Title
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Journal
Physical Mesomechanics, 6

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Publication Date
2001-09-20

LBNL-48943
Preprint

ERNEST ORLANDO LAWRENCE BERKELEY NATIONAL LABORATORY

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September 2001
To be submitted for publication

DISCLAIMER

LBNL-48943

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This work was supported by the Director, Office of Science, Office of Advanced Scientific Computing Research, Mathematical, Information, and Computational Sciences Division, of the U.S. Department of Energy under Contract No. DE-AC03-76SF00098.

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DAMAGE ACCUMULATION: A NON-LOCAL MODEL

ABSTRACT. A concise discussion is presented of a new non-local formulation developed in the last few years of the damage accumulation problem . According to the new approach the classical L. M. Kachanov formulation is modified, taking into account the microinhomogeneity of the material. It is shown that the microinhomogeneity leads to a peculiar nonlinear diffusion-like damage-extension process. This process changes the mathematical formulation of the damage accumulation problem making it non-local, and leading to a nonlinear parabolic differential equation for the damage (or integrity), contrary to the ordinary differential equation which appears in the classical formulation. The main feature of the process according to new formulation is that the localized damage at first extends to the non-damaged part of the structure, and only after such extension begins to grow quickly and rather uniformly. The extension time occupies an essential part of the lifetime of the damaged structure. As in the classical formulation the fracture corresponds to the end of the existence of the solution of the equation for the damage. If the extension of the damaged part of the structure occupies an essential part of the lifetime, the critical value of damage becomes insignificant.

Key Words: damage accumulation, integrity, non-local theories, blow-up, fracture criteria, continuous damage mechanics

1. INTRODUCTION

The papers of L. M. Kachanov (1958, 1961) opened a new branch of fracture mechanics: *continuum damage mechanics.* Following the paper by Odqvist and Hult (1964), and the book of Odqvist (1966), continuum damage mechanics has developed and found many important applications. Ultimately it reached its modern shape which got a comprehensive exposition in particular in the illuminating books of Kachanov (1986), Lemaitre (1992), and Krajcinovic (1996).

The basis quantity for this theory is the *damage* factor ω or, what was introduced by L. M. Kachanov himself, the *integrity* factor $\psi = 1 - \omega$. (We will also use a related quantity $\phi = 1/\psi$ – the *overloading* factor.) For the damage growth rate a *kinetic equation* is proposed, which we will write in the form

1.1 $$\frac{d\omega}{dt} = \frac{1}{\tau} f(\sigma, \omega, T),$$

where σ is the actual stress in load-supporting elements, the material constant τ is a characteristic time, t is time, and T the absolute temperature. Later we will consider the isothermic processes only. The actual stress σ and the damage parameter ω are related in the classical model by a finite equation

1.2 $$\sigma = \frac{\sigma_0}{\psi} = \frac{\sigma_0}{1-\omega}$$

where σ_0 is the stress in the pristine material at $\omega = 0$.

Under prescribed load variation $\sigma_0(t)$, in particular at $\sigma_0 =$ const, equation (1.1) becomes an ordinary differential equation for the function $\omega(t)$. Generally speaking the solution to this equation satisfying the initial condition $\omega(0) = 0$ exists in a finite time interval $0 \leq t < t_0$ only. At $t = t_0$ the function $\omega(t)$ becomes equal to 1, the stress σ goes to infinity; for $t > t_0$ the solution does not exist. Namely, reaching of this ultimate state is identified with fracture, and the time t_0 with the lifetime of the structure.

An alternative model of damage accumulation was proposed more recently (Barenblatt, 1993, Barenblatt and Prostokishin, 1993). This model was based on the observation that a peculiar nonlinear diffusion-like process of the damage transfer appears due to microinhomogeneity of the material, both inherent due to the technology of material production and induced by micro-defects which appear in the process of damage accumulation. Due to this effect the ordinary differential equation (1.1) is replaced by a nonlinear parabolic differential equation of the form

1.3 $$\partial_t \omega = \frac{1}{\tau} f(\sigma, \omega, T) + \frac{\lambda^2}{\tau} \partial_x (D(\omega) \partial_x \omega).$$

Here D is an essentially positive dimensionless function of its argument, τ is a characteristic time, and λ is a characteristic length scale dictated by the material microinhomogeneity, and perhaps also by the local damage. If this length scale is small, equation (1.3) reduces to the ordinary differential equation and we return to the classic theory.

The solution to equation (1.3) satisfying a certain initial condition $u(x, 0) = u_0(x)$ (initial damage distribution along the bar) also exists for a finite time interval $0 \leq t < t_0$ only, at $t = t_0$ there is a "blow-up", a phenomenon well known to mathematicians. In the referenced papers of the present author the proposed model was presented rather as a mathematical possibility needed to modify the classical model, replacing it with a non-local one. However, the recent experimental works of V. C. Li (see Li and Maalej (1996), Li (1997)) confirmed qualitatively the basic features of the new model for the the damage accumulation in fiber-reinforced concrete. The mathematical paper by Bertsch and Bisegna (1997) was very important because it was shown there that the new model is well-posed mathematically, at least for a certain important class of kinetic equations. These works stimulated the author to present the model for discussion in the solid mechanics community.

2. DISCUSSION OF THE CLASSICAL MODEL

We restrict ourselves to the one-dimensional model: i.e. we consider a prismatic bar under static tension and constant temperature. According to our impression the construction of a non-formal, rich in content theory for a multidimensional case requires much more experimental information than is available now and is therefore premature. We will use three related quantities: Kachanov's original *integrity* factor ψ, the *damage* factor $\omega = 1 - \psi$, and the *overloading* factor $\phi = 1/\psi$. The basic kinetic equation traditionally is written in the form

2.1 $$\partial_t \omega = \frac{1}{\tau} \, q(\omega, \sigma, T).$$

As before, σ is the average actual stress in the load supporting elements of the material. The process is assumed, as a rule, tacitly to be homogeneous along the bar. The function q is assumed to be a universal function for a given material. As examples of the function q we will consider, following the literature, the Arrhenius-type kinetic law with stress dependent activation energy

2.2 $$q = \psi^p \exp\left(-\frac{U - \gamma\sigma}{kT}\right)$$

and the power law

2.3 $$q = \psi^p \exp\left(\frac{\sigma}{\sigma_1}\right)^n$$

Here k is the Boltzmann universal constant; p ("reaction order"), n, γ, U and σ_1 are kinetic constants which are universal for a given material. Parameter n is assumed to be large, $n >> 1$, as well as dimensionless parameters

U/kT, $\gamma\sigma_0/kT$. This means that stress σ dependence of the damage accumulation rate is much stronger than direct integrity ψ (damage ω) dependence which is governed by the pre-exponential factor ψ^p: usually the "reaction order" p is of the order of one. The simplest interpretation of the integrity factor used in the literature is the ratio of the non-destroyed part of the cross-section area S_n to the bulk area S, and the stress σ is related to ψ, ω and ϕ as

2.4 $$\sigma = \frac{\sigma_0}{\psi} = \frac{\sigma_0}{1-\omega} = \sigma_0\phi \, .$$

Let us consider the example of the power law kinetic equation (2.3). From (2.1), (2.3), (2.4) we obtain

2.5 $$\frac{d\psi}{\psi^{p-n}} = -\frac{1}{\tau}\frac{\sigma_0^n}{\sigma_1^n}\,dt \, , \qquad \frac{p-n+1}{\tau}\left(\frac{\sigma_0}{\sigma_1}\right)^n t = \psi_0^{-p+n+1} - \psi^{-p+n+1}$$

where ψ_0 is the value of the integrity factor at $t=0$, and the value of the integrity factor at t is equal to

2.6 $$\psi = \psi_0\left[1-\left(\frac{\sigma_0}{\sigma_1}\right)^n \cdot \frac{t}{\tau} \cdot \frac{p-n+1}{\psi_0^{-p+n+1}}\right]^{n+1-p}$$

The lifetime according to the classical model corresponds to reaching the state when $\psi = 0$:

2.7 $$t_0 = \psi_0^{n+1-p}\frac{\tau}{p-n+1}\left(\frac{\sigma_0}{\sigma_1}\right)^n \, .$$

The constant n is very large, of the order of 10, therefore the lifetime according to the classical theory strongly depends on the initial value of the integrity. However the integrity ψ (and the damage ω) have a rather vague definition. There is no indication of a procedure which would allow determining these quantities at least tentatively. The physical definition of the integrity is the properly averaged fraction of non-broken bonds in the material near a given point. This definition, however, does not give a definite macroscopic procedure for the integrity measurement. So, the customer, purchasing some structure having even the integrity 0.9, seemingly close to one, does not know what the initial integrity is, and will be disappointed to learn that its lifetime will be three times less than s/he expected.

The next point worth mentioning in this discussion is that according to the classical model the process of the damage accumulation in a prismatic bar goes independently in all its cross-sections. Therefore the lifetime is entirely determined by the cross-section having the largest damage: in the classical model there is no mechanism of transfer of damage from a cross-section to its neighbors.

3. THE NON-LOCAL MODEL

Real materials — metallic alloys, polymers, composites, in particular concretes, especially fiber-reinforced concretes, possess inhomogeneous microstructure. In metals, the microstructure is formed by grains and intercrystalline matter, while in polymers there always exist a supramolecular microstructure of various orders. Moreover, there exist in the materials some aggregates of coherent microstructural elements, i.e. clusters of elements whose size is larger than the size of a single microstructural element, as exemplified by the clusters of microstructural elements joined by stronger bonds. In particular, in the fiber-reinforced concrete the clusters are formed by the microstructural elements attached to a fiber.

Damage accumulation consists of breaking bonds between the elements of microstructure and/or inside these elements. An instructive example of such a process is cavitation and micro-crack formation at grain boundary facets as studied by V.Tvergaard and E. Van der Giessen (Tvergaard (1991), Van der Giessen and Tvergaard (1991)). Due to microinhomogeneity (different sizes of grains and clusters, variation of their properties, etc.), the damage accumulation process will be microinhomogeneous. In particular, the formation of cracks at the boundaries of clusters increases the local stresses induced by the action of an applied tensile load, and so increases the local damage accumulation rate. Conversely, the local inhibition of crack formation in stronger aggregates and clusters stiffened by fibers reduces the local damage accumulation rate. Therefore, the *local* integrity or damage values across any cross-sections of the bar are stochastic quantities. We assume the bar cross-section to be sufficiently large, so that it can be considered as a representative sampling. We will operate with damage (integrity) averaged in the statistical sense, i.e. over the ensemble.

This statistically averaged damage or integrity can be non-uniformly distributed along the bar axis x and time dependent:

$$\omega = \omega(x,t), \qquad \psi = \psi(x,t), \qquad \phi = \phi(x,t).$$

We assume that locally at every point of the cross-section the kinetic equation (2.1) is valid. Now let's average the relation (2.1) over the cross-section area: due to the ergodicity hypothesis, which is plausible under the conditions considered, statistical averaging can be replaced by averaging over the cross-section area S. We obtain for the integrity ψ the equation

3.1
$$\partial_t \psi = -\frac{1}{\tau}\,\frac{1}{S}\int_S q(\psi,\sigma,T)ds\ .$$

(The cross-section area is also denoted by S.) For statistical averaging the time differentiation and the averaging are interchangeable, therefore we replaced averaged $\partial_t \psi$ by the time derivative of the averaged integrity. However, the function $q(\psi, \sigma, T)$ is a strongly nonlinear function of σ, so that we have no right to replace the integral in the right-hand side of (3.1) by the value of q, corresponding to the average value of σ in the present cross-section, if the damage is inhomogeneous along the x-axis. Indeed, in this case the microstresses, influencing the damage accumulation rate in a given section are different on both sides of it.

To calculate the right-hand side of (3.1) the following artifice will be used: it will be replaced by an integral over the bar axis by introducing the weight function $\rho(\xi - x)$:

3.2 $$\frac{1}{S}\int_S q(\psi, \sigma, T)ds = \int_{-\infty}^{\infty} q(\psi(x), \sigma(\xi, t), T)\rho(\xi - x)d\xi \, .$$

The physical sense of the weight function is that it determines the relative amount of the elements of the cross-sectional area, where the damage accumulation rate corresponds to average stress in a neighboring cross-section having a certain "shifted" coordinate ξ: this can always be done. Here the fact is used that the average damage (integrity), and consequently the average actual stress are continuously varying over the bar length, so the actual stress at a certain point of the cross-section, increased or reduced by the influence of microstructure, corresponds to the average stress in a certain neighboring cross-section. In fact, the replacement (3.2) is the introduction of non-locality.

We emphasize that it was assumed that actual stress dependence of the damage accumulation rate is much stronger than direct damage dependence (see (2.2), (2.3)), therefore in (3.2) we left the integrity ψ untouched by non-locality: $\psi = \psi(x, t)$. Furthermore, under the assumption of strong actual stress-dependence of damage accumulation rate, the weight-function is rapidly decreasing with growing modulus of its argument. Therefore the limits in the integral over the bar axis on the right-hand side of (3.2) can be taken as infinite.

The weight function $\rho(\xi - x)$ satisfies the relation

3.3 $$\int_{-\infty}^{\infty} \rho(\xi - x)d\xi = 1$$

which follows from the relation (3.2) for an ideally homogeneous case when ω and σ are constant over the cross-section and along the bar length. In this case the classical model of the damage accumulation is obtained. We introduce furthermore two length scales ℓ and λ according to the relations

3.4 $$\ell = \int_{-\infty}^{\infty} \rho(\xi - x)(\xi - x)d\xi$$

(the length scale ℓ is equal to zero if the weight function is symmetric), and

3.5 $$\lambda^2 = \tfrac{1}{2}\int_{-\infty}^{\infty} (\xi - x)^2 \rho(\xi - x)d\xi \, .$$

The length scale λ gives a characteristic size of clusters, aggregates of coherent microstructural elements mentioned above.In principle both length scales ℓ and λ can depend on the current integrity ψ.

Now expand the function $q(\omega(x,t),\sigma(\xi,t),T)$ under the integral sign in (3.2) by a Taylor series near $\xi = x$:

3.6

$$q(\psi(x),\sigma(\xi,t),T) = q(\psi(x,t),\sigma(x,t),T)+(\xi-x)(\partial_\xi q)_{\xi=x}+\tfrac{1}{2}(\xi-x)^2(\partial^2_{\xi\xi}q)_{\xi=x}+\dots$$

and neglect (a strong hypothesis!) the remaining terms. Substitute (3.6) into (3.9) and (3.1) to obtain

3.7 $$\partial_t\psi = -\frac{1}{\tau}q(\psi(x,t),\sigma(x,t),T) - \left(\frac{\ell}{\tau}\,\partial_\sigma q\right)\partial_x\sigma - \partial_x\left\{\left[\frac{\lambda^2}{\tau}\,\partial_\sigma q\right]\partial_x\sigma\right\}.$$

We remind the reader that here we use the non-constructive definition of the integrity ψ: the average relative amount of unbroken bounds in the vicinity of a given point. Now we need a relation between ψ and actual stress σ. In fact, the simplest version of such a connection is one used in the classical model — the relation (2.4). It is not necessary: the arbitrary relation

3.8 $$\frac{\sigma}{\sigma_0} = g(\psi), \qquad \partial_\psi\sigma = -\sigma_0 h(\psi)$$

where $g(\psi)$ is a monotonically decaying function such that $g(0)=\infty$, $g(1)=1$ is appropriate, but a certain relation of the type (3.8) should be assumed. From (3.7) and (3.8) we obtain a nonlinear parabolic equation for the integrity ψ,

3.9 $$\partial_t\psi = -\frac{1}{\tau}q(\psi,\sigma,T) + \frac{\ell}{\tau}(\sigma_0\partial_\sigma q)h(\psi)\partial_x\psi + \partial_x\left[\frac{\lambda^2}{\tau}\sigma_0\partial_\sigma q h(\psi)\right]\partial_x\psi\,.$$

Now we assume for the sake of simplicity only that the function $\rho(\xi - x)$ is a symmetric one, so that the length scale ℓ is equal to zero. The qualitative properties of the model under this assumption remain untouched, and we do not yet have sufficient experimental information which allows us to confirm or to disprove this assumption. Also it is convenient at this stage to pass to the new variable $\phi = 1/\psi$ — the overloading. In assumption (2.4) the actual stress σ is proportional to the overloading: $\sigma = \sigma_0\phi$. We obtain (using for definiteness sake the assumption $\sigma = \sigma_0\phi$) the equation for the overloading ϕ

3.10 $$\partial_t\phi = \phi^2\frac{1}{\tau}\,q\left[\frac{1}{\phi},\sigma_0\phi,T\right] + \phi^2\partial_x\left[\left(\frac{\lambda^2}{\tau}\,\sigma_0\partial_\sigma q\right)\partial_x\phi\right].$$

Obviously the damage (overloading) under ordinary conditions cannot heal: therefore equation (3.10) is valid only if the right-hand side is positive. Otherwise

3.11 $$\partial_t\omega = \partial_t\phi = 0\,.$$

We specify equation (3.10) for the case of Arrhenius-type kinetics (2.2) and $p = 1$

3.12
$$\partial_t \phi = \phi e^{\mu\phi} + \varepsilon\mu\phi^2 \partial_\xi \left[\frac{1}{\phi} e^{\mu\phi} \partial_\xi \phi \right] .$$

Here the following dimensionless parameters and variables are introduced

3.13
$$\theta = \mu(\frac{t}{\tau}) \exp\left(-\frac{U}{kT}\right) , \qquad \mu = \frac{\gamma\sigma_0}{kT} , \qquad \xi = \frac{x}{L} , \qquad \varepsilon = \frac{\lambda^2}{L^2}$$

where L is a characteristic length size. For the case of Arrhenius kinetics and $p = 0$ the equation (3.10) assumes the form

3.14
$$\partial_\theta \phi = \phi^2 e^{\mu\phi} + \varepsilon\phi^2 \partial^2_{\xi\xi} e^{\mu\phi} .$$

For the case of power-type kinetics (2.3) and $p = 1$ the equation (3.10) assumes the form

3.15
$$\partial_\theta \phi = \phi^{n+1} + \left(\frac{\varepsilon n}{n-1}\right) \phi^2 \partial^2_{\xi\xi} \phi^{n-1} .$$

Here $\theta = t/\tau$. It is instructive that for these equations there exists the phenomenon of blow-up: in a finite time the solutions of the initial-value problem cease to exist (see Bertsch and Bisegna (1997)).

4. MATHEMATICAL FORMULATION. QUALITATIVE PROPERTIES OF THE SOLUTIONS

Summing up, for the non-local model the following mathematical formulation is obtained:

4.1
$$\partial_\theta \phi = \begin{cases} I(\phi) & \text{if } I(\phi) \geq 0 \\ 0 & \text{if } I(\phi) < 0 \end{cases}$$

Here the operator $I(\phi)$ has the form, for the Arrhenius-type kinetics (2.2), $p = 1$:

4.2
$$I(\phi) = \phi \; e^{\mu\phi} + \varepsilon\mu\phi^2 \partial_\xi \left(\frac{1}{\phi} \; \partial_\xi \; e^{\mu\phi} \right)$$

for the Arrhenius-type kinetics and $p = 0$:

4.3
$$I(\phi) = \phi^2 \; e^{\mu\phi} + \varepsilon\phi^2 \partial_{\xi\xi} e^{\mu\phi}$$

for the power law kinetics (2.3),

4.4 $$I(\phi) = \phi^{n+1} + \varepsilon \frac{n}{n-1} \phi^2 \partial^2_{\xi\xi} \phi^{n-1} .$$

Initial conditions correspond to a certain distribution of overloading (inverse integrity) over the bar length

4.5 $$\phi(\xi, 0) = \phi_0(\xi)$$

where $\phi_0(\xi)$ is a function larger or equal to 1.

Here $\varepsilon = \lambda^2/L^2$ is a basic dimensionless parameter of the problem. If $\varepsilon = 0$ we return to the classical model, see Section 2, where the damage accumulation is described by a simple ordinary differential equation. The formulated problem for localized initial damage was investigated numerically (Barenblatt and Prostokishin (1993)) and analytically (Bertsch and Bisegna (1997)). It was shown that at very small ε the evolution of damage (overloading) is similar to the classical model: the overloading grows practically independently in all damaged sections, and the damaged part of the bar does not extend. When the parameter ε is not very small, the situation becomes drastically different. The process of the damage accumulation in this case consists of two stages. At the first stage the damaged region extends along the bar, but the damage in all sections does not exceed the maximal initial damage. This stage occupies a basic part of the process. At the second stage the extension of the damaged region of the bar is stopped, the damage grows in the whole damaged region which was reached during the first stage, and after a rather small time the solution blows up. This blow-up is natural to identify with fracture.

It is remarkable that in the experiments of V. C. Li (Li and Maaley (1996), Li (1997)) with damaged specimens of fiber-reinforced concrete, similar evolution of the damaged region around local initial notch was observed. At first the damaged region was extending: it was clearly seen on the side surface of the specimen. The boundary of the damaged region at first was curved, but soon became plane, normal to the axis of the bar. The failure of the specimen followed a substantial extension of the damaged region where the damage is more or less uniformly distributed.

5. CONCLUSION

A new non-local model of the damage accumulation is proposed, which generalizes the classical model of continuous damage mechanics. Subsequent numerical, analytical and direct experimental investigations supported the new model at least qualitatively. Hopefully, future investigations will supply the model with the flesh of specific information which will allow its use in predicting the lifetime of the structures.

Acknowledgment: This work was supported by the Director, Office of Science, Office of Advanced Scientific Computing Research, Mathematical, Information, and Computational Sciences Division of the U.S. Department of Energy under Contract No. DE–AC03–76SF00098.

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Prepared for the U.S. Department of Energy under Contract No. DE-AC03-76SF00098