Title
ON THE CONVERGENCE OF THE MAXIMUM LIKELIHOOD ESTIMATOR METHOD OF TOMOGRAPHIC IMAGE RECONSTRUCTION

Authors
Llacer, J.
Veklerov, E.
Hoffman, E.J.

Publication Date
1986-06-01

Lawrence Berkeley Laboratory

UNIVERSITY OF CALIFORNIA

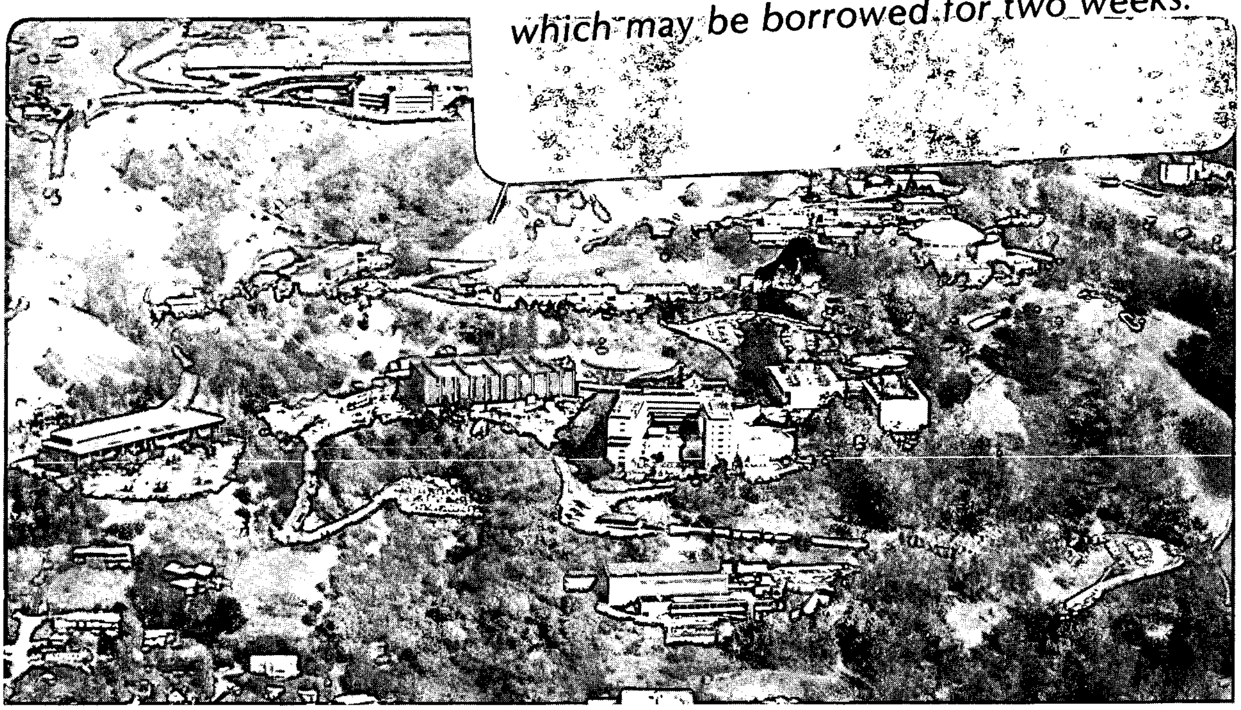
Engineering Division

Presented at the Society of Photo-Optical Instrumentation Engineers' Medical Imaging I Conference, Newport Beach, CA, February 1-6, 1987; to be published in the Proceedings

ON THE CONVERGENCE OF THE MAXIMUM LIKELIHOOD ESTIMATOR METHOD OF TOMOGRAPHIC IMAGE RECONSTRUCTION

J. Llacer, E. Veklerov, and E.J. Hoffman

June 1986



Prepared for the U.S. Department of Energy under Contract DE-AC03-76SF00098

DISCLAIMER

On the Convergence of the Maximum Likelihood Estimator Method of Tomographic Image Reconstruction

Jorge Llacer and Eugene Veklerov
Lawrence Berkeley Laboratory, University of California
Mailstop 29-100, Berkeley, CA 94720 U.S.A.
and
Edward J. Hoffman
Dept. of Radiological Sciences, School of Medicine
University of California
Los Angeles, CA 90024 U.S.A.

Abstract

The Maximum Likelihood Estimator (MLE) method of image reconstruction has been reported to exhibit image deterioration in regions of expected uniform activity as the number of iterations increases beyond a certain point. This apparent instability raises questions as to the usefulness of a method that yields images at different stages of the reconstruction that could have different medical interpretations. In this paper we look in some detail into the question of convergence of MLE solutions at a large number of iterations and show that the MLE method converges towards the image that it was designed to yield, i.e. the image which has the maximum likelihood to have generated the specific projection data resulting from a measurement. We also show that the maximum likelihood image can be a very deteriorated version of the true source image and that only as the number of counts in the projection data becomes very high, will the maximum likelihood image converge towards an acceptable reconstruction.

Introduction

Since its introduction into the emission tomography literature by Shepp and Vardi[1], the MLE method of image reconstruction has received a substantial amount of attention due to its apparent ability to yield images with excellent signal-to-noise ratio, good sharpness and increased contrast[1-5].

It has also become apparent[6-8], that the algorithm results in images that deteriorate by processes variously named noise artifact[6], ringing[7] or image breakup[8] as the number of iterations increases beyond a certain point. In a previous paper[8], one of us and co-workers examined the behavior of the MLE algorithm when reconstructing images from a non-tomographic, two-plane, 128 element BGO camera and arrived at the conclusion that the observed image deterioration was inherent in the nature of the Poisson distribution used as basis for the MLE algorithm. In order to understand in sufficient detail the mechanisms at work, we simulated a very simple camera (6 elements in two planes) and followed, step by step, the progress of an MLE reconstruction. We have extended that work to tomographic positron emission images and, in particular, we have simulated the ECAT-III tomograph[9] of U.C.L.A. for this study. The principal aim of our work has been to ascertain whether, indeed, the deteriorated images at high iteration numbers are converging towards a maximum likelihood image and under what conditions the maximum likelihood image may be a good representation of the source.

We will begin by presenting the theoretical basis of our analysis method, show reconstruction results from a phantom which exhibits strong image deterioration, describe the results of analyzing those images and discuss our conclusions about the convergence properties of MLE reconstructions.

Theoretical Basis

The likelihood function that one seeks to maximize in the MLE method of Ref. 1 is defined as:

$$L(\lambda) = P(n^*|\lambda) = \sum_{A} \prod_{\substack{b=1...B \\ d=1...D}} e^{-\lambda(b,d)} \left[\lambda(b,d)^{n(b,d)} / n(b,d)!\right] \tag{1}$$

where the sum is over all arrays A of n(b,d)'s with n*(d) observed counts in the dth detector tube. The notation is that of Ref. 1, which we use throughout this paper. The variables n(b,d) are the number of emissions in box b detected in tube d and are independent Poisson variables with mean

$$En(b,d) = \lambda(b,d) = \lambda(b)\, p(b,d) \tag{2}$$

where λ(b) is the activity in pixel b and p(b,d) is the probability of detecting one count emitted by b in detector tube d. The vector of observed counts n*(d) is defined as:

$$n^*(d) = \sum_{b=1}^{B} n(b,d) \tag{3}$$

From the description of the members of arrays A in Eq. 1, it is clear that the algorithm attempts to find an image that maximizes the likelihood function $L(\lambda)$ within all possible sets of values of $n(b,d)$ that satisfy the observed counts $n^*(d)$. This observation can be restated as: the MLE algorithm will find the image which has the highest probability of having given a particular set of detected events. The formulation of the MLE algorithm does not insure, however, that the projection of the resulting image will have some norm of minimum differences with the experimental projection data $n^*(d)$.

Our analysis of reconstruction will first verify that the MLE algorithm does indeed converge towards an image which has the highest probability of having given a vector of measurements $n^*(d)$. It will then become evident that the maximum likelihood image can be quite different from the true source image that generated the projection data. Finally, we will show that the difference decreases when the number of counts in the measurement increases.

Following our previous work[8], we consider that after a number of iterations i, we have found an image $\lambda^i(b)$. The projection of that image into the detector tube space is:

$$n^{*i}(d) = \sum_b p(b,d)\,\lambda^i(b) \tag{4}$$

The probability that the image obtained after i iterations has resulted in a measurement $n^*(d)$ will be:

$$p^i = \prod_d P[n^*(d) \mid n^{*i}(d)] \tag{5}$$

which should increase with increasing number of iterations. $P[j|m]$ is the probability of obtaining j counts when the mean of the distribution is m. P corresponds to the Poisson distribution in our case.

As part of the analysis we also calculate the root mean square error:

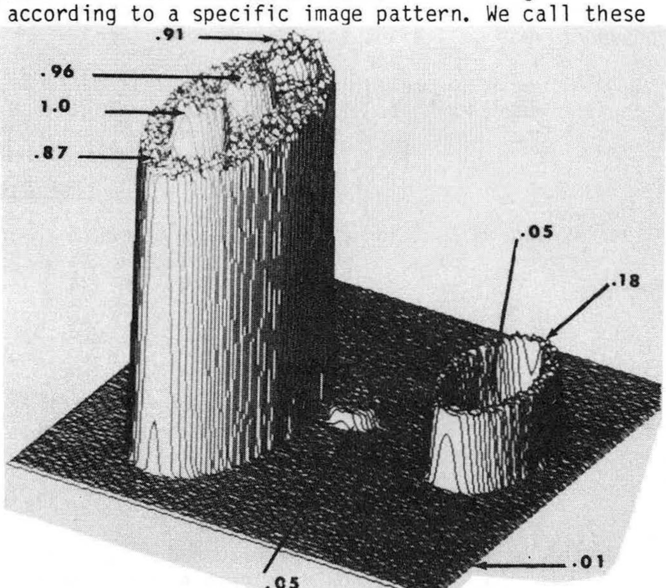
$$\left\{ \sum_d [n^*(d) - n^{*i}(d)]^2 \Big/ \sum_d n^*(d) \right\}^{1/2} \tag{6}$$

as the number of iterations increases.

Results of Reconstructions

In order to evaluate the behavior of the MLE algorithm exclusively, without being sensitive to effects due to the possible inaccuracies of the probability matrix $p(b,d)$ in representing a particular detector instrument, we have generated and reconstructed images by computer in a self-consistent manner: the images were obtained first by random assignment of events to pixels according to a specific image pattern. We call these images the source images throughout this paper. The corresponding projections were then obtained by assigning each event in each pixel to a tube d by a random process using a previously computed set of probabilities $p(b,d)$ that may correspond to a model of some specific instrument. The same set of probabilities was used subsequently in the reconstructions.

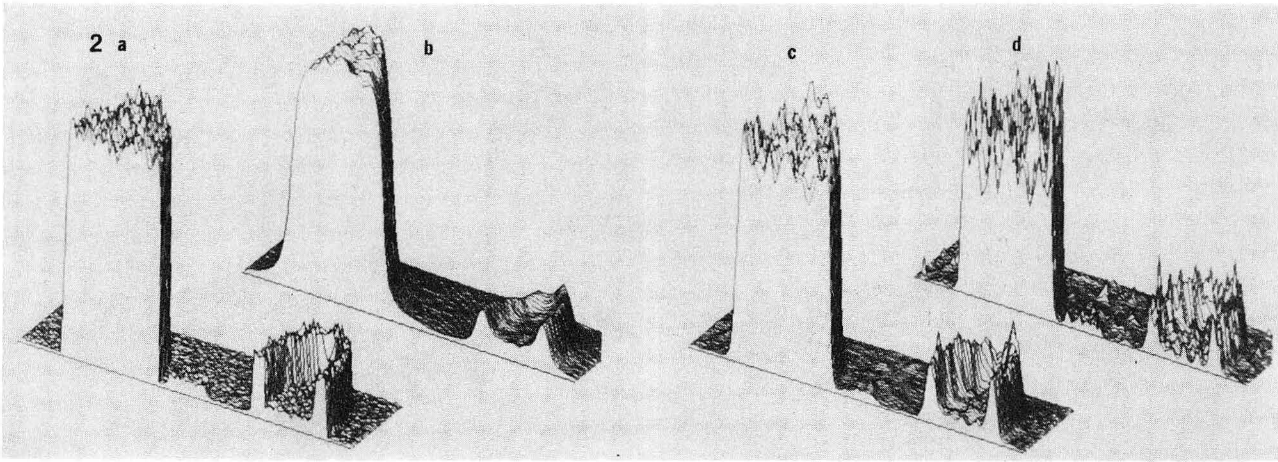
Figure 1 shows the source image used for the experiments for the particular case of 32 million counts in the image. The relative activities per unit area are shown in the figure. The interior of the elliptical shell has a relative activity of .05. Random background has been simulated by a relative activity of .01 over the entire surface of the image. The image plane has been discretized into 128x128 pixels. The matrix of $p(b,d)$ values used for projection generation and image reconstruction was obtained by the Shepp and Vardi prescription of Ref. 1, simulating one ring of the U.C.L.A. ECAT-III, 512 detector tomograph[9]. We obtain 512 projections with 96 bins each. A random number generator with a period of $2^{32} - 1$ was used in the random process calculations.



XBB 8612-10391A

Fig. 1 Source image with 32 million counts obtained by a random process with probabilities corresponding to the relative activities indicated. Image has been discretized in 128 x 128 pixels.

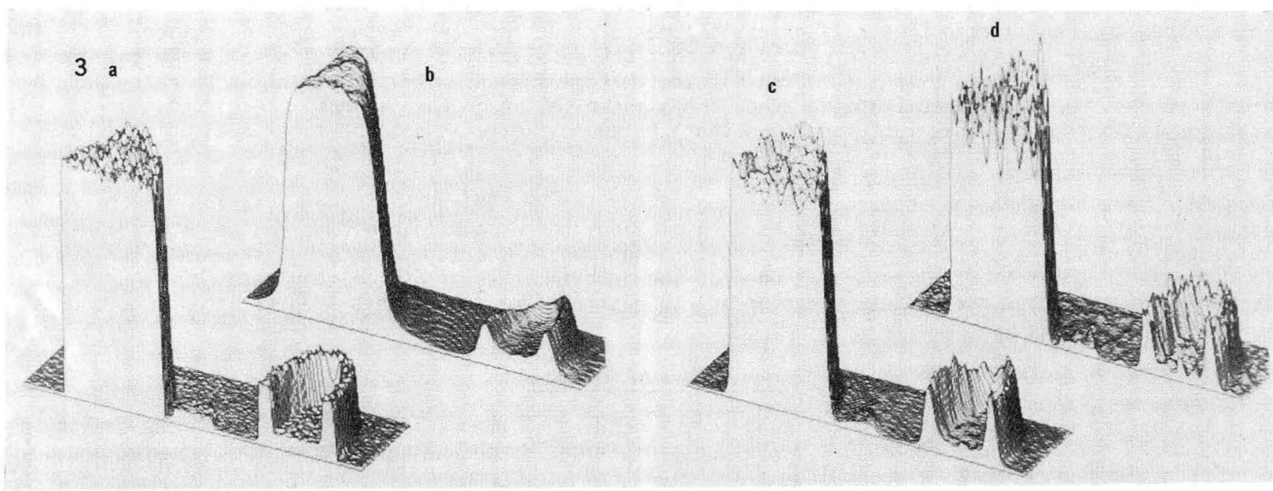
The reconstruction experiments have been carried out with three different levels of counts in the image: 2 million (2M), 8 million (8M) and 32 million (32M) counts. The signal-to-noise ratio in the source image pixels increases by factors of 2 in succession. The reconstructions were carried out to iteration 500 in all cases using single precision (32 bit) floating point arithmetic. Tests indicated that the results were practically identical to those obtained in double precision up to approx 500 iterations. Double precision arithmetic appears necessary above that number. The change in likelihood function was monitored continuously during the reconstructions and a monotonic increase in likelihood was observed.



XBB 860-10551A

Fig. 2 a) Cut through a line in the source image of 2 million counts. b) ditto for a corresponding MLE reconstruction after 9 iterations. c) ditto after 45 iterations. d) ditto after 200 iterations.

Examination of the reconstructed images shows that the reconstructions improve towards a reasonable representation of the source images up to approx iteration 45 and that image deterioration of a degree depending on the statistical quality of the original projection data is observable beyond that point. Figures 2a-d show cuts through a line in the source image and in MLE reconstructions with 9, 45 and 200 iterations, respectively, for the case with 2M counts. Figures 3a-d show results with the 8M images and Figs. 4a-d are for the 32M count case. The qualitative improvement up to iteration 45 and a changing degree of deterioration at 200 iterations is clearly observable.

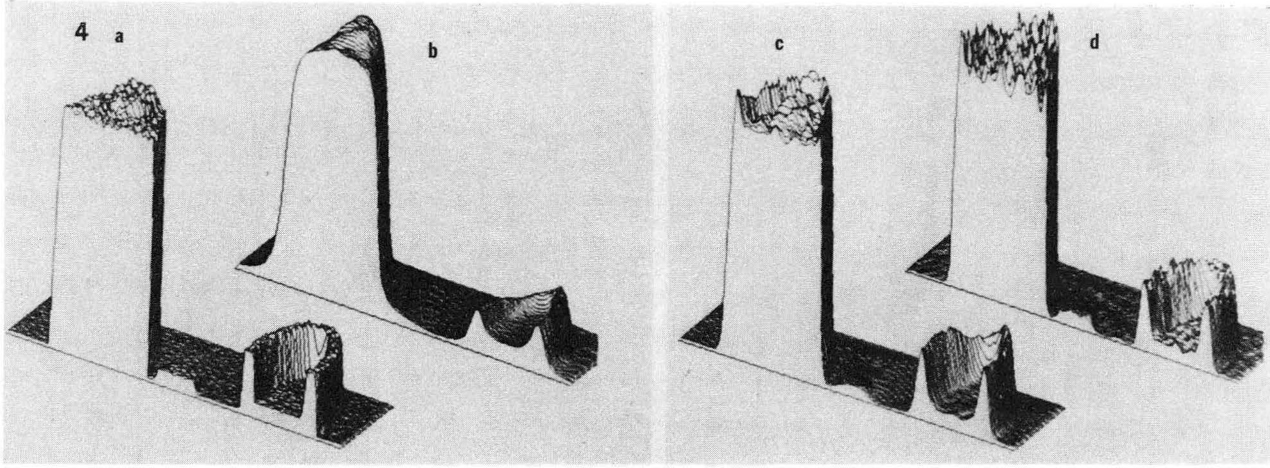


XBB 860-10552A

Fig. 3 a) Cut through a line in the source image of 8 million counts. b) ditto for reconstruction after 9 iterations. c) ditto after 45 iterations. d) ditto after 200 iterations.

Effect of Input Data Discretization

The question arises of whether the observed image deterioration could be the result of discretizing the image before calculating the projection data. Comments regarding this possibility have been made by Herman et al in a written comment to a paper by Vardi, Shepp and Kaufman[10]. The comment of Herman et al is based on experience reported by Gray and co-workers[11]. A test of that possibility should be carried out by fabricating an appropriate phantom and making measurements with a true detector ring. Although we intend to do that in the near future, we have conducted a preliminary test by discretizing one image (2M counts) in both 64x64 pixels and 256x256 pixels using the same sequence of random numbers to generate event coordinates in the imaging plane. We then generated matrices of p(b,d) values with the appropriate 4096 and 65536 columns, respectively and used those matrices to generate two different sets of projection data. Reconstructions of the two sets of data have been carried out using only the 64x64 matrix of p(b,d) values. The results of the two reconstructions, carried out to 200 iterations each, are basically indistinguishable. We conclude, at this time, that the degree of discretization of the input data does not affect the main results reported in this paper.



XBB 860-10553A

Fig. 4 a) Cut through a line in the source image of 32 million counts. b) ditto for reconstruction after 9 iterations. c) ditto after 45 iterations. d) ditto after 200 iterations.

Analysis of the Reconstructed Images

Equation 4 has been used to calculate projections $n^{*i}(d)$ starting from the reconstructed images $\lambda^i(b)$. The probabilities p^i of Eq. 5 (converted to sum of logs) have been calculated next using the input projection data $n^*(d)$ and the projections of Eq. 4. Figures 5a-c show the total probabilities obtained for the source images (horizontal line) and for the reconstructions at different number of iterations for the 2M, 8M and 32M cases. It is consistently found, as expected, that the probability that the image obtained at iteration i would have given the input projection data $n^*(d)$ in a measurement increases with iteration number. This indicates that the algorithm is working correctly and that the deteriorated images are more likely to have given the input projection data than the images with better appearance. It is interesting to note that the probability numbers for the source images are not the highest, although they approach that condition as the number of counts increases. This fact indicates that the source image is not the most likely distribution to have yielded the projection data in a measurement.

Figures 6a-c show the root mean square error of Eq. 6 for the 2M, 8M and 32M cases, respectively, as a function of iteration number. As above, the horizontal lines correspond to the data for the source images. Evidently, the mean square error decreases as the iteration number increases, although the algorithm does not seek to minimize that functional. As in the case of the probabilities of Fig. 5, the mean square error of the source image approaches the values corresponding to higher iteration numbers when the number of counts in the data increases.

Discussion of Results

Once it has become evident that the likelihood of the converging images is increasing while the quality of the image deteriorates beyond acceptability, the question arises as to why the "assymptotic" images are so different from the source images except with an increasing number of counts in the projection data. A point to be examined is the one of discretization of the image plane in the reconstruction phase. An interesting discussion between Vardi, Shepp and Kaufman, on one side, and Herman, Censor, Gordon and Lewitt, on the other, has been reported[10].

We wish first to examine whether condition c of the Theorem in Sec. 2.1 of Ref. 10 is fulfilled. The condition states that the maximum of the likelihood function is unique if and only if the grid of the image plane is such that the D vectors

$$[p(1,d), \ldots p(B,d)], \quad d = 1, \ldots, D \tag{7}$$

span E_B, the B-dimensional Euclidean space. The implication of this condition, discussed in Sec. 2.2 of Ref. 10, is that if the total number of pixels (boxes) B is greater than the number of detector tubes D, then the MLE is certainly non-unique.

For the reconstructions described in this paper, the number of tubes carrying information is approx D = 32,000. The 128x128 reconstructions shown correspond to a number B = 16,384. We would expect that B is sufficiently smaller than D to insure uniqueness. To be on the safe side, we have also carried out reconstructions in a plane of 64x64 pixels, with a value of B = 4096. The same kind of image deterioration was also observed in these images. We feel it is safe to state that the image deterioration is not due to non-uniqueness of the maximum of the likelihood functions.

We have also reconstructed the same kind of images in a 256 x 256 plane, with a number B = 65,536, clearly in violation of condition c. The images were very similar to those reported above at all the stages of the reconstruction. This may indicate that, even if the maximum of the MLE is not unique, the different maxima are not very different from each other. Image deterioration was qualitatively similar to that of the 128x128 or 64x64 reconstructions. This fact appears to support the idea that discretization of the image plane before developing the algorithm does not cause the image deterioration observed. The reconstructions at 256x256 have a discretization which is 16 times finer than those at 64x64 and they behave alike. This point, however, deserves more scrutiny as it has some definite relationship with the discussion that follows.

In our previous examination of the behavior of the MLE algorithm in simpler imaging situations[8] we came to two principal conclusions which can be rephrased here:

1) The Poisson-based MLE algorithm will generate an image which favors the matching of tubes in the projection data containing few counts at the expense of those containing many counts. This is due to the fact that the derivative of the Poisson function with respect to its mean decreases as the mean increases, for a number of experimental counts in the vicinity of the mean. Thus, more likelihood is gained by accurately matching tubes with low counts than by matching tubes with high counts.
2) There is a "pivot" effect at the edges of regions of high activity brought about by small p(b,d)'s linking the discrete pixels and tubes at those edges. Small statistical errors in some tubes at the edge can cause overshoots in the interior of the high activity region which do not get corrected in the iterative process because of the bias of the Poisson function, as indicated above.

The justification for these arguments appears in Ref. 8 for simpler image reconstruction problems. We feel that the arguments can be applied to tomographic reconstruction; we are proceeding with an attempt to prove them for the latter case.

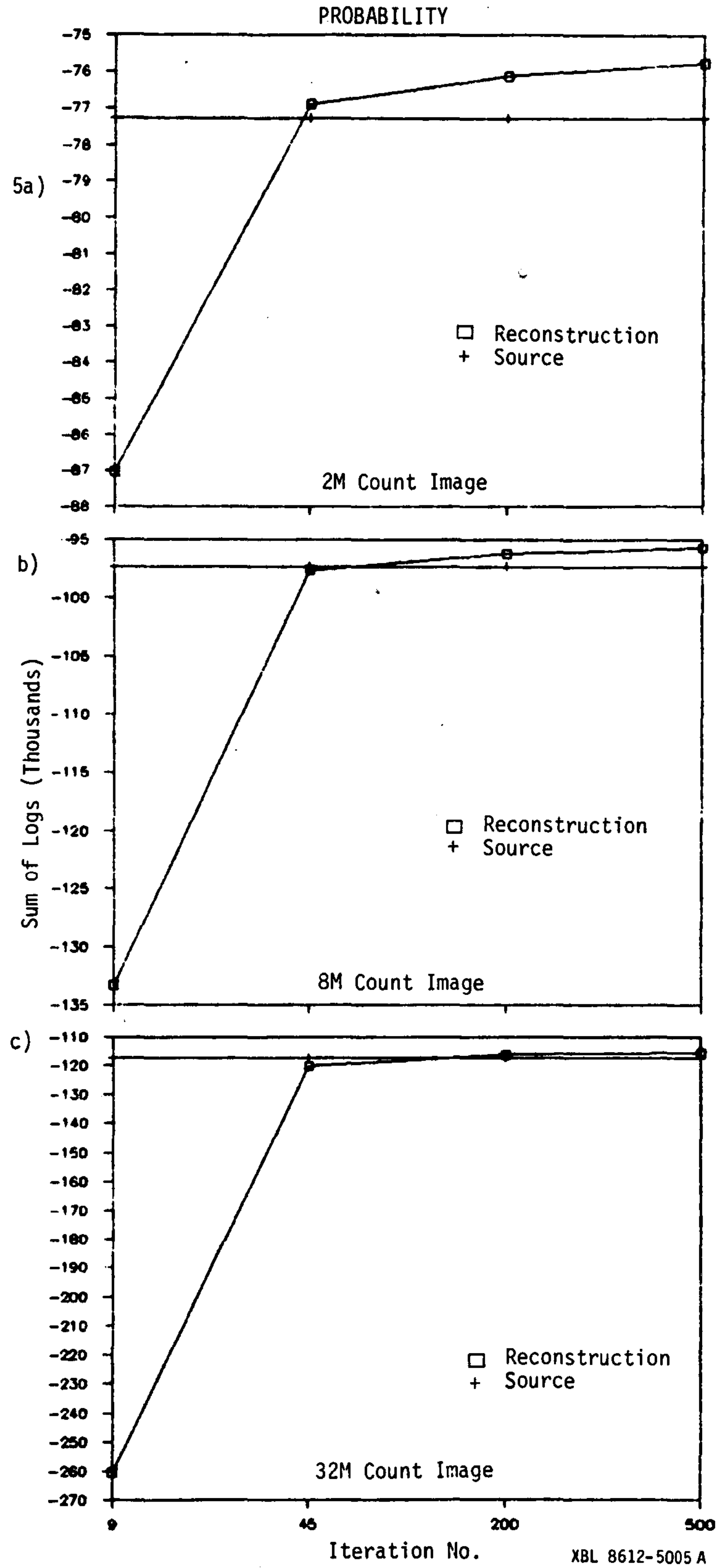


Fig. 5 a) Probability that the image obtained at a certain no. of iterations has yielded the particular set of projection data used as input for the reconstruction; horizontal line indicates the probability for the source image that truly generated the input data. Case with 2 million counts in the source image. b) Case with 8 million counts. c) Case with 32 million counts.

Conclusion

In trying to draw a conclusion from the observations reported in this paper, it is interesting to begin by considering that in the case of a very large number of counts in the projection data, the final images obtained by the MLE and an iterative minimum least squares error algorithm, for example, will be the same. Both functionals are optimized in the limit of very large number of counts. In practical imaging situations with a limited number of counts, however, one can expect different results with different algorithms. It appears that the MLE algorithm does not take very kindly to the statistical fluctuations in the projection data specifically by the effects registered in the uniform regions of high activity in the source. It is very well behaved in the regions of zero or near zero activity since projection tubes with few counts will be matched very well by the algorithm. We are continuing work on the characterization of the MLE reconstructions and a search for a possible functional that may overcome some of the observed difficulties.

Acknowledgments

The idea of looking at the source images and comparing them with the reconstructed results was suggested to one author (J. Llacer) by L.A. Shepp. This work was supported in part by the National Cancer Institute (CA-39501) and the U.S. Department of Energy under Contract No. DE-AC03-76SF00098.

References

1. L.A. Shepp and Y. Vardi, "Maximum Likelihood Reconstruction for Emission Tomography", IEEE Trans. Med. Imaging, MI-1, No. 2, 113 (1982).

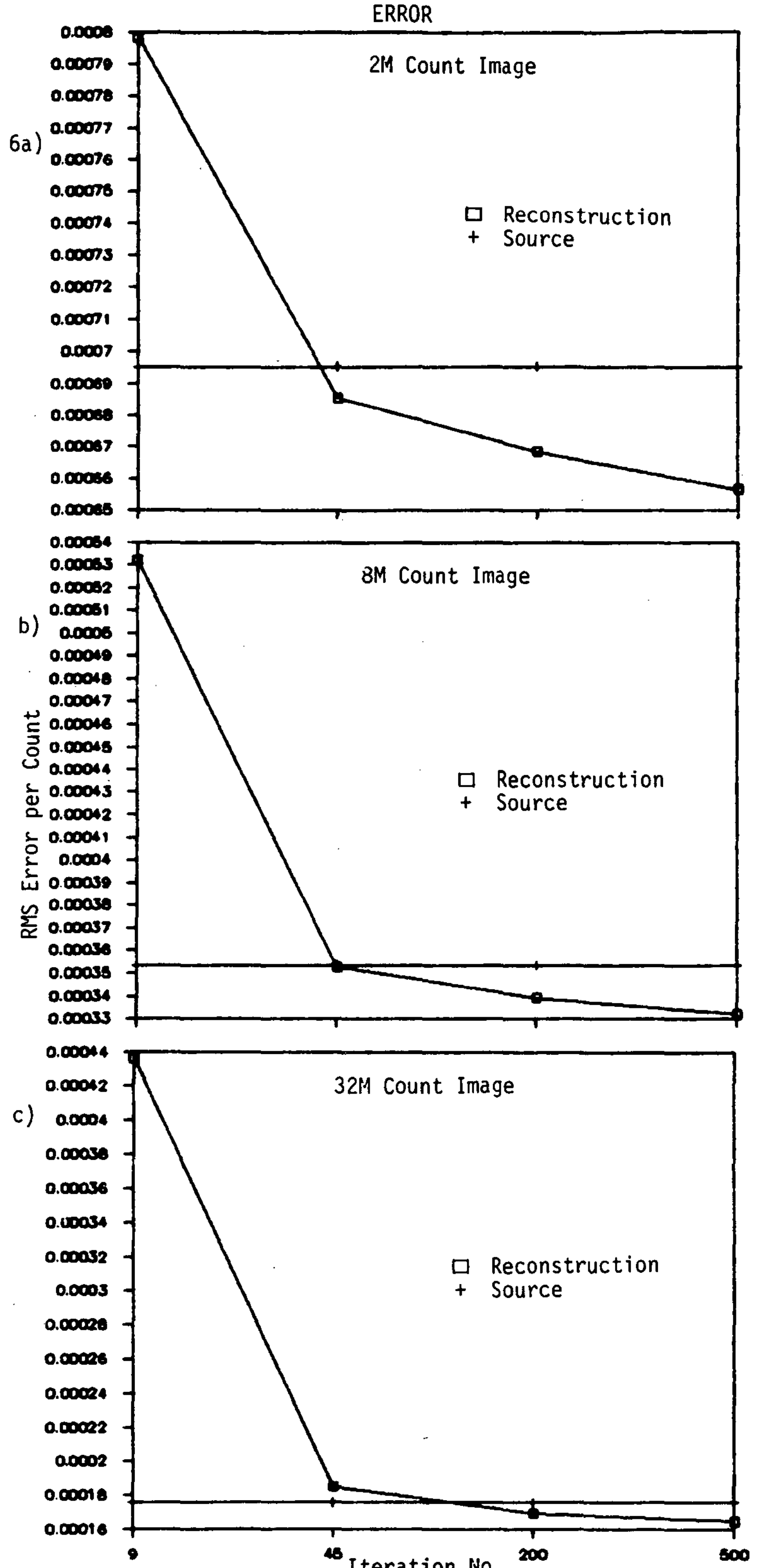
2. L.A. Shepp et al, "Maximum Likelihood PET with Real Data", IEEE Trans. Nucl. Sci., NS-31, No. 2, 910 (1984)

3. D.L. Snyder and D.G. Politte, "Image Reconstruction from List-Mode Data in an Emission Tomography System Having Time-of-Flight Measurements", IEEE Trans. Nucl. Sci., NS-30, No. 3, 1843 (1983).

4. C.E. Floyd, R. Jaszczak and R.E. Coleman, "Inverse Monte Carlo: a Unified Reconstruction Algorithm for SPECT", IEEE Trans Nucl Sci, NS-32, No. 1, 779 (1985).

5. J. Llacer, S. Andreae, E. Veklerov and E.J. Hoffman, "Towards a Practical Implementation of the MLE Algorithm for Positron Emission Tomography", IEEE Trans. Nucl. Sci., NS-33, No. 1, 468 (1986).

6. D.L. Snyder and M.I. Miller, "The Use of Sieves to Stabilize Images Produced with the EM Algorithm for Emission Tomography" IEEE Trans. Nucl. Sci., NS-32, No. 5, 3864 (1985).



XBL 8612-5006A

Fig. 6 a) Mean square error between the projections obtained from the image at certain number of iterations and the particular set of projection data used as input for the reconstruction. The horizontal line indicates the mean square error for the source image. Case with 2 million counts. b) Case with 8 million counts. c) Case with 32 million counts.

7. E. Tanaka and M. Yamamoto, "Utilization of Non-Negativity Constraints in Convolutional Reconstruction of Emission Tomograms", Proc. 9th Inter'l. Conf. on Information Processing in Medical Imaging, Washington, D.C. (1985). Martinus Nijhoff Pub., Dordrecht, The Netherlands, 379.

8. J. Llacer, S. Andreae and A. Chatterjee, "On the Applicability of the Maximum Likelihood Estimator Algorithm for Image Recovery in Accelerated Positron Emitter Beam Injection", SPIE, 671, 269 (1986).

9. E.J. Hoffman, A.R. Ricci, L.M.A.M. Van der Stee and M.E. Phelps, "ECAT-III, Basic Design Considerations", IEEE Trans. Nucl. Sci., NS-30, No. 1, 729 (1983).

10. Y. Vardi, L.A. Shepp and L. Kaufman, "A Statistical Model for Positron Emission Tomography", Journal of the American Statistical Association, 80, No. 389, 8 (1985).

11. J.E. Gray, D.P. Hanson, G.T. Herman, R.M. Lewitt, R.A. Reynolds, R.A. Robb, B. Smith and H. Tuy, "Reconstruction Algorithms for Dose Reduction in X-Ray Computed Tomography", Medical Imaging Processing Group Technical Report No. MIPG63, U. of Pennsylvania, Dept. of Radiology (1982).

LAWRENCE BERKELEY LABORATORY
TECHNICAL INFORMATION DEPARTMENT
UNIVERSITY OF CALIFORNIA
BERKELEY, CALIFORNIA 94720