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Verifying depth-zero supercuspidal L -packets for inner forms of GS_{p_4}

A dissertation submitted in partial satisfaction of the
requirements for the degree
Doctor of Philosophy

in

Mathematics

by

Jaime Lust

Committee in charge:

Professor Wee Teck Gan, Chair
Professor Ronald Graham
Professor Aneesh Manohar
Professor Cristian Popescu
Professor Nolan Wallach

2010

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Chair

University of California, San Diego

2010

DEDICATION

For Philip, Kiku, Sarah, Micket, and Sophie.

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PUBLICATIONS

J. Lust, *Verifying depth-zero supercuspidal L -packets for inner forms of $GS\!p_4$* , in preparation.

ABSTRACT OF THE DISSERTATION

Verifying depth-zero supercuspidal L -packets for inner forms of GS_{p_4}

by

Jaime Lust

Doctor of Philosophy in Mathematics

University of California San Diego, 2010

Professor Wee Teck Gan, Chair

In [GT], Gan and Takeda prove the local Langlands conjecture for GS_{p_4} . In [GTan], Gan and Tantano prove the local Langlands conjecture for $GU_2(D)$, the inner form of GS_{p_4} . Under these parametrizations, the L -packets L_ϕ^{GT} attached to a parameter ϕ do not give an explicit realization of supercuspidal representations as compactly induced from open compact mod center subgroups. For tame regular discrete L -parameters ϕ of GS_{p_4} and its inner form, we extend the construction of DeBacker and Reeder given in [DR] to attach L -packets L_ϕ^{DR} of depth-zero compactly induced supercuspidal representations. We then show that the two parametrizations agree, namely for tame regular discrete L -parameters ϕ , we have $L_\phi^{GT} = L_\phi^{DR}$.

1 Introduction

In the 1960's, Robert Langlands made a broad series of conjectures relating the structure of the absolute Galois group of a local field k to the representation theory of linear reductive groups over k . Roughly, the local Langlands conjectures predict that irreducible representations of a linear reductive group G over k should be parametrized by certain homomorphisms of the absolute Galois group of k to the Langlands dual group ${}^L G$.

More precisely, there should be a surjective finite-to-one map from the set of equivalence classes of irreducible smooth representations of G to the set of equivalence classes of admissible homomorphisms

$$\phi : W'_k \longrightarrow {}^L G.$$

Here, $W'_k = W_k \times SL_2(\mathbb{C})$ is the Weil-Deligne group, where the Weil group W_k is a dense subgroup of the absolute Galois group of k . As terminology, an L -parameter ϕ corresponds under the map to an L -packet L_ϕ of representations of G . This map should satisfy certain desired properties. One such property is the preservation of local factors such as L -functions and ϵ -factors attached to both sides of the correspondence.

Around 2000, Harris and Taylor [HT], and separately Henniart [He2], proved the local Langlands conjecture for GL_n . Cases for small n were established earlier by Kutzko and Henniart. Rogawski [Ro] proved the conjecture for U_2 and U_3 . In 2007, Gan and Takeda [GT] proved the local Langlands conjecture for GSp_4 . Also, Gan and Tantano [GTan] proved the local Langlands conjecture for $GU_2(D)$, the inner form of GSp_4 . Forthcoming work of Arthur will prove the local Langlands conjecture for the classical groups.

For supercuspidal representations, there is another conjectural classification which is independent of the local Langlands conjectures. It is conjectured that any irreducible supercuspidal representation π of G can be compactly induced from an open compact

mod center subgroup. More precisely, we would like to write down a nice family of compact open mod center subgroups K of G and representations σ of K , such that all irreducible supercuspidal representations are of the form

$$\pi \cong \text{c} - \text{Ind}_K^G \sigma.$$

Recently much work has been done in this direction. In 1993 Bushnell and Kutzko [BK1] showed that any irreducible supercuspidal representation of GL_n is compactly induced. In 2001, J.K. Yu [Y] constructed a family (K, σ) for any connected reductive group G . In 2007, J.-L. Kim [Ki] proved that, for p large, the family constructed by Yu exhausts all supercuspidal representations of G . Independently, for $p \neq 2$, Stevens [St] constructed another family (K, σ) for $G = U_n, Sp_{2n}$ or SO_n . He also proved that for these groups G , his family exhausts all supercuspidal representations of G .

It is not obvious how to relate the construction of supercuspidal representations via compact induction to the classification given by the local Langlands correspondence. There is a series of papers by Bushnell and Henniart [BH1], [BH2] devoted to answering this question for GL_n . For tame parameters of GL_n , these results were established earlier by Henniart [He1]. My thesis is towards answering this question for GSp_4 and its inner form $GU_2(D)$.

We consider the subset of tame regular discrete parameters of GSp_4 . For pure inner forms of unramified p -adic groups, DeBacker and Reeder [DR] give a parametrization of tame regular discrete Langlands parameters. For any tame regular discrete parameter they explicitly construct an L -packet of depth-zero supercuspidal representations. This construction applies to GSp_4 but not to $GU_2(D)$. We extend their construction to give L -packets L_ϕ^{DR} of depth zero supercuspidal representations for both GSp_4 and $GU_2(D)$, that agree with DeBacker and Reeder for GSp_4 . We then show that the L -packets we construct are the same as given by the local Langlands conjectures for GSp_4 and $GU_2(D)$.

We show

Theorem 1.1. *Let ϕ be a tame regular discrete L -parameter. Let L_ϕ^{DR} be the L -packet of depth-zero supercuspidal representations of $GSp_4(k)$ or $GU_2(D)$ corresponding to ϕ as in Definition 5.4. Let L_ϕ^{GT} be the L -packet of supercuspidal representations of $GSp_4(k)$ or $GU_2(D)$ corresponding to ϕ via the local Langlands conjecture for GSp_4 or $GU_2(D)$. Then*

$$L_\phi^{DR} = L_\phi^{GT}.$$

In [GT] and [GTan], the local Langlands classification is characterized by the preservation of L -factors and ϵ -factors. To prove Theorem 1.1, we need to show that certain L -functions have poles at $s = 0$. By theory of Shahidi [Sh], this question is equivalent to studying the reducibility points of the generalized principal series

$$I(s, \pi \boxtimes \sigma) = \text{Ind}_{M \cdot N}^G \delta_{M \cdot N}^{1/2} \pi \boxtimes \sigma |\det|^s.$$

Here, π is an irreducible representation of $GSpin_5(k) \cong GSp_4(k)$ or $GSpin_{4,1}(k) \cong GU_2(D)$, and σ a representation of $GL_{2m}(k)$, where

- (i) $G = GSpin_{4m+5}(k)$ or $GSpin_{2m+4,2m+1}(k)$,
- (ii) $M = GSpin_5(k) \times GL_{2m}(k)$ or $GSpin_{4,1}(k) \times GL_{2m}(k)$.

We need to show

Theorem 1.2. *Let π be a depth zero supercuspidal representation of $GSpin_5(k)$ or $GSpin_{4,1}(k)$ corresponding to a tame regular discrete Langlands parameter $\phi = \phi_1 \oplus \cdots \oplus \phi_r$, $r = 1, 2$, as in Definition 5.4. Let $\sigma \cong \sigma_{\phi_i}$, where $1 \leq i \leq r$, be the depth-zero supercuspidal representation of $GL_{2m}(k)$ attached to the Langlands parameter ϕ_i via the local Langlands correspondence for GL_{2m} . Then the generalized principal series $I(s, \pi \boxtimes \sigma)$ reduces at a unique $s_0 > 0$.*

The proof of Theorem 1.2 is achieved by studying a Hecke algebra $\mathcal{H}(G//\mathcal{P}, \rho)$ associated to this family of induced representations. More precisely, using the theory of types and covers developed by Bushnell and Kutzko [BK2], we find an open compact subgroup $\mathcal{P}$ of G and representation ρ of $\mathcal{P}$ such that irreducible representations in the Bernstein component of $I(s, \pi \boxtimes \sigma)$ are parametrized by simple $\mathcal{H}(G//\mathcal{P}, \rho)$ -modules. In this situation $c - \text{Ind}_{\mathcal{P}}^G \rho$ is a depth-zero representation, and Morris [Mo] describes explicit generators and relations for the Hecke algebra $\mathcal{H}(G//\mathcal{P}, \rho) \cong \text{End}_G(c - \text{Ind}_{\mathcal{P}}^G \rho)$. We have $\mathcal{H}(G, \mathcal{P}, \rho) = \langle T_0, T_1, T_2 \rangle$, subject to the relations

- (i) $T_0 * T_i = T_i * T_0$, $i = 1, 2$,
- (ii) $T_i^2 = (p_i - 1)T_i + p_i$, $i = 1, 2$.

Notice that $\mathcal{H}(G, \mathcal{P}, \rho)$ is a Hecke algebra of type $\tilde{A}_1$ tensored with a polynomial algebra. For T_i , $i = 1, 2$, we show there is a subalgebra of $\mathcal{H}(G//\mathcal{P}, \rho)$ which is canonically isomorphic to the endomorphism ring of an induced representation over a finite field,

with T_i identified as the unique non-identity generator. In this way computation of the parameters p_i reduce to computations over a finite field, which can be done following a theorem of Lusztig [Lu]. We show the parameters $p_i, i = 1, 2$, of the Hecke algebra $\mathcal{H}(G//\mathcal{P}, \rho)$ are unequal. Theorem 1.2 then follows using a results of Matsumoto [Ma] and Harish-Chandra [Sil].

We begin by giving a short description of the various groups used in this thesis in Section 2. In Section 3, we describe tame regular discrete L -parameters for GSp_4 . In Section 4 we give an exposition of the local Langlands conjectures for GSp_4 and $GU_2(D)$. Then, in Section 5, we extend the DeBacker-Reeder construction to give L -packets L_ϕ^{DR} of depth-zero supercuspidal representations for $GSpin_5(k)$ and $GSpin_{4,1}(k)$. In Section 6, we introduce the generalized principal series $I(s, \pi \boxtimes \sigma)$. In Sections 7 and 8 we give a presentation of $\mathcal{H}(G//\mathcal{P}, \rho)$ and compute the parameters p_i using results of Morris and Lusztig. Then in Sections 9 and 10, we state the main theorem and show that the L -packets L_ϕ^{GT} and L_ϕ^{DR} agree.

2 Structure theory

Let k be a p -adic field, namely a non-archimedian local field of characteristic 0 with residue field $\mathfrak{f}$ of characteristic p . (In the following sections $\mathfrak{f} = \mathbb{F}_q$ will denote a finite field, but we will not assume so in this section.) Let $F = k$ or $\mathfrak{f}$.

2.1 General linear groups

If V is an n -dimensional vector space over F , then

$$GL(V) = \text{Aut}_F(V).$$

2.2 General symplectic groups

Let $V_1 = Ff_1 \oplus Ff_2$ be the 2 dimensional vector space over F equipped with the non-degenerate alternating bilinear form $\langle \ , \ \rangle$ given by

$$\langle f_1, f_2 \rangle = -\langle f_2, f_1 \rangle = 1, \quad \langle f_i, f_i \rangle = 0.$$

Then $V = V_1^{\oplus n}$ is a symplectic space of dimension $2n$ over F . Let

$$GSp_{2n} := GSp(V) = \{g \in GL(V) : \langle gv_1, gv_2 \rangle = \lambda(g)\langle v_1, v_2 \rangle\}$$

where $\lambda(g) \in F^\times$ is a scalar. The scalar $\lambda(g)$ is multiplicative and

$$\text{sim} : GSp(V) \longrightarrow F^\times, \quad \text{sim}(g) = \lambda(g)$$

is the similitude character of $GSp(V)$. We have

$$Sp(V) = \{g \in GSp(V) : \lambda(g) = 1\}.$$

We now define $GU_{2n}(D)$, an inner form of GSp_{4n} . Let D be the quaternion division algebra over k . Let $V_2 = De_1 \oplus De_2$ be the 2-dimensional vector space over D equipped with a Hermitian form with inner product given by

$$\langle e_1, e_2 \rangle = \langle e_2, e_1 \rangle = 1, \quad \langle e_i, e_i \rangle = 0.$$

Also,

$$\langle dv_1, d'v_2 \rangle = \tau(d)\langle v_1, v_2 \rangle d'$$

where τ is the standard involution on D . Then $V = V_2^{\oplus n}$ is a Hermitian space of dimension $2n$ over D and

$$GU_{2n}(D) = \{g \in \text{Aut}_D(V) : \langle gv_1, gv_2 \rangle = \mu(g)\langle v_1, v_2 \rangle\}$$

where $\mu(g) \in k^\times$ is a scalar.

2.3 General spin groups

In this section we define general spin groups $GSpin_m$, and related groups.

Let V be an m dimensional vector space over F equipped with a non-degenerate quadratic form $\langle \cdot, \cdot \rangle$. Then

$$GO(V) = \{g \in GL(V) : \langle gv_1, gv_2 \rangle = \lambda(g)\langle v_1, v_2 \rangle\}$$

where $\lambda \in F^\times$ is a scalar. The similtude character of $GO(V)$ is

$$\text{sim} : GO(V) \longrightarrow F^\times, \quad \text{sim}(g) = \lambda(g).$$

Let $GSO(V)$ be the connected component of $GO(V)$, as the latter is not a connected algebraic group. We have

$$SO(V) = \{g \in GSO(V) : \lambda(g) = 1\}.$$

Let

$$C(V) = C^+(V) \oplus C^-(V)$$

be the Clifford algebra of V with its 2-grading. There is a canonical embedding $V \hookrightarrow C^-(V)$. Then

$$GSpin(V) = \{g \in C^+(V)^\times \mid gg^t = \nu(g), \text{ and } gVg^{-1} = V\}$$

where ι is the main involution of $C(V)$ and $\nu(g)$ is a scalar. The scalar $\nu(g)$ is multiplicative and

$$\text{sim} : GSpin(V) \longrightarrow F^\times, \quad \text{sim}(g) = \nu(g)$$

is the similitude character of $GSpin(V)$. Denote by

$$Spin(V) = \{g \in GSpin(V) : \nu(g) = 1\}.$$

As algebraic groups, we have the exact sequence

$$1 \longrightarrow Z^0 \longrightarrow GSpin(V) \longrightarrow SO(V) \longrightarrow 1,$$

where Z^0 is the connected center of $GSpin(V)$.

Over a p -adic field k , there are two quadratic forms of dimension m and discriminant 1. Let $V^+ = H^{2n} \oplus \langle 1 \rangle$ or $V^+ = H^{2n}$ be the split quadratic space of dimension $m = 2n + 1$ or $2n$ and discriminant 1. Here $H = ke_1 \oplus ke_2$ is the hyperbolic plane with inner form given by

$$\langle e_1, e_2 \rangle = \langle e_2, e_1 \rangle = 1, \quad \langle e_i, e_i \rangle = 0.$$

Let $V^- = H^{2(n-1)} \oplus D_0$ be the non-split quadratic space of dimension $2n + 1$ and discriminant 1, where D_0 is the subset of the quaternion k -algebra D of elements of reduced trace 0 and the quadratic form on D_0 is given by the reduced norm. Let

$$GSO_m = GSO(V^+), \quad SO_m = GSO(V^+).$$

Define

$$GSpin_m = GSpin(V^+),$$

and

$$GSpin_{n+2,n-1} = GSpin(V^-).$$

Over a field $\mathfrak{f}$ of characteristic p , let $V^+ = H^{2n} \oplus \langle 1 \rangle$ or $V^+ = H^{2n}$ be the quadratic space of dimension $m = 2n + 1$ or $2n$ over $\mathfrak{f}$ where $H = \mathfrak{f}e_1 \oplus \mathfrak{f}e_2$ is the hyperbolic plane with inner product given as above. Let

$$GSpin_m = GSpin(V^+).$$

For $\mathfrak{f} = \mathbb{F}_q$, let $V^- = H^{2(n-1)} \oplus \mathbb{F}_{q^2}$ where $\mathbb{F}_{q^2}$ is the unique quadratic extension of $\mathbb{F}_q$ with quadratic form given by $N_{\mathbb{F}_{q^2}/\mathbb{F}_q}$. Let

$${}^2GSpin_{2n} = GSpin(V^-).$$

2.4 Root datum

In this section let G be a reductive p -adic group and T a maximal split torus of G . The text [Sp2] is a good reference for root datum. Let

$$X = \text{Hom}(T, GL_1), \quad X^\vee = \text{Hom}(GL_1, T),$$

be the group of algebraic characters of T and the group of algebraic cocharacters of T , respectively. Let Φ denote the roots of T , the set of non-zero weights of T acting via the adjoint representation Ad in the Lie algebra $\mathfrak{g}$ of G . Let

$$W_0 = N_G(T)/T$$

be the Weyl group of T . For $a \in \Phi$, let G_a denote the centralizer of the subtorus $(\text{Ker } a)^0$ of T . Let s_a be the image in W_0 of an element in $N_{G_a}(T) \setminus C_{G_a}(T)$. Via its action on T , W_0 acts on $\mathbb{R} \otimes X$. Then, for s_a in W_0 , also denote by s_a its induced action on $\mathbb{R} \otimes X$. Let $\langle \cdot, \cdot \rangle : X \times X^\vee \rightarrow \mathbb{Z}$ be the canonical pairing. There exists a unique $a^\vee \in X^\vee$ with $\langle a, a^\vee \rangle = 2$ such that for $x \in X$

$$s_a = x - \langle x, a^\vee \rangle a.$$

Define the coroots $\Phi^\vee$ of T as

$$\Phi^\vee = \{a^\vee : a \in \Phi\}.$$

The quadruple

$$\Psi = (X, \Phi, X^\vee, \Phi^\vee)$$

is the root datum for G .

In the following, if $(X, \Phi, X^\vee, \Phi^\vee)$ is the root datum for a group G , then there is a maximal split torus T of G such that $X, X^\vee$ are the character and cocharacter lattices of T , respectively, and $\Phi, \Phi^\vee$ are the roots and coroots of T . Instead of listing the roots Φ and coroots $\Phi^\vee$, we give Δ a set of simple roots for T that generate Φ , and $\Delta^\vee$ a set of simple coroots for T that generate $\Phi^\vee$.

The root datum for GL_n can be given by

$$X = \mathbb{Z}e_0 \oplus \mathbb{Z}e_1 \oplus \cdots \oplus \mathbb{Z}e_n, \quad X^\vee = \mathbb{Z}e_0^* \oplus \mathbb{Z}e_1^* \oplus \cdots \oplus \mathbb{Z}e_n^*,$$

$$\Delta = \{a_1 = e_1 - e_2, a_2 = e_2 - e_3, \dots, a_{n-1} = e_{n-1} - e_n\},$$

$$\Delta^\vee = \{a_1^\vee = e_1^* - e_2^*, a_2^\vee = e_2^* - e_3^*, \dots, a_{n-1}^\vee = e_{n-1}^* - e_n^*\}.$$

The root datum for $GSpin_{2n+1}$ can be given by [AS, Prop 2.1]

$$X = \mathbb{Z}e_0 \oplus \mathbb{Z}e_1 \oplus \dots \oplus \mathbb{Z}e_n, \quad X^\vee = \mathbb{Z}e_0^* \oplus \mathbb{Z}e_1^* \oplus \dots \oplus \mathbb{Z}e_n^*,$$

$$\Delta = \{a_1 = e_1 - e_2, a_2 = e_2 - e_3, \dots, a_{n-1} = e_{n-1} - e_n, a_n = e_n\},$$

$$\Delta^\vee = \{a_1^\vee = e_1^* - e_2^*, a_2^\vee = e_2^* - e_3^*, \dots, a_{n-1}^\vee = e_{n-1}^* - e_n^*, a_n^\vee = 2e_n^* - e_0^*\}.$$

The root datum for $GSpin_{2n}$ can be given by [AS, Prop 2.1]

$$X = \mathbb{Z}e_0 \oplus \mathbb{Z}e_1 \oplus \dots \oplus \mathbb{Z}e_n, \quad X^\vee = \mathbb{Z}e_0^* \oplus \mathbb{Z}e_1^* \oplus \dots \oplus \mathbb{Z}e_n^*,$$

$$\Delta = \{a_1 = e_1 - e_2, a_2 = e_2 - e_3, \dots, a_{n-1} = e_{n-1} - e_n, a_n = e_{n-1} + e_n\},$$

$$\Delta^\vee = \{a_1^\vee = e_1^* - e_2^*, a_2^\vee = e_2^* - e_3^*, \dots, a_{n-1}^\vee = e_{n-1}^* - e_n^*, a_n^\vee = e_{n-1}^* + e_n^* - e_0^*\}.$$

The root datum for GSp_{2n} can be given by

$$X = \mathbb{Z}e_0 \oplus \mathbb{Z}e_1 \oplus \dots \oplus \mathbb{Z}e_n, \quad X^\vee = \mathbb{Z}e_0^* \oplus \mathbb{Z}e_1^* \oplus \dots \oplus \mathbb{Z}e_n^*,$$

$$\Delta = \{a_1 = e_1 - e_2, a_2 = e_2 - e_3, \dots, a_{n-1} = e_{n-1} - e_n, a_n = 2e_n - e_0\},$$

$$\Delta^\vee = \{a_1^\vee = e_1^* - e_2^*, a_2^\vee = e_2^* - e_3^*, \dots, a_{n-1}^\vee = e_{n-1}^* - e_n^*, a_n^\vee = e_n^*\},$$

The root datum for GSO_{2n} can be given by

$$X = \mathbb{Z}e_0 \oplus \mathbb{Z}e_1 \oplus \dots \oplus \mathbb{Z}e_n, \quad X^\vee = \mathbb{Z}e_0^* \oplus \mathbb{Z}e_1^* \oplus \dots \oplus \mathbb{Z}e_n^*,$$

$$\Delta = \{a_1 = e_1 - e_2, a_2 = e_2 - e_3, \dots, a_{n-1} = e_{n-1} - e_n, a_n = e_{n-1} + e_n - e_0\},$$

$$\Delta^\vee = \{a_1^\vee = e_1^* - e_2^*, a_2^\vee = e_2^* - e_3^*, \dots, a_{n-1}^\vee = e_{n-1}^* - e_n^*, a_n^\vee = e_{n-1}^* + e_n^*\}.$$

If $\Psi = (X, \Phi, X^\vee, \Phi^\vee)$ is the root datum for a group G , then we define the dual group of G , written $G^\vee$, as the group with root datum dual to that of G . Namely, the root datum of $G^\vee$ is

$$\Psi^\vee = (X^\vee, \Phi^\vee, X, \Phi).$$

We observe

$$GL_n^\vee = GL_n, \quad GSpin_{2n+1}^\vee = GSp_{2n}, \quad GSpin_{2n}^\vee = GSO_{2n}.$$

Given a quadratic space V , if one has the decomposition $V = V_1 \oplus V_2$, with V_i nondegenerate quadratic subspaces, then $SO(V_1) \times SO(V_2) \subset SO(V)$. If we restrict the covering

$$1 \longrightarrow Z^0 \longrightarrow GSpin(V) \longrightarrow SO(V) \longrightarrow 1$$

to the subgroup $SO(V_1) \times SO(V_2)$ we get

$$1 \longrightarrow Z^0 \longrightarrow GSpin(V_1) \times GSpin(V_2)/\Delta GL_1 \longrightarrow SO(V_1) \times SO(V_2) \longrightarrow 1.$$

Precisely, let

$$(GSpin_{2m} \times GSpin_{2n+1})/\Delta GL_1 = (GSpin_{2m} \times GSpin_{2n+1})/\{h_0^*(\lambda)g_0^*(\lambda) : \lambda \in GL_1\}$$

where h_0^* and g_0^* are given in the following lemma. Let

$$(GSO_{2m} \times GSp_{2n})^\circ = \{(g_1, g_2) \in GSO_{2m} \times GSp_{2n} : \text{sim}(g_1) = \text{sim}(g_2)\}.$$

Lemma 2.1. *The root datum for $(GSpin_{2m} \times GSpin_{2n+1})/\Delta GL_1$ is given by*

$$X = \mathbb{Z}e_0 \oplus \cdots \oplus \mathbb{Z}e_{m+n}, \quad X^\vee = \mathbb{Z}e_0^* \oplus \cdots \oplus \mathbb{Z}e_{m+n}^*,$$

$$\Delta = \{e_1 - e_2, \dots, e_{m-1} - e_n, e_{m-1} + e_m\} \cup \{e_{m+1} - e_{m+2}, \dots, e_{m+n-1} - e_{m+n}, e_{m+n}\},$$

$$\begin{aligned} \Delta^\vee = & \{e_1^* - e_2^*, \dots, e_{m-1}^* - e_m^*, e_{m-1}^* + e_m^* - e_0^*\} \\ & \cup \{e_{m+1}^* - e_{m+2}^*, \dots, e_{m+n-1}^* - e_{m+n}^*, 2e_{m+n}^* - e_0^*\}. \end{aligned}$$

Also, $(GSO_{2m} \times GSp_{2n})^\circ = ((GSpin_{2m} \times GSpin_{2n+1})/\Delta GL_1)^\vee$.

Proof. We work with the root datum for $GSpin_{2m}$ and $GSpin_{2n+1}$ given above using the letter h for $GSpin_{2m}$ and g for $GSpin_{2n+1}$. The character lattice for $GSpin_{2m} \times GSpin_{2n+1}$ is the $\mathbb{Z}$ -span of

$$h_0, h_1, \dots, h_m, g_0, g_1, \dots, g_n.$$

The characters for $G = (GSpin_{2m} \times GSpin_{2n+1})/\Delta GL_1$ are those which are trivial on

$$\{h_0^*(\lambda)g_0^*(\lambda) : \lambda \in GL_1\}.$$

The character lattice for G is the $\mathbb{Z}$ -span of

$$h_0 - g_0, h_1, \dots, h_m, g_1, \dots, g_n.$$

Using the $\mathbb{Z}$ pairing of the root datum, the cocharacter lattice is the $\mathbb{Z}$ -span of

$$\overline{h_0^*} = \overline{g_0^*}, \overline{h_1^*}, \dots, \overline{h_m^*}, \overline{g_1^*}, \dots, \overline{g_n^*}.$$

Set

$$e_0 = h_0 - g_0, e_1 = h_1, \dots, e_m = h_m, e_{m+1} = g_1, e_{m+n} = g_n$$

and

$$e_0^* = \overline{g_0^*}, e_1^* = \overline{h_1^*}, \dots, e_m^* = \overline{h_m^*}, e_{m+1}^* = \overline{g_1^*}, \dots, e_{m+n}^* = \overline{g_n^*}.$$

Using this notation we see that the roots and coroots for G are those given in the statement of the lemma.

Similarly, we work with the root datum for GSO_{2m} given above using the letter h and the root datum for GSp_{2n} given above using the letter g . The characters for $G' = (GSO_{2m} \times GSp_{2n})^\circ$ are equivalence classes of characters for $GSO_{2m} \times GSp_{2n}$. Two characters are equivalent if they have the same value on all elements of G' . The character lattice for G' is the $\mathbb{Z}$ -span of

$$\overline{1/2(h_0 + g_0)} = \overline{h_0} = \overline{g_0}, \overline{h_1}, \dots, \overline{h_m}, \overline{g_1}, \dots, \overline{g_n}.$$

Using the $\mathbb{Z}$ pairing of the root datum, the cocharacter lattice is the $\mathbb{Z}$ -span of

$$h_0^* + g_0^*, h_1^*, \dots, h_m^*, g_1^*, \dots, g_n^*.$$

Setting

$$e_0 = \overline{1/2(h_0 + g_0)}, e_1 = \overline{h_1}, \dots, e_m = \overline{h_m}, e_{m+1} = \overline{g_1}, \dots, e_{m+n} = \overline{g_n}$$

and

$$e_0^* = h_0^* + g_0^*, e_1^* = h_1^*, \dots, e_m^* = h_m^*, e_{m+1}^* = g_1^*, \dots, e_{m+n}^* = g_n^*,$$

we see that the roots of G' are the coroots of G , and the coroots of G' are the roots of G . $\square$

The center of $GSpin_{2n}$ is not connected, and in the following we will have need to work with groups with connected center. Let

$$GSpin_{2n}^\sim = (GL_1 \times GSpin_{2n}) / \{(1, 1), (-1, \zeta_0)\},$$

where $\zeta_0 = e_1^*(-1)e_2^*(-1)\dots e_n^*(-1)$ is an element in the center of $GSpin_{2n}$. The root datum for $GSpin_{2n}^\sim$ can be given by [AS, 2.6]

$$X = \mathbb{Z}E_{-1} \oplus \mathbb{Z}E_0 \oplus \dots \oplus \mathbb{Z}E_n, \quad X^\vee = \mathbb{Z}E_{-1}^* \oplus \mathbb{Z}E_0^* \oplus \dots \oplus \mathbb{Z}E_n^*,$$

$$\begin{aligned}\Delta &= \{E_1 - E_2, \dots, E_{n-1} - E_n, E_{n-1} + E_n - E_{-1}\}, \\ \Delta^\vee &= \{E_1^* - E_2^*, \dots, E_{n-1}^* - E_n^*, E_{n-1}^* + E_n^* - E_0^*\}.\end{aligned}$$

The center of $GSpin_{2n}^\sim$ is the set of elements which that belong to the kernel of all the simple roots, namely

$$\{E_0^*(\mu)E_1^*(\nu) \dots E_n^*(\nu)E_{-1}^*(\nu^2) : \mu, \nu \in GL_1\} \simeq GL_1 \times GL_1,$$

which is connected.

Let

$$(GSpin_{2m}^\sim \times GSpin_{2n+1})/\Delta GL_1 = (GSpin_{2m}^\sim \times GSpin_{2n+1})/\{E_0^*(\lambda)e_0^*(\lambda) : \lambda \in GL_1\}.$$

Lemma 2.2. *The root datum for $(GSpin_{2m}^\sim \times GSpin_{2n+1})/\Delta GL_1$ is given by*

$$X = \mathbb{Z}E_{-1} \oplus \mathbb{Z}E_0 \oplus \dots \oplus \mathbb{Z}E_{m+n}, \quad X^\vee = \mathbb{Z}E_{-1}^* \oplus \mathbb{Z}E_0^* \oplus \dots \oplus \mathbb{Z}E_{m+n}^*,$$

$$\begin{aligned}\Delta &= \{E_1 - E_2, \dots, E_{m-1} - E_n, E_{m-1} + E_m - E_{-1}\} \\ &\quad \cup \{E_{m+1} - E_{m+2}, \dots, E_{m+n-1} - E_{m+n}, E_{m+n}\}, \\ \Delta^\vee &= \{E_1^* - E_2^*, \dots, E_{m-1}^* - E_m^*, E_{m-1}^* + E_m^* - E_0^*\} \\ &\quad \cup \{E_{m+1}^* - E_{m+2}^*, \dots, E_{m+n-1}^* - E_{m+n}^*, 2E_{m+n}^* - E_0^*\}.\end{aligned}$$

Proof. The proof is similar to that of Lemma 2.1. □

The center of $(GSpin_{2m}^\sim \times GSpin_{2n+1})/\Delta GL_1$ is given by

$$\{E_0^*(\mu)E_1^*(\nu) \dots E_m^*(\nu)E_{-1}^*(\nu^2) : \mu, \nu \in GL_1\} \simeq GL_1 \times GL_1,$$

which is connected.

3 Tame regular discrete L -parameters

We first set some notation. Let k be a non-archimedean local field of characteristic zero with finite residue field $\mathfrak{f}$ of characteristic p . Let $q = |\mathfrak{f}|$. Let $\bar{k}$ be a fixed algebraic closure of k , K the maximal unramified extension of k in $\bar{k}$, and k_m the unramified extension of k degree m in $\bar{k}$. Let $\bar{\mathfrak{f}}$ be the residue field of K . Then $\bar{\mathfrak{f}}$ is an algebraic closure of $\mathfrak{f}$. Let $\mathfrak{f}_m$ be the degree m extension of $\mathfrak{f}$ in $\bar{\mathfrak{f}}$.

Let $\mathcal{I} = \text{Gal}(\bar{k}/K)$ be the inertia subgroup of the Galois group $\text{Gal}(\bar{k}/k)$. We have

$$\text{Gal}(\bar{k}/k)/\mathcal{I} \cong \text{Gal}(K/k) \cong \text{Gal}(\bar{\mathfrak{f}}/\mathfrak{f}),$$

where

$$\text{Gal}(\bar{\mathfrak{f}}/\mathfrak{f}) = \varprojlim_{m \geq 1} \text{Gal}(\mathfrak{f}_m/\mathfrak{f})$$

and $\text{Gal}(k_m/k) \cong \text{Gal}(\mathfrak{f}_m/\mathfrak{f}) \cong \mathbb{Z}/m\mathbb{Z}$. A geometric Frobenius

$$\text{Frob} \in \text{Gal}(\bar{k}/k)$$

is one whose image in $\text{Gal}(\bar{\mathfrak{f}}/\mathfrak{f})$ is the automorphism which acts on $\mathfrak{f}_m$ as the inverse of $x \mapsto x^q$ for all m . We define W_k , the Weil group of k , as

$$\begin{array}{ccccccc} 1 & \longrightarrow & \mathcal{I} & \longrightarrow & W_k & \longrightarrow & \mathbb{Z} \longrightarrow 1 \\ & & \parallel & & \downarrow & & \downarrow \\ 1 & \longrightarrow & \mathcal{I} & \longrightarrow & \text{Gal}(\bar{k}/k) & \longrightarrow & \text{Gal}(\bar{\mathfrak{f}}/\mathfrak{f}) \longrightarrow 1 \end{array}$$

Fix a choice of geometric Frobenius. Then $\langle \text{Im}(\text{Frob}) \rangle \subset \text{Gal}(\bar{\mathfrak{f}}/\mathfrak{f})$ is a dense subgroup and

$$W_k = \mathcal{I} \rtimes \langle \text{Frob} \rangle.$$

The Weil-Deligne group W'_k is defined as

$$W'_k = W_k \times SL_2(\mathbb{C}).$$

Let G be a quasi-split connected reductive k -group, and T a maximally k -split torus contained in a k -Borel subgroup B of G . Let X and $X^\vee$ be the groups of algebraic characters and cocharacters of T (defined over $\bar{k}$), respectively. Let Φ^+ and $\check{\Phi}^+$ be the sets of positive roots and coroots of T , respectively. We have

$$(X, X^\vee, \Phi^+, \check{\Phi}^+)$$

is a based root datum for G . Up to isomorphism, there is a unique complex reductive group $\hat{G}$ with based root datum $(X^\vee, X, \check{\Phi}^+, \Phi^+)$ dual to that of G . The Galois group $\text{Gal}(\bar{k}/k)$ acts on X and $X^\vee$ preserving Φ^+ and $\check{\Phi}^+$. Denote by $\text{Gal}(E/k)$ the kernel of this action. This action extends to an automorphism of $\hat{G}$. Define the L -group of G as

$${}^L G := \text{Gal}(E/k) \ltimes \hat{G}.$$

If G' is not quasi-split, then G' is the inner form of a quasi-split group G . The L -group of G' is defined as the L -group of G .

By a Langlands parameter we mean a continuous homomorphism

$$\phi : W'_k \longrightarrow {}^L G$$

such that the projection onto $\text{Gal}(E/k)$ is the composition

$$W_k \hookrightarrow \text{Gal}(\bar{k}/k) \longrightarrow \text{Gal}(E/k),$$

and such that the projection of $\phi(\text{Frob})$ onto $\hat{G}$ is semisimple. A parameter is said to be admissible if it satisfies these conditions. For $GS p_4(k)$, since it is a split group, the action of $\text{Gal}(E/k)$ on $\hat{G} = GS p_4(\mathbb{C})$ is trivial, so we will consider admissible homomorphisms

$$\phi : W'_k \longrightarrow GS p_4(\mathbb{C}).$$

Definition 3.1. Let ϕ be a Langlands parameter.

- (i) We say that ϕ is tame if ϕ is trivial on the wild inertia group $\mathcal{I}^+$, the maximal pro- p subgroup of $\mathcal{I}$.
- (ii) We say the ϕ is regular if the centralizer in $\hat{G}$ of $\phi(\mathcal{I})$ is a maximal torus $\hat{T}$ in $\hat{G}$.

- (iii) We say that ϕ is discrete if the identity component of the centralizer in $\hat{G}$ of $\phi(W_k)$ is equal to the identity component of the center $\hat{Z}$ of $\hat{G}$.

For k_{2d} the unramified extension of k of degree $2d$, the Weil group of k_{2d} is

$$W_{k_{2d}} = \langle \text{Frob}^{2d} \rangle \ltimes \mathcal{I}.$$

Lemma 3.2. *Fix $\lambda : W_k \longrightarrow \mathbb{C}^\times$. Let $V(\eta) = \text{Ind}_{W_{k_{2d}}}^{W_k} \eta$, where $\eta : W_{k_{2d}} \longrightarrow \mathbb{C}^\times$ is a continuous character. Then*

- (i) $V(\eta)$ is tame $\iff \eta$ is trivial on the wild inertia subgroup $\mathcal{I}^+$,
- (ii) $V(\eta)$ is irreducible $\iff \eta, \eta^{\text{Frob}}, \dots, \eta^{\text{Frob}^{2d-1}}$ are pairwise distinct,
- (iii) Under (ii), $V(\eta)$ is symplectic with similitude character $\lambda \iff \eta \cdot \eta^{\text{Frob}^d} \cong \lambda|_{W_{k_{2d}}}$ and $\eta(\text{Frob}^{2d}) = -\lambda(\text{Frob}^d)$.

Proof. (i) Since $\mathcal{I} \subset W_{k_{2d}}$, by the definition of induced representation $\phi : W_k \longrightarrow GL(V(\eta))$ is trivial on $\mathcal{I}^+$ if and only if η is trivial on $\mathcal{I}^+$.

(ii) A set of coset representative for $W_{k_{2d}} \setminus W_k / W_{k_{2d}}$ is $\{\text{Frob}, \text{Frob}^2, \dots, \text{Frob}^{2d}\}$. Define $\eta^{\text{Frob}^i} : W_{k_{2d}} \longrightarrow \mathbb{C}^\times$ by

$$\eta^{\text{Frob}^i}(w) = \eta((\text{Frob}^i)^{-1} w \text{Frob}^i).$$

By Mackey's irreducibility criterion, $V(\eta)$ is irreducible if and only if

- (a) η is irreducible, and
- (b) the conjugates of η , namely $\eta, \eta^{\text{Frob}}, \dots, \eta^{\text{Frob}^{2d-1}}$, are pairwise distinct.

Therefore, since η is 1-dimensional, $V(\eta)$ is irreducible if and only (b) is satisfied.

(iii) Assume (ii) is satisfied. Let λ be a character of W_k . There exists a W_k -equivariant map

$$B : V(\eta) \otimes V(\eta) \longrightarrow \lambda$$

if and only if $V(\eta) \cong V(\eta)^\vee \otimes \lambda$. Since $V(\eta)$ is irreducible, any such nonzero map B would be nondegenerate. Now,

$$V(\eta)^\vee \otimes \lambda \cong V(\eta^\vee) \otimes \lambda \cong V(\eta^{-1}) \otimes \lambda \cong V(\eta^{-1} \cdot \lambda|_{W_{k_{2d}}}),$$

where $\eta^\vee = \eta^{-1}$. By Frobenius reciprocity,

$$\mathrm{Hom}_{W_k}(V(\eta^{-1} \cdot \lambda|_{W_{k_{2d}}}), V(\eta)) = \mathrm{Hom}_{W_{k_{2d}}}(\eta^{-1} \cdot \lambda|_{W_{k_{2d}}}, V(\eta)|_{W_{k_{2d}}}).$$

Also by Frobenius reciprocity

$$V(\eta)|_{W_{k_{2d}}} \cong \eta \oplus \eta^{\mathrm{Frob}} \oplus \dots \oplus \eta^{\mathrm{Frob}^{2d-1}}.$$

Therefore,

$$V(\eta) \cong V(\eta)^\vee \otimes \lambda \iff \eta^{-1} \cdot \lambda|_{W_{k_{2d}}} \cong \eta^{\mathrm{Frob}^i} \iff \lambda|_{W_{k_{2d}}} \cong \eta \cdot \eta^{\mathrm{Frob}^i}$$

for some $0 \leq i \leq 2d-1$.

Assume such a W_k -equivariant map B exists. Then $\lambda|_{W_{k_{2d}}} \cong \eta \cdot \eta^{\mathrm{Frob}^i}$ for some $0 \leq i \leq 2d-1$. Since λ is a character of W_k , for any power l ,

$$(\lambda|_{W_{k_{2d}}})^{\mathrm{Frob}^l}(w) = \lambda(\mathrm{Frob}^{-l}w\mathrm{Frob}^l) = \lambda(\mathrm{Frob}^{-l})\lambda(w)\lambda(\mathrm{Frob}^l) = \lambda|_{W_{k_{2d}}}(w).$$

Then

$$\lambda|_{W_{k_{2d}}} = (\lambda|_{W_{k_{2d}}})^{\mathrm{Frob}^{-i}} \cong (\eta \cdot \eta^{\mathrm{Frob}^i})^{\mathrm{Frob}^{-i}} = \eta^{\mathrm{Frob}^{-i}} \cdot \eta.$$

This implies

$$\eta \cdot \eta^{\mathrm{Frob}^i} \cong \eta^{\mathrm{Frob}^{-i}} \cdot \eta \implies \eta^{\mathrm{Frob}^{2i}} = \eta.$$

Since $0 \leq i \leq 2d-1$, this is only possible if $i = 0$ or d since $\eta, \eta^{\mathrm{Frob}}, \dots, \eta^{\mathrm{Frob}^{2d-1}}$ are pairwise distinct.

Assume $\eta \cdot \eta^{\mathrm{Frob}^d} \cong \lambda|_{W_{k_{2d}}}$. Extend $\eta \cdot \eta^{\mathrm{Frob}^d}$ to a character of W_k such that the extension is isomorphic to λ . Then there exists a nondegenerate W_k -equivariant map $B : V(\eta) \otimes V(\eta) \longrightarrow \lambda$. Therefore such a map B exists if and only if $\eta \cdot \eta^{\mathrm{Frob}^d} \cong \lambda|_{W_{k_{2d}}}$.

We need to determine under what conditions B is symplectic. By Frobenius reciprocity

$$V(\eta)|_{W_{k_{2d}}} = \bigoplus_{i=0}^{2d-1} \eta^{\mathrm{Frob}^i}.$$

Then

$$B : \left(\bigoplus_{i=0}^{2d-1} \eta^{\mathrm{Frob}^i} \right) \otimes \left(\bigoplus_{i=0}^{2d-1} \eta^{\mathrm{Frob}^i} \right) \longrightarrow \lambda|_{W_{k_{2d}}}$$

is a $W_{k_{2d}}$ -equivariant map. With respect to this basis, by restriction,

$$B : \eta^{\mathrm{Frob}^i} \otimes \eta^{\mathrm{Frob}^j} \longrightarrow \lambda|_{W_{k_{2d}}}$$

is a nonzero $W_{k_{2d}}$ -invariant map if the entry t_{ij} in the matrix of the form B is nonzero. So

$$t_{ij} \neq 0 \iff \eta^{\text{Frob}^i} \cdot \eta^{\text{Frob}^j} \cong \lambda|_{W_{k_{2d}}} \iff |j - i| = d.$$

We can choose eigenvectors in the subspaces η and η^{Frob^d} of $V(\eta)|_{W_{k_{2d}}}$ such that the matrix of $\phi(\text{Frob}^d)$ on $\eta \oplus \eta^{\text{Frob}^d}$ has the form $\begin{pmatrix} 0 & 1 \\ \eta(\text{Frob}^{2d}) & 0 \end{pmatrix}$. This matrix preserves

$\begin{pmatrix} 0 & t_{0d} \\ t_{d0} & 0 \end{pmatrix}$ up to λ , where as B is either symplectic or orthogonal, either $t_{0d} = -t_{d0}$ or $t_{0d} = t_{d0}$. Therefore $\begin{pmatrix} 0 & t_{0d} \\ t_{d0} & 0 \end{pmatrix}$ is preserved up to λ if and only if

$$\eta(\text{Frob}^{2d}) = \pm \lambda(\text{Frob}^d).$$

Hence, B is symplectic if and only if $\eta(\text{Frob}^{2d}) = -\lambda(\text{Frob}^d)$. $\square$

Lemma 3.3. *All tame regular discrete Langlands parameters for $GSp_{2n}(k)$ with similitude character λ are of the form $\phi : W_k \longrightarrow GSp(V)$ such that*

$$V = V(\eta_1) \oplus \cdots \oplus V(\eta_s) \text{ where } V(\eta_i) = \text{Ind}_{W_{k_{2d_i}}}^{W_k} \eta_i$$

satisfying the conditions of Lemma 3.2, where k_{2d_i} are unramified extensions of k of degree $2d_i$ such that $d_1 + \cdots + d_s = n$. In addition,

(i) if $k_{2d_i} = k_{2d_j}$, η_i is not equal to any conjugate $\eta_j^{\text{Frob}^k}$, $0 \leq k \leq d_j$, of η_j so the $V(\eta_i)$ are pairwise non-isomorphic and

(ii) the similitude character of $V(\eta_i)$ is λ for all i .

Proof. Let $\phi : W_k \longrightarrow GSp(V)$, where V is a $2n$ dimensional complex vector space, be a tame regular discrete Langlands parameter with similitude character λ . As ϕ is discrete, the identity component of the centralizer in $GSp(V)$ of $\phi(W_k)$ is equal to the identity component of the center of $GSp(V)$, which implies that V is multiplicity free. We have,

$$V = \bigoplus V_i$$

where any two V_i are pairwise non-isomorphic and each V_i is symplectic with similitude character λ . Therefore, it suffices to show any irreducible, tame, regular $\phi : W_k \rightarrow GSp(V)$ with similitude character λ is of the form $V(\eta)$ with η as in Lemma 3.2.

Let ϕ be such a parameter where $\dim V = 2d$. As ϕ is tame, it factors through the tame inertia group $\mathcal{I}_t = \mathcal{I}/\mathcal{I}^+ \simeq \varprojlim \mathfrak{f}_n^\times$, where $\mathfrak{f}_n$ is the degree n extension of $\mathfrak{f}$ the residue field of k . Then, ϕ factors through $\mathfrak{f}_n^\times$ for some $n \geq 1$ and since $\mathfrak{f}_n^\times$ is cyclic, $\phi(\mathcal{I}_t)$ is cyclic. There is a basis of V such that

$$\phi|_{\mathcal{I}_t} = \bigoplus_i \chi_i,$$

where the χ_i are characters. Since ϕ is regular the centralizer of $\phi(\mathcal{I})$ is a maximal torus, so the χ_i are pairwise distinct. Since V is irreducible, Frob permutes the set $\{\chi_i\}$ transitively. Note that Frob^{2d} induces the trivial permutation on $\{\chi_i\}$. Choose $\eta \in \{\chi_i\}$. Then

$$\text{Stab}_{W_k} \eta = \langle \text{Frob}^{2d} \rangle \ltimes \mathcal{I} = W_{k_{2d}}.$$

By Frobenius reciprocity

$$\dim(\text{Hom}_{W_{k_{2d}}}(\eta, \phi|_{W_{k_{2d}}})) = \dim(\text{Hom}_{W_k}(\text{Ind}_{W_{k_{2d}}}^{W_k} \eta, \phi)).$$

Therefore

$$\phi \cong \text{Ind}_{W_{k_{2d}}}^{W_k} \eta =: V(\eta).$$

We have that $V(\eta)$ is tame, irreducible, and symplectic with similitude character λ , and therefore satisfies the conditions of Lemma 3.2. $\square$

4 Local Langlands for GSp_4 and $GU_2(D)$

In [GT], Gan and Takeda prove the local Langlands conjecture for GSp_4 . They define a surjective finite-one-map

$$L : \Pi(GSp_4) \longrightarrow \Phi(GSp_4)$$

from the set of isomorphism classes of irreducible smooth representations of $GSp_4(k)$ to the set of equivalence classes of admissible homomorphisms

$$\phi : W'_k \longrightarrow GSp_4(\mathbb{C})$$

taken up to $GSp_4(\mathbb{C})$ conjugacy. Let L_ϕ^{GT} be the fiber of L over ϕ , so that

$$L_\phi^{GT} \mapsto \phi.$$

The map L satisfies many expected and desired properties such as the preservation of L -functions and ϵ -factors attached to both sides of the correspondence. Specifically, L satisfies [GT, Main Theorem (i), (ii), (iii), (iv), (v), (vi), (viii)]

Theorem 4.1. (i) π is a (essentially) discrete series representation if and only if its L -parameter ϕ_π does not factor through any proper Levi subgroup of $GSp_4(\mathbb{C})$.

(ii) For an L -parameter ϕ , its fiber L_ϕ can be naturally parametrized by the set of irreducible characters of the component group

$$A_\phi = \pi_0(C_{GSp_4(\mathbb{C})}(Im(\phi))).$$

This component group is either trivial or equal to $\mathbb{Z}/2\mathbb{Z}$. When $A_\phi = \mathbb{Z}/2\mathbb{Z}$, exactly one of the two representations in L_ϕ is generic and it is the one indexed by the trivial character of A_ϕ .

- (iii) *The similitude character $\text{sim}(\phi_\pi)$ of ϕ_π is equal to the central character ω_π of π (via local class field theory). Here, $\text{sim} : GSp_4(\mathbb{C}) \rightarrow \mathbb{C}^\times$ is the similitude character of $GSp_4(\mathbb{C})$.*
- (iv) *The L -parameter of $\pi \otimes (\chi \circ \lambda)$ is equal to $\phi_\pi \otimes \chi$. Here, $\lambda : GSp_4(k) \rightarrow k^\times$ is the similitude character of $GSp_4(k)$, and we have regarded χ as both a character of $k^\times$ and a character of W_k by local class field theory.*
- (v) *Suppose that π is a generic representation or a non-supercuspidal representation. Then for any irreducible representation σ of $GL_r(k)$, with L -parameter ϕ_σ , we have:*

$$\begin{aligned}\gamma(s, \pi \times \sigma, \psi) &= \gamma(s, \phi_\pi \otimes \phi_\sigma, \psi) \\ L(s, \pi \times \sigma) &= L(s, \phi_\pi \otimes \phi_\sigma) \\ \epsilon(s, \pi \times \sigma, \psi) &= \epsilon(s, \phi_\pi \otimes \phi_\sigma, \psi).\end{aligned}$$

Here the functions on the RHS are the local factors of Artin type associated to the relevant representations of $W_k \times SL_2(\mathbb{C})$, whereas those on the LHS are the local factors attached by Shahidi to the generic representations of $GSp_4 \times GL_r$ and extended to all non-generic non-supercuspidal representations by using the Langlands classification and multiplicativity.

- (vi) *Suppose that π is a non-generic supercuspidal representation. For any irreducible supercuspidal representation σ of $GL_r(k)$ with L -parameter ϕ_σ , let $\mu(s, \pi \boxtimes \sigma)$ denote the Plancherel measure associated to the family of induced representations $I_P(\pi \boxtimes \sigma, s)$ on $GSpin_{2r+5}$, where we have regarded $\pi \boxtimes \sigma$ as a representation of the Levi subgroup $GSpin_5 \times GL_r$. Then $\mu(s, \pi \boxtimes \sigma)$ is equal to*

$$\gamma(s, \phi_\pi^\vee \otimes \phi_\sigma, \psi) \cdot \gamma(-s, \phi_\pi \otimes \phi_\sigma^\vee, \bar{\psi}) \cdot \gamma(2s, \text{Sym}^2 \phi_\sigma \otimes \text{sim} \phi_\pi^{-1}, \psi) \cdot \gamma(-2s, \text{Sym}^2 \phi_\sigma^\vee \otimes \text{sim} \phi_\pi, \bar{\psi}).$$

- (viii) *The map L is uniquely determined by the properties (i), (iii), (v) and (vi), with $r \leq 2$ in (v) and (vi).*

In [GTan], Gan and Tantano extend the local Langlands correspondence for GSp_4 to an analogous result for the inner form $GU_2(D)$. Let $\Pi(GU_2(D))$ denote the set of isomorphism classes of irreducible smooth representations of $GU_2(D)$. The set of L -parameters for $GU_2(D)$, $\Phi(GU_2(D))$, are a subset

$$\Phi(GU_2(D)) \subset \Phi(GSp_4).$$

Up to conjugacy, $GU_2(D)$ has a unique minimal parabolic k -subgroup P . The Levi factor of P is $M = D^\times \times GL_1$ and the unipotent radical of P is abelian. A parabolic subgroup is relevant for $GU_2(D)$ if it is (in the conjugacy class of) $P^\vee(\mathbb{C})$, the dual parabolic subgroup determined by P in the dual group $GSp_4(\mathbb{C}) = GU_2(D)^\vee$, and otherwise is irrelevant. An L-parameter ϕ in $\Phi(GSp_4)$ is said to be relevant for $GU_2(D)$ if it does not factor through any irrelevant parabolic subgroup. The set $\Phi(GU_2(D))$ is defined as the set of relevant L-parameters for $GU_2(D)$. Gan and Tantano define a natural surjective finite-to-one map

$$L : \Pi(GU_2(D)) \longrightarrow \Phi(GU_2(D))$$

that satisfies analogous properties to those given in the description of the local Langlands correspondence for GSp_4 . They show the map L is uniquely characterized by these properties.

We would like a natural way to parametrize the L -packets L_ϕ^{GT} for $GU_2(D)$. As expected in the local Langlands correspondence for inner forms, consider the modified component group

$$B_\phi = \pi_0(C_{Sp_4(\mathbb{C})}(Im(\phi))).$$

We have

$$Z_{Sp_4(\mathbb{C})} = \langle \pm 1 \rangle \rightarrow B_\phi \rightarrow A_\phi \rightarrow 0$$

so we have an injection of the group of irreducible characters

$$\text{Irr}(A_\phi) \hookrightarrow \text{Irr}(B_\phi).$$

This injection identifies $\text{Irr}(A_\phi)$ as the subgroup of characters of B_ϕ which are trivial on the image of the center $Z_{Sp_4(\mathbb{C})}$ of $Sp_4(\mathbb{C})$. A parameter ϕ is relevant for $GU_2(D)$ if and only if $\text{Irr}(B_\phi) \neq \text{Irr}(A_\phi)$. Then, among other properties, the surjection L satisfies [GTan, Main Theorem (ii)]:

- (ii) For an L-parameter ϕ , its fiber L_ϕ can be naturally parametrized by the set $\text{Irr}(B_\phi) \setminus \text{Irr}(A_\phi)$. This set has size either one or two.

5 Construction of representations

5.1 DeBacker-Reeder L -packets

Given a tame regular discrete Langlands parameter ϕ of an unramified p -adic group, by explicit construction, DeBacker and Reeder associate an L -packet of depth-zero supercuspidal representations. These representations are distributed among the pure inner forms of the p -adic group. In the following, for

$$G = GSp_4 \cong GSpin_5,$$

we extend their construction to associate to a TRD parameter ϕ an L -packet of depth-zero supercuspidal representations distributed between GSp_4 and its inner form $GU_2(D)$.

Here is how the representations constructed by DeBacker and Reeder are distributed among the pure inner forms of a group G . The pure inner forms of G are parametrized by classes in the Galois cohomology set $H^1(k, G)$. We have that

$$H^1(k, GSp_4) = 0.$$

So, in this situation, the L -packet associated to ϕ consists of representations living only on GSp_4 .

Members of the L -packet $\Pi(\phi)$ corresponding to ϕ are parametrized by the irreducible characters of the component group $\text{Irr}(A_\phi)$, where

$$A_\phi = \pi_0(C_{\widehat{G}}(\text{Im}(\phi))).$$

Given a Langlands parameter ϕ , the center $Z(^L G)$ of $^L G$ is contained in $C_{\widehat{G}}(\phi)$. Any $\rho \in \text{Irr}(A_\phi)$ determines a character on $\pi_0(Z(^L G))$. Then, via Kottwitz' isomorphism [Ko],

$$\pi_0(Z(^L G)) \simeq H^1(k, G),$$

this character determines a class $\omega_\rho \in H^1(k, G)$. The correspondence $\rho \rightarrow \omega_\rho$ is the method by which the representations in the L -packet $\Pi(\phi)$ are distributed among the different pure inner forms of G . So,

$$\Pi(\phi) = \coprod_{\omega \in H^1(k, G)} \Pi(\phi, \omega)$$

where

$$\Pi(\phi, \omega) = \{\pi(\phi, \rho) : \rho \in \text{Irr}(A_\phi), \omega_\rho = \omega\}$$

and $\pi(\phi, \rho)$ is a representation of G^ω .

For $G = GSp_4$, we extend DeBacker and Reeder's construction as follows. The inner forms of G are parametrized by classes in the Galois cohomology set $H^1(k, G_{ad})$. We have that

$$H^1(k, PGSp_4) = \{\pm 1\}.$$

The group GSp_4 has one inner form, namely $GU_2(D)$. Given a TRD parameter ϕ , we construct an L -packet associated to ϕ consisting of representations living on GSp_4 and $GU_2(D)$. Members of the L -packet $\Pi(\phi)$ corresponding to ϕ are parametrized by the irreducible characters of the component group $\text{Irr}(B_\phi)$, where

$$B_\phi = \pi_0(C_{\widehat{G_{ad}}}(\text{Im}(\phi))).$$

By restriction, any $\rho \in \text{Irr}(B_\phi)$ determines a character on $\pi_0(Z({}^L G_{ad}))$. Then, via Kottwitz' isomorphism, this character determines a class $\omega_\rho \in H^1(k, G_{ad})$. Using the correspondence $\rho \rightarrow \omega_\rho$ we distribute the representations in the L -packet $\Pi(\phi)$ between GSp_4 and $GU_2(D)$. Set

$$\Pi(\phi) = \coprod_{\omega \in H^1(k, G_{ad})} \Pi(\phi, \omega)$$

where

$$\Pi(\phi, \omega) = \{\pi(\phi, \rho) : \rho \in \text{Irr}(B_\phi), \omega_\rho = \omega\}.$$

We now construct the representations $\pi(\phi, \rho)$.

5.2 Affine Weyl groups and Bruhat-Tits theory

In this section let G be a quasi-split k -group such that G becomes split over the maximal unramified extension K of k . Let T be a maximally k -split torus of G such that

T splits over K . We have $k = K^{\text{Frob}}$. Let $F \in \text{Aut}(G(K))$ be the automorphism given by the action of Frob on $G(K)$. A reference for this section is [T].

Let $(X, \Phi, X^\vee, \Phi^\vee)$ be the root datum consisting of the character lattice X , cocharacter lattice $X^\vee$, roots Φ , and coroots $\Phi^\vee$ of $T(K)$. Let

$$\mathcal{A} = \mathbb{R} \otimes X^\vee.$$

For $u\varpi^n \in K^\times$, where $u \in \mathfrak{o}_K^\times, n \in \mathbb{Z}$,

$$\omega : K \rightarrow \mathbb{Z}, \quad \omega(u\varpi^n) = n, \quad \omega(0) = \infty$$

is a nontrivial discrete valuation on K . Fix a hyperspecial vertex, which we denote by 0 in $\mathcal{A}$. For $a \in \Phi$, this determines a K -isomorphism $\chi_a : \mathbb{G}_A \rightarrow U_a(K)$ from the additive group $\mathbb{G}_A$ to the root subgroup $U_a(K)$. This isomorphism χ_a is well-defined up to precomposing by multiplication by $\mathfrak{o}_K^\times$. For $a \in \Phi, l \in \mathbb{Z}$ set

$$X_{a+l} = \chi_a(\omega^{-1}[l, \infty]).$$

These groups form a filtration of the root subgroups U_a . Let

$$\Phi_{af} = \{\alpha = a + l \mid a \in \Phi, l \in \mathbb{Z}\}$$

be the set of affine functions on $\mathcal{A}$ given by

$$\alpha : x \mapsto a(x - 0) + l, \quad x \in \mathcal{A}.$$

There is a mapping of Φ_{af} onto the subgroups X_α of the root subgroups U_a given by $\alpha \mapsto X_\alpha$. Any $n \in N(K)$, the normalizer of $T(K)$ in $G(K)$, acts on $\mathcal{A}$ by an affine transformation such that

$$n^{-1}X_\alpha n = X_{\alpha \circ n}.$$

The affine space $\mathcal{A}$ endowed with the structure given above is called an apartment. Denote by H_α the affine hyperplane on which α vanishes. We have that $\mathcal{A}$ is a disjoint union of facets where two points $x, y \in \mathcal{A}$ are in the same facet if, for every $\alpha \in \Phi_{af}$ either x and y both lie on H_α or are both strictly on the same side of H_α . A chamber is an open facet.

If Δ is a set of simple roots for Φ , let $\alpha_0 = -\tilde{a} + 1$, where $\tilde{a}$ is the highest root. Then

$$\Pi = \{\alpha_0\} \cup \Delta$$

is a set of simple roots for the affine root system Φ_{af} . The simple affine roots Π determine a particular chamber C , called the fundamental chamber,

$$C = \{x \in \mathcal{A} | 1 > \langle \tilde{a}, x \rangle > 0, \langle a_i, x \rangle > 0, a_i \in \Delta\}.$$

where $\langle \cdot, \cdot \rangle$ is the canonical pairing between X and $X^\vee$. The vertices of the local Dynkin diagram of $G(K)$, indexed by the set of simple affine roots Π , are in correspondence with the vertices of the fundamental chamber C as follows. Let $x_i \longleftrightarrow \alpha_i$, where x_i is the unique vertex of C such that x_i is not in H_{α_i} , the affine hyperplane on which α_i vanishes.

For $\alpha = a + l \in \Phi_{af}$, let s_α be the linear transformation on $\mathcal{A}$ given by

$$s_\alpha(x) = x - \langle a, x \rangle a^\vee - la^\vee, \quad x \in \mathcal{A}.$$

So s_α is the reflection in the hyperplane H_α . Set

$$W' = \langle s_\alpha | \alpha \in \Phi_{af} \rangle$$

to be the group generated by the affine reflections. We call W' the affine Weyl group. Let Ω be the stabilizer of the fundamental chamber C . We have W' acts on $\mathcal{A}$ permuting the set of chambers simply transitively. Let

$$W = W' \rtimes \Omega.$$

As a group of linear transformations on $\mathcal{A}$, the Weyl group W_0 can be identified with the stabilizer of the point 0 in $\mathcal{A}$. The lattice $X^\vee$ acts on $\mathcal{A}$ by translations given by

$$T(m) \cdot x = x + m, \quad m \in X^\vee, x \in \mathcal{A}.$$

Let $R^\vee$ be the sublattice of $X^\vee$ generated by $\Phi^\vee$, then we have

$$W' = W_0 \rtimes R^\vee, \quad W = W_0 \rtimes X^\vee.$$

Let ${}^0T$ be the maximal bounded subgroup of T . Via evaluation at ϖ , we can identify $X^\vee$ with the normal subgroup $T/{}^0T \triangleleft N_G(T)/{}^0T$. We have

$$W \simeq N_G(T)/{}^0T.$$

As in [DR, §2], the inner forms of the k -group G are classified by $H^1(K/k, G_{ad})$. Denote by $Z^1(K/k, G_{ad})$ the set of 1-cocycles c of $\text{Gal}(K/k)$ in $G(K)$. A 1-cocycle is a continuous map $c : \text{Gal}(K/k) \rightarrow G(K)$ such that

$$c(\gamma\delta) = c(\gamma)(\gamma \cdot c(\delta)), \quad \gamma, \delta \in \text{Gal}(K/k).$$

Recall $\text{Gal}(K/k)$ is topologically generated by the geometric Frobenius element Frob whose action on $G(K)$ is denoted by F . Any cocycle in $Z^1(K/k, G_{ad})$ is determined by its value on Frob . We have $H^1(K/k, G_{ad})$ is the quotient of $Z^1(K/k, G_{ad})$ under the action

$$g * c(\text{Frob}) = gc(\text{Frob})F(g)^{-1}.$$

Let $c \in Z^1(F, G_{ad})$ be a cocycle in $[c] \in H^1(K/k, G_{ad})$. This determines a k -form $G(K)^{F_c}$, the fixed points of $G(K)$ under the action

$$F_c = \text{Ad}(c(\text{Frob})) \circ F \in \text{Aut} G.$$

For $g \in G$ we have

$$\text{Ad}(g) \circ F_c = F_{g*c} \circ \text{Ad}(g)$$

so $\text{Ad}(g)$ induces an isomorphism $G^{F_c} \simeq G^{F_{g*c}}$. Therefore the isomorphism class of G^{F_c} is determined by $[c] \in H^1(K/k, G_{ad})$. If we pick a representative u for the class $[c]$ such that

$$u(\text{Frob}) \in N_{G(K)}(T(K))$$

then the action of F_u preserves the apartment $\mathcal{A}_T$. We now show how an apartment for a maximally k -split torus S in G^{F_u} can be identified with a subset of the apartment $\mathcal{A}_T$.

For the non-twisted Frobenius F , the apartment $\mathcal{A}_S$ associated to $S = T(k) \subset G(k)$ can be identified (as an affine space) with a subset of the apartment $\mathcal{A}_T$ associated to $T(K) \subset G(K)$ as follows. The Galois group $\text{Gal}(K/k)$ acts on $\mathcal{A}_T$ and $\mathcal{A}_S$ is identified with the fixed point set of this action. Since $u(\text{Frob}) \in N_{G(K)}(T(K))$, u acts on $\mathcal{A}_T$ as given above. If F_u is a twisted Frobenius, $\mathcal{A}_S$ is the fixed point set of $\mathcal{A}_T$ under the action of $u(\text{Frob})$ composed with the action of Frob on $\mathcal{A}_T$. Since K/k is unramified,

$$\Phi_{af}(G, S, k)$$

consists of all nonconstant restrictions $\alpha|_{\mathcal{A}_S}$, with $\alpha \in \Phi_{af}(G, T, K)$. The action of $N(k)$ on $\mathcal{A}_S$ is inherited from the action of $N(K)$ on $\mathcal{A}_T$ and is described in [T, 1.10]. We have

$$W'_S = \langle s_\alpha \mid \alpha \in \Phi_{af}(G, S, k) \rangle$$

is the affine Weyl group of the affine root system $\Phi_{af}(G, S, k)$. Let Φ_S be the set of nonzero weights of S acting via the adjoint representation in the Lie algebra $\mathfrak{g}$ of G . If G is semisimple, W'_S is the affine Weyl group of a reduced root system Φ' whose elements

are proportional to Φ_S . However, the root system Φ' is not necessarily proportional to Φ_S . The vertices of the relative local Dynkin diagram for $G(k)$ are in correspondence with the orbits of the twisted $\text{Gal}(K/k)$ action on the vertices of the local Dynkin diagram for $G(K)$.

As a set, the Bruhat-Tits building

$$\mathcal{B}(G(K)) = \bigcup_{g \in G(K)} g \cdot \mathcal{A}$$

is the union of $G(K)$ translates of $\mathcal{A}$. The translates $g \cdot \mathcal{A}$ are not disjoint but are glued together in a certain way. The building $\mathcal{B}(G)$ is a simplicial complex with left $G(K)$ action. The same properties hold for the building $\mathcal{B}(G(k))$. Then we have the relation

$$\mathcal{B}(G(k)) = \mathcal{B}(G(K)^F) = (\mathcal{B}(G(K)))^F.$$

The same holds for any twisted Frobenius action F_u on $G(K)$, namely

$$\mathcal{B}(G(K)^{F_u}) = (\mathcal{B}(G(K)))^{F_u}.$$

We will mainly use Bruhat-Tits theory to identify certain compact open subgroups of $G(K)$. A facet $J \in \mathcal{B}(G)$ determines a certain compact open subgroup K_J of G , called a parahoric subgroup. If Ω is a subset of $\mathcal{B}(G)$, denote by G^Ω the group of all elements of G fixing Ω pointwise. We have that $K_J \subset G^J$ is a normal subgroup of G^J with finite quotient. K_J is profinite and fits into an exact sequence

$$1 \longrightarrow K_J^+ \longrightarrow K_J \longrightarrow \mathbf{G}_J \longrightarrow 1,$$

where K_J^+ is the pro-unipotent radical of K_J , and $\mathbf{G}_J = \mathbf{G}_J(\bar{\mathbf{f}})$ is the group of $\bar{\mathbf{f}}$ -points of a connected reductive group over the residue field $\bar{\mathbf{f}}$.

Recall the vertices of the local Dynkin diagram of $G(K)$, indexed by the set of simple affine roots Π , are in correspondence with the vertices of the fundamental chamber C . We say the parahoric subgroup of $G(K)$ corresponding to $\alpha_i \in \Pi$ is the parahoric subgroup of $G(K)$ corresponding to x_i . Let J be a facet in C . We can determine the root datum of the group $\mathbf{G}_J$ associated to J as follows. The reduction mod $\mathfrak{p}$ of T , denoted $\mathbf{T}$, is a maximal $\bar{\mathbf{f}}$ -split torus of the reduction mod $\mathfrak{p}$ of G^J . The character group of $\mathbf{T}$ is canonically isomorphic with the character group of T . We have that, $\mathbf{T} \subset \mathbf{G}_J$. The positive roots of $\mathbf{G}_J$ are

$$\Phi_J^+ = \{a \in \Phi^+ : \langle a, x \rangle \in \mathbb{Z} \text{ for all } x \in J\},$$

where Φ^+ is a set of positive roots for G given by a choice of simple roots Δ . The coroot associated with a root $a \in \Phi_J^+$ is the same for $\mathbf{G}_J$ as for G .

5.3 The DeBacker-Reeder construction

In this section we outline the process given in [DR] to construct an isomorphism class $[\pi(\phi, \rho)]$ of irreducible depth-zero supercuspidal representations associated to a TRD parameter ϕ and a character $\rho \in \text{Irr}(A_\phi)$.

DeBacker and Reeder consider quasi-split k -groups $\underline{G}$, such that $\underline{G}$ is K -split. As we will only need the split case and it will simplify notation, we will take $\underline{G}$ to be K -split. For notation we use the following conventions. Denote by $G = \underline{G}(K)$ the K -rational points of $\underline{G}$, and $G^F = \underline{G}(k)$ the k -rational points of $\underline{G}$. Also, identify an $\mathfrak{f}$ -group $\mathbf{G}$ with its group of $\bar{f}$ -rational points. Denote $\mathbf{G}^F = \mathbf{G}(\mathfrak{f})$. Let ϕ be a TRD parameter for G .

Let T be a maximal k -torus of G such that T is K -split. The dual group of T is the complex torus $\hat{T} = X \otimes \mathbb{C}^\times$ which is a maximal torus in $\hat{G}$ satisfying

$$X^*(\hat{T}) = X_*(T) = X^\vee, \quad X_*(\hat{T}) = X^*(T) = X.$$

As ϕ is regular, $\phi(\text{Frob}) \in N_{\hat{G}}(\hat{T})$. Denote by $\hat{w}$ the image of $\phi(\text{Frob})$ in $\hat{W}_0 = N_{\hat{G}}(\hat{T})/\hat{T}$. For any $\hat{\sigma} \in \text{Aut}(X)$, we define $\sigma \in \text{Aut}(X^\vee)$ by

$$\langle \eta, \sigma\lambda \rangle = \langle \hat{\sigma}\eta, \lambda \rangle, \quad \eta \in X, \lambda \in X^\vee.$$

Therefore, $\hat{w}$ induces a dual automorphism w of $X^\vee$.

Given $\lambda \in X^\vee$, let $t_\lambda \in W$ be the corresponding translation. As an automorphism of $X^\vee$, w is a linear transformation on $\mathcal{A}$, so define

$$\sigma_\lambda = t_\lambda w : x \mapsto \lambda + wx.$$

For $j : G \rightarrow G_{ad}$ the adjoint quotient, let

$$X_{ad}^\vee = X_*(j(T)), \quad \mathcal{A}_{ad} = X_{ad}^\vee \otimes \mathbb{R},$$

and let W_{ad} be the affine Weyl group of $j(T)$ in G_{ad} . We also write

$$j : X^\vee \rightarrow X_{ad}^\vee, \quad j : W \rightarrow W_{ad}$$

for the maps induced by j . Since ϕ is a discrete parameter,

$$(X^\vee)^w = X_*(Z^\circ), \quad (X_{ad}^\vee)^w = \{0\},$$

where Z° is the identity component of the center Z of G . The operator $I - w$ acts invertibly on $\mathcal{A}_{ad}$, so σ_λ has a unique fixed point there, namely

$$x_\lambda = (I - w)^{-1} t_{j\lambda} \cdot 0.$$

Denote by $\tilde{x}_\lambda$ the preimage of x_λ in $\mathcal{A}^{\sigma_\lambda}$. Let J_λ be the facet of $\mathcal{A}$ containing $\tilde{x}_\lambda$. Choose an alcove C_λ in $\mathcal{A}$ that contains J_λ in its closure. There is a unique element $w_\lambda \in W_\lambda$ such that $\sigma_\lambda \cdot C_\lambda = w_\lambda \cdot C_\lambda$, where W_λ is the subgroup of W generated by reflections in the hyperplanes containing J_λ . If we let $y_\lambda = w_\lambda^{-1} t_\lambda w$, then we have two expressions for σ_λ , namely

$$t_\lambda w = \sigma_\lambda = w_\lambda y_\lambda.$$

Let K_λ be the parahoric subgroup of G given by J_λ , and $\mathbf{G}_\lambda = K_\lambda / K_\lambda^+$.

We have that

$$C_{\hat{G}}(Im(\phi)) = \hat{T}^{\hat{w}}.$$

Recall $A_\phi = \pi_0(C_{\hat{G}}(Im(\phi)))$. Any $\lambda \in X^\vee$ determines a character $\rho_\lambda \in \text{Irr}(A_\phi)$ by the restriction map $X^\vee \rightarrow \text{Hom}(\hat{T}^{\hat{w}}, \mathbb{C}^\times)$ which induces an isomorphism [DR, 4.1]

$$[X^\vee / (1 - w)X^\vee]_{\text{tor}} \xrightarrow{\sim} \text{Irr}(A_\phi), \quad \lambda \rightarrow \rho_\lambda.$$

Let X_w be the preimage in $X^\vee$ of $[X^\vee / (1 - w)X^\vee]_{\text{tor}}$. For $\lambda \in X_w$, define $[\bar{\lambda}]$ as the image of $[\lambda]$ under the map

$$[X^\vee / (1 - w)X^\vee]_{\text{tor}} \rightarrow [X_*(Z(G))]_{\text{tor}} \simeq H^1(k, G), \quad (5.1)$$

where the first map is projection and the second map is Kottwitz' isomorphism [DR, 2.4.3]. By [DR, 2.7], there exists a lift $u_\lambda \in N_G(T)$ of y_λ such that

$$[u_\lambda] = [\bar{\lambda}] \in H^1(k, G).$$

DeBacker and Reeder define

$$F_\lambda = \text{Ad}(u_\lambda) \circ F.$$

This determines the inner form G^{F_λ} on which the representation they construct will live.

Choose a lift $\dot{w}$ of w to an F -stable element of $N_G(T) \cap K_0$ where 0 is the fixed hyperspecial vertex 0 in $\mathcal{A}$. Let

$$F_w = \text{Ad}(\dot{w}) \circ F.$$

By [DR, 2.3.1] there exists an element $p_\lambda \in G_\lambda$ such that if

$$T_\lambda := \text{Ad}(p_\lambda)T$$

then $\text{Ad}(p_\lambda)$ intertwines (T, F_w) with (T_λ, F_λ) .

In the next section we use the local Langlands correspondence for tori to obtain a character χ_ϕ of T^{F_w} . DeBacker and Reeder use this character to give an irreducible cuspidal representation of G_λ which they in turn use to construct an irreducible depth-zero supercuspidal representation $\pi_\lambda \in [\pi(\phi, \rho)]$. This process is described in [DR, §4.4]. We emulate this process to construct our representations π_λ as given in Section 5.5 below, and for this part our process is very similar to theirs.

5.4 Depth-zero characters

In [MP1], Moy and Prasad define the depth of a representation of a group G . An irreducible smooth representation (π, V) of G has depth-zero if there exists a parahoric subgroup K of G such that $V^{K^+} \neq 0$. Using data from our TRD parameter ϕ we construct a depth-zero character of T^{F_w} . We slightly modify ϕ to obtain a parameter ϕ' of T^{F_w} . The L -group of T^{F_w} is

$${}^L(T^{F_w}) = \langle \hat{w} \rangle \ltimes \hat{T}.$$

We have that the inclusion $\hat{T} \hookrightarrow \hat{G}$ induces a bijection $\hat{T}/(1 - \hat{w})\hat{T} \rightarrow \hat{G}/\hat{G}'$ where $\hat{G}'$ is the derived group of $\hat{G}$. Define $\phi' : W_k \rightarrow {}^L(T^{F_w})$ by

$$\phi'(\mathcal{I}) = \phi(\mathcal{I}), \quad \phi'(\text{Frob}) = \hat{w} \ltimes u \in \langle \hat{w} \rangle \ltimes \hat{T},$$

where $u \in \hat{T}$ is any element whose class in $\hat{T}/(1 - \hat{w})\hat{T}$ corresponds to the image of $\phi(\text{Frob})$ in $\hat{G}/\hat{G}'$.

As given in [DR, 4.3], by the Langlands correspondence for unramified tori, the parameter ϕ' determines a depth-zero character of T^{F_w} . As ϕ is discrete, $T^{F_w} = (X^\vee)^w \times {}^0T^{F_w}$. Let

$$\chi_\phi : T^{F_w} = [X^\vee]^w \times {}^0T^{F_w} \rightarrow \mathbb{C}^\times,$$

be written as

$$\chi_\phi = \chi_u \otimes \chi_s,$$

where χ_u is a character of $[X^\vee]^w$ and χ_s is a character of ${}^0T^{F_w}$ defined as follows. We have $\phi'(\text{Frob}) = \hat{w} \ltimes u$ as above. Identify $\hat{T}/(1-\hat{w})\hat{T}$ with the character group of $[X^\vee]^w$. Then u corresponds to

$$\chi_u \in \text{Hom}([X^\vee]^w, \mathbb{C}^\times), \quad \chi_u(\lambda) = \lambda(u).$$

Let

$$\langle s \rangle = \phi|_{\mathcal{I}_t} = \phi'|_{\mathcal{I}_t}.$$

If α, β are automorphisms of abelian groups A, B , let $\text{Hom}_{\alpha, \beta}(A, B)$ be the set of homomorphisms $f : A \rightarrow B$ such that $f \circ \alpha = \beta \circ f$. Let d be the order of w . The map $s \mapsto \chi_s$ is a canonical bijection

$$\text{Hom}_{\text{Ad Frob}, \hat{w}}(\mathcal{I}_t, \hat{T}) \simeq \text{Hom}_{\text{Frob}, \hat{w}}(\mathfrak{f}_d^\times, \hat{T}) \xrightarrow{\sim} \text{Hom}_{F_w, \text{Id}}(X^\vee \otimes \mathfrak{f}_d^\times, \mathbb{C}^\times) \simeq \text{Hom}(\mathbb{T}^{F_w}, \mathbb{C}^\times),$$

where the second isomorphism is given by

$$\chi_s(\lambda \otimes a) = \lambda(s(a)), \quad \lambda \in X^\vee, a \in \mathfrak{f}_d^\times.$$

Then, χ_s is viewed as a character of ${}^0T^{F_w}$ via inflation.

Using a character such as $\chi_s : \mathbb{T}^{F_w} \rightarrow \mathbb{C}^\times$, we can define a cuspidal representation of a reductive group over a finite field. Let $F : \mathbf{G} \rightarrow \mathbf{G}$ be a Frobenius map of a connected reductive group $\mathbf{G}$ over a finite field. Let $\mathbf{S}$ be an F stable maximal torus of $\mathbf{G}^F$ and χ a character of $\mathbf{S}^F$. By a cohomological construction, Deligne and Lusztig [DL] define the virtual character, called a Deligne-Lusztig character,

$$R_{\mathbf{S}, \chi}^{\mathbf{G}}.$$

They show, for all characters χ of all F stable maximal tori $\mathbf{S}$ of $\mathbf{G}$, the character of any irreducible representation of $\mathbf{G}^F$ appears among these virtual characters. If an (irreducible) character appears in $R_{\mathbf{S}, \chi}^{\mathbf{G}}$ for a F -minisotropic torus $\mathbf{S}$ and does not appear in any $R_{\mathbf{T}, \chi}^{\mathbf{G}}$ where the torus $\mathbf{T}$ is contained in a proper F -stable Levi subgroup of $\mathbf{G}$ then it is cuspidal. A character χ of $\mathbf{S}^F$ is in general position if no non-identity element of $(N_{\mathbf{G}}(\mathbf{S})/\mathbf{S})^F$ fixes χ . If χ is in general position then $R_{\mathbf{S}, \chi}^{\mathbf{G}}$ is irreducible. If $\mathbf{S}$ is a F -minisotropic torus of $\mathbf{G}$ and χ is a character of $\mathbf{S}^F$ in general position, then

$$\epsilon_{\mathbf{T}} \epsilon_{\mathbf{G}} R_{\mathbf{S}, \chi}^{\mathbf{G}}$$

is the character of an irreducible cuspidal representation of G^F . Note that T is F_w -minisotropic, and since ϕ is a regular parameter, χ_ϕ is in general position.

Deligne-Lusztig characters can be parametrized by pairs (S^*, t) where S^* is an F^* -stable maximal torus of the dual group G^* with dual Frobenius F^* , and $t \in S^{*F^*}$. We will also sometimes use the notation

$$R_{S^*}^G(t)$$

to denote $R_{S, \chi}^G$ if χ is the character of S^F given by t .

5.5 Computations

In this section let $G = GSpin_5$. Then $\hat{G} = GSp_4$, $G_{ad} = SO_5$, and $\widehat{G_{ad}} = Sp_4$. The root datum $(X, \Phi, X^\vee, \Phi^\vee)$ of $GSpin_5$ can be described as follows [AS, Prop 2.1]. We have that,

$$X = \mathbb{Z}e_0 \oplus \mathbb{Z}e_1 \oplus \mathbb{Z}e_2, \quad X^\vee = \mathbb{Z}e_0^* \oplus \mathbb{Z}e_1^* \oplus \mathbb{Z}e_2^*,$$

are the character and cocharacter lattices, respectively. The roots Φ and coroots $\Phi^\vee$ are generated, respectively by the simple roots and simple coroots

$$\Delta = \{a_1 = e_1 - e_2, a_2 = e_2\}, \quad \Delta^\vee = \{a_1^\vee = e_1^* - e_2^*, a_2^\vee = 2e_2^* - e_0^*\}.$$

Then

$$T(K) = \left\{ \prod_{j=0}^2 e_j^*(\lambda_j) : \lambda \in K^\times \right\}$$

is the set of K points of a maximal K -split torus T of G .

Let $T_{ad} = T/Z$ where Z is the center of G . Then $X^*(T_{ad}) \hookrightarrow X^*(T)$ is the submodule $\bigoplus a_i e_i$ such that $a_0 = 0$. This submodule contains the simple roots $e_1 - e_2$ and e_2 . Also,

$$X_*(T) \rightarrow X_*(T_{ad}) = \left(\bigoplus_i \mathbb{Z}e_i^* \right) / \mathbb{Z}e_0^*.$$

The simple coroots for T_{ad} are $e_1^* - e_2^*$ and $2e_2^*$, the image of the coroots for T . We see that

$$X_{ad} = \mathbb{Z}e_1 \oplus \mathbb{Z}e_2, \quad X_{ad}^\vee = \mathbb{Z}e_1^* \oplus \mathbb{Z}e_2^*, \quad \Delta = \{e_1 - e_2, e_2\}, \quad \Delta^\vee = \{e_1^* - e_2^*, 2e_2^*\}$$

gives a root datum for SO_5 . We have

$$\mathcal{A} = \mathbb{R}e_0^* \oplus \mathbb{R}e_1^* \oplus \mathbb{R}e_2^*$$

and

$$\mathcal{A}_{ad} = \mathbb{R}e_1^* \oplus \mathbb{R}e_2^*,$$

where the projection $j : \mathcal{A} \rightarrow \mathcal{A}_{ad}$ is given by

$$(ce_0^*, ae_1^*, be_2^*) \mapsto (ae_1^*, be_2^*).$$

The fundamental chamber C is given by the inequalities

$$1 - e_2 > e_1 > e_2 > 0.$$

Let ϕ be a tame regular discrete parameter for G . A description of such parameters was given in Section 3. Denote by $\hat{w}$ the image of $\phi(\text{Frob})$ in $\hat{W}_0 = N_{\hat{G}}(\hat{T})/\hat{T}$. If ϕ is irreducible, $\hat{w}$ is a Coxeter element. Precisely, if s_1 is the simple reflection corresponding to the simple root $e_1 - e_2$ for $GS p_4$ and s_2 is the simple reflection corresponding to the simple root $2e_2 - e_0$ for $GS p_4$, then

$$\hat{w} = s_1 s_2 \quad \text{or} \quad s_2 s_1.$$

If $\phi = \phi_1 \oplus \phi_2$ then

$$\hat{w} = s_1 s_2 s_1 s_2.$$

If ϕ is irreducible, or $\phi = \phi_1 \oplus \phi_2$ respectively, as automorphisms of X ,

$$\hat{w}e_0 = e_0 + e_2, \quad \hat{w}e_1 = -e_2, \quad \hat{w}e_2 = e_1,$$

or

$$\hat{w}e_0 = e_0 + e_1 + e_2, \quad \hat{w}e_1 = -e_1, \quad \hat{w}e_2 = -e_2.$$

We have $\widehat{T}_{ad} = \hat{T} \cap Sp_4$ is a maximal torus of Sp_4 . Since

$$\widehat{W}_0 = N_{GS p_4(\mathbb{C})}(\hat{T})/\hat{T} \cong N_{Sp_4(\mathbb{C})}(\hat{T}_0)/\hat{T}_0 = \widehat{W}_{0ad}$$

are canonically isomorphic, let $\hat{w}$ also denote the corresponding element in $N_{Sp_4(\mathbb{C})}(\hat{T}_0)/\hat{T}_0$.

We will now apply the DeBacker-Reeder construction as given above to the group SO_5 . The automorphism $w \in \text{Aut}(X_{ad}^\vee)$ is defined by the equations

$$\langle e_i, we_j^* \rangle = \langle \hat{w}e_i, e_j^* \rangle$$

where $i, j = 1, 2$. Then,

$$we_1^* = e_2^*, \quad we_2^* = -e_1^*,$$

or

$$we_1^* = -e_1^*, \quad we_2^* = -e_2^*$$

for ϕ irreducible or $\phi = \phi_1 \oplus \phi_2$, respectively. One can compute

$$[X^\vee/(1-w)X^\vee]_{\text{tor}} = \{\bar{0}, \bar{e}_1^* = \bar{e}_2^*\} \cong \text{Irr}(B_\phi), \quad \phi \text{ irreducible}$$

$$[X^\vee/(1-w')X^\vee]_{\text{tor}} = \{\bar{0}, \overline{e_1^* + e_2^*}, \bar{e}_1^*, \bar{e}_2^*\} \cong \text{Irr}(B_\phi), \quad \phi = \phi_1 \oplus \phi_2.$$

We then have the following

ϕ	ρ_λ	x_λ	$\Phi_{x_\lambda}^+$	$[u_\lambda]$
irred	$\bar{0}$	0	$\{e_1 - e_2, e_2, e_1, e_1 + e_2\}$	1
irred	$\bar{e}_1^*$	$1/2(e_1^* + e_2^*)$	$\{e_1 + e_2, e_1 + e_2\}$	-1
$\phi_1 \oplus \phi_2$	$\bar{0}$	0	$\{e_1 - e_2, e_2, e_1, e_1 + e_2\}$	1
$\phi_1 \oplus \phi_2$	$\overline{e_1^* + e_2^*}$	$1/2(e_1^* + e_2^*)$	$\{e_1 - e_2, e_1 + e_2\}$	1
$\phi_1 \oplus \phi_2$	$\bar{e}_1^*$	$1/2(e_1^*)$	$\{e_2\}$	-1
$\phi_1 \oplus \phi_2$	$\bar{e}_2^*$	$1/2(e_2^*)$	$\{e_1\}$	-1

The DeBacker-Reeder construction also gives us an element $p_\lambda \in SO_5(K)$ such that the map $T_{ad} \rightarrow \text{Ad}(p_\lambda)T_{ad}$ satisfies

$$F_\lambda \circ \text{Ad}(p_\lambda) = \text{Ad}(p_\lambda) \circ F_w.$$

Also, we have an element

$$\dot{w} \in N_{SO_5(k)}(T_{ad}(k)) \cap K_0,$$

where K_0 is the parahoric subgroup attached to the F -fixed hyperspecial vertex 0 of the fundamental chamber C of SO_5 .

We will use this data to construct representations of $GSpin_5(k)$ and its inner form $GSpin_{4,1}(k)$. Let X_w be the preimage in $X^\vee$ of $[X^\vee/(1-w)X^\vee]_{\text{tor}}$, and let $X_{w,ad}$ be the preimage in $X_{ad}^\vee$ of $[X_{ad}^\vee/(1-w)X_{ad}^\vee]_{\text{tor}}$. We have

$$X^\vee \longrightarrow X_{ad}^\vee \longrightarrow 0,$$

and for $\lambda \in X_{w,ad}$, there is a unique lift $\dot{\lambda}$ of λ to X_w . Let

$$\sigma_{\dot{\lambda}} = t_{\dot{\lambda}} w.$$

We also have $\mathcal{A} \longrightarrow \mathcal{A}_{ad} \longrightarrow 0$. Let $\tilde{x}_\lambda$ be the pre-image of x_λ in $\mathcal{A}^{\sigma_\lambda}$. Let J_λ be the unique facet containing $\tilde{x}_\lambda$. Let K_λ be the parahoric subgroup of $GSpin_5(K)$ determined by J_λ . The positive roots of

$$G_\lambda := K_\lambda / K_\lambda^+$$

are given by $\Phi_{x_\lambda}^+$. The class $[u_\lambda] \in H^1(k, G_{ad})$ determines the inner form of $GSpin_5$ containing K_λ .

In the following table note that

$${}^2GSpin_4 \cong GL_2(\mathbb{F}_{q^2})^\circ = \{g \in GL_2(\mathbb{F}_{q^2}) : \det(g) \in \mathbb{F}_q^\times\}, \quad {}^2GSpin_2 \cong (\mathbb{F}_{q^2}^\times)_{\{N=1\}}.$$

We have that

ϕ	ρ_λ	$G_\lambda^{F_\lambda}$	G^{F_λ}
irred	(1)	$GSpin_5(\mathfrak{f})$	$GSpin_5(k)$
irred	(-1)	${}^2GSpin_4(\mathfrak{f})$	$GSpin_{4,1}(k)$
$\phi_1 \oplus \phi_2$	(1, 1)	$GSpin_5(\mathfrak{f})$	$GSpin_5(k)$
$\phi_1 \oplus \phi_2$	(-1, -1)	$GSpin_4(\mathfrak{f})$	$GSpin_5(k)$
$\phi_1 \oplus \phi_2$	(-1, 1)	$[({}^2GSpin_2 \times GSpin_3)/\Delta GL_1](\mathfrak{f})$	$GSpin_{4,1}(k)$
$\phi_1 \oplus \phi_2$	(1, -1)	$[({}^2GSpin_2 \times GSpin_3)/\Delta GL_1](\mathfrak{f})$	$GSpin_{4,1}(k)$

We have the short exact sequence

$$1 \rightarrow GL_1 \rightarrow GSpin_5 \rightarrow SO_5 \rightarrow 1$$

which gives

$$1 \rightarrow GL_1(K) \rightarrow GSpin_5(K) \rightarrow SO_5(K) \rightarrow H^1(K, GL_1) = 0$$

so that

$$GSpin_5(K) \rightarrow SO_5(K) \rightarrow 1$$

is surjective. Take the element $p_\lambda \in SO_5(K)$ and pull it back to an element we will also denote $p_\lambda \in GSpin_5(K)$. We will use the notation $p_{\lambda,sc}$ if we need to distinguish the pullback from its image in $SO_5(K)$. Any two such choices of pullback p_λ differ by an element of the center of $GSpin_5(K)$. Denote also by $\dot{w}$ an element of K_0^F , where K_0 is the parahoric subgroup of $GSpin_5(K)$ attached to the hyperspecial vertex $0 \in \mathcal{A}$, that projects onto $\dot{w}$ as given above. Let

$$F_w = \text{Ad}(\dot{w}) \circ F.$$

The parameter ϕ determines a character χ_ϕ of T^{F_w} . Define

$$T_\lambda := \text{Ad}(p_\lambda)T.$$

If $j : G \rightarrow G_{ad}$ denotes the adjoint quotient, then $GSpin_5(K)$ acts on $\mathcal{B}(SO_5(K))$ via the map j . By [DR, 4.4.1], $p_\lambda \cdot x_\lambda = x_\lambda$. Therefore $p_{\lambda,sc} \cdot x_\lambda = x_\lambda$ so that

$${}^0T_\lambda \subset K_\lambda$$

where ${}^0T_\lambda$ is the maximal compact subgroup of T_λ .

We have that ${}^0T_\lambda = T_\lambda \cap G_\lambda$ and that via the reduction mod $\mathfrak{p}$ map, ${}^0T_\lambda$ projects onto $\mathbb{T}_\lambda$, an F_λ -minisotropic maximal torus in $\mathbb{G}_\lambda$. Conjugate the character χ_ϕ of T^{F_w} to get a character χ_λ of $T_\lambda^{F_\lambda}$:

$$\chi_\lambda = \chi_\phi \circ \text{Ad}(p_\lambda)^{-1}.$$

As ϕ is a tame parameter, χ_λ is a depth-zero character. Since ϕ is a regular parameter, χ_λ has trivial stabilizer in $[N_{G_\lambda}(T_\lambda)/T_\lambda]^{F_\lambda}$ so χ_λ is F_λ -regular. The restriction of χ_λ to ${}^0T_\lambda^{F_\lambda}$ factors through a character $\chi_\lambda^0 \in \text{Irr}(\mathbb{T}_\lambda^{F_\lambda})$ which is in general position. By Deligne-Lusztig induction we have an irreducible cuspidal representation

$$\kappa_\lambda^0 = \epsilon_{\mathbb{T}} \epsilon_G R_{\mathbb{T}_\lambda, \chi_\lambda^0}^{\mathbb{G}_\lambda}$$

of $\mathbb{G}_\lambda^{F_\lambda}$. Inflate κ_λ^0 to a representation of $K_\lambda^{F_\lambda}$ and define an extension to $Z^F \cdot K_\lambda^{F_\lambda}$ by

$$\kappa_\lambda = \chi_\lambda \otimes \kappa_\lambda^0$$

where Z is the center of G . This is well-defined as $(Z \cap K_\lambda)^{F_\lambda}$ acts on κ_λ by the restriction of the scalar character χ_λ^0 . Define

$$\pi_\lambda = c - \text{Ind}_{Z^F K_\lambda^{F_\lambda}}^{G^{F_\lambda}} (\chi_\lambda \otimes \kappa_\lambda^0).$$

Lemma 5.1. *The representation π_λ of G^{F_λ} is irreducible supercuspidal.*

Proof. This is [DR, 4.5.1]. By [MP2, 6.6] we only need to show that κ_λ remains irreducible when induced to the group

$$(G_\lambda^*)^{F_\lambda} = \{g \in G^{F_\lambda} : g \cdot J_\lambda = J_\lambda\},$$

where $(G_\lambda^*)^{F_\lambda}$ is the normalizer of $K_\lambda^{F_\lambda}$ in G^{F_λ} . As in [DR, 4.5.1], by [DL, Thm 6.8] and [D], if $h \in (G_\lambda^*)^{F_\lambda}$ is such that $\text{Ad}(h)_* \kappa_\lambda = \kappa_\lambda$, then there exists $l \in G_\lambda^{F_\lambda}$ such that $lh \in N_G(T_\lambda)^{F_\lambda}$ and fixes χ_λ . Then $p_\lambda^{-1}lh p_\lambda \in N_G(T)^{F_w}$ and fixes χ_ϕ . By [DR, 4.3.1], since ϕ is a regular parameter, the projection of $p_\lambda^{-1}lh p_\lambda$ to $N_G(T)/T$ is the identity. Therefore

$$lh \in T_\lambda^{F_\lambda} \cap (G_\lambda^*)^{F_\lambda} \subset Z^{F_\lambda} G_\lambda^{F_\lambda},$$

and since $l \in G_\lambda^{F_\lambda}$, we have $h \in Z^{F_\lambda} G_\lambda^{F_\lambda}$. Therefore by [MP2, 6.6], π_λ is irreducible and supercuspidal. $\square$

5.6 The L -packets L_ϕ^{DR}

Lemma 5.2. *The $GSpin_5(K)$ -orbit $[u_\lambda, \pi_\lambda] = \text{Ad}(GSpin_5(K)) \cdot (u_\lambda, \pi_\lambda)$ depends only on the character $\rho_\lambda \in \text{Irr}(B_\phi)$.*

Proof. Let $X_{w,ad}$ denote the preimage in $X_{ad}^\vee$ of $[X_{ad}^\vee/(1-w)X_{ad}^\vee]_{\text{tor}} \simeq \text{Irr}(B_\phi)$. Given a TRD paramter ϕ and a $\lambda \in X_{w,ad}$ using the DeBacker Reeder construction applied to SO_5 choices of $C_\lambda, u_\lambda, p_\lambda, \dot{w}$ were made. In defining the representation π_λ we made choices of lifts $\dot{w}_{sc}$ and $p_{\lambda,sc}$ in $GSpin_5(K)$. Given a TRD parameter ϕ , for $\lambda, \mu \in X_{w,ad}$, make choices

$$(C_\lambda, u_\lambda, p_{\lambda,sc}, \dot{w}_{\lambda,sc}), \quad (C_\mu, u_\mu, p_{\mu,sc}, \dot{w}_{\mu,sc})$$

respectively. We will show that $\rho_\lambda = \rho_\mu$ if and only if there exists a $g \in GSpin_5(K)$ such that

$$g * u_\lambda = u_\mu, \quad g \cdot J_\lambda = J_\mu, \quad \text{Ad}(g)_* \kappa_\lambda \simeq \kappa_\mu.$$

This is what we mean by the statement the $GSpin_5(K)$ -orbit $[u_\lambda, \pi_\lambda]$ depends only on ρ_λ . By [MP2, 6.2] those three conditions are equivalent to having a $g \in G$ such that $\text{Ad}(g) \cdot (u_\lambda, \pi_\lambda) = (u_\mu, \pi_\mu)$.

By [DR, 4.5.2] there exists a $g_{ad} \in SO_5(K)$ such that

$$g_{ad} * u_\lambda = u_\mu, \quad g_{ad} \cdot x_\lambda = x_\mu, \quad \text{Ad}(g_{ad})(T_{ad})_\lambda = \text{Ad}(s_{ad})(T_{ad})_\mu,$$

where $s_{ad} \in (T_{ad})_\mu$. As $GSpin_5(K) \rightarrow SO_5(K)$ is surjective, choose a lift g of g_{ad} to $GSpin_5(K)$. Then

$$g * u_\lambda = g_{ad} * u_\lambda = u_\mu.$$

An element $g \in GSpin_5(K)$ acts on $\mathcal{B}(SO_5(K))$ via the adjoint quotient $j : GSpin_5(K) \rightarrow SO_5(K)$, so

$$g \cdot x_\lambda = g_{ad} \cdot x_\lambda = x_\mu.$$

Therefore $g \cdot J_\lambda = J_\mu$. Also choose a lift s of s_{ad} to $GSpin_5(K)$. Then,

$$\text{Ad}(g)T_\lambda = \text{Ad}(s)T_\mu, \quad \chi_\lambda \circ \text{Ad}(g)^{-1} = \chi_\mu \circ \text{Ad}(s)^{-1},$$

where $s \in T_\mu$. This shows $\text{Ad}(g)_* \kappa_\lambda \simeq \kappa_\mu$. $\square$

Lemma 5.3. *Given a TRD parameter ϕ for $GSpin_5$ and $\rho_\lambda \in \text{Irr}(B_\phi)$, let $[u_\lambda, \pi_\lambda]$ be the associated $GSpin_5(K)$ -orbit of representations. For any $\rho_\lambda \in \text{Irr}(A_\phi) \subset \text{Irr}(B_\phi)$, let $[u_\lambda, \pi_\lambda]_{DR}$ be the $GSpin_5(K)$ -orbit of representations associated to ϕ and ρ_λ by DeBacker and Reeder. Then,*

$$[u_\lambda, \pi_\lambda] = [u_\lambda, \pi_\lambda]_{DR}.$$

Proof. Let ϕ be a TRD parameter for $GSpin_5$ and let $\rho_\lambda \in \text{Irr}(A_\phi)$. We will show that for each step in the DeBacker and Reeder construction of a representation π_λ associated to ρ_λ , we can make concurrent choices in our construction so that

$$\pi_\lambda \in [u_\lambda, \pi_\lambda]_{DR} \iff \pi_\lambda \in [u_\lambda, \pi_\lambda].$$

- (i) Let X_w be the preimage in $X^\vee$ of $[X^\vee/(1-w)X^\vee]_{\text{tor}}$, and let $X_{w,ad}$ be the preimage in $X_{ad}^\vee$ of $[X_{ad}^\vee/(1-w)X_{ad}^\vee]_{\text{tor}}$. If $\lambda \in X_w$ gives the character $\rho_\lambda \in \text{Irr}(A_\phi)$, let $\bar{\lambda}$ denote the image of λ under the projection

$$X^\vee \rightarrow X_{ad}^\vee \rightarrow 0.$$

Then $\bar{\lambda} \in X_{ad}^\vee$ also gives the character $\rho_\lambda \in \text{Irr}(A_\phi) \subset \text{Irr}(B_\phi)$. If $\lambda \in X_{w,ad}$ gives the character $\rho_\lambda \in \text{Irr}(A_\phi) \subset \text{Irr}(B_\phi)$, there is a unique lift of λ to X_w , which we denote by $\dot{\lambda}$. We have that $\dot{\lambda}$ gives the character $\rho_\lambda \in \text{Irr}(A_\phi)$.

- (ii) We have

$$x_{\dot{\lambda}} = x_{\bar{\lambda}}$$

where $x_{\dot{\lambda}} \in \mathcal{A}_{ad}$ is the unique fixed point of $\sigma_{\dot{\lambda}} = t_{\dot{\lambda}}w \in W$ acting via the adjoint quotient j on $\mathcal{A}_{ad}$, and $x_{\bar{\lambda}} \in \mathcal{A}_{ad}$ is the unique fixed point under the action $\sigma_{\bar{\lambda}} =$

$t_{\bar{\lambda}}w \in W_{ad}$ on $\mathcal{A}_{ad}$. Under both constructions we have the same lift $\tilde{x}_\lambda$ in the same fundamental chamber $C_\lambda \subset \mathcal{A}$ and hence the same parahoric subgroup $K_\lambda \subset G$.

- (iii) We have $y_{\check{\lambda}} = w_{\check{\lambda}}^{-1}t_{\check{\lambda}}w \in W$ and $y_{\bar{\lambda}} = w_{\bar{\lambda}}^{-1}t_{\bar{\lambda}}w \in W_{ad}$ where under the map induced by the adjoint quotient $j : W \rightarrow W_{ad}$, we have

$$j(y_{\check{\lambda}}) = y_{\bar{\lambda}}.$$

From [DR, 2.7.2], there exists a lift $u_{\check{\lambda}}$ of $y_{\check{\lambda}}$ to $N_{GSpin_5(K)}(T)$ and a lift $u_{\bar{\lambda}}$ of $y_{\bar{\lambda}}$ to $N_{SO_5(K)}(T)$. For a general such u_λ , we have that u_λ is constructed using an element $x \in W'$ such that

$$C_\lambda = x \cdot C.$$

As W' permutes the set of chambers simply transitively, there is a unique such x . As

$$j(C_{\check{\lambda}}) = C_{\bar{\lambda}},$$

we have that such an x is the same element in both cases under the canonical isomorphism $W' = W_0 \ltimes R^\vee \cong W_{0ad} \ltimes R_{ad}^\vee = W'_{ad}$. Here $R^\vee \subset X^\vee$ is the sublattice generated by the coroots $\Phi^\vee$, and $R_{ad}^\vee \subset X_{ad}^\vee$ is the sublattice generated by the coroots $\Phi^\vee \subset X_{ad}^\vee$. There is a choice made in [DR, 2.7.2]. If we choose a lift $\dot{x}$ of x to $N_{GSpin_5(K)}(T)$ and a lift $\dot{x}_{ad}$ of x to $N_{SO_5(K)}(T)$ such that $j(\dot{x}) = \dot{x}_{ad}$, then

$$j(u_{\check{\lambda}}) = u_{\bar{\lambda}}.$$

Therefore any lift of $u_{\bar{\lambda}}$ to $GSpin_5(K)$ will give the same conjugation action on $GSpin_5(K)$ as given by $u_{\check{\lambda}}$.

- (iv) As for x in (iii), we can view that elements $w_{\check{\lambda}} \in W'$ and $w_{\bar{\lambda}} \in W'_{ad}$ as the same element under the isomorphism given above. Make a choices of lifts such that $j(\dot{w}_{\check{\lambda}}) = \dot{w}_{\bar{\lambda}}$. As in [DR, pg.19], there exists $p_{\check{\lambda}} \in GSpin_5(K)$, $p_{\bar{\lambda}} \in SO_5(K)$, such that

$$p_{\check{\lambda}}^{-1}u_{\check{\lambda}}F(p_{\check{\lambda}})u_{\check{\lambda}}^{-1} = \dot{w}_{\check{\lambda}}, \quad p_{\bar{\lambda}}^{-1}u_{\bar{\lambda}}F(p_{\bar{\lambda}})u_{\bar{\lambda}}^{-1} = \dot{w}_{\bar{\lambda}}.$$

Applying the map j to the first equation we see

$$j(p_{\check{\lambda}}^{-1})u_{\bar{\lambda}}F(j(p_{\check{\lambda}}))u_{\bar{\lambda}}^{-1} = \dot{w}_{\bar{\lambda}}.$$

Then for an $h = j(p_{\check{\lambda}})p_{\bar{\lambda}}^{-1} \in GSO_5(K)$ we have

$$h * u_{\bar{\lambda}} = u_{\bar{\lambda}}, \quad \text{Ad}(h)_*(T_{\bar{\lambda}}, \chi_{\bar{\lambda}}, \kappa_{\bar{\lambda}}) = (T_{j(\check{\lambda})}, \chi_{j(\check{\lambda})}, \kappa_{j(\check{\lambda})}),$$

so that using the two sets of data gives rise to two representations in the same $SO_5(K)$ orbit. There exists an lift of $\dot{p}_{\bar{\lambda}}$ to $GSpin_5(K)$ such that

$$\dot{p}_{\bar{\lambda}}^{-1} u_{\dot{\lambda}} F(\dot{p}_{\bar{\lambda}}) u_{\dot{\lambda}}^{-1} = \dot{w}_{\dot{\lambda}}.$$

Then for $h = \dot{p}_{\bar{\lambda}} \dot{p}_{\dot{\lambda}}^{-1} \in GSpin_5(K)$ we have

$$h * u_{\dot{\lambda}} = u_{\dot{\lambda}}, \quad \text{Ad}(h)_*(T_{\dot{\lambda}}, \chi_{\dot{\lambda}}, \kappa_{\dot{\lambda}}) = (T_{\dot{\lambda}}, \chi_{\dot{\lambda}}, \kappa_{\dot{\lambda}}),$$

so that using the two sets of data gives rise to two representations in the same $GSpin_5(K)$ orbit.

□

We now define the L -packet of depth-zero supercuspidal representations we attach to a TRD ϕ . This definition is an extension of [DR, 4.5.3].

Definition 5.4. Let ϕ be a tame regular discrete parameter for $GSpin_5(k)$. Let

$$r : X_{w,ad} \longrightarrow H^1(k, SO_5), \quad \lambda \mapsto [\bar{\lambda}]$$

be as in (5.1) in Section 5.3. For each $\omega \in H^1(k, SO_5)$ define

$$\Pi(\phi, \omega) := \{[u_{\lambda}, \pi_{\lambda}] : [\bar{\lambda}] = \omega \in H^1(k, SO_5)\}.$$

Remark. This is well-defined by Lemma 5.2.

Let $L_{\phi}^{DR} = \Pi(\phi, 1)$ or $\Pi(\phi, -1)$ denote the L -packet attached to $GSpin_5(k)$ or $GSpin_{4,1}(k)$, respectively.

6 The Bernstein component of

$$I(s, \pi \boxtimes \sigma)$$

6.1 DeBacker-Reeder for GL_{2m}

Let $\phi = \phi_1 \oplus \cdots \oplus \phi_r$, $r = 1, 2$, be a tame regular discrete Langlands parameter for $GSpin_5$. Then, for $1 \leq i \leq r$,

$$\phi_i : W_k \longrightarrow GSp_{2m}(\mathbb{C}) \hookrightarrow GL_{2m}(\mathbb{C}) = GL_{2m}(k)^\vee$$

is also Langlands parameter for $GL_{2m}(k)$. By the local Langlands correspondence for GL_{2m} , let

$$\phi_i \longleftrightarrow L_{\phi_i} = \{\sigma_{\phi_i}\},$$

where the L-packets for GL_{2m} are always of size one.

By [Hel], the representation σ_{ϕ_i} is the irreducible depth-zero supercuspidal representation attached to ϕ_i by DeBacker and Reeder. We now describe these representations. Let T be the maximal split torus in $GL_{2m}(k)$, and $\hat{T} \subset GL_{2m}^\vee$ the dual torus. We have $\hat{w}$, the image of $\phi(\text{Frob})$ in $N_{GL_{2m}^\vee}(\hat{T})/\hat{T}$, is a Coxeter element. Namely, if $m = 1$ then

$$\hat{w} = s_1$$

where s_1 is the reflection corresponding to the simple root $e_1 - e_2$. If $m = 2$,

$$\hat{w} \text{ is conjugate to } s_1 s_2 s_3$$

where s_1 is the reflection corresponding to the simple root $e_1 - e_2$, s_2 is the reflection corresponding to the simple root $e_2 - e_3$, and s_3 is the reflection corresponding to the simple root $e_3 - e_4$. Let $F_w = \text{Ad}(\hat{w}) \circ F$, where $\hat{w}$ is as in Section 5.3. Let χ_ϕ be the character of T^{F_w} given in Section 5.4. Then the restriction of the character χ_ϕ to ${}^0T^{F_w}$

is given by the element s where $\langle s \rangle = \phi(\mathcal{I}_t) \subset GL_{2m}(\mathbb{C})$. Let $\Omega_{p'}$ denote the set of all complex roots of unity of order prime to p . As in [Ca, Prop 3.1.3], we fix an identification of $\bar{\mathfrak{f}}^\times$ and $\Omega_{p'}$. Then viewed as an element of $\bar{\mathfrak{f}}^\times$,

$$s = \text{diag}(\tau, \tau^q, \dots, \tau^{q^{2m-1}}), \quad \tau \in \mathbb{F}_{q^{2m}}^\times \setminus \mathbb{F}_{q^m}^\times.$$

Any maximal compact subgroup of $GL(V)$, where $V = ke_1 \oplus \dots \oplus ke_{2m}$, is conjugate to

$$GL_{2m}(\mathfrak{o}),$$

the stabilizer of the lattice

$$\mathfrak{o}e_1 \oplus \dots \oplus \mathfrak{o}e_{2m} \subset V.$$

We have

$$1 \longrightarrow GL_{2m}(\mathfrak{o})^+ \longrightarrow GL_{2m}(\mathfrak{o}) \longrightarrow GL_{2m}(\mathfrak{f}) \longrightarrow 1,$$

where $GL_{2m}(\mathfrak{o})^+$ is the pro-unipotent radical of the profinite group $GL_{2m}(\mathfrak{o})$. For any $\lambda \in X_w$, where X_w is the inverse image in $X_*(T)$ of $[X_*(T)/(1-w)X_*(T)]_{\text{tor}}$, there exists $p_\lambda \in GL_{2m}(K)$ that intertwines (T_λ, F_λ) and (T, F_w) , where $T_\lambda = \text{Ad}(p_\lambda)T$. Let

$$\chi_\lambda = \chi_\phi \circ \text{Ad}(p_\lambda)^{-1}.$$

To λ we also associate $F_\lambda = \text{Ad}(u_\lambda) \circ F$, such that

$${}^0T^{F_\lambda} \subset K_\lambda^{F_\lambda} \subset GL_{2m}(K)^{F_\lambda}.$$

Any such $K_\lambda^{F_\lambda}$ is conjugate to $GL_{2m}(\mathfrak{o})$. We can take $u_\lambda = I, p_\lambda = \dot{w}$. Then

$$\sigma_{\phi_i} = c - \text{Ind}_{k^\times \cdot GL_{2m}(\mathfrak{o})}^{GL_{2m}(k)} \chi_\lambda \otimes \pm R_{\mathbf{T}^*}^{GL_{2m}(\mathfrak{f})}(t),$$

where t is conjugate to s as given above.

6.2 The generalized principal series $I(s, \pi \boxtimes \sigma)$

From now on, let $G = GSpin_{4m+5}(k)$ or $GSpin_{2m+4, 2m+1}(k)$. In the following, we will always assume $m = 1$ or 2 . For simplicity of notation, let $n = 2m + 2$.

If

$$G = GSpin_{4m+5}(k),$$

let $P = M \cdot N$ be the maximal parabolic subgroup of G with Levi factor

$$M = GSpin_5(k) \times GL_{2m}(k) \cong GSp_4(k) \times GL_{2m}(k).$$

If

$$G = GSpin_{2m+4, 2m+1}(k),$$

let $P = M \cdot N$ be the maximal parabolic subgroup of G with Levi factor

$$M = GSpin_{4,1}(k) \times GL_{2m}(k) \cong GU_2(D) \times GL_{2m}(k).$$

Note that in each case P corresponds to the simple root $a_{2m} = e_{2m} - e_{2m+1}$. If π is an irreducible representation of $GSpin_5(k)$ or $GSpin_{4,1}(k)$, and σ a representation of $GL_{2m}(k)$ we can form the generalized principal series representation

$$I(s, \pi \boxtimes \sigma) = \text{Ind}_P^{G\delta_P^{1/2}} \pi \boxtimes \sigma |\det|^s.$$

From now on let $I(s, \pi \boxtimes \sigma)$ be the generalized principal series where given a TRD parameter ϕ , $\pi \in L_\phi^{DR}$ where L_ϕ^{DR} is the L-packet of representations of $GSpin_5(k)$ or $GSpin_{4,1}(k)$ as given in Definition 5.4, and $\sigma = \sigma_{\phi_i}$ as given in Section 6.1. These representations are of the form

$$c - \text{Ind}_{ZK_\lambda}^{G(k)} \chi_\lambda \otimes \pm R_{\mathbb{T}_\lambda^*}^{G(f)}(t)$$

where $G(k)$ and $G(f)$ are listed in the tables below. The character χ_λ of $\mathbb{T}_\lambda^{F_\lambda}$ is given by an element t in the F_λ^* -stable dual torus $\mathbb{T}_\lambda^{*F_\lambda}$. We also list the element s in the F_w^* -stable torus $\mathbb{T}^{*F_w}$ such that t is conjugate to s . We have

$$\tau \in \mathbb{F}_{q^4}^\times \setminus \mathbb{F}_{q^2}^\times,$$

and

$$\tau_1, \tau_2 \in \mathbb{F}_{q^2}^\times \setminus \mathbb{F}_q^\times, \quad N_{\mathbb{F}_{q^2}^\times/\mathbb{F}_q^\times}(\tau_1) = N_{\mathbb{F}_{q^2}^\times/\mathbb{F}_q^\times}(\tau_2) \quad \text{and} \quad \tau_1 \neq \tau_2, \tau_2^q.$$

If ϕ is irreducible these representations are given by

$\pm R_{\mathbb{T}_\lambda^*}(t)$	s	$G(f)$	$G(k)$
R_π^+	$(\tau, \tau^q, \tau^{q^3}, \tau^{q^2})$	$GSpin_5(f)$	$GSpin_5(k)$
$R_\pi^\dagger$	$(\tau, \tau^q, \tau^{q^3}, \tau^{q^2})$	${}^2GSpin_4(f)$	$GSpin_{4,1}(k)$
$R_\sigma^{+\text{or}\dagger}$	$(\tau, \tau^q, \tau^{q^2}, \tau^{q^3})$	$GL_4(f)$	$GL_4(k)$

If $\phi = \phi_1 \oplus \phi_2$ these representations are given by

$\pm R_{\tau_\lambda^*}(t)$	s	$\mathbb{G}(\mathfrak{f})$	$G(k)$
R_π^+	$(\tau_1, \tau_2, \tau_2^q, \tau_1^q)$	$GSpin_5(\mathfrak{f})$	$GSpin_5(k)$
R_π^-	$(\tau_1, \tau_2, \tau_2^q, \tau_1^q)$	$GSpin_4(\mathfrak{f})$	$GSpin_5(k)$
$R_\pi^\dagger$	$(\tau_1, \tau_2, \tau_2^q, \tau_1^q)$	$[(^2GSpin_2 \times GSpin_3)/\Delta GL_1](\mathfrak{f})$	$GSpin_{4,1}(k)$
$R_\pi^\ddagger$	$(\tau_2, \tau_1, \tau_1^q, \tau_2^q)$	$[(^2GSpin_2 \times GSpin_3)/\Delta GL_1](\mathfrak{f})$	$GSpin_{4,1}(k)$
$R_\sigma^{+, -, \text{or} \dagger}$	(τ_1, τ_1^q)	$GL_2(\mathfrak{f})$	$GL_2(k)$
$R_\sigma^{+, -, \text{or} \ddagger}$	(τ_2, τ_2^q)	$GL_2(\mathfrak{f})$	$GL_2(k)$

6.3 Definition of $\mathcal{P}$

The vertices of the local Dynkin diagram for $G = GSpin_{4m+5}$ are in correspondence with the set

$$\Pi^+ = \{\alpha_{0+} = -e_1 - e_2 + 1, \alpha_{1+} = e_1 - e_2, \alpha_{2+} = e_2 - e_3, \dots, \alpha_{n-1+} = e_{n-1} - e_n, \alpha_{n+} = e_n\}.$$

By Bruhat-Tits theory, let $\mathcal{K}_{i+}$ be the parahoric subgroup of $GSpin_{4m+5}(k)$ corresponding to the root α_{i+} in the local Dynkin diagram. Let

$$\mathcal{P}^+ = \mathcal{K}_{0+} \cap \mathcal{K}_{2m+}.$$

Another choice of simple roots for $GSpin_9$ is

$$\Delta^- = \{e_3 - e_4, e_4 - e_1, e_1 - e_2, e_2\}.$$

Let $\alpha_{0-} = -e_3 - e_4 + 1$ where $-e_3 - e_4$ is the lowest root. Let $\mathcal{K}_{i-}$ be the parahoric subgroup of $GSpin_9(k)$ corresponding to the root α_{i-} in the local Dynkin diagram corresponding to the set

$$\Pi^- = \{\alpha_{0-} = -e_3 - e_4 + 1, \alpha_{1-} = e_3 - e_4, \alpha_{2-} = e_4 - e_1, \alpha_{3-} = e_1 - e_2, \alpha_{4-} = e_2\}.$$

Let

$$\mathcal{P}^- = \mathcal{K}_{2-} \cap \mathcal{K}_{4-}.$$

The vertices of the relative local Dynkin diagram for $G = GSpin_{2m+4, 2m+1}$ are in correspondence with the set

$$\{\alpha_0 = 1 - e_1, \alpha_1 = e_1 - e_2, \alpha_2 = e_2 - e_3, \dots, \alpha_{2m} = e_{2m} - e_{2m+1}, \alpha_{2m+1} = e_{2m+1}\}.$$

Note that this is a set of simple affine roots for an affine Weyl group of type C_{2m+1} . Another choice of simple roots for S , where S is the maximal k -split torus of $GSpin_{8,5}$ gives us another set of simple affine roots

$$\Pi^\dagger = \{\alpha_{0^\dagger} = 1 - e_5, \alpha_{1^\dagger} = e_5 - e_1, \alpha_{2^\dagger} = e_1 - e_2, \alpha_{3^\dagger} = e_2 - e_3, \alpha_{4^\dagger} = e_3 - e_4, \alpha_{5^\dagger} = e_4\}.$$

Let

$$\mathcal{P}^\dagger = \mathcal{K}_{1^\dagger} \cap \mathcal{K}_{5^\dagger}.$$

The vertices of the local Dynkin diagram for $GSpin_{6,3}$ are in correspondence with the set

$$\Pi^\ddagger = \{\alpha_{0^\ddagger} = 1 - e_1, \alpha_{1^\ddagger} = e_1 - e_2, \alpha_{2^\ddagger} = e_2 - e_3, \alpha_{3^\ddagger} = e_3\}.$$

Let

$$\mathcal{P}^\ddagger = \mathcal{K}_{0^\ddagger} \cap \mathcal{K}_{2^\ddagger}.$$

Denote $\mathcal{P} = \mathcal{P}^+, \mathcal{P}^-, \mathcal{P}^\dagger$, or $\mathcal{P}^\ddagger$ depending on the context, so

$$\mathcal{Q} = \mathcal{P} \cap M = K_\lambda \times GL_{2m}(\mathfrak{o}).$$

Let $M = G_\lambda \times GL_{2m}(\mathfrak{f})$ be the reductive component of the reduction mod $\mathfrak{p}$ of $\mathcal{P}$ which is equal to the reductive component of the reduction mod $\mathfrak{p}$ of $\mathcal{Q}$. If $\mathcal{P} = \mathcal{P}^+$, denote by

$$\rho = R_\pi^+ \boxtimes R_\sigma^+,$$

a representation of M , where the R^+ are given in the tables in the previous section. For $\mathcal{P} = \mathcal{P}^-, \mathcal{P}^\dagger$, or $\mathcal{P}^\ddagger$ make the analogous definition for a representation ρ of M . For notation, we will sometimes write $\rho = R_\pi \boxtimes R_\sigma$. We can view ρ as a representation of $\mathcal{P}$ via inflation. We can also view ρ as a representation of $\mathcal{Q}$ via inflation. We denote this representation of $\mathcal{Q}$ by ρ_M .

We will show that representations in the Bernstein component of $I(s, \pi \boxtimes \sigma)$ are parametrized by modules of the Hecke algebra $\mathcal{H}(G//\mathcal{P}, \rho)$.

6.4 The Hecke Algebra $\mathcal{H}(G//\mathcal{P}, \rho)$

Let $(\check{\rho}, W^\vee)$ denote the contragredient representation of the representation (ρ, W) of $\mathcal{P}$. The following definitions apply for any compact open subgroup of G . Let

$$\mathcal{H}(G//\mathcal{P}, \rho) = \{f \in \mathcal{C}_c^\infty(G, \text{End}_{\mathbb{C}}(W^\vee)) \mid f(k_1 g k_2) = \check{\rho}(k_1) f(g) \check{\rho}(k_2), \quad k_i \in \mathcal{P}, g \in G\},$$

where

$$\mathcal{C}_c^\infty(G, \text{End}_{\mathbb{C}}(W^\vee)) = \{f : G \rightarrow \text{End}_{\mathbb{C}}(W^\vee) \mid f \text{ locally constant with compact support}\}.$$

With the convolution operator

$$f_1 * f_2(g) = \int_G f_1(x) f_2(x^{-1}g) dx, \quad f_i \in \mathcal{H}(G//\mathcal{P}, \rho),$$

$\mathcal{H}(G//\mathcal{P}, \rho)$ is an associative $\mathbb{C}$ -algebra with 1. As algebras,

$$\mathcal{H}(G//\mathcal{P}, \rho) \cong \text{End}_G(c - \text{Ind}_{\mathcal{P}}^G(\rho)).$$

Here

$$c - \text{Ind}_{\mathcal{P}}^G(\rho) = \{f : G \rightarrow W \mid f \text{ has compact support, } f(kg) = \rho(k)f(g), \ k \in \mathcal{P}, g \in G\},$$

on which G acts by right translation.

6.5 Bernstein components

A full account of this material is given in [BK2]. Denote by $\mathfrak{R}(G)$ the category of smooth complex representations of G . One may partition the set of equivalence classes of irreducible objects of $\mathfrak{R}(G)$ using parabolic induction. For any smooth irreducible representation π of G , one can find a Levi subgroup L of G and an irreducible supercuspidal representation σ of L such that π is a composition factor of $\text{Ind}_{L,N}^G \delta^{1/2} \sigma$. The pair (L, σ) is unique up to conjugacy in G and is called the support of π . Two pairs (L_1, σ_1) and (L_2, σ_2) are inertially equivalent if there exist $g \in G$ and an unramified quasicharacter χ of L_2 such that

$$L_2 = L_1^g \quad \text{and} \quad \sigma_1^g \cong \sigma_2 \otimes \chi,$$

where $L_1^g = g^{-1}L_1g$ and $\sigma_1^g : x \mapsto \sigma_1(gxg^{-1})$, for $x \in L_1^g$. Then, $[L, \sigma]_G$, called the inertial support of π , denotes the inertial equivalence class of (L, σ) . Also, $\mathfrak{B}(G)$ denotes the set of all inertial equivalence classes in G . For an inertial equivalence class $\mathfrak{s}$, define the objects of a full subcategory $\mathfrak{R}^{\mathfrak{s}}(G)$ of $\mathfrak{R}(G)$ to be the smooth representations of G all of whose irreducible subquotients have inertial support $\mathfrak{s}$. Then, [Be, 2.10] gives

$$\mathfrak{R}(G) = \prod_{\mathfrak{s} \in \mathfrak{B}(G)} \mathfrak{R}^{\mathfrak{s}}(G).$$

The Bernstein component of $I(s, \pi \boxtimes \sigma)$ is $\mathfrak{R}^{[M, \pi \boxtimes \sigma]}_G(G)$. This is the set of smooth representations κ of G all of whose irreducible subquotients have inertial support $[M, \pi \boxtimes \sigma]_G$. Any unramified quasicharacter χ of $GSpin_5 \times GL_{2m}$ or $GSpin_{4,1} \times GL_{2m}$ is of the form

$$\chi = |\text{sim}|^t \boxtimes |\det|^s, \quad s, t \in \mathbb{C},$$

where sim and $\det$ are k -rational characters of $GSpin_5$ or $GSpin_{4,1}$ and GL_{2m} , respectively. So $\mathfrak{R}^{[M, \pi \boxtimes \sigma]}_G(G)$ is the set of all smooth reps κ of G such that all the irreducible subquotients of κ are a composition factor of a representation equivalent to

$$\text{Ind}_P^G \delta_P^{1/2} \pi |\text{sim}|^t \boxtimes \sigma |\det|^s, \quad \text{for some } s, t \in \mathbb{C}.$$

Let K be a compact open subgroup G , and (ρ, W) an irreducible smooth representation of K . If (π, V) is a smooth representation of G , the space of ρ -invariants V_ρ of V is

$$V_\rho = \text{Hom}_K(W, V).$$

Now, V_ρ is naturally isomorphic to $\text{Hom}_G(c\text{-Ind}_K^G \rho, V)$. It is a left $\mathcal{H}(G//K, \rho)$ -module via the right action of $\mathcal{H}(G//K, \rho)$ on $c\text{-Ind}_K^G(\rho)$. This module structure is given by

$$f \cdot \varphi : w \mapsto \int_G \pi(g) \varphi(\check{f}(g^{-1})w) dg, \quad f \in \mathcal{H}(G//K, \rho), \varphi \in V_\rho, w \in W.$$

Then we have the functor

$$M_\rho : \mathfrak{R}(G) \longrightarrow \mathcal{H}(G//K, \rho)\text{-Mod}, \quad (\pi, V) \mapsto V_\rho.$$

Write $V[\rho]$ for the G -subspace of V generated by the ρ -isotypic subspace V^ρ of V . Here, V^ρ is the sum of all irreducible K -subspaces of V which are equivalent to ρ . Let $\mathfrak{R}_\rho(G)$ be the full subcategory of $\mathfrak{R}(G)$ whose objects are the representations (π, V) such that $V[\rho] = V$.

By [BK2, Def 4.1], the pair (K, ρ) is a type in G if the category $\mathfrak{R}_\rho(G)$ is closed under subquotients in $\mathfrak{R}(G)$. By [BK2, Thm 4.3]

Theorem 6.1. *The following are equivalent:*

- (i) *the category $\mathfrak{R}_\rho(G)$ is closed under subquotients in $\mathfrak{R}(G)$,*
- (ii) *the functor M_ρ restricted to $\mathfrak{R}_\rho(G)$ is an equivalence of categories,*

(iii) there is a finite subset $\mathfrak{S}$ of $\mathfrak{B}(G)$ such that

$$\mathfrak{R}_\rho(G) = \prod_{\mathfrak{s} \in \mathfrak{S}} \mathfrak{R}^{\mathfrak{s}}(G),$$

as subcategories of $\mathfrak{R}(G)$.

If the set $\mathfrak{S}$ contains only one element $\mathfrak{s}$, then (K, ρ) is said to be an $\mathfrak{s}$ -type in G . For a definition of G -cover, see [BK2, 8.1]. What we will need is [BK2, Thm 8.3]

Theorem 6.2. *Let M be a Levi subgroup of G . Suppose that (J_M, τ_M) is an $\mathfrak{s}_M$ -type in M and that (J, τ) is a G -cover of (J_M, τ_M) . Then (J, τ) is a $\mathfrak{s}$ -type in G .*

Then, towards relating the Bernstein component of $\text{Ind}(s, \pi \boxtimes \sigma)$ with $\mathcal{H}(G, \mathcal{P}, \rho)$:

Lemma 6.3. *The pair $(\mathcal{Q}, \rho_M)$ is a type for the inertial class $[M, \pi \boxtimes \sigma]_M$ in M and $(\mathcal{P}, \rho)$ is a G -cover for it. Therefore, the pair $(\mathcal{P}, \rho)$ is a type for the inertial class $[M, \pi \boxtimes \sigma]_G$ in G .*

Proof. Since $((\chi_\lambda \otimes R_\pi) \boxtimes (\chi_\lambda \otimes R_\sigma))|_{\mathcal{Q}} = \rho_M$ is irreducible, by [BK2, Prop 5.4] $(\mathcal{Q}, \rho_M)$ is a type for the inertial class $[M, \pi \boxtimes \sigma]_M$ in M . It is shown in [Mo1, pg. 149] that $(\mathcal{P}, \rho)$ is a G -cover for $(\mathcal{Q}, \rho_M)$. Then, by [BK2, Thm 8.3], $(\mathcal{P}, \rho)$ is a type for the inertial class $[M, \pi \boxtimes \sigma]_G$ in G . $\square$

Corollary 6.4. *The Hecke algebra of the Bernstein component of $\text{Ind}(s, \pi \boxtimes \sigma)$ is $\mathcal{H}(G//\mathcal{P}, \rho)$.*

Proof. We have

$$\mathfrak{R}^{[M, \pi \boxtimes \sigma]_G}(G) = \mathfrak{R}_\rho(G) \cong \mathcal{H}(G//\mathcal{P}, \rho)\text{-Mod},$$

where $[M, \pi \boxtimes \sigma]_G$ is the inertial support of $\text{Ind}(s, \pi \boxtimes \sigma)$. The last equivalence is given by the functor M_ρ restricted to $\mathfrak{R}_\rho(G)$. $\square$

7 A presentation of $\mathcal{H}(G//\mathcal{P}, \rho)$

In [Mo2], Morris describes explicit generators and relations for $\mathcal{H}(G//\mathcal{P}, \rho)$ when ρ is an irreducible cuspidal representation of the reductive quotient of a parahoric subgroup $\mathcal{P}$ of G . Let Π be a set of simple affine roots in correspondence with the vertices of the relative local Dynkin diagram for G . For $\Theta \subset \Pi$ define $W_\Theta = \langle s_\alpha \mid \alpha \in \Theta \rangle$. Define

$$S_\Theta = \{w \in N_W(W_\Theta) \mid w\Theta = \Theta\},$$

where $N_W(W_\Theta)$ is the normalizer of W_Θ in W . If $\mathcal{P}$ corresponds to $\Theta \subset \Pi$, let

$$\mathfrak{M} = \langle Z_c, U_a \mid a \text{ is the gradient of some } \alpha \in \Theta \rangle,$$

where Z_c is the maximal compact subgroup of the center Z . Let $\mathcal{Q} = \mathfrak{M} \cap \mathcal{P}$. For $g \in G, p \in \mathcal{P}$, let $g\rho$ be the representation of ${}^g\mathcal{P}$ defined by $g\rho(gpg^{-1}) = \rho(p)$. Then, by [Mo2, §4] $\mathcal{H}(G//\mathcal{P}, \rho)$ is supported on double cosets $\mathcal{P}\dot{w}\mathcal{P}$ where $\dot{w} \in N_G(T) \cap N_G(\mathcal{Q})$ such that

- (i) under the induced action on $\mathbf{M}$, ${}^{\dot{w}}\mathbf{M} = \mathbf{M}$;
- (i) as representations of $\mathbf{M}$, $\dot{w}\rho \cong \rho$;
- (ii) $\dot{w}$ projects to an element $w \in S_\Theta$.

Suppose that $J \subset \Pi$ such that $wJ = \Theta$ for some $w \in W$, and $\alpha \in \Pi$. Let t be the longest element in the Weyl group W_J corresponding to the spherical root system obtained from J such that $t^2 = 1$ and $t(J) = -J$. Let u be the longest element in the Weyl group $W_{J \cup \{\alpha\}}$ corresponding to the spherical root system obtained from $J \cup \{\alpha\}$ such that $u^2 = 1$ and $u(J \cup \{\alpha\}) = -(J \cup \{\alpha\})$. Set

$$v[\alpha, J] = u \cdot t \in W_{J \cup \alpha} \subset W' \subset W.$$

We now specialize to the case of interest. Denote by

$$\Theta = \Theta^+ \subset \Pi^+, \Theta^- \subset \Pi^-, \Theta^\dagger \subset \Pi^\dagger, \Theta^\ddagger \subset \Pi^\ddagger :$$

$$\Theta^+ = \{\alpha_{0^+}, \dots, \alpha_{2m+2^+}\} \setminus \{\alpha_{0^+}, \alpha_{2m^+}\}, \quad \Theta^- = \{\alpha_{0^-}, \dots, \alpha_{4^-}\} \setminus \{\alpha_{2^-}, \alpha_{4^-}\},$$

$$\Theta^\dagger = \{\alpha_{0^\dagger}, \dots, \alpha_{5^\dagger}\} \setminus \{\alpha_{1^\dagger}, \alpha_{5^\dagger}\}, \quad \Theta^\ddagger = \{\alpha_{0^\ddagger}, \dots, \alpha_{3^\ddagger}\} \setminus \{\alpha_{0^\ddagger}, \alpha_{2^\ddagger}\}.$$

Lemma 7.1. (i) $S_{\Theta^+} = \langle v[\alpha_{0^+}, \Theta^+], v[\alpha_{2m^+}, \Theta^+], T(e_0^*) \rangle;$

(ii) $S_{\Theta^-} = \langle v[\alpha_{2^-}, \Theta^-], v[\alpha_{4^-}, \Theta^-], \nu \rangle$ where $\nu = T(e_3^*)s_{\alpha_{1^-}}s_{\alpha_{2^-}}s_{\alpha_{3^-}}s_{\alpha_{4^-}}s_{\alpha_{3^-}}s_{\alpha_{2^-}}s_{\alpha_{1^-}}$, is a diagram automorphism preserving the fundamental chamber corresponding to Π^- such that $\nu^2 = T(e_0^*)$ is translation in the central direction;

(iii) $S_{\Theta^\dagger} = \langle v[\alpha_{1^\dagger}, \Theta^\dagger], v[\alpha_{5^\dagger}, \Theta^\dagger], T(e_0^*) \rangle;$

(iv) $S_{\Theta^\ddagger} = \langle v[\alpha_{0^\ddagger}, \Theta^\ddagger], v[\alpha_{2^\ddagger}, \Theta^\ddagger], T(e_0^*) \rangle.$

In all cases the elements $v[\alpha_i, \Theta]$ are involutions, and $T(e_0^*)$ is translation by e_0^* .

Proof. (i) We will show that $v[\alpha_i, \Theta^+]\Theta^+ = \Theta^+$, $i = 0^+, 2m^+$. For $J \subset \Pi$, corresponding to a connected piece of the extended Dynkin diagram of type B_n, C_n, D_n (n even), $u(\alpha_j) = -\alpha_j$ for $\alpha_j \in J$. Here, u is the longest element in the Weyl group W_J defined above. As the piece of the extended Dynkin diagram corresponding to $\Theta^+ \cup \{\alpha_{2m}\}$ is of type B_{2m+2} ,

$$v[\alpha_{2m^+}, \Theta^+]\Theta^+ = \Theta^+.$$

By [Ho, Lem 10], $v[\alpha_0, \Theta^+] = v[\alpha_0, \{\alpha_1, \dots, \alpha_{2m-1}\}]$, for $b \in \{\alpha_{2m+1}, \alpha_{2m+2}\}$, $v[\alpha_0, \Theta^+](b) = b$, for $b \in \{\alpha_1, \dots, \alpha_{2m-1}\}$, $v[\alpha_0, \Theta^+](b) = b$, so

$$v[\alpha_0, \Theta^+]\Theta^+ = \Theta^+$$

($\{\alpha_1, \dots, \alpha_{2m-1}\} \cup \{\alpha_0\}$ corresponds to a diagram of type D_{2m}). Since $v[\alpha_i, \Theta^+]\Theta^+ = \Theta^+$ for $i = 0^+, 2m^+$, By [Mo2, Lem 2.4(c)] they are involutions.

Now, by [Mo2, Lem 2.5], if $w \in W$ such that $w\Theta = \Theta$, we can find $J_1, \dots, J_{r+2}$, where $J_i \subset \Pi$ and $\Theta = J_1 = J_{r+2}$, and $\alpha_1, \dots, \alpha_r \in \Pi$ such that $v[\alpha_i, J_i]J_i = J_{i+1}$, $1 \leq i \leq r$, and

$$w = \nu v[\alpha_r, J_r] \dots v[\alpha_1, J_1]$$

where $\nu \in \Omega$, $\nu J_{r+1} = J_{r+2} = \Theta$. Here,

$$\nu \in \Omega = \{w \in W \mid wC = C\}$$

the set of diagram automorphisms that fix the fundamental chamber C . For $GSpin_{4m+5}$, for the simple roots given by Δ^+ , the set $\Omega = \langle \nu \rangle$, where

$$\nu = T(e_1^*)s_{\alpha_1} \cdots s_{\alpha_{n-1}}s_{\alpha_n}s_{\alpha_{n-1}} \cdots s_{\alpha_1}$$

is a diagram automorphism preserving the fundamental chamber given by Δ^+ . We have $\nu^2 = T(e_0^*)$ is translation in the central direction. The action of ν on the simple affine roots is as follows,

$$\nu \cdot \alpha_0 = \alpha_1, \nu \cdot \alpha_1 = \alpha_0, \nu \cdot \alpha_i = \alpha_i, i > 1.$$

Therefore, ν does not preserve Θ^+ . However, $\nu^2 = T(e_0^*)$ does preserve Θ^+ . Since $v[\alpha_i, \Theta^+]\Theta^+ = \Theta^+$ for all $\alpha_i \in \Pi^+ \setminus \Theta^+$, $J_1 = J_2 = \cdots = J_{r+1} = \Theta^+$. Therefore, if $w\Theta^+ = \Theta^+$, w is a word in $v[\alpha_0, \Theta^+], v[\alpha_{2m}, \Theta^+]$, and $T(e_0^*)$.

(ii) For the rest of the cases, the proof will as in (i). As the piece of the extended Dynkin diagram corresponding to $\Theta^- \cup \{\alpha_{2-}\}$ is of type D_4 ,

$$v[\alpha_{2-}, \Theta^-]\Theta^- = \Theta^-.$$

By [Ho, Lem 10], $v[\alpha_{4-}, \Theta^-] = v[\alpha_{4-}, \{\alpha_{3-}\}]$ and

$$v[\alpha_{4-}, \Theta^-]\Theta^- = \Theta^-$$

($\{\alpha_{3-}\} \cup \{\alpha_{4-}\}$ corresponds to a diagram of type B_2). By [Mo2, Lem 2.4(c)] $v[\alpha_i, \Theta^-], i = 0^-, 2^-$, are involutions. When the simple roots are given by Δ^- , the set of diagram automorphisms $\Omega = \langle \nu \rangle$, where

$$\nu = T(e_3^*)s_{\alpha_{1-}}s_{\alpha_{2-}}s_{\alpha_{3-}}s_{\alpha_{4-}}s_{\alpha_{3-}}s_{\alpha_{2-}}s_{\alpha_{1-}}$$

is a diagram automorphism preserving the fundamental chamber given by Δ^- such that $\nu^2 = T(e_0^*)$ is translation in the central direction. The action of ν on the simple roots is as follows,

$$\nu \cdot \alpha_{0-} = \alpha_{1-}, \nu \cdot \alpha_{1-} = \alpha_{0-}, \nu \cdot \alpha_{i-} = \alpha_{i-}, i > 1.$$

Therefore, ν does preserve Θ^- . Since $v[\alpha_i, \Theta^-]\Theta^- = \Theta^-$ for all $\alpha_i \in \Pi^- \setminus \Theta^-$, $J_1 = J_2 = \cdots = J_{r+1} = \Theta^-$. Therefore, if $w\Theta^- = \Theta^-$, w is a word in $v[\alpha_{2-}, \Theta^-], v[\alpha_{4-}, \Theta^-]$, and ν .

(iii) As the piece of the extended Dynkin diagram corresponding to $\Theta^\dagger \cup \{\alpha_{1\dagger}\}$ is of type B_5 ,

$$v[\alpha_{1\dagger}, \Theta^\dagger]\Theta^\dagger = \Theta^\dagger.$$

By [Ho, Lem 10], $v[\alpha_{5^\dagger}, \Theta^\dagger] = v[\alpha_{5^\dagger}, \{\alpha_{2^\dagger}, \alpha_{3^\dagger}, \alpha_{4^\dagger}\}]$ and

$$v[\alpha_{5^\dagger}, \Theta^\dagger]\Theta^\dagger = \Theta^\dagger$$

($\{\alpha_{2^\dagger}, \alpha_{3^\dagger}, \alpha_{4^\dagger}\} \cup \{\alpha_{5^\dagger}\}$ corresponds to a diagram of type B_4). By [M2, Lem 2.4(c)] $v[\alpha_i, \Theta^\dagger], i = 1^\dagger, 5^\dagger$, are involutions. In this case $\Omega = \langle T(e_0^*) \rangle$ and $T(e_0^*)$ preserves $\Theta^\dagger$. Since $v[\alpha_i, \Theta^\dagger]\Theta^\dagger = \Theta^\dagger$ for all $\alpha_i \in \Pi^\dagger \setminus \Theta^\dagger$, $J_1 = J_2 = \dots = J_{r+1} = \Theta^\dagger$. Therefore, if $w\Theta^\dagger = \Theta^\dagger$, w is a word in $v[\alpha_{1^\dagger}, \Theta^\dagger], v[\alpha_{5^\dagger}, \Theta^\dagger]$, and $T(e_0^*)$.

(iv) As the piece of the extended Dynkin diagram corresponding to $\Theta^\dagger \cup \{\alpha_{2^\dagger}\}$ is of type B_3 ,

$$v[\alpha_{2^\dagger}, \Theta^\dagger]\Theta^\dagger = \Theta^\dagger.$$

By [Ho, Lem 10], $v[\alpha_{0^\dagger}, \Theta^\dagger] = v[\alpha_{0^\dagger}, \{\alpha_{1^\dagger}\}]$ and

$$v[\alpha_{0^\dagger}, \Theta^\dagger]\Theta^\dagger = \Theta^\dagger$$

($\{\alpha_{1^\dagger}\} \cup \{\alpha_{0^\dagger}\}$ corresponds to a diagram of type B_2). By [Mo2, Lem 2.4(c)] $v[\alpha_i, \Theta^\dagger], i = 0^\dagger, 2^\dagger$, are involutions. In this case $\Omega = \langle T(e_0^*) \rangle$ and $T(e_0^*)$ preserves $\Theta^\dagger$. Since $v[\alpha_i, \Theta^\dagger]\Theta^\dagger = \Theta^\dagger$ for all $\alpha_i \in \Pi^\dagger \setminus \Theta^\dagger$, $J_1 = J_2 = \dots = J_{r+1} = \Theta^\dagger$. Therefore, if $w\Theta^\dagger = \Theta^\dagger$, w is a word in $v[\alpha_{0^\dagger}, \Theta^\dagger], v[\alpha_{2^\dagger}, \Theta^\dagger]$, and $T(e_0^*)$. $\square$

Lemma 7.2. (i) For $i = 0^+, 2m^+$ if $\Theta = \Theta^+$, $i = 2^-, 4^-$ if $\Theta = \Theta^-$, $i = 1^\dagger, 5^\dagger$ if $\Theta = \Theta^\dagger$, and $i = 0^\ddagger, 2^\ddagger$ if $\Theta = \Theta^\ddagger$, we have

$$v[\alpha_i, \Theta] \cdot \rho \cong \rho,$$

for $\rho = R_\pi \boxtimes R_\sigma$ a representation of $\mathbf{M}$ as given in Section 6.3.

(ii) In addition, for $i = 2^-, 4^-$,

$$\nu \cdot \rho \not\cong \rho,$$

for $\rho = R_\pi^- \boxtimes R_\sigma^-$ a representation of $\mathbf{M} = GSpin_4(\mathfrak{f}) \times GL_2(\mathfrak{f})$.

Proof. (i) For $i = 0^+, 2m^+$ if $\Theta = \Theta^+$, $i = 2^-, 4^-$ if $\Theta = \Theta^-$, $i = 1^\dagger, 5^\dagger$ if $\Theta = \Theta^\dagger$, and $i = 0^\ddagger, 2^\ddagger$ if $\Theta = \Theta^\ddagger$, $v[\alpha_i, \Theta]$ acts on the root datum for $\mathbf{M}$, $\Psi = (X, \Phi, X^\vee, \Phi^\vee)$, preserving the set of positive roots Φ^+ . So $v[\alpha_i, \Theta]$ gives an automorphism of the based root datum for $\mathbf{M}$

$$\Psi_0 = (X, \Phi^+, X^\vee, (\Phi^+)^vee).$$

Let $\mathbf{B}$ be the Borel subgroup of $\mathbf{M}$ given by the positive roots Φ^+ , and for each $a \in \Delta$ let $u_a \neq e$ be a fixed element in the root subgroup U_a . By [Sp1, Prop 2.13] $\text{Aut } \Psi_0(\mathbf{M})$ is isomorphic to the group

$$\text{Aut}(\mathbf{M}, \mathbf{B}, \mathbf{T}, \{u_a\}_{a \in \Delta})$$

of automorphisms of $\mathbf{M}$ which stabilize $\mathbf{B}$, $\mathbf{T}$ and the set of u_a . Therefore to determine the automorphism of $\mathbf{M}$ given by $v[\alpha_i, \Theta]$, we need only find an automorphism of $\mathbf{M}$ that stabilizes the set of u_a and gives the same action on $\mathbf{T}$ as $v[\alpha_i, \Theta]$.

The action of $v[\alpha_i, \Theta^+]$, $i = 0^+, 2m^+$, on the maximal torus T of $G = GSpin_{4m+5}(k)$ is given by

$$\begin{aligned} & v[\alpha_i, \Theta^+] \cdot \left(\prod_{j=0}^{2m+2} e_j^*(\lambda_j) \right) \\ &= e_0^*(\lambda_0 \lambda_1 \dots \lambda_{2m}) e_1^*(\lambda_{2m}^{-1}) e_2^*(\lambda_{2m-1}^{-1}) \dots e_{2m}^*(\lambda_1^{-1}) e_{2m+1}^*(\lambda_{2m+1}) \dots e_{2m+2}^*(\lambda_{2m+2}), \end{aligned}$$

for $\lambda_j \in GL_1(k)$. This factors through an action on $\mathbf{T} \subset \mathbf{M}$. If $A \in GL_{2m}(\mathfrak{f})$, $B \in GSpin_5(\mathfrak{f})$ the action of $v[\alpha_i, \Theta^+]$ on $\mathbf{M}$ is given by

$$(A, B) \mapsto (w_0({}^t A^{-1}) w_0^{-1}, (\det A) B),$$

where $w_0 \in GL_{2m}$ is the matrix with 1 on the antidiagonal and 0 elsewhere.

For $i = 2^-, 4^-$, $v[\alpha_i, \Theta^-]$ acts on the maximal torus T of $G = GSpin_9(k)$ by

$$v[\alpha_i, \Theta^-] \cdot \left(\prod_{j=0}^4 e_j^*(\lambda_j) \right) = e_0^*(\lambda_0 \lambda_1 \lambda_2) e_1^*(\lambda_2^{-1}) e_2^*(\lambda_1^{-1}) e_3^*(\lambda_3) e_4^*(\lambda_4).$$

Then, if $A \in GL_2(\mathfrak{f})$, $B \in GSpin_4(\mathfrak{f})$ the action of $v[\alpha_i, \Theta^-]$ on $\mathbf{M}$ is given by

$$(A, B) \mapsto (w_0({}^t A^{-1}) w_0^{-1}, (\det A) B).$$

For $i = 1^\dagger, 5^\dagger$, $v[\alpha_i, \Theta^\dagger]$ acts on the maximal k -split torus S of $G = GSpin_{8,5}(k)$ by

$$v[\alpha_i, \Theta^\dagger] \cdot \left(\prod_{j=0}^5 e_j^*(\lambda_j) \right) = e_0^*(\lambda_0 \lambda_1 \lambda_2 \lambda_3 \lambda_4) e_1^*(\lambda_4^{-1}) e_2^*(\lambda_3^{-1}) e_3^*(\lambda_2^{-1}) e_4^*(\lambda_1^{-1}) e_5^*(\lambda_5).$$

This factors through an action on $\mathbf{T} \subset GL_4(\mathfrak{f})$, and by [Sp1, Prop 2.13], $v[\alpha_i, \Theta]$ acts on $A \in GL_4(\mathfrak{f})$ as

$$A \mapsto w_0({}^t A^{-1}) w_0^{-1}.$$

If $A \in GL_4(\mathfrak{f})$, $B \in {}^2GSpin_4(\mathfrak{f})$ the action of $v[\alpha_i, \Theta^\dagger]$ on $\mathbf{M}$ is given by

$$(A, B) \mapsto (w_0({}^t A^{-1}) w_0^{-1}, (\det A) B).$$

For $i = 0^\dagger, 2^\dagger$, $v[\alpha_i, \Theta^\dagger]$ acts on the maximal k -split torus S of $G = GSpin_{6,3}(k)$ by

$$v[\alpha_i, \Theta^\dagger] \cdot \left(\prod_{j=0}^3 e_j^*(\lambda_j) \right) = e_0^*(\lambda_0 \lambda_1 \lambda_2) e_1^*(\lambda_2^{-1}) e_2^*(\lambda_1^{-1}) e_3^*(\lambda_3).$$

As in the previous case we find, if $A \in GL_2(\mathfrak{f})$, $B \in [(^2GSpin_2 \times GSpin_3)/\Delta GL_1](\mathfrak{f})$ the action of $v[\alpha_i, \Theta^\dagger]$ on $\mathbf{M}$ is given by

$$(A, B) \mapsto (w_0({}^t A^{-1}) w_0^{-1}, (\det A) B).$$

By [Bu, 4.1.1], if $R'(A) = R_\sigma({}^t A^{-1})$ then $R' \cong R_\sigma^\vee$. For $i = 0, 2m$ if $\Theta = \Theta^+$, $i = 2^-, 4^-$ if $\Theta = \Theta^-$, $i = 1^\dagger, 5^\dagger$ if $\Theta = \Theta^\dagger$, and $i = 0^\dagger, 2^\dagger$ if $\Theta = \Theta^\dagger$,

$$\rho(v[\alpha_i, \Theta] \cdot (A, B)) = R_\sigma({}^t A^{-1}) \omega_{R_\pi}(\det A) R_\pi(B) = R_\sigma^\vee(A) \omega_{R_\pi}(\det A) R_\pi(B).$$

For $m = 1$, as R_π and R_σ are constructed using data from the same TRD parameter $\phi = \phi_1 \oplus \phi_2$, we have $\omega_{R_\pi} = \omega_{R_\sigma}$. As R_σ is a representation of $GL_2(\mathfrak{f})$, by [Bu, 4.1.1],

$$R_\sigma^\vee(\omega_{R_\sigma} \circ \det) \cong R_\sigma.$$

For $m = 2$, both R_π and R_σ are constructed using data from the irreducible TRD parameter ϕ . For

$$\phi(\mathcal{I}_t) = \langle s \rangle \subset GSp_4(\mathbb{C}) \hookrightarrow GL_4(\mathbb{C}),$$

we have

$$s = \text{diag}(\tau, \tau^q, \tau^{q^2}, \tau^{q^3}), \quad \tau \tau^{q^2} = \tau^q \tau^{q^3} =: c.$$

As in Section 5.4, for $R_\pi = R_{T_\lambda, \chi_\lambda}$, the character χ_λ of $T_\lambda^{F_\lambda} \subset GSpin_5(\mathfrak{f})$ or ${}^2GSpin_4(\mathfrak{f})$ is represented by s . Similarly, for $R_\pi = R_{T_\lambda, \chi_\lambda}$, the character χ_λ of $T_\lambda^{F_\lambda} \subset GL_4(\mathfrak{f})$ is represented by s . We have the character $\omega_\pi \circ \det$ of $GL_4(\mathfrak{f})$ restricted to $T_\lambda^{F_\lambda}$ is represented by the element

$$\text{diag}(c, c, c, c).$$

Then, as $R_\pi^\vee = R_{T_\lambda, \chi_\lambda^{-1}}$, by [Ca, 7.2.8] the representation $R_\sigma^\vee \otimes (\omega_\pi \circ \det)$ of $GL_4(\mathfrak{f})$ is represented by the element

$$s^{-1}c.$$

We have

$$s_2 s_3 s_1 s_2 \cdot s^{-1}c = s$$

for $s_2 s_3 s_1 s_2 \in N_{GL_4(\mathbb{C})}(\hat{T})/\hat{T}$ where $s_2 s_3 s_1 s_2$ commutes with the Coxeter element $\hat{w} = s_1 s_2 s_3 \in N_{GL_4(\mathbb{C})}(\hat{T})/\hat{T}$. Then by [Ca, 7.3.4], $R_\sigma^\vee(\omega_{R_\pi} \circ \det) \cong R_\sigma$.

Therefore, for $i = 0, 2m$ if $\Theta = \Theta^+$, $i = 2^-, 4^-$ if $\Theta = \Theta^-$, $i = 1^\dagger, 5^\dagger$ if $\Theta = \Theta^\dagger$, and $i = 0^\ddagger, 2^\ddagger$ if $\Theta = \Theta^\ddagger$,

$$v[\alpha_i, \Theta] \cdot \rho \cong \rho.$$

(ii) We will now show that the action of ν on $M = GSpin_4(\mathfrak{f}) \times GL_2(\mathfrak{f})$ does not preserve $\rho = R_\pi^- \boxtimes R_\sigma^-$. We have ν acts on the simple roots in Θ^- by,

$$\nu \cdot \alpha_{0^-} = \alpha_{1^-}, \nu \cdot \alpha_{1^-} = \alpha_{0^-}, \nu \cdot \alpha_{2^-} = \alpha_{2^-}, \nu \cdot \alpha_{3^-} = \alpha_{3^-}, \nu \cdot \alpha_{4^-} = \alpha_{4^-}.$$

Note that ν fixes R_σ^- , but it suffices to show $\nu \cdot R_\pi^- \not\cong R_\pi^-$, which we now do.

Up to conjugation, there is only one minisotropic torus in $GSpin_4(\mathfrak{f})$. So there exists $g \in GSpin_4(\mathfrak{f})$ such that $\nu \cdot T_\lambda = g T_\lambda g^{-1}$. Then by [D] there exists a lift $\dot{g}$ of g to $\mathcal{P}$ such that $\nu \cdot T_\lambda = \dot{g} T_\lambda \dot{g}^{-1}$. By replacing ν by $\text{Ad}(\dot{g}^{-1}) \circ \nu$ we can assume ν fixes T_λ . Therefore

$$\nu \in N_{GSpin_5(K)}(T_\lambda)/T_\lambda$$

gives a non-trivial action on T_λ . In fact, since ν induces a non-trivial diagram automorphism for the diagram of $GSpin_4$,

$$\nu \in N_{GSpin_5(K)}(T_\lambda)/T_\lambda \setminus N_{GSpin_4(K)}(T_\lambda)/T_\lambda.$$

Then, since for $R_\pi^- = R_{T_\lambda, \chi_\lambda}$ the character χ_λ is in general position, by [Ca, 7.3.4]

$$\nu \cdot \rho \not\cong \rho.$$

□

Define

$$W(\Theta, \rho) = \{w \in S_\Theta | w\rho \simeq \rho\}.$$

The elements of $W(\Theta, \rho)$ parametrize a basis for $\mathcal{H}(G//\mathcal{P}, \rho)$, with relations given by Theorem (7.12) of [Mo2].

Corollary 7.3.

$$\mathcal{H}(G//\mathcal{P}, \rho) = \langle T_a, T_b, T_c \rangle,$$

$\{a, b\} = \{0^+, 2m^+\}, \{2^-, 4^-\}, \{1^\dagger, 5^\dagger\}$ or $\{0^\ddagger, 2^\ddagger\}$, subject to the relations

$$(i) \ T_c * T_i = T_i * T_c,$$

$$(ii) \ T_i^2 = (p_i - 1)T_i + p_i,$$

where $i \in \{0^+, 2m^+\}, \{2^-, 4^-\}, \{1^\dagger, 5^\dagger\}$ or $\{0^\ddagger, 2^\ddagger\}$.

Proof. We have shown

$$W(\Theta, \rho) = \langle v[\alpha_a, \Theta], v[\alpha_b, \Theta], T(e_0^*) \rangle,$$

for $\{a, b\} = \{0^+, 2m^+\}, \{2^-, 4^-\}, \{1^\dagger, 5^\dagger\}$ or $\{0^\ddagger, 2^\ddagger\}$. By [Mo2, Thm 7.12], $\mathcal{H}(G//\mathcal{P}, \rho)$ is generated by three elements T_a, T_b, T_c subject to the given relations. Note that the cocycle $\mu(c, i)$ is trivial for all i as $c = T(e_0^*)$ is a translation. Also, $T_c * T_i = T_i * T_c$ for all i as $T(e_0^*)v[\alpha_i, \Theta] = v[\alpha_i, \Theta]T(e_0^*)$ (in W) for all i . This is as $wT(e_0^*)w^{-1}(x) = T(w(e_0^*))(x) = T(e_0^*)(x)$ for $x \in \mathcal{A}, w \in W_0$, so all simple reflections commute with $T(e_0^*)$ in W . $\square$

8 Calculation of parameters p_i

8.1 A theorem of Lusztig

In this section, we will compute the parameters p_i in Corollary 7.3. The support of T_i is $\mathcal{P}\dot{w}_i\mathcal{P}$ where $\dot{w}_i \in N_G(T)$ projects to

$$\begin{aligned} v[\alpha_i, \Theta^+] &\in S_{\Theta^+}, i = 0, 2m, \text{ if } \mathcal{P} = \mathcal{P}^+, & v[\alpha_i, \Theta^-] &\in S_{\Theta^-}, i = 2^-, 4^-, \text{ if } \mathcal{P} = \mathcal{P}^-, \\ v[\alpha_i, \Theta^\dagger] &\in S_{\Theta^\dagger}, i = 1^\dagger, 5^\dagger, \text{ if } \mathcal{P} = \mathcal{P}^\dagger, & v[\alpha_i, \Theta^\ddagger] &\in S_{\Theta^\ddagger}, i = 0^\ddagger, 2^\ddagger, \text{ if } \mathcal{P} = \mathcal{P}^\ddagger. \end{aligned}$$

The elements $\dot{w}_i$ lie in $\mathcal{K}_i$. So T_i is supported in $\mathcal{K}_i$, for $i = 0, 2m$ if $\mathcal{P} = \mathcal{P}^+$, in $\mathcal{K}_i$ for $i = 2^-, 4^-$ if $\mathcal{P} = \mathcal{P}^-$, in $\mathcal{K}_i$ for $i = 1^\dagger, 5^\dagger$ if $\mathcal{P} = \mathcal{P}^\dagger$, and in $\mathcal{K}_i$ for $i = 0^\ddagger, 2^\ddagger$ if $\mathcal{P} = \mathcal{P}^\ddagger$. Let $\mathbf{G}_i$ be the quotient of $\mathcal{K}_i$ by its pro-unipotent radical. Let $\mathbf{P}_i = \mathbf{M}_i\mathbf{N}_i$ be the image of $\mathcal{P}$ in $\mathbf{G}_i$. Recall that

$$\mathbf{M}_i \cong \mathbf{G}_\lambda^{F_\lambda} \times GL_{2m}(\mathbf{f}),$$

so that ρ is a representation of $\mathbf{M}_i$. Consider

$$V_i = \text{Ind}_{\mathbf{P}_i}^{\mathbf{G}_i}(\rho).$$

The algebra of right $\mathbf{G}_i$ -endomorphisms of V_i ,

$$\text{End}_{\mathbf{G}_i}(V_i) = \mathcal{H}(\mathbf{G}_i, \rho),$$

is isomorphic to the Hecke algebra

$$\mathcal{H}(\mathbf{G}_i, \rho) = \{f : \mathbf{G}_i \longrightarrow \text{End}_{\mathbb{C}}(W^\vee) \mid f(p_1gp_2) = \check{\rho}(p_1)f(g)\check{\rho}(p_2), p_1, p_2 \in \mathbf{P}_i, g \in \mathbf{G}_i\},$$

where $(\check{\rho}, W^\vee)$ is the contragredient representation of (ρ, W) . The finite Hecke algebra $\mathcal{H}(\mathbf{G}_i, \rho)$ can be canonically identified with a subalgebra of $\mathcal{H}(G//\mathcal{P}, \rho)$. This is because

$$\text{Ind}_{\mathcal{P}}^{\mathcal{K}_i}(\rho) \cong \text{infl}_{\mathbf{G}_i}^{\mathcal{K}_i}(\text{Ind}_{\mathbf{P}_i}^{\mathbf{G}_i}(\rho))$$

and $\mathcal{H}(G//\mathcal{K}_i, \rho)$ can be canonically identified with a subalgebra of $\mathcal{H}(G//\mathcal{P}, \rho)$. By [Mo2, 6.5] $\mathcal{H}(\mathbf{G}_i, \rho)$ is two dimensional. It is generated by a function T_e supported on $\mathbf{P}_i$ and a function T_i supported on $\mathbf{P}_i w_i \mathbf{P}_i$. Here, w_i is the image of w_i in $\mathbf{G}_i$. We have $w_i \in N_{\mathbf{G}_i}(\mathbf{M}_i) \cap N_{\mathbf{G}_i}(\mathbf{T})$, and $w_i \cdot \rho \cong \rho$. The function T_i satisfies

$$T_i^2 = (p_i - 1)T_i + p_i.$$

Since the endomorphism algebra of V_i is two dimensional it has two irreducible summands:

$$\text{Ind}_{\mathbf{P}_i}^{\mathbf{G}_i}(\rho) = \rho_1 \oplus \rho_2.$$

By [HL], the parameter p_i is the quotient of the degrees of the two irreducible summands.

We will use a theorem of Lusztig, [Lu, Thm 8.6], to compute the parameters p_i .

8.2 Identification of $\mathbf{G}_i(\bar{\mathbf{f}})$

We will now describe $\mathbf{G}_i(\bar{\mathbf{f}})$ in the cases of concern. With the identification of the character group of T with the character group of $\mathbf{T}$, we have the character and cocharacter lattices for $\mathbf{T}$ are those for T ,

$$X = \mathbb{Z}e_0 \oplus \mathbb{Z}e_1 \oplus \cdots \oplus \mathbb{Z}e_n, \quad X^\vee = \mathbb{Z}e_0^* \oplus \mathbb{Z}e_1^* \oplus \cdots \oplus \mathbb{Z}e_n^*,$$

where $n = 2m + 2$. When the simple roots for T are given by Δ^+ , the fundamental chamber C^+ in the apartment $\mathcal{A} = X^\vee \otimes \mathbb{R}$ for $G = GSpin_{4m+5}$ is defined by the inequalities

$$1 - e_2 > e_1 > e_2 > \cdots > e_n > 0.$$

When the simple roots for T are given by Δ^- , the fundamental chamber C^- in the apartment $\mathcal{A}$ for $G = GSpin_9$ is defined by the inequalities

$$1 - e_4 > e_3 > e_4 > e_1 > e_2 > 0.$$

When the simple roots for T are given by

$$\Delta^\dagger = \{\beta_{1\dagger} = e_6 - e_5, \beta_{2\dagger} = e_5 - e_1, \beta_{3\dagger} = e_1 - e_2, \beta_{4\dagger} = e_2 - e_3, \beta_{5\dagger} = e_3 - e_4, \beta_{6\dagger} = e_4\},$$

the fundamental chamber $C^\dagger$ in the apartment $\mathcal{A}$ for $G = GSpin_{13}$ is defined by the inequalities

$$1 - e_5 > e_6 > e_5 > e_1 > e_2 > e_3 > e_4 > 0.$$

When the simple roots for T are given by

$$\Delta^\dagger = \{\beta_{1\dagger} = e_4 - e_1, \beta_{2\dagger} = e_1 - e_2, \beta_{3\dagger} = e_2 - e_3, \beta_{4\dagger} = e_3\},$$

the fundamental chamber $C^\dagger$ in the apartment $\mathcal{A}$ for $G = GSpin_9$ is defined by the inequalities

$$1 - e_1 > e_4 > e_1 > e_2 > e_3 > 0.$$

Let x_i be the vertex of the fundamental chamber C corresponding to the root α_i , (or β_i) in the local Dynkin diagram Π for G . The positive roots of $\mathbf{G}_i$ are

$$\Phi_{x_i}^+ = \{a \in \Phi^+ : \langle a, x_i \rangle \in \mathbb{Z}\},$$

where Φ^+ is the set of positive roots for T given by Δ^+ if $\mathcal{P} = \mathcal{P}^+$, by Δ^- if $\mathcal{P} = \mathcal{P}^-$, by $\Delta^\dagger$ if $\mathcal{P} = \mathcal{P}^\dagger$, and by $\Delta^\ddagger$ if $\mathcal{P} = \mathcal{P}^\ddagger$. The coroot associated with a root $a \in \Phi_{x_i}^+$ is the same for $\mathbf{G}_i$ as for G . We will also list $\mathbf{M}_i$ and the dual groups $\mathbf{M}_i^*, \mathbf{G}_i^*$ in each case.

When $\mathcal{P} = \mathcal{P}^+$, the reductive component of reduction mod $\mathfrak{p}$ of $\mathcal{P}$ is

$$\mathbf{M}_{0+} = \mathbf{M}_{2m+} = GL_{2m} \times GSpin_5, \quad \mathbf{M}_{0+}^* = \mathbf{M}_{2m+}^* = GL_{2m} \times GSp_4.$$

The root α_{0+} corresponds to the vertex $x_0 = 0$, so the set of positive roots for $\mathbf{G}_{0+}$ is

$$\Phi_{x_0}^+ = \{e_i \pm e_j \mid 1 \leq i < j \leq n\} \cup \{e_i \mid 1 \leq i \leq n\}.$$

Then

$$\mathbf{G}_{0+} = GSpin_{4m+5}, \quad \mathbf{G}_{0+}^* = GSp_{4m+4}.$$

The root α_{2m+} corresponds to the vertex $x_{2m} = 1/2(e_1^* + e_2^* + \cdots + e_{2m}^*)$, so

$$\Phi_{x_{2m}}^+ = \{e_i \pm e_j \mid 1 \leq i < j \leq 2m\} \cup \{e_{2m+1} - e_{2m+2}, e_{2m+2}, e_{2m+1}, e_{2m+1} + e_{2m+2}\}.$$

Therefore,

$$\mathbf{G}_{2m+} = (GSpin_{4m} \times GSpin_5)/\Delta GL_1, \quad \mathbf{G}_{2m+}^* = (GSO_{4m} \times GSp_4)^\circ.$$

When $\mathcal{P} = \mathcal{P}^-$,

$$\mathbf{M}_{2-} = \mathbf{M}_{4-} = GL_2 \times GSpin_4, \quad \mathbf{M}_{2-}^* = \mathbf{M}_{4-}^* = GL_2 \times GSO_4.$$

The root α_{2-} corresponds to the vertex $x_{2-} = 1/2(e_3^* + e_4^*)$, so the set of positive roots for $\mathbf{G}_{2-}$ is

$$\Phi_{x_{2-}}^+ = \{e_1 - e_2, e_2, e_1, e_1 + e_2\} \cup \{e_3 - e_4, e_3 + e_4\}.$$

Then we have

$$\mathbf{G}_{2-} = (GSpin_5 \times GSpin_4)/\Delta GL_1, \quad \mathbf{G}_{2-}^* = (GSp_4 \times GSO_4)^\circ.$$

The root α_{4-} corresponds to the vertex $x_{4-} = 1/2(e_1^* + e_2^* + e_3^* + e_4^*)$, so

$$\Phi_{x_{4-}}^+ = \{e_3 \pm e_4, e_4 \pm e_1, e_1 \pm e_2\}.$$

Therefore

$$\mathbf{G}_{4-} = GSpin_8, \quad \mathbf{G}_{4-}^* = GSO_8.$$

When $\mathcal{P} = \mathcal{P}^\dagger$,

$$\mathbf{M}_{1^\dagger}(\bar{\mathbf{f}}) = \mathbf{M}_{5^\dagger}(\bar{\mathbf{f}}) = GL_4 \times GSpin_4, \quad \mathbf{M}_{1^\dagger}^*(\bar{\mathbf{f}}) = \mathbf{M}_{5^\dagger}^*(\bar{\mathbf{f}}) = GL_4 \times GSO_4.$$

The root $\beta_{2^\dagger}$ corresponds to the vertex $x_{2^\dagger} = 1/2(e_5^* + e_6^*)$, so the set of positive roots for $\mathbf{G}_{1^\dagger}(\bar{\mathbf{f}})$ is

$$\Phi_{x_{2^\dagger}}^+ = \{e_i \pm e_j \mid 1 \leq i < j \leq 4\} \cup \{e_1, e_2, e_3, e_4\} \cup \{e_6 \pm e_5\}.$$

Then we have

$$\mathbf{G}_{1^\dagger}(\bar{\mathbf{f}}) = (GSpin_9 \times GSpin_4)/\Delta GL_1, \quad \mathbf{G}_{1^\dagger}^*(\bar{\mathbf{f}}) = (GSp_8 \times GSO_4)^\circ.$$

The root $\beta_{6^\dagger}$ corresponds to the vertex $x_{6^\dagger} = 1/2(e_1^* + e_2^* + e_3^* + e_4^* + e_5^* + e_6^*)$. We have

$$\mathbf{G}_{5^\dagger}(\bar{\mathbf{f}}) = GSpin_{12}, \quad \mathbf{G}_{5^\dagger}^*(\bar{\mathbf{f}}) = GSO_{12}.$$

When $\mathcal{P} = \mathcal{P}^\ddagger$,

$$\mathbf{M}_{0^\ddagger}(\bar{\mathbf{f}}) = \mathbf{M}_{2^\ddagger}(\bar{\mathbf{f}}) = GL_2 \times (GSpin_2 \times GSpin_3)/\Delta GL_1,$$

$$\mathbf{M}_{0^\ddagger}^*(\bar{\mathbf{f}}) = \mathbf{M}_{2^\ddagger}^*(\bar{\mathbf{f}}) = GL_2 \times (GSO_2 \times GSp_2)^\circ.$$

The affine roots $1 - e_4 - e_1$ and $\beta_{1^\ddagger} = e_4 - e_1$ corresponds to the edge of $C^\ddagger$ containing the vertices 0 and $1/2(e_4^*)$, so the set of positive roots for $\mathbf{G}_{0^\ddagger}(\bar{\mathbf{f}})$ is

$$\{e_1 \pm e_2, e_2 \pm e_3, e_1 \pm e_3, e_1, e_2, e_3\}.$$

Then we have

$$\mathbf{G}_{0^\ddagger}(\bar{\mathbf{f}}) = (GSpin_7 \times GSpin_2)/\Delta GL_1, \quad \mathbf{G}_{0^\ddagger}^*(\bar{\mathbf{f}}) = (GSp_6 \times GSO_2)^\circ.$$

The root $\beta_{3\ddagger}$ corresponds to the vertex $x_{3\ddagger} = 1/2(e_4^* + e_1^* + e_2^*)$, so

$$\Phi_{x_{3\ddagger}}^+ = \{e_4 \pm e_1, e_1 \pm e_2, e_4 \pm e_2\} \cup \{e_3\}.$$

Therefore

$$\mathbf{G}_{2\ddagger}(\bar{\mathbf{f}}) = (GSpin_3 \times GSpin_6)/\Delta GL_1, \quad \mathbf{G}_{2\ddagger}^*(\bar{\mathbf{f}}) = (GSp_2 \times GSO_6)^\circ.$$

To compute the parameters p_i , [Lu, 8.6] applies only when $\mathbf{G}_i$ has a connected center. To use this theorem in the cases where $\mathbf{G}_i$ does not have connected center, we will map $\mathbf{G}_i \hookrightarrow \mathbf{G}'_i$ where $\mathbf{G}'_i$ has connected center in all cases. Let $\mathbf{M}'_i$ be the Levi component of the parabolic subgroup $\mathbf{P}'_i \subset \mathbf{G}'_i$ such that $\mathbf{M}_i \subset \mathbf{M}'_i$ and $\mathbf{P}_i \subset \mathbf{P}'_i$. Set

i	$\mathbf{G}'_i(\bar{\mathbf{f}})$	$\mathbf{M}'_i(\bar{\mathbf{f}})$
0^+	$GSpin_{4m+5}$	$GL_{2m} \times GSpin_5$
$2m^+$	$(GSpin_{4m}^\sim \times GSpin_5)/\Delta GL_1$	$GL_1 \times GL_{2m} \times GSpin_5$
2^-	$(GSpin_5 \times GSpin_4^\sim)/\Delta GL_1$	$GL_2 \times GSpin_4^\sim$
4^-	$GSpin_8^\sim$	$GL_2 \times GSpin_4^\sim$
$1^\dagger$	$(GSpin_9 \times GSpin_4^\sim)/\Delta GL_1$	$GL_4 \times GSpin_4^\sim$
$5^\dagger$	$GSpin_{12}^\sim$	$GL_4 \times GSpin_4^\sim$
$0^\ddagger$	$(GSpin_7 \times GSpin_2)/\Delta GL_1$	$GL_2 \times (GSpin_2 \times GSpin_3)/\Delta GL_1$
$2^\ddagger$	$(GSpin_3 \times GSpin_6^\sim)/\Delta GL_1$	$GL_1 \times GL_2 \times (GSpin_2 \times GSpin_3)/\Delta GL_1$

Note that in each case $\mathbf{G}'_i$ has a connected center. The center of $GSpin_{4m+5}$ is

$$\{e_0^*(\mu) : \mu \in GL_1\} \simeq GL_1.$$

The center of $(GSpin_{2n}^\sim \times GSpin_{2m+1})/\Delta GL_1$ is

$$\{E_0^*(\mu)E_1^*(\nu) \dots E_n^*(\nu)E_{-1}^*(\nu^2) : \mu, \nu \in GL_1\} \simeq GL_1 \times GL_1.$$

The center of $GSpin_{2n}^\sim$ is

$$\{E_0^*(\mu)E_1^*(\nu) \dots E_n^*(\nu)E_{-1}^*(\nu^2) : \mu, \nu \in GL_1\} \simeq GL_1 \times GL_1.$$

The center of $(GSpin_7 \times GSpin_2)/\Delta GL_1$ is

$$\{e_0^*(\mu)e_4^*(\nu) : \mu, \nu \in GL_1\} \simeq GL_1 \times GL_1.$$

8.3 Calculation of p_i

Recall that there exists an element $\overline{p_\lambda} \in \mathbf{G}_\lambda(\bar{\mathbf{f}})$ such that

$$\mathbf{T}_\lambda^{F_\lambda} = \text{Ad}(\overline{p_\lambda})\mathbf{T}^{F_w},$$

- (i) if $\mathbf{G}_\lambda(\bar{\mathbf{f}}) = GL_{2m}(\bar{\mathbf{f}})$ then $\hat{w}$ is a Coxeter element,
- (ii) if $\mathbf{G}_\lambda(\bar{\mathbf{f}}) = \mathbf{G}^+(\bar{\mathbf{f}}) = GSpin_5(\bar{\mathbf{f}})$ or $\mathbf{G}^\dagger(\bar{\mathbf{f}}) = GSpin_4(\bar{\mathbf{f}})$ attached to a irreducible TRD parameter ϕ then $\hat{w}$ is a Coxeter element
- (iii) if $\mathbf{G}_\lambda(\bar{\mathbf{f}}) = \mathbf{G}^\dagger(\bar{\mathbf{f}}) = [(GSpin_2 \times GSpin_3)/\Delta GL_1](\bar{\mathbf{f}})$, $\mathbf{G}^+(\bar{\mathbf{f}}) = GSpin_5(\bar{\mathbf{f}})$ or $\mathbf{G}^-(\bar{\mathbf{f}}) = GSpin_4(\bar{\mathbf{f}})$ attached to a TRD parameter $\phi_1 \oplus \phi_2$, then $\hat{w}$ acts on $\mathbf{T}^*$ by

$$\hat{w} \cdot \text{diag}(t_1, t_2, t_3, t_4) = \text{diag}(t_4, t_3, t_2, t_1).$$

Let

$$\mathbf{T}_i = \mathbf{T}_{GL_{2m}} \times \mathbf{T}_{\mathbf{G}_\lambda} \subset \mathbf{M}_i,$$

where $\mathbf{T}_{GL_{2m}}$ is the split maximal torus in $GL_{2m}(\bar{\mathbf{f}})$ and $\mathbf{T}_{\mathbf{G}_\lambda}$ is the split maximal torus in $\mathbf{G}_\lambda(\bar{\mathbf{f}})$. Denote by

$$\mathbf{S}_i^{F_\lambda} = \text{Ad}(\overline{p_{\lambda(GL_{2m})}}, \overline{p_{\lambda(\mathbf{G}_\lambda)}})\mathbf{T}_i^{F_w}$$

where

$$F_\lambda = F_{\lambda(GL_{2m})} \otimes F_{\lambda(\mathbf{G}_\lambda)}, \quad F_w = F_{w(GL_{2m})} \otimes F_{w(\mathbf{G}_\lambda)}.$$

We will also denote by F_λ, F_w the twisted Frobenius which acts on $\mathbf{G}'_i$ such that the restriction of to $\mathbf{G}_i$ is F_λ, F_w . Also, F_λ^*, F_w^* will denote the dual Frobenius which acts on $\mathbf{G}_i^{I*}$.

The character $\chi_\lambda = \chi_{\lambda(GL_{2m})} \otimes \chi_{\lambda(\mathbf{G}_\lambda)}$ of the $F_\lambda = F_{\lambda(GL_{2m})} \otimes F_{\lambda(\mathbf{G}_\lambda)}$ stable torus $\mathbf{S}_i$ can be viewed as a regular element in a dual F_λ^* stable torus $\mathbf{S}_i^*$ of the dual group $\mathbf{G}_i^*$. Let t correspond to $\chi_\lambda = \chi_{\lambda(GL_{2m})} \otimes \chi_{\lambda(\mathbf{G}_\lambda)}$. There is a surjective map of dual groups

$$\psi : \mathbf{G}_i^{I*} \longrightarrow \mathbf{G}_i^*.$$

Since χ'_λ , the character given by t' , is in general position, we have that t' defines an irreducible representation ρ' of $\mathbf{M}'_i(\mathbf{f})$.

In the following table we list the groups $\mathbf{M}_i(\mathbf{f})$ and $\mathbf{M}'_i(\mathbf{f})$. As notation

$$\mathbf{G}^\dagger = ({}^2GSpin_2 \times GSpin_3)/\Delta GL_1.$$

i	$M_i(\mathfrak{f})$	$M'_i(\mathfrak{f})$
0^+	$GL_{2m} \times GSpin_5$	$GL_{2m} \times GSpin_5$
$2m^+$	$GL_{2m} \times GSpin_5$	$GL_1 \times GL_{2m} \times GSpin_5$
2^-	$GL_2 \times GSpin_4$	$GL_2 \times GSpin_4^\sim$
4^-	$GL_2 \times GSpin_4$	$GL_2 \times GSpin_4^\sim$
$1^\dagger$	$GL_4 \times {}^2GSpin_4$	$GL_4 \times {}^2GSpin_4^\sim$
$5^\dagger$	$GL_4 \times {}^2GSpin_4$	$GL_4 \times {}^2GSpin_4^\sim$
$0^\ddagger$	$GL_2 \times \mathbf{G}^\ddagger$	$GL_2 \times \mathbf{G}^\ddagger$
$2^\ddagger$	$GL_2 \times \mathbf{G}^\ddagger$	$GL_1 \times GL_2 \times \mathbf{G}^\ddagger$

Lemma 8.1. *The restriction of ρ' to $M_i(\mathfrak{f})$ is isomorphic to ρ .*

Proof. For $i = 0^+, 0^\ddagger$, $M'_i = M_i$ so we will not consider these cases. We will use a formula for Deligne-Lusztig characters given in Carter. We first define some notation. Let $\mathbf{G}$ a reductive $\mathfrak{f}$ -group, F a Frobenius automorphism of $\mathbf{G}$, and $\mathbf{T}$ an F -stable maximal torus of $\mathbf{G}$. Let $\mathbf{B}$ be a (not necessarily F -stable) Borel subgroup of $\mathbf{G}$ containing $\mathbf{T}$. We have $\mathbf{B} = \mathbf{U}\mathbf{T}$ where $\mathbf{U} = R_u(\mathbf{B})$. Lang's map is a surjective map $L : \mathbf{G}^F \rightarrow \mathbf{G}^F$ defined by

$$L(g) = g^{-1}F(g).$$

Let $\tilde{X}$ be the algebraic subset of $\mathbf{G}$ defined by

$$\tilde{X} = L^{-1}(\mathbf{U}).$$

We have an action of $\mathbf{G}^F$ on $\tilde{X}$ by left multiplication. For a semisimple element $s \in \mathbf{G}^F$, write $C^0(s)$ for the connected centralizer of s in $\mathbf{G}^F$. If

$$\tilde{Y}_s = C^0(s) \cap \tilde{X},$$

by [Ca, 7.1.11] we have

$$\mathcal{L}(g, \tilde{Y}_s) = |(\tilde{Y}_s)^g|$$

for $g \in C^0(s)$. Then for $x, s, u \in \mathbf{G}^F$ such that $s \in x\mathbf{T}^F x^{-1}$ and u is unipotent

$$\mathcal{Q}_{x\mathbf{T}^F x^{-1}}^{C^0(s)}(u) = \frac{1}{|\mathbf{T}^F|} \mathcal{L}(u, \tilde{Y}_s) = \frac{1}{|\mathbf{T}^F|} |(\tilde{Y}_s)^u|.$$

Then, the character formula [Ca, 7.2.8] says: for $g \in \mathbf{G}^F$ where g has the Jordan decomposition $g = su = us$ where s is semisimple and u unipotent we have

$$R_{\mathbf{T}, \theta}(g) = \frac{1}{|C^0(s)^F|} \sum_{x \in G^F, x^{-1}sx \in \mathbf{T}^F} \theta(x^{-1}sx) \mathcal{Q}_{x\mathbf{T}x^{-1}}^{C^0(s)}(u).$$

Let $R_{\mathbf{S}'_i}(t')$ be the virtual character (up to a sign) of the representation of $\mathbf{M}'_i{}^{F_\lambda}$ given by t' . Then $\mathbf{S}'_i{}^{F_\lambda} \subset \mathbf{S}_i'^{F_\lambda}$ where t gives the character of χ_λ of $\mathbf{S}_i^{F_\lambda}$ and t' gives a character χ'_λ of $\mathbf{S}_i'^{F_\lambda}$ such that $\chi'_\lambda(t) = \chi_\lambda(t)$ for $t \in \mathbf{S}_i^{F_\lambda}$. Let $R|_{\mathbf{M}_i}$ be the restriction of $R_{\mathbf{S}'_i}(t')$ to $\mathbf{M}_i^{F_\lambda}$. Let $g = su \in \mathbf{M}_i^{F_\lambda}$. Then for $x \in \mathbf{M}_i'^{F_\lambda}$ such that $x^{-1}sx \in \mathbf{S}_i^{F_\lambda}$

$$\chi'_\lambda(x^{-1}sx) = \chi_\lambda(x^{-1}sx).$$

We have

$$|C_{\mathbf{M}'_i}^0(s)^{F_\lambda}| = (q-1)|C_{\mathbf{M}_i}^0(s)^{F_\lambda}|$$

and

$$|\mathbf{S}_i'^{F_\lambda}| = (q-1)|\mathbf{S}_i^{F_\lambda}|.$$

Also,

$$|\{x \in \mathbf{M}_i'^{F_\lambda} : x^{-1}sx \in \mathbf{T}_i'^{F_\lambda}\}| = (q-1)|\{x \in \mathbf{M}_i^{F_\lambda} : x^{-1}sx \in \mathbf{T}_i^{F_\lambda}\}|.$$

Let

$$\tilde{Y}'_s = C_{\mathbf{M}'_i(\tilde{\mathbf{f}})}^0(s) \cap \tilde{X}'$$

where $\tilde{X}' = L^{-1}(\mathbf{U}'_i)$ for $L : \mathbf{M}'_i(\tilde{\mathbf{f}}) \rightarrow \mathbf{M}'_i(\tilde{\mathbf{f}})$ and $\mathbf{B}' = \mathbf{U}'_i\mathbf{S}'_i$ a Borel subgroup of $\mathbf{M}'_i$ containing $\mathbf{S}'_i$. Similarly, let

$$\tilde{Y}_s = C_{\mathbf{M}_i(\tilde{\mathbf{f}})}^0(s) \cap \tilde{X}$$

where $\tilde{X} = L^{-1}(\mathbf{U}_i)$ for $L : \mathbf{M}_i(\tilde{\mathbf{f}}) \rightarrow \mathbf{M}_i(\tilde{\mathbf{f}})$ and $\mathbf{B} = \mathbf{U}_i\mathbf{S}_i$. Then

$$|(\tilde{Y}'_s)^u| = (q-1)|(\tilde{Y}_s)^u|.$$

If we put all these facts together, by [Ca, 7.2.8]

$$R|_{\mathbf{M}_i}(g) = R_{\mathbf{S}_i}(t)(g), \quad g \in \mathbf{M}_i,$$

so they are the same virtual character. Since $\epsilon_{\mathbf{M}'_i}\epsilon_{\mathbf{S}'_i} = \epsilon_{\mathbf{M}_i}\epsilon_{\mathbf{S}_i}$ we have

$$\rho'|_{\mathbf{M}_i(\mathbf{f})} \cong \rho.$$

□

Lemma 8.2. *The homomorphism $\psi : \mathbf{G}_i'^* \longrightarrow \mathbf{G}_i^*$ central kernel.*

Proof. For $i = 0^+, 0^\dagger$, $\mathbf{T}_i'^* = \mathbf{T}_i^*$, and for $i = 2m^+, 2^\dagger$, the kernel of ψ is $\{(g, 1) \mid g \in GL_1\}$ which is in the center of $GL_1 \times \mathbf{M}_i^*$. For the remaining cases it suffices to show the homomorphism $\psi|_{(GSpin_{2n})^*} : \mathbf{T}^* \subset (GSpin_{2n})^* \longrightarrow \mathbf{T}^* \subset (GSpin_{2n})^*$ has kernel contained in the center of $\mathbf{G}_i'^*$. The center of $(GSpin_{2n})^*$,

$$\{E_{-1}(\mu)E_1(\nu) \dots E_n(\nu)E_0(\nu^2); \mu, \nu \in GL_1\},$$

is given by all elements of the split torus $\mathbf{T}_i'^*$ that belong to the kernel of all the simple roots. We have

$$\psi|_{(GSpin_{2n})^*} : \mathbf{T}^* = \prod_{j=-1}^n E_j(\lambda_j) \longrightarrow \mathbf{T}^* = \prod_{j=0}^n E_j(\lambda_j), \quad \lambda_j \in GL_1,$$

is given by

$$E_{-1}(\lambda_{-1})E_0(\lambda_0) \dots E_n(\lambda_n) \mapsto E_0(\lambda_0) \dots E_n(\lambda_n),$$

which has kernel contained in the center of $\mathbf{G}_i'^*$. □

Lemma 8.3. *The group $C_{\mathbf{G}_i'^*(\bar{\mathfrak{f}})}(t')$ is connected, reductive with root system*

- (i) type $(A_1)^2$ and Weyl group $W_{A_1}^2$ if $i = 2m^+ (m = 2), 1^\dagger$;
- (ii) type $(A_2)^2$ and Weyl group $W_{A_2}^m$ if $i = 0^+ (m = 2), 5^\dagger$;
- (iii) type A_1 and Weyl group W_{A_1} if $i = 2m^+ (m = 1), 2^-, 2^\dagger (j = 1), 0^\dagger (j = 2)$;
- (iv) type A_2 and Weyl group W_{A_2} if $i = 0^+ (m = 1), 4^-, 0^\dagger (j = 1), 2^\dagger (j = 2)$.

In each case there is one orbit on the simple roots for $W_{\mathbf{G}_i'^*}$, the Weyl group of $C_{\mathbf{G}_i'^*(\bar{\mathfrak{f}})}(t')$, under the action of the dual Frobenius F_λ^* on $\mathbf{G}_i'^*$. In addition, $C_{\mathbf{M}_i'^*(\bar{\mathfrak{f}})}(t') = \mathbf{S}_i'^*$.

Proof. Over $\bar{\mathfrak{f}}$, there exists $x \in \mathbf{M}_i^*$ such that $x\mathbf{S}_i^*x'^{-1} = \mathbf{T}_i^*$. Again over $\bar{\mathfrak{f}}$, there exists $x' \in \mathbf{M}_i'^*$ such that $\psi(x') = x$ and $x'\mathbf{S}_i'^*x'^{-1} = \mathbf{T}_i'^*$. Let $s = x^{-1}tx$ and $s' = x'^{-1}t'x'$, where $\psi(s') = s$. We have an isomorphism of reductive groups

$$\varphi : C_{\mathbf{G}_i'^*}(s') \rightarrow C_{\mathbf{G}_i'^*}(t'), \quad \varphi(z) = x'zx'^{-1}.$$

So it suffices to compute $C_{\mathbf{G}_i'^*}(s')$ with its F_w^* structure, which we do in the following.

As the center of G'_i is connected, centralizers of semisimple elements in G'^*_i are connected. Therefore $C_{G'^*_i}(s')$ is generated by T'^*_i and the root groups U_a such that $a(s') = 1$. Since the map

$$\psi : G'^*_i \rightarrow G^*_i$$

has central kernel, it suffices to compute the root groups U_a such that $a(s) = 1$.

The elements s are listed in the tables in Section 6.2. For $\rho = R_\sigma^+ \otimes R_\pi^+$

$$s = (\text{diag}(\tau, \tau^q, \tau^{q^2}, \tau^{q^3}), \text{diag}(\tau, \tau^q, \tau^{q^3}, \tau^{q^2})) \in GL_{2m}(\bar{\mathfrak{f}}) \times GSp_4(\bar{\mathfrak{f}}) \ (m = 2),$$

or

$$s = (\text{diag}(\tau_1, \tau_1^q), \text{diag}(\tau_1, \tau_2, \tau_2^q, \tau_1^q)) \in GL_{2m}(\bar{\mathfrak{f}}) \times GSp_4(\bar{\mathfrak{f}}) \ (m = 1).$$

For $\rho = R_\sigma^- \otimes R_\pi^-$

$$s = (\text{diag}(\tau_1, \tau_1^q), \text{diag}(\tau_1, \tau_2, \tau_2^q, \tau_1^q)) \in GL_2(\bar{\mathfrak{f}}) \times GSO_4(\bar{\mathfrak{f}}).$$

For $\rho = R_\sigma^\dagger \otimes R_\pi^\dagger$

$$s = (\text{diag}(\tau, \tau^q, \tau^{q^2}, \tau^{q^3}), \text{diag}(\tau, \tau^q, \tau^{q^3}, \tau^{q^2})) \in GL_4(\bar{\mathfrak{f}}) \times GSO_4(\bar{\mathfrak{f}}).$$

For $\rho = R_\sigma^\dagger \otimes R_\pi^\dagger$

$$s = (\text{diag}(\tau_j, \tau_j^q), \text{diag}(\tau_1, \tau_2, \tau_2^q, \tau_1^q)) \in GL_2 \times (GSO_2(\bar{\mathfrak{f}}) \times GSp_2(\bar{\mathfrak{f}}))^\circ$$

where $j = 1$ or 2 . Note that in each case s is F_w^* stable.

Let I be the root system of $W_{G'^*_i}$, the Weyl group of $C_{G'^*_i}(s')$. Then in the following table we display the different possibilities.

i	$G_i^*(\bar{f})$	I	$W_{G_i'^*}$
4^+	$(GSO_8 \times GSp_4)^\circ$	$\{\pm(e_1^* + e_3^* - e_0^*), \pm(e_2^* + e_4^* - e_0^*)\}$	$W_{A_1}^2$
0^+	GSp_{12}	$\langle e_1^* - e_5^*, e_3^* + e_5^* - e_0^*, e_2^* - e_6^*, e_4^* + e_6^* - e_0^* \rangle$	$W_{A_2}^2$
2^+	$(GSO_4 \times GSp_4)^\circ$	$\{\pm(e_1^* + e_2^* - e_0^*)\}$	W_{A_1}
0^+	GSp_8	$\langle e_1^* - e_3^*, e_2^* + e_3^* - e_0^* \rangle$	W_{A_2}
2^-	$(GSp_4 \times GSO_4)^\circ$	$\{\pm(e_1^* + e_2^* - e_0^*)\}$	W_{A_1}
4^-	GSO_8	$\langle e_1^* - e_3^*, e_2^* + e_3^* - e_0^* \rangle$	W_{A_2}
$1^\dagger$	$(GSp_8 \times GSO_4)^\circ$	$\{\pm(e_1^* + e_3^* - e_0^*), \pm(e_2^* + e_4^* - e_0^*)\}$	$W_{A_1}^2$
$5^\dagger$	GSO_{12}	$\langle e_1^* - e_5^*, e_3^* + e_5^* - e_0^*, e_2^* - e_6^*, e_4^* + e_6^* - e_0^* \rangle$	$W_{A_2}^2$
$0^\ddagger$	$(GSp_6 \times GSO_2)^\circ$	$j = 1, \quad \langle e_1^* - e_3^*, e_2^* + e_3^* - e_0^* \rangle$	W_{A_2}
$2^\ddagger$	$(GSp_2 \times GSO_6)^\circ$	$j = 1, \quad \{\pm(e_1^* + e_2^* - e_0^*)\}$	W_{A_1}
$0^\ddagger$	$(GSp_6 \times GSO_2)^\circ$	$j = 2, \quad \{\pm(e_1^* + e_2^* - e_0^*)\}$	W_{A_1}
$2^\ddagger$	$(GSp_2 \times GSO_6)^\circ$	$j = 2, \quad \langle e_1^* - e_4^*, e_2^* + e_4^* - e_0^* \rangle$	W_{A_2}

Note that in each case F_w^* acts transitively on the simple roots in I . For $i = 4^+$, F_w^* interchanges $e_1^* + e_3^* - e_0^*$ and $e_2^* + e_4^* - e_0^*$. For $i = 0^+$,

$$F_w^*(e_1^* - e_5^*) = e_4^* + e_6^* - e_0^*, \quad F_w^*(e_3^* + e_5^* - e_0^*) = e_2^* - e_6^*,$$

$$F_w^*(e_2^* - e_6^*) = e_1^* + e_5^*, \quad F_w^*(e_4^* + e_6^* - e_0^*) = e_3^* + e_5^* - e_0^*.$$

For $i = 2^+$, F_w^* fixes $e_1^* + e_2^* - e_0^*$. For $i = 0^+$, F_w^* interchanges $e_1^* - e_3^*$ and $e_2^* + e_3^* - e_0^*$. The other cases are similar.

As χ_λ is in general position, s is not fixed by any element of the Weyl group $(N_{M_i^*}(\mathbb{T}_i^*)/\mathbb{T}_i^*)^{F_w^*}$ so $C_{M_i^*}(s) = \mathbb{T}_i^*$. Since $\psi : \mathbb{T}_i'^* \rightarrow \mathbb{T}_i^*$ has central kernel, we have that $C_{M_i'^*}(s') = \mathbb{T}_i'^*$ which implies

$$C_{M_i'^*}(t') = S_i'^*.$$

□

Lemma 8.4. *We have $p_i' = p_i$.*

Proof. By [Lu, Thm 8.6],

$$\text{End}_{G'_i(f)}(\text{Ind}_{P'_i(f)}^{G'_i(f)} \rho') = \langle T_e, T_i \rangle$$

as an algebra, where T_e is supported on P'_i and T_i satisfies the relation

$$T_i^2 = (p'_i - 1)T_i + p'_i.$$

Here, T_i corresponds to the unique F_w^* -orbit on I . Therefore, since $\text{End}_{G'_i}(\text{Ind}_{P'_i}^{G'_i} \rho')$ has dimension two, we have that

$$\text{Ind}_{P'_i}^{G'_i} \rho' = \rho'_1 \oplus \rho'_2,$$

where ρ'_1 and ρ'_2 are distinct irreducible representations of G'_i . As $\rho'|_{M_i} = \rho$, by Mackey's induction restriction theorem,

$$\text{Ind}_{P'_i}^{G'_i}(\rho')|_{G_i} \cong \text{Ind}_{P_i}^{G_i}(\rho) \implies (\rho'_1 \oplus \rho'_2)|_{G_i} = \rho_1 \oplus \rho_2.$$

Therefore the quotient of the degrees of ρ'_1 and ρ'_2 is equal to the quotient of the degrees of ρ_1 and ρ_2 , so

$$p'_i = p_i.$$

□

Lemma 8.5. *We have:*

- (i) $p_i = q^2$ if $i = 2m^+ (m = 2), 1^\dagger$;
- (ii) $p_i = q^6$ if $i = 0^+ (m = 2), 5^\dagger$;
- (iii) $p_i = q$ if $i = 2m^+ (m = 1), 2^-, 2^\dagger (j = 1), 0^\dagger (j = 2)$;
- (iv) $p_i = q^3$ if $i = 0^+ (m = 1), 4^-, 0^\dagger (j = 1), 2^\dagger (j = 2)$.

Proof. Apply formula 8.2.3 of [Lu] to the pair $(W_{A_1}^n, 1)$. This calculation is done in [KM, Appendix], and we have

$$p'_i = q^n.$$

Apply formula 8.2.3 of [Lu] to the pair $(W_{A_2}^n, 1)$. This calculation is done in [Sa, §5] and we have

$$p'_i = q^{3n}.$$

From the previous lemma, $p'_i = p_i$ in each case. □

Corollary 8.6. *The parameters p_i of $\mathcal{H}(G//\mathcal{P}, \rho)$ are unequal.*

9 Reducibility of generalized principal series

9.1 A result of Matsumoto

Let (W, S) be a Coxeter system of type $\tilde{A}_1$, so that W is generated by

$$S = \{s_1, s_2\}$$

where $s_1^2 = s_2^2 = 1$ and $s_1 s_2$ has infinite order. Let R_+ denote the set of positive roots generated by the simple affine roots $s \in S$. Then the length function l on W is given by

$$l(w) = |R_+ \cap w^{-1}R_-|.$$

Let q be a real, positive valued, quasi-multiplicative function on W . Thus q satisfies $q(uv) = q(u)q(v)$ if $l(uv) = l(u)l(v)$, and $q(w) = 1$ if $l(w) = 0$. We have that q is completely determined by its values on S and satisfies $q(s) = q(s')$ if $s, s' \in S$ are conjugate in W . For k a commutative ring with unit and $w \in W$, denote by $\epsilon_w : W \rightarrow k$ the characteristic function of w . Let $k(W, q)$ be the k -algebra generated by $\epsilon_{s_1}, \epsilon_{s_2}$ subject to the relations

$$\begin{aligned} \epsilon_{s_i} * \epsilon_w &= \epsilon_{s_i w}, \quad l(s_i w) > l(w), \\ \epsilon_{s_i}^2 &= (q(s_i) - 1)\epsilon_{s_i} + q(s_i)\epsilon_e, \quad i = 1, 2. \end{aligned}$$

This is the Hecke algebra of type $\tilde{A}_1$ associated to the quasi-multiplicative function q .

In [Ma], Matsumoto gives the Plancherel measure for $k(W, q)$. Let $L^1(W, q)$ be the Banach space of L^1 integrable functions $f \in k(W, q)$ with respect to a fixed Haar measure on W . Matsumoto gives a measure dm on the unitary dual of $k(W, q)$ such that for $f \in L^1(W, q)$

$$f(e) = \int_{\pi \in \widehat{k(W, q)}} \text{Tr}(\pi(f)) \, dm(\pi).$$

Denote by $q_1 = q(s_1)$, $q_2 = q(s_2)$, and assume $q_2 \geq q_1 \geq 1$. Matsumoto defines representations π_ξ of $k(W, q)$, indexed by $\xi \in \mathbb{C}^\times$, such that all irreducible finite dimensional representations of $k(W, q)$ and all irreducible unitary representations of $k(W, q)$ occur in the composition series of such representations. We have:

- (i) For $|\xi| = 1$, the representations π_ξ are irreducible and unitary.
- (ii) For $\xi \in \mathbb{R}$ such that

$$1 < \xi \leq \sqrt{q_1 q_2}, \quad -\sqrt{q_2/q_1} \leq \xi < -1,$$

we have π_ξ is unitary. For $\xi \neq \sqrt{q_1 q_2}$, $-\sqrt{q_2/q_1}$, π_ξ is irreducible.

- (iii) For

$$\xi = \sqrt{q_1 q_2}, \quad -\sqrt{q_2/q_1},$$

the composition series of π_ξ is of length two.

Let $\chi_{\xi'}$ denote the irreducible subrepresentation of $\pi_{\sqrt{q_1 q_2}}$ and $\chi_{\xi''}$ the irreducible subrepresentation of $\pi_{-\sqrt{q_2/q_1}}$. Let S^1 be the multiplicative group of complex numbers of modulus 1, and $d\xi$ be the Haar measure on S^1 such that $\int_{S^1} d\xi = 1$. There is a function φ_ξ on W such that one has

$$f * \varphi_\xi(e) = \text{Tr}(\pi_\xi(f))$$

for all $f \in k(W, q)$, and a meromorphic function c_1 on $\mathbb{C}^\times$ such that we have the following.

For all $f \in L^1(W, q)$,

$$\begin{aligned} f(e) &= \frac{1}{2} \int_{S^1} \text{Tr}(\pi_\xi(f)) |c_1(\xi)|^{-2} d\xi \\ &\quad + \frac{1 - q_1^{-1} q_2^{-1}}{(1 + q_1^{-1})(1 + q_2^{-1})} \text{Tr}(\chi_{\xi'}(f)) + \frac{1 - q_1 q_2^{-1}}{(1 + q_1)(1 + q_2^{-1})} \text{Tr}(\chi_{\xi''}(f)). \end{aligned}$$

This is the Plancherel formula for $k(W, q)$.

We have

Theorem 9.1. (*Matsumoto*) *A Hecke algebra of type $\tilde{A}_1$ has two complementary series if and only if $q_1 \neq q_2$.*

9.2 Main theorem

To proceed we will need to use information about $\mu(s, \pi \boxtimes \sigma)$, the Plancherel measure associated to the generalized principal series $I(s, \pi \boxtimes \sigma)$. If $P = M \cdot N$ is the maximal parabolic of $GSpin_{4m+5}$ or $GSpin_{2m+4, 2m+1}$ defined in Section 6.2, let $\bar{P} = M \cdot \bar{N}$ be the opposite parabolic, and $I_{\bar{P}}(s, \pi \boxtimes \sigma)$ the corresponding generalized principal series representation. There is a local intertwining operator

$$A(s, \pi \boxtimes \sigma, N, \bar{N}) : I(s, \pi \boxtimes \sigma) \longrightarrow I_{\bar{P}}(s, \pi \boxtimes \sigma).$$

The composite $A(s, \pi \boxtimes \sigma, \bar{N}, N) \circ A(s, \pi \boxtimes \sigma, N, \bar{N})$ is a scalar operator on $I(s, \pi \boxtimes \sigma)$ and the Plancherel measure is the meromorphic function defined by

$$\mu(s, \pi \boxtimes \sigma)^{-1} = A(s, \pi \boxtimes \sigma, \bar{N}, N) \circ A(s, \pi \boxtimes \sigma, N, \bar{N}).$$

From [Si, 5.3-5.4] we have

Proposition 9.2. (*Harish-Chandra*) *If $\pi \boxtimes \sigma$ is a unitary supercuspidal representation of $GSpin_5(k) \times GL_r(k)$ or $GSpin_{4,1}(k) \times GL_4(k)$, then for s varying over the real numbers:*

- (i) *If $\sigma \not\cong \sigma^\vee(\omega_\pi \circ \det)$, then $\mu(0, \pi \boxtimes \sigma) \neq 0$ and $I_{P_r}(s, \pi \boxtimes \sigma)$ is irreducible for all $s \in \mathbb{R}$. In this case only $I(0, \pi \boxtimes \sigma)$ is unitary.*
- (ii) *If $\sigma \cong \sigma^\vee(\omega_\pi \circ \det)$, then there is a unique real $s_0 \geq 0$ such that $I_{P_r}(s_0, \pi \boxtimes \sigma)$ is reducible. Moreover, $s_0 > 0$ if and only if $\mu(0, \pi \boxtimes \sigma) = 0$, in which case s_0 is the unique pole of $\mu(s, \pi \boxtimes \sigma)$ on the positive real axis.*
 - (a) *When $s_0 = 0$, for all $s \neq 0$ we have $I(s, \pi \boxtimes \sigma)$ is irreducible and non-unitary. The representation $I(s_0, \pi \boxtimes \sigma)$ is of length 2, with irreducible subquotients tempered representations.*
 - (b) *When $s_0 > 0$, we have $I(s, \pi \boxtimes \sigma)$ is only reducible for $s = \pm s_0$. For $|s| < |s_0|$, $I(s, \pi \boxtimes \sigma)$ is unitary. The representation $I(s_0, \pi \boxtimes \sigma)$ is of length 2, with unique irreducible submodule a discrete series representation.*

Lemma 9.3. *Let $\pi \boxtimes \sigma$ be a unitary supercuspidal representation where π is a supercuspidal representation of $GSpin_5(k)$ or $GSpin_{4,1}(k)$ corresponding to a TRD Langlands parameter $\phi = \phi_1 \oplus \cdots \oplus \phi_r$, $r = 1, 2$, as in Definition 5.4. Also $\sigma \cong \sigma_{\phi_i}$, where $1 \leq i \leq r$,*

is the depth-zero supercuspidal representation of $GL_{2m}(k)$ attached to the Langlands parameter ϕ_i via the local Langlands correspondence for GL_{2m} . Then there are at most two twists $|\det|^{iv_j}$ of σ such that

$$\sigma|\det|^{iv_1} \not\cong \sigma|\det|^{iv_2}$$

and a representation

$$\mathrm{Ind}_P^G \delta_P^{1/2} \pi \boxtimes \sigma|\det|^{u+iv_j}$$

in the Bernstein component of the generalized principal series $I(s, \pi \boxtimes \sigma)$ could reduce for some $u > 0$.

Proof. Recall the Bernstein component of $I(s, \pi \boxtimes \sigma)$ is the set of all smooth representations κ of G such that any irreducible subquotient of κ is a composition factor of a representation equivalent to $\mathrm{Ind}_P^G \delta_P^{1/2} \pi |\mathrm{sim}|^t \boxtimes \sigma|\det|^s$ for some $s, t \in \mathbb{C}$. By Proposition 9.2, $I(s, \pi \boxtimes \sigma|\det|^{iv})$ can reduce for some real $s > 0$ only if

$$\sigma|\det|^{iv} \cong (\sigma|\det|^{iv})^\vee (\omega_\pi \circ \det).$$

Suppose σ_0 satisfies $\sigma_0 \cong \sigma_0^\vee (\omega_\pi \circ \det)$. If $\sigma_0|\det|^{iw}$ satisfies

$$\sigma_0|\det|^{iw} \cong (\sigma_0|\det|^{iw})^\vee (\omega_\pi \circ \det),$$

then

$$\sigma_0|\det|^{2iw} \cong \sigma_0.$$

Then for $g \in Z(GL_{2m})$, $|\det(g)|^{2iw} = 1$ so we must have

$$w = \frac{-k\pi}{2m\log(q)}$$

for some integer k such that $0 \leq k \leq 2$. By the local Langlands conjecture for GL_{2m} , if $\chi : k^\times \rightarrow \mathbb{C}^\times$, then

$$\sigma_0(\chi \circ \det) \cong \sigma_0 \iff \phi_{\sigma_0} \otimes \chi \cong \phi_{\sigma_0},$$

where χ is viewed as a character of W_k via local class field theory. We have $\phi_{\sigma_0} = \mathrm{Ind}_{W_{k2m}}^{W_k} \eta$. Then

$$\begin{aligned} \phi_{\sigma_0} \otimes \chi \cong \phi_{\sigma_0} &\iff \mathrm{Ind}_{W_{k2m}}^{W_k} (\eta \cdot \chi|_{W_{k2m}}) \cong \mathrm{Ind}_{W_{k2m}}^{W_k} \eta \\ &\iff \chi|_{W_{k2m}} = 1 \iff \chi^{2m} = 1. \end{aligned}$$

Therefore if $|\cdot|^{iw}$ has order dividing $2m$, then $\sigma_0|\det|^{iw} \cong \sigma_0$. Since $|\cdot|^{iw}$ must have order dividing $4m$, $|\cdot|^{iw}$ gives a nontrivial twist of σ_0 only if it has order $4m$. Any two

such twists differ by a character of order dividing $2m$, so there is at most one twist $|\cdot|^{iw}$ of σ_0 such that $\sigma_0|\det|^{iw} \cong (\sigma_0|\det|^{iw})^\vee(\omega_\pi \circ \det)$. $\square$

We now come to the main theorem:

Theorem 9.4. *Let π be a depth-zero supercuspidal representation of $GSpin_5(k)$ or $GSpin_{4,1}(k)$ corresponding to a tame regular discrete Langlands parameter $\phi = \phi_1 \oplus \cdots \oplus \phi_r$, $r = 1, 2$, as in Definition 5.4. Let $\sigma \cong \sigma_{\phi_i}$, where $1 \leq i \leq r$, be the depth-zero supercuspidal representation of $GL_{2m}(k)$ attached to the Langlands parameter ϕ_i via the local Langlands correspondence for GL_{2m} . Then the generalized principal series $I(s, \pi \boxtimes \sigma)$ reduces at a unique $s_0 > 0$.*

Proof. Assume $\pi \boxtimes \sigma$ is unitary. Recall that representations in the Bernstein component $\mathfrak{R}^{[M, \pi \boxtimes \sigma]G}(G)$ of $I(s, \pi \boxtimes \sigma)$ are parametrized by $\mathcal{H}(G//\mathcal{P}, \rho)$ -modules via the map

$$M_\rho : \mathfrak{R}^{[M, \pi \boxtimes \sigma]G}(G) \rightarrow \mathcal{H}(G//\mathcal{P}, \rho) - \text{Mod}, \quad (\kappa, V) \mapsto V_\rho.$$

For an irreducible representation $\kappa = \text{Ind}_P^G \delta_P^{1/2} \pi |\text{sim}|^t \boxtimes \sigma |\det|^s$ in $\mathfrak{R}^{[M, \pi \boxtimes \sigma]G}(G)$, the function

$$T_c \in \mathcal{H}(G//\mathcal{P}, \rho)$$

acts on $M_\rho(\kappa)$ by the scalar

$$C \omega_\pi(n) |\text{sim}(n)|^t$$

where $n = e_0^*(\varpi^{-1}) \in Z(GSpin_5)$ and C is a nonzero constant independent of t . Therefore irreducible representations in the Bernstein component of $I(s, \pi \boxtimes \sigma)$ which are a composition factor of some

$$\text{Ind}_P^G \delta_P^{1/2} \pi |\text{sim}|^t \boxtimes \sigma |\det|^{u+iv_j}, \quad t = 0,$$

are parametrized by simple modules of a Hecke algebra of type $\tilde{A}_1$. This Hecke algebra of type $\tilde{A}_1$ has unequal parameters by Corollary 8.6 and therefore has to have two complementary series by Theorem 9.1. By Lemma 9.3 (up to isomorphism) there are only two possible v_j such that the representation $\text{Ind}_P^G \delta_P^{1/2} \pi \boxtimes \sigma |\det|^{u+iv_j}$ could reduce for some $u > 0$. In the course of the proof of Lemma 7.2 we showed that for $\rho = R_\pi \boxtimes R_\sigma$,

$$R_\sigma \cong R_\sigma^\vee(\omega_{R_\pi} \circ \det).$$

Then since

$$\sigma = \text{Ind}(\chi_\lambda \otimes R_\sigma),$$

we have

$$\sigma \cong \sigma^\vee(\omega_\pi \circ \det).$$

Therefore, by Proposition 9.2, $I(s, \pi \boxtimes \sigma)$ could reduce for some real $s > 0$. Since there must be two complementary series, $I(s, \pi \boxtimes \sigma)$ does reduce for a unique real $s_0 > 0$. $\square$

10 The L -packets agree

When $\pi \boxtimes \sigma$ is irreducible and generic as a representation of M , we can apply Shahidi's theory of L-functions. Here π is a generic representation of $GSpin_5(k)$. The dual parabolic subgroup of P is $P^\vee = M^\vee \cdot N^\vee \subset GSpin_{4m+5}^\vee = GSp_{4m+4}(\mathbb{C})$, where

$$M^\vee = GSp_4(\mathbb{C}) \times GL_{2m}(\mathbb{C}).$$

Under the adjoint action of $M^\vee$, $\mathfrak{n}^\vee = Lie(N^\vee)$ decomposes as $r_1 \oplus r_2$, where each r_i is a maximal isotypic component for the action of the central torus in $M^\vee$. If

$$\mathfrak{n}^\vee = \begin{bmatrix} 0 & X & Y \\ & 0 & X^T \\ & & 0 \end{bmatrix} \quad \text{where } X \in M_{2m \times 4}(\mathbb{C}) : XJ = X, \ Y \in M_2(\mathbb{C}) : Y = Y^T,$$

(where J is the form preserved by $GSp_4(\mathbb{C})$) and

$$M^\vee = \begin{bmatrix} A & & \\ & B & \\ & & (A^T)^{-1} \text{sim}(B) \end{bmatrix} \quad \text{where } A \in GL_{2m}(\mathbb{C}), \ B \in GSp_4(\mathbb{C}),$$

then $r_1((B, A)) : X \mapsto A \cdot X \cdot B^{-1}$ and $r_2((B, A)) : Y \mapsto (A \cdot Y \cdot A^T) \text{sim}(B)^{-1}$. Therefore

$$r_1 = std^\vee \boxtimes std \quad \text{and} \quad r_2 = \text{sim}^{-1} \otimes Sym^2,$$

where std is the standard representation and sim is the similitude character of $GSp_4(\mathbb{C})$.

If $\bar{P}$ is the opposite parabolic and $\bar{P}^\vee$ is the dual of the opposite parabolic, then on the opposite nilpotent radical $\bar{\mathfrak{n}}^\vee$, the adjoint action of $M^\vee$ is the dual representation $r_1^\vee \oplus r_2^\vee$. Shahidi decomposed the Plancherel measure as a product of gamma factors. More precisely [Sh, Thm. 3.5]:

Proposition 10.1. *Suppose that $\pi \boxtimes \sigma$ is a generic representation of $M(k) = GSp_4(k) \times GL_{2m}(k)$. Then*

$$\mu(s, \pi \boxtimes \sigma) = \gamma(s, \pi \boxtimes \sigma, r_1, \psi) \gamma(s, \pi \boxtimes \sigma, r_1^\vee, \psi) \gamma(2s, \pi \boxtimes \sigma, r_2, \psi) \gamma(2s, \pi \boxtimes \sigma, r_2^\vee, \psi).$$

The local factors satisfy

$$\gamma(s, \pi \boxtimes \sigma, r_i, \psi) = \epsilon(s, \pi \times \sigma, r_i, \psi) \cdot \frac{L(1-s, (\pi \times \sigma)^\vee, r_i)}{L(s, \pi \times \sigma, r_i)},$$

where the factors on the right hand side were defined by Shahidi to satisfy the given decomposition of the Plancherel measure.

Collecting [GT, Cor 9.3], [GT, Thm 9.4], and [GTan, §8] we have

Proposition 10.2. *Let π be an irreducible supercuspidal representation of $GSpin_5(k)$ or $GSpin_{4,1}(k)$ with parameter ϕ_π given by the local Langlands conjectures for GSp_4 and its inner form. Then if σ is any irreducible supercuspidal representation of $GL_{2m}(k)$ with parameter ϕ_σ , the Plancherel measure $\mu(s, \pi \boxtimes \sigma)$ is equal to*

$$\begin{aligned} & \gamma(s, \phi_\pi \otimes \phi_\sigma, r_1, \psi) \gamma(s, \phi_\pi \otimes \phi_\sigma, r_1^\vee, \bar{\psi}) \gamma(2s, \phi_\pi \otimes \phi_\sigma, r_2, \psi) \gamma(2s, \phi_\pi \otimes \phi_\sigma, r_2^\vee, \bar{\psi}) = \\ & \gamma(s, \phi_\pi^\vee \otimes \phi_\sigma, \psi) \gamma(-s, \phi_\pi \otimes \phi_\sigma^\vee, \bar{\psi}) \gamma(2s, Sym^2 \phi_\sigma \otimes \text{sim} \phi_\pi^{-1}, \psi) \gamma(-2s, Sym^2 \phi_\sigma^\vee \otimes \text{sim} \phi_\pi, \bar{\psi}). \end{aligned}$$

Here

$$\gamma(s, \phi_\pi \otimes \phi_\sigma, r_i, \psi) = \epsilon(s, \phi_\pi \otimes \phi_\sigma, r_i, \psi) \cdot \frac{L(1-s, (\phi_\pi \otimes \phi_\sigma)^\vee, r_i)}{L(s, \phi_\pi \otimes \phi_\sigma, r_i)},$$

are the local factors of Artin type associated to the given representations of the Weil-Deligne group W'_k . For a representation $\phi_1 \otimes \phi_2$ of W'_k the Artin L-function $L(s, \pi_1 \otimes \pi_2)$ is given by

$$L(s, \phi_1 \otimes \phi_2) = \frac{1}{\det(I - q^{-s}(\phi_1 \otimes \phi_2)(\text{Frob})|_{(V_{\phi_1} \otimes V_{\phi_2})^I})}.$$

Lemma 10.3. *Let π be an irreducible supercuspidal representation of $GSpin_5(k)$ or $GSpin_{4,1}(k)$ with L-parameter ϕ_π given by the local Langlands conjectures for GSp_4 and its inner form. Let σ be an irreducible supercuspidal representation of $GL_{2m}(k)$ such that its L-parameter ϕ_σ factors through $GSp_{2m}(\mathbb{C})$ with similitude character $\text{sim} \phi_\pi$. Then*

$$\mu(0, \pi \boxtimes \sigma) = 0 \implies \text{Hom}_{W_k}(\phi_\pi, \phi_\sigma) \neq 0.$$

Proof. Let π be a representation of $GSpin_5(k)$ or $GSpin_{4,1}(k)$ with parameter ϕ_π and σ a representation of $GL_{2m}(k)$ with parameter ϕ_σ as in the statement of the lemma. By

Proposition 10.2 we have

$$\begin{aligned}
\mu(s, \pi \otimes \sigma) &= \gamma(s, \phi_\pi^\vee \otimes \phi_\sigma, \psi) \gamma(-s, \phi_\pi \otimes \phi_\sigma^\vee, \bar{\psi}) \\
&\quad \gamma(2s, \text{Sym}^2 \phi_\sigma \otimes \text{sim} \phi_\pi^{-1}, \psi) \gamma(-2s, \text{Sym}^2 \phi_\sigma^\vee \otimes \text{sim} \phi_\pi, \bar{\psi}) \\
&= \epsilon - \text{factors} \cdot \frac{L(1-s, [\phi_\pi^\vee \otimes \phi_\sigma]^\vee)}{L(s, \phi_\pi^\vee \otimes \phi_\sigma)} \cdot \frac{L(1+s, [\phi_\pi \otimes \phi_\sigma^\vee]^\vee)}{L(-s, \phi_\pi \otimes \phi_\sigma^\vee)} \\
&\quad \cdot \frac{L(1-2s, [\text{Sym}^2 \phi_\sigma \otimes \text{sim} \phi_\pi^{-1}]^\vee)}{L(2s, \text{Sym}^2 \phi_\sigma \otimes \text{sim} \phi_\pi^{-1})} \cdot \frac{L(1+2s, [\text{Sym}^2 \phi_\sigma^\vee \otimes \text{sim} \phi_\pi]^\vee)}{L(-2s, \text{Sym}^2 \phi_\sigma^\vee \otimes \text{sim} \phi_\pi)}.
\end{aligned}$$

Let $\mu(0, \pi \boxtimes \sigma) = 0$. From the expression for Artin L-functions given above, we can see that none of the numerators has a zero at $s = 0$. We have that ϕ_σ is irreducible and symplectic with similitude character $\text{sim} \phi_\pi$. By Schurs lemma it cannot also be orthogonal with similitude character $\text{sim} \phi_\pi$. Therefore neither $\text{Sym}^2 \phi_\sigma \otimes \text{sim} \phi_\pi^{-1}$ or $\text{Sym}^2 \phi_\sigma^\vee \otimes \text{sim} \phi_\pi$ can contain a nonzero fixed vector under W_k and neither of the last two denominators has a pole. This forces one of the first two denominators to have a pole. Therefore $\phi_\pi^\vee \otimes \phi_\sigma$ or $\phi_\pi \otimes \phi_\sigma^\vee$ contains the trivial representation and

$$\text{Hom}_{W_k}(\phi_\pi, \phi_\sigma) \neq 0.$$

□

We can now prove:

Theorem 10.4. *Let ϕ be a tame regular discrete L-parameter. Let L_ϕ^{DR} be the L-packet of depth-zero supercuspidal representations of $GSp_4(k)$ or $GU_2(D)$ corresponding to ϕ as in Definition 5.4. Let L_ϕ^{GT} be the L-packet of supercuspidal representations of $GSp_4(k)$ or $GU_2(D)$ corresponding to ϕ via the local Langlands conjecture for GSp_4 or $GU_2(D)$. Then*

$$L_\phi^{DR} = L_\phi^{GT}.$$

Proof. In the following assume that all representations π are unitary. Let $\phi = \phi_1 \oplus \cdots \oplus \phi_r$, $r = 1, 2$, be a tame regular discrete Langlands parameter. Let π_ϕ be a representation of $GSp_4(k)$ or $GU_2(D)$ corresponding to ϕ as in Definition 5.4. Under the correspondence defined by Gan and Takeda for GSp_4 , or Gan and Tantano for $GU_2(D)$, π_ϕ corresponds to some L-parameter we call ϕ' . Let $\sigma = \sigma_{\phi_i}$, $1 \leq i \leq r$, be the depth-zero supercuspidal representation of GL_{2m} attached to ϕ_i via the local Langlands correspondence for GL_{2m} . Note that if $r = 1$ then $m = 2$, and if $r = 2$ then $m = 1$.

By Theorem 9.4, $I(s, \pi_\phi \boxtimes \sigma)$ reduces for some $s_0 > 0$. By Proposition 9.2 this implies $\mu(0, \pi_\phi \boxtimes \sigma) = 0$. Then, by Lemma 10.3

$$\mathrm{Hom}_{W_k}(\phi_i, \phi') \neq 0.$$

This holds for $1 \leq i \leq r$. Since for $r = 2$, $\phi_1 \not\cong \phi_2$,

$$\phi' = \phi$$

in all cases. Therefore $L_\phi^{DR} = L_\phi^{GT}$. $\square$

Corollary 10.5. *The parametrization of DeBacker and Reeder of depth-zero supercuspidal representations of $GSp_4(k)$ arising from tame regular discrete Langlands parameters coincides with the parametrization of Gan and Takeda.*

Proof. Let ϕ be a TRD parameter for $GSp_4(k)$. By Lemma 5.3 and Theorem 10.4, the L -packet L_ϕ^{DR} of representations attached to ϕ by DeBacker and Reeder agrees with the L -packet L_ϕ^{GT} given by the local Langlands conjecture for GSp_4 . For L -packets of size two, by [GT, Main Thm (ii)] L_ϕ^{GT} contains exactly one generic representation indexed by the trivial character of A_ϕ . By [DR, 6.2.1] the generic representation in L_ϕ^{DR} is also indexed by the trivial character of A_ϕ . Therefore the parametrizations agree. $\square$

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