in the following section, where it will be obtained from the exact solution using an independent method.

3 Exact evolution of the spin 4-vector

To obtain the exact solution for the spin dynamics, we begin with the following covariant form of the Thomas-BMT equation for the spin 4-vector S^μ, using proper time τ as the independent variable:

$$\frac{dS^\mu}{d\tau} = \frac{q}{mc}\left[(G+1)F^{\mu\nu}S_\nu + \frac{1}{c^2}G\frac{dx^\mu}{d\tau}S_\alpha F^{\alpha\beta}\frac{dx_\beta}{d\tau}\right]. \tag{37}$$

Here $F^{\mu\nu}$ denotes the antisymmetric Faraday tensor, and x^μ is the 4-vector with components (ct, x, y, z). To simplify the units, we take $\tau \mapsto c\tau$ to have units of length, we take $F^{\mu\nu} \mapsto qF^{\mu\nu}/mc^2$ to have units of inverse length, and we define $p^0 = \gamma$, $p^j = \gamma\beta_j$ $j = 1, 2, 3$. Then we obtain:

$$\frac{dS^\mu}{d\tau} = (G+1)F^{\mu\nu}S_\nu + Gp^\mu S_\alpha F^{\alpha\beta}p_\beta. \tag{38}$$

We will re-express (38) in a matrix-vector form, as follows. First, we let $\eta = \mathrm{diag}(1, -1, -1, -1)$ denote the 4×4 matrix defining the Lorentz metric, with components $\eta = [\eta_{\mu\nu}]$. For each 4-vector quantity (such as p^μ), we define a column vector p with components $p = [p^\mu]$, where $\mu = 0, 1, 2, 3$. Likewise, we define a row vector $p^\dagger = p^T\eta$ with components $p^\dagger = [p_\mu]$, so that $p^\dagger p = p^\mu p_\mu = 1$. Likewise, define a 4×4 matrix F with components $F = [F^\mu{}_\nu] = [F^{\mu\rho}\eta_{\rho\nu}]$ and adjoint $F^\dagger = \eta F^T\eta$. Then (38) becomes:

$$\frac{dS}{d\tau} = (G+1)FS - Gpp^\dagger FS = \Lambda S, \tag{39}$$

where the 4×4 matrix Λ is clearly:

$$\Lambda = (G+1)F - Gpp^\dagger F. \tag{40}$$

Next, let us recall how the momentum 4-vector p evolves in a uniform electromagnetic field given by F. The Lorentz equation becomes simply:

$$\frac{dp}{d\tau} = Fp \Rightarrow p(\tau) = e^{\tau F}p_0. \tag{41}$$

Then the adjoint $p^\dagger$ evolves as:

$$\frac{dp^\dagger}{d\tau} = -p^\dagger F \Rightarrow p^\dagger(\tau) = p_0^\dagger e^{-\tau F}, \tag{42}$$

and we see that:

$$\Lambda(\tau) = (G+1)F - G(e^{\tau F} p_0 p_0^\dagger e^{-\tau F})F = e^{\tau F}\left\{(G+1)F - G p_0 p_0^\dagger F\right\} e^{-\tau F}, \tag{43}$$

where we used the fact that the matrices $e^{-\tau F}$ and F commute. Therefore we see that:

$$\Lambda(\tau) = e^{\tau F}\Lambda(0)e^{-\tau F}. \tag{44}$$

Now let us define:

$$\tilde{S} = e^{-\tau F} S. \tag{45}$$

Then we see that:

$$\frac{d\tilde{S}}{d\tau} = -F e^{-\tau F} S + e^{-\tau F}\frac{dS}{d\tau} = -F\tilde{S} + e^{-\tau F}\Lambda(\tau)S \tag{46}$$

$$= -F\tilde{S} + \Lambda(0)e^{-\tau F} S = -F\tilde{S} + \Lambda(0)\tilde{S}, \tag{47}$$

with initial condition $\tilde{S}(0) = S(0)$. The solution for $\tilde{S}$ is clearly given by:

$$\tilde{S}(\tau) = \exp\left\{\tau(-F + \Lambda(0))\right\} S(0). \tag{48}$$

It follows that the solution for S is given by:

$$S(\tau) = e^{\tau F}\tilde{S}(\tau) = e^{\tau F} e^{\tau(-F+\Lambda(0))} S(0). \tag{49}$$

The generator in the second factor can be written explicitly as:

$$-F + \Lambda(0) = GF - G p_0 p_0^\dagger F = G(I - p_0 p_0^\dagger)F. \tag{50}$$

Thus, we have the final result:

$$S(\tau) = e^{\tau F} e^{\tau G(I - p_0 p_0^\dagger)F} S(0). \tag{51}$$

The solution is required to satisfy the following identities:

$$S^\dagger(\tau)S(\tau) = S^\dagger(0)S(0), \qquad p^\dagger(\tau)S(\tau) = 0. \tag{52}$$

This follows from the equation of motion (39-41). We can see this as follows. First note that:

$$\frac{d}{d\tau}(p^\dagger S) = -p^\dagger F S + p^\dagger\left[(G+1)FS - G p p^\dagger F S\right]$$

$$= G p^\dagger F S - G(p^\dagger p)p^\dagger F S = 0, \tag{53}$$

provided $p^\dagger p = 1$. Next, let's note that:

$$\frac{dS^\dagger}{d\tau} = -(G+1)S^\dagger F + G S^\dagger F p p^\dagger. \tag{54}$$

It follows that:

$$\frac{d}{d\tau}(S^\dagger S) = \frac{dS^\dagger}{d\tau}S + S^\dagger \frac{dS}{d\tau} = -(G+1)S^\dagger FS + GS^\dagger Fpp^\dagger S + (G+1)S^\dagger FS - GS^\dagger pp^\dagger FS. \tag{55}$$

The first and third terms on the rhs clearly cancel. In the second and fourth terms, we have the quantities $p^\dagger S = 0$ and $S^\dagger p = 0$, respectively. As a result, the derivative vanishes, and we obtain (52).

There is an advantage to replacing the quantity Λ appearing in (39-40) with the quantity:

$$\tilde{\Lambda} = F + G(I - pp^\dagger)F(I - pp^\dagger). \tag{56}$$

This has the advantage that the adjoint is trivial:

$$\tilde{\Lambda}^\dagger = -\tilde{\Lambda}, \tag{57}$$

while since $p^\dagger S = 0$,

$$\tilde{\Lambda}S = \Big(F + G(I - pp^\dagger)F(I - pp^\dagger)\Big)\, S = FS + G(I - pp^\dagger)FS = (G+1)FS - Gpp^\dagger FS = \Lambda S, \tag{58}$$

leaving the equation of motion (39) unaffected. It follows that the solution for the spin 4-vector can be written in the alternative form:

$$S(\tau) = e^{\tau F} e^{\tau G(I - p_0 p_0^\dagger)F(I - p_0 p_0^\dagger)} S(0). \tag{59}$$

Although expression (59) looks more complicated than the previous expression (51), in this case each matrix factor on the rhs is a Lorentz transformation. Since the quantities $S^\dagger S$ and $p^\dagger S$ are Lorentz scalars, the identities (52) follow immediately.

3.1 Evaluation in the particle rest-frame

Evaluation of the spin 3-vector $\vec{\mathcal{S}}$ in the particle rest frame can be obtained by applying a Lorentz boost opposite the instantaneous particle velocity. Let L denote a 4×4 matrix whose components are given by $L = [L^\mu{}_\nu]$, where $L^\mu{}_\nu$ are the tensor components of the Lorentz transformation. The matrix L for a pure Lorentz boost opposite the momentum p is known in explicit form. It is given by:

$$L = \begin{pmatrix} p^0 & -\vec{p}^T \\ -\vec{p} & I + (p^0 - 1)\frac{\vec{p}\vec{p}^T}{\vec{p}^T\vec{p}} \end{pmatrix}. \tag{60}$$

It is straighforward to verify that:

$$L^T \eta L = \eta, \qquad Lp = (1,0,0,0)^T. \tag{61}$$

In particular, for each value of τ:

$$L(\tau)p(\tau) = (1,0,0,0)^T. \tag{62}$$

For each value of τ, we define another Lorentz transformation $\hat{L}(\tau)$ by:

$$\hat{L}(\tau) = L(0)e^{-\tau F}. \tag{63}$$

Since $e^{-\tau F}$ is a Lorentz transformation, so is (63) and we see that:

$$\hat{L}(\tau)p(\tau) = L(0)e^{-\tau F}e^{\tau F}p_0 = L(0)p_0 = (1,0,0,0)^T. \tag{64}$$

From (62) and (64) it follows that:

$$L(\tau) = R(\tau)\hat{L}(\tau), \tag{65}$$

for some rotation matrix $R(\tau)$. This implies that:

$$S_{rest}(\tau) = R(\tau)L(0)e^{\tau G(I-p_0p_0^\dagger)F(I-p_0p_0^\dagger)}L(0)^{-1}S_{rest}(0). \tag{66}$$

Then we have:

$$L(0)e^{\tau G(I-p_0p_0^\dagger)F(I-p_0p_0^\dagger)}L(0)^{-1} = \exp\left(GL(0)(I-p_0p_0^\dagger)F(I-p_0p_0^\dagger)L(0)^{-1}\right). \tag{67}$$

The argument of the exponential is transformed into the initial rest frame of the particle, the frame in which $p_0 = (1,0,0,0)^T$, and it is straightforward to verify that:

$$L(0)(I-p_0p_0^\dagger)F(I-p_0p_0^\dagger)L(0)^{-1} = -\vec{B}_{rest}\cdot\vec{J}, \tag{68}$$

where $\vec{B}_{rest}$ denotes the magnetic field in this initial rest frame. Therefore we have:

$$S_{rest}(\tau) = R(\tau)e^{-\tau G\vec{B}_{rest}\cdot\vec{J}}S_{rest}(0). \tag{69}$$

It remains to determine the generator of the rotation $R(\tau)$. If we can do this, then (69) descends naturally to a Lie-algebraic expression for the $SO(3)$ spin map of the rest-frame spin 3-vector $\vec{S}$. We have:

$$R(\tau) = L(\tau)\hat{L}(\tau)^{-1} = L(\tau)e^{\tau F}L(0)^{-1}. \tag{70}$$

We note that $R(\tau)$ is exactly the spin rotation that would be obtained in the special case that $G=0$. The case $G=0$ can be solved directly, in the original formulation.

3.2 Exact solution for a longitudinal electric and magnetic field

In this section, we will make use of the following two integrals over particle orbits:

$$\Delta\tau = \int_{z_i}^{z_f}\frac{1}{p_z(z)}dz = \frac{2\theta}{k}, \tag{71}$$

$$\int_{z_i}^{z_f}\frac{1}{p_z(z)(1+\gamma(z))}dz = \frac{2}{\lambda|P_{\mathrm{mech}}|}\left\{\tan^{-1}\left(\frac{1-P_t+\lambda z+P_z}{|P_{\mathrm{mech}}|}\right)\right\}\Big|_{z_i}^{z_f} = \frac{2\mu}{\lambda}, \tag{72}$$

In (71), θ is the same parameter defined in (8), and $\Delta\tau$ denotes the proper time elapsed between locations z_i and z_f. In (72), we have defined the parameter μ.

Consider the Thomas-BMT precession vector $\vec{\Omega}$ in the special case that $G=0$. We have:

$$\vec{\Omega}_{(G=0)} = -\frac{k\hat{z}}{p_z} + \frac{\lambda}{p_z(1+\gamma)}(\vec{p}_\perp\times\hat{z}) = \vec{\Omega}_0 + \vec{\Omega}_r. \tag{73}$$

The solution for $\vec{\Omega}_0$ is given easily by:

$$M_0 = \exp\left(-k\hat{z}\int_{z_i}^{z_f}\frac{1}{p_z(z)}dz\cdot\vec{L}\right) = e^{-2\theta\hat{z}\cdot\vec{L}}. \tag{74}$$

Then the solution for $\vec{\Omega}_{(G=0)}$ is given by $M_{(G=0)} = M_0 M_r$, where:

$$\frac{d}{dz}M_r = (\vec{\Omega}_{\text{int}}\cdot\vec{L})M_r, \qquad M_r(z_i) = I, \qquad \vec{\Omega}_{\text{int}} = M_0^T\vec{\Omega}_r. \tag{75}$$

However, the transverse mechanical momenta evolve as:

$$\vec{p}_\perp(z) = e^{-2\theta\hat{z}\cdot\vec{L}}\vec{p}_\perp(0), \tag{76}$$

and it follows that:

$$\vec{\Omega}_{\text{int}} = e^{2\theta\hat{z}\cdot\vec{L}}\vec{\Omega}_r = \frac{\lambda}{p_z(1+\gamma)}\left(e^{2\theta\hat{z}\cdot\vec{L}}\vec{p}_\perp\times\hat{z}\right) = \frac{\lambda}{p_z(1+\gamma)}\left(\vec{p}_\perp(0)\times\hat{z}\right). \tag{77}$$

Since the direction of this vector is again independent of z, we have the solution:

$$M_r = \exp\left(\lambda(\vec{p}_\perp(0)\times\hat{z})\int_{z_i}^{z_f}\frac{1}{p_z(z)(1+\gamma(z))}dz\cdot\vec{L}\right) = \exp\left(2\mu(\vec{p}_\perp(0)\times\hat{z})\cdot\vec{L}\right). \tag{78}$$

Finally, we must evaluate $\vec{B}_{rest}$. Note that:

$$\begin{aligned}
\vec{B}_{rest} &= p^0\vec{B} - \left(\vec{p}\times\vec{E}\right) - \frac{(p^0-1)}{||\vec{p}||^2}\left(\vec{B}\cdot\vec{p}\right)\vec{p} \\
&= p^0\vec{B} - \left(\vec{p}\times\vec{E}\right) - \frac{1}{p^0+1}\left(\vec{B}\cdot\vec{p}\right)\vec{p} \\
&= kp^0\hat{z} - \lambda\left(\vec{p}\times\hat{z}\right) - \frac{1}{p^0+1}\left(kp^3\right)\vec{p} \\
&= k\hat{z} + \frac{k}{p^0+1}\left(|p_\perp|^2\hat{z} - p^3\vec{p}_\perp\right) - \lambda\left(\vec{p}\times\hat{z}\right).
\end{aligned} \tag{79}$$

The same expression can also be evaluated at any location z along the particle orbit, and we denote the result as $\vec{B}_{rest}(z)$. A direct comparison with the Thomas-BMT equation shows that, at each z:

$$\vec{\Omega}(z) = \vec{\Omega}_{(G=0)}(z) - \frac{G}{p_z(z)}\vec{B}_{rest}(z). \tag{80}$$

Likewise, a tedious but straightforward calculation shows that:

$$\vec{\Omega}_{(G=0)}\times\vec{B}_{rest} = \frac{k\lambda}{p^3(1+p^0)}\vec{p}_\perp + \frac{k^2}{p^0+1}(\hat{z}\times\vec{p}_\perp) + \frac{k\lambda}{(p^0+1)^2}\left[(p^3)\vec{p}_\perp - |\vec{p}_\perp|^2\hat{z}\right], \tag{81}$$

with an identical expression for $d\vec{B}_{rest}/dz$, so that:

$$\frac{d\vec{B}_{rest}}{dz} = \vec{\Omega}_{(G=0)}\times\vec{B}_{rest}. \tag{82}$$

This shows that the evolution of $\vec{B}_{rest}$ along an orbit is given by:

$$\vec{B}_{rest}(z) = M_0 M_r\vec{B}_{rest}(z_i). \tag{83}$$

The full solution is then given by:

$$M = e^{-2\theta \hat{z}\cdot\vec{L}} e^{2\mu(\vec{p}_\perp\times\hat{z})\cdot\vec{L}} e^{-2G\theta\vec{B}_{rest}/k\cdot\vec{L}}. \tag{84}$$

For a particle on a design orbit with $\vec{p}_\perp = 0$, it follows that $\vec{B}_{rest} = k\hat{z}$ and

$$M_d = e^{-2\theta\hat{z}\cdot\vec{L}} e^{-2G\theta\hat{z}\cdot\vec{L}} = e^{-2\theta(1+G)\hat{z}\cdot\vec{L}}, \tag{85}$$

which coincides with the result obtained in (20-22). We can now verify that (84) satisfies the spin map evolution equation, where all z-dependence is contained in the functions θ and μ. Taking the derivative of (84) gives:

$$\begin{aligned}
\frac{dM}{dz} &= \left(-2\frac{d\theta}{dz}\hat{z}\cdot\vec{L}\right) M + e^{-2\theta\hat{z}\cdot\vec{L}} \left(2\frac{d\mu}{dz}(\vec{p}_\perp\times\hat{z})\cdot\vec{L}\right) e^{2\mu(\vec{p}_\perp\times\hat{z})\cdot\vec{L}} e^{-2G\theta\vec{B}_{rest}/k\cdot\vec{L}} \\
&\quad + M_0 M_r \left(-2G\frac{d\theta}{dz}\frac{\vec{B}_{rest}}{k}\cdot\vec{L}\right) e^{-2G\theta\vec{B}_{rest}/k\cdot\vec{L}} \\
&= \left(-\frac{k}{p^3(z)}\hat{z}\cdot\vec{L}\right) M + \left(2\frac{d\mu}{dz}(\vec{p}_\perp(z)\times\hat{z})\cdot\vec{L}\right) M + \left(-2G\frac{d\theta}{dz}\frac{\vec{B}_{rest}(z)}{k}\cdot\vec{L}\right) M \\
&= \left\{\left(-\frac{k\hat{z}}{p^3(z)} + \frac{\lambda(\vec{p}_\perp(z)\times\hat{z})}{p^3(z)(1+p^0(z))} - \frac{G}{p^3(z)}\vec{B}_{rest}(z)\right)\cdot\vec{L}\right\} M \\
&= \left\{\left(\vec{\Omega}_{(G=0)}(z) - \frac{G}{p^3(z)}\vec{B}_{rest}(z)\right)\cdot\vec{L}\right\} M = \left\{\vec{\Omega}(z)\cdot\vec{L}\right\} M,
\end{aligned} \tag{86}$$

as required.

3.3 Application to paraxial expansion

The result takes the form:

$$M = M_d M_r, \qquad M_r = e^{2\theta G\hat{z}\cdot\vec{L}} e^{\vec{a}\cdot\vec{L}} e^{-2\theta G(\hat{z}+\vec{b})\cdot\vec{L}}, \tag{87}$$

where

$$\vec{a} = 2\mu(\vec{p}_\perp\times\hat{z}), \qquad \vec{b} = -\frac{p^3}{p^0+1}\vec{p}_\perp - \frac{\lambda}{k}\left(\vec{p}_\perp\times\hat{z}\right). \tag{88}$$

We use the factorization:

$$e^{-2\theta G(\hat{z}+\epsilon\vec{b})\cdot\vec{L}} = e^{\epsilon\vec{d}\cdot\vec{L}} e^{-2\theta G\hat{z}\cdot\vec{L}} e^{-\epsilon\vec{d}\cdot\vec{L}} + O(\epsilon^2), \qquad \vec{d} = \hat{z}\times\vec{b}. \tag{89}$$

This follows since to $O(\epsilon)$:

$$\begin{aligned}
e^{\epsilon\vec{d}\cdot\vec{L}} e^{-2\theta G\hat{z}\cdot\vec{L}} e^{-\epsilon\vec{d}\cdot\vec{L}} &= \exp\left(-2\theta G e^{\epsilon\vec{d}\cdot\vec{L}}\hat{z}\cdot\vec{L}\right) = \exp\left(-2\theta G\{\hat{z}+\epsilon(\vec{d}\times\hat{z})\}\cdot\vec{L}\right) &(90)\\
&= \exp\left(-2\theta G\{\hat{z}+\epsilon\vec{b}\}\cdot\vec{L}\right), &(91)
\end{aligned}$$

where we used the fact that $\vec{b}\cdot\hat{z} = 0$. Therefore to $O(\epsilon)$:

$$\begin{aligned}
M_r &= e^{2\theta G\hat{z}\cdot\vec{L}} e^{\epsilon\vec{a}\cdot\vec{L}} e^{\epsilon\vec{d}\cdot\vec{L}} e^{-2\theta G\hat{z}\cdot\vec{L}} e^{-\epsilon\vec{d}\cdot\vec{L}} &(92)\\
&= e^{2\theta G\hat{z}\cdot\vec{L}} e^{\epsilon(\vec{a}+\vec{d})\cdot\vec{L}} e^{-2\theta G\hat{z}\cdot\vec{L}} e^{-\epsilon\vec{d}\cdot\vec{L}} &(93)\\
&= \exp\left(\epsilon e^{2\theta G\hat{z}\cdot\vec{L}}(\vec{a}+\vec{d})\cdot\vec{L}\right) e^{-\epsilon\vec{d}\cdot\vec{L}} = e^{\epsilon\vec{\lambda}\cdot\vec{L}}, &(94)
\end{aligned}$$

where

$$\vec{\lambda} = e^{2\theta G\hat{z}\cdot\vec{L}}(\vec{a}+\vec{d}) - \vec{d} = \left[e^{2\theta G\hat{z}\cdot\vec{L}}(2\mu\vec{p}_\perp - \vec{b}) + \vec{b}\right] \times \hat{z}. \tag{95}$$

The latter vector can be written out explicitly. It is enough to put:

$$\vec{p}^{\,\prime}_\perp = e^{2\theta G\hat{z}\cdot\vec{L}}\vec{p}_\perp, \tag{96}$$

so

$$p'_x = p_x\cos(2G\theta) - p_y\sin(2G\theta), \qquad p'_y = p_y\cos(2G\theta) + p_x\sin(2G\theta). \tag{97}$$

Then the vector in brackets becomes:

$$2\mu\vec{p}^{\,\prime}_\perp + \frac{p^3}{p^0+1}\vec{p}^{\,\prime}_\perp + \frac{\lambda}{k}(\vec{p}^{\,\prime}_\perp \times \hat{z}) - \frac{p^3}{p^0+1}\vec{p}_\perp - \frac{\lambda}{k}\left(\vec{p}_\perp \times \hat{z}\right), \tag{98}$$

giving $\vec{\lambda}$ in components:

$$\lambda_x = 2\mu p'_y + \frac{p^3}{p^0+1}(p'_y - p_y) - \frac{\lambda}{k}(p'_x - p_x), \tag{99}$$

$$\lambda_y = -\left\{2\mu p'_x + \frac{p^3}{p^0+1}(p'_x - p_x) + \frac{\lambda}{k}(p'_y - p_y)\right\}, \tag{100}$$

$$\lambda_z = 0. \tag{101}$$

This shows that, in the paraxial approximation, the map M_r takes the form:

$$M_r = e^{A\Delta\zeta}, \tag{102}$$

where the matrix A is given by $A_{ij} = \partial\lambda_i/\partial\zeta_j$, with:

$$A = \begin{pmatrix} \alpha G & F & \alpha F & -G \\ -\alpha F & G & \alpha G & F \\ 0 & 0 & 0 & 0 \end{pmatrix}, \tag{103a}$$

where

$$F = \frac{p^3}{1+p^0}\sin(2G\theta) + \frac{\lambda(1-\cos(2G\theta))}{2\alpha} + 2\mu\sin(2G\theta), \tag{103b}$$

$$G = \frac{p^3}{1+p^0}(1-\cos(2G\theta)) - \frac{\lambda\sin(2G\theta)}{2\alpha} - 2\mu\cos(2G\theta), \tag{103c}$$

and in the paraxial approximation:

$$\mu = \frac{\lambda}{2}\int_{z_i}^{z_f} \frac{dz}{\sqrt{(P_t-\lambda z)^2-1}(1-P_t+\lambda z)} = \frac{1}{2}\frac{\sqrt{(P_t-\lambda z)^2-1}}{1-P_t+\lambda z}\Bigg|_{z_i}^{z_f} = \frac{\nu(z)}{2}\Bigg|_{z_i}^{z_f}. \tag{104}$$

Using this in (103), we reproduce the result in (33).

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