UNIVERSITY OF CALIFORNIA SAN DIEGO

The Chow rings of some moduli spaces of curves and surfaces

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The Dissertation of Samir Canning is approved, and it is acceptable in quality and form for publication on microfilm and electronically.

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DEDICATION

To my parents, Robert and Deval, my grandparents, Anil, Bindu, William, and Edna, and my brother Krishna.

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ABSTRACT OF THE DISSERTATION

The Chow rings of some moduli spaces of curves and surfaces

by

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We study the Chow rings of the Hurwitz spaces parametrizing degree 3, 4, and 5 covers of the projective line, the Chow rings of the moduli spaces of curves of genus 7, 8, and 9, and the Chow rings of moduli spaces of elliptic surfaces. We prove a stabilization result for the Chow rings of the Hurwitz spaces, and completely determine the Chow ring for degree 3 covers. We use these results to compute the Chow rings of the moduli spaces of curves of genus 7, 8, and 9. Then, we compute the Chow rings of moduli spaces of elliptic surfaces. We show that they satisfy a stability property, and that they satisfy vanishing and dimension properties predicted by Oprea–Pandharipande.

Introduction

0.1 Moduli problems and tautological classes

In algebraic geometry, we study algebraic varieties, which are geometric objects defined by polynomial equations. A fascinating phenomenon is that there are algebro-geometric spaces, called moduli spaces, whose points correspond in a natural way to algebraic varieties. Moreover, we can often study these moduli spaces using algebro-geometric techniques. The geometric properties of moduli spaces are interesting by themselves, and they may also shed light on the varieties that the moduli space parametrizes.

The focus of this thesis is the intersection theory of some moduli spaces of curves and surfaces. We begin with some definitions and basic examples to see how moduli theory "tautologically" produces interesting questions in intersection theory.

Definition 0.1.1. Let S be a scheme, F be a contravariant functor from the category of schemes over S to the category of sets. Suppose that M represents F; that is, there is a natural isomorphism

$$F \to \mathrm{Hom}(-, M).$$

We then call F a moduli problem and M the moduli space associated to F.

The natural isomorphism $F \to \mathrm{Hom}(-, M)$ furnishes a universal object $U \to M$, which comes from the point in $F(M)$ corresponding to the identity map in $\mathrm{Hom}(M, M)$. In general, we think of $F(T)$ as the set of "families" of certain types of schemes over T. The universal object $U \to M$ is the universal family, and every family over T is pulled back from the universal family via a morphism $T \to M$.

Example 0.1.2. Consider the functor F from schemes over $\operatorname{Spec}\mathbb{C}$ to the category of sets sending a scheme T to the set of isomorphism classes of quotients

$$\mathcal{O}_T^{\oplus n} \to Q,$$

where Q is a locally free sheaf of rank $n-k$ on T. The functor F is representable by the Grassmannian $G(k,n)$, which we think of as parametrizing k-dimensional subspaces (equivalently $n-k$-dimensional quotient spaces) of an n-dimensional vector space. The universal object over $G(k,n)$ is the tautological sequence

$$0 \to \mathcal{S} \to \mathcal{O}_{G(k,n)}^{\oplus n} \to \mathcal{Q} \to 0, \tag{0.1.1}$$

where $\mathcal{S}$ is locally free of rank k and $\mathcal{Q}$ is locally free of rank $n-k$.

Mumford observed that moduli problems F give rise to interesting objects in the intersection theory of the moduli space M [Mum83]. In his words,

> *Whenever a variety or topological space is defined by some universal property, one expects that by virtue of its defining property, it possesses certain cohomology classes called tautological classes.*

Example 0.1.3. The cohomology and Chow ring of the Grassmannian have the Chern classes of the tautological bundles $c_i(\mathcal{S})$ and $c_j(\mathcal{Q})$. From the tautological exact sequence (0.1.1), we obtain the relation

$$c(\mathcal{S})c(\mathcal{Q}) = 1. \tag{0.1.2}$$

In fact, the Chow and cohomology rings of the Grassmannian $G(k,n)$ are generated by the classes $c_i(\mathcal{S})$, and relations can be obtained from equation (0.1.2) by noting that $c_j(\mathcal{Q})$ vanishes for $j \geq n-k+1$. See [EH16, Theorem 5.26] for a complete description of the Chow ring of $G(k,n)$.

With the Grassmannian as the guiding example, Mumford [Mum83] defined the tautological ring of the moduli space of curves $\mathcal{M}_g$ as a subring of the Chow ring. The tautological ring of $\mathcal{M}_g$ will play a central role in this thesis, as will the tautological rings of a few other moduli spaces. In order to define these rings, we will use equivariant intersection theory.

0.2 Equivariant intersection theory

Unfortunately (or not, depending on your point of view), the moduli problems we will deal with in this thesis are almost never representable by schemes. We need to enlarge the spaces we work with to include algebraic stacks. One could work with the associated coarse moduli spaces when they exist, but these are often singular, and it is difficult to study intersection theory on singular varieties. In fact, Mumford ran into this problem when defining the Chow ring of $\mathcal{M}_g$. He showed that the singularities of the coarse moduli variety of $\mathcal{M}_g$ are nice enough so that an intersection product could still be defined, at least with rational coefficients.

Nowadays, there are other approaches to intersection theory on algebraic stacks, and we can avoid any discussion of the coarse moduli spaces. In this thesis, we will use the approach developed by Totaro [Tot99] and Edidin–Graham [EG98] called equivariant intersection theory. Equivariant intersection theory is inspired by equivariant cohomology, which is defined as follows. Let G be a topological group acting on a topological space X. Let BG denote the classifying space for principal G-bundles. It comes equipped with a universal family,

$$\pi : EG \to BG.$$

The space EG is the total space of the universal principal G-bundle, and π is the quotient map for a free action of G on EG. The group G acts diagonally on $EG \times X$. One then defines the equivariant cohomology of X as

$$H_G^*(X, \mathbb{Z}) := H^*(EG \times X/G, \mathbb{Z}).$$

Restricting to the setting where X is a complex algebraic variety and G is an algebraic group, we would like to compare the cohomology of the quotient stack $\mathcal{X} := [X/G]$ with $H_G^*(X, \mathbb{Z})$. The cohomology of a stack $\mathcal{Y}$ over $\operatorname{Spec} \mathbb{C}$ is defined functorially: a cohomology class in $H^*(\mathcal{Y}, \mathbb{Z})$ is the data of a cohomology class $c(t) \in H^*(T, \mathbb{Z})$ for every scheme T over $\mathbb{C}$ and every object $t \in \mathcal{Y}(T)$ satisfying natural compatibility conditions. The comparison between $H^*(\mathcal{X}, \mathbb{Z})$ and $H_G^*(X, \mathbb{Z})$ is difficult, however, because EG and BG are not algebraic varieties. In particular, the G-torsor $X \times EG \to X \times EG/G$ is not an object of the stack $[X/G]$.

Totaro [Tot99] showed, however, that EG and BG can be approximated by algebraic varieties in the following sense. Let V be a representation of G and let $U \subset V$ be an open subset such that G acts freely on U, and such that $\operatorname{codim}(V \smallsetminus U) > i$. Then

$$H^k(U/G) \cong H^k(BG)$$

and

$$H^k(X \times U/G) \cong H_G^k(X).$$

for $k \leq 2i$. Using these approximations, one can show that

$$H^*(\mathcal{X}, \mathbb{Z}) \cong H_G^*(X, \mathbb{Z}).$$

The above isomorphisms are independent of the choice of representation V. This observation motivates the definition of equivariant Chow groups of quotient stacks.

Definition 0.2.1 (Totaro [Tot99], Edidin–Graham [EG98])**.** With notation as above, define

$$\mathrm{CH}_i^G(X) = \mathrm{CH}_{i+\dim(U)-\dim(G)}(X \times U/G).$$

By definition, the equivariant Chow groups of X are the Chow groups of a scheme, so they enjoy the usual properties of Chow groups of schemes. In particular, if X is smooth, then there is an intersection product. We can then define the Chow ring $\mathrm{CH}^*(\mathcal{X})$ to be the equivariant Chow ring $\mathrm{CH}_*^G(X)$.

This theory works with integral coefficients, but it is typically much more difficult to compute with integral coefficients than with rational coefficients. For most of this thesis we will work with $A^*(\mathcal{X}) := \mathrm{CH}^*(\mathcal{X}) \otimes \mathbb{Q}$.

The definition of the Chow ring of a quotient stack as an equivariant Chow ring agrees with other proposed definitions of Chow rings of stacks, including Mumford's definition for so-called Q-varieties [Mum83] and Vistoli's for Deligne–Mumford stacks [Vis89b]. See [EG98] for proofs.

We end this section with an example of the computation of an equivariant Chow ring.

Example 0.2.2. Let $\mathbb{G}_m$ be the multiplicative group over a field k. We compute the Chow ring of the classifying stack $\mathrm{B}\mathbb{G}_\mathrm{m} := [\operatorname{Spec} k/\mathbb{G}_m]$. The scaling action of $\mathbb{G}_m$ on $\mathbb{A}^{n+1}$ is free on the complement of the origin $\mathbb{A}^{n+1} \smallsetminus 0$. The quotient is a familiar variety: $(\mathbb{A}^{n+1} \smallsetminus 0)/\mathbb{G}_m = \mathbb{P}^n$. The Chow ring of $\mathbb{P}^n$ is simply $\mathbb{Z}[c_1]/(c_1^{n+1})$, where c_1 is the first Chern class of the tautological line bundle. Taking the limit as $n \to \infty$, we see that

$$\mathrm{CH}^*(\mathrm{B}\mathbb{G}_\mathrm{m}) = \mathbb{Z}[c_1].$$

This computation demonstrates that, unlike the case of smooth schemes, the Chow rings of stacks can be nonzero in arbitrarily high codimension.

0.3 Some moduli spaces of curves and surfaces, and their tautological rings

Now that we know how to define Chow rings of stacks, we will discuss the most pertinent moduli problems that appear in this thesis and their tautological and Chow rings. Recall that $A^*(X)$ denotes the Chow ring with $\mathbb{Q}$-coefficients of X.

The first moduli space is $\mathcal{M}_g$. The moduli space of curves $\mathcal{M}_g$ is the stack whose objects over a scheme T are smooth proper morphisms $C \to T$ of relative dimension 1 such that the geometric fibers are of genus g. There is a universal family

$$f : \mathcal{C} \to \mathcal{M}_g.$$

The sheaf of relative differentials ω_f furnishes natural classes called the kappa classes in $A^*(\mathcal{M}_g)$:

$$\kappa_i = f_*(c_1(\omega_f)^{i+1}).$$

Definition 0.3.1. The tautological ring $R^*(\mathcal{M}_g)$ is the $\mathbb{Q}$-subalgebra of $A^*(\mathcal{M}_g)$ generated by the kappa classes.

Immediately, there are two natural questions

- **Question 1:** Does $A^*(\mathcal{M}_g) = R^*(\mathcal{M}_g)$? If they exist, what can we say about the support of non-tautological classes?

- **Question 2:** What is the structure of $R^*(\mathcal{M}_g)$?

The answer to Question 1 is no in general. Van Zelm [vZ18] proved that $A^*(\mathcal{M}_{12}) \neq R^*(\mathcal{M}_{12})$. Question 2 is the subject of Faber's conjectures [Fab99], which have been verified computationally for $g \leq 23$ using the Faber–Zagier relations [Fab99, Pan18]. In particular, we know a complete set of generators and relations for the tautological ring of

$R^*(\mathcal{M}_g)$ when $g \leq 23$. We will give a positive answer to Question 1 when $7 \leq g \leq 9$ in Chapter 4.

The next moduli space is the Hurwitz space $\mathcal{H}_{k,g}$, which is in fact the key player in our study of $\mathcal{M}_g$. The objects of the Hurwitz space over a scheme T are smooth proper morphisms $C \to P \to T$ such that $C \to P$ is finite and flat of degree k, $C \to T$ is a relative curve of genus g, and $P \to T$ is a $\mathbb{P}^1$-fibration. We have the universal diagram

$$\begin{tikzcd} \mathcal{C} \arrow[r, "\alpha"] \arrow[dr, "f"'] & \mathcal{P} \arrow[d, "\pi"] \\ & \mathcal{H}_{k,g}. \end{tikzcd}$$

Definition 0.3.2. The tautological ring of the Hurwitz space is the $\mathbb{Q}$-subalgebra $R^*(\mathcal{H}_{k,g}) \subseteq A^*(\mathcal{H}_{k,g})$ generated by classes of the form

$$f_*(c_1(\omega_f)^i \cdot \alpha^* c_1(\omega_\pi)^j) = \pi_*(\alpha_*(c_1(\omega_f)^i) \cdot c_1(\omega_\pi)^j).$$

Note that when we set $j = 0$, we recover the pullbacks of the kappa classes from $\mathcal{M}_g$.

We can then ask the same questions for $\mathcal{H}_{k,g}$ that we asked for $\mathcal{M}_g$. We also have the following additional question.

- **Question 3:** What is the relationship between $R^*(\mathcal{H}_{k,g})$ and $R^*(\mathcal{M}_g)$?

We answer Question 3 for $k \leq 5$ by showing that classes in $R^*(\mathcal{H}_{k,g})$ push forward to $R^*(\mathcal{M}_g)$ in Chapter 2.

Finally, we turn our attention to surfaces, specifically K3 surfaces. The moduli spaces we study are moduli spaces of lattice polarized K3 surfaces. The cohomology lattice of a K3 surface is isomorphic to the lattice

$$E_8(-1)^2 \oplus U^3$$

where E_8 is the unique postive definite unimodular lattice of rank 8 and U is the hyperbolic lattice. For any sublattice $\Lambda \subset E_8(-1)^2 \oplus U^3$ there are moduli spaces $\mathcal{F}_\Lambda$ parametrizing K3 surfaces whose Picard groups contain the lattice Λ and such that Λ contains the class of a quasi-polarization. After choosing a basis of Λ, we think of the moduli problem as follows: the objects of $\mathcal{F}_\Lambda$ over a scheme T are the data of a family of K3 surfaces

$$X \to T$$

together with a choice of line bundles $H_1, \dots H_r$ on X, corresponding to the basis elements of the lattice. There are forgetful morphisms

$$\mathcal{F}_{\Lambda'} \hookrightarrow \mathcal{F}_\Lambda$$

for any lattice $\Lambda \subset \Lambda'$. We call the subvarieties $\mathcal{F}_{\Lambda'}$ Noether-Lefschetz loci of $\mathcal{F}_\Lambda$.

The stack $\mathcal{F}_\Lambda$ comes equipped with a universal K3 surface

$$\pi_\Lambda : \mathcal{X}_\Lambda \to \mathcal{F}_\Lambda.$$

and universal bundles $\mathcal{H}_1, \dots \mathcal{H}_r$ on $\mathcal{X}_\Lambda$, well-defined up to pullbacks from $\mathcal{F}_\Lambda$, corresponding to the chosen basis. Let $\mathcal{T}_{\pi_\Lambda}$ denote the relative tangent bundle. Following [MOP17], we define the κ-classes

$$\kappa^{\Lambda}_{a_1,\dots,a_r,b} := \pi_{\Lambda *}\left(c_1(\mathcal{H}_1)^{a_1} \cdots c_1(\mathcal{H}_r)^{a_r} \cdot c_2(\mathcal{T}_{\pi_\Lambda})^b\right).$$

Definition 0.3.3 (Marian–Oprea–Pandharipande)**.** The tautological ring $R^*(\mathcal{F}_\Lambda)$ is the subring of $A^*(\mathcal{F}_\Lambda)$ generated by pushforwards from the Noether–Lefschetz loci of all κ-classes.

In this thesis, we will focus on $\mathcal{F}_U$, the moduli space of hyperbolically polarized

K3 surfaces. We can ask the analogues of Questions 1 and 2 for $\mathcal{F}_U$. We will show that $A^*(\mathcal{F}_U) = R^*(\mathcal{F}_U)$ and completely determine the structure of $A^*(\mathcal{F}_U)$ in Chapter 5.

0.4 Structure of the thesis

This thesis is made up of five papers, each with its own chapter. It is designed so that each chapter can be read separately. In particular, all the necessary notations are (re)introduced in each chapter. The main results of each chapter are presented in the introduction to each chapter. The first three chapters are about tautological and Chow rings of Hurwitz spaces. The fourth chapter is about Chow rings of moduli spaces of low genus curves. The final chapter is about Chow rings of moduli spaces of elliptic surfaces, with a particular focus on elliptic K3 surfaces.

Chapter 1

Tautological classes on low-degree Hurwitz spaces

1.1 Introduction

When studying intersection theory of moduli spaces, one often introduces certain natural or "tautological" classes coming from the universal family. In the words of Mumford [Mum83]:

> *Whenever a variety or topological space is defined by some universal property, one expects that by virtue of its defining property, it possesses certain cohomology classes called tautological classes.*

For example, in the case of the moduli space of curves $\mathcal{M}_g$, the tautological classes Mumford proposes to study are the *kappa classes*, defined as follows. Let $f : \mathcal{C} \to \mathcal{M}_g$ be the universal curve; then $\kappa_i := f_*(c_1(\omega_f)^{i+1}) \in A^i(\mathcal{M}_g)$, the Chow ring of $\mathcal{M}_g$. The *tautological ring*, denoted $R^*(\mathcal{M}_g) \subseteq A^*(\mathcal{M}_g)$, is the subring of the rational Chow ring generated by the kappa classes.

In this paper, we study the intersection theory of the Hurwitz space $\mathcal{H}_{k,g}$, the moduli space of degree k, genus g covers of $\mathbb{P}^1$, up to automorphisms of the target. Following Mumford's philosophy, let us begin by introducing a notion of tautological classes. Let $\mathcal{C}$ be the universal curve and $\mathcal{P}$ the universal $\mathbb{P}^1$-fibration over the Hurwitz space $\mathcal{H}_{k,g}$:

$$\begin{tikzcd} \mathcal{C} \arrow[r, "\alpha"] \arrow[dr, "f"'] & \mathcal{P} \arrow[d, "\pi"] \\ & \mathcal{H}_{k,g}. \end{tikzcd}$$

We define the tautological subring of the Hurwitz space $R^*(\mathcal{H}_{k,g}) \subseteq A^*(\mathcal{H}_{k,g})$ to be the subring generated by classes of the form $f_*(c_1(\omega_f)^i \cdot \alpha^* c_1(\omega_\pi)^j) = \pi_*(\alpha_*(c_1(\omega_f)^i) \cdot c_1(\omega_\pi)^j)$.

In general, determining the full Chow ring of a moduli space — such as $\mathcal{M}_g$ or $\mathcal{H}_{k,g}$ — may be quite difficult. Having established a notion of tautological classes, however, it makes sense to split the study of the intersection theory of a moduli space into two parts:

- **Question 1:** To what extent are classes tautological? If they exist, what can we say about the support of non-tautological classes?

- **Question 2:** What is the structure of the tautological ring? Although the full Chow ring may be complicated, one hopes that the tautological ring has a more easily described structure.

In this paper, we provide an answer to Question 1 for $\mathcal{H}_{k,g}$ with $k \leq 5$. The ground work we develop here will also be important for addressing Question 2, which we undertake in subsequent work [CL21a].

Before stating our results, we highlight some known results about the Chow ring of $\mathcal{M}_g$ related to Question 1 for context.

(1a) (codimension 1) Codimension 1 classes are tautological: $A^1(\mathcal{M}_g) = R^1(\mathcal{M}_g)$ [Har83].

(1b) (low genus) For $g \leq 6$, all classes are tautological: $A^*(\mathcal{M}_g) = R^*(\mathcal{M}_g)$ [Mum83, Fab90a, Fab90b, Iza95, PV15b].

(1c) (bielliptics) In genus 12, the fundamental class of the bielliptic locus B_{12} is *not* tautological: $[B_{12}] \notin R^*(\mathcal{M}_{12})$ [vZ18].

Remark 1.1.1. Building upon the results for Hurwitz spaces in this paper and its sequel [CL21a], we extend (1b) to prove $A^*(\mathcal{M}_g) = R^*(\mathcal{M}_g)$ for all $g \leq 9$ in [CL21b].

Meanwhile, for the Hurwitz space $\mathcal{H}_{k,g}$, the previously known results regarding tautological classes are as follows:

(2a) (codimension 1) Codimension 1 classes are tautological $A^1(\mathcal{H}_{k,g}) = R^1(\mathcal{H}_{k,g})$ for $k \leq 5$ [DP15] and $k > g - 1$ [Mul20]. The general case remains an open conjecture known as the *Picard rank conjecture.*

(2b) (low degree) For $k \leq 3$, all classes are tautological: $A^*(\mathcal{H}_{k,g}) = R^*(\mathcal{H}_{k,g})$. In the case $k = 2$, it is well-known that $A^*(\mathcal{H}_{2,g}) = \mathbb{Q}$; the case $k = 3$ is due to Patel–Vakil [PV15a].

Our main theorems make significant progress towards answering Question 1 for the next open cases: the Hurwitz spaces $\mathcal{H}_{4,g}$ and $\mathcal{H}_{5,g}$. A degree 4 cover $C \to \mathbb{P}^1$ can factor as two degree two covers $C \to C' \to \mathbb{P}^1$. Let $\mathcal{H}_{4,g}^{\mathrm{nf}} \subset \mathcal{H}_{4,g}$ denote the open locus of *non-factoring covers*, or equivalently covers whose monodromy group is not contained in the dihedral group D_4. By $R^*(\mathcal{H}_{4,g}^{\mathrm{nf}})$ we mean the image of $R^*(\mathcal{H}_{4,g})$ under the restriction map $A^*(\mathcal{H}_{4,g}) \to A^*(\mathcal{H}_{4,g}^{\mathrm{nf}})$.

Theorem 1.1.2. *If they exist, any non-tautological classes on $\mathcal{H}_{4,g}$ are supported on the locus of factoring covers or have codimension at least $(g+3)/4-4$. In other words,*

$$A^i(\mathcal{H}_{4,g}^{\mathrm{nf}}) = R^i(\mathcal{H}_{4,g}^{\mathrm{nf}}) \quad \textit{for all } i < (g+3)/4-4.$$

Remark 1.1.3. The fact that there may be non-tautological classes on the locus of factoring covers should be compared with (1c). In fact, using van Zelm's result that $[B_{12}]$ is not tautological, we establish in [CL21a, Remark 1.10] that $\mathcal{H}_{4,12}$ indeed possesses non-tautological classes supported on the factoring locus.

In degree 5, covers cannot factor, and we obtain the following result.

Theorem 1.1.4. *If they exist, any non-tautological classes on $\mathcal{H}_{5,g}$ have codimension at least $(g+4)/5-16$. In other words,*

$$A^i(\mathcal{H}_{5,g}) = R^i(\mathcal{H}_{5,g}) \quad \textit{for all } i < (g+4)/5-16.$$

Theorems 1.1.2 and 1.1.4 are reminiscent of the Madsen–Weiss theorem [MW07], which proves Mumford's conjecture that the stable cohomology of $\mathcal{M}_g$ is a polynomial ring in the kappa classes. Edidin [Edi13, Question 3.34] asked if the analogue of the Madsen–Weiss theorem holds in the Chow ring $A^*(\mathcal{M}_g)$, but very little is known about this question. We view Theorems 1.1.2 and 1.1.4 as providing some evidence toward a positive answer to Edidin's question. Unlike the case of the stable cohomology of $\mathcal{M}_g$, we will show in [CL21a] that there are many interesting relations among the tautological classes on $\mathcal{H}_{k,g}$ when $3 \leq k \leq 5$.

Sketch of the proof

There are three key ingredients to proving Theorems 1.1.2 and 1.1.4. The set up we develop will also be essential for later results determining structure of the tautological ring in [CL21a]. We shall therefore carry them out in the case $k = 3$ as well, which fits into the same framework.

(1) Useful generators: We first explain how structure theorems of Casnati–Ekedahl for finite covers give rise to a collection of classes on $\mathcal{H}_{k,g}$, which we term Casnati–Ekedahl (CE) classes. In Theorem 1.3.10, we show that all CE classes are tautological and that they generate the tautological ring. An interesting consequence of this is that, for fixed k, i, $\dim R^i(\mathcal{H}_{k,g})$ is bounded above, independent of g. (This part works for any k; see Remark 1.3.11).

(2) The good open: For $k = 3, 4, 5$, we define a "good open" $\mathcal{H}'_{k,g} \subseteq \mathcal{H}_{k,g}$. Using our interpretation of the Casnati–Ekedahl structure theorems, we show that this "good open" possesses an open embedding inside a vector bundle $\mathcal{X}'_{k,g}$ over a moduli space $\mathcal{B}'_{k,g}$ of pairs of vector bundles on $\mathbb{P}^1$. The pullbacks of classes along $A^*(\mathcal{B}'_{k,g}) = A^*(\mathcal{X}'_{k,g}) \to A^*(\mathcal{H}'_{k,g})$ are CE classes (essentially by the definition of CE classes). It follows that $A^*(\mathcal{H}'_{k,g})$ is generated by tautological classes.

(3) Codimension bounds: By excision, there is a surjection $A^*(\mathcal{H}_{k,g}) \to A^*(\mathcal{H}'_{k,g})$ whose kernel is generated by classes supported on the complement of $\mathcal{H}'_{k,g}$. Thus, the final step is to bound the codimension of the complement of $\mathcal{H}'_{k,g}$. When $k = 3$, it turns out $\mathcal{H}'_{k,g} = \mathcal{H}_{k,g}$, so we recover the result of Patel–Vakil [PV15a] that all classes are tautological. When $k = 4$, the complement of $\mathcal{H}'_{k,g}$ contains the locus of covers that factor through a

double cover of a low-genus curve. Thus, the complement has codimension 2. However, it turns out that the *non-factoring* covers in the complement of $\mathcal{H}'_{k,g}$ have codimension at least $(g+3)/4-4$. This leads to the proof of Theorem 1.1.2 at the end of Section 1.5.2. Finally, for $k=5$, there are no factoring covers, and we show that the complement of $\mathcal{H}'_{k,g}$ has codimension at least $(g+4)/5-16$. With this, we conclude the proof of Theorem 1.1.4 at the end of Section 1.5.3.

1.2 Notation and conventions

We will work over an algebraically closed field of characteristic 0 or characteristic $p>5$. All schemes in this paper will be taken over this fixed field.

1.2.1 Projective bundles

We follow the subspace convention for projective bundles: given a scheme (or stack) X and a vector bundle E of rank r on X, we set

$$\mathbb{P}E := \operatorname{Proj}(\operatorname{Sym}^\bullet E^\vee),$$

so we have the tautological inclusion

$$\mathcal{O}_{\mathbb{P}E}(-1) \hookrightarrow \gamma^*E,$$

where $\gamma : \mathbb{P}E \to X$ is the structure map. Set $\zeta := c_1(\mathcal{O}_{\mathbb{P}E}(1))$. With this convention, the Chow ring of $\mathbb{P}E$ is given by

$$A^*(\mathbb{P}E) = A^*(X)[\zeta]/\langle \zeta^r + \zeta^{r-1}c_1(E) + \ldots + c_r(E)\rangle. \tag{1.2.1}$$

We call this the *projective bundle theorem*. Note that $1, \zeta, \zeta^2, \ldots, \zeta^{r-1}$ form a basis for $A^*(\mathbb{P}E)$ as an $A^*(X)$-module. Since

$$\gamma_*\zeta^i = \begin{cases} 0 & \text{if } i \leq r-2 \\ 1 & \text{if } i = r-1, \end{cases}$$

this determines the γ_* of all classes from $\mathbb{P}E$.

1.2.2 (Equivariant) Intersection Theory

Let X be a scheme and suppose $Z \subseteq X$ is a closed subscheme of codimension c and U is its open complement. We denote the Chow ring of X with rational coefficients by $A^*(X)$. The *excision property* of Chow is the right exact sequence

$$A^{*-c}(Z) \to A^*(X) \to A^*(U) \to 0.$$

If one knows $A^*(X)$, then to find the Chow ring of an open $U \subset X$, one must describe the image of $A^{*-c}(Z) \to A^*(X)$. If $\widetilde{Z} \to Z$ is proper and surjective, then pushforward $A_*(\widetilde{Z}) \to A_*(Z)$ is surjective, see [Vis89a, Lemma 1.2]. Given a graded ring $R = \bigoplus R^i$, let

$$\operatorname{Trun}^d R := R / \oplus_{i \geq d} R^d$$

denote the degree d trunction. With this notation, if the complement of $U \subseteq X$ has codimension c, then the excision property implies

$$\operatorname{Trun}^c A^*(X) \xrightarrow{\sim} \operatorname{Trun}^c A^*(U). \tag{1.2.2}$$

Chow rings also satisfy the *homotopy property*: if $V \to X$ is a vector bundle, then the pullback map $A^*(X) \to A^*(V)$ is an isomorphism. This property motivates the definition of equivariant Chow groups as developed by Edidin-Graham in [EG98]. Again, we will be using rational coefficients for our equivariant Chow rings. Let V be a representation of G and suppose G acts freely on $U \subset V$ and the codimension of $V \smallsetminus U$ is greater than c. If X is a smooth scheme and G is a linear algebraic group acting on X, Edidin and Graham defined

$$A^c_G(X) := A^c((X \times U)/G),$$

and showed that the graded ring $A^*_G(X)$ possesses an intersection product. For quotient stacks, one has $A^*([X/G]) \cong A^*_G(X)$ by [EG98, Proposition 19], which may suffice as the definition of the Chow rings of all stacks appearing in this paper.

By Edidin-Graham [EG98, Proposition 5], there is also an excision sequence for equivariant Chow groups. Let $Z \subseteq X$ be a G-invariant closed subscheme of codimension c and U its complement. Then there is an exact sequence

$$A^{*-c}_G(Z) \to A^*_G(X) \to A^*_G(U) \to 0.$$

The following lemma is a useful consequence of the excision sequence. See also [Vis87, Theorem 2] for a much more general statement.

Lemma 1.2.1. *Suppose $P \to X$ is a principal $\mathbb{G}_m$-bundle. Then $A^*(P) = A^*(X)/\langle c_1(L)\rangle$, where L is the corresponding line bundle.*

Proof. By the correspondence between principal $\mathbb{G}_m$-bundles and line bundles over X, P is the complement of the zero section of the line bundle $L \to X$. The excision sequence gives

$$A^{*-1}(X) \to A^*(L) \to A^*(P) \to 0.$$

Under the identification of $A^*(L)$ with $A^*(X)$, the first map in the above exact sequence is multiplication by $c_1(L)$, from which the result follows. $\square$

1.2.3 The Hurwitz space

Given a scheme S, an S point of the *parametrized Hurwitz scheme* $\mathcal{H}^\dagger_{k,g}$ is the data of a finite, flat map $C \to \mathbb{P}^1 \times S$, of constant degree k so that the composition $C \to \mathbb{P}^1 \times S \to S$ is smooth with geometrically connected fibers. (We do not impose the condition that a cover $C \to \mathbb{P}^1$ be simply branched.)

The *unparametrized Hurwitz stack* is the PGL_2 quotient of the parametrized Hurwitz scheme. There is also a natural action of SL_2 on $\mathcal{H}^\dagger_{k,g}$ (via $\mathrm{SL}_2 \subset \mathrm{GL}_2 \to \mathrm{PGL}_2$). The natural map $[\mathcal{H}^\dagger_{k,g}/\mathrm{SL}_2] \to [\mathcal{H}^\dagger_{k,g}/\mathrm{PGL}_2]$ is a μ_2 banded gerbe. It is a general fact that *with rational coefficients*, the pullback map along a gerbe banded by a finite group is an isomorphism [PV15b, Section 2.3]. In particular, since we work with rational coefficients throughout, $A^*([\mathcal{H}^\dagger_{k,g}/\mathrm{PGL}_2]) \cong A^*([\mathcal{H}^\dagger_{k,g}/\mathrm{SL}_2])$. It thus suffices to prove all statements for the SL_2 quotient $[\mathcal{H}^\dagger_{k,g}/\mathrm{SL}_2]$, which we shall denote by $\mathcal{H}_{k,g}$ from now on.

Explicitly, the SL_2 quotient $\mathcal{H}_{k,g}$ is the stack whose objects over a scheme S are families $(C \to P \to S)$ where $P = \mathbb{P}V \to S$ is the projectivization of a rank 2 vector bundle V with trivial determinant, $C \to P$ is a finite, flat, finitely presented morphism of constant degree k, and the composition $C \to S$ has smooth fibers of genus g. The benefit of working with $\mathcal{H}_{k,g}$ is that the SL_2 quotient is equipped with a universal $\mathbb{P}^1$-bundle $\mathcal{P} \to \mathcal{H}_{k,g}$ that has a relative degree one line bundle $\mathcal{O}_{\mathcal{P}}(1)$ (a $\mathbb{P}^1$-fibration does not). Working with this $\mathbb{P}^1$-bundle simplifies our intersection theory calculations.

1.3 The Casnati–Ekedahl structure theorem

The main objective of this section is to give a description of stacks of low-degree covers using structure theorems of Casnati–Ekedahl. The descriptions in Sections 1.3.1–1.3.3 are likely well-known but have not previously been spelled out in the language of stacks except in the degree 3 case [BV12], as we shall need them. On a first pass, the reader may wish to skip forward to Section 1.3.4, where we introduce natural classes coming from these structure theorems and prove that they generate the tautological ring.

Generalizing earlier results of Schreyer [Sch86] and Miranda [Mir85], Casnati–Ekedahl [CE96] proved a general structure theorem for degree k, Gorenstein covers of integral schemes. Given a degree k cover $\alpha : X \to Y$ where Y is integral, one obtains an exact sequence

$$0 \to \mathcal{O}_Y \to \alpha_*\mathcal{O}_X \to E_\alpha^\vee \to 0, \tag{1.3.1}$$

where E_α is a vector bundle of rank $k-1$ on Y. When α is Gorenstein, $\alpha_*\mathcal{O}_X \cong (\alpha_*\omega_\alpha)^\vee$ by Serre duality. Pulling back and using adjunction, we therefore obtain a map

$$\omega_\alpha^\vee \to (\alpha^*\alpha_*\omega_\alpha)^\vee \to \alpha^*E_\alpha^\vee, \tag{1.3.2}$$

which induces a map $X \to \mathbb{P}E^\vee$ that factors $\alpha : X \to Y$.

Example 1.3.1 (Covers of $\mathbb{P}^1$)**.** If $\alpha : C \to \mathbb{P}^1$ is a degree k, genus g cover, then we have

$$\deg(E_\alpha^\vee) = \deg(\alpha_*\mathcal{O}_C) = \chi(\alpha_*\mathcal{O}_C) - k = \chi(\mathcal{O}_C) - k = 1 - g - k,$$

so $\deg(E_\alpha) = g + k - 1$. The map $C \to \mathbb{P}E_\alpha^\vee$ factors the canonical embedding $C \hookrightarrow \mathbb{P}^{g-1}$, where the map $\mathbb{P}E_\alpha^\vee \to \mathbb{P}^{g-1}$ is given by the line bundle $\mathcal{O}_{\mathbb{P}E_\alpha^\vee}(1) \otimes \omega_{\mathbb{P}^1}$. Each linear space in the image of $\mathbb{P}E_\alpha^\vee \to \mathbb{P}^{g-1}$ is the span of the image of the corresponding fiber of $C \to \mathbb{P}^1$.

The Casnati–Ekedahl structure theorem below gives a resolution of the ideal sheaf of X inside of $\mathbb{P}E_\alpha^\vee$ [CE96]; see also [CN07].

Theorem 1.3.2 (Casnati–Ekedahl, Theorem 2.1 of [CE96])**.** *Let X and Y be schemes, Y integral and let $\alpha : X \to Y$ be a Gorenstein cover of degree $k \geq 3$. There exists a unique $\mathbb{P}^{k-2}$-bundle $\gamma : \mathbb{P} \to Y$ and an embedding $i : X \hookrightarrow \mathbb{P}$ such that $\alpha = \gamma \circ i$ and $X_y := \alpha^{-1}(y) \subset \gamma^{-1}(y) \cong \mathbb{P}^{k-2}$ is a nondegenerate arithmetically Gorenstein subscheme* *for each $y \in Y$. Moreover, the following properties hold.*

1. $\mathbb{P} \cong \mathbb{P}E_\alpha^\vee$ *where* $E_\alpha^\vee := \operatorname{coker}(\mathcal{O}_Y \to \alpha_*\mathcal{O}_X)$.

2. *The composition* $\alpha^*E_\alpha \to \alpha^*\alpha_*\omega_\alpha \to \omega_\alpha$ *is surjective (dually,* (1.3.2) *does not drop rank) and the ramification divisor* R *satisfies* $\mathcal{O}_X(R) \cong \omega_\alpha \cong \mathcal{O}_X(1) := i^*\mathcal{O}_{\mathbb{P}E_\alpha^\vee}(1)$.

3. *There exists an exact sequence of locally free* $\mathcal{O}_\mathbb{P}$ *sheaves*

$$0 \to \gamma^*F_{k-2}(-k) \to \gamma^*F_{k-3}(-k+2) \to \cdots \to \gamma^*F_1(-2) \to \mathcal{O}_\mathbb{P} \to \mathcal{O}_X \to 0. \quad (1.3.3)$$

   where F_i *is locally free on* Y. *The restriction of the exact sequence above to a fiber gives a minimal free resolution of* $X_y := \alpha^{-1}(y)$. *This sequence is unique up to unique isomorphism. Moreover the resolution is self-dual, meaning there is a canonical isomorphism* $\mathcal{H}om_{\mathcal{O}_\mathbb{P}}(F_i, F_{k-2}) \cong F_{k-2-i}$. *The ranks of the* F_i *are*

$$\operatorname{rank} F_i = \frac{i(k-2-i)}{k-1}\binom{k}{i+1}.$$

4. *If* $\mathbb{P} \cong \mathbb{P}E'^\vee$, *then* $E' \cong E$ *if and only if* $F_{k-2} \cong \det E'$ *in the resolution* (1.3.3) *computed with respect to the polarization* $\mathcal{O}_{\mathbb{P}E'^\vee}(1)$.

Remark 1.3.3. There is a canonical isomorphism $F_{k-2} \cong \det E_\alpha$, which we describe here. Following [CE96, p. 446], let A_1 be the image of $\gamma^*F_1(-2) \to \mathcal{O}_\mathbb{P}$, and for $2 \leq i \leq k-3$, let A_i denote the image of $\gamma^*F_i(-i-1) \to \gamma^*F_{i-1}(-i)$. We set A_{k-2} to be $\gamma^*F_{k-2}(-k)$. We have exact sequences

$$0 \to A_1 \to \mathcal{O}_\mathbb{P} \to \mathcal{O}_X \to 0 \quad (1.3.4)$$

and

$$0 \to A_{i+1} \to \gamma^*F_i(-i-1) \to A_i \to 0. \quad (1.3.5)$$

First, we claim that

$$R^j\gamma_*\gamma^*F_i(-i-1) \cong \begin{cases} F_{k-2}\otimes \det E^\vee & \text{if } i=j=k-2 \\ 0 & \text{otherwise.} \end{cases}$$

This is very similar to the calculations of [CE96, p. 446], but twisted up by one. To prove the first case above, we note that the dualizing sheaf of γ is $\omega_\gamma = (\gamma^*\det E)(-k+1)$, and apply Serre duality for γ, which is of relative dimension $k-2$. The other cases follow from the theorem on cohomology and base change and the well-known cohomology of line bundles on projective space. Tensoring the exact sequences of (1.3.5) by $\mathcal{O}_{\mathbb{P}}(1)$ and pushing forward by γ, the boundary maps provide us with isomorphisms

$$\gamma_*A_1(1) \cong R^1\gamma_*A_2(1) \cong R^2\gamma_*A_3(1) \cong \cdots \cong R^{k-1}\gamma_*(\gamma^*F_{k-2}(-k+1)) = 0.$$

Similarly, we have

$$R^1\gamma_*A_1(1) \cong R^2\gamma_*A_2(1) \cong \cdots \cong R^{k-2}\gamma_*(\gamma^*F_{k-2}(-k+1)) \cong F_{k-2}\otimes \det E^\vee.$$

On the other hand, tensoring (1.3.4) with $\mathcal{O}_{\mathbb{P}}(1)$ and pushing forward by γ we obtain

$$0 \to E \to \alpha_*\mathcal{O}_X(1) \to R^1\gamma_*A_1(1) \to 0.$$

Recall that $\mathcal{O}_X(1) \cong \omega_\alpha$, so dualizing (1.3.1) we see that the cokernel of the left map is $\mathcal{O}_Y$. By the universal property of cokernel, we obtain an isomorphism

$$\mathcal{O}_Y \to R^1\gamma_*A_1(1) \cong F_{k-2}\otimes \det E^\vee,$$

or equivalently, an isomorphism $F_{k-2} \cong \det E$.

In the cases $k = 3, 4, 5$, using self-duality, only pullbacks of the bundles E_α and F_1 and determinants and tensor products thereof appear in the resolution (1.3.3). We set $F_\alpha := F_1$. Twisting up (1.3.3) by $\mathcal{O}_{\mathbb{P}}(2)$ and pushing forward by γ, we see that

$$F_\alpha = \ker(\operatorname{Sym}^2 E_\alpha \twoheadrightarrow \alpha_* \omega_\alpha^{\otimes 2}).$$

In these low degrees $k = 3, 4, 5$, there is a special map δ_α in the resolution (1.3.3) from which one can reconstruct the cover. Furthermore, as we shall explain, it is an *open condition* on a space of global sections of all such maps δ to define a finite cover. This is what distinguishes $k = 3, 4, 5$ and lies at the core of why our methods work in these low degrees. Below we present an equivalence of categories between the category of degree k, Gorenstein covers of a scheme S and a category of certain linear algebraic data on S. The main content of this step is to point out the "essential data" of a cover, which we may remember instead of the entire resolution. For the case of triple covers, this was done by Bolognesi–Vistoli [BV12]. We give a slightly different perspective below.

1.3.1 The category of triple covers

Let $\operatorname{Trip}(S)$ denote the category of Gorenstein triple covers of a scheme S: the objects are Gorenstein triple covers $\alpha : X \to S$ and the arrows are isomorphisms over S. Specializing (1.3.3) to the case $k = 3$, associated to a cover $\alpha : X \to S$, one obtains a rank 2 vector bundle E_α and an exact sequence

$$0 \to \mathcal{O}_{\mathbb{P}E_\alpha^\vee}(-3) \otimes \gamma^* \det E_\alpha \xrightarrow{\delta_\alpha} \mathcal{O}_{\mathbb{P}E_\alpha^\vee} \to \mathcal{O}_X \to 0.$$

Conversely, from the above sequence, we can recover the cover $\alpha : X \to S$. Indeed, the map δ_α is a global section in $H^0(\mathbb{P}E_\alpha^\vee, \mathcal{O}_{\mathbb{P}E_\alpha^\vee}(3) \otimes \gamma^* \det E_\alpha^\vee)$, whose zero locus inside of $\mathbb{P}E_\alpha^\vee$ is X. Meanwhile, given any rank 2 vector bundle E on S, it is an open condition on the space of sections $H^0(\mathbb{P}E^\vee, \mathcal{O}_{\mathbb{P}E^\vee}(3) \otimes \gamma^* \det E^\vee)$ for the vanishing of a section δ to define a finite triple cover: δ must not be the zero polynomial on any fiber of $\mathbb{P}E \to S$. Equivalently, if

$$\Phi : H^0(S, \operatorname{Sym}^3 E \otimes \det E^\vee) \xrightarrow{\sim} H^0(\mathbb{P}E^\vee, \mathcal{O}_{\mathbb{P}E^\vee}(3) \otimes \gamma^* \det E^\vee) \tag{1.3.6}$$

denotes the natural isomorphism, then $V(\delta) \subset \mathbb{P}E^\vee$ is a Gorenstein triple cover so long as $\Phi^{-1}(\delta)$ is non-vanishing.

This "essential data" is captured by a category $\mathrm{Trip}'(S)$ we now define. The objects of $\mathrm{Trip}'(S)$ are pairs (E, η) where E is a rank 2 vector bundle and $\eta \in H^0(S, \operatorname{Sym}^3 E \otimes \det E^\vee)$ is non-vanshing on S. An arrow $(E_1, \eta_1) \to (E_2, \eta_2)$ in $\mathrm{Trip}'(S)$ is an isomorphism $E_1 \to E_2$ that sends η_1 into η_2. There is a functor $\mathrm{Trip}(S) \to \mathrm{Trip}'(S)$ that sends $\alpha : X \to S$ to the pair $(E_\alpha, \Phi^{-1}(\delta_\alpha))$. There is also a functor $\mathrm{Trip}'(S) \to \mathrm{Trip}(S)$ that sends a pair (E, η) to the triple cover $V(\Phi(\eta)) \subset \mathbb{P}E^\vee \to S$. The following is essentially a restatement of [CE96, Theorem 3.4], which was proved earlier by Miranda [Mir85].

Theorem 1.3.4 (Miranda, Casnati–Ekedahl)**.** *The functors above define an equivalence of categories* $\mathrm{Trip}(S) \cong \mathrm{Trip}'(S)$.

1.3.2 The category of quadruple covers

Let $\mathrm{Quad}(S)$ denote the category whose objects are Gorenstein quadruple covers $\alpha : X \to S$ and whose arrows are isomorphisms over S. Associated to a degree 4 cover $\alpha : X \to S$, there is a rank 3 vector bundle E_α and a rank 2 vector bundle F_α and a resolution

$$0 \to \gamma^* \det E_\alpha(-4) \to \gamma^* F_\alpha(-2) \xrightarrow{\delta_\alpha} \mathcal{O}_{\mathbb{P}E_\alpha^\vee} \to \mathcal{O}_X \to 0. \tag{1.3.7}$$

The section $\delta_\alpha \in H^0(\mathbb{P}E_\alpha^\vee, \mathcal{O}_{\mathbb{P}E_\alpha^\vee}(2) \otimes \gamma^* F^\vee)$ corresponds to a relative pencil of quadrics. The cover X can be recovered as the vanishing locus of δ_α. By comparing (1.3.7) with the

Koszul resolution of δ_α,

$$0 \to \gamma^* \det F_\alpha(-4) \to \gamma^* F_\alpha(-2) \xrightarrow{\delta_\alpha} \mathcal{O}_{\mathbb{P}E_\alpha^\vee} \to \mathcal{O}_X \to 0, \tag{1.3.8}$$

the uniqueness of Theorem 1.3.2 (3) induces a distinguished isomorphism $\phi_\alpha : \det F_\alpha \cong \det E_\alpha$ (see [CE96, p. 450]).

We now define a category $\mathrm{Quad}'(S)$ of the corresponding linear algebraic data of a quadruple cover. Given vector bundles E, F on S, there is a natural isomorphism

$$\Phi : H^0(S, F^\vee \otimes \operatorname{Sym}^2 E) \xrightarrow{\sim} H^0(\mathbb{P}E^\vee, \gamma^* F^\vee \otimes \mathcal{O}_{\mathbb{P}E^\vee}(2)). \tag{1.3.9}$$

Definition 1.3.5. Let E and F be vector bundles of ranks 3 and 2 respectively on S. We say that a section $\eta \in H^0(S, F^\vee \otimes \operatorname{Sym}^2 E)$ *has the right codimension* at $s \in S$ if the vanishing locus of $\Phi(\eta)$ restricted to the fiber over $s \in S$ is zero dimensional.

The objects of $\mathrm{Quad}'(S)$ are tuples (E, F, ϕ, η) where E and F are vector bundles of ranks 3 and 2 respectively, $\phi : \det F \cong \det E$ is an isomorphism and $\eta \in H^0(S, F^\vee \otimes \operatorname{Sym}^2 E)$ has the right codimension at all $s \in S$. An arrow in $\mathrm{Quad}'(S)$ is a pair of isomorphisms $\xi : E_1 \to E_2$, and $\psi : F_1 \to F_2$, such that the following diagrams commute

$$\begin{array}{ccc} F_1 & \xrightarrow{\eta_1} & \operatorname{Sym}^2 E_1 \\ \downarrow{\scriptstyle \psi} & & \downarrow{\scriptstyle \operatorname{Sym}^2 \xi} \\ F_2 & \xrightarrow{\eta_2} & \operatorname{Sym}^2 E_2 \end{array} \qquad \begin{array}{ccc} \det F_1 & \xrightarrow{\phi_1} & \det E_1 \\ \downarrow{\scriptstyle \det \psi} & & \downarrow{\scriptstyle \det \xi} \\ \det F_2 & \xrightarrow{\phi_2} & \det E_2. \end{array}$$

There is a functor $\mathrm{Quad}(S) \to \mathrm{Quad}'(S)$ that sends $\alpha : X \to S$ to $(E_\alpha, F_\alpha, \phi_\alpha, \eta_\alpha)$ where $\eta_\alpha := \Phi^{-1}(\delta_\alpha)$. There is also a functor $\mathrm{Quad}'(S) \to \mathrm{Quad}(S)$ that sends a tuple (E, F, ϕ, η) to the quadruple cover $V(\Phi(\eta)) \subset \mathbb{P}E^\vee \to S$. The following is essentially a restatement of [CE96, Theorem 4.4].

Theorem 1.3.6 (Casnati–Ekedahl)**.** *The functors above define an equivalence of categories* $\mathrm{Quad}(S) \cong \mathrm{Quad}'(S)$*.*

Proof. Work of Casnati–Ekedahl established that the composition

$$\mathrm{Quad}(S) \to \mathrm{Quad}'(S) \to \mathrm{Quad}(S)$$

is equivalent to the identity, as $V(\delta_\alpha) \to S$ is naturally identified with the cover $\alpha : X \to S$.

We must provide a natural isomorphism of $\mathrm{Quad}'(S) \to \mathrm{Quad}(S) \to \mathrm{Quad}'(S)$ with the identity on $\mathrm{Quad}'(S)$. Suppose we are given $(E, F, \phi, \eta) \in \mathrm{Quad}'(S)$. We want to define an arrow $(E, F, \phi, \eta) \to (E_\alpha, F_\alpha, \phi_\alpha, \eta_\alpha)$. Let $X = V(\Phi(\eta)) \subset \mathbb{P}E^\vee$, and $\alpha : X \to S$. The Koszul resolution of $\Phi(\eta)$ is

$$0 \to (\gamma^* \det F)(-4) \to \gamma^* F(-2) \xrightarrow{\Phi(\eta)} \mathcal{O}_{\mathbb{P}E^\vee} \to \mathcal{O}_X \to 0$$

and is exact since η has the right codimension at all $s \in S$. We break this into two sequences

$$0 \to (\gamma^* \det F)(-4) \to \gamma^* F(-2) \to A \to 0 \tag{1.3.10}$$

and

$$0 \to A \to \mathcal{O}_{\mathbb{P}E^\vee} \to \mathcal{O}_X \to 0. \tag{1.3.11}$$

Pushing forward (1.3.11) we get a short exact sequence on S:

$$0 \to \mathcal{O}_S \to \alpha_* \mathcal{O}_X \to R^1\gamma_* A \to 0.$$

Using (1.3.10), we obtain isomorphisms

$$R^1\gamma_* A \cong R^2\gamma_*(\gamma^* \det F)(-4) \cong \det F \otimes R^2\gamma_* \mathcal{O}_{\mathbb{P}E^\vee}(-4).$$

Because the dualizing sheaf of γ is $\omega_\gamma = \mathcal{O}_{\mathbb{P}E^\vee}(-3) \otimes \gamma^* \det E$, using Serre duality, we obtain an isomorphism $R^2\gamma_* \mathcal{O}_{\mathbb{P}E^\vee}(-4) \cong \det E^\vee \otimes E^\vee$. Now the universal property of cokernel produces an isomorphism

$$E_\alpha^\vee = \operatorname{coker}(\mathcal{O}_S \to \alpha_*\mathcal{O}_X) \xrightarrow{\sim} R^1\gamma_*A \cong \det F \otimes \det E^\vee \otimes E^\vee.$$

Meanwhile ϕ determines an isomorphism $\det F \otimes \det E^\vee \cong \mathcal{O}_S$. Composing with this, and dualizing, we obtain an isomorphism $\xi : E \to E_\alpha$. Next, we have a commuting diagram

$$\begin{tikzcd} 0 \arrow[r] & F \arrow[r, "\eta"] \arrow[d, dashed, "\psi"'] & \operatorname{Sym}^2 E \arrow[r] \arrow[d, "\operatorname{Sym}^2 \xi"] & \alpha_*\mathcal{O}_X(2) \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & F_\alpha \arrow[r, "\eta_\alpha"] & \operatorname{Sym}^2 E_\alpha \arrow[r] & \alpha_*\omega_\alpha^{\otimes 2} \arrow[r] & 0 \end{tikzcd}$$

where the left vertical map is induced by the universal property of kernel. Note that for any $t \in \mathcal{O}_S^\times(S)$, the diagram

$$\begin{tikzcd} F \arrow[r, "\eta"] \arrow[d, "t^2\cdot\psi"'] & \operatorname{Sym}^2 E \arrow[d, "\operatorname{Sym}^2(t\cdot\xi)"] \\ F_\alpha \arrow[r, "\eta_\alpha"] & \operatorname{Sym}^2 E_\alpha \end{tikzcd} \tag{1.3.12}$$

also commutes. Finally, the cover α determines an isomorphism $\phi_\alpha : \det F_\alpha \cong \det E_\alpha$. It may not agree with ϕ, but since the maps below involve isomorphisms of line bundles, there exists some $t \in \mathcal{O}_S^\times(S)$ such that the following diagram commutes

$$\begin{tikzcd} \det F \arrow[r, "\phi"] \arrow[d, "t\cdot\det\psi"'] & \det E \arrow[d, "\det\xi"] \\ \det F_\alpha \arrow[r, "\phi_\alpha"] & \det E_\alpha. \end{tikzcd}$$

Since E is rank 3 and F is rank 2, this implies the diagram

$$\begin{tikzcd} \det F \arrow[r, "\phi"] \arrow[d, "\det(t^2\cdot\psi)"'] & \det E \arrow[d, "\det(t\cdot\xi)"] \\ \det F_\alpha \arrow[r, "\phi_\alpha"] & \det E_\alpha \end{tikzcd} \tag{1.3.13}$$

also commutes. Thus, the pair of isomorphisms $t \cdot \xi : E \to E_\alpha$ and $t^2 \cdot \psi : F \to F_\alpha$ determine an arrow $(E, F, \phi, \eta) \to (E_\alpha, F_\alpha, \phi_\alpha, \eta_\alpha)$. $\square$

1.3.3 The category of regular pentagonal covers

By the Casnati–Ekedahl theorem, each degree 5 Gorenstein cover $\alpha : X \to S$ determines a resolution

$$0 \to \gamma^* \det E_\alpha(-5) \to \gamma^*(F_\alpha^\vee \otimes \det E_\alpha)(-3) \xrightarrow{\delta_\alpha} \gamma^* F_\alpha(-2) \to \mathcal{O}_{\mathbb{P}} \to \mathcal{O}_X \to 0, \quad (1.3.14)$$

where E_α has rank 4 and F_α has rank 5. Casnati showed that the map δ_α is alternating in the sense that it can be identified with a section of $\wedge^2 \pi^* F_\alpha \otimes \gamma^* \det E_\alpha^\vee(1)$. For any pair of vector bundles E and F, via push-pull, we have an identification

$$\Phi : H^0(S, \mathcal{H}om(E^\vee \otimes \det E, \wedge^2 F)) \xrightarrow{\sim} H^0(\mathbb{P}E^\vee, \gamma^*(\wedge^2 F \otimes \det E^\vee)(1)). \quad (1.3.15)$$

Hence, δ_α corresponds to a map $\eta_\alpha := \Phi^{-1}(\delta_\alpha) : E_\alpha^\vee \otimes \det E_\alpha \to \wedge^2 F_\alpha$. Throughout this section we shall write $E' := E^\vee \otimes \det E$. A degree 5 cover $\alpha : X \to S$ is called *regular* if η_α is injective as a map of vector bundles (i.e. the cokernel of η_α is locally free). Casnati notes that if $\alpha^{-1}(s)$ is a local complete intersection scheme for all $s \in S$, then α is regular, so all covers we need will be regular. We let $\mathrm{Pent}(S)$ denote the category whose objects are regular, degree 5 Gorenstein covers $\alpha : X \to S$ and arrows are isomorphisms over S.

Regular degree 5 covers have a nice geometric description. Indeed, if the cover is regular, then η_α corresponds to an injective map $E'_\alpha \to \wedge^2 F_\alpha$, which induces an embedding

$$\mathbb{P}E'_\alpha \hookrightarrow \mathbb{P}(\wedge^2 F_\alpha). \quad (1.3.16)$$

Given a section $\delta \in H^0(\mathbb{P}E^\vee, \gamma^*(\wedge^2 F \otimes \det E^\vee)(1))$, we let $D(\delta) \subset \mathbb{P}E^\vee$ be the subscheme defined by the vanishing of 4×4 Pfaffians of δ. When α is regular, we can recover $X = D(\delta_\alpha)$, which is also the same as the scheme defined by the 3×3 minors of δ_α (Proposition 3.5 of [Cas96]). These 3×3 minors are pullbacks to $\mathbb{P}E'_\alpha$ along (1.3.16) of the equations that define the Grassmannian bundle $G(2, F_\alpha) \subset \mathbb{P}(\wedge^2 F_\alpha)$ under its relative Plücker embedding. Using a resolution of the relative Grassmannian, Casnati obtains another resolution of $\mathcal{O}_X$ in equation (3.5.2) of [Cas96]. Comparing this resolution with (1.3.14), the uniqueness of Theorem 1.3.2 (2) induces a distinguished isomorphism $\epsilon : F_\alpha \otimes \det F_\alpha^\vee \otimes (\det E_\alpha)^{\otimes 2} \to F_\alpha$ (see p. 467 of [Cas96]). Moreover, both of these vector bundles arise as subbundles of $\operatorname{Sym}^2 E_\alpha$ and the projectivization of ϵ induces the identity on points (as it must be the restriction of the identity on $\mathbb{P}(\operatorname{Sym}^2 E_\alpha)$). Hence, we obtain an isomorphism of line bundles

$$\mathcal{O}_{\mathbb{P}F_\alpha}(1) \otimes \det F_\alpha^\vee \otimes (\det E_\alpha)^{\otimes 2} \cong \mathcal{O}_{\mathbb{P}(F_\alpha \otimes \det F_\alpha^\vee \otimes \det E_\alpha^2)}(1) \cong \epsilon^* \mathcal{O}_{\mathbb{P}F_\alpha}(1) = \mathcal{O}_{\mathbb{P}F_\alpha}(1).$$

which induces a distinguished isomorphism $\phi_\alpha : (\det E_\alpha)^{\otimes 2} \cong \det F_\alpha$.

Now we define a category $\mathrm{Pent}'(S)$ that keeps track of the associated linear algebraic data of regular degree 5 covers.

Definition 1.3.7. Suppose we are given vector bundles E and F on S of ranks 4 and 5. Let $\eta \in H^0(S, \mathcal{H}om(E', \wedge^2 F))$ be a global section. We say η has the *right codimension* if every fiber of $D(\Phi(\eta)) \subset \mathbb{P}E^\vee \to S$ is 0-dimensional and $\eta : E' \to \wedge^2 F$ is injective with locally free cokernel.

We define $\mathrm{Pent}'(S)$ to be the category whose objects are tuples (E, F, ϕ, η) where E and F are vector bundles on S of ranks 4 and 5 respectively, ϕ is an isomorphism $(\det E)^{\otimes 2} \cong \det F$ and $\eta \in H^0(S, \mathcal{H}om(E^\vee \otimes \det E, \wedge^2 F))$ has the right codimension. An arrow $(E_1, F_1, \phi_1, \eta_1) \to (E_2, F_2, \phi_2, \eta_2)$ in $\mathrm{Pent}'(S)$ is pair of isomorphisms $\xi : E_1 \to E_2$ and $\psi : F_1 \to F_2$ such that the following two diagrams commute

$$\begin{tikzcd} E_1' \arrow[r, "\eta_1"] \arrow[d, "\det\xi \otimes (\xi^{-1})^\vee"'] & \wedge^2 F_1 \arrow[d, "\wedge^2\psi"] \\ E_2' \arrow[r, "\eta_2"] & \wedge^2 F_2 \end{tikzcd} \qquad \begin{tikzcd} \det E_1^{\otimes 2} \arrow[r, "\phi_1"] \arrow[d, "(\det\xi)^{\otimes 2}"'] & \det F_1 \arrow[d, "\det\psi"] \\ \det E_2^{\otimes 2} \arrow[r, "\phi_2"] & \det F_2. \end{tikzcd}$$

There is a functor $\mathrm{Pent}(S) \to \mathrm{Pent}'(S)$ that sends $\alpha : X \to S$ to the tuple $(E_\alpha, F_\alpha, \phi_\alpha, \eta_\alpha)$. There is also a functor $\mathrm{Pent}'(S) \to \mathrm{Pent}(S)$ that sends a tuple (E, F, ϕ, η) to the degree 5 cover $D(\Phi(\eta)) \subset \mathbb{P}E^\vee \to S$. The following is essentially a restatement of [Cas96, Theorem 3.8].

Theorem 1.3.8 (Casnati)**.** *The above functors define an equivalence of categories between* $\mathrm{Pent}(S)$ *and* $\mathrm{Pent}'(S)$.

Proof. The fact that $\mathrm{Pent}(S) \to \mathrm{Pent}'(S) \to \mathrm{Pent}(S)$ is equivalent to the identity was established by Casnati. We provide further details here that $\mathrm{Pent}'(S) \to \mathrm{Pent}(S) \to \mathrm{Pent}'(S)$ is naturally isomorphic to the identity on $\mathrm{Pent}'(S)$. Let $(E, F, \phi, \eta) \in \mathrm{Pent}(S)$ be given and let $X = D(\Phi(\eta))$ and $\alpha : X \to S$. By (3.5.2) of [Cas96], $\mathcal{O}_X$ admits a resolution

$$\begin{aligned} 0 \to \gamma^*(\det F^{-2} \otimes \det E^5(-5) &\to \gamma^*(F^\vee \otimes \det F^{-1} \otimes \det E^3)(-3) \\ &\to \gamma^*(F \otimes \det F^{-1} \otimes \det E^2)(-2) \to \mathcal{O}_{\mathbb{P}E^\vee} \to \mathcal{O}_X \to 0. \end{aligned}$$

Let A_1 be the image of $\gamma^*(F \otimes \det F^{-1} \otimes \det E^2)(-2) \to \mathcal{O}_{\mathbb{P}E^\vee}$. When we push forward the above equation by γ, we obtain

$$0 \to \mathcal{O}_S \to \alpha_*\mathcal{O}_X \to R^1\gamma_*A_1 \to 0.$$

We use a similar method as in Remark 1.3.3 to produce isomorphisms

$$E_\alpha \cong R^1\gamma_*A_1 \cong R^2\gamma_*A_2 \cong R^3\gamma_*(\gamma^*(\det F^{-2} \otimes \det E^5))(-5) \cong \det F^{-2} \det E^4 \otimes E^\vee.$$

Using ϕ, we turn this into an isomorphism $E_\alpha^\vee \cong E^\vee$, which we dualize to define $\xi : E \cong E_\alpha$. Using the uniqueness of the CE resolution, we also get an isomorphism $F \otimes \det F^{-1} \otimes \det E^2 \to F_\alpha$. Making use of ϕ again, we obtain an isomorphism $\psi : F \cong F_\alpha$. This in turn induces a map $G(2, F) \to G(2, F_\alpha)$ which sends X into X. Since the points of X span each fiber of $\mathbb{P}E' \cong \mathbb{P}E'_\alpha$, the following diagram of linear maps of spaces commutes

$$\begin{tikzcd}
\mathbb{P}E' \arrow[r] \arrow[d] & \mathbb{P}(\wedge^2 F) \arrow[d] \\
\mathbb{P}E'_\alpha \arrow[r] & \mathbb{P}(\wedge^2 F_\alpha).
\end{tikzcd}$$

In other words, there exists $t \in \mathcal{O}_S^\times(S)$ such that the first diagram below commutes, and, since E has rank 4, so does the second:

$$\begin{tikzcd}
E' \arrow[r, "\eta"] \arrow[d, "t\cdot\det\xi\otimes(\xi^{-1})^\vee"'] & \wedge^2 F \arrow[d, "\wedge^2\psi"] \\
E'_\alpha \arrow[r, "\eta_\alpha"] & \wedge^2 F_\alpha
\end{tikzcd} \qquad
\begin{tikzcd}
E' \arrow[r, "\eta"] \arrow[d, "\det(t\cdot\xi)\otimes((t\cdot\xi)^{-1})^\vee"'] & \wedge^2 F \arrow[d, "\wedge^2(t\cdot\psi)"] \\
E'_\alpha \arrow[r, "\eta_\alpha"] & \wedge^2 F_\alpha.
\end{tikzcd} \tag{1.3.17}$$

Finally, we must compare ϕ and ϕ_α. Since all the maps involved are isomorphisms of line bundles, there exists some $x \in \mathcal{O}_S^\times(S)$ such that the first diagram below commutes; recalling that E is rank 4 and F is rank 5, hence so does the second:

$$\begin{tikzcd}
\det E^{\otimes 2} \arrow[r, "\phi"] \arrow[d, "x\cdot\det(t\cdot\xi)^2"'] & \det F \arrow[d, "\det(t\cdot\psi)"] \\
\det E_\alpha^{\otimes 2} \arrow[r, "\phi_\alpha"] & \det F_\alpha
\end{tikzcd} \qquad
\begin{tikzcd}
\det E^{\otimes 2} \arrow[r, "\phi"] \arrow[d, "\det(x^2t\cdot\xi)^2"'] & \det F \arrow[d, "\det(x^3t\cdot\psi)"] \\
\det E_\alpha^{\otimes 2} \arrow[r, "\phi_\alpha"] & \det F_\alpha.
\end{tikzcd}$$

Finally, note that

$$\begin{tikzcd}
E' \arrow[r, "\eta"] \arrow[d, "\det(x^2t\cdot\xi)\otimes((x^2t\cdot\xi)^{-1})^\vee"'] & \wedge^2 F \arrow[d, "\wedge^2(x^3t\cdot\psi)"] \\
E'_\alpha \arrow[r, "\eta_\alpha"] & \wedge^2 F_\alpha.
\end{tikzcd}$$

also commutes, as it just rescales both vertical maps of the second diagram in (1.3.17) by x^6. Hence, pair of isomorphisms $x^2t\cdot\xi : E \to E_\alpha$ and $x^3t\cdot\psi : F \to F_\alpha$ define an arrow $(E, F, \phi, \eta) \to (E_\alpha, F_\alpha, \phi_\alpha, \eta_\alpha)$ in $\mathrm{Pent}'(S)$. □

1.3.4 Casnati–Ekedahl classes

We now define some preferred generators for $R^*(\mathcal{H}_{k,g})$ using the Chern classes of vector bundles appearing in the Casnati–Ekedahl resolution. Let $\pi : \mathcal{P} \to \mathcal{H}_{k,g}$

denote the universal $\mathbb{P}^1$-bundle and $\alpha : \mathcal{C} \to \mathcal{P}$ the universal degree k cover. We define $z := -\frac{1}{2}c_1(\omega_\pi) = c_1(\mathcal{O}_{\mathcal{P}}(1))$ and

$$c_2 := c_2(\pi_* \mathcal{O}_{\mathcal{P}}(1)) \qquad \Rightarrow \qquad z^2 + \pi^* c_2 = 0, \tag{1.3.18}$$

where the equality on the right follows from (1.2.1). Define $\mathcal{E}^\vee := E_\alpha^\vee$ to be the cokernel of $\mathcal{O}_{\mathcal{P}} \to \alpha_* \mathcal{O}_{\mathcal{C}}$, which is a rank $k-1$ vector bundle on $\mathcal{P}$. For $i = 1, \ldots, k-1$, we define classes $a_i \in A^i(\mathcal{H}_{k,g})$ and $a_i' \in A^{i-1}(\mathcal{H}_{k,g})$ by the formula

$$a_i := \pi_*(z \cdot c_i(\mathcal{E})), \quad a_i' := \pi_*(c_i(\mathcal{E})) \qquad \Rightarrow \qquad c_i(\mathcal{E}) = \pi^* a_i + \pi^* a_i' z. \tag{1.3.19}$$

By Example 1.3.1, $\mathcal{E}$ has relative degree $g+k-1$ on the fibers of $\mathcal{P} \to \mathcal{H}_{k,g}$, so $a_1' = g+k-1$. By the Casnati–Ekedahl structure theorem, the universal curve $\mathcal{C}$ embeds in $\mathbb{P}\mathcal{E}^\vee$. We have the associated Casnati–Ekedahl resolution

$$0 \to \gamma^* \mathcal{F}_{k-2}(-k) \to \gamma^* \mathcal{F}_{k-3}(-k+2) \to \cdots \to \gamma^* \mathcal{F}_1(-2) \to \mathcal{O}_{\mathbb{P}\mathcal{E}^\vee} \to \mathcal{O}_{\mathcal{C}} \to 0.$$

For each bundle $\mathcal{F}_j$, we define

$$f_{j,i} := \pi_*(z \cdot c_i(\mathcal{F}_j)), \quad f_{j,i}' := \pi_*(c_i(\mathcal{F}_j)) \qquad \Rightarrow \qquad c_i(\mathcal{F}_j) = \pi^* f_{j,i} + \pi^* f_{j,i}' z.$$

Definition 1.3.9. We define $c_2, a_i, a_i', f_{j,i}, f_{j,i}'$ to be the *Casnati–Ekedahl classes*, abbreviated CE classes.

Theorem 1.3.10. *The CE classes are tautological and they generate the tautological ring* $R^*(\mathcal{H}_{k,g})$.

Remark 1.3.11. The ranks of the $\mathcal{F}_i$ depend only on i and k, so this bounds the number of generators of $R^*(\mathcal{H}_{k,g})$ and their degrees in terms of k (independent of g).

Proof. First, we show that the Casnati–Ekedahl classes are tautological. Let us call a class on $\mathcal{P}$ *pre-tautological* if it is a polynomial in z and classes of the form $\alpha_*(c_1(\omega_f)^j)$. By the push-pull formula, the π pushforward of a pre-tautological class is tautological. Therefore, our goal is to show that the Chern classes of $\mathcal{E}$ and $\mathcal{F}_i$ are pre-tautological.

By Grothendieck–Riemann–Roch and the splitting principle, we have that the Chern classes of $\alpha_*(\omega_\alpha^{\otimes i}) = \alpha_*(\omega_f^{\otimes i}) \otimes (\omega_\pi^\vee)^{\otimes i}$ are pre-tautological. In particular, the Chern classes of $\mathcal{E}$ are pre-tautological by its defining exact sequence. By the construction of the Casnati–Ekedahl sequence, $\mathcal{F}_1$ is the kernel of a surjective map $\operatorname{Sym}^2 \mathcal{E} \twoheadrightarrow \alpha_*(\omega_\alpha^{\otimes 2})$, so the Chern classes of $\mathcal{F}_1$ are pre-tautological. Similarly, following the construction of $\mathcal{F}_i$ on [CE96, p. 445-446] and using the splitting principle, we inductively see that the Chern classes of all $\mathcal{F}_i$ are pre-tautological.

Next, we must show that all tautological classes are polynomials in Casnati–Ekedahl classes. We have a diagram

$$\begin{tikzcd} \mathcal{C} \arrow[r, "\iota"] \arrow[dr, "\alpha"'] \arrow[ddr, bend right, "f"'] & \mathbb{P}\mathcal{E}^\vee \arrow[d, "\gamma"] \\ & \mathcal{P} \arrow[d, "\pi"] \\ & \mathcal{H}_{k,g} \end{tikzcd} \tag{1.3.20}$$

First, note that

$$f_*(c_1(\omega_f)^i \cdot \alpha^*(\omega_\pi)^j) = \pi_*(\alpha_*(c_1(\omega_\alpha) + \alpha^* c_1(\omega_\pi))^i \cdot c_1(\omega_\pi)^j),$$

so using push-pull, it will suffice to show that $\pi_*(\alpha_*(c_1(\omega_\alpha)^i) \cdot z^j)$ is a polynomial in CE classes for all pairs i, j. Now, let $\zeta := c_1(\mathcal{O}_{\mathbb{P}\mathcal{E}^\vee}(1))$ and note that $\iota^*\zeta = c_1(\omega_\alpha)$. We have

$$\alpha_*(c_1(\omega_\alpha)^i) = \gamma_*\iota_*(\iota^*\zeta^i) = \gamma_*([\mathcal{C}] \cdot \zeta^i).$$

Grothendieck–Riemann–Roch for $\iota : \mathcal{C} \hookrightarrow \mathbb{P}\mathcal{E}^\vee$ tells us that $[\mathcal{C}] = \mathrm{ch}_{k-2}(\iota_*\mathcal{O}_\mathcal{C})$. By additivity of Chern characters in exact sequences, the later is a polynomial in ζ and the Chern classes of $\mathcal{F}_i$. Using the projective bundle theorem (1.2.1), $\gamma_*([\mathcal{C}] \cdot \zeta^i)$ is therefore a polynomial in the Chern classes of $\mathcal{E}$ and the $\mathcal{F}_i$. The π push forward of such a polynomial times any power of z is a polynomial in the CE classes (essentially from the definition of the CE classes). $\square$

Using the idea in the proof above, we explain how to rewrite the κ-classes in terms of CE classes.

Example 1.3.12 (κ-classes)**.** Let us retain notation as in (1.3.20). Writing ζ for the hyperplane class of $\mathbb{P}\mathcal{E}^\vee$ and z for the hyperplane class on $\mathcal{P}$, we have

$$c_1(\omega_f) = c_1(\omega_\alpha) + c_1(\omega_\pi) = \iota^*(\zeta - 2z).$$

By the push-pull formula, we have

$$\kappa_i = f_*(c_1(\omega_f)^{i+1}) = \pi_*\gamma_*\iota_*(\iota^*(\zeta - 2z)^{i+1}) = \pi_*\gamma_*([\mathcal{C}] \cdot (\zeta - 2z)^{i+1}). \tag{1.3.21}$$

Meanwhile, the fundamental class of $\mathcal{C} \subset \mathbb{P}\mathcal{E}^\vee$ is

$$\begin{aligned} [\mathcal{C}] &= \sum_{i=1}^{k-3} (-1)^{i-1} \operatorname{ch}_{k-2}(\mathcal{F}_i(-i-1)) + (-1)^{k-2} \operatorname{ch}_{k-2}(\mathcal{F}_{k-2}(-k)) \\ &= \begin{cases} -\operatorname{ch}_1(\det \mathcal{E}(-3)) = c_1(\det \mathcal{E}^\vee(3)) & \text{if } k = 3 \\ -\operatorname{ch}_2(\mathcal{F}(-2)) + \operatorname{ch}_2(\det \mathcal{E}(-4)) = c_2(\mathcal{F}^\vee(2)) & \text{if } k = 4 \\ -\operatorname{ch}_3(\mathcal{F}(-2)) + \operatorname{ch}_3((\mathcal{F}^\vee \otimes \det \mathcal{E})(-3)) - \operatorname{ch}_3(\det \mathcal{E}(-5)) & \text{if } k = 5. \end{cases} \end{aligned} \tag{1.3.22}$$

Using (1.3.21) and (1.3.22), it is straightforward to compute κ_i in terms of the CE classes using a computer.

In degree $k = 3$, the CE classes are c_2, a_1, a_2, a_2'. In degrees $k = 4, 5$, self-duality of the Casnati–Ekedahl resolution implies that all CE classes are expressible in terms of c_2 the a_i, a'_i and the $b_i := f_{1,i}$ and $b'_i := f'_{1,i}$. These classes are all pulled back from a moduli space $\mathcal{B}_{k,g}$ of (pairs of) vector bundles on $\mathbb{P}^1$, which we shall construct in the next section.

1.4 Pairs of vector bundles on $\mathbb{P}^1$-bundles

By the results of Casnati–Ekedahl and Casnati in the previous section, there is a correspondence between covers of $\mathbb{P}^1$ and certain linear algebraic data. In this section, following ideas of Bolognesi–Vistoli [BV12], we construct moduli stacks parametrizing the associated linear algebraic data and describe the Chow rings of these stacks. In [BV12], Bolognesi–Vistoli gave a quotient stack presentation for the moduli stack parametrizing globally generated vector bundles on $\mathbb{P}^1$-fibrations. As explained in Section 1.2.3, we will instead make use of SL_2 quotients, since they have the same rational Chow ring as the PGL_2 quotient.

Definition 1.4.1. Let r, d be nonnegative integers.

1. The objects of $\mathcal{V}^{\dagger}_{r,d}$ are pairs (S, E) where E is a locally free sheaf of rank r on $\mathbb{P}^1 \times S$ whose restriction to each of the fibers of $\mathbb{P}^1 \times S \to S$ is globally generated of degree d. A morphism between objects (S, E) and (S', E') is a Cartesian diagram

$$\begin{tikzcd} \mathbb{P}^1 \times S' \arrow[r, "F"] \arrow[d] & \mathbb{P}^1 \times S \arrow[d] \\ S' \arrow[r] & S \end{tikzcd}$$

together with an isomorphism $\phi : F^*E \to E'$.

2. We define $\mathcal{V}_{r,d}$ to be the SL_2 quotient of $\mathcal{V}^{\dagger}_{r,d}$. Explicitly, the objects of $\mathcal{V}_{r,d}$ are triples (S, V, E) where S is a k-scheme, V is a rank 2 vector bundle on S with trivial determinant, and E is a rank r vector bundle on $\mathbb{P}V$ whose restrictions to the fibers of $\mathbb{P}V \to S$ are globally generated of degree d. A morphism between objects (S, V, E)

and (S', V', E') is a Cartesian diagram

$$\begin{tikzcd} \mathbb{P}V' \arrow[r, "F"] \arrow[d] & \mathbb{P}V \arrow[d] \\ S' \arrow[r] & S \end{tikzcd}$$

together with an isomorphism $\phi : F^*E \to E'$.

Bolognesi–Vistoli gave a presentation for $\mathcal{V}^\dagger_{r,d}$ as a quotient stack, which we briefly summarize here. Let $M_{r,d}$ be the affine space that represents the functor which sends a scheme S to the set of matrices of size $(r+d) \times d$ with entries in $H^0(\mathbb{P}^1_S, \mathcal{O}_{\mathbb{P}^1_S}(1))$. We can identify such a matrix with the associated map

$$\mathcal{O}_{\mathbb{P}^1_S}(-1)^d \to \mathcal{O}_{\mathbb{P}^1_S}^{r+d}.$$

Let $\Omega_{r,d} \subset M_{r,d}$ denote the open subscheme parametrizing injective maps with locally free cokernel. The group GL_d acts $M_{r,d}$ by multiplication on the left, GL_{r+d} by multiplication on the right. Bolognesi–Vistoli establish [BV12, Theorem 4.4] that

$$\mathcal{V}^\dagger_{r,d} \cong [\Omega_{r,d}/\operatorname{GL}_d \times \operatorname{GL}_{d+r}].$$

The group SL_2 acts by change of coordinates on $H^0(\mathbb{P}^1_S, \mathcal{O}_{\mathbb{P}^1_S}(1))$. This commutes with the $\mathrm{GL}_d \times \mathrm{GL}_{d+r}$ action. Thus we obtain the following.

Proposition 1.4.2. *There is an isomorphism of fibered categories*

$$\mathcal{V}_{r,d} \cong [\mathcal{V}^\dagger_{r,d}/\operatorname{SL}_2] \cong [\Omega_{r,d}/\operatorname{GL}_d \times \operatorname{GL}_{r+d} \times \operatorname{SL}_2].$$

Remark 1.4.3. Bolognesi–Vistoli also describe the PGL_2 quotient of $\mathcal{V}^\dagger_{r,d}$, which is slightly more subtle. This distinction is important in their work which concerns *integral* coefficients.

To parametrize the linear algebraic data associated to a low degree cover of $\mathbb{P}^1$, we are interested in products of the form $\mathcal{V}_{r,d} \times_{\mathrm{BSL}_2} \mathcal{V}_{s,e}$, which parametrize a pair of vector bundles on the same $\mathbb{P}^1$-bundle. Let $G_{r,d,s,e} := \mathrm{GL}_d \times \mathrm{GL}_{r+d} \times \mathrm{GL}_e \times \mathrm{GL}_{s+e}$. The group $G_{r,d,s,e} \times \mathrm{SL}_2$ acts on $M_{r,d}$ via the projection $G_{r,d,s,e} \times \mathrm{SL}_2 \to \mathrm{GL}_d \times \mathrm{GL}_{r+d} \times \mathrm{SL}_2$; and similarly on $M_{s,e}$ via the projection $G_{r,d,s,e} \times \mathrm{SL}_2 \to \mathrm{GL}_e \times \mathrm{GL}_{s+e} \times \mathrm{SL}_2$. By Proposition 1.4.2, it follows that

$$\mathcal{V}_{r,d} \times_{\mathrm{BSL}_2} \mathcal{V}_{s,e} = [\Omega_{r,d} \times \Omega_{s,e} / G_{r,d,s,e} \times \mathrm{SL}_2]. \tag{1.4.1}$$

Let T_d and T_{r+d} denote the universal vector bundles on BGL_d and BGL_{r+d}; similarly, let S_e and S_{s+e} be the universal vector bundles on BGL_e and BGL_{s+e}. The Chow ring of $\mathrm{B}(G_{r,d,s,e} \times \mathrm{SL}_2)$ is the free $\mathbb{Q}$-algebra on the Chern classes of $T_d, T_{r+d}, S_e, S_{s+e}$, together with the universal second Chern class c_2 on BSL_2. Let us denote these classes by

$$\begin{aligned} t_i &= c_i(T_d) \qquad \text{and} \qquad u_i = c_i(T_{r+d}) \\ v_i &= c_i(S_e) \qquad \text{and} \qquad w_i = c_i(S_{s+e}). \end{aligned}$$

Since $\Omega_{r,d} \times \Omega_{s,e}$ is open inside the affine space $M_{r,d} \times M_{s,e}$, the excision and homotopy properties imply

$$A^*(\mathcal{V}_{r,d} \times_{\mathrm{BSL}_2} \mathcal{V}_{s,e}) \text{ is generated by the restrictions of the } t_i, u_i, v_i, w_i. \tag{1.4.2}$$

We now identify the restrictions of the tautological bundles T_d and T_{d+r} in terms of the universal rank r, degree d vector bundle on $\mathbb{P}^1$. Let $\pi : \mathcal{P} \to \mathcal{V}_{r,d}$ be the universal $\mathbb{P}^1$-bundle. We write $z := c_1(\mathcal{O}_{\mathcal{P}}(1)) \in A^1(\mathcal{P})$. We have $c_2 = c_2(\pi_* \mathcal{O}_{\mathcal{P}}(1)) \in A^2(\mathcal{V}_{r,d})$, the universal second Chern class (pulled back via the natural map $\mathcal{V}_{r,d} \to \mathrm{BSL}_2$). Note that

$c_1(\pi_*\mathcal{O}_{\mathcal{P}}(1)) = 0$, so by Equation (1.2.1),

$$A^*(\mathcal{P}) = A^*(\mathcal{V}_{r,d})[z]/(z^2 + \pi^*c_2).$$

Let $\mathcal{E}$ be the universal rank r, degree d vector bundle on $\mathcal{P}$. The Chern classes of $\mathcal{E}$ may thus be written as

$$c_i(\mathcal{E}) = \pi^*a_i + (\pi^*a_i')z \qquad \text{where} \qquad a_i \in A^i(\mathcal{V}_{r,d}), \quad a_i' \in A^{i-1}(\mathcal{V}_{r,d}).$$

Note that $a_1' = d$. Let $\gamma : \mathcal{V}_{r,d} \times_{\mathrm{BSL}_2} \mathcal{V}_{s,e} \to B(G_{r,d,s,e} \times \mathrm{SL}_2)$ be the structure map. Then by [Lar21b, Lemma 3.2] (noting that $\det(\pi_*\mathcal{O}_{\mathcal{P}}(1))$ is trivial), we have

$$\gamma^*T_d = \pi_*\mathcal{E}(-1) \qquad \text{and} \qquad \gamma^*T_{r+d} = \pi_*\mathcal{E}. \tag{1.4.3}$$

Since $R^1\pi_*\mathcal{E}(-1)$ and $R^1\pi_*\mathcal{E}$ are zero, Grothendieck–Riemann–Roch says that the Chern characters of $\pi_*\mathcal{E}(-1)$ and $\pi_*\mathcal{E}$ are push forwards by π of polynomials in the $c_i(\mathcal{E})$ and z. The push forward of such a polynomial is a polynomial in the a_i, a_i' and c_2. In particular, the restrictions of t_i and u_i to $A^*(\mathcal{V}_{r,d})$ are polynomials in $a_1, \ldots, a_r, a_2', \ldots, a_r'$ and c_2.

1.4.1 The rational Chow ring

Let us denote the universal rank s vector bundle from the second factor of $\mathcal{V}_{r,d} \times_{\mathrm{BSL}_2} \mathcal{V}_{s,e}$ by $\mathcal{F}$ on $\mathcal{P}$ and its Chern classes by

$$c_i(\mathcal{F}) = \pi^*b_i + (\pi^*b_i')z \qquad \text{where} \qquad b_i \in A^i(\mathcal{V}_{s,e}), \quad b_i' \in A^{i-1}(\mathcal{V}_{s,e}).$$

It follows from Equation (1.4.2) and the discussion following (applied to both factors of the product) that the a_i, a_i', b_i, b_i' and c_2 are generators for $A^*(\mathcal{V}_{r,d} \times_{\mathrm{BSL}_2} \mathcal{V}_{s,d})$. We now show that there are no relations among these generators in low degrees. This is a generalization of [Lar21b, Theorem 4.1], which shows $A^*(\mathcal{V}_{r,d})$ is generated by the a_i, a_i', and c_2 with no relations in degrees less than $d+1$.

Theorem 1.4.4. *The rational Chow ring of* $\mathcal{V}_{r,d} \times_{\mathrm{BSL}_2} \mathcal{V}_{s,e}$ *is generated as a* $\mathbb{Q}$*-algebra by*

$$c_2, a_1, \dots, a_r, a_2', \dots, a_r', b_1, \dots, b_s, b_2', \dots, b_s',$$

and all relations have degree at least $\min(d,e)+1$. *In the notation of Equation* (1.2.2),

$$\begin{aligned}&\mathrm{Trun}^{\min(d,e)+1} A^*(\mathcal{V}_{r,d} \times_{\mathrm{BSL}_2} \mathcal{V}_{s,e}) \\ &\qquad = \mathrm{Trun}^{\min(d,e)+1} \mathbb{Q}[c_2, a_1, \dots, a_r, a_2', \dots, a_r', b_1, \dots, b_s, b_2', \dots, b_s'].\end{aligned}$$

Remark 1.4.5. (1) The codimension of the complement of $\Omega_{r,d} \subset M_{r,d}$ is r, so the Theorem does *not* follow immediately from dimension counting and excision if $\min(d,e) > \min(r,s)$.

(2) If $s=0$, the fact that there are no b_i classes follows immediately from [Lar21b, Theorem 4.1].

Proof. Let

$$M := [M_{r,d}/G_{r,d,s,e} \times \mathrm{SL}_2] \quad \text{and} \quad N := [M_{s,e}/G_{r,d,s,e} \times \mathrm{SL}_2].$$

Equation (1.4.1) says that $\mathcal{V}_{r,d} \times_{\mathrm{BSL}_2} \mathcal{V}_{s,e}$ is an open inside the vector bundle $M \oplus N$ over $B := \mathrm{B}(G_{r,d,s,e} \times \mathrm{SL}_2)$. The complement consists of two components, namely

$$X := [\Omega_{r,d}^c \times M_{s,e}/G_{r,d,s,e} \times \mathrm{SL}_2] \quad \text{and} \quad Y := [M_{r,d} \times \Omega_{s,e}^c/G_{r,d,s,e} \times \mathrm{SL}_2].$$

One readily checks that $X \subset \mathcal{V}_{r,d} \times_{\mathrm{BSL}_2} \mathcal{V}_{s,e}$ is irreducible of codimension r and $Y \subset \mathcal{V}_{r,d} \times_{\mathrm{BSL}_2} \mathcal{V}_{s,e}$ is irreducible of codimension s (see [BV12, Remark 4.3]). Excision gives a right-exact sequence

$$A^{*-r}(X) \oplus A^{*-s}(Y) \to A^*(M \oplus N) \to A^*(\mathcal{V}_{r,d} \times_{\mathrm{BSL}_2} \mathcal{V}_{s,e}) \to 0. \tag{1.4.4}$$

From this it is clear that there are no relations among the restrictions to $\mathcal{V}_{r,d} \times_{\mathrm{BSL}_2} \mathcal{V}_{s,e}$ of the Chern classes of T_d, T_{d+r}, S_e and S_{s+e} in degrees less than $\min(r,s)$. We now describe relations among the restrictions of these Chern classes in degrees $\min(r,s)$ up to $\min(d,e)$. (If $\min(d,e) < \min(r,s)$ we are already done.) In particular, shall conclude that

$$\begin{aligned}&\mathrm{Trun}^{\min(d,e)+1} A^*(\mathcal{V}_{r,d} \times_{\mathrm{BSL}_2} \mathcal{V}_{s,e}) \\ &\quad = \mathrm{Trun}^{\min(d,e)+1} \mathbb{Q}[c_2, t_1, \ldots, t_{r-1}, u_1, \ldots, u_r, v_1, \ldots v_{s-1}, w_1, \ldots, w_s].\end{aligned} \tag{1.4.5}$$

Since the classes in the statement of the theorem are generators and have the same degrees as those above, the statement in the theorem must hold for dimension reasons.

It suffices to understand the image of $A^{*-r}(X) \to A^*(M \oplus N)$, the other factor being similar. For this we resolve X (resp. Y) as in the proof of [Lar21b, Theorem 4.1], taking the N factor (resp. M factor) "along for the ride".

Using the excision sequence (1.4.4) and arguing exactly as in [Lar21b, Theorem 4.1], we have

$$A^*(\mathcal{V}_{r,d} \times_{\mathrm{BSL}_2} \mathcal{V}_{s,e}) = \frac{\mathbb{Q}[c_2, t_1, \ldots, t_d, u_1, \ldots, u_{r+d}, v_1, \ldots, v_e, w_1, \ldots, w_{s+e}]}{\langle f_{i,j} : 0 \le i \le 1, 0 \le j \le d-1\rangle + \langle g_{i,j} : 0 \le i \le 1, 0 \le j \le e-1\rangle}. \tag{1.4.6}$$

where

$$\begin{aligned} f_{1,j-1} &= -t_{j+r} + u_{j+r} + \ldots & f_{0,j} &= -(r+d)t_{j+r} + (d-j)u_{j+r} + \ldots \\ g_{1,j-1} &= -v_{j+s} + w_{j+s} + \ldots & g_{0,j} &= -(s+e)v_{j+s} + (e-j)w_{j+s} + \ldots. \end{aligned}$$

Hence, in $A^*(\mathcal{V}_{r,d} \times_{\mathrm{BSL}_2} \mathcal{V}_{s,e})$, the classes t_n for $r \le n \le d$ and u_m for $r+1 \le m \le d$ are expressible as polynomials in $c_2, t_1, \ldots, t_{r-1}, u_1, \ldots, u_r$. Moreover, after eliminating these higher degree generators, the $f_{i,j}$ produce no additional relations in degrees less than or equal to d among the restrictions to $\mathcal{V}_{r,d} \times_{\mathrm{BSL}_2} \mathcal{V}_{s,e}$ of

$$c_2, t_1, \ldots, t_{r-1}, u_1, \ldots, u_r, v_1, \ldots, v_e, w_1, \ldots, w_{s+e}.$$

With the analogous calculation for the $g_{i,j}$, equation (1.4.6) then implies (1.4.5), and hence the statement of the theorem. $\square$

1.4.2 Splitting loci

Every vector bundle E on $\mathbb{P}^1$ splits as a direct sum of line bundles, $E \cong \mathcal{O}(e_1) \oplus \cdots \oplus \mathcal{O}(e_r)$ for integers $e_1 \leq \cdots \leq e_r$. We call the non-decreasing sequence of integers $\vec{e} = (e_1, \ldots, e_r)$ the *splitting type* of E and will often abbreviate the corresponding sum of line bundles by $\mathcal{O}(\vec{e}) := \mathcal{O}(e_1) \oplus \cdots \oplus \mathcal{O}(e_r)$. Given a family of vector bundles $\mathcal{E}$ on a $\mathbb{P}^1$-bundle $\pi : \mathcal{P} \to B$, the base B is stratified by locally closed subvarieties

$$\{b \in B : \mathcal{E}_{\pi^{-1}(b)} \cong \mathcal{O}(\vec{e})\},$$

which we call the *splitting locus for $\vec{e}$*. A subscheme structure on splitting loci is defined in [Lar21c, Section 2], though it will not be necessary here.

The splitting type $\vec{e}$ of E is equivalent to the data of the ranks of cohomology groups $h^0(\mathbb{P}^1, E(m))$ for all $m \in \mathbb{Z}$. Conversely, the locus of points $b \in B$ where the fibers of $\mathcal{E}$ satisfy some cohomological condition is a union of splitting loci. For example, the locus in B where $\mathcal{E}$ fails to be globally generated on fibers is the union of splitting loci for splitting types $\vec{e}$ with $e_1 \leq -1$. Similarly, $\operatorname{Supp} R^1\pi_*\mathcal{E}(-2)$ is the union of all splitting loci with $e_1 \leq 0$.

Following the argument in [BV12, Lemma 5.1], the codimension in $\mathcal{V}_{r,d}$ of the splitting locus where the universal $\mathcal{E}$ over $\mathcal{V}_{r,d}$ has splitting type $\vec{e}$ on fibers of $\mathcal{P} \to \mathcal{V}_{r,d}$

is $h^1(\mathbb{P}^1, \mathcal{E}nd(\mathcal{O}(\vec{e})))$. If we have a $\mathbb{P}^1$-bundle equipped with two vector bundles, we can consider the intersections of splitting loci for both bundles. The simultaneous splitting locus in $\mathcal{V}_{r,d} \times_{\mathrm{BSL}_2} \mathcal{V}_{s,e}$ where $\mathcal{E}$ has splitting type $\vec{e}$ and $\mathcal{F}$ has splitting type $\vec{f}$ is equal to the product of the $\vec{e}$ splitting locus in $\mathcal{V}_{r,d}$ with the $\vec{f}$ splitting locus in $\mathcal{V}_{s,e}$, and therefore has codimension

$$h^1(\mathbb{P}^1, \mathcal{E}nd(\mathcal{O}(\vec{e}))) + h^1(\mathbb{P}^1, \mathcal{E}nd(\mathcal{O}(\vec{f}))). \tag{1.4.7}$$

1.5 The good opens and codimension bounds

For each $k = 3, 4, 5$ and genus g, we will define a stack $\mathcal{B}_{k,g}$ parametrizing the vector bundles associated to a degree k, genus g cover of $\mathbb{P}^1$. The stack $\mathcal{B}_{k,g}$ will come equipped with a universal $\mathbb{P}^1$-bundle $\pi : \mathcal{P} \to \mathcal{B}_{k,g}$. Then, we will define a vector bundle $\mathcal{U}_{k,g}$ on $\mathcal{P}$ whose sections on a fiber of $\mathcal{P} \to \mathcal{B}_{k,g}$ is the relevant space of sections in the linear algebraic data of covers appearing in Section 1.3.

Before treating the case for each k in depth, we briefly outline our construction of certain open substacks of the Hurwitz stack. For $k = 3$, we shall define $\mathcal{B}'_{3,g} \subseteq \mathcal{B}_{3,g}$ to be the open substack over which $\mathcal{U}_{3,g}$ is globally generated on the fibers of $\pi : \mathcal{P} \to \mathcal{B}$. For $k = 4, 5$, let us define the open substack

$$\mathcal{B}'_{k,g} := \mathcal{B}_{k,g} \smallsetminus \operatorname{Supp}(R^1\pi_*\mathcal{U}_{k,g}). \tag{1.5.1}$$

By the theorem on cohomology and base change, the restriction of $\pi_*\mathcal{U}_{k,g}$ to $\mathcal{B}'_{k,g}$ is locally free with fibers given by the relevant space of sections in the linear algebraic data of covers appearing in Section 1.3. We denote the total space of this vector bundle on $\mathcal{B}'_{k,g}$ by

$$\mathcal{X}'_{k,g} := \pi_*\mathcal{U}_{k,g}|_{\mathcal{B}'_{k,g}}.$$

As discussed in Section 1.4.2, the complement of $\mathcal{B}'_{k,g}$ is a union of splitting loci.

The splitting loci in the complement are determined by the condition that $\mathcal{U}_{k,g}$ has a summand of degree -2 or less. It will also be convenient to work with a slightly smaller open substack

$$\mathcal{B}_{k,g}^{\circ} := \mathcal{B}_{k,g} \smallsetminus \mathrm{Supp}(R^1\pi_*(\mathcal{U}_{k,g} \otimes \mathcal{O}_{\mathcal{P}}(-2))), \tag{1.5.2}$$

whose complement consists of splitting loci where $\mathcal{U}_{k,g}$ has a summand of degree 0 or less. (Having a cut-off in terms of a degree 0 summand will be cleaner than a cut-off in terms of a -2 summand for our next step of bounding the codimension of the complement. The open $\mathcal{B}_{k,g}^{\circ}$ also plays an important role in our sequel [CL21a], where the slightly stronger positivity condition on the fibers of $\mathcal{U}_{k,g}$ will be needed.)

Pulling back these open substacks along the natural map $\mathcal{H}_{k,g} \to \mathcal{B}_{k,g}$ defines open substacks of the Hurwitz space as in the diagram below

$$\begin{tikzcd} \mathcal{H}_{k,g}^{\circ} \arrow[r] \arrow[d] & \mathcal{H}'_{k,g} \arrow[r] \arrow[d] & \mathcal{H}_{k,g} \arrow[d] \\ \mathcal{B}_{k,g}^{\circ} \arrow[r] & \mathcal{B}'_{k,g} \arrow[r] & \mathcal{B}_{k,g}. \end{tikzcd} \tag{1.5.3}$$

In all cases, we shall see that $\mathcal{H}'_{k,g}$ is an open substack inside the vector bundle $\mathcal{X}'_{k,g}$ over $\mathcal{B}'_{k,g}$. In particular, we will find that the Chow ring of $\mathcal{H}'_{k,g}$ is generated by tautological classes. To turn this into meaningful results for the Chow ring of $\mathcal{H}_{k,g}$ we must describe the complement of $\mathcal{H}'_{k,g} \subseteq \mathcal{H}_{k,g}$. When $k = 3$, it turns out $\mathcal{H}'_{k,g} = \mathcal{H}_{k,g}$. For $k = 4$, the complement of $\mathcal{H}'_{4,g}$ contains covers that factor through an intermediate curve of low genus. Nevertheless, we show that the complement of $\mathcal{H}'_{k,g}$ intersects $\mathcal{H}_{4,g}^{\mathrm{nf}}$ in high codimension. Finally, for $k = 5$, we show that the complement of $\mathcal{H}'_{5,g}$ has high codimension.

Of course, the complement of $\mathcal{H}'_{k,g}$ is contained in the complement of $\mathcal{H}_{k,g}^{\circ}$, so it will suffice to bound the codimension of the complement of $\mathcal{H}_{k,g}^{\circ}$ (which in turn provides a lower bound on the codimension of the complement of $\mathcal{H}'_{k,g}$). For arbitrary g, the coefficient of g in the bounds we obtain will be sharp (and the same for $\mathcal{H}_{k,g}^{\circ}$ and $\mathcal{H}'_{k,g}$). In any particular case, however, one may find slight improvements on our bounds by enumerating the splitting loci in the complement $\mathcal{H}'_{k,g}$, and calculating their codimensions.

Along the way, we also bound the codimension of the complement of $\mathcal{B}^\circ_{k,g}$. We point out some immediate corollaries regarding the Chow rings of $\mathcal{B}^\circ_{k,g}$, which will be used in our subsequent work [CL21a].

1.5.1 Degree 3

As a warm-up, we first explain our set-up in degree 3, as it is simplest. The results in this subsection are not new (they have already been established by Bolognesi–Vistoli [BV12] and Patel–Vakil [PV15a]) but spelling them out in our language will be instructive; it will also be useful in our subsequent work [CL21a].

In Section 1.4, we gave a construction for $\mathcal{V}_{r,d}$ as the moduli space of vector bundles on $\mathbb{P}^1$-bundles. As discussed in Section 1.3.1, the linear algebraic data of a degree 3, genus g cover involves a rank 2, degree $g+2$ vector bundle E on $\mathbb{P}^1$ and section of $\det E^\vee \otimes \operatorname{Sym}^3 E$. We set

$$\mathcal{B}_{3,g} := \mathcal{V}_{2,g+2} \qquad \text{and} \qquad \mathcal{U}_{3,g} := \det \mathcal{E}^\vee \otimes \operatorname{Sym}^3 \mathcal{E},$$

where $\mathcal{E}$ is the universal rank 2 bundle on $\pi : \mathcal{P} \to \mathcal{V}_{2,g+2}$. There is a natural map $\mathcal{H}_{3,g} \to \mathcal{B}_{3,g}$ that sends a family of triple covers $C \xrightarrow{\alpha} P \to S$ in $\mathcal{H}_{3,g}(S)$ to the associated rank 2 vector bundle E_α on $P \to S$ in $\mathcal{B}_{3,g}(S)$. If $C \xrightarrow{\alpha} \mathbb{P}^1$ is an integral triple cover and $E_\alpha = \mathcal{O}(e_1) \oplus \mathcal{O}(e_2)$ is the associated rank 2 vector bundle on $\mathbb{P}^1$, then by [BV12, Proposition 2.2], we have $e_1, e_2 \geq \frac{g+2}{3}$. Equivalently, every summand of $\det E_\alpha^\vee \otimes \operatorname{Sym}^3 E_\alpha$ is non-negative. Hence, the map $\mathcal{H}_{3,g} \to \mathcal{B}_{3,g}$ factors through the substack $\mathcal{B}'_{3,g} \subseteq \mathcal{B}_{3,g}$ over which $\mathcal{U}_{3,g}$ is globally generated on fibers of $\mathcal{P} \to \mathcal{B}_{3,g}$. In particular, $\mathcal{H}'_{3,g} = \mathcal{H}_{3,g}$. We define $\mathcal{X}'_{3,g} := \pi_* \mathcal{U}_{3,g}|_{\mathcal{B}'_{3,g}}$, which is a vector bundle on $\mathcal{B}'_{3,g}$ by the theorem on cohomology and base change.

Lemma 1.5.1. *There is an open inclusion $\mathcal{H}_{3,g} = \mathcal{H}'_{3,g} \to \mathcal{X}'_{3,g}$. In particular, $A^*(\mathcal{H}_{3,g})$ is* *generated by the CE classes c_2, a_1, a_2, a_2', and therefore $A^*(\mathcal{H}_{3,g}) = R^*(\mathcal{H}_{3,g})$.*

Proof. The first sentence was essentially observed in [BV12, p. 12]. We include an explanation using our notation. Given a scheme S, the objects of $\mathcal{X}_{3,g}(S)$ are tuples $(P \to S, E, \eta)$ where $(P \to S, E)$ is an object of $\mathcal{B}'_{3,g}(S)$ and $\eta \in H^0(P, \operatorname{Sym}^3 E \otimes \det E^\vee)$. We define an open substack $\mathcal{Y}'_{3,g} \subset \mathcal{X}'_{3,g}$ by the condition that $V(\Phi(\eta)) \subset \mathbb{P}E^\vee \to S$ is a family of smooth curves, where Φ is as in (1.3.6). Considering the Hilbert polynomial of $V(\Phi(\eta))$, one sees that the fibers have arithmetic genus g. Theorem 1.3.4 now shows that there is an equivalence $\mathcal{H}'_{3,g} \cong \mathcal{Y}'_{3,g}$.

By excision, the Chow ring of $\mathcal{H}_{3,g} = \mathcal{H}'_{3,g}$ is generated by restrictions of classes on $\mathcal{X}'_{3,g}$. Since $\mathcal{X}'_{3,g}$ is a vector bundle over $\mathcal{B}'_{3,g}$, their Chow rings are isomorphic, so the statement about generators follows from Theorem 1.4.4. $\square$

1.5.2 Degree 4

By Casnati–Ekedahl's characterization of quadruple covers (Theorem 1.3.6), the linear algebraic data of a quadruple cover of $\mathbb{P}^1$ is equivalent to the data of: a rank 3 vector bundle E; a rank 2 vector bundle F; an isomorphism $\det F \cong \det E$; and a global section of $F^\vee \otimes \operatorname{Sym}^2 E$ on $\mathbb{P}^1$ having the right codimension. By Example 1.3.1, $\deg(E) = \deg(F) = g + 3$. The stacks $\mathcal{V}_{2,g+3}$ and $\mathcal{V}_{3,g+3}$ both admit natural morphisms to BSL_2, and the fiber product $\mathcal{V}_{3,g+3} \times_{\mathrm{BSL}_2} \mathcal{V}_{2,g+3}$ is the stack whose objects are quadruples (S, V, E, F) where S is a k-scheme, V is a rank 2-vector bundle with trivial determinant, E is a rank 3 vector bundle on $\mathbb{P}V$ whose restriction to the fibers of $\mathbb{P}V \to S$ is globally generated of degree $g + 3$, and F is a rank 2 vector bundle on $\mathbb{P}V$ whose restriction to the fibers of $\mathbb{P}V \to S$ is globally generated of degree $g + 3$.

The additional data of an isomorphism $\det F \cong \det E$ is captured by a $\mathbb{G}_m$ torsor over $\mathcal{V}_{3,g+3} \times_{\mathrm{BSL}_2} \mathcal{V}_{2,g+3}$ defined as follows. Let $\mathcal{E}$ be the universal rank 3 bundle and $\mathcal{F}$ be the universal rank 2 bundle on the universal $\mathbb{P}^1$-bundle $\pi : \mathcal{P} \to \mathcal{V}_{3,g+3} \times_{\mathrm{BSL}_2} \mathcal{V}_{2,g+3}$. Since $\det \mathcal{E}^\vee \otimes \det \mathcal{F}$ has degree 0 on each fiber of $\pi : \mathcal{P} \to \mathcal{V}_{3,g+3} \times_{\mathrm{BSL}_2} \mathcal{V}_{2,g+3}$, the theorem on cohomology and base change shows that $\mathcal{L} := \pi_*(\det \mathcal{E}^\vee \otimes \det \mathcal{F})$ is a line bundle with $\pi^*\mathcal{L} \cong \det \mathcal{E}^\vee \otimes \det \mathcal{F}$.

Definition 1.5.2. With notation as above, define the stack $\mathcal{B}_{4,g}$ to be the $\mathbb{G}_m$-torsor over $\mathcal{V}_{3,g+3} \times_{\mathrm{BSL}_2} \mathcal{V}_{2,g+3}$ given by the complement of the zero section of the line bundle $\mathcal{L}$.

The objects of $\mathcal{B}_{4,g}$ are tuples (S, V, E, F, ϕ) where (S, V, E, F) is an object of $\mathcal{V}_{3,g+3} \times_{\mathrm{BSL}_2} \mathcal{V}_{2,g+3}$ and ϕ is an isomorphism $\det F \cong \det E$. Recalling the notation of Section 1.3.2, given an object $C \xrightarrow{\alpha} P \to S$ of $\mathcal{H}_{4,g}(S)$, the restriction of E_α and F_α to fibers of $P \to S$ are both known to be globally generated (see Proposition 1.5.6). Hence, there is a natural map $\mathcal{H}_{4,g} \to \mathcal{B}_{4,g}$ that sends the family $C \xrightarrow{\alpha} P \xrightarrow{\pi} S$ to the tuple $(S, \pi_*\mathcal{O}_P(1)^\vee, E_\alpha, F_\alpha, \phi_\alpha)$.

By slight abuse of notation, let us denote the pullback to $\mathcal{B}_{4,g}$ of the universal $\mathbb{P}^1$-bundle by $\pi : \mathcal{P} \to \mathcal{B}_{4,g}$, and the universal rank 3 and 2 vector bundles on it by $\mathcal{E}$ and $\mathcal{F}$. Let $z = \mathcal{O}_{\mathcal{P}}(1)$ and write

$$c_i(\mathcal{E}) = \pi^* a_i + (\pi^* a_i')z \qquad \text{and} \qquad c_i(\mathcal{F}) = \pi^* b_i + (\pi^* b_i')z.$$

for $a_i, b_i \in A^i(\mathcal{B}_{4,g})$ and $a_i', b_i' \in A^{i-1}(\mathcal{B}_{4,g})$. Note that $a_1' = b_1' = g + 3$. Moreover, by definition of $\mathcal{B}_{4,g}$, we have $c_1(\det \mathcal{E}^\vee \otimes \det \mathcal{F}) = 0$, so $a_1 = b_1$. Further, by Lemma 1.2.1, we have

$$A^*(\mathcal{B}_{4,g}) = A^*(\mathcal{V}_{3,g+3} \times_{\mathrm{BSL}_2} \mathcal{V}_{2,g+3})/\langle c_1(\mathcal{L})\rangle = A^*(\mathcal{V}_{3,g+3} \times_{\mathrm{BSL}_2} \mathcal{V}_{2,g+3})/\langle a_1 - b_1\rangle. \quad (1.5.4)$$

Thus, Theorem 1.4.4 shows that $c_2, a_1, a_2, a_3, a_2', a_3', b_2', b_2$ generate $A^*(\mathcal{B}_{4,g})$ and

$$\mathrm{Trun}^{g+4} A^*(\mathcal{B}_{4,g}) = \mathrm{Trun}^{g+4} \mathbb{Q}[c_2, a_1, a_2, a_3, a_2', a_3', b_2', b_2]. \quad (1.5.5)$$

Next, we define $\mathcal{U}_{4,g} := \mathcal{F}^\vee \otimes \mathrm{Sym}^2 \mathcal{E}$ on $\mathcal{P}$. We then define $\mathcal{B}_{4,g}'$ and $\mathcal{B}_{4,g}^\circ$ by (1.5.1) and (1.5.2) respectively. Correspondingly, the open substacks $\mathcal{H}^\circ_{4,g} \subseteq \mathcal{H}'_{4,g} \subseteq \mathcal{H}_{4,g}$ are described by

$$\{S \to \mathcal{H}^\circ_{4,g}\} = \{S \to \mathcal{H}_{4,g} : R^1(\pi_S)_*(\mathcal{F}_S^\vee \otimes \operatorname{Sym}^2 \mathcal{E}_S \otimes \mathcal{O}_{\mathcal{P}_S}(-2)) = 0\}$$
$$\{S \to \mathcal{H}'_{4,g}\} = \{S \to \mathcal{H}_{4,g} : R^1(\pi_S)_*(\mathcal{F}_S^\vee \otimes \operatorname{Sym}^2 \mathcal{E}_S) = 0\}.$$

The key property of $\mathcal{H}'_{4,g}$ is that the map $\mathcal{H}'_{4,g} \to \mathcal{B}'_{4,g}$ factors through an open inclusion in the total space of a vector bundle $\mathcal{X}'_{4,g} := \pi_*\mathcal{U}_{4,g}|_{\mathcal{B}'_{4,g}}$.

Lemma 1.5.3. *There is an open inclusion $\mathcal{H}'_{4,g} \to \mathcal{X}'_{4,g}$. In particular, $A^*(\mathcal{H}'_{4,g}) = R^*(\mathcal{H}'_{4,g})$ is generated by the CE classes $c_2, a_1, a_2, a_3, a'_2, a'_3, b'_2, b_2$.*

Proof. The objects of $\mathcal{X}'_{4,g}$ are tuples (S, V, E, F, ϕ, η) where $(S, V, E, F, \phi) \in \mathcal{B}'_{4,g}$ and $\eta \in H^0(\mathbb{P}V, F^\vee \otimes \operatorname{Sym}^2 E)$. Letting Φ be as in (1.3.9), we define $\mathcal{Y}'_{4,g} \subset \mathcal{X}'_{4,g}$ to be the open substack defined by the condition that $V(\Phi(\eta)) \subset \mathbb{P}E^\vee \to S$ is a family of smooth curves. Considering the Hilbert polynomial of $V(\Phi(\eta))$, using (1.3.7), we see that the fibers have arithmetic genus g. Using Theorem 1.3.6, we see that $\mathcal{H}'_{4,g}$ is equivalent to $\mathcal{Y}'_{4,g}$

By excision, the Chow ring of $\mathcal{H}'_{4,g}$ is generated by restriction of classes from $\mathcal{X}'_{4,g}$. Since $\mathcal{X}'_{4,g}$ is a vector bundle over $\mathcal{B}'_{4,g}$, their Chow rings are isomorphic, so the statement about generators follows from (1.5.4). □

Lemma 1.5.4. *The codimension of $\operatorname{Supp}(R^1\pi_*(\mathcal{U}_{4,g} \otimes \mathcal{O}_{\mathcal{P}}(-2)))$ is at least $\frac{g+3}{4} - 4$. That is, the codimension of the complement of $\mathcal{B}^\circ_{4,g} \subseteq \mathcal{B}_{4,g}$ has codimension at least $\frac{g+3}{4} - 4$.*

Proof. By equation (1.4.7), the codimension of the support of $R^1\pi_*(\mathcal{F}^\vee \otimes \operatorname{Sym}^2 \mathcal{E} \otimes \mathcal{O}_{\mathcal{P}}(-2))$ is the minimum value of $h^1(\mathbb{P}^1, \mathcal{E}nd(\mathcal{O}(\vec{e}))) + h^1(\mathbb{P}^1, \mathcal{E}nd(\mathcal{O}(\vec{f}))$ as we range over splitting types $\vec{e} = (e_1, e_2, e_3)$ with $e_1 \leq e_2 \leq e_3$ and $\vec{f} = (f_1, f_2)$ with $f_1 \leq f_2$ and

$$h^1(\mathbb{P}^1, \mathcal{O}(\vec{f})^\vee \otimes (\operatorname{Sym}^2 \mathcal{O}(\vec{e})) \otimes \mathcal{O}_{\mathbb{P}^1}(-2)) > 0 \qquad \Leftrightarrow \qquad 2e_1 \leq f_2.$$

We have

$$h^1(\mathbb{P}^1, \mathcal{E}nd(\mathcal{O}(\vec{e}))) + h^1(\mathbb{P}^1, \mathcal{E}nd(\mathcal{O}(\vec{f}))) \geq 2e_3 - 2e_1 - 3 + f_2 - f_1 - 1.$$

To find the minimum, we consider the function of 5 real variables

$$f(x_1, x_2, x_3, y_1, y_2) := 2x_3 - 2x_1 + y_2 - y_1$$

on the compact region D defined by

$$0 \leq x_1 \leq x_2 \leq x_3, \quad x_1 + x_2 + x_3 = 1, \quad y_1 \leq y_2, \quad y_1 + y_2 = 1, \quad 2x_1 \leq y_2.$$

Since f is piecewise linear, its extreme values are attained where multiple boundary conditions intersect at a point. Code provided at [CL21c] performs the linear algebra to locate such points and evaluates f at the them to determine its minimum. The minimum is $\frac{1}{4}$, attained at $(\frac{1}{4}, \frac{3}{8}, \frac{3}{8}, \frac{1}{2}, \frac{1}{2})$. Thus,

$$\dim \operatorname{Supp} R^1\pi_*(\mathcal{U}_{4,g} \otimes \mathcal{O}_{\mathcal{P}}(-2)) \geq (g+3) \cdot \min_D(f) - 4 = \frac{g+3}{4} - 4. \qquad \square$$

For later use, let us note an immediate consequence of the previous lemma: Using excision and (1.5.4), we see

$$\operatorname{Trun}^{(g+3)/4-4} A^*(\mathcal{B}_{4,g}^\circ) = \operatorname{Trun}^{(g+3)/4-4} \mathbb{Q}[c_2, a_1, a_2, a_3, a_2', a_3', b_2', b_2]. \tag{1.5.6}$$

Just because the complement of $\mathcal{B}_{4,g}^\circ$ has high codimension inside $\mathcal{B}_{4,g}$ does *not* mean that the complement of $\mathcal{H}_{4,g}^\circ$ will have high codimension in $\mathcal{H}_{4,g}$. The condition for $\alpha : C \to \mathbb{P}^1$ to be in $\mathcal{H}_{4,g}^\circ$ is that $h^1(\mathbb{P}^1, F_\alpha^\vee \otimes \operatorname{Sym}^2 E_\alpha) = 0$. We shall refer to this as "our cohomological condition." Our cohomological condition fails for factoring covers, as we explain now. Suppose $\alpha : C \to \mathbb{P}^1$ factors as $C \xrightarrow{\beta} C' \xrightarrow{h} \mathbb{P}^1$ where C' has genus g'. We claim $E_\alpha = \mathcal{O}(g'+1) \oplus E'$ for some rank 2 bundle E'. Indeed, because β is a double cover, we have

$$\beta_* \mathcal{O}_C \cong \mathcal{O}_{C'} \oplus L,$$

where L is a line bundle on C'. Pushing forward again by h,

$$\alpha_* \mathcal{O}_C \cong \mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-g'-1) \oplus h_* L.$$

This establishes that E_α has an $\mathcal{O}(g'+1)$ summand. In particular, since some summand of F has degree at least $\frac{g+3}{2}$,

$$h^1(\mathbb{P}^1, F^\vee \otimes \operatorname{Sym}^2 E) \geq \frac{g+3}{2} - 2(g'+1) - 1.$$

Thus, covers that factor with g' small are never in $\mathcal{H}_{4,g}^\circ$. More precisely, if a factoring cover does satisfy our cohomological condition, then the genus of the intermediate curve must satisfy $2(g'+1) \geq \frac{g+3}{2}$.

Lemma 1.5.5. *The locus of degree 4 covers $C \to \mathbb{P}^1$ that factor $C \to C' \to \mathbb{P}^1$ where C' has genus g' has codimension $2(g'+1)$ in $\mathcal{H}_{4,g}$. Hence, the complement of $\mathcal{H}_{4,g}^\circ \cap \mathcal{H}_{4,g}^{\mathrm{nf}} \subset \mathcal{H}_{4,g}^\circ$ has codimension at least $\frac{g+3}{2}$.*

Proof. The dimension of the Hurwitz stack is the degree of the branch locus minus $3 = \dim \operatorname{Aut}(\mathbb{P}^1)$, giving $\dim \mathcal{H}_{4,g} = 2g+3$. Meanwhile, by Riemann–Hurwitz, the dimension of the space of genus g double covers of a fixed curve C' of genus g' is $2g-2-2(2g'-2)$. The dimension of the stack of genus g' double covers of $\mathbb{P}^1$ modulo $\operatorname{Aut}(\mathbb{P}^1)$ is $2g'-1$. Therefore, the dimension of the space of degree 4 covers that factor through a curve of genus g' is

$$2g-2-2(2g'-2)+2g'-1=2g+1-2g'=\dim\mathcal{H}_{4,g}-2(g'+1). \qquad \square$$

Covers that factor through a curve of low g' are therefore loci of fixed codimension that fail our cohomological condition. For this reason, in degree 4, our techniques will only prove that certain Chow groups of the locus of *non-factoring* covers are generated by tautological classes. Below, we collect some results about the splitting types of the vector bundles associated to a degree 4 cover. These facts were known to Schreyer [Sch86] (though Schreyer's notation differs from ours). We include proofs here as they demonstrate the geometric meaning of splitting types.

Proposition 1.5.6. *Suppose $\alpha : C \to \mathbb{P}^1$ is a degree 4 cover and $E_\alpha = \mathcal{O}(e_1)\oplus\mathcal{O}(e_2)\oplus\mathcal{O}(e_3)$ with $e_1 \leq e_2 \leq e_3$, and $F = \mathcal{O}(f_1)\oplus\mathcal{O}(f_2)$ with $f_1 \leq f_2$. The following are true:*

1. *$e_1+e_2+e_3 = f_1+f_2 = g+3$ and with $e_1 \geq 1$ if C irreducible.*

2. *If C is irreducible, $2e_1 \geq f_1$, and $2e_2 \geq f_2$. Hence F is globally generated.*

3. *If α does not factor then $e_1+e_3-f_2 \geq 0$.*

Proof. (1) follows from Example 1.3.1 and fact that $\det E_\alpha \cong \det F_\alpha$. If C is irreducible, we have $h^0(\mathbb{P}^1, E_\alpha^\vee) = h^0(\mathbb{P}^1, \alpha_*\mathcal{O}_C) - 1 = 0$, so $e_1 \geq 1$.

The remaining conditions can be seen from the description of C as the intersection of two relative quadrics on $\mathbb{P}E_\alpha^\vee$. Let us choose a splitting $E = \mathcal{O}(e_1)\oplus\mathcal{O}(e_2)\oplus\mathcal{O}(e_3)$ and corresponding coordinates X, Y, Z on $\mathbb{P}E^\vee$. The two quadrics that define C are of the form

$$p = p_{1,1}X^2 + p_{1,2}XY + p_{2,2}Y^2 + p_{1,3}XZ + p_{2,3}YZ + p_{3,3}Z^2 \tag{1.5.7}$$

$$q = q_{1,1}X^2 + q_{1,2}XY + q_{2,2}Y^2 + q_{1,3}XZ + q_{2,3}YZ + q_{3,3}, Z^2 \tag{1.5.8}$$

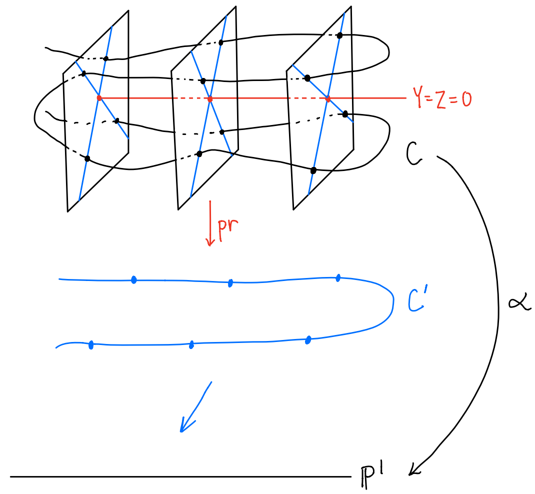


Figure 1.1. A factoring degree 4 cover.

where $p_{i,j}$ is a polynomial on $\mathbb{P}^1$ of degree $e_i + e_j - f_1$ and $q_{i,j}$ is a polynomial on $\mathbb{P}^1$ of degree $e_i + e_j - f_2$. If this degree is negative, then we mean this coefficient is zero.

(2) If $2e_1 < f_1$, then $p_{1,1} = q_{1,1} = 0$ and $C = V(p, q)$ would contain the curve $Y = Z = 0$, forcing C to be reducible. If $2e_2 < f_2$, then $q_{1,1} = q_{1,2} = q_{2,2} = 0$ so Z divides q. If C were irreducible, it would be contained in one of the linear components of $V(q)$ but this is impossible. The global generation of F follows because the inequalities imply $f_1 = g + 3 - f_2 \geq e_1 \geq 1$.

(3) If $e_1 + e_3 - f_2 \leq -1$, then we show α factors. This inequality implies

$$2e_1 - f_2 \leq e_1 + e_2 - f_2 \leq e_1 + e_3 - f_2 \leq -1,$$

so the coefficients $p_{1,1}, p_{1,2}$, and $p_{1,3}$ vanish. Therefore, p is a combination of Y^2, YZ, and Z^2. Hence, $V(p)$ is reducible in every fiber and contains the point $[1, 0, 0]$ in each fiber. In other words, each fiber of $C \to \mathbb{P}^1$ consists of two pairs of points colinear with $[1, 0, 0]$. Projection away from the line $Y = Z = 0$ defines a double cover $C \to C'$ that factors α. $\square$

The simultaneous splitting loci of the universal $\mathcal{E}$ and $\mathcal{F}$ over $\mathcal{H}_{4,g}$ give rise to a stratification of $\mathcal{H}_{4,g}$. In [DP15, Remark 4.2], Deopurkar–Patel show that the codimension of the splitting locus where $\mathcal{E}$ has splitting type $\vec{e}$ and $\mathcal{F}$ has splitting type $\vec{f}$ is

$$h^1(\mathbb{P}^1, \mathcal{E}nd(\mathcal{O}(\vec{e}))) + h^1(\mathbb{P}^1, \mathcal{E}nd(\mathcal{O}(\vec{f}))) - h^1(\mathbb{P}^1, \mathcal{O}(\vec{f})^\vee \otimes \operatorname{Sym}^2 \mathcal{O}(\vec{e})). \tag{1.5.9}$$

Note that this differs from (1.4.7) by $h^1(\mathbb{P}^1, \mathcal{O}(\vec{f})^\vee \otimes \operatorname{Sym}^2 \mathcal{O}(\vec{e}))$!

Example 1.5.7 ($g = 6$)**.** We have $\dim \mathcal{H}_{4,6} = \dim \mathcal{M}_6 = 15$. Using Proposition 1.5.6 (2), we see that the non-empty strata are

1. $\vec{e} = (3,3,3), \vec{f} = (4,5)$, (codimension 0): The generic stratum.

2. $\vec{e} = (2,3,4), \vec{f} = (4,5)$, (codimension 1): By Casnati-Del Centina [CDC02], the bielliptic locus is contained in this stratum as the locus where $p_{1,2} = 0$ and $p_{1,3} = 0$. Note that $\deg(p_{1,2}) = 0$ and $\deg(p_{1,3}) = 1$, so this represents 3 conditions, making the bielliptic locus codimension 4 inside $\mathcal{H}_{4,6}$.

3. $\vec{e} = (3,3,3), \vec{f} = (3,6)$, (codimension 2): This stratum consists of trigonal curves. We have $\mathbb{P}E^\vee \cong \mathbb{P}^1 \times \mathbb{P}^2$. Since $\deg(q_{i,j}) = 0$ and $\deg(p_{i,j}) = 3$ for all i, j, the projection onto the $\mathbb{P}^2$ factor realizes C as a degree 3 cover of a conic in $\mathbb{P}^2$.

4. $\vec{e} = (2,3,4), \vec{f} = (3,6)$, (codimension 2): Curves with a g^2_5. We have $p_{1,1} = 0$ and $\deg(q_{1,1}) = 1$, so the curve meets the line $Y = Z = 0$ in $\mathbb{P}E^\vee$ in one point, say $\nu \in C$. The canonical line bundle on C is the restriction of $\mathcal{O}_{\mathbb{P}E^\vee}(1) \otimes \omega_{\mathbb{P}^1}$, which contracts the line $Y = Z = 0$ in the map $\mathbb{P}E^\vee \to \mathbb{P}^5$. Thus, ν is contained in each of the planes spanned by the image of a fiber of α under the canoncial. Hence, the g^1_4 plus ν is a g^2_5. The locus of genus 6 curves possessing a g^2_5 is codimension 3 in $\mathcal{M}_6$, but this stratum has codimension 2 in $\mathcal{H}_{4,6}$ because projection from any point on a plane quintic gives a g^1_4.

5. $\vec{e} = (1, 4, 4), \vec{f} = (2, 7)$, (codimension 2): Hyperelliptic curves

The open $\mathcal{H}'_{4,6}$ is the union of strata (1), (2), and (3), while $\mathcal{H}^{\circ}_{4,6}$ contains only the generic stratum (1). The image in $\mathcal{M}_g$ of $\mathcal{H}'_{4,6}$ under the forgetful map is what Penev–Vakil [PV15b] call the "Mukai general locus" of genus 6 curves.

Using the numerical results of Lemma 1.5.6, we show that the codimension of *non-factoring* covers that fail our cohomological condition grows as a positive fraction of the genus.

Lemma 1.5.8. *The locus of non-factoring degree* 4 *covers* $\alpha : C \to \mathbb{P}^1$ *such that*

$$h^1(\mathbb{P}^1, F_\alpha^\vee \otimes \operatorname{Sym}^2 E_\alpha \otimes \mathcal{O}(-2)) > 0$$

has codimension at least $\frac{g+3}{4} - 4$. *That is, the codimension the complement of* $\mathcal{H}^{\circ}_{4,g} \cap \mathcal{H}^{\mathrm{nf}}_{4,g} \subset \mathcal{H}^{\mathrm{nf}}_{4,g}$ *is at least* $\frac{g+3}{4} - 4$.

Proof. By equation (1.5.9), the codimension of the locus of covers α with $E_\alpha = \mathcal{O}(\vec{e})$ and $F_\alpha = \mathcal{O}(\vec{f})$ is

$$\begin{aligned} u(\vec{e}, \vec{f}) &:= h^1(\mathbb{P}^1, \mathcal{E}nd(\mathcal{O}(\vec{e}))) + h^1(\mathbb{P}^1, \mathcal{E}nd(\mathcal{O}(\vec{f}))) - h^1(\mathbb{P}^1, \mathcal{O}(\vec{f})^\vee \otimes \operatorname{Sym}^2 \mathcal{O}(\vec{e})) \\ &\geq 2e_3 - 2e_1 + f_2 - f_1 - 4 - h^1(\mathbb{P}^1, \mathcal{O}(\vec{f})^\vee \otimes \operatorname{Sym}^2 \mathcal{O}(\vec{e})). \end{aligned}$$

Assuming α does not factor, Proposition 1.5.6 (2) and (3) show that the only summands of $\mathcal{O}(\vec{f})^\vee \otimes \operatorname{Sym}^2 \mathcal{O}(\vec{e})$ that can contribute to $h^1(\mathbb{P}^1, \mathcal{O}(\vec{f})^\vee \otimes \operatorname{Sym}^2 \mathcal{O}(\vec{e}))$ are $\mathcal{O}(2e_1 - f_2)$ and $\mathcal{O}(e_1 + e_2 - f_2)$. Thus, our task is to bound the function

$$2e_3 - 2e_1 + f_2 - f_1 - 4 - \max\{0, f_2 - 2e_1 - 1\} - \max\{0, f_2 - e_1 - e_2 - 1\}$$

from below on the region where the conditions of Proposition 1.5.6 hold and $2e_1 - f_2 \leq 0$, which is equivalent to $h^1(\mathbb{P}^1, \mathcal{O}(\vec{f})^\vee \otimes \operatorname{Sym}^2 \mathcal{O}(\vec{e}) \otimes \mathcal{O}(-2)) > 0$.

Let us introduce a function of 5 real variables

$$f(x_1, x_2, x_3, y_1, y_2) := 2x_3 - 2x_1 + y_2 - y_1 - \max\{0, y_2 - 2x_1\} - \max\{0, y_2 - x_1 - x_2\}$$

so that

$$u(\vec{e}, \vec{f}) \geq (g+3) f\left(\frac{e_1}{g+3}, \frac{e_2}{g+3}, \frac{e_3}{g+3}, \frac{f_1}{g+3}, \frac{f_2}{g+3}\right) - 4.$$

We wish to minimize f on the compact region defined by

$$x_1 + x_2 + x_3 = 1, \quad y_1 + y_2 = 1, \quad 0 \leq x_1 \leq x_2 \leq x_3, \quad 0 \leq y_1 \leq 2x_1 \leq y_2 \leq 2x_2, x_1 + x_3.$$

These correspond to the conditions from Proposition 1.5.6, together with the condition that $2e_1 \leq f_2$, which must be satisfied if the cohomological condition is failed. Since f is piecewise linear, its extreme values are attained where multiple boundary conditions (including those where the function changes) intersect at a point. A program provied in [CL21c] performs the linear algebra to locate such points and evaluates f at them to determine its minimum. The minimum is $\frac{1}{4}$, attained at $(x_1, x_2, x_3, y_1, y_2) = (\frac{1}{4}, \frac{3}{8}, \frac{3}{8}, \frac{1}{2}, \frac{1}{2})$. It follows that, if $h^1(\mathbb{P}^1, \mathcal{O}(\vec{f})^\vee \otimes \operatorname{Sym}^2 \mathcal{O}(\vec{e}) \otimes \mathcal{O}(-2)) > 0$, then $u(\vec{e}, \vec{f}) \geq \frac{g+3}{4} - 4$. $\square$

Remark 1.5.9. Aaron Landesman points out that our above Lemma 1.5.8 parallels [Bha05, Lemma 11] of Bhargava. Bhargava's two cases $a_{11} = 0$ or $a_{11} = a_{12} = 0$ correspond to the fact that either $\mathcal{O}(2e_1 - f_2)$ or $\mathcal{O}(2e_1 - f_2)$ and $\mathcal{O}(e_1 + e_2 - f_2)$ are the only possible negative summands of $F_\alpha^\vee \otimes \operatorname{Sym}^2 E_\alpha$ for a non-factoring cover α.

Lemmas 1.5.5 and 1.5.8 together should be thought of as saying that $\mathcal{H}_{4,g}^\circ$ and $\mathcal{H}_{4,g}^{\mathrm{nf}}$ are "good approximations" of each other. We can now complete the proof of Theorem 1.1.2.

Proof of Theorem 1.1.2. Suppose $i < \frac{g+3}{4} - 4$. Consider the restriction maps

$$A^i(\mathcal{H}_{4,g}^{\mathrm{nf}}) \xrightarrow{\sim} A^i(\mathcal{H}_{4,g}^{\mathrm{nf}} \cap \mathcal{H}_{4,g}^{\circ}) \xleftarrow{\sim} A^i(\mathcal{H}_{4,g}^{\circ}).$$

Lemma 1.5.8 says the arrow on the left is an isomorphism; Lemma 1.5.5 says the arrow on the right is an isomorphism. In turn then, we also have

$$R^i(\mathcal{H}_{4,g}^{\mathrm{nf}}) \xrightarrow{\sim} R^i(\mathcal{H}_{4,g}^{\mathrm{nf}} \cap \mathcal{H}_{4,g}^{\circ}) \xleftarrow{\sim} R^i(\mathcal{H}_{4,g}^{\circ}),$$

where R^i of an open substack of $\mathcal{H}_{4,g}$ means the image of tautological classes under the restriction to that open. Since $\mathcal{H}_{4,g}^{\circ} \subset \mathcal{H}'_{4,g}$, Lemma 1.5.3 implies $A^i(\mathcal{H}_{4,g}^{\circ}) = R^i(\mathcal{H}_{4,g}^{\circ})$. Thus, we conclude

$$A^i(\mathcal{H}_{4,g}^{\mathrm{nf}}) = A^i(\mathcal{H}_{4,g}^{\circ}) = R^i(\mathcal{H}_{4,g}^{\circ}) = R^i(\mathcal{H}_{4,g}^{\mathrm{nf}}). \qquad \square$$

1.5.3 Degree 5

Using Casnati's characterization of regular degree 5 covers (Theorem 1.3.8), a regular degree 5 cover of is equivalent to the data of a rank 4 vector bundle E; a rank 5 vector bundle F; an isomorphism $(\det E)^{\otimes 2} \cong \det F$; and a global section of $\mathcal{H}om(E^\vee \otimes \det E, \wedge^2 F)$ satisfying certain conditions. By Example 1.3.1, if a cover $\alpha : C \to \mathbb{P}^1$ has genus g, then $\deg(E_\alpha) = g + 4$. In turn, $\deg(F_\alpha) = 2\deg(E_\alpha) = 2g + 8$. To build the appropriate base stack, we start with $\mathcal{V}_{4,g+4} \times_{\mathrm{BSL}_2} \mathcal{V}_{5,2g+8}$ which parametrizes tuples (S, V, E, F) where V is a rank 2 vector bundle on S with trivial determinant, and E and F are vector bundles of the appropriate ranks and degrees on $\mathbb{P}V$. We let $\mathcal{E}$ denote the universal rank 4 vector bundle and $\mathcal{F}$ the universal rank 5 bundle on the universal $\mathbb{P}^1$-bundle $\pi : \mathcal{P} \to \mathcal{V}_{4,g+4} \times_{\mathrm{BSL}_2} \mathcal{V}_{5,2g+8}$. Since $\det \mathcal{E}^{\otimes 2} \otimes \det \mathcal{F}^\vee$ is a line bundle of degree 0 on each fiber of π, we have $\det \mathcal{E}^{\otimes 2} \otimes \det \mathcal{F}^\vee \cong \pi^*\mathcal{L}$ where $\mathcal{L} := \pi_*(\det \mathcal{E}^{\otimes 2} \otimes \det \mathcal{F}^\vee)$, which is a line bundle by cohomology and base change.

Definition 1.5.10. With notation as above, we define the stack $\mathcal{B}_{5,g}$ as the $\mathbb{G}_m$-torsor over $\mathcal{V}_{5,g+4} \times_{\mathrm{BSL}_2} \mathcal{V}_{5,2g+8}$ given by the complement of the zero section of the line bundle $\mathcal{L}$.

By slight abuse of notation, we continue to denote the universal $\pi : \mathbb{P}^1$-bundle by $\mathcal{P} \to \mathcal{B}_{5,g}$ and the universal rank 4 and 5 vector bundles on it by $\mathcal{E}$ and $\mathcal{F}$. Let $z = \mathcal{O}_{\mathcal{P}}(1)$ and write

$$c_i(\mathcal{E}) = \pi^* a_i + (\pi^* a_i')z \qquad \text{and} \qquad c_i(\mathcal{F}) = \pi^* b_i + (\pi^* b_i')z.$$

for $a_i, b_i \in A^i(\mathcal{B}_{5,g})$ and $a_i', b_i' \in A^{i-1}(\mathcal{B}_{5,g})$. Note that $2a_1' = b_1' = 2(g+4)$. Moreover, by definition of $\mathcal{B}_{5,g}$, we have $c_1(\det \mathcal{E}^{\otimes 2} \otimes \det \mathcal{F}^\vee) = 0$, so $b_1 = 2a_1$. Using Lemma 1.2.1 and Theorem 1.4.4 as in the previous subsection, we have

$$\mathrm{Trun}^{g+5} A^*(\mathcal{B}_{5,g}) = \mathrm{Trun}^{g+5}\, \mathbb{Q}[c_2, a_1, \ldots, a_4, a_2', \ldots, a_4', b_2, \ldots, b_5, b_2', \ldots, b_5']. \tag{1.5.10}$$

We define $\mathcal{U}_{5,g} := \mathcal{H}om(\mathcal{E}^\vee \otimes \det \mathcal{E}, \wedge^2 \mathcal{F})$, and $\mathcal{B}_{5,g}'$ and $\mathcal{B}_{5,g}^\circ$ as in (1.5.1) and (1.5.2), respectively. Given a map $S \to \mathcal{H}_{5,g}$, let $\pi_S : \mathcal{P}_S \to S$ denote the $\mathbb{P}^1$-bundle and let $\mathcal{E}_S$ (resp. $\mathcal{F}_S$) be the rank 4 (resp. rank 5) vector bundle on $\mathcal{P}_S$ associated to the family in the sense of Casnati–Ekedahl. The open substacks $\mathcal{H}_{5,g}^\circ \subseteq \mathcal{H}_{5,g}' \subseteq \mathcal{H}_{5,g}$ are defined by

$$\begin{aligned}\{S \to \mathcal{H}_{5,g}^\circ\} = \{S \to \mathcal{H}_{5,g} : R^1(\pi_S)_*(\mathcal{H}om(\mathcal{E}_S^\vee \otimes \det \mathcal{E}_S, \wedge^2 \mathcal{F}_S) \otimes \mathcal{O}_{\mathcal{P}_S}(-2)) = 0 \\ \text{and } \mathcal{F}_S \text{ globally generated on fibers of } \pi_S\}. \\ \{S \to \mathcal{H}_{5,g}'\} = \{S \to \mathcal{H}_{5,g} : R^1(\pi_S)_*(\mathcal{H}om(\mathcal{E}_S^\vee \otimes \det \mathcal{E}_S, \wedge^2 \mathcal{F}_S)) = 0 \\ \text{and } \mathcal{F}_S \text{ globally generated on fibers of } \pi_S\}.\end{aligned}$$

The important feature of the open $\mathcal{H}_{5,g}'$ is that it can be realized as an open inside the vector bundle $\mathcal{X}_{5,g}' := \pi_* \mathcal{U}_{5,g}|_{\mathcal{B}_{5,g}'}$ over $\mathcal{B}_{5,g}'$.

Lemma 1.5.11. *There is an open inclusion $\mathcal{H}'_{5,g} \to \mathcal{X}'_{5,g}$. In particular, $A^*(\mathcal{H}'_{5,g}) = R^*(\mathcal{H}'_{5,g})$ is generated by the CE classes $c_2, a_1, \ldots, a_4, a'_2, \ldots, a'_4, b_2, \ldots, b_5, b'_2, \ldots, b'_5$.*

Proof. The objects of $\mathcal{X}'_{5,g}$ are tuples (S, V, E, F, ϕ, η) where $(S, V, E, F, \phi) \in \mathcal{B}'_{5,g}$ and $\eta \in H^0(\mathbb{P}V, \mathcal{H}om(E^\vee \otimes \det E, \wedge^2 F))$. Using the notation of Section 1.3.3, we define $\mathcal{Y}'_{5,g} \subset \mathcal{X}'_{5,g}$ to be the open substack defined by the condition that $D(\Phi(\eta)) \subset \mathbb{P}E^\vee \to S$ is a family of smooth curves. Considering their Hilbert polynomials as determined by the resolution (1.3.14), we see that the fibers of $D(\Phi(\eta)) \to S$ have arithmetic genus g. Applying Theorem 1.3.8, we see that $\mathcal{H}'_{5,g}$ is equivalent to $\mathcal{Y}'_{5,g}$

By excision, the Chow ring of $\mathcal{H}'_{5,g}$ is generated by restriction of classes from $\mathcal{X}'_{5,g}$. Since $\mathcal{X}'_{5,g}$ is a vector bundle over $\mathcal{B}'_{5,g}$, their Chow rings are isomorphic, so the statement about generators follows from Theorem 1.4.4. □

Now we show that the complements of the opens we have defined have high codimension.

Lemma 1.5.12. *The support of $R^1\pi_*(\mathcal{U}_{5,g} \otimes \mathcal{O}_{\mathcal{P}}(-2))$ has codimension at least $\frac{g+4}{5} - 16$. That is, the codimension of the complement of $\mathcal{B}^\circ_{5,g} \subset \mathcal{B}_{5,g}$ is at least $\frac{g+4}{5} - 16$.*

Proof. By (1.4.7), the codimension of the support of $R^1\pi_*(\mathcal{H}om(\mathcal{E}^\vee \otimes \det \mathcal{E}, \wedge^2 \mathcal{F}) \otimes \mathcal{O}_{\mathcal{P}}(-2))$ is the minimum value of

$$h^1(\mathbb{P}^1, \mathcal{E}nd(\mathcal{O}(\vec{e}))) + h^1(\mathbb{P}^1, \mathcal{E}nd(\mathcal{O}(\vec{f})))$$

as we range over splitting types $\vec{e}$ of degree $g + 4$ and $\vec{f}$ of degree $2g + 8$ so that

$$h^1(\mathbb{P}^1, \mathcal{H}om(\mathcal{O}(\vec{e})^\vee \otimes \det \mathcal{O}(\vec{e}), \wedge^2 \mathcal{O}(\vec{f})) \otimes \mathcal{O}_{\mathbb{P}^1}(-2)) > 0 \quad \Leftrightarrow \quad e_1 + f_1 + f_2 - (g + 4) \leq 0.$$

Similar to the proof of Lemma 1.5.4, we may find this minimum by finding the minimum of the function

$$f(x_1,\dots,x_4,y_1,\dots,y_5) = 3x_4 + x_3 - x_2 - 3x_1 + 4y_5 + 2y_4 - 2y_2 - 4y_1$$

on the compact region D defined by

$$0 \le x_1 \le \cdots \le x_4, \quad x_1 + \ldots + x_4 = 1, \quad 0 \le y_1 \le \cdots \le y_5, \quad y_1 + \ldots + y_5 = 2$$
$$x_1 + y_1 + y_2 - 1 \le 0.$$

Using our code [CL21c], we find that the minimum of the linear function f over D is $\frac{1}{5}$ attained at $(\frac{1}{5}, \frac{4}{15}, \frac{4}{15}, \frac{4}{15}, \frac{2}{5}, \frac{2}{5}, \frac{2}{5}, \frac{2}{5}, \frac{2}{5})$. Thus,

$$\dim \operatorname{Supp} R^1\pi_*(\mathcal{U}_{5,g} \otimes \mathcal{O}_{\mathcal{P}}(-2)) \ge (g+4)\cdot \min_D(f) - 16 = \frac{g+4}{5} - 16. \qquad \square$$

For later use, let us note an immediate consequence of the previous lemma: Using excision and (1.5.10), we see

$$\operatorname{Trun}^{(g+4)/5-16} A^*(\mathcal{B}^\circ_{5,g}) = \operatorname{Trun}^{(g+4)/5-16} \mathbb{Q}[c_2, a_1, \dots, a_4, a_2', \dots, a_4', b_2, \dots, b_5, b_2', \dots, b_5']. \tag{1.5.11}$$

Lemma 1.5.13. *The codimension of the locus of smooth degree* 5 *covers* α *such that*

$$h^1(\mathcal{H}om(E_\alpha^\vee \otimes \det E_\alpha, \wedge^2 F_\alpha) \otimes \mathcal{O}_{\mathbb{P}^1}(-2)) > 0$$

has codimension at least $\frac{g+4}{5} - 16$*. That is, the codimension of the complement of* $\mathcal{H}^\circ_{5,g}$ *inside* $\mathcal{H}_{5,g}$ *is at least* $\frac{g+4}{5} - 16$*.*

Proof. The cohomological statement depends only on the splitting type of E_α and F_α. In the proof of [DP15, Proposition 5.2], Deopurkar–Patel show that the codimension of the locus of covers such that $E_\alpha \cong \mathcal{O}(\vec{e})$ and $F_\alpha \cong \mathcal{O}(\vec{f})$ has codimension

$$\begin{aligned} u(\vec{e},\vec{f}) := h^1(\mathbb{P}^1, \mathcal{E}nd(\mathcal{O}(\vec{e}))) + h^1(\mathbb{P}^1, \mathcal{E}nd(\mathcal{O}(\vec{f}))) \\ - h^1(\mathbb{P}^1, \mathcal{O}(\vec{e}) \otimes \wedge^2\mathcal{O}(\vec{f}) \otimes \mathcal{O}_{\mathbb{P}^1}(-g-4)). \end{aligned} \tag{1.5.12}$$

A cover with these discrete invariants corresponds to a global section η of

$$\mathcal{H}om(\mathcal{O}(\vec{e})^\vee \otimes \det \mathcal{O}(\vec{e}), \wedge^2\mathcal{O}(\vec{f})) = \mathcal{O}(\vec{e}) \otimes \wedge^2\mathcal{O}(\vec{f}) \otimes \mathcal{O}_{\mathbb{P}^1}(-g-4).$$

Such a global section can be represented by a skew-symmetric matrix

$$M_\eta = \begin{pmatrix} 0 & L_{1,2} & L_{1,3} & L_{1,4} & L_{1,5} \\ -L_{1,2} & 0 & L_{2,3} & L_{2,4} & L_{2,5} \\ -L_{1,3} & -L_{2,3} & 0 & L_{3,4} & L_{3,5} \\ -L_{1,4} & -L_{2,4} & -L_{3,4} & 0 & L_{4,5} \\ -L_{1,5} & -L_{2,5} & -L_{3,5} & -L_{4,5} & 0 \end{pmatrix}, \tag{1.5.13}$$

where $L_{i,j} \in H^0(\mathcal{O}(f_i + f_j) \otimes \mathcal{O}(\vec{e}) \otimes \mathcal{O}(-g-4))$. The corresponding curve $C \subset \mathbb{P}E^\vee$ is cut out by the 4×4 Pfaffians of the main minors of M_η. The Pfaffian of the submatrix obtained by deleting the last row and column is

$$Q_5 = L_{1,2}L_{3,4} - L_{1,3}L_{2,4} + L_{2,3}L_{1,4}.$$

If Q_5 is reducible, then C is reducible. Indeed, if C were irreducible, it would be contained in one component of Q_5, forcing every fiber to be contained in a hyperplane, violating the Geometric-Riemann-Roch theorem. Therefore, as observed in [DP15, p. 21], $L_{1,2}$ and $L_{1,3}$

cannot both be identically zero, and so

$$f_1 + f_3 + e_4 - (g+4) \geq 0. \tag{1.5.14}$$

Let $X_1, \ldots, X_4$ be coordinates on $\mathbb{P}E^\vee$ corresponding to a choice of splitting $E \cong \mathcal{O}(\vec{e})$, so we think of $L_{i,j}$ as a linear form in $X_1, \ldots, X_4$ where the coefficient of X_k is a section of $\mathcal{O}(f_i+f_j)\otimes\mathcal{O}(e_k)\otimes\mathcal{O}(-g-4)$, i.e. a homogeneous polynomial of degree $f_i+f_j+e_k-(g+4)$ on $\mathbb{P}^1$. If Q_5 is irreducible, it cannot be divisible by X_4. Observe that if $f_i+f_j+e_3-(g+4) < 0$, then the coefficients of X_k for $k \leq 3$ vanish, so X_4 divides $L_{i,j}$. If X_4 divides $L_{1,2}$, $L_{1,3}$ and $L_{1,4}$, then X_4 divides Q_5 and Q_5 is reducible. To prevent this, we must have

$$f_1 + f_4 + e_3 - (g+4) \geq 0. \tag{1.5.15}$$

Similarly, if X_4 divides $L_{1,2}$, $L_{1,3}$ and $L_{2,3}$, then X_4 divides Q_5 and Q_5 is reducible. To prevent this, we must have

$$f_2 + f_3 + e_3 - (g+4) \geq 0. \tag{1.5.16}$$

For splitting types satisfying (1.5.14), (1.5.15), and (1.5.16), at most 11 of the 40 summands of the form $\mathcal{O}(e_i + f_j + f_k - (g+4))$ in $\mathcal{O}(\vec{e}) \otimes \wedge^2\mathcal{O}(\vec{f}) \otimes \mathcal{O}_{\mathbb{P}^1}(-g-4)$ can be negative. For these allowed splitting types, we have

$$\begin{aligned} u(\vec{e},\vec{f}) = h^1(\mathbb{P}^1, \mathcal{E}nd(\mathcal{O}(\vec{e}))) + h^1(\mathbb{P}^1, \mathcal{E}nd(\mathcal{O}(\vec{f}))) \\ - \sum_{i=1}^{4} \max\{0, g+3-f_1-f_2-e_i\} - \sum_{i=1}^{3} \max\{0, g+3-f_1-f_3-e_i\} \\ - \sum_{i=1}^{2} \max\{0, g+3-f_1-f_4-e_i\} - \sum_{i=1}^{2} \max\{0, g+3-f_2-f_3-e_i\}. \end{aligned}$$

We seek a lower bound on $u(\vec{e},\vec{f})$ given that $\mathcal{O}(\vec{e})\otimes\wedge^2\mathcal{O}(\vec{f})\otimes\mathcal{O}(-g-4)$ has a non-positive summand, i.e. in the region where $e_1 + f_1 + f_2 - (g+4) \leq 0$. Note that

$$h^1(\mathbb{P}^1, \mathcal{E}nd(\mathcal{O}(\vec{e})) \geq 3e_4 + e_3 - e_2 - 3e_1 - 6$$
$$h^1(\mathbb{P}^1, \mathcal{E}nd(\mathcal{O}(\vec{f})) \geq 4f_5 + 2f_4 - 2f_2 - 4f_1 - 10.$$

Let us define a function of 9 real variables

$$\begin{aligned} f(x_1, \ldots, x_4, y_1, \ldots, y_5) := {} & 3x_4 + x_3 - x_2 - 3x_1 + 4y_5 + 2y_4 - 2y_2 - 4y_1 \\ & - \sum_{i=1}^{4} \max\{0, 1 - y_1 - y_2 - x_i\} - \sum_{i=1}^{3} \max\{0, 1 - y_1 - y_3 - x_i\} \\ & - \sum_{i=1}^{2} \max\{0, 1 - y_1 - y_4 - x_i\} - \sum_{i=1}^{2} \max\{0, 1 - y_2 - y_3 - x_i\} \end{aligned}$$

so that

$$u(\vec{e}, \vec{f}) \geq (g+4) f\left(\frac{e_1}{g+4}, \ldots, \frac{e_4}{g+4}, \frac{f_1}{g+4}, \ldots, \frac{f_5}{g+4}\right) - 16.$$

Now we wish to find the minimum of f on the compact region defined by

$$0 \leq x_1 \leq \cdots \leq x_4, \quad x_1 + \ldots + x_4 = 1, \quad 0 \leq y_1 \leq \cdots \leq y_5, \quad y_1 + \ldots + y_5 = 2$$
$$y_1 + y_3 + x_4 - 1 \geq 0, \quad y_1 + y_4 + x_3 - 1 \geq 0, \qquad y_2 + y_3 + x_3 - 1 \geq 0$$
$$x_1 + y_1 + y_2 - 1 \leq 0.$$

Since f is piecewise linear, its extreme values are attained at points where multiple boundary conditions (including those where the linear function changes) intersect to give a single point. Our code [CL21c] performs the linear algebra to locate such points and determines that the minimum is $\frac{1}{5}$, which is attained at $(\frac{1}{5}, \frac{4}{15}, \frac{4}{15}, \frac{4}{15}, \frac{2}{5}, \frac{2}{5}, \frac{2}{5}, \frac{2}{5}, \frac{2}{5})$. It follows that if $\vec{e}$ and $\vec{f}$ satisfy $h^1(\mathbb{P}^1, \mathcal{O}(\vec{e}) \otimes \wedge^2 \mathcal{O}(\vec{f}) \otimes \mathcal{O}(-g-4) \otimes \mathcal{O}(-2)) > 0$ then $u(\vec{e}, \vec{f}) \geq \frac{g+4}{5} - 16$. $\square$

We now prove Theorem 1.1.4 to complete the $k = 5$ case.

Proof of Theorem 1.1.4. Suppose $i < \frac{g+4}{5} - 16$. Then, by excision and Lemma 1.5.13, the restriction map $A^i(\mathcal{H}_{5,g}) \to A^i(\mathcal{H}_{5,g}^\circ)$ is an isomorphism. Hence, $R^i(\mathcal{H}_{5,g}) \to R^i(\mathcal{H}_{5,g}^\circ)$ is also an isomorphism. Since $\mathcal{H}_{5,g}^\circ \subseteq \mathcal{H}_{5,g}'$, Lemma 1.5.11 tells us that $A^i(\mathcal{H}_{5,g}^\circ) = R^i(\mathcal{H}_{5,g}^\circ)$. Hence, we have shown

$$A^i(\mathcal{H}_{5,g}) = A^i(\mathcal{H}_{5,g}^\circ) = R^i(\mathcal{H}_{5,g}^\circ) = R^i(\mathcal{H}_{5,g}). \qquad \square$$

1.6 Conclusion and preview of subsequent work

At this point, we have established that, for $k \leq 5$, a large portion of the Chow ring of the (non-factoring) Hurwitz space $\mathcal{H}_{k,g}^{\mathrm{nf}}$ is tautological. We did so by showing that $\mathcal{H}_{k,g}^{\mathrm{nf}}$ is closely approximated by an open substack $\mathcal{H}_{k,g}^\circ$ which, in turn, can be realized as an open substack of a vector bundle $\mathcal{X}_{r,d}^\circ = \mathcal{X}_{r,d}'|_{\mathcal{B}_{r,d}^\circ}$ over the stack $\mathcal{B}_{r,d}^\circ$. By (1.5.6) and (1.5.11), we understand $A^*(\mathcal{B}_{r,d}^\circ) \cong A^*(\mathcal{X}_{r,d}^\circ)$ well. First of all, we know $A^*(\mathcal{B}_{r,d}^\circ)$ is generated by classes which pullback to the CE classes on $\mathcal{H}_{k,g}^\circ$; this is how we saw $A^*(\mathcal{H}_{k,g}^\circ) = R^*(\mathcal{H}_{k,g}^\circ)$. However, we actually know a bit more: the generators we list for $A^*(\mathcal{B}_{r,d}^\circ) \cong A^*(\mathcal{X}_{r,d}^\circ)$ satisfy no relations in low degrees. In other words, all relations that their pullbacks to $\mathcal{H}_{k,g}^\circ$ satisfy come from performing excision on the complement of $\mathcal{H}_{k,g}^\circ \subset \mathcal{X}_{r,d}^\circ$.

Determining these relations will be the focus of our subsequent work [CL21a]. The central innovation there is to find an appropriate resolution of the complement of $\mathcal{H}_{k,g}^\circ \subset \mathcal{X}_{r,d}^\circ$, which allows us to determine *all* relations in degrees up to roughly g/k. Furthermore, we will prove that the relations we find among the restrictions of CE classes to $\mathcal{H}_{k,g}^\circ$ actually hold on *all* of $\mathcal{H}_{k,g}$. Using the codimension bounds we established in Section 5 here, results about the structure of $A^*(\mathcal{H}_{k,g}^\circ) = R^*(\mathcal{H}_{k,g}^\circ)$ will then translate into results about the structure of $A^*(\mathcal{H}_{k,g}^{\mathrm{nf}})$ and $R^*(\mathcal{H}_{k,g})$ in degrees up to roughly g/k.

This chapter, in full, has been submitted for publication. It is coauthored with Larson, Hannah. The dissertation author was co-primary investigator and author of this paper.

Chapter 2

Chow rings of low-degree Hurwitz spaces

2.1 Introduction

Intersection theory on the moduli space of curves $\mathcal{M}_g$ has received much attention since Mumford's famous paper [Mum83], in which he introduced the Chow ring of $\mathcal{M}_g$. Based on Harer's result [Har85] that the cohomology of the moduli space of curves is independent of the genus g in degrees small relative to g, Mumford conjectured that the stable cohomology ring is isomorphic to $\mathbb{Q}[\kappa_1, \kappa_2, \kappa_3, \ldots]$. Madsen–Weiss [MW07] later proved Mumford's conjecture. It is unknown whether there is an analogous stabilization result in the Chow ring of $\mathcal{M}_g$. Upon restricting attention to the tautological ring, however, more is known.

The *tautological subring* $R^*(\mathcal{M}_g) \subseteq A^*(\mathcal{M}_g)$ is defined to be the subring of the rational Chow ring generated by the kappa classes. There are many conjectures concerning the relations and structure of the tautological ring. Prominent among them is Faber's conjecture [Fab99, Conjecture 1], which states that the tautological ring should be Gorenstein with socle in codimension $g-2$ and generated by the first $\lfloor g/3 \rfloor$ kappa classes with no relations in degree less than $\lfloor g/3 \rfloor$. The Gorenstein part of Faber's conjecture is unknown, although it has been shown to hold when $g \leq 23$ by a direct computer calculation of Faber. The second portion of Faber's conjecture has been proved: Ionel [Ion05] proved that the tautological ring is generated by $\kappa_1, \kappa_2, \ldots, \kappa_{\lfloor g/3 \rfloor}$, and Boldsen [Bol12] proved that there are no relations among the κ-classes in degrees less than $\lfloor g/3 \rfloor$. In other words, there is a surjection

$$\mathbb{Q}[\kappa_1, \kappa_2, \ldots, \kappa_{\lfloor g/3 \rfloor}] \twoheadrightarrow R^*(\mathcal{M}_g), \tag{2.1.1}$$

which is an isomorphism in degrees less than $\lfloor g/3 \rfloor$.

In this paper, we study the Chow rings of low-degree Hurwitz spaces. Our main theorem is a stabilization result of a similar flavor to (2.1.1). Let $\mathcal{H}_{k,g}$ be the Hurwitz stack parametrizing degree k, genus g covers of $\mathbb{P}^1$ up to automorphisms of the target. Let

$\mathcal{C}$ be the universal curve and $\mathcal{P}$ the universal $\mathbb{P}^1$-fibration over the Hurwitz space $\mathcal{H}_{k,g}$:

$$\begin{tikzcd}
\mathcal{C} \arrow[r, "\alpha"] \arrow[dr, "f"'] & \mathcal{P} \arrow[d, "\pi"] \\
& \mathcal{H}_{k,g}.
\end{tikzcd}$$

We define the *tautological subring* of the Hurwitz space $R^*(\mathcal{H}_{k,g}) \subseteq A^*(\mathcal{H}_{k,g})$ to be the subring generated by classes of the form $f_*(c_1(\omega_f)^i \cdot \alpha^* c_1(\omega_\pi)^j) = \pi_*(\alpha_*(c_1(\omega_f)^i) \cdot c_1(\omega_\pi)^j)$. Let $\mathcal{E}^\vee$ be the cokernel of the map $\mathcal{O}_{\mathscr{P}} \to \alpha_* \mathcal{O}_{\mathcal{C}}$ (the universal "Tschirnhausen bundle"). Set $z = -\frac{1}{2}c_1(\omega_\pi)$"$= c_1(\mathcal{O}_{\mathcal{P}}(1))$". Our theorem will be stated in terms of the tautological classes $c_2 = -\pi_*(z^3) \in A^2(\mathcal{H}_{k,g})$ and

$$a_i = \pi_*(c_i(\mathcal{E}) \cdot z) \in A^i(\mathcal{H}_{k,g}) \qquad \text{and} \qquad a_i' = \pi_*(c_i(\mathcal{E})) \in A^{i-1}(\mathcal{H}_{k,g}).$$

When $k = 3, 4, 5$, our main theorem gives a minimal set of generators for $R^*(\mathcal{H}_{k,g})$ and determines all relations among them in degrees up to roughly g/k. In contrast with the case of $\mathcal{M}_g$ in (2.1.1), the tautological ring of $\mathcal{H}_{k,g}$ does *not* require a growing number of generators as g increases. In degree 3, we determine the full Chow ring of $\mathcal{H}_{3,g}$. When $k = 3, 5$, our results imply that the dimensions of the Chow groups of $\mathcal{H}_{k,g}$ are independent of g for g sufficiently large. In degree 4, *factoring covers* — i.e. covers $C \to \mathbb{P}^1$ that factor as a composition of two double covers $C \to C' \to \mathbb{P}^1$ — present a difficulty. We instead obtain stabilization results for the Chow groups of $\mathcal{H}_{4,g}^{\mathrm{nf}} \subseteq \mathcal{H}_{4,g}$, the open substack parametrizing non-factoring covers, or equivalently covers whose monodromy group is not contained in the dihedral group D_4.

Theorem 2.1.1. *Let $g \geq 2$ be an integer.*

1. *The rational Chow ring of* $\mathcal{H}_{3,g}$ *is*

$$A^*(\mathcal{H}_{3,g}) = R^*(\mathcal{H}_{3,g}) = \begin{cases} \mathbb{Q} & \text{if } g = 2 \\ \mathbb{Q}[a_1]/(a_1^2) & \text{if } g = 3,4,5 \\ \mathbb{Q}[a_1]/(a_1^3) & \text{if } g \geq 6. \end{cases}$$

2. *Let* $r_i = r_i(g)$ *be defined as in Section 2.5.4. For each* g *there is a map*

$$\frac{\mathbb{Q}[a_1, a_2', a_3']}{\langle r_1, r_2, r_3, r_4 \rangle} \twoheadrightarrow R^*(\mathcal{H}_{4,g}) \subseteq A^*(\mathcal{H}_{4,g}) \to A^*(\mathcal{H}_{4,g}^{\mathrm{nf}}),$$

   such that the composition is an isomorphism in degrees up to $\frac{g+3}{4} - 4$*. Furthermore, the dimension of the Chow group* $A^i(\mathcal{H}_{4,g}^{\mathrm{nf}})$ *is independent of* g *for* $g > 4i + 12$*. When* $g > 4i + 12$*, the dimensions are given by*

$$\dim A^i(\mathcal{H}_{4,g}^{\mathrm{nf}}) = \dim R^i(\mathcal{H}_{4,g}) = \begin{cases} 2 & i = 1,4 \\ 4 & i = 2 \\ 3 & i = 3 \\ 1 & i \geq 5. \end{cases}$$

3. *Let* $r_i = r_i(g)$ *be as defined in Section 2.6.4. There is a map*

$$\frac{\mathbb{Q}[a_1, a_2', a_2, c_2]}{\langle r_1, r_2, r_3, r_4, r_5 \rangle} \twoheadrightarrow R^*(\mathcal{H}_{5,g}) \subseteq A^*(\mathcal{H}_{5,g})$$

   such that the composition is an isomorphism in degrees $\leq \frac{g+4}{5} - 16$*. Furthermore, the dimension of the Chow group* $A^i(\mathcal{H}_{5,g})$ *is independent of* g *for* $g > 5i + 76$*. When*

$g > 5i + 76$, the dimensions are given by

$$\dim A^i(\mathcal{H}_{5,g}) = \dim R^i(\mathcal{H}_{5,g}) = \begin{cases} 2 & i = 1, i \geq 7 \\ 5 & i = 2 \\ 6 & i = 3 \\ 7 & i = 4 \\ 4 & i = 5 \\ 3 & i = 6. \end{cases}$$

Remark 2.1.2. Angelina Zheng recently computed the rational cohomology of $\mathcal{H}_{3,5}$ in [Zhe20], and, in subsequent work [Zhe21], finds the stable rational cohomology of $\mathcal{H}_{3,g}$. Together, our results prove that the cycle class map is injective. The corresponding statement for $\mathcal{M}_g$ is unknown, but when $g \leq 6$ it follows from the fact that the tautological ring is the entire Chow ring.

Remark 2.1.3. Note that Theorem 2.1.1(2) implies that the restriction map $R^i(\mathcal{H}_{4,g}) \to R^i(\mathcal{H}_{4,g}^{\mathrm{nf}})$ is an isomorphism for $i < \frac{g+3}{4} - 4$. This implies an interesting vanishing result: Any tautological class of codimension $i < \frac{g+3}{4} - 4$ supported on the locus of factoring covers is zero.

Remark 2.1.4. In [Bha04a, Bha04b, Bha08, Bha05, Bha10], Bhargava famously applied structure theorems for degree 3, 4, and 5 covers to counting number fields. As in Bhargava's work, our techniques rely on special aspects of structure theorems that do not seem to extend to covers of degree $k \geq 6$. Our need to throw out factoring covers in order to obtain asymptotic results for the full Chow ring seems to parallel the fact that, when quartic covers are counted by discriminant, the D_4 covers constitute a positive proportion of all covers [Bha05, Theorem 4].

Remark 2.1.5. Ellenberg-Venkatesh-Westerland [EVW16] have studied stability in the *homology* of Hurwitz spaces of G covers (which in particular separates out factoring covers). Like the work of Harer and Madsen-Weiss, their techniques are topological. On the other hand, ours are algebro-geometric: they are about the Chow groups rather than (co)homology and they work in characteristic $p > 5$ without the use of a comparison theorem.

Remark 2.1.6. For g suitably large, our proof of Theorem 2.1.1 (2) shows $\dim R^i(\mathcal{H}_{4,g}) \leq 1$ for all $i \geq 5$, and similarly in (3) that $\dim R^i(\mathcal{H}_{5,g}) \leq 2$ for all $i \geq 7$. Hence, $R^*(\mathcal{H}_{4,g})$ and $R^*(\mathcal{H}_{5,g})$ are *not* Gorenstein because there cannot be a perfect pairing for dimension reasons. On the other hand, $A^*(\mathcal{H}_{3,g}) = R^*(\mathcal{H}_{3,g})$ is Gorenstein.

Our method of proof is to study a large open substack $\mathcal{H}^\circ_{k,g} \subset \mathcal{H}_{k,g}$, which can be represented as an open substack of a vector bundle $\mathcal{X}^\circ_{k,g}$ over a certain moduli stack of vector bundles on $\mathbb{P}^1$. The fact that the moduli space admits such a description comes from the structure theorms of degree $3, 4, 5$ covers and is precisely what is so special about these low-degree cases. We then determine the Chow ring of $\mathcal{H}^\circ_{k,g}$ via excision on the complement of $\mathcal{H}^\circ_{k,g}$ inside $\mathcal{X}^\circ_{k,g}$. This complement is a "discriminant locus" parametrizing singular covers and maps that are not even finite. The stability of the Chow groups we find fits in with the philosophy of Vakil–Wood [VW15] about discriminants and suggests some possible variations on their theme. The key point, which is reflected in the ampleness assumptions in some of the conjectures from [VW15], is that the covers we parametrize correspond to sections of a vector bundle that becomes "more positive" as the genus of the curve grows. We compute generators for the Chow ring of the discriminant locus by constructing a resolution whose Chow ring we can compute. See Figure 2.3 in Section 2.5.3 for a picture summarizing our method.

We also give formulas in Section 2.7 that express other natural classes on $\mathcal{H}_{k,g}$ — namely the κ-classes pulled back from $\mathcal{M}_g$ and the classes corresponding to covers with certain ramification profiles — in terms of the generators from Theorem 2.1.1. We give two applications of these formulas. First, we show that for $k = 4, 5$, "the push forward of tautological classes on $\mathcal{H}_{k,g}$ are tautological on $\mathcal{M}_g$." (The case $k = 3$ already follows from Patel–Vakil's result that $A^*(\mathcal{H}_{3,g}) = R^*(\mathcal{H}_{3,g})$ is generated by κ_1 when $g > 3$, and and all classes on $\mathcal{M}_3$ are tautological.) Note that for $k > 3$, there are tautological classes on $\mathcal{H}_{k,g}$ that are *not* pullbacks of tautological classes on $\mathcal{M}_g$: Theorem 2.1.1 implies $\dim R^1(\mathcal{H}_{k,g}) > 1$, so it cannot be spanned by the pullback of κ_1. Hence, our claim regarding pushforwards is not a priori true. To set the stage for the theorem, let $\beta : \mathcal{H}_{k,g} \to \mathcal{M}_g$ be the forgetful morphism. Define $\mathcal{M}_g^k \subset \mathcal{M}_g$ to be the locus of curves of gonality $\leq k$. There is a proper morphism $\beta' : \mathcal{H}_{k,g} \setminus \beta^{-1}(\mathcal{M}_g^{k-1}) \to \mathcal{M}_g \smallsetminus \mathcal{M}_g^{k-1}$. We define a class to be tautological on $\mathcal{M}_g \smallsetminus \mathcal{M}_g^{k-1}$ if it is the restriction of a tautological class on $\mathcal{M}_g$.

Theorem 2.1.7. *Let $g \geq 2$ be an integer and $k \in \{3, 4, 5\}$. The β' push forward of classes in $R^*(\mathcal{H}_{k,g})$ are tautological on $\mathcal{M}_g \smallsetminus \mathcal{M}_g^{k-1}$.*

Remark 2.1.8. Theorem 2.1.7 is a key tool in recent work of the authors [CL21b], which proves that the Chow rings of $\mathcal{M}_7, \mathcal{M}_8$ and $\mathcal{M}_9$ are tautological. Because the tautological ring has been computed in these cases by Faber [Fab99], this work settles the next open case in the program suggested by Mumford [Mum83] of determining the Chow ring of $\mathcal{M}_g$ for small g.

Remark 2.1.9. We emphasize that when $k = 4$, there can be non-tautological classes in low codimension supported on the locus of factoring covers. In particular, the fundamental class of the bielliptic locus on $\mathcal{M}_{12}$ is not tautological by a theorem of van Zelm [vZ18], so Theorem 2.1.7 implies $R^*(\mathcal{H}_{4,g}) \neq A^*(\mathcal{H}_{4,g})$ for $g = 12$.

The second application of our formulas is to vanishing results for the Chow groups of the simply branched Hurwitz space $\mathcal{H}_{k,g}^s \subseteq \mathcal{H}_{k,g}$. The Hurwitz space Picard rank conjecture [HM98, Conjecture 2.49] says that $A^1(\mathcal{H}^s_{k,g}) = \mathrm{Pic}(\mathcal{H}^s_{k,g}) \otimes \mathbb{Q} = 0$. This conjecture is known for $k \leq 5$ [DP15], and for $k > g-1$ [Mul20]. In the cases $k = 2, 3$, the stronger vanishing result $A^i(\mathcal{H}^s_{k,g}) = 0$ holds for all $i > 0$. The following theorem provides further evidence for a generalization of the Hurwitz space Picard rank conjecture to higher codimension cycles.

Theorem 2.1.10. *Let $g \geq 2$ be an integer. The rational Chow groups of the simply-branched Hurwitz space satisfy*

$$A^i(\mathcal{H}^s_{4,g}) = 0 \qquad \text{for } 1 \leq i < \frac{g+3}{4} - 4$$
$$A^i(\mathcal{H}^s_{5,g}) = 0 \qquad \text{for } 1 \leq i < \frac{g+4}{5} - 16.$$

The paper is structured as follows. In Section 2.2, we introduce some notational conventions and some basic ideas from (equivariant) intersection theory that we will use throughout the paper. We prove a lemma, the "Trapezoid Lemma", which establishes a useful set up where one can determine all relations coming from certain excisions with an appropriate resolution. In Section 2.3, we introduce certain bundles of principal parts, which will be used throughout the remainder of the paper. Loosely speaking, these bundles help detect singularities and ramification behavior. As we shall see in the later sections of the paper, constructing a suitable principal parts bundle often requires geometric insights and can be somewhat involved. In Sections 2.4, 2.5, and 2.6, we use principal parts bundles and the Trapezoid Lemma to produce relations among tautological classes in $A^*(\mathcal{H}_{3,g})$, $A^*(\mathcal{H}_{4,g})$, and $A^*(\mathcal{H}_{5,g})$, respectively. From these calculations, we obtain the proof Theorem 2.1.1. Finally, in Section 2.7, we rewrite the κ-classes and classes that parametrize covers with certain ramification behavior in terms of our preferred generators. These calculations allow us to prove Theorems 2.1.7 and 2.1.10.

Several of the calculations in this paper were using the Macaulay2 [GS] package

Schubert2 [GSS+]. All of the code used in this paper is provided in a Github repository [CL21c]. Whenever there is a reference to a calculation done with a computer, one can find the code to perform that calculation in the Github repository.

2.2 Conventions and some intersection theory

We will work over an algebraically closed field of characteristic 0 or characteristic $p > 5$. All schemes in this paper will be taken over this fixed field.

2.2.1 Projective and Grassmann bundles

We follow the subspace convention for projective bundles: given a scheme (or stack) X and a vector bundle E of rank r on X, set

$$\mathbb{P}E := \operatorname{Proj}(\operatorname{Sym}^\bullet E^\vee),$$

so we have the tautological inclusion

$$\mathcal{O}_{\mathbb{P}E}(-1) \hookrightarrow \gamma^* E,$$

where $\gamma : \mathbb{P}E \to X$ is the structure map. Set $\zeta := c_1(\mathcal{O}_{\mathbb{P}E}(1))$. With this convention, the Chow ring of $\mathbb{P}E$ is given by

$$A^*(\mathbb{P}E) = A^*(X)[\zeta]/\langle \zeta^r + \zeta^{r-1}c_1(E) + \ldots + c_r(E)\rangle. \tag{2.2.1}$$

We call this the *projective bundle theorem*. Note that $1, \zeta, \zeta^2, \ldots, \zeta^{r-1}$ form a basis for $A^*(\mathbb{P}E)$ as an $A^*(X)$-module. Since

$$\gamma_*\zeta^i = \begin{cases} 0 & \text{if } i \leq r-2 \\ 1 & \text{if } i = r-1, \end{cases}$$

this determines the γ_* of all classes from $\mathbb{P}E$.

More generally, we define the Grassmann bundle $G(n, E)$ of n-dimensional subspaces in E, which is equipped with a tautological sequence

$$0 \to S \to \gamma^* E \to Q \to 0$$

where $\gamma : G(n, E) \to X$ is the structure map and S has rank n. The relative tangent bundle of $G(n, E) \to X$ is $\mathcal{H}om(S, Q)$. The Chow ring $A^*(G(n, E))$ is generated as an $A^*(X)$-*algebra* by the classes $\zeta_i = c_i(Q)$. Of particular interest to us will be Grassmann bundles $A^*(G(2, E))$ when the rank of E is either 4 or 5. If the rank of E is 4, $A^*(G(2, E))$ is generated as a $A^*(X)$-*module* by $\zeta_1^i \zeta_2^j$ for $0 \leq i \leq 2$, $0 \leq j \leq 2$, $0 \leq i + j \leq 2$. If the rank of E is 5, $A^*(G(2, E))$ is generated as a $A^*(X)$ *module* by $\zeta_1^i \zeta_2^j \zeta_3^k$ for $0 \leq i \leq 2$, $0 \leq j \leq 2$, $0 \leq k \leq 2$ and $0 \leq i + j + k \leq 2$. See [GSS12] for a much more general discussion on the Chow rings of flag bundles. In particular, these bases seem to be the preferred ones of the Macaulay2 [GS] package Schubert2 [GSS$^+$], which is what we use for calculations in this paper.

2.2.2 The Trapezoid Lemma

Let $\tau : V \to B$ be a rank r vector bundle. If σ is a section of V which vanishes in codimension r, then the vanishing locus of σ has fundamental class $c_r(V) \in A^r(B)$. The identity induces a section of $\tau^* V$ on the total space of V whose vanishing locus is the zero section. Thus, a special case of this fact is that the zero section in the total space of a vector bundle has class $c_r(\tau^* V) = \tau^* c_r(V) \in A^r(V) \cong \tau^* A^r(B)$. More generally, suppose $\rho : X \to B$ is another vector bundle on B and we are given a map of vector bundles $\phi : X \to V$ over B. Composing ϕ after the section induced by the identity on the total space of X defines a section of $\rho^* V$ on the total space of X. We call the vanishing locus K of this section the *preimage under* ϕ *of the zero section in*

V. If ϕ is a surjection of vector bundles, then K is simply the total space of the kernel subbundle. If K has codimension r inside the total space of W, then its fundamental class is $[K] = c_r(\rho^*V) = \rho^*c_r(V) \in A^r(X) \cong \rho^*A^r(B)$.

A basic tool we shall use repeatedly is the following.

Lemma 2.2.1 ("Trapezoid push forwards")**.** *Suppose $\widetilde{B} \to B$ is proper (e.g. a tower of Grassmann bundles). Let X be a vector bundle on B and let V be a vector bundle of rank r on $\widetilde{B}$. Suppose that we are given a map of vector bundles $\phi : \sigma^*X \to V$ on $\widetilde{B}$. Let $K \subset \sigma^*X$ be the preimage under ϕ of the zero section in V, and suppose that K has codimension r. We call this a trapezoid diagram:*

$$\begin{tikzcd} K \arrow[r, "\iota"] \arrow[dr, "\rho''"'] & \sigma^*X \arrow[r, "\sigma'"] \arrow[d, "\rho'"] & X \arrow[d, "\rho"] \\ & \widetilde{B} \arrow[r, "\sigma"'] & B. \end{tikzcd}$$

The image of $(\sigma' \circ \iota)_* : A_*(K) \to A_*(X)$ contains the ideal generated by $\rho^*(\sigma_*(c_r(V) \cdot \alpha_i))$ as $\alpha_i \in A^*(\widetilde{B})$ ranges over generators for $A^*(\widetilde{B})$ as a $A^*(B)$-module. Equality holds if ϕ is a surjection. In other words, we have a surjective map of rings

$$A^*(B)/\langle \sigma_*(c_r(V) \cdot \alpha_i)) \rangle \to A^*(X \smallsetminus \sigma'(\iota(K))),$$

which is an isomorphism when ϕ is a surjection of vector bundles.

Proof. The pullback maps $(\rho')^*$ and ρ^* are isomorphisms on Chow rings. The fundamental class of K in σ^*X is $(\rho')^*c_r(V)$, since it is defined by the vanishing of a section of $(\rho')^*V$. Consider classes in $A^*(K)$ of the form $(\rho'')^*\alpha$, where $\alpha \in A^*(\widetilde{B})$. The effect of $(\sigma' \circ \iota)_*$ on such classes is

$$\sigma'_*\iota_*(\rho'')^*\alpha = \sigma'_*\iota_*\iota^*(\rho')^*\alpha = \sigma'_*([K] \cdot (\rho')^*\alpha) = \sigma'_*(\rho')^*(c_r(V) \cdot \alpha) = \rho^*\sigma_*(c_r(V) \cdot \alpha). \quad (2.2.2)$$

The last step uses that flat pull back and proper push forward commute in fiber diagram.

If $\alpha = \sum_i (\sigma^* \beta_i) \cdot \alpha_i$, then the projection formula gives

$$\rho^* \sigma_*(c_r(V) \cdot \alpha) = \sum_i \rho^*(\beta_i) \cdot \rho^*(\sigma_*(c_r(V) \cdot \alpha_i)).$$

If K is a vector bundle, then *every* class in $A^*(K)$ has the form $(\rho'')^* \alpha$ for some $\alpha \in A^*(\widetilde{B})$. Thus, if K is a vector bundle, the image of $(\sigma' \circ \iota)_*$ is generated over $A^*(X) \cong \rho^* A^*(B)$ by the classes $\rho^*(\sigma_*(c_r(V) \cdot \alpha_i))$, as α_i runs over generators for $A^*(\widetilde{B})$ as a $A^*(B)$-module. $\square$

2.2.3 The Hurwitz space

Given a scheme S, an S point of the *parametrized Hurwitz scheme* $\mathcal{H}^{\dagger}_{k,g}$ is the data of a finite, flat map $C \to \mathbb{P}^1 \times S$, of constant degree k so that the composition $C \to \mathbb{P}^1 \times S \to S$ is smooth with geometrically connected fibers. (We do not impose the condition that a cover $C \to \mathbb{P}^1$ be simply branched.)

The *unparametrized Hurwitz stack* is the PGL_2 quotient of the parametrized Hurwitz scheme. There is also a natural action of SL_2 on $\mathcal{H}^{\dagger}_{k,g}$ (via $\mathrm{SL}_2 \subset \mathrm{GL}_2 \to \mathrm{PGL}_2$). The natural map $[\mathcal{H}^{\dagger}_{k,g} / \mathrm{SL}_2] \to [\mathcal{H}^{\dagger}_{k,g} / \mathrm{PGL}_2]$ is a μ_2 banded gerbe. It is a general fact that *with rational coefficients*, the pullback map along a gerbe banded by a finite group is an isomorphism [PV15b, Section 2.3]. In particular, since we work with rational coefficients throughout, $A^*([\mathcal{H}^{\dagger}_{k,g} / \mathrm{PGL}_2]) \cong A^*([\mathcal{H}^{\dagger}_{k,g} / \mathrm{SL}_2])$. It thus suffices to prove all statements for the SL_2 quotient $[\mathcal{H}^{\dagger}_{k,g} / \mathrm{SL}_2]$, which we denote by $\mathcal{H}_{k,g}$ from now on.

Explicitly, $\mathcal{H}_{k,g} = [\mathcal{H}^{\dagger}_{k,g} / \mathrm{SL}_2]$ is the stack whose objects over a scheme S are families $(C \to P \to S)$ where $P = \mathbb{P}V \to S$ is the projectivization of a rank 2 vector bundle V with trivial determinant, $C \to P$ is a finite flat finitely presented morphism of constant degree k, and the composition $C \to S$ has smooth fibers of genus g. The benefit of working with $\mathcal{H}_{k,g}$ is that the SL_2 quotient is equipped with a universal $\mathbb{P}^1$-bundle $\mathcal{P} \to \mathcal{H}_{k,g}$ that has a relative degree one line bundle $\mathcal{O}_{\mathcal{P}}(1)$ (a $\mathbb{P}^1$-fibration does not). Working with this $\mathbb{P}^1$-bundle simplifies our intersection theory calculations.

We shall also work with $\mathcal{H}_{k,g}^{\mathrm{nf}}$, the open substack of $\mathcal{H}_{k,g}$ parametrizing covers that do not factor through a lower genus curve. When k is prime, $\mathcal{H}_{k,g}^{\mathrm{nf}} = \mathcal{H}_{k,g}$. In Section 2.7 of the paper, we will consider the open substack $\mathcal{H}_{k,g}^{s} \subset \mathcal{H}_{k,g}$, which parametrizes covers that are simply branched. Note that $\mathcal{H}_{k,g}^{s} \subseteq \mathcal{H}_{k,g}^{\mathrm{nf}}$.

2.3 Relative bundles of principal parts

In this section, we collect some background on bundles of principal parts, which will be used to produce relations among tautological classes in Sections 2.4, 2.5, 2.6, and to compute classes of certain ramification strata in Section 2.7. For the basics, we follow the exposition in Eisenbud-Harris [EH16].

2.3.1 Basic properties

Let $b : Y \to Z$ be a smooth proper morphism. Let $\Delta_{Y/Z} \subset Y \times_Z Y$ be the relative diagonal. With p_1 and p_2 the projection maps, we obtain the following commutative diagram:

$$\begin{tikzcd}
\Delta_{Y/Z} \arrow[dr] \arrow[drr, bend left] \arrow[ddr, bend right] & & \\
& Y \times_Z Y \arrow[r, "p_1"] \arrow[d, "p_2"'] & Y \arrow[d, "b"] \\
& Y \arrow[r, "b"'] & Z.
\end{tikzcd}$$

Definition 2.3.1. Let $\mathcal{W}$ be a vector bundle on Y and let $\mathcal{I}_{\Delta_{Y/Z}}$ denote the ideal sheaf of the diagonal in $Y \times_Z Y$. The bundle of relative m^{th} order principal parts $P^m_{Y/Z}(\mathcal{W})$ is defined as

$$P^m_{Y/Z}(\mathcal{W}) = p_{2*}(p_1^*\mathcal{W} \otimes \mathcal{O}_{Y\times_Z Y}/\mathcal{I}^{m+1}_{\Delta_{Y/Z}}).$$

The following explains all the basic properties of bundles of principal parts that we need. Parts (1) and (2) are Theorem 11.2 in [EH16]. Let $m\Delta_{Y/Z}$ be the closed subscheme of $Y \times_Z Y$ defined by the ideal sheaf $\mathcal{I}^m_{\Delta_{Y/Z}}$. Part (3) below follows because the restriction of p_2 to the thickened diagonal $m\Delta_{Y/Z} \to Y$ is finite, so the push forward is exact.

Proposition 2.3.2. *With notation as above,*

1. *The quotient map* $p_1^*\mathcal{W} \to p_1^*\mathcal{W} \otimes \mathcal{O}_{Y\times_Z Y}/\mathcal{I}^{m+1}_{\Delta_{Y/Z}}$ *pushes forward to a map*

$$b^*b_*\mathcal{W} \cong p_{2*}p_1^*\mathcal{W} \to P^m_{Y/Z}(\mathcal{W}),$$

   which, fiber by fiber, associates to a global section δ *of* $\mathcal{W}$ *a section* δ' *whose value at* $z \in Z$ *is the restriction of* δ *to an* m^{th} *order neighborhood of* z *in the fiber* $b^{-1}b(z)$.

2. $P^0_{Y/Z}(\mathcal{W}) = \mathcal{W}$. *For* $m > 1$, *the filtration of the fibers* $P^m_{Y/Z}(\mathcal{W})_y$ *by order of vanishing at* y *gives a filtration of* $P^m_{Y/Z}(\mathcal{W})$ *by subbundles that are kernels of the natural surjections* $P^m_{Y/Z}(\mathcal{W}) \to P^k_{Y/Z}(\mathcal{W})$ *for* $k < m$. *The graded pieces of the filtration are identified by the exact sequences*

$$0 \to \mathcal{W} \otimes \mathrm{Sym}^m(\Omega_{Y/Z}) \to P^m_{Y/Z}(\mathcal{W}) \to P^{m-1}_{Y/Z}(\mathcal{W}) \to 0.$$

3. *A short exact sequence* $0 \to K \to \mathcal{W} \to \mathcal{W}' \to 0$ *of vector bundles on* Y *induces an exact sequence of principal parts bundles*

$$0 \to P^m_{Y/Z}(K) \to P^m_{Y/Z}(\mathcal{W}) \to P^m_{Y/Z}(\mathcal{W}') \to 0$$

We will need to know when the map from part (1) is surjective.

Lemma 2.3.3. *Suppose* $\mathcal{W}$ *is a relatively very ample line bundle on* Y. *Then the evaluation map* $b^*b_*\mathcal{W} \to P^1_{Y/Z}(\mathcal{W})$ *is surjective.*

Proof. The statement can be checked fiber by fiber over Z. Then, it follows from the fact that very ample line bundles separate points and tangent vectors. $\square$

Together with the above lemma, the following two lemmas will help us establish when evaluation maps are surjective in our particular setting.

Lemma 2.3.4. *Let $E = \mathcal{O}(e_1) \oplus \cdots \oplus \mathcal{O}(e_r)$ be a vector bundle on $\mathbb{P}^1$ with $e_1 \leq \cdots \leq e_r$ and let $\gamma : \mathbb{P}E^\vee \to \mathbb{P}^1$ be the projectivization. The line bundle $L = \gamma^*\mathcal{O}_{\mathbb{P}^1}(a) \otimes \mathcal{O}_{\mathbb{P}E^\vee}(m)$ is very ample if and only if $m \geq 1$ and $a + me_1 \geq 1$, equivalently if and only if $h^1(\mathbb{P}^1, \gamma_*L \otimes \mathcal{O}_{\mathbb{P}^1}(-2)) = 0$.*

Proof. First, note that L is the pullback of $\mathcal{O}(1)$ under a degree m relative Veronese embedding $\mathbb{P}E^\vee \hookrightarrow \mathbb{P}(\mathcal{O}(a) \otimes \operatorname{Sym}^m E)^\vee$. The $\mathcal{O}(1)$ on the projective bundle $\mathbb{P}(\mathcal{O}_{\mathbb{P}^1}(a) \otimes \operatorname{Sym}^m E)^\vee$ is very ample if and only if all summands of $\mathcal{O}_{\mathbb{P}^1}(a) \otimes \operatorname{Sym}^m E = \gamma_*L$ have positive degree (see [EH16, Section 9.1.1]). These summands have degrees of the form $a + e_{i_1} + \ldots + e_{i_m}$, all of which are at least $a + me_1$. □

Lemma 2.3.5. *Suppose $\mathcal{E}$ is a vector bundle on a $\mathbb{P}^1$-bundle $\pi : \mathcal{P} \to B$ and let $\gamma : \mathbb{P}\mathcal{E}^\vee \to \mathcal{P}$ be the projectivization. Suppose $\mathcal{W} = (\gamma^*A) \otimes \mathcal{O}_{\mathbb{P}\mathcal{E}^\vee}(m)$ for some $m \geq 1$ and vector bundle A on $\mathcal{P}$. If $R^1\pi_*[\gamma_*\mathcal{W} \otimes \mathcal{O}_{\mathcal{P}}(-2)] = 0$, then the evaluation map*

$$(\pi \circ \gamma)^*(\pi \circ \gamma)_*\mathcal{W} \to P^1_{\mathbb{P}\mathcal{E}^\vee/B}(\mathcal{W})$$

is surjective.

Proof. It suffices to check surjectivity in each of the fibers over B, so we are reduced to the case that B is a point. Now we may assume A splits as a sum of line bundles, say $A \cong \mathcal{O}(a_1) \oplus \cdots \oplus \mathcal{O}(a_r)$. By cohomology and base change, we have $h^1(\mathbb{P}^1, \gamma_*\mathcal{W} \otimes \mathcal{O}_{\mathbb{P}^1}(-2)) = 0$, which implies $h^1(\mathbb{P}^1, \gamma_*(\gamma^*\mathcal{O}(a_i) \otimes \mathcal{O}_{\mathbb{P}E^\vee}(m)) \otimes \mathcal{O}(-2)) = 0$ for each i. By Lemma 2.3.4, we have that $\mathcal{W}$ is a sum of very ample line bundles (over B). The bundle of principal parts respects direct sums, so the evaluation map is surjective by Lemma 2.3.3. □

The following lemma should be thought of as saying "pulled back sections have vanishing vertical derivatives."

Lemma 2.3.6. *Let $X \xrightarrow{a} Y \xrightarrow{b} Z$ be a tower of schemes with a and b smooth, and let $\mathcal{W}$ be a vector bundle on Y. For each m there is a natural map $a^*P^m_{Y/Z}(\mathcal{W}) \to P^m_{X/Z}(a^*\mathcal{W})$. This map fits in an exact sequence*

$$0 \to a^*P^m_{Y/Z}(\mathcal{W}) \to P^m_{X/Z}(a^*\mathcal{W}) \to F_m \to 0,$$

where $F_1 \cong \Omega_{X/Y} \otimes a^*\mathcal{W}$ and F_m for $m > 1$ is filtered as

$$0 \to \operatorname{Sym}^{m-1} \Omega_{X/Z} \otimes \Omega_{X/Y} \otimes a^*\mathcal{W} \to F_m \to F_{m-1} \to 0.$$

In particular, the evaluation map

$$b^*b_*\mathcal{W} \to P^m_{Y/Z}(\mathcal{W})$$

gives rise to a composition

$$a^*b^*b_*\mathcal{W} \to a^*P^m_{Y/Z}(\mathcal{W}) \to P^m_{X/Z}(a^*\mathcal{W}),$$

which, fiber by fiber, gives the Taylor expansion of sections of $\mathcal{W}$ along the "horizontal" pulled back directions.

Proof. We begin by constructing the map $a^*P^m_{X/Z}(\mathcal{W}) \to P^m_{Y/Z}(a^*\mathcal{W})$. Consider the following commutative diagram:

$$\begin{tikzcd}
& X \arrow[dl] \arrow[d, "a"'] & X \times_Z X \arrow[l, "p_2"'] \arrow[r, "p_1"] \arrow[d, "a \times a"] & X \arrow[d, "a"] \arrow[dr] & \\
Z & Y \arrow[l, "b"] & Y \times_Z Y \arrow[l, "q_2"] \arrow[r, "q_1"'] & Y \arrow[r, "b"'] & Z
\end{tikzcd}$$

Let $\Delta_Y \subset Y \times_Z Y$ denote the relative diagonal, and similarly for $\Delta_X \subset X \times_Z X$. By definition, we have

$$a^*P^m_{Y/Z}(\mathcal{W}) = a^*(q_{1*}(\mathcal{O}_{Y\times_Z Y}/\mathcal{I}^{m+1}_{\Delta_Y} \otimes q_2^*\mathcal{W})).$$

The natural transformation of functors $a^*q_{1*} \to p_{1*}(a\times a)^*$ induces a map

$$a^*P^m_{Y/Z}(\mathcal{W}) \to p_{1*}((a\times a)^*(\mathcal{O}_{Y\times_Z Y}/\mathcal{I}^{m+1}_{\Delta_Y}) \otimes (a\times a)^*q_2^*\mathcal{W})).$$

The transform $(q_2\circ(a\times a))^* \to (a\circ p_2)^*$ induces a map

$$p_{1*}((a\times a)^*(\mathcal{O}_{Y\times_Z Y}/\mathcal{I}^{m+1}_{\Delta_Y}) \otimes (a\times a)^*q_2^*\mathcal{W})) \to p_{1*}((a\times a)^*(\mathcal{O}_{Y\times_Z Y}/\mathcal{I}^{m+1}_{\Delta_Y}) \otimes p_2^*a^*\mathcal{W}).$$

The natural morphism of sheaves $\mathcal{O}_{Y\times_Z Y} \to (a\times a)_*\mathcal{O}_{X\times_Z X}$ induces a map on quotients $\mathcal{O}_{Y\times_Z Y}/\mathcal{I}^{m+1}_{\Delta_Y} \to (a\times a)_*(\mathcal{O}_{X\times_Z X}/\mathcal{I}^{m+1}_{\Delta_X})$. By adjunction, we obtain a map

$$(a\times a)^*(\mathcal{O}_{Y\times_Z Y}/\mathcal{I}^{m+1}_{\Delta_Y}) \to \mathcal{O}_{X\times_Z X}/\mathcal{I}^{m+1}_{\Delta_X}.$$

Then we have a morphism

$$p_{1*}((a\times a)^*(\mathcal{O}_{Y\times_Z Y}/\mathcal{I}^{m+1}_{\Delta_Y}) \otimes p_2^*a^*\mathcal{W}) \to p_{1*}(\mathcal{O}_{X\times_Z X}/\mathcal{I}^{m+1}_{\Delta_X} \otimes p_2^*(a^*\mathcal{W})) = P^m_{X/Z}(a^*\mathcal{W}).$$

By construction, the maps $a^*P^m_{Y/Z}(\mathcal{W}) \to P^m_{X/Z}(a^*\mathcal{W})$ are compatible with the filtrations on the fibers by order of vanishing, so we obtain an induced map on the graded pieces of the filtrations:

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{Sym}^m(a^*\Omega_{Y/Z}) \otimes a^*\mathcal{W} \arrow[r] \arrow[d, hook] & a^*P^m_{Y/Z}(\mathcal{W}) \arrow[r] \arrow[d] & a^*P^{m-1}_{Y/Z}(\mathcal{W}) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathrm{Sym}^m(\Omega_{X/Z}) \otimes a^*\mathcal{W} \arrow[r] & P^m_{X/Z}(a^*\mathcal{W}) \arrow[r] & P^{m-1}_{X/Z}(a^*\mathcal{W}) \arrow[r] & 0
\end{tikzcd}$$

When $m = 1$, the right vertical map is the identity on $a^*\mathcal{W}$. Hence, $a^*P^1_{Y/Z}(\mathcal{W}) \to P^1_{X/Z}(a^*\mathcal{W})$ is injective. By the snake lemma, the cokernel is isomorphic to the cokernel of the left vertical map, which in turn is $\Omega_{X/Y} \otimes a^*\mathcal{W}$ because a and b are smooth and $\mathcal{W}$ is locally free. For $m > 1$, we may assume by induction that the right vertical map is injective, hence the center vertical map is injective. The filtration of the cokernel F_m of the center vertical map follows by induction and the snake lemma. $\square$

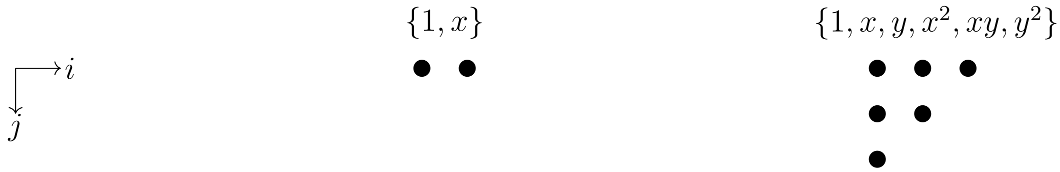
2.3.2 Directional refinements

Much of the exposition in this subsection is based on unpublished notes of Ravi Vakil. Suppose we have a tower $X \xrightarrow{a} Y \xrightarrow{b} Z$ and $a^*\Omega_{Y/Z}$ admits a filtration on X

$$0 \to \Omega_y \to a^*\Omega_{Y/Z} \to \Omega_x \to 0. \tag{2.3.1}$$

For example, take $X = \mathbb{P}(\Omega_{Y/Z})$ or $G(n, \Omega_{Y/Z})$ with the filtration given by the tautological sequence. First, suppose Ω_x and Ω_y are rank 1. The filtration (2.3.1) is the same as saying we can choose local coordinates x, y at each point of Y where y is well-defined up to $(x, y)^2$, and x is only defined modulo y. The vanishing of y defines a distinguished "x-direction" on the tangent space $T_{Y/Z}$ at each point, which is dual to the quotient $a^*\Omega_{Y/Z} \to \Omega_x$.

The goal of this section is to define principal parts bundles that measure certain parts of a Taylor expansion with respect to these local coordinates. These principal parts bundles will be indexed by *admissible sets* S of monomials in x and y (defined below). If $x^iy^j \in S$, then $P^S_{Y/Z}(\mathcal{W})$ will keep track of the coefficient of x^iy^j in the Taylor expansion of a section of $\mathcal{W}$. For example, $S = \{1, x\}$ will correspond to a quotient of $a^*P^1_{Y/Z}(\mathcal{W})$ that measures only derivatives in the x-direction. The set $S = \{1, x, y, x^2, xy, y^2\}$ corresponds to the pullback of the usual second order principal parts. It is helpful to visualize these sets with diagrams as below, where we place a dot at coordinate (i, j) if $x^iy^j \in S$.



More generally, if Ω_x and Ω_y have any ranks, the quotient Ω_x is dual to a distinguished subspace of $T_{Y/Z}$. The construction below will build bundles $P^S_{Y/Z}(\mathcal{W})$ such that if $x^i y^j \in S$, then $P^S_{Y/Z}(\mathcal{W})$ tracks the coefficients of all monomials corresponding to $\operatorname{Sym}^i \Omega_x \otimes \operatorname{Sym}^j \Omega_y$. In other words, $P^S_{Y/Z}(\mathcal{W})$ will admit a filtration with quotients $\operatorname{Sym}^i \Omega_x \otimes \operatorname{Sym}^j \Omega_y \otimes \mathcal{W}$ for each (i, j) such that $x^i y^j \in S$. Each dot in the diagram corresponds to a piece of this filtration. Only diagrams of certain shapes are allowed.

Definition 2.3.7. A set S is *admissible* if the following hold

- If $x^i y^j \in S$, then $x^{i-1} y^j \in S$ (if $i - 1 \geq 0$). That is, for each dot in the diagram, the dot to its left is also in the diagram if possible.

- If $x^i y^j \in S$, then $x^{i-2} y^{j+1}$ (if $i - 2 \geq 0$). That is, for each dot in the diagram, the dot two to the left and one down is also in the diagram if possible.

Equivalently, the diagram associated to S is built, via intersections and unions, from triangular collections of lattice points bounded by the axes and a line of slope 1 or slope $\frac{1}{2}$.

To build the principal parts bundles $P^S_{Y/Z}(W)$, let us consider the diagram

$$\begin{tikzcd}
\widetilde{\Delta} \arrow[r] \arrow[d, "\iota"] & \Delta \arrow[d] & \\
X \times_Z Y \arrow[r, "\widetilde{a}"] \arrow[d, "\widetilde{p}_2"'] & Y \times_Z Y \arrow[r, "p_1"] \arrow[d, "p_2"'] & Y \arrow[d, "b"] \\
X \arrow[r, "a"] & Y \arrow[r, "b"] & Z,
\end{tikzcd}$$

where $\Delta = \Delta_{Y/Z} \subset Y \times_Z Y$ is the diagonal and all squares are fibered squares. The composition of vertical maps give isomorphisms $\Delta \cong Y$ and $\widetilde{\Delta} \cong X$. There is an identification $\iota_*\Omega_{\widetilde{\Delta}/Z} \cong \mathcal{I}_{\widetilde{\Delta}}/\mathcal{I}^2_{\widetilde{\Delta}}$. Using (2.3.1) and the isomorphism $\widetilde{\Delta} \cong X$, we obtain an injection

$$\iota_*\Omega_y \to \iota_* a^*\Omega_{Y/Z} \to \iota_*\Omega_{X/Z} \cong \mathcal{I}_{\widetilde{\Delta}}/\mathcal{I}^2_{\widetilde{\Delta}},$$

which determines a subsheaf $\mathcal{J} \subset \mathcal{I}_{\widetilde{\Delta}} =: \mathcal{I}$. The sheaf $\mathcal{I}$ corresponds to the monomials $\{x^i y^j : i + j \geq 1\}$ (see (2.3.3) below). The subsheaf $\mathcal{J}$ corresponds to the monomials $\{x^i y^j : i + j \geq 2 \text{ or } j \geq 1\}$ (see (2.3.4) below). The condition $i + j \geq 2$ says $\mathcal{I}^2 \subset \mathcal{J}$. The condition $j \geq 1$ says $\mathcal{J} \subset \mathcal{I}$ and it "picks out our y-coordinate(s) to first order."

In the next paragraph, we will explain how to construct an ideal $\mathcal{I}_S$, via intersections and unions of $\mathcal{I}$ and $\mathcal{J}$, corresponding to monomials not in S. Our refined principal parts bundles will then be defined as

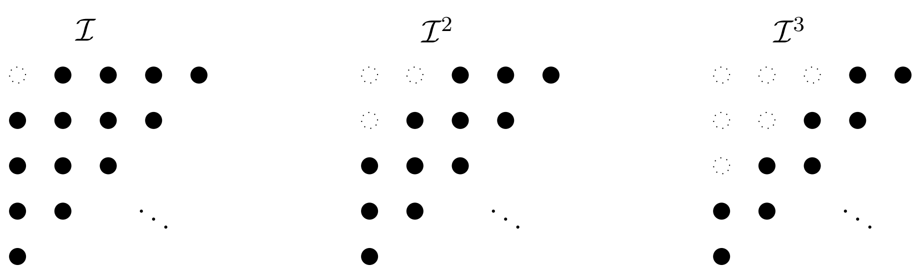
$$P^S_{Y/Z}(\mathcal{W}) := \widetilde{p}_{2*}\left(\widetilde{a}^* p_1^* \mathcal{W} \otimes \mathcal{O}_{X \times_Z Y}/\mathcal{I}_S\right),$$

The bundle $P^S_{Y/Z}(\mathcal{W})$ is defined on X and will be a quotient of $a^* P^m_{Y/Z}(\mathcal{W})$ for $m = \max\{i + j : x^i y^j \in S\}$. In particular, there are restricted evaluation maps

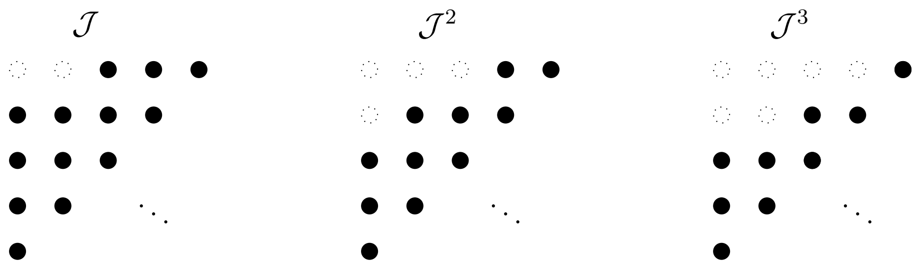
$$a^* b^* b_* \mathcal{W} \to a^* P^m_{Y/Z}(\mathcal{W}) \to P^S_{Y/Z}(\mathcal{W}), \tag{2.3.2}$$

which we think of as Taylor expansions only along certain directions specified by S.

To start, we shall have $\mathcal{I}_{\{1,x,y\}} := \mathcal{I}$ and $\mathcal{I}_{\{1,x\}} := \mathcal{J}$. Powers of these ideals correspond to regions below lines of slope 1 and $\frac{1}{2}$ respectively.



(2.3.3)



(2.3.4)

To say that S is admissible is to say that $\mathcal{I}_S$ is built by taking unions and intersections such half planes, which corresponds to intersections and unions of $\mathcal{I}$ and $\mathcal{J}$. We list below the principal parts bundles we require in the remainder of the paper and their associated ideal $\mathcal{I}_S$.

1. $S = \{1, x\}$ with $\mathcal{I}_S = \mathcal{J}$, which we call the bundle of *restricted principal parts*.

2. $S = \{1, x, y, x^2\}$ with $\mathcal{I}_S = \mathcal{J}^2$ will arise in triple point calculations.

3. $S = \{1, x, y, x^2, xy\}$ with $\mathcal{I}_S = \mathcal{I}^3 + \mathcal{J}^3$ arises when finding quadruple points in a pencil of conics.

4. $S = \{1, x, y, x^2, xy, x^3\}$ with $\mathcal{I}_S = \mathcal{J}^3$ will arise in finding quadruple points in pentagonal covers.

Diagrams corresponding to these sets appear at the end of the next subsection. Given two admissible sets $S \subset S'$, there is a natural surjection $P^{S'}_{Y/Z}(\mathcal{W}) \to P^{S}_{Y/Z}(\mathcal{W})$, which corresponds to truncating Taylor series. This determines the order(s) that the terms $\mathrm{Sym}^i \Omega_x \otimes \mathrm{Sym}^j \Omega_y \otimes \mathcal{W}$ corresponding to $x^i y^j \in S'$ may appear as quotients in a filtration: a term corresponding to $x^i y^j \in S'$ is a well-defined subbundle of $P^{S'}_{Y/Z}(\mathcal{W})$ if $S' \smallsetminus x^i y^j$ is an admissible set.

2.3.3 Bundle-induced refinements

Now suppose that $a^*\mathcal{W}$ admits a filtration on X by

$$0 \to K \to a^*\mathcal{W} \to \mathcal{W}' \to 0, \tag{2.3.5}$$

where $\mathcal{W}'$ is a vector bundle, and hence so is K. Exactness of principal parts for X over Z gives an exact sequence

$$0 \to P^m_{X/Z}(K) \to P^m_{X/Z}(a^*\mathcal{W}) \to P^m_{X/Z}(\mathcal{W}') \to 0.$$

We are interested in the restriction of this filtration to $a^*P^m_{Y/Z}(\mathcal{W}) \subset P^m_{X/Z}(a^*\mathcal{W})$. First, we need the following fact.

Lemma 2.3.8. *The intersection of the two subbundles*

$$P^m_{X/Z}(K) \subset P^m_{X/Z}(a^*\mathcal{W}) \qquad \textit{and} \qquad a^*P^m_{Y/Z}(\mathcal{W}) \subset P^m_{X/Z}(a^*\mathcal{W}) \tag{2.3.6}$$

is a subbundle.

Proof. We proceed by induction. For $m = 0$, the claim is just that K is a subbundle of $a^*\mathcal{W}$. The question is local, so we can assume that the vanishing order filtration exact sequences

$$0 \to \operatorname{Sym}^m \Omega_{X/Z} \otimes a^*\mathcal{W} \to P^m_{X/Z}(a^*\mathcal{W}) \to P^{m-1}_{X/Z}(a^*\mathcal{W}) \to 0,$$

are split. By induction and the (locally split) exact sequences,

$$0 \to \operatorname{Sym}^m \Omega_{X/Z} \otimes K \to P^m_{X/Z}(K) \to P^{m-1}_{X/Z}(K) \to 0$$

and

$$0 \to a^* \operatorname{Sym}^m \Omega_{Y/Z} \otimes a^*\mathcal{W} \to a^*P^m_{Y/Z}(\mathcal{W}) \to a^*P^{m-1}_{Y/Z}(\mathcal{W}) \to 0$$

it suffices to show that the intersection of $\operatorname{Sym}^m \Omega_{X/Z} \otimes K$ and $a^* \operatorname{Sym}^m \Omega_{Y/Z} \otimes a^*\mathcal{W}$ is a subbundle of $\operatorname{Sym}^m \Omega_{X/Z} \otimes a^*\mathcal{W}$. But this intersection is given by $a^* \operatorname{Sym}^m \Omega_{Y/Z} \otimes K$, which is a subbundle. $\square$

Definition 2.3.9. We define $\underline{P}^m_{Y/Z}(K)$ to be the intersection of the two subbundles in (2.3.6). This subbundle tracks principal parts of K in the directions of Y/Z. We include the underline to remind ourselves that this bundle is defined on X since K is defined on X. We define $Q^m_{Y/Z}(\mathcal{W}')$ to be the cokernel of $\underline{P}^m_{Y/Z}(K) \to a^*P^m_{Y/Z}(\mathcal{W})$.

When $K = a^*K'$ for a bundle K' on Y, the bundle $\underline{P}^m_{Y/Z}(K)$ is just the bundle $a^*P^m_{Y/Z}(K')$.

The vanishing order filtrations from Proposition 2.3.2 of $P^m_{X/Z}(K)$ and $a^*P^m_{Y/Z}(\mathcal{W})$ restrict to a vanishing order filtration on $\underline{P}^m_{Y/Z}(K)$, which in turn induces a vanishing order filtration on $Q^m_{Y/Z}(\mathcal{W}')$. We describe this for $m = 1$ below for future use.

Lemma 2.3.10. *The bundle $Q^1_{Y/Z}(\mathcal{W}')$ is equipped with a surjection $a^*P^1_{Y/Z}(\mathcal{W}) \to Q^1_{Y/Z}(\mathcal{W}')$ and a filtration*

$$0 \to a^*\Omega_{Y/Z} \otimes \mathcal{W}' \to Q^1_{Y/Z}(\mathcal{W}') \to \mathcal{W}' \to 0$$

A section $\mathcal{O}_Y \xrightarrow{\delta} \mathcal{W}$ on X induces a section $\mathcal{O}_X \xrightarrow{\delta'} a^*P^1_{Y/Z}(\mathcal{W}) \to Q^1_{Y/Z}(\mathcal{W}')$ that records the values and "horizontal derivatives" of δ in the quotient $\mathcal{W}'$.

2.3.4 Directional and bundle-induced refinements

The principal parts bundles constructed in this subsection will not be needed until Section 2.7. Here, we suppose that we have filtrations as in (2.3.1) and (2.3.5). We have an inclusion $\underline{P}^m_{Y/Z}(K) \hookrightarrow a^*P^m_{Y/Z}(\mathcal{W})$ as well as a quotient $a^*P^m_{Y/Z}(\mathcal{W}) \to P^S_{Y/Z}(\mathcal{W})$. We define $\underline{P}^S_{Y/Z}(K)$ to be image of the composition

$$\underline{P}^m_{Y/Z}(K) \hookrightarrow a^*P^m_{Y/Z}(\mathcal{W}) \to P^S_{Y/Z}(\mathcal{W}),$$

which tracks the principal parts of K in the Y/Z directions specified by S.

Given two admissible sets $S \subset S'$, there is a quotient $\underline{P}^{S'}_{Y/Z}(K) \to \underline{P}^{S}_{Y/Z}(K)$. Let $V \subset \underline{P}^{S'}_{Y/Z}(K)$ be the kernel. We define $P^{S\subset S'}_{Y/Z}(\mathcal{W} \to \mathcal{W}')$ to be the cokernel of the composition

$$V \hookrightarrow \underline{P}^{S'}_{Y/Z}(K) \hookrightarrow P^{S'}_{Y/Z}(\mathcal{W}).$$

The bundle $P^{S\subset S'}_{Y/Z}(\mathcal{W} \to \mathcal{W}')$ tracks the principal parts associated to S on $\mathcal{W}$ and then the principal parts associated to the rest of S' but just in the $\mathcal{W}'$ quotient. We visualize $P^{S\subset S'}_{Y/Z}(\mathcal{W} \to \mathcal{W}')$ by a decorated diagram of shape S', where the dots are filled in the subshape S and half filled (representing values just in $\mathcal{W}'$) in the remainder $S' \smallsetminus S$ (colored in blue below). A preview of the cases we shall need later are pictured below.

(6.4A) $S = \{1, x\}$ and $S' = \{1, x, y, x^2\}$, for triple points in Section 2.7.3.

● ● ◐
◐

(6.4B) $S = \{1, x\}$ and $S' = \{1, x, y, x^2, xy\}$, for quadruple points in Lemma 2.7.7.

● ● ◐
◐ ◐

(6.4C) $S = \{1, x\}$ and $S' = \{1, x, y, x^2, xy, x^3\}$, for quadruple points in Lemma 2.7.13.

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Revisiting Definition 2.3.9, $Q^1_{Y/Z}(\mathcal{W}') = P^{\varnothing\subset\{1,x,y\}}(\mathcal{W} \to \mathcal{W}')$ would be represented by

◐ ◐
◐

2.4 The Chow ring in degree 3

2.4.1 Set up

We begin by recalling the linear algebraic data associated to a degree 3 cover as developed by Miranda and Casnsati–Ekedahl [Mir85, CE96]. For more details in our context, see [BV12] and [CL21d, Section 3.1]. Given a degree 3, genus g cover, $\alpha : C \to \mathbb{P}^1$, define $E_\alpha := (\alpha_* \mathcal{O}_C / \mathcal{O}_{\mathbb{P}^1})^\vee$, which is a rank 2, degree $g+2$ vector bundle on $\mathbb{P}^1$. There is a natural embedding $C \subset \mathbb{P}E_\alpha^\vee$ and C is the zero locus of a section of

$$H^0(\mathbb{P}E_\alpha^\vee, \gamma^* \det E_\alpha^\vee \otimes \mathcal{O}_{\mathbb{P}E_\alpha^\vee}(3)) \cong H^0(\mathbb{P}^1, \det E_\alpha^\vee \otimes \operatorname{Sym}^3 E_\alpha),$$

where $\gamma : \mathbb{P}E_\alpha^\vee \to \mathbb{P}^1$ is the structure map. Conversely, given a globally generated, rank 2, degree $g+2$ vector bundle E on $\mathbb{P}^1$ with $\operatorname{Sym}^3 E \otimes \det E^\vee$ globally generated, the vanishing of a general section $\delta \in H^0(\mathbb{P}E^\vee, \gamma^* \det E^\vee \otimes \mathcal{O}_{\mathbb{P}E^\vee}(3))$ defines a smooth, genus g triple cover $\alpha : C = V(\delta) \subset \mathbb{P}E^\vee \to \mathbb{P}^1$ such that $E_\alpha \cong E$. First, let us give a characterization of which sections do not yield covers parametrized by $\mathcal{H}_{3,g}$.

Lemma 2.4.1. *Let E be a rank 2, degree $g+2$ vector bundle on $\mathbb{P}^1$ such that $\operatorname{Sym}^3 E \otimes \det E^\vee$ is globally generated. Let $\delta \in H^0(\mathbb{P}E, \gamma^* \det E^\vee \otimes \mathcal{O}_{\mathbb{P}E_\alpha}(3))$. Suppose that the zero locus $C = V(\delta) \subseteq \mathbb{P}E^\vee$ is not a smooth, irreducible genus g triple cover of $\mathbb{P}^1$. Then there exists a point $p \in C$ such that $\dim T_p C = 2$.*

Proof. If $\delta = 0$, then C is 2-dimensional and the claim follows. Now suppose $\delta \neq 0$. We will show that C is connected, which implies that if C fails to be an irreducible triple cover, it must have a point with 2 dimensional tangent space. If $\operatorname{Sym}^3 E \otimes \det E^\vee$ is globally generated, then both summands of $E = \mathcal{O}(e_1) \oplus \mathcal{O}(e_2)$ have degree at least $\frac{g+2}{3}$. Hence, $h^0(\mathbb{P}^1, E^\vee) = 0$. If $C \to \mathbb{P}^1$ is finite we have $h^0(C, \mathcal{O}_C) = h^0(\mathbb{P}^1, \alpha_* \mathcal{O}_C) = h^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}) + h^0(\mathbb{P}^1, E_\alpha^\vee) = 1$, so C is connected. Now suppose C has a positive dimensional fiber over $\mathbb{P}^1$. Any curve in the class $\mathcal{O}_{\mathbb{P}E^\vee}(3) \otimes \gamma^* \det E^\vee$ has a component that meets

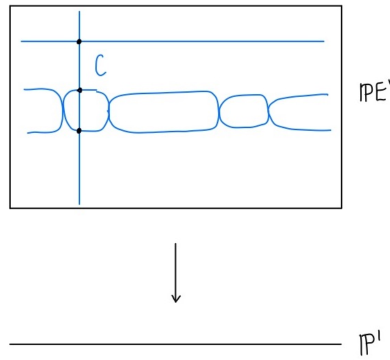

Figure 2.1. A singular triple cover.

every fiber, thus C is again connected. $\square$

We now recall some notation and constructions from [CL21d]. The association of $\alpha : C \to \mathbb{P}^1$ with E_α gives rise to a map of $\mathcal{H}_{3,g}$ to the moduli stack $\mathcal{B}_{3,g}$ of rank 2, degree $g+3$, globally generated vector bundles on $\mathbb{P}^1$-bundles. Let $\pi : \mathcal{P} \to \mathcal{B}_{3,g}$ be the universal $\mathbb{P}^1$-bundle and let $\mathcal{E}$ be the universal rank 2 vector bundle on $\mathcal{P}$. Continuing the notation of [CL21d], let $z = c_1(\mathcal{O}_{\mathcal{P}}(1))$ and define classes $a_i \in A^i(\mathcal{B}_{3,g})$ and $a_i' \in A^{i-1}(\mathcal{B}_{3,g})$ by the formula

$$c_i(\mathcal{E}) = \pi^* a_i + \pi^* a_i' z.$$

We also define $c_2 = -\pi_*(z^3) \in A^2(\mathcal{B}_{3,g})$, which is the pullback of the universal second Chern class on BSL_2. By [CL21d, Theorem 4.4],

$$a_1, a_2, a_2', c_2 \text{ generate } A^*(\mathcal{B}_{3,g}) \text{ and satisfy no relations in degrees up to } g+2. \tag{2.4.1}$$

Now, let $\gamma : \mathbb{P}\mathcal{E}^\vee \to \mathcal{P}$ and define $\mathcal{W} := \mathcal{O}_{\mathbb{P}\mathcal{E}^\vee}(3) \otimes \gamma^* \det \mathcal{E}^\vee$, which is a line bundle on $\mathbb{P}\mathcal{E}^\vee$. Consider the bundle $\mathcal{U}_{3,g} := \gamma_* \mathcal{W} = \mathrm{Sym}^3 \mathcal{E} \otimes \det \mathcal{E}^\vee$. We define

$$\mathcal{B}_{3,g}' := \mathcal{B}_{3,g} \smallsetminus \mathrm{Supp}\, R^1\pi_* \mathcal{U}_{3,g}(-1). \tag{2.4.2}$$

Equivalently, $\mathcal{B}'_{3,g}$ is the open locus where $\mathcal{U}_{3,g}$ is globally generated on the fibers of π. By the theorem on cohomology and base change

$$\mathcal{X}'_{3,g} := \pi_*\mathcal{U}_{3,g}|_{\mathcal{B}'_{3,g}}$$

is a vector bundle. In [CL21d, Lemma 5.1], we showed that the map $\mathcal{H}_{3,g} \to \mathcal{B}_{3,g}$ factors through an open embedding $\mathcal{H}_{3,g} \to \mathcal{X}'_{3,g}$. Hence, the Chow ring of $\mathcal{H}_{3,g}$ is generated by the pullbacks of the classes a_1, a'_2, a_2, c_2 from $\mathcal{B}'_{3,g}$. We must determine the relations among these classes that come from excising $\operatorname{Supp} R^1\pi_*\mathcal{U}_{3,g}(-1)$ from $\mathcal{B}_{3,g}$ and from excising

$$\Delta_{3,g} := \mathcal{X}'_{3,g} \smallsetminus \mathcal{H}_{3,g}.$$

In other words, we shall compute the Chow ring $A^*(\mathcal{H}_{3,g})$ by computing the image of the left-hand map in the excision sequence

$$A^{*-1}(\Delta_{3,g}) \to A^*(\mathcal{X}'_{3,g}) \to A^*(\mathcal{H}_{3,g}) \to 0.$$

2.4.2 Resolution and excision

We begin by constructing a space $\widetilde{\Delta}_{3,g}$, which corresponds to triple covers (or worse) with a marked singular point. By forgetting the marked point, we will obtain a proper surjective morphism $\widetilde{\Delta}_{3,g} \to \Delta_{3,g}$ by Lemma 2.4.1. Because our Chow rings are taken with rational coefficients, pushforward induces a surjection on Chow groups $A^*(\widetilde{\Delta}_{3,g}) \to A^*(\Delta_{3,g})$. Thus, it will suffice to describe the image of $A^{*-1}(\widetilde{\Delta}_{3,g}) \to A^*(\mathcal{X}'_{3,g})$.

To build $\widetilde{\Delta}_{3,g}$, we use the machinery of bundles of relative principal parts. By Proposition 2.3.2 part (1), there is an evaluation map

$$\gamma^*\pi^*\mathcal{X}'_{3,g} = (\pi \circ \gamma)^*(\pi \circ \gamma)_*\mathcal{W} \to P^1_{\mathbb{P}\mathcal{E}^\vee/\mathcal{B}'_{3,g}}(\mathcal{W}). \tag{2.4.3}$$

A geometric point of $\gamma^*\pi^*\mathcal{X}'_{3,g}$ is the data of (E, δ, p) where E is a geometric point of $\mathcal{B}'_{3,g}$, δ a section of $\mathcal{O}_{\mathbb{P}E}(3) \otimes \gamma^* \det E^\vee$, and p a point of $\mathbb{P}E^\vee$. Such a point lies in the kernel of the evaluation map (2.4.3) precisely when $\delta(p) = 0$ and the first order derivatives of δ also vanish at p, which is to say $V(\delta) \subset \mathbb{P}E^\vee$ has 2-dimensional tangent space at p. A similar description works in arbitrary families. We define $\widetilde{\Delta}_{3,g}$ to be the preimage of the zero section of (2.4.3) so we obtain a "trapezoid" diagram:

$$\begin{tikzcd} \widetilde{\Delta}_{3,g} \arrow[r, hook, "i"] \arrow[dr, "\rho''"'] & \gamma^*\pi^*\mathcal{X}'_{3,g} \arrow[r, "\gamma'"] \arrow[d, "\rho'"] & \pi^*\mathcal{X}'_{3,g} \arrow[r, "\pi'"] \arrow[d] & \mathcal{X}'_{3,g} \arrow[d, "\rho"] \\ & \mathbb{P}\mathcal{E}^\vee \arrow[r, "\gamma"'] & \mathcal{P} \arrow[r, "\pi"'] & \mathcal{B}'_{3,g}. \end{tikzcd} \tag{2.4.4}$$

We can thus determine information about the Chow ring of $\mathcal{H}_{3,g} = \mathcal{X}'_{3,g} \smallsetminus (\pi' \circ \gamma' \circ i)(\widetilde{\Delta}_{3,g})$ using the Trapezoid Lemma 2.2.1.

Lemma 2.4.2. *The rational Chow ring of $\mathcal{H}_{3,g}$ is a quotient of $\mathbb{Q}[a_1]/(a_1^3)$. Moreover,*

1. *For all $g \geq 3$, we have $A^1(\mathcal{H}_{3,g}) = \mathbb{Q}a_1$.*
2. *For all $g \geq 6$, we have $A^2(\mathcal{H}_{3,g}) = \mathbb{Q}a_1^2$.*

Proof. Let $z = c_1(\mathcal{O}_{\mathcal{P}}(1))$ and $\zeta = c_1(\mathcal{O}_{\mathbb{P}\mathcal{E}^\vee}(1))$, so $z^i\zeta^j$ for $0 \leq i, j \leq 1$ form a basis for $A^*(\mathbb{P}\mathcal{E}^\vee)$ as a $A^*(\mathcal{B}'_{3,g})$ module. Let I be the ideal generated by $(\pi \circ \gamma)_*(c_3(P^1_{\mathbb{P}\mathcal{E}^\vee/\mathcal{B}'_{3,g}}(\mathcal{W})) \cdot z^i\zeta^j)$ for $0 \leq i, j \leq 1$. We compute expressions for these push forwards in terms of a_1, a_2, a_2', c_2, and we find $\mathbb{Q}[a_1, a_2', a_2, c_2]/I \cong \mathbb{Q}[a_1]/(a_1^3)$. The code to do the above computations is provided at [CL21c]. For example, when $i = j = 0$, because $\widetilde{\Delta}_{3,g} \to \Delta_{3,g}$ is generically one-to-one, this allows us to find

$$[\Delta_{3,g}] = \pi'_*\gamma'_*[\widetilde{\Delta}_{3,g}] = \rho^*(\pi \circ \gamma)_*(c_3(P^1_{\mathbb{P}\mathcal{E}^\vee/\mathcal{B}'_{3,g}}(\mathcal{W})) = (8g + 12)a_1 - 9a_2'. \tag{2.4.5}$$

By the Trapezoid Lemma 2.2.1, we have that $A^*(\mathcal{H}_{3,g})$ is a quotient of $A^*(\mathcal{B}'_{3,g})/I$. Since $A^*(\mathcal{B}'_{3,g})$ is a quotient of $\mathbb{Q}[a_1, a_2, a_2', c_2]$, it follows that $A^*(\mathcal{H}_{3,g})$ is a quotient of $\mathbb{Q}[a_1]/(a_1^3)$.

First, note that the complement of $\mathcal{B}'_{3,g}$ inside $\mathcal{B}_{3,g}$ is the union of splitting loci where $E = \mathcal{O}(e_1) \oplus \mathcal{O}(e_2)$ for $3e_1 < g+2$ (see [CL21d, Section 4.2] for a review of splitting loci in our context). The codimension of the (e_1, e_2) splitting locus with $e_1 \leq e_2$ is $\max\{0, e_2 - e_1 - 1\}$. Using this, one readily checks that the complement of $\mathcal{B}'_{3,g}$ has codimension at least 2 for $g \geq 3$ and at least 3 for $g \geq 6$. Thus, by (2.4.1), for $g \geq 3$, the only relations in codimension 1 come from the push forwards of classes on $\widetilde{\Delta}_{3,g}$. Further, for $g \geq 6$, the only relations in codimension 2 come from the push forwards of classes supported on $\widetilde{\Delta}_{3,g}$.

To prove (1) and (2), it suffices to show that I already accounts for all such relations in codimension 1 when $g \geq 3$ and for all such relations in codimension 2 when $g \geq 6$. Precisely, let $\mathcal{Z} \subset \mathbb{P}\mathcal{E}^\vee$ be the locus where the map (2.4.3) fails to be surjective on fibers. We will show that

$$A^0(\widetilde{\Delta}_{3,g}) = A^0(\widetilde{\Delta}_{3,g} \smallsetminus \rho''^{-1}(\mathcal{Z})) \cong \rho''^* A^0(\mathbb{P}\mathcal{E}^\vee \smallsetminus \mathcal{Z}) = \rho''^* A^0(\mathbb{P}\mathcal{E}^\vee) \tag{2.4.6}$$

and when $g \neq 4$, that

$$A^1(\widetilde{\Delta}_{3,g}) = A^1(\widetilde{\Delta}_{3,g} \smallsetminus \rho''^{-1}(\mathcal{Z})) \cong \rho''^* A^1(\mathbb{P}\mathcal{E}^\vee \smallsetminus \mathcal{Z}) = \rho''^* A^1(\mathbb{P}\mathcal{E}^\vee). \tag{2.4.7}$$

The middle isomorphism follows in both cases from the fact that $\widetilde{\Delta}_{3,g} \smallsetminus \rho''^{-1}(\mathcal{Z})$ is a vector bundle over $\mathbb{P}\mathcal{E}^\vee \smallsetminus \mathcal{Z}$. To show the other equalities we use excision.

We claim that the map (2.4.3) always has rank at least 2. To see this, consider the diagram

$$\begin{tikzcd} \gamma^*\pi^*\pi_*\gamma_*\mathcal{W} \arrow[r] \arrow[d] & P^1_{\mathbb{P}\mathcal{E}^\vee/\mathcal{B}'_{3,g}}(\mathcal{W}) \arrow[d] \\ \gamma^*\gamma_*\mathcal{W} \arrow[r] & P^1_{\mathbb{P}\mathcal{E}^\vee/\mathcal{P}}(\mathcal{W}) \end{tikzcd} \tag{2.4.8}$$

The left vertical map is a surjection because $\gamma_*\mathcal{W}$ is relatively globally generated along

$\mathcal{P} \to \mathcal{B}'_{3,g}$ (by definition of $\mathcal{B}'_{3,g}$, see (2.4.2)); the bottom horizontal map is surjective by Lemma 2.3.3 because $\mathcal{W}$ is relatively very ample on $\mathbb{P}\mathcal{E}^\vee$ over $\mathcal{P}$. Thus, the top horizontal map must have rank at least $2 = \operatorname{rank}(P^1_{\mathbb{P}\mathcal{E}^\vee/\mathcal{P}}(\mathcal{W}))$. It follows that

$$\operatorname{codim}(\rho''^{-1}(\mathcal{Z}) \subset \widetilde{\Delta}_{3,g}) = \operatorname{codim}(\mathcal{Z} \subset \mathbb{P}\mathcal{E}^\vee) - 1. \tag{2.4.9}$$

By the argument in Lemma 2.3.5, $\mathcal{Z}$ is the locus where $\mathcal{W}$ fails to induce a relative embedding on $\mathbb{P}\mathcal{E}^\vee$ over $\mathcal{B}'_{3,g}$. By Lemma 2.3.4, the restriction to a fiber over $\mathcal{B}'_{3,g}$, say $\mathcal{W}|_{\mathbb{P}E^\vee} \cong \mathcal{O}_{\mathbb{P}E^\vee}(3) \otimes \gamma^*\mathcal{O}_{\mathbb{P}^1}(-g-2)$ fails to be very ample if and only if $E \cong \mathcal{O}(e_1) \oplus \mathcal{O}(e_2)$ with $3e_1 \leq g+2$. Moreover, in this case, the linear system fails to induce an embedding precisely along the directrix of $\mathbb{P}E^\vee$. By definition of $\mathcal{B}'_{3,g}$, we always have $3e_1 \geq g+2$. Thus, $\gamma(\mathcal{Z})$ is contained in at most one splitting locus, which is nonempty if and only if $g \equiv 1 \pmod 3$. In particular:

1. if $g = 4$, then $\gamma(\mathcal{Z})$ is the splitting locus $(e_1, e_2) = (2, 4)$, which has codimension 1
2. if $g = 7$, then $\gamma(\mathcal{Z})$ is the splitting locus $(e_1, e_2) = (3, 6)$, which has codimension 2
3. if $g \neq 4, 7$, then $\gamma(\mathcal{Z})$ has codimension at least 3

Since the directrix has codimension 1, it follows that

$$\operatorname{codim}(\mathcal{Z} \subset \mathbb{P}\mathcal{E}^\vee) = \begin{cases} 2 & \text{if } g = 4 \\ 3 & \text{if } g = 7 \\ \geq 4 & \text{otherwise.} \end{cases}$$

By (2.4.9), we see then that $\rho''^{-1}(\mathcal{Z})$ has suitably high codimension so that (2.4.6) is satisfied for all g and (2.4.7) is satisfied for $g \neq 4$. $\square$

This completes the proof of Theorem 2.1.1(1) when $g \geq 6$.

2.4.3 Low genus calculations

The lemmas in this section show that the remaining Chow groups not already determined by Lemma 2.4.2 vanish. This is due to certain geometric phenomena that occur in low codimension when the genus is small.

Lemma 2.4.3. *When $g = 2$, we have $A^*(\mathcal{H}_{3,2}) = 0$.*

Proof. When $g = 2$, the complement of $\mathcal{B}'_{3,2} \subset \mathcal{B}_{3,2}$ is the $(1,3)$ splitting locus, which has codimension 1. As a consequence, a_1 and a_2' satisfy a relation on $\mathcal{B}'_{3,2}$. Using [Lar21c, Lemma 5.1], we calculate the class of the $(1,3)$ splitting locus as the degree 1 piece of a ratio of total Chern classes below, which we compute with the code [CL21c]:

$$0 = s_{1,3} = \left[\frac{c((\pi_*\mathcal{E}(-2) \otimes \pi_*\mathcal{O}_{\mathcal{P}}(1))^\vee)}{c((\pi_*\mathcal{E}(-1))^\vee)}\right]_1 = a_2' - 2a_1$$

on $\mathcal{B}'_{3,2}$. Specializing (2.4.5) to $g = 2$, we also have the additional relation $0 = [\Delta_{3,2}] = 28a_1 - 9a_2'$ in $A^1(\mathcal{H}_{3,g})$, so we conclude $a_1 = a_2' = 0$ and hence, the Chow ring is trivial. $\square$

Lemma 2.4.4. *For $g = 3, 4, 5$, we have $A^2(\mathcal{H}_{3,g}) = 0$.*

Proof. We first explain the case $g = 3$. Here, the complement of $\mathcal{B}'_{3,3}$ inside $\mathcal{B}_{3,3}$ is the closure of the splitting locus $(e_1, e_2) = (1, 4)$, which has codimension 2. The universal formulas for classes of splitting loci [Lar21c] say that the class of this unbalanced splitting locus is the degree 2 piece of a ratio of total Chern classes, which we computed in the code [CL21c],

$$s_{1,4} = \left[\frac{c((\pi_*\mathcal{E}(-2) \otimes \pi_*\mathcal{O}_{\mathcal{P}}(1))^\vee)}{c((\pi_*\mathcal{E}(-1))^\vee)}\right]_2 = 3a_1^2 + \frac{1}{2}a_2 - \frac{5}{2}a_1a_2' + \frac{1}{2}a_2'^2 + 3c_2.$$

It follows that $A^*(\mathcal{H}_{3,3})$ is a quotient of $\mathbb{Q}[a_1, a_2, a_2', c_2]/(I + \langle s_{1,4}\rangle)$. We checked in the code [CL21c] that the codimension 2 piece of this ring is zero.

The case $g = 5$ is very similar so we explain it next. The complement of $\mathcal{B}'_{3,5}$ inside $\mathcal{B}_{3,5}$ is the closure of the splitting locus $(e_1, e_2) = (2, 5)$, which has codimension 2. The class of this splitting locus is computed similarly:

$$s_{2,5} = \left[\frac{c((\pi_*\mathcal{E}(-3) \otimes \pi_*\mathcal{O}_{\mathcal{P}}(1))^\vee)}{c((\pi_*\mathcal{E}(-2))^\vee)}\right]_2 = 6a_1^2 + \frac{1}{2}a_2 - \frac{7}{2}a_1a_2' + \frac{1}{2}a_2'^2 + 6c_2.$$

Therefore, $A^*(\mathcal{H}_{5,3})$ is a quotient $\mathbb{Q}[a_1, a_2, a_2', c_2]/(I + \langle s_{2,5}\rangle)$, whose codimension 2 piece we also checked is zero [CL21c].

In the case $g = 4$, our additional relation will come from $\rho''^{-1}(\mathcal{Z}) \subset \widetilde{\Delta}_{3,4}$, which has codimension 1, and whose push forward therefore determines a class that is zero in $A^2(\mathcal{H}_{3,4})$. By (2.4.8), we have that $\rho''^{-1}(\mathcal{Z})$ is the transverse intersection of $\rho'^{-1}(\mathcal{Z})$ with the kernel subbundle of $\gamma^*\pi^*\pi_*\gamma_*\mathcal{W} \to P^1_{\mathbb{P}\mathcal{E}^\vee/\mathcal{P}}(\mathcal{W})$. That is, our possible additional relation is given by

$$s := \pi'_*\gamma'_*i_*[\rho''^{-1}(\mathcal{Z})] = \gamma'_*\pi'_*(\rho'^*[\mathcal{Z}] \cdot \rho'^*c_2(P^1_{\mathbb{P}\mathcal{E}^\vee/\mathcal{P}}(\mathcal{W}))) = \rho^*\gamma_*\pi_*([\mathcal{Z}] \cdot c_2(P^1_{\mathbb{P}\mathcal{E}^\vee/\mathcal{P}}(\mathcal{W}))). \tag{2.4.10}$$

It remains to compute $[\mathcal{Z}]$, which we do now. Let $\Sigma = \gamma(\mathcal{Z}) \subset \mathcal{B}'_{3,4}$ be the $(2, 4)$ splitting locus. Using the formulas for classes of splitting loci [Lar21c], we compute

$$[\Sigma] = s_{1,4} = \left[\frac{c(((\pi_*\mathcal{E}(-3) \otimes \pi_*\mathcal{O}_{\mathcal{P}}(1))^\vee)}{c(((\pi_*\mathcal{E}(-2))^\vee)}\right]_1 = a_2' - 3a_1.$$

Over Σ, there is a sequence

$$0 \to \pi^*\mathcal{M}(-2) \to \mathcal{E}^\vee|_\Sigma \to \pi^*\mathcal{N}(-4) \to 0 \tag{2.4.11}$$

for line bundles $\mathcal{M}$ and $\mathcal{N}$ on Σ. Let $m = c_1(\mathcal{M})$ and $n = c_1(\mathcal{N})$. The directrix over Σ is $\mathcal{Z} = \mathbb{P}(\pi^*\mathcal{M}(-2)) \subset \mathbb{P}\mathcal{E}^\vee|_\Sigma$. By [EH16, Proposition 9.13], the fundamental class of $\mathcal{Z}$ inside $\mathbb{P}\mathcal{E}^\vee|_\Sigma$ is $\zeta + c_1(\pi^*\mathcal{N}(-4)) = \zeta + n - 4z$. Considering Chern classes in the exact sequence (2.4.11), we learn (recall $a_1' = g+2 = 6$)

$$-a_1|_\Sigma - 6z = c_1(\mathcal{E}^\vee|_\Sigma) = m - 4z + n - 2z \qquad \Rightarrow \qquad m + n = -a_1|_\Sigma$$

and

$$\begin{aligned} a_2|_\Sigma + (a_2'|_\Sigma)\cdot z = c_2(\mathcal{E}^\vee|_\Sigma) &= (m-4z)(n-2z) \\ &= mn - c_2 - (2m+4n)z \qquad \Rightarrow \qquad 2m + 4n = -a_2'|_\Sigma. \end{aligned}$$

In particular, $n = \left(a_1 - \frac{a_2'}{2}\right)\Big|_\Sigma$. Hence, the fundamental class of $\mathcal{Z}$ inside all of $\mathbb{P}\mathcal{E}^\vee$ is

$$[\mathcal{Z}] = (\zeta + a_1 - \tfrac{a_2'}{2} - 4z)\cdot[\Sigma] = (\zeta + a_1 - \tfrac{a_2'}{2} - 4z)(a_2' - 3a_1).$$

This allows us to compute s in (2.4.10), and our code confirms that the codimension 2 piece of $\mathbb{Q}[a_1, a_2, a_2', c_2]/(I + \langle s\rangle)$ is zero [CL21c]. $\square$

Together, Lemmas 2.4.2, 2.4.3 and 2.4.4 determine the rational Chow ring of $\mathcal{H}_{3,g}$ for all g:

$$A^*(\mathcal{H}_{3,g}) = \begin{cases} \mathbb{Q} & \text{if } g = 2 \\ \mathbb{Q}[a_1]/(a_1^2) & \text{if } g = 3,4,5 \\ \mathbb{Q}[a_1]/(a_1^3) & \text{if } g \geq 6. \end{cases}$$

This completes the proof of Theorem 2.1.1(1).

2.5 The Chow ring in degree 4

2.5.1 Set up

We begin by briefly recalling the linear algebraic data associated to a degree 4 cover, as developed by Casnati–Ekedahl [CE96]. For more details in our context, see [CL21d, Section 3.2]. Given a degree 4 cover $\alpha : C \to \mathbb{P}^1$, we associate two vector bundles on $\mathbb{P}^1$:

$$E_\alpha := (\alpha_* \mathcal{O}_C/\mathcal{O}_{\mathbb{P}^1})^\vee = \ker(\alpha_*\omega_\alpha \to \mathcal{O}_{\mathbb{P}^1}) \qquad \text{and} \qquad F_\alpha := \ker(\operatorname{Sym}^2 E_\alpha \to \alpha_*\omega_\alpha^{\otimes 2}).$$

The first is rank 3 and the second is rank 2. If C has genus g, then both bundles have degree $g+3$. Geometrically, the curve C is embedded in $\gamma : \mathbb{P}E_\alpha^\vee \to \mathbb{P}^1$ as the zero locus of a section

$$\delta_\alpha \in H^0(\mathbb{P}E_\alpha^\vee, \mathcal{O}_{\mathbb{P}E_\alpha^\vee}(2) \otimes \gamma^* F_\alpha^\vee).$$

In each fiber of γ, the four points are the base locus of a pencil of conics parametrized by F_α.

Conversely, given vector bundles E, F of ranks 3 and 2, both of degree $g+3$, we wish to characterize when a section

$$\delta \in H^0(\mathbb{P}E^\vee, \mathcal{O}_{\mathbb{P}E^\vee}(2) \otimes \gamma^* F^\vee)$$

fails to produce a smooth degree 4, genus g cover.

Lemma 2.5.1. *Suppose E, F, δ are as above with $F^\vee \otimes \operatorname{Sym}^2 E$ globally generated. If the zero locus $C = V(\delta)$ is not an irreducible, smooth quadruple cover of $\mathbb{P}^1$, then there is a point $p \in C$ such that $\dim T_pC \geq 2$.*

Proof. If C is connected or has a component of dimension at least 2 then the lemma is immediate. Suppose C is 1-dimensional and disconnected. We first rule out the case in which C has at least 2 connected components, both mapping finitely onto $\mathbb{P}^1$. In this case, $\alpha_*\mathcal{O}_C$ has more than one $\mathcal{O}$ factor; then E has a degree 0 summand, so $\operatorname{Sym}^2 E \otimes F^\vee$ would have a negative degree summand, which we are assuming is not the case.

Next suppose C has a component C_0 which does not map finitely onto $\mathbb{P}^1$. Then

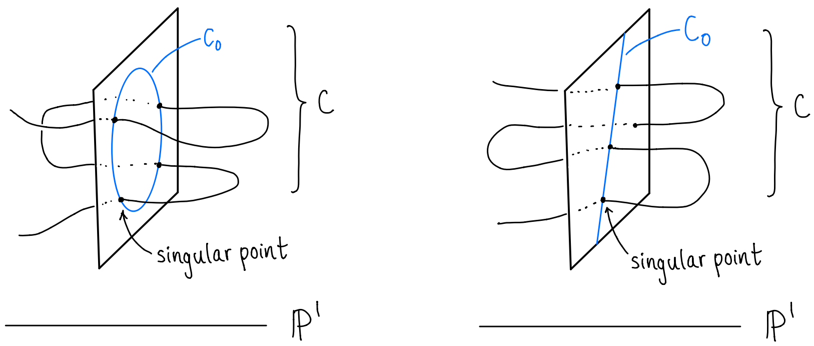

Figure 2.2. Possible 1-dimensional fibers.

C_0 must be contained in a fiber of $\gamma : \mathbb{P}E^\vee \to \mathbb{P}^1$. The restriction of the zero locus of δ to a fiber is the intersection of two (possibly singular) conics in $\mathbb{P}^2$. The only way for such an intersection to have a 1-dimensional component is for the conics to have a common component C_0. Hence, some fiber of C is equal to C_0 union a finite subscheme of length less than 4 (length 1 if C_0 is a line, empty if C_0 is a conic). Since the generic fiber consists of 4 points, some of those 4 points must specialize to C_0, which means C is singular at those points on C_0 (and C is connected). □

The association of $\alpha : C \to \mathbb{P}^1$ with the pair (E_α, F_α) gives rise to map $\mathcal{H}_{4,g}$ to the moduli stack $\mathcal{B}_{4,g}$ of pairs of vector bundles on $\mathbb{P}^1$-bundles, as defined in [CL21d, Definition 5.2]. Let $\pi : \mathcal{P} \to \mathcal{B}_{4,g}$ be the universal $\mathbb{P}^1$-bundle. Let $\mathcal{E}$ be the universal rank 3 vector bundle on $\mathcal{P}$, and $\mathcal{F}$ the universal rank 2 bundle on $\mathcal{P}$. Continuing the notation of [CL21d], let $z = c_1(\mathcal{O}_{\mathcal{P}}(1))$ and define classes $a_i, b_i \in A^i(\mathcal{B}_{4,g})$ and $a_i', b_i' \in A^{i-1}(\mathcal{B}_{4,g})$ by the formula

$$c_i(\mathcal{E}) = \pi^* a_i + \pi^* a_i' z \qquad \text{and} \qquad c_i(\mathcal{F}) = \pi^* b_i + \pi^* b_i' z.$$

(Note that there is a "determinant compatibility" condition which implies $a_1 = b_1$, see [CL21d, Equation 5.4].) We also define $c_2 = -\pi_*(z^3) \in A^2(\mathcal{B}_{4,g})$, which is the pullback of the universal second Chern class on BSL_2. By [CL21d, Equation 5.5],

$$a_1, a_2, a_3, a_2', a_3', b_2, b_2', c_2 \text{ generate } A^*(\mathcal{B}_{4,g}) \text{ and satisfy} \qquad (2.5.1)$$
$$\text{no relations in degrees up to } g+3.$$

We call the pullbacks of $\mathcal{E}$ and $\mathcal{F}$ to $\mathcal{H}_{4,g}$ the *CE bundles* (these are the bundles appearing in the Casnati–Ekedahl resolution for the universal curve). We call the pullbacks to $\mathcal{H}_{4,g}$ of the associated classes in (2.5.1) the *CE classes*. By [CL21d, Theorem 3.10], the CE classes are tautological and generate the tautological ring.

Up to this point, the set up has been quite similar to degree 3. However, unlike in degree 3, the full Hurwitz stack $\mathcal{H}_{4,g}$ *cannot* be realized as an open substack of a vector bundle over an open substack of $\mathcal{B}_{4,g}$. This is why we were unable to determine the full Chow ring of $\mathcal{H}_{4,g}$ with our techniques. We now proceed in two steps. First, in Section 2.5.2 we shall produce several relations among CE classes on $\mathcal{H}_{4,g}$ using principal parts bundles. Then, in Section 2.5.3, we shall define an open substack $\mathcal{H}_{4,g}^\circ \subset \mathcal{H}_{4,g}$, which does lie inside a vector bundle over an open substack $\mathcal{B}_{4,g}^\circ \subset \mathcal{B}_{4,g}$, and use it to demonstrate that we have found all relations in degrees up to roughly $g/4$. It may help to think of $\mathcal{H}_{4,g}^\circ$ as an open substack that is "large enough to witness the independence of many CE classes."

2.5.2 Relations among CE classes

Let $\mathcal{E}$ and $\mathcal{F}$ be the CE bundles on the universal $\mathbb{P}^1$-bundle $\pi : \mathcal{P} \to \mathcal{H}_{4,g}$. Let $\gamma : \mathbb{P}\mathcal{E}^\vee \to \mathcal{P}$ be the structure map. We define a rank 2 vector bundle on $\mathbb{P}\mathcal{E}^\vee$ by $\mathcal{W} := \mathcal{O}_{\mathbb{P}\mathcal{E}^\vee}(2) \otimes \mathcal{F}^\vee$. By the Casnati–Ekedahl theorem in degree 4, see [CL21d, Equation 3.7] or [CE96], the universal curve $\mathcal{C} \subset \mathbb{P}\mathcal{E}^\vee$ determines a global section δ^{univ} of $\mathcal{W}$, whose vanishing locus is $V(\delta^{\mathrm{univ}}) = \mathcal{C} \subset \mathbb{P}\mathcal{E}^\vee$.

The global section δ^{univ} induces a global section $\delta^{\mathrm{univ}\prime}$ of the principal parts bundle

$P^1_{\mathbb{P}\mathcal{E}^\vee/\mathcal{H}_{4,g}}(\mathcal{W})$ on $\mathbb{P}\mathcal{E}^\vee$, which records the value and derivatives of δ^{univ}. Now consider the tower

$$G(2, T_{\mathbb{P}\mathcal{E}^\vee/\mathcal{H}_{4,g}}) \xrightarrow[a]{} \mathbb{P}\mathcal{E}^\vee \xrightarrow[\gamma]{} \mathcal{P} \xrightarrow[\pi]{} \mathcal{H}_{4,g},$$

where $G(2, T_{\mathbb{P}\mathcal{E}^\vee/\mathcal{H}_{4,g}})$ parametrizes 2 dimensional subspaces of the vertical tangent space of $\mathbb{P}\mathcal{E}^\vee$ over $\mathcal{H}_{4,g}$. Dualizing the tautological sequence on $G(2, T_{\mathbb{P}\mathcal{E}^\vee/\mathcal{H}^\circ_{4,g}})$ we obtain a filtration

$$0 \to \Omega_y \to a^*\Omega_{\mathbb{P}\mathcal{E}^\vee/\mathcal{H}_{4,g}} \to \Omega_x \to 0,$$

where Ω_y is rank 1 and Ω_x is rank 2. Let $P^{\{1,x\}}_{\mathbb{P}\mathcal{E}^\vee/\mathcal{H}_{4,g}}(\mathcal{W})$ be the bundle of restricted principal parts as defined in Section 2.3.2.

On $G(2, T_{\mathbb{P}\mathcal{E}^\vee/\mathcal{H}_{4,g}})$, we obtain a global section, call it $\delta^{\mathrm{univ}''}$, of $P^{\{1,x\}}_{\mathbb{P}\mathcal{E}^\vee/\mathcal{H}_{4,g}}(\mathcal{W})$ by composing the section $a^*\delta^{\mathrm{univ}'}$ with the quotient $a^*P^1_{\mathbb{P}\mathcal{E}^\vee/\mathcal{H}_{4,g}}(\mathcal{W}) \to P^{\{1,x\}}_{\mathbb{P}\mathcal{E}^\vee/\mathcal{H}_{4,g}}(\mathcal{W})$. The vanishing locus of $\delta^{\mathrm{univ}''}$ is the space of pairs (p, S) where $p \in V(\delta^{\mathrm{univ}}) \subset \mathbb{P}\mathcal{E}^\vee$ and S is a two-dimensional subspace of the tangent space of the fiber of $V(\delta^{\mathrm{univ}}) \to \mathcal{H}_{4,g}$ through p. But $V(\delta^{\mathrm{univ}}) = \mathcal{C} \to \mathcal{H}_{4,g}$ is smooth of relative dimension 1. Thus, $\delta^{\mathrm{univ}''}$ must be non-vanishing on $G(2, T_{\mathbb{P}\mathcal{E}^\vee/\mathcal{H}_{4,g}})$.

Since $P^{\{1,x\}}_{\mathbb{P}\mathcal{E}^\vee/\mathcal{H}_{4,g}}(\mathcal{W})$ has a non-vanishing global section, its top Chern class, we have

$$c_6(P^{\{1,x\}}_{\mathbb{P}\mathcal{E}^\vee/\mathcal{H}_{4,g}}(\mathcal{W})) = 0.$$

Moreover, the push forward of this class times any class on $G(2, T_{\mathbb{P}\mathcal{E}^\vee/\mathcal{H}_{4,g}})$ is also zero. Such relations are generated by the following classes.

Lemma 2.5.2. *Let $\tau = c_1(\Omega_y^\vee)$ where $\Omega_y^\vee$ is the tautological quotient on $G(2, T_{\mathbb{P}\mathcal{E}^\vee/\mathcal{H}_{4,g}})$. Let $\zeta = \mathcal{O}_{\mathbb{P}\mathcal{E}^\vee}(1)$ and $z = c_1(\mathcal{O}_{\mathcal{P}}(1))$. Then all classes of the form*

$$\pi_*\gamma_*a_*(c_6(P^{\{1,x\}}_{\mathbb{P}\mathcal{E}^\vee/\mathcal{H}_{4,g}}(\mathcal{W})) \cdot \tau^i\zeta^j z^k) \tag{2.5.2}$$

are zero in $R^*(\mathcal{H}_{4,g}) \subseteq A^*(\mathcal{H}_{4,g})$.

It is straightforward for a computer to compute such push forwards as polynomials in the CE classes. We describe the ideal these push forwards generate in Section 2.5.4

2.5.3 All relations in low codimension

We now recall the construction of our "large open" substack $\mathcal{H}^\circ_{4,g} \subset \mathcal{H}_{4,g}$. We start with $\mathcal{B}_{4,g}$, the moduli space of pairs of vector bundles E of rank 3, degree $g+3$ and F of rank 2, degree $g+3$ on $\mathbb{P}^1$-bundles together with an isomorphism of their determinants (see [CL21d, Section 5.2]). Now, working over $\mathcal{B}_{4,g}$, let $\mathcal{E}$ and $\mathcal{F}$ be the universal bundles on $\pi : \mathcal{P} \to \mathcal{B}_{4,g}$ and let $\gamma : \mathbb{P}\mathcal{E}^\vee \to \mathcal{P}$ be the structure map. Define $\mathcal{W} := \gamma^*\mathcal{F}^\vee \otimes \mathcal{O}_{\mathbb{P}\mathcal{E}^\vee}(2)$, and let $\mathcal{U}_{4,g} := \gamma_*\mathcal{W} = \mathcal{F}^\vee \otimes \operatorname{Sym}^2 \mathcal{E}$. We consider an open substack $\mathcal{B}^\circ_{4,g} \subset \mathcal{B}_{4,g}$, defined by a certain positivity condition for the bundle $\mathcal{U}_{4,g}$

$$\mathcal{B}^\circ_{4,g} := \mathcal{B}_{4,g} \smallsetminus \operatorname{Supp} R^1\pi_*(\mathcal{U}_{4,g}(-2)). \tag{2.5.3}$$

Let $\mathcal{H}^\circ_{4,g}$ denote the base change of $\mathcal{H}_{4,g} \to \mathcal{B}_{4,g}$ along the open embedding $\mathcal{B}^\circ_{4,g} \hookrightarrow \mathcal{B}_{4,g}$.

Remark 2.5.3. We note that the complement of $\mathcal{H}^\circ_{4,g} \subset \mathcal{H}_{4,g}$ (represented in blue in the right of Figure 2.3) contains covers that factor through a curve of low genus (see [CL21d, p. 21-22]). Thus, the codimension of the complement of $\mathcal{H}^\circ_{4,g} \subset \mathcal{H}_{4,g}$ is 2. However, upon restricting to non-factoring covers, the codimension of the complement of $\mathcal{H}^\circ_{4,g} \cap \mathcal{H}^{\mathrm{nf}}_{4,g} \subset \mathcal{H}^{\mathrm{nf}}_{4,g}$ has codimension at least $\frac{g+3}{4} - 4$ [CL21d, Lemma 5.5]. In this sense, $\mathcal{H}^{\mathrm{nf}}_{4,g}$ and $\mathcal{H}^\circ_{4,g}$ are "good approximations" to each other. This is what allows us to find stabilization results for $\dim A^i(\mathcal{H}^{\mathrm{nf}}_{4,g})$.

Over $\mathcal{B}^\circ_{4,g}$, we see that $\mathcal{X}^\circ_{4,g} := \pi_*\mathcal{U}_{4,g}|_{\mathcal{B}^\circ_{4,g}}$ is a vector bundle whose fibers correspond to sections of $\mathcal{U}_{4,g}$. The open $\mathcal{H}^\circ_{4,g}$ is contained in the open $\mathcal{H}'_{4,g}$ of [CL21d, Lemma 5.3],

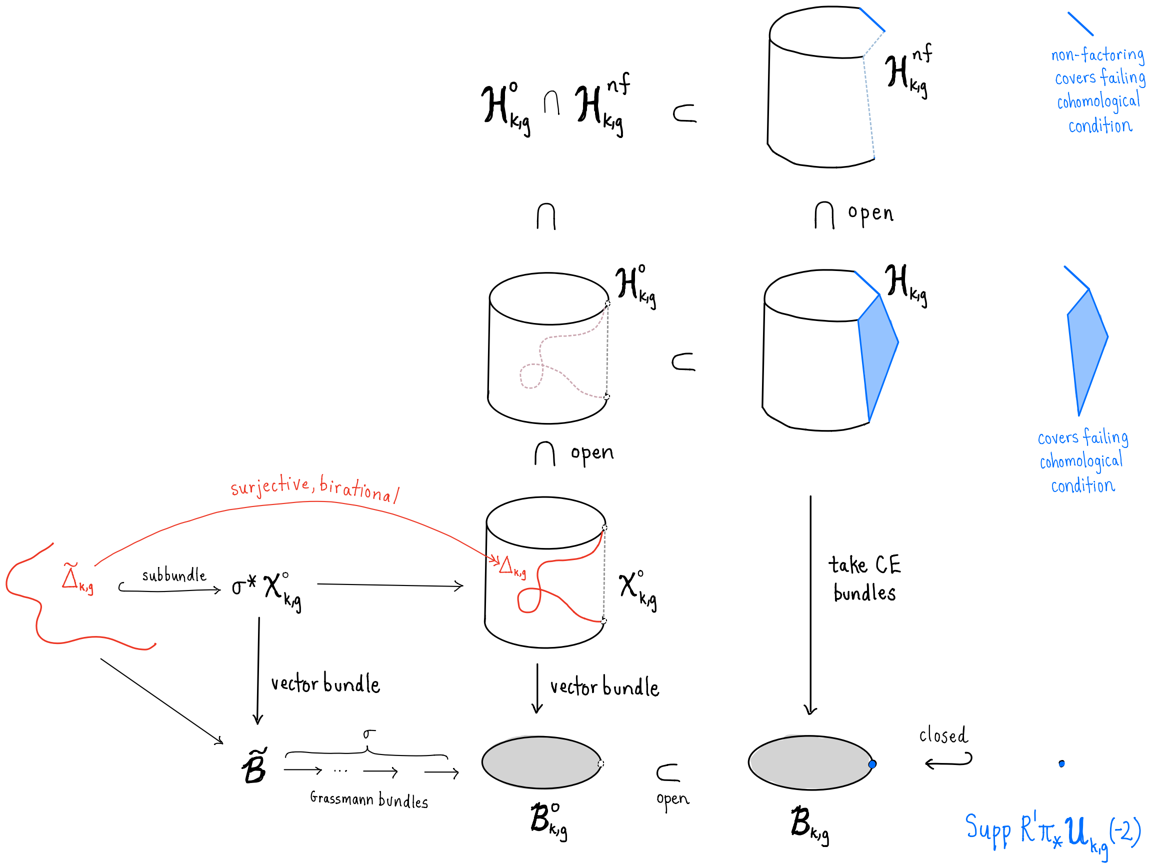

Figure 2.3. Summary of the method.

so that lemma implies $\mathcal{H}^\circ_{4,g} \to \mathcal{B}^\circ_{4,g}$ factors through an open embedding in $\mathcal{X}^\circ_{4,g}$. We define

$$\Delta_{4,g} := \mathcal{X}^\circ_{4,g} \smallsetminus \mathcal{H}^\circ_{4,g},$$

represented in red in the middle column of Figure 2.3. Now we wish to use the excision to determine the Chow ring of $\mathcal{H}^\circ_{4,g}$ in degrees up to $\frac{g+3}{4} - 4$. Note that $A^*(\mathcal{X}^\circ_{4,g}) \cong A^*(\mathcal{B}^\circ_{4,g})$, and we have already determined the latter in degrees up to $\frac{g+3}{4} - 4$ by [CL21d, Equation 5.6].

The next step is to construct a space $\widetilde{\Delta}_{4,g}$ (pictured in red on the far left of Figure 2.3), which surjects properly onto $\Delta_{4,g}$. With rational coefficients, the push forward $\widetilde{\Delta}_{4,g} \to \Delta_{4,g}$ will be surjective on Chow groups. Thus, pushing forward all classes from

$\widetilde{\Delta}_{4,g}$ will produce all relations needed to describe $\mathcal{H}_{4,g}^\circ$ as a quotient of $A^*(\mathcal{X}_{4,g}^\circ) \cong A^*(\mathcal{B}_{4,g}^\circ)$.

Each geometric point of $\mathcal{X}_{4,g}^\circ$ corresponds to a tuple (E, F, δ) where E, F are vector bundles on $\mathbb{P}^1$ and $\delta \in H^0(\mathbb{P}E^\vee, F^\vee \otimes \mathcal{O}_{\mathbb{P}E^\vee}(2))$. We now use restricted bundles of relative principal parts for $\mathbb{P}\mathcal{E}^\vee \to \mathcal{B}_{4,g}^\circ$ to define a space parametrizing triples

$$((E, F, \delta) \in \mathcal{X}_{4,g}^\circ,\ p \in V(\delta),\ S \subset T_pV(\delta) \text{ of dimension 2}).$$

Let $a : G(2, T_{\mathbb{P}\mathcal{E}^\vee/\mathcal{B}_{4,g}^\circ}) \to \mathbb{P}\mathcal{E}^\vee$ be the Grassmann bundle of 2-planes in the relative tangent bundle. Dualizing the tautological sequence on $G(2, T_{\mathbb{P}\mathcal{E}^\vee/\mathcal{B}_{4,g}^\circ})$ we obtain a filtration

$$0 \to \Omega_y \to a^*\Omega_{\mathbb{P}\mathcal{E}^\vee/\mathcal{B}_{4,g}^\circ} \to \Omega_x \to 0,$$

where Ω_y is rank 1 and Ω_x is rank 2. Using the bundle of restricted principal parts constructed in Section 2.3.2, we obtain an evaluation map

$$a^*\gamma^*\pi^*\pi_*\gamma_*\mathcal{W} \cong a^*\gamma^*\pi^*\mathcal{X}_{4,g}^\circ \to P^1_{\mathbb{P}\mathcal{E}^\vee/\mathcal{B}_{4,g}^\circ}(\mathcal{W}) \to P^{\{1,x\}}_{\mathbb{P}\mathcal{E}^\vee/\mathcal{B}_{4,g}^\circ}(\mathcal{W}), \tag{2.5.4}$$

which we claim is surjective. The rightmost map from principal parts to restricted principal parts is always a surjection. Thus, it suffices to show that the map $\gamma^*\pi^*\mathcal{X}_{4,g}^\circ \to P^1_{\mathbb{P}\mathcal{E}^\vee/\mathcal{B}_{4,g}^\circ}(\mathcal{W})$ is surjective. By definition of $\mathcal{B}_{4,g}^\circ$ (see (2.5.3)), we have $R^1\pi_*[(\gamma_*\mathcal{W}) \otimes \mathcal{O}_{\mathcal{P}}(-2)] = 0$, so the surjectivity follows from Lemma 2.3.5.

We define $\widetilde{\Delta}_{4,g}$ to be the kernel bundle of (2.5.4). We have the following "trapezoid" diagram:

$$\begin{tikzcd}
\widetilde{\Delta}_{4,g} \arrow[r, hook] \arrow[dr, "\rho''"'] & a^*\gamma^*\pi^*\mathcal{X}_{4,g}^\circ \arrow[r, "a'"] \arrow[d, "\rho'"] & \gamma^*\pi^*\mathcal{X}_{4,g}^\circ \arrow[r, "\gamma'"] \arrow[d] & \pi^*\mathcal{X}_{4,g}^\circ \arrow[r, "\pi'"] \arrow[d] & \mathcal{X}_{4,g}^\circ \arrow[d, "\rho"] \\
 & G(2, T_{\mathbb{P}\mathcal{E}^\vee/\mathcal{B}_{4,g}^\circ}) \arrow[r, "a"'] & \mathbb{P}\mathcal{E}^\vee \arrow[r, "\gamma"'] & \mathcal{P} \arrow[r, "\pi"'] & \mathcal{B}_{4,g}^\circ
\end{tikzcd} \tag{2.5.5}$$

Proposition 2.5.4. *Let $\tau = c_1(\Omega_y^\vee)$ where $\Omega_y^\vee$ is the tautological quotient line bundle on $G(2, T_{\mathbb{P}\mathcal{E}^\vee/\mathcal{B}^\circ_{4,g}})$. Let $\zeta = \mathcal{O}_{\mathbb{P}\mathcal{E}^\vee}(1)$ and $z = c_1(\mathcal{O}_{\mathcal{P}}(1))$. Let I be the ideal generated by*

$$\pi_*\gamma_*a_*(c_6(P^{\{1,x\}}_{\mathbb{P}\mathcal{E}^\vee/\mathcal{B}^\circ_{4,g}}(\mathcal{W}))\cdot \tau^i\zeta^j z^k) \qquad \text{for } 0 \leq i,j \leq 2,\ 0 \leq k \leq 1. \tag{2.5.6}$$

Then $A^*(\mathcal{H}^\circ_{4,g}) \cong A^*(\mathcal{B}^\circ_{4,g})/I$. Together with $a_1 = b_1$, the classes in (2.5.2) therefore generate all relations among the CE classes on $\mathcal{H}_{4,g}$ in degrees less than $\frac{g+3}{4} - 4$.

Proof. By Lemma 2.5.1, $\widetilde{\Delta}_{4,g}$ surjects onto $\Delta_{4,g}$, so we may apply the Trapezoid Lemma 2.2.1. Since $T_{\mathbb{P}\mathcal{E}^\vee/\mathcal{B}^\circ_{4,g}}$ has rank 3, the Grassmann bundle $G(2, T_{\mathbb{P}\mathcal{E}^\vee/\mathcal{B}^\circ_{4,g}})$ is just the projectivization of $T^\vee_{\mathbb{P}\mathcal{E}^\vee/\mathcal{B}^\circ_{4,g}}$; hence its Chow ring is generated as a module over $A^*(\mathbb{P}\mathcal{E}^\vee)$ by τ^i for $0 \leq i \leq 2$. Similarly $A^*(\mathbb{P}\mathcal{E}^\vee)$ is generated as a module over $A^*(\mathcal{P})$ by ζ^j for $0 \leq j \leq 2$ and $A^*(\mathcal{P})$ is generated as a module over $A^*(\mathcal{B}^\circ_{4,g})$ by z^k for $0 \leq k \leq 1$. Thus, the Trapezoid Lemma 2.2.1 implies that the classes in (2.5.6) generate all relations among the pullbacks of classes on $\mathcal{B}^\circ_{4,g}$. In particular, setting $i = j = k = 0$, we obtain

$$[\Delta_{4,g}] = \pi'_*\gamma'_*a'_*[\widetilde{\Delta}_{4,g}] = \rho^*(\pi\circ\gamma\circ a)_*(c_6(P^1_{\mathbb{P}\mathcal{E}^\vee/\mathcal{B}^\circ_{4,g}}(\mathcal{W}))) = (8g+20)a_1 - 8a'_2 - b'_2. \tag{2.5.7}$$

To see the second claim, note that the classes in (2.5.6) pullback to the classes in (2.5.2). By [CL21d, Equation 5.6], the generators $a_1 = b_1, a_2, a'_2, a_3, a'_3, b_2, b'_2, c_2$ of $A^*(\mathcal{B}^\circ_{4,g})$ satisfy no relations in codimension less than $\frac{g+3}{4} - 4$ (besides $a_1 = b_1$). Since one can only obtain more relations under restriction $A^*(\mathcal{H}_{4,g}) \to A^*(\mathcal{H}^\circ_{4,g})$, we have found all relations among CE classes in degrees less than $\frac{g+3}{4} - 4$. □

2.5.4 Presentation of the ring and stabilization

We use the code [CL21c] compute the classes in (2.5.2). Let I be the ideal they generate in the $\mathbb{Q}$-algebra on the CE classes. It turns out that modulo I, all CE classes are expressible in terms of a_1, a_2', a_3'. In particular,

$$\mathbb{Q}[c_2, a_1, a_2, a_3, a_2', a_3', b_2', b_2]/I \cong \mathbb{Q}[a_1, a_2', a_3']/\langle r_1, r_2, r_3, r_4\rangle, \tag{2.5.8}$$

where

$$\begin{aligned}
r_1 &= (2g^3 + 9g^2 + 10g)a_1^3 - (8g^2 + 24g + 8)a_1 a_3' \\
r_2 &= (12g^3 + 42g^2 + 36g)a_1^2 a_2' - (22g^3 + 121g^2 + 187g + 66)a_1 a_3' - (24g^2 + 24g)a_2' a_3' \\
r_3 &= (432g^3 + 1512g^2 + 1296g)a_1 a_2'^2 - (1450g^3 + 8001g^2 + 13115g + 5442)a_1 a_3' \\
&\qquad - (1584g^3 + 5544g^2 + 3936g)a_2' a_3' \\
r_4 &= (14344g^6 + 165692g^5 + 747682g^4 + 1636869g^3 + 1719009g^2 + 677844g - 540)a_1^2 a_3' \\
&\quad - (17280g^4 + 112320g^3 + 224640g^2 + 129600g)a_2'^2 a_3' + (352g^5 + 1440g^4 + 1448g^3 + 120g^2)a_3'^2.
\end{aligned}$$

Remark 2.5.5. In contrast with the degree 3 case, brute force computations show that there is no presentation of the Chow ring whose relations do not involve g.

Corollary 2.5.6. *Suppose $g \geq 2$.*

1. *$R^1(\mathcal{H}_{4,g})$ is spanned by $\{a_1, a_2'\}$.*
2. *$R^2(\mathcal{H}_{4,g})$ is spanned by $\{a_1^2, a_1 a_2', a_2'^2, a_3'\}$.*
3. *$R^3(\mathcal{H}_{4,g})$ is spanned by $\{a_1 a_3', a_2'^3, a_2' a_3'\}$.*
4. *$R^4(\mathcal{H}_{4,g})$ is spanned by $\{a_2'^4, a_3'^2\}$.*
5. *For $i \geq 5$, $R^i(\mathcal{H}_{4,g})$ is spanned by $\{a_2'^i\}$.*

For $g > 4i + 12$, the spanning set of $R^i(\mathcal{H}_{4,g})$ given above is a basis.

Proof. Our code [CL21c] verifies that the lists above are bases for $\mathbb{Q}[a_1, a_2', a_3']/\langle r_1, r_2, r_3, r_4\rangle$ in degrees $i \leq 9$. In particular, for $5 \leq i \leq 10$, every monomial in a_1, a_2', a_3' of degree i

is a multiple of $a_2'^i$. By inspection, $a_2'^i$ is not in the ideal $\langle r_1, r_2, r_3, r_4\rangle$ for any i, so $a_2'^i$ is non-zero for all i. For $i \geq 11$, every monomial of degree i in a_1, a_2', a_3' is expressible as a product of monomials having degrees between 5 and 10. It follows that every monomial of degree $i \geq 11$ is a multiple of $a_2'^i$.

Proposition 2.5.4 states that I provides all relations among the CE classes in degrees less than $\frac{g+3}{4} - 4$. That is, the left-hand side of (2.5.8) maps to $R^*(\mathcal{H}_{4,g})$ isomorphically in degrees $i < \frac{g+3}{4} - 4$. Hence, a basis for the degree i piece of $\mathbb{Q}[a_1, a_2', a_3']/\langle r_1, r_2, r_3, r_4\rangle$ is a basis for $R^i(\mathcal{H}_{4,g})$ when $i < \frac{g+3}{4} - 4$, equivalently when $g > 4i + 12$. □

Proof of Theorem 2.1.1(2). Consider the equation

$$\frac{\mathbb{Q}[a_1, a_2', a_3']}{\langle r_1, r_2, r_3, r_4\rangle} \to R^*(\mathcal{H}_{4,g}) \to R^*(\mathcal{H}^\circ_{4,g}) = A^*(\mathcal{H}^\circ_{4,g}).$$

The first map exists and is surjective by Proposition 2.5.2 and the presentation (2.5.8). Meanwhile, Lemma 2.5.4 establishes that the composition is an isomorphism in degrees less than $\frac{g+3}{4} - 4$. Therefore, the first map can have no kernel in codimension less than $\frac{g+3}{4} - 4$.

Finally, in [CL21d, Lemmas 5.5 and 5.8], we showed that $\mathcal{H}^\circ_{4,g}$ and $\mathcal{H}^{\mathrm{nf}}_{4,g}$ are "good approximations of each other" in the sense that the codimension the complement of $\mathcal{H}^\circ_{4,g} \cap \mathcal{H}^{\mathrm{nf}}_{4,g} \subset \mathcal{H}^{\mathrm{nf}}_{4,g}$ and of $\mathcal{H}^\circ_{4,g} \cap \mathcal{H}^{\mathrm{nf}}_{4,g} \subset \mathcal{H}^\circ_{4,g}$ are both at least $\frac{g+3}{4} - 4$. Therefore, by excision there is an isomorphism $A^i(\mathcal{H}^\circ_{4,g}) \cong A^i(\mathcal{H}^{\mathrm{nf}}_{4,g})$. In particular, we have $\dim A^i(\mathcal{H}^{\mathrm{nf}}_{4,g}) = \dim A^i(\mathcal{H}^\circ_{4,g}) = \dim R^i(\mathcal{H}_{4,g})$. The calculation of $\dim R^i(\mathcal{H}_{4,g})$ follows from Corollary 2.5.6. □

2.6 The Chow ring in degree 5

2.6.1 Set up

We begin by recalling the linear algebraic data associated to degree 5 covers, as developed by Casnati [Cas96]. For more details in our context, see [CL21d, Section 3.3]. To a degree 5, cover $\alpha : C \to \mathbb{P}^1$, we again associate two vector bundles on $\mathbb{P}^1$:

$$E_\alpha := (\alpha_*\mathcal{O}_C/\mathcal{O}_{\mathbb{P}^1})^\vee = \ker(\alpha_*\omega_\alpha \to \mathcal{O}_{\mathbb{P}^1}) \qquad \text{and} \qquad F_\alpha := \ker(\operatorname{Sym}^2 E_\alpha \to \alpha_*\omega_\alpha^{\otimes 2}).$$

If C has genus g, then E_α has degree $g+4$, and rank 4, while F_α has degree $2g+8$ and rank 5. Geometrically, the curve C is embedded in $\gamma : \mathbb{P}(E_\alpha^\vee \otimes \det E_\alpha) \to \mathbb{P}^1$, which further maps to $\mathbb{P}(\wedge^2 F_\alpha)$ via an associated section

$$\eta \in H^0(\mathbb{P}^1, \mathcal{H}om(E_\alpha^\vee \otimes \det E_\alpha, \wedge^2 F_\alpha)).$$

The curve C is obtained as the intersection of the image of $\mathbb{P}(E_\alpha^\vee \otimes \det E_\alpha)$ with the Grassmann bundle $G(2, F_\alpha) \subset \mathbb{P}(\wedge^2 F_\alpha)$.

Conversely, suppose we are given a rank 4, degree $g+4$ vector bundle E and a rank 5, degree $2g+8$ vector bundle F on $\mathbb{P}^1$. We write $E' := E^\vee \otimes \det E$ and $\gamma : \mathbb{P}E' \to \mathbb{P}^1$. We characterize which sections η fail to produce a smooth degree 5, genus g cover. Let

$$\Phi : H^0(\mathbb{P}^1, \mathcal{H}om(E^\vee \otimes \det E, \wedge^2 F) \xrightarrow{\sim} H^0(\mathbb{P}E', \gamma^* \wedge^2 F \otimes \mathcal{O}_{\mathbb{P}E'}(1)).$$

Lemma 2.6.1. *Let E and F be as above, with $\mathcal{H}om(E', \wedge^2 F)$ globally generated. Suppose we have a map $\eta : E' \to \wedge^2 F$.*

1. *If η is not injective on fibers then the subscheme $D(\Phi(\eta)) \subset \mathbb{P}E'$ cut by the 4×4 Pfaffians of $\Phi(\eta)$ is not smooth of dimension 1.*

2. *If $\eta : E' \to \wedge^2 F$ is injective on fibers, the intersection $C = \eta(\mathbb{P}E') \cap G(2,F)$ fails to be a smooth, irreducible genus g, degree 5 cover of $\mathbb{P}^1$ if and only if there exists $p \in C$ so that $\dim T_pC \geq 2$.*

Proof. (1) Suppose $\eta(e_1) = 0$ for e_1 a vector in the fiber of E' over $0 \in \mathbb{P}^1$, where $\mathbb{P}^1$ has coordinate t. We can choose coordinates X_1, X_2, X_3, X_4 on $\mathbb{P}E'$ so that $\mathrm{span}(e_1) \in \mathbb{P}E'|_0 \subset \mathbb{P}E'$ is defined by vanishing of t and X_2, X_3, X_4. Since $\eta(e_1)$ vanishes at $t = 0$, all entries of a matrix representative M_η for $\Phi(\eta)$ as in [CL21d, Equation 5.13] would have coefficient of X_1 divisible by t. In particular, the quadrics Q_i that define the Pfaffian locus $C = D(\Phi(\eta))$ of η lie in the ideal $(t) + (X_2, X_3, X_4)^2$. Hence, T_pC contains the entire vertical tangent space of $\mathbb{P}E' \to \mathbb{P}^1$, and therefore has dimension at least 3.

(2) If $\eta(\mathbb{P}E') \cap G(2,F) \subset \mathbb{P}(\wedge^2 F)$ is connected, or has a component of dimension ≥ 2, then we are done, so we suppose $\dim C = 1$. The general fiber of C over $\mathbb{P}^1$ consists of 5 points. The global generation of $\mathcal{H}om(E', \wedge^2 F)$ implies all summands of E have positive degree, so $h^0(\mathbb{P}^1, E^\vee) = 0$. Hence, if C has the right codimension in each fiber, then $h^0(C, \mathcal{O}_C) = h^0(\mathbb{P}^1, E^\vee) + 1 = 1$ so C is connected.

Now suppose that C has a component C_0 that is contained in a fiber. We claim C is connected (and thus has a two dimension tangent space at some point on C_0). Suppose that the fiber over $x \in \mathbb{P}^1$ is the union of a one dimensional component C_0 together with a finite scheme Γ. The image $\eta(\mathbb{P}E'|_x)$ is the intersection of six hyperplanes H_i in the fiber $\mathbb{P}(\wedge^2\mathcal{F})|_x \cong \mathbb{P}^9$. Thus the fiber of C over x is the intersection of six hyperplanes H_i and the Grassmannian $G(2, F|_x)$ in its Plücker embedding. Because the Plücker embedding is nondegenerate, we can arrange it so that $H_1 \cap \cdots \cap H_5 \cap G(2, F|_x)$ has pure dimension 1, i.e. the excess dimension appears only after intersecting with H_6; see [EH16, Section 13.3.6] for a similar argument due to Vogel.

To obtain the excess component C_0 in the final intersection, we must have that

$$H_1 \cap \cdots \cap H_5 \cap G(2, F|_x) = C_0 \cup \Phi$$

with $C_0 \subset H_6$. Note that the reducible curve $C_0 \cup \Phi$ must have degree $5 = \deg G(2, F|_x)$, so each component has degree at most 4. Therefore, the finite scheme $\Gamma = \Phi \cap H_6$ has degree at most 4. Because the general fiber of C over $\mathbb{P}^1$ consists of a degree five zero dimensional subscheme, it follows that some of the five points in the general fiber must specialize into C_0, and the intersection $\eta(\mathbb{P}E') \cap G(2, F)$ is singular there. $\square$

The association of $\alpha : C \to \mathbb{P}^1$ with the pair (E_α, F_α) gives rise to a map $\mathcal{H}_{5,g} \to \mathcal{B}_{5,g}$, where $\mathcal{B}_{5,g}$ is the moduli stack of pairs of vector bundles on $\mathbb{P}^1$-bundles, as defined in [CL21d, Definition 5.10]. Let $\pi : \mathcal{P} \to \mathcal{B}_{5,g}$ be the universal $\mathbb{P}^1$-bundle and let $\mathcal{E}$ be the universal rank 4 vector bundle on $\mathcal{P}$. Continuing the notation of [CL21d], let $z = c_1(\mathcal{O}_{\mathcal{P}}(1))$ and define classes $a_i, b_i \in A^i(\mathcal{B}_{5,g})$ and $a_i', b_i' \in A^{i-1}(\mathcal{B}_{5,g})$ by the formula

$$c_i(\mathcal{E}) = \pi^* a_i + \pi^* a_i' z \qquad \text{and} \qquad c_i(\mathcal{F}) = \pi^* b_i + \pi^* b_i' z.$$

(Note that there is a "determinant compatibility condition" which implies $2a_1 = b_1$, see [CL21d, p. 25].) We also define $c_2 = -\pi_*(z^3) \in A^2(\mathcal{B}_{5,g})$, which is the pullback of the universal second Chern class on BSL_2.

By [CL21d, Equation 5.10]

$$a_1, a_2, a_3, a_4, a_2', a_3', a_4', b_2, b_3, b_4, b_5, b_2', b_3', b_4', b_5', c_2 \text{ generate } A^*(\mathcal{B}_{5,g}) \text{ and satisfy} \quad (2.6.1)$$
$$\text{no relations in degrees up to } g+4.$$

We call the pullbacks of $\mathcal{E}$ and $\mathcal{F}$ to $\mathcal{H}_{5,g}$ the *CE bundles*, just like in the degree 4 case. Similarly, the pullbacks of the classes appearing in 2.6.1 to $\mathcal{H}_{5,g}$ are called the *CE classes*. By [CL21d, Theorem 3.10], the CE classes are tautological and generate the tautological ring.

In order to prove Theorem 2.1.1(3), we proceed in two steps, just like we did in degree 4. First, in Section 2.6.2, we construct a certain bundle of principal parts, and it to find relations among the CE classes. In Section 2.6.3, we define an open substack $\mathcal{H}^\circ_{5,g} \subset \mathcal{H}_{5,g}$, which is an open substack of a vector bundle over $\mathcal{B}^\circ_{5,g} \subset \mathcal{B}_{5,g}$, and use it to demonstrate that we have found all relations in degrees up to roughly $g/5$. Just like in degree 4, the method is summarized by Figure 2.3, but this time there are no factoring covers, so one can ignore the top row.

2.6.2 The construction of the bundle of principal parts and relations

In this section, we will perform a construction that starts with the data $(\mathcal{P} \to B, \mathcal{E}, \mathcal{F}, \eta)$ associated to degree 5 covers and produces a vector bundle called $RQ^1_{\mathbb{P}\mathcal{E}'/B}(\mathcal{W}')$ whose sections help us detect when the associated subscheme $D(\Phi(\eta)) \subset \mathbb{P}\mathcal{E}'$ defined by the vanishing of Pfaffians fails to be smooth of relative dimension 1 over B. The formation of this bundle commutes with base change. We will use this construction to produce relations among CE classes in the Chow ring of $\mathcal{H}_{5,g}$.

Suppose we are given the data $(\mathcal{P} \to B, \mathcal{E}, \mathcal{F}, \eta)$ where $\mathcal{P} \to B$ is a $\mathbb{P}^1$-bundle, $\mathcal{E}$ is a rank 4 vector bundle on $\mathcal{P}$, $\mathcal{F}$ is a rank 5 vector bundle on $\mathcal{P}$, and $\eta \in H^0(\mathcal{P}, \mathcal{H}om(\mathcal{E}^\vee \otimes \det \mathcal{E}, \wedge^2 \mathcal{F}))$. Set $\mathcal{E}' = \mathcal{E}^\vee \otimes \det \mathcal{E}$. Furthermore, we will assume that $\eta : \mathcal{E}' \to \wedge^2 \mathcal{F}$ is injective with locally free cokernel. It thus induces an inclusion $\mathbb{P}\eta : \mathbb{P}\mathcal{E}' \to \mathbb{P}(\wedge^2 \mathcal{F})$.

To set up this construction, let $\mathcal{Y} := G(2, \mathcal{F}) \times_{\mathcal{P}} \mathbb{P}\mathcal{E}'$ and let $p_1 : \mathcal{Y} \to G(2, \mathcal{F})$ and $p_2 : \mathcal{Y} \to \mathbb{P}\mathcal{E}'$ be the projection maps, so we have the diagram below.

$$\begin{tikzcd}
\mathcal{X} \arrow[r, "p_1"] \arrow[d, hook] \arrow[dd, bend right=60, "p_2"'] & G(2,\mathcal{F}) \arrow[d, hook, "i"] \\
\mathbb{P}\mathcal{E}' \times_{\mathcal{P}} \mathbb{P}(\wedge^2\mathcal{F}) \arrow[r, "q_1"] \arrow[d, "q_2"'] & \mathbb{P}(\wedge^2\mathcal{F}) \arrow[d, "\epsilon"] \\
\mathbb{P}\mathcal{E}' \arrow[r, "\gamma"'] & \mathcal{P} \arrow[d, "\pi"] \\
& B
\end{tikzcd}$$

These spaces come equipped with tautological sequences, which we label as follows. On

$G(2, \mathcal{F})$, we have an exact sequence

$$0 \to \mathcal{T} \to i^*\epsilon^*\mathcal{F} \to \mathcal{R} \to 0,$$

where $\mathcal{T}$ is rank 2 and $\mathcal{R}$ is rank 3. Meanwhile, on $\mathbb{P}(\wedge^2\mathcal{F})$, we have an exact sequence

$$0 \to \mathcal{O}_{\mathbb{P}(\wedge^2\mathcal{F})}(-1) \to \epsilon^*(\wedge^2\mathcal{F}) \to \mathcal{U}_9 \to 0 \tag{2.6.2}$$

where $\mathcal{U}_9$ is the tautological rank 9 quotient bundle. Noting that the Plücker embedding satisfies $i^*\mathcal{O}_{\mathbb{P}(\wedge^2\mathcal{F})}(-1) = \wedge^2\mathcal{T}$, the restriction of (2.6.2) to $G(2, \mathcal{F})$ takes the form

$$0 \to \wedge^2\mathcal{T} \to i^*\epsilon^*(\wedge^2\mathcal{F}) \to i^*\mathcal{U}_9 \to 0. \tag{2.6.3}$$

It follows that the map $i^*\epsilon^*(\wedge^2\mathcal{F}) \to \wedge^2\mathcal{R}$ descends to a map

$$i^*\mathcal{U}_9 \to \wedge^2\mathcal{R}. \tag{2.6.4}$$

Remark 2.6.2. The tensor product of (2.6.4) with $i^*\mathcal{O}_{\mathbb{P}(\wedge^2\mathcal{F})}(1)$ is the natural map from the restriction of the tangent bundle to the normal bundle, $i^*T_{\mathbb{P}(\wedge^2\mathcal{F})} \to N_{G(2,\mathcal{F})/\mathbb{P}(\wedge^2\mathcal{F})}$.

We define

$$\mathcal{W} := \mathcal{H}om(\mathcal{O}_{\mathbb{P}\mathcal{E}'}(-1), \gamma^*(\wedge^2\mathcal{F})) = \mathcal{O}_{\mathbb{P}\mathcal{E}^\vee}(1) \otimes \gamma^*(\wedge^2\mathcal{F}) \otimes \det \mathcal{E},$$

which is a rank 10 vector bundle on $\mathbb{P}\mathcal{E}'$. The composition

$$\mathcal{O}_{\mathbb{P}\mathcal{E}'}(-1) \to \gamma^*\mathcal{E}' \xrightarrow{\gamma^*\eta} \gamma^*(\wedge^2\mathcal{F})$$

defines a section δ of $\mathcal{W}$. Pulling back to $\mathbb{P}\mathcal{E}' \times_{\mathcal{P}} \mathbb{P}(\wedge^2\mathcal{F})$, consider the further composition

$$q_2^*\mathcal{O}_{\mathbb{P}\mathcal{E}'}(-1) \to q_2^*\gamma^*\mathcal{E}' \xrightarrow{q_1^*\epsilon^*\eta} q_1^*\epsilon^*(\wedge^2\mathcal{F}) \to q_1^*\mathcal{U}_9. \tag{2.6.5}$$

The vanishing locus of this composition is precisely the graph of $\mathbb{P}\eta$ inside $\mathbb{P}\mathcal{E}' \times_{\mathcal{P}} \mathbb{P}(\wedge^2\mathcal{F})$. Restricting (2.6.5) to $\mathcal{Y}$, we obtain a section, which we call $\overline{\delta}$, of the rank 9 vector bundle

$$\mathcal{W}' := \mathcal{H}om(p_2^*\mathcal{O}_{\mathbb{P}\mathcal{E}'}(-1), p_1^*i^*\mathcal{U}_9).$$

The vanishing $V(\overline{\delta}) \subset \mathcal{Y}$ is the intersection of the graph of $\mathbb{P}\eta$ with $\mathcal{Y}$ and is therefore identified with the intersection $G(2,\mathcal{F}) \cap \mathbb{P}\eta(\mathbb{P}\mathcal{E}')$. Viewed inside $\mathbb{P}\mathcal{E}'$, this intersection is equal to the desired associated subscheme $D(\Phi(\eta)) \subset \mathbb{P}\mathcal{E}'$.

Remark 2.6.3. The subscheme $D(\Phi(\eta)) \subseteq \mathbb{P}\mathcal{E}'$ is not in general the zero locus of a section of a vector bundle. However, we have found how to realize this scheme as the zero locus of a section of a vector bundle on $\mathcal{Y}$, basically by using the fact that the graph of $\mathbb{P}\eta$ is defined by the zero locus of a section of a vector bundle.

Next, we are going to construct a certain restricted principal parts bundle from $\mathcal{W}'$ that will detect when fibers of $\mathcal{C} = V(\overline{\delta}) \to B$ have vertical tangent space of dimension 2 or more. Before giving the construction, let us describe the geometric picture on a single fiber $\mathbb{P}^1$ of $\mathcal{P} \to B$. Let E and F be vector bundles on $\mathbb{P}^1$ of ranks 4 and 5 respectively and suppose $\eta : E' \to \wedge^2 F$ is an injection of vector bundles with locally free cokernel. Let $p \in \mathbb{P}E'$. The intersection $G(2,F) \cap \eta(\mathbb{P}E')$ has a two dimensional tangent space at $\eta(p) \in G(2,F)$ if and only if there exists a two dimensional subspace $S \subset T_p\mathbb{P}E'$ such that the differential of the projectivization of η sends S into $T_qG(2,F) \subset T_q\mathbb{P}(\wedge^2 F)$. Equivalently, the composition $S \subset T_p\mathbb{P}E' \xrightarrow{\mathrm{d}\mathbb{P}\eta} T_{\eta(p)}\mathbb{P}(\wedge^2F) \to N_{G(2,F)/\mathbb{P}(\wedge^2F)}|_{\eta(p)}$ is zero (see Figure 2.4).

First consider $Q^1_{\mathbb{P}\mathcal{E}'/B}(\mathcal{W}')$ (see Definition 2.3.9), which comes equipped with a

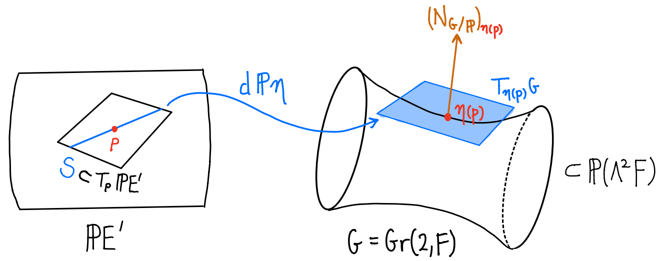

Figure 2.4. Does $d\mathbb{P}\eta$ send S into $T_{\eta(p)}G$?

filtration

$$0 \to p_2^*\Omega_{\mathbb{P}\mathcal{E}'/B} \otimes \mathcal{W}' \to Q^1_{\mathbb{P}\mathcal{E}'/B}(\mathcal{W}') \to \mathcal{W}' \to 0. \tag{2.6.6}$$

Given any section δ of $\mathcal{W}$, there is an induced section of $Q^1_{\mathbb{P}\mathcal{E}'/B}(\mathcal{W}')$, which records the values and first order changes of the induced section $\overline{\delta}$ of $\mathcal{W}'$ as we move across $\mathbb{P}\mathcal{E}'$. Now let $\widetilde{\mathcal{X}} := G(2, p_2^*T_{\mathbb{P}\mathcal{E}'/B}) \xrightarrow{a} \mathcal{X}$, which comes equipped with a tautological sequence

$$0 \to \Omega_x^\vee \to a^*p_2^*T_{\mathbb{P}\mathcal{E}'/B} \to \Omega_y^\vee \to 0,$$

where Ω_x and Ω_y are both rank 2. Dualizing the left map gives

$$a^*p_2^*\Omega_{\mathbb{P}\mathcal{E}'/B} \to \Omega_x. \tag{2.6.7}$$

Meanwhile, tensoring the p_1^* of (2.6.4) with $p_2^*\mathcal{O}_{\mathbb{P}\mathcal{E}'}(1)$, we have a quotient

$$\mathcal{W}' \to p_2^*\mathcal{O}_{\mathbb{P}\mathcal{E}'}(1) \otimes p_1^*(\wedge^2\mathcal{R}). \tag{2.6.8}$$

Remark 2.6.4. If one has an injection $\eta : \mathcal{E}' \to \wedge^2\mathcal{F}$, then one has an isomorphism of $p_2^*\mathcal{O}_{\mathbb{P}\mathcal{E}'}(1)$ with $p_1^*i^*\mathcal{O}_{\mathbb{P}(\wedge^2\mathcal{F})}(1)$ on $V(\overline{\delta})$ (coming from (2.6.5)). By Remark 2.6.2, the restriction of (2.6.8) to $V(\overline{\delta})$ then agrees with the restriction of $p_1^*i^*T_{\mathbb{P}(\wedge^2\mathcal{F})} \to p_1^*N_{G(2,\mathcal{F})/\mathbb{P}(\wedge^2\mathcal{F})}$ to $V(\overline{\delta})$. This was the geometric intuition behind the definition we are about to make.

Pulling back (2.6.8) to $\widetilde{\mathcal{X}}$ and tensoring with (2.6.7), we obtain a quotient

$$a^*(p_2^*\Omega_{\mathbb{P}\mathcal{E}'/B} \otimes \mathcal{W}') \to \Omega_x \otimes a^*(p_2^*\mathcal{O}_{\mathbb{P}\mathcal{E}'}(1) \otimes p_1^*(\wedge^2\mathcal{R})). \tag{2.6.9}$$

Note that the term on the left of (2.6.9) is the a^* of the term on the left of (2.6.6) (the "derivatives part" of the principal parts bundle). Let $RQ^1_{\mathbb{P}\mathcal{E}'/B}(\mathcal{W}')$ be the quotient of $a^*Q^1_{\mathbb{P}\mathcal{E}'/B}(\mathcal{W}')$ by the kernel of (2.6.9). This bundle comes equipped with a filtration

$$0 \to \Omega_x \otimes a^*(p_2^*\mathcal{O}_{\mathbb{P}\mathcal{E}'}(1) \otimes p_1^*(\wedge^2\mathcal{R})) \to RQ^1_{\mathbb{P}\mathcal{E}'/B}(\mathcal{W}') \to \mathcal{W}' \to 0 \tag{2.6.10}$$

and has rank 15. The bundle $RQ^1_{\mathbb{P}\mathcal{E}'/B}(\mathcal{W}')$ remembers derivatives just in the "x-directions" (i.e. along a distinguished 2-plane) and remembers their values under the quotient (2.6.8). Considering Remark 2.6.4 and Figure 2.4, this is telling us to what extent vectors in the subspace S corresponding to "x-directions" leave $T_{\eta(p)}G(2,F)$. This will be spelled out in local coordinates in the lemma below.

The global section $\overline{\delta}$ of $\mathcal{W}'$ induces a global section $\overline{\delta}'$ of $Q^1_{\mathbb{P}\mathcal{E}'/B}(\mathcal{W}')$, which in turn gives rise to a global section $\overline{\delta}''$ of $RQ^1_{\mathbb{P}\mathcal{E}'/B}(\mathcal{W}')$. The following lemma describes the geometric condition for such an induced section to vanish at a geometric point of $\widetilde{\mathcal{X}}$.

Lemma 2.6.5. *Let E and F be vector bundles on $\mathbb{P}^1$ of ranks 4 and 5 respectively. Let $Y = \mathbb{P}E' \times_{\mathbb{P}^1} G(2,F)$ and let W, W', R, $Q^1_{\mathbb{P}E'}(W')$ and $RQ^1_{\mathbb{P}E'}(W')$ be defined analogously to the constructions above (working over a point instead of B). Suppose $\eta : E' \to \wedge^2 F$ is an injection of vector bundles. Then the following are true:*

1. *The induced section $\overline{\delta}$ of W' corresponding to η vanishes at $(p,q) \in Y$ if and only if the projectivization of η sends p to q.*

2. *The induced section $\overline{\delta}''$ of $RQ^1_{\mathbb{P}E'}(W')$ corresponding to η vanishes at $(p,q,S) \in \widetilde{Y}$ if and only if the differential of the projectivization of η sends the subspace $S \subset T_p\mathbb{P}E'$ into the subspace $T_qG(2,F) \subset T_q\mathbb{P}(\wedge^2 F)$.*

Hence, given any family $(\mathcal{P} \to B, \mathcal{E}, \mathcal{F}, \eta)$, the image of the vanishing of the induced section $\overline{\delta}''$ of $RQ^1_{\mathbb{P}\mathcal{E}'}(\mathcal{W}')$ is the locus in B over which fibers of $D(\Phi(\eta)) \to B$ fail to be smooth of relative dimension 1.

Proof. (1) Let t be a coordinate on $\mathbb{P}^1$, and let $p \in \mathbb{P}E'$ and $q \in G(2,F)$ be points in the fiber over $0 \in \mathbb{P}^1$. To say η sends p to q is to say that η sends the subspace of $E'|_0$ corresponding to p into the subspace of $\wedge^2 F|_0$ corresponding to q. Hence, by the definition of the tautological sequences, η sends p to q if and only if the composition

$$p_2^*\mathcal{O}_{\mathbb{P}E'}(-1) \to p_2^*\gamma^*E' \to p_1^*i^*\epsilon^*(\wedge^2 F) \to p_1^*i^*U_9$$

vanishes at (p,q), which is to say $\overline{\delta}$ vanishes.

(2) Trivializing E and F over an open $0 \in U \subset \mathbb{P}^1$, we may choose a basis $e_1, \ldots, e_4$ for E so that $p = \operatorname{span}(e_1)$ and a basis $f_1, \ldots, f_5$ for F so that $q = \operatorname{span}(f_1 \wedge f_2)$. Let $\eta_{k,ij}$ be the coefficient of $f_i \wedge f_j$ in $\eta(e_k)$, so $\eta_{k,ij}$ is a polynomial in t. In these local coordinates, to say η sends p to q is to say that $\eta_{1,ij}|_{t=0} = 0$ for $ij \neq 12$.

The map $p_1^* \wedge^2 F \to \wedge^2 R$ corresponds to projection onto the span of $f_3 \wedge f_4, f_3 \wedge f_5$, and $f_4 \wedge f_5$. If η sends p to q, then the induced section $\overline{\delta}$ of W' already vanishes. Therefore, the value of $\overline{\delta}''$ at (p,q) lands in the subbundle $p_2^*\Omega_{\mathbb{P}E'/B} \otimes p_2^*\mathcal{O}_{\mathbb{P}E'}(1) \otimes p_1^*(\wedge^2 R) \subset Q^1_{\mathbb{P}E'}(W')$. This "value" of $\overline{\delta}''$ at (p,q) records the first order information of $\eta_{1,ij}$ for $ij = 34, 35, 45$ as p deforms.

First order deformations of p are of the form $\operatorname{span}(e_1) \mapsto \operatorname{span}(e_1 + \epsilon(ae_2 + be_3 + ce_4))|_{t=\epsilon d}$, where $\epsilon^2 = 0$. Here, a, b, c, d are coordinates on the tangent space at p (a, b, c are vertical coordinates and d is the horizontal coordinate). The coefficient of $f_i \wedge f_j$ in $\eta(e_1 + \epsilon(ae_2 + be_3 + ce_4))|_{t=\epsilon d}$ is

$$\eta_{1,ij} + \left(d \left(\frac{d}{dt}\eta_{1,ij} \right)\bigg|_{t=0} + a\eta_{2,ij}|_{t=0} + b\eta_{3,ij}|_{t=0} + c\eta_{4,ij}|_{t=0} \right)\epsilon \qquad \text{for } ij = 34, 35, 45. \tag{2.6.11}$$

Locally, $a\epsilon, b\epsilon, c\epsilon, d\epsilon$ are our basis for $\Omega_{\mathbb{P}E}$ and $f_i \wedge f_j$ for $ij = 34, 35, 45$ is our basis for $\wedge^2 R$. The "value" we wish to extract in the fiber of $p_2^*\Omega_{\mathbb{P}E'/B} \otimes p_2^*\mathcal{O}_{\mathbb{P}E'}(1) \otimes p_1^*(\wedge^2 R)$ over (p, q) is the coefficients of $a\epsilon, b\epsilon, c\epsilon$, and $d\epsilon$ in (2.6.11) for $ij = 34, 35, 45$.

Now suppose η is injective on fibers, so $\mathbb{P}\eta$ is well-defined. In particular, $\eta_{1,12}|_{t=0} \neq 0$. With respect to a, b, c, d the differential of $\mathbb{P}\eta$, from $T_p\mathbb{P}E' \to T_q\mathbb{P}(\wedge^2 F)$, is represented by a 9×4 matrix

$$\frac{1}{\eta_{1,12}|_{t=0}} \begin{pmatrix} \frac{d}{dt}\eta_{1,13}|_{t=0} & \eta_{2,13}|_{t=0} & \eta_{3,13}|_{t=0} & \eta_{4,13}|_{t=0} \\ \frac{d}{dt}\eta_{1,14}|_{t=0} & \eta_{2,14}|_{t=0} & \eta_{3,14}|_{t=0} & \eta_{4,14}|_{t=0} \\ \vdots & & & \vdots \\ \frac{d}{dt}\eta_{1,45}|_{t=0} & \eta_{2,45}|_{t=0} & \eta_{3,45}|_{t=0} & \eta_{4,45}|_{t=0} \end{pmatrix}. \tag{2.6.12}$$

The subspace $T_qG(2, F) \subset T_q\mathbb{P}(\wedge^2 F)$ corresponds to the first 6 coordinates. (A first order deformation of $f_1 \wedge f_2$ remains a pure wedge to first order if and only if the $f_i \wedge f_j$ with non-zero coefficient in the deformation have one of i, j is equal to 1 or 2. See also Remark 2.6.4.) Thus, $\mathbb{P}\eta$ sends $T_p\mathbb{P}E'$ into $T_qG(2, F)$ if and only if the bottom three rows of (2.6.12) vanish, which occurs if and only if the coefficients of a, b, c, d in (2.6.11) vanish. More generally, a tangent vector in $T_p\mathbb{P}E'$ is sent into $T_qG(2, F)$ if and only if (2.6.11) vanishes (for $ij = 34, 35, 45$) when the corresponding values of a, b, c, d are plugged in. Plugging in values for a, b, c, d in a given two dimensional subspace S of $T_p\mathbb{P}E'$ then corresponds to the "value" of η in $S^\vee \otimes p_2^*\mathcal{O}_{\mathbb{P}E'}(1) \otimes p_1^*(\wedge^2 R)$ over (p, q). By the filtration (2.6.10), this "value" is zero if and only if $\overline{\delta}''$ vanishes at $(p, q, S) \in \widetilde{X}$.

Since the formation of these (refined) principal parts bundles commutes with base change, the claim regarding families follows. $\square$

We now apply the above construction in the case $B = \mathcal{H}_{5,g}$ and $\eta = \eta^{\text{univ}}$, the section associated to the universal cover $\mathcal{C} \to \mathcal{P}$. By Lemma 2.6.5 and the fact that the universal curve $\mathcal{C} = V(\overline{\delta}'')$ is smooth of relative dimension 1 over $\mathcal{H}_{5,g}$, the global section

$\overline{\delta}''$ of $RQ^1_{\mathbb{P}\mathcal{E}'/\mathcal{H}_{5,g}}(\mathcal{W}')$ is nowhere vanishing. We therefore have the following lemma, which gives a source of relations among the CE classes on $\mathcal{H}_{5,g}$.

Lemma 2.6.6. *Let $z = c_1(\mathcal{O}_{\mathcal{P}}(1)), \zeta = c_1(\mathcal{O}_{\mathbb{P}\mathcal{E}'}(1)), \sigma_i = c_i(\mathcal{R})$, and $s_i = c_i(\Omega_y^\vee)$. All classes of the form (some pullbacks omitted for ease of notation):*

$$a_* p_{2*} \gamma_* \pi_*(c_{15}(RQ^1_{\mathbb{P}\mathcal{E}'/\mathcal{H}_{5,g}}(\mathcal{W}')) \cdot s_1^{l_1} s_2^{l_2} \sigma_1^{k_1} \sigma_2^{k_2} \sigma_3^{k_3} \zeta^j z^i)$$

are zero in $R^*(\mathcal{H}_{5,g}) \subseteq A^*(\mathcal{H}_{5,g})$.

2.6.3 All relations in low codimension

We recall the construction of an open substack $\mathcal{H}^\circ_{5,g} \subset \mathcal{H}_{5,g}$ and what we already know about its Chow ring from [CL21d]. We start with $\mathcal{B}_{5,g}$, the moduli space of pairs of vector bundles E of rank 4, degree $g+4$ and F of rank 5, degree $g+5$ on $\mathbb{P}^1$-bundles together with an isomorphism of $\det E^{\otimes 2}$ and $\det F$ (see [CL21d, Section 5.3]). Let $\mathcal{E}$ and $\mathcal{F}$ be the universal bundles on $\pi : \mathcal{P} \to \mathcal{B}_{5,g}$ and let $\gamma : \mathbb{P}\mathcal{E}^\vee \to \mathcal{P}$ be the structure map. Define $\mathcal{U}_{5,g} := \mathcal{H}om(\mathcal{E}^\vee \otimes \det \mathcal{E}, \wedge^2 \mathcal{F})$. We consider an open substack $\mathcal{B}^\circ_{5,g} \subset \mathcal{B}_{5,g}$, defined by a certain positivity condition for the bundle $\mathcal{U}_{5,g}$

$$\mathcal{B}^\circ_{5,g} := \mathcal{B}_{5,g} \smallsetminus \operatorname{Supp} R^1\pi_*(\mathcal{U}_{5,g}(-2)). \tag{2.6.13}$$

Let $\mathcal{H}^\circ_{5,g}$ denote the base change of $\mathcal{H}_{5,g} \to \mathcal{B}_{5,g}$ along the open embedding $\mathcal{B}^\circ_{5,g} \hookrightarrow \mathcal{B}_{5,g}$.

Over $\mathcal{B}^\circ_{5,g}$, we see that $\mathcal{X}^\circ_{5,g} := \pi_* \mathcal{U}_{5,g}|_{\mathcal{B}^\circ_{5,g}}$ is a vector bundle whose fibers correspond to sections of $\mathcal{U}_{5,g}$. The open $\mathcal{H}^\circ_{5,g}$ is contained in the open $\mathcal{H}'_{5,g}$ of [CL21d, Lemma 5.11], so that lemma implies $\mathcal{H}^\circ_{5,g} \to \mathcal{B}^\circ_{5,g}$ factors through an open embedding in $\mathcal{X}^\circ_{5,g}$. We define

$$\Delta_{5,g} := \mathcal{X}^\circ_{5,g} \smallsetminus \mathcal{H}^\circ_{5,g},$$

represented in red in the middle column of Figure 2.3. Now we wish to use excision to determine the Chow ring of $\mathcal{H}^\circ_{5,g}$ in degrees up to $\frac{g+4}{5} - 16$. We already understand $A^*(\mathcal{X}^\circ_{5,g}) \cong A^*(\mathcal{B}^\circ_{5,g})$ in degrees up to $\frac{g+4}{5} - 16$ by [CL21d, Equation 5.11]. Lemma 2.6.1 says we need to remove the locus of non-injective maps and the locus of injective maps such that the induced intersection of $\mathbb{P}\mathcal{E}'$ and $G(2, \mathcal{F})$ has a singular point.

We begin by computing the relations obtained from removing the locus of non-injective maps $\mathcal{E}' \to \wedge^2 \mathcal{F}$, i.e. maps that drop rank along some point on $\mathcal{P}$. Consider the projective bundle $\gamma : \mathbb{P}\mathcal{E}' \to \mathcal{P} \to \mathcal{B}^\circ_{5,g}$, and let $\mathcal{W} := \mathcal{O}_{\mathbb{P}\mathcal{E}'}(1) \otimes \gamma^*(\wedge^2 \mathcal{F})$. We have that $\gamma_* \mathcal{W} = \mathcal{H}om(\mathcal{E}', \wedge^2 \mathcal{F}) = \mathcal{U}_{5,g}$, so by the definition of $\mathcal{B}^\circ_{5,g}$ (see (2.6.13)) and Lemma 2.3.5, the map

$$\gamma^* \pi^* \mathcal{X}^\circ_{5,g} = \gamma^* \pi^* \pi_* \gamma_* \mathcal{W} \to P^1_{\mathbb{P}\mathcal{E}'/\mathcal{B}^\circ_{5,g}}(\mathcal{W}) \qquad \text{is surjective.} \tag{2.6.14}$$

Composing with the surjection $P^1_{\mathbb{P}\mathcal{E}'/\mathcal{B}^\circ_{5,g}}(\mathcal{W}) \to \mathcal{W}$, we obtain a surjection $\gamma^* \pi^* \mathcal{X}^\circ_{5,g} \to \mathcal{W}$, whose kernel we define to be $\widetilde{\mathcal{X}}^{\text{ni}}$. The fiber of $\widetilde{\mathcal{X}}^{\text{ni}}$ at a point $p \in \mathbb{P}\mathcal{E}'$ corresponds to maps of $\mathcal{E}' \to \wedge^2 \mathcal{F}$ (on the fiber over $\pi(\gamma(p))$) whose kernel contains the subspace referred to by p.

We then have the following trapezoid diagram:

$$\begin{tikzcd}
\widetilde{\mathcal{X}}^{\text{ni}} \arrow[r, hook] \arrow[dr] & \gamma^*\pi^*\mathcal{X}^\circ_{5,g} \arrow[r] \arrow[d] & \pi^*\mathcal{X}^\circ_{5,g} \arrow[r] \arrow[d] & \mathcal{X}^\circ_{5,g} \arrow[d] \\
& \mathbb{P}\mathcal{E}' \arrow[r, "\gamma"'] & \mathcal{P} \arrow[r, "\pi"'] & \mathcal{B}^\circ_{5,g}
\end{tikzcd}$$

Thus, Lemma 2.2.1 yields:

Proposition 2.6.7. *The image of the pushforward map $A_*(\widetilde{\mathcal{X}}^{\text{ni}}) \to A_*(\mathcal{X}_{5,g})$ is equal to the ideal generated by*

$$\rho^* \pi_* \gamma_* (c_{10}(\mathcal{W})) \cdot \zeta^j z^i), \qquad 0 \leq j \leq 3, \ 0 \leq i \leq 1. \tag{2.6.15}$$

where $\zeta = c_1(\mathcal{O}_{\mathbb{P}\mathcal{E}'}(1))$ and $z = c_1(\mathcal{O}_{\mathcal{P}}(1))$.

Next, we excise the locus of injective maps such that the induced intersection of $\mathbb{P}\mathcal{E}'$ and $G(2, \mathcal{F})$ has a singular point. From the construction in Section 2.6.2 applied to the case $B = \mathcal{B}_{5,g}^{\circ}$, we have a rank 15 vector bundle $RQ^1_{\mathbb{P}\mathcal{E}'/\mathcal{B}_{5,g}^{\circ}}(\mathcal{W}')$ on $\widetilde{\mathcal{X}}$, which comes equipped with a series of surjections (see Lemma 2.3.10 for the first map; the second map comes from the construction of $RQ^1_{\mathbb{P}\mathcal{E}'/\mathcal{B}_{5,g}^{\circ}}(\mathcal{W}')$, which was made just after (2.6.9)):

$$a^*p_1^*P^1_{\mathbb{P}\mathcal{E}'/\mathcal{B}_{5,g}^{\circ}}(\mathcal{W}) \to a^*Q^1_{\mathbb{P}\mathcal{E}'/\mathcal{B}_{5,g}^{\circ}}(\mathcal{W}') \to RQ^1_{\mathbb{P}\mathcal{E}'/\mathcal{B}_{5,g}^{\circ}}(\mathcal{W}'). \tag{2.6.16}$$

Applying $a^*p_2^*$ to (2.6.14) and composing the result with (2.6.16), we obtain a surjection

$$a^*p_2^*\gamma^*\pi^*\mathcal{X}_{5,g}^{\circ} \to RQ^1_{\mathbb{P}\mathcal{E}'/\mathcal{B}_{5,g}^{\circ}}(\mathcal{W}'). \tag{2.6.17}$$

Define $\widetilde{\Delta}_{5,g}$ to be the kernel of (2.6.17), so that we obtain a trapezoid diagram:

$$\begin{tikzcd}
\widetilde{\Delta}_{5,g} \arrow[r, hook, "i"] \arrow[dr, "\rho''"'] & \sigma^*p_2^*\gamma^*\pi^*\mathcal{X}_{5,g}^{\circ} \arrow[r] \arrow[d, "\rho'"] & p_2^*\gamma^*\pi^*\mathcal{X}_{5,g}^{\circ} \arrow[r] \arrow[d] & \gamma^*\pi^*\mathcal{X}_{5,g}^{\circ} \arrow[r] \arrow[d] & \pi^*\mathcal{X}_{5,g}^{\circ} \arrow[r] \arrow[d] & \mathcal{X}_{5,g}^{\circ} \arrow[d, "\rho"] \\
& \widetilde{\mathcal{X}} \arrow[r, "a"'] & \mathcal{X} \arrow[r, "p_2"'] & \mathbb{P}\mathcal{E}' \arrow[r, "\gamma"'] & \mathcal{P} \arrow[r, "\pi"'] & \mathcal{B}_{5,g}^{\circ}
\end{tikzcd}$$

Lemma 2.6.8. *Let $z = c_1(\mathcal{O}_{\mathcal{P}}(1)), \zeta = c_1(\mathcal{O}_{\mathbb{P}\mathcal{E}'}(1)), \sigma_i = c_i(\mathcal{R})$, and $s_i = c_i(\Omega_y^{\vee})$. The image of the push forward $A_*(\widetilde{\Delta}_{5,g}) \to A_*(\mathcal{X}_{5,g}^{\circ})$ is the ideal generated by*

$$\rho^*a_*p_{2*}\gamma_*\pi_*(c_{15}(RQ^1_{\mathbb{P}\mathcal{E}'/\mathcal{B}_{5,g}^{\circ}}(\mathcal{W}')) \cdot s_1^{l_1}s_2^{l_2}\sigma_1^{k_1}\sigma_2^{k_2}\sigma_3^{k_3}\zeta^j z^i) \tag{2.6.18}$$

for $0 \leq j \leq 3$, $0 \leq i \leq 1$, $0 \leq l_1, l_2 \leq 2$ with $l_1 + l_2 \leq 2$, and $0 \leq k_1, k_2, k_3 \leq 2$ with $k_1 + k_2 + k_3 \leq 2$.

Proof. The monomials $s_1^{l_1}s_2^{l_2}\sigma_1^{k_1}\sigma_2^{k_2}\sigma_3^{k_3}\zeta^j z^i$ with exponents satisfying the inequalities in the statement of the lemma generate $A^*(\widetilde{\mathcal{X}})$ as an $A^*(\mathcal{B}_{5,g}^{\circ})$ module (see the last paragraph of Section 2.2.1). The result now follows from the Trapezoid Lemma 2.2.1. In codimension 1, for example, since $\widetilde{\Delta}_{5,g} \to \Delta_{5,g}$ is generically one-to-one, we see

$$[\Delta_{5,g}] = \rho^*(\pi \circ \gamma \circ p_2 \circ a)_*(c_{15}(RQ^1_{\mathbb{P}\mathcal{E}'/\mathcal{B}^\circ_{5,g}}(\mathcal{W}'))) = (10g + 36)a_1 - 7a'_2 - b'_2. \qquad (2.6.19)$$

$\square$

Lemma 2.6.9. *Let I be the ideal generated by the classes in* (2.6.15) *and* (2.6.18)*. Then $A^*(\mathcal{H}^\circ_{5,g}) = A^*(\mathcal{B}^\circ_{5,g})/I$. In fact, I is generated by the classes in* (2.6.18)*, so Lemma 2.6.6 determines all relations among CE classes in codimension up to $\frac{g+4}{5} - 16$.*

Proof. By Lemmas 2.6.5 and 2.6.1, we have that $\Delta_{5,g}$ is the union of the image of $\widetilde{\Delta}_{5,g}$ in $\mathcal{X}^\circ_{5,g}$ with the image of $\widetilde{\mathcal{X}}^{\mathrm{ni}}$ in $\mathcal{X}^\circ_{5,g}$. The first claim now follows from excision, the fact that push forward is surjective with rational coefficients, and Lemmas 2.6.7 and 2.6.8.

Meanwhile, direct computation [CL21c] shows that I is generated by the classes in (2.6.18). Since ρ is flat, the classes in (2.6.18) equal the classes of Lemma 2.6.6. Next, [CL21d, Equation 5.11] says that our generators on $\mathcal{B}^\circ_{5,g}$ satisfy no relations in codimension less than $\frac{g+4}{5} - 16$. Thus, we have determined all relations among CE classes in codimension up to $\frac{g+4}{5} - 16$ $\square$

2.6.4 Presentation of the ring and stabilization

Modulo the relations in Lemma 2.6.9, it turns out $R^*(\mathcal{H}_{5,g})$ is generated by $a_1, a'_2 \in R^1(\mathcal{H}_{5,g})$ and $a_2, c_2 \in R^2(\mathcal{H}_{5,g})$, as we now explain. Let I be the ideal generated by the classes in (2.6.15) and (2.6.18) in the $\mathbb{Q}$-algebra on the CE classes. Using Macaulay, we determined a simplified presentation

$$\mathbb{Q}[c_2, a_1, \ldots, a_4, a'_2, \ldots, a'_4, b_2, \ldots, b_5, b'_2, \ldots, b'_5]/I \cong \mathbb{Q}[a_1, a'_2, a_2, c_2]/\langle r_1, r_2, r_3, r_4, r_5 \rangle, \qquad (2.6.20)$$

where

$$\begin{aligned}
r_1 &= (1064g+3610)a_1^3 - 1074a_1^2a_2' + (-2148g-7272)a_1a_2 + 2160a_2a_2' + \\
&\quad + (-1064g^3 - 10830g^2 - 36680g - 41360)a_1c_2 + (1074g^2 + 7272g + 12288)a_2'c_2 \\
r_2 &= (-6412g-21255)a_1^3 + 6207a_1^2a_2' + (12414g+40896)a_1a_2 + (-11880)a_2a_2' + \\
&\quad + (6412g^3 + 63765g^2 + 211540g + 234480)a_1c_2 + (-6207g^2 - 40896g - 68184)a_2'c_2 \\
r_3 &= (-22845g-67763)a_1^4 + 18141a_1^3a_2' + (54423g+146550)a_1^2a_2 - 35640a_1a_2a_2 \\
&\quad + (45690g^3 + 406578g^2 + 1184220g + 1123060)a_1^2c_2 \\
&\quad - (54423g^2 + 293100g + 372648)a_1a_2'c_2 + (17820g + 24840)a_2'^2c_2 \\
&\quad - (17820g+24840)a_2^2 - (18141g^3 + 146550g^2 + 372648g + 283824)a_2c_2 \\
&\quad - (4569g^5 + 67763g^4 + 394740g^3 + 1123060g^2 + 1546176g + 810432)c_2^2 \\
r_4 &= 133a_1^4 - 537a_1^2a_2 + (-798g^2 - 5415g - 9170)a_1^2c_2 + (1074g + 3636)a_1a_2'c_2 \\
&\quad - 540a_2'^2c_2 + 540a_2^2 + (537g^2 + 3636g + 6144)a_2c_2 \\
&\quad + (133g^4 + 1805g^3 + 9170g^2 + 20680g + 17472)c_2^2 \\
r_5 &= (-18545g-68407)a_1^4 + 15261a_1^3a_2' + (45783g+175866)a_1^2a_2 - 31320a_1a_2a_2' \\
&\quad + (37090g^3 + 410442g^2 + 1499460g + 1811300)a_1^2c_2 \\
&\quad + (-45783g^2 - 351732g - 662976)a_1a_2'c_2 + (15660g + 72360)a_2'^2c_2 \\
&\quad + (-15660g - 72360)a_2^2 + (-15261g^3 - 175866g^2 - 662976g - 822096)a_2c_2 \\
&\quad + (-3709g^5 - 68407g^4 - 499820g^3 - 1811300g^2 - 3260256g - 2334528)c_2^2.
\end{aligned}$$

As a corollary of the above presentation, we can use Macaulay2 to determine a spanning set for each group $R^i(\mathcal{H}_{5,g})$, which is actually a basis when g is sufficiently large relative to i. We will use these spanning sets in Section 2.7.3 to prove another collection of classes are additive generators.

Corollary 2.6.10. *Suppose $g \geq 2$.*

1. *$R^1(\mathcal{H}_{5,g})$ is spanned by $\{a_1, a_2'\}$.*

2. $R^2(\mathcal{H}_{5,g})$ *is spanned by* $\{a_1^2, a_1a_2', a_2, a_2'^2, c_2\}$.

3. $R^3(\mathcal{H}_{5,g})$ *is spanned by* $\{a_1^2a_2', a_1a_2'^2, a_1c_2, a_2a_2', a_2'c_2\}$.

4. $R^4(\mathcal{H}_{5,g})$ *is spanned by* $\{a_1^2c_2, a_1a_2'^3, a_1a_2'c_2, a_2c_2, a_2'^4, a_2'^2, a_2'^2c_2, c_2^2\}$.

5. $R^5(\mathcal{H}_{5,g})$ *is spanned by* $\{a_1a_2'^4, a_1c_2^2, a_2'^5, a_2'c_2^2\}$

6. $R^6(\mathcal{H}_{5,g})$ *is spanned by* $\{a_1a_2'^5, a_2'^6, c_2^3\}$

7. For $i \geq 7$, $R^7(\mathcal{H}_{5,g})$ *is spanned by* $\{a_1a_2'^{i-1}, a_2'^i\}$.

The above spanning set for $R^i(\mathcal{H}_{5,g})$ *is a basis when* $g > 5i + 76$.

Proof. Let S^i denote the degree i group of the graded ring $\mathbb{Q}[a_1, a_2', a_2, c_2]/\langle r_1, r_2, r_3, r_4, r_5\rangle$. By Proposition 2.6.9 and Equation (2.6.20), S^i surjects onto $R^i(\mathcal{H}_{5,g})$ and is an isomorphism in degrees $i < \frac{g+4}{5} - 16$, equivalently when $g > 5i + 76$.

Using Macaulay, we check that the set listed in the lemma is a basis of S^i for $i \leq 14$. For $7 \leq i \leq 14$, in particular, we see that $a_2'^i$ and $a_2'^{i-1}a_1$ form a basis for the group S^i. For $i \geq 15$, every monomial of degree i in a_1, a_2', a_2, c_2 is expressible as a product of two monomials, both of degree at least 7. Then the product of two such monomials is in the span of $a_2'^i, a_2'^{i-1}a_1$ and $a_2'^{i-2}a_1^2 = a_2'^{i-7}(a_2'^5a_1^2)$. The last monomial is already in the span of the first two because S^7 is spanned by $a_2'^7, a_2'^6a_1$. It follows that $a_2'^i$ and $a_2'^{i-1}a_1$ span S^i for all $i \geq 15$. Meanwhile, no monomial of the form $a_2'^i$ or $a_2'^{i-1}a_1$ appears in the relations $r_1, \ldots, r_5$. Hence, no combination of $a_2'^i$ and $a_2'^{i-1}a_1$ lies in $\langle r_1, \ldots, r_5\rangle$, so $a_1a_2'^{i-1}$ and $a_2'^i$ are independent for all i. $\square$

Proof of Theorem 2.1.1(3). Consider the equation

$$\frac{\mathbb{Q}[a_1, a_2', a_2, c_2]}{\langle r_1, r_2, r_3, r_4, r_5\rangle} \to R^*(\mathcal{H}_{5,g}) \to R^*(\mathcal{H}_{5,g}^\circ) = A^*(\mathcal{H}_{5,g}^\circ).$$

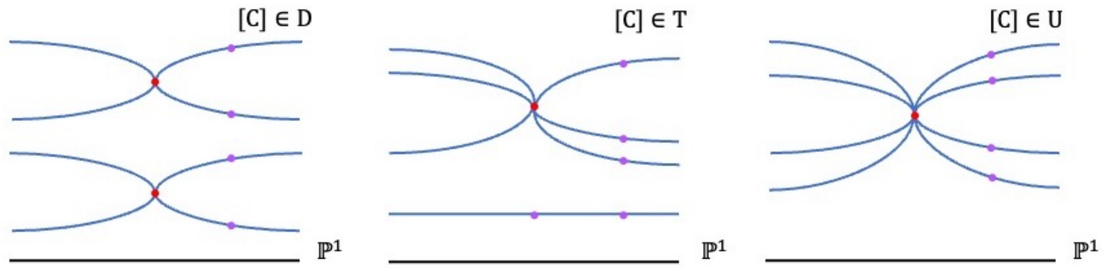

Figure 2.5. Covers in T, D, and U.

The first map exists and is surjective by Proposition 2.6.6. Meanwhile, Lemma 2.6.9 establishes that the composition is an isomorphism in degrees less than $\frac{g+4}{5} - 16$. Therefore, the first map can have no kernel in codimension less than $\frac{g+4}{5} - 16$. Finally, for $i < \frac{g+4}{5} - 16$, we have $A^i(\mathcal{H}_{5,g}) = R^i(\mathcal{H}_{5,g})$ by [CL21d, Theorem 1.4]. The dimension of $R^i(\mathcal{H}_{5,g})$ follows from Corollary 2.6.10. □

2.7 Applications to the moduli space of curves and a generalized Picard rank conjecture

In this section, we express the Chow rings we have computed in terms of some natural classes associated to the Hurwitz spaces. We use those expressions to prove Theorems 2.1.7 and 2.1.10. The natural classes we discuss can be defined on $\mathcal{H}_{k,g}$ for any k. They are the kappa classes and loci parametrizing covers with certain ramification profiles.

Definition 2.7.1. We define the following three closed loci in $\mathcal{H}_{k,g}$:

1. $T := \overline{\{[\alpha : C \to \mathbb{P}^1] : \alpha^{-1}(q) = 3p_1 + p_2 \cdots + p_{k-2}, \text{ for some } q \text{ and distinct } p_i\}}$
2. $D := \overline{\{[\alpha : C \to \mathbb{P}^1] : \alpha^{-1}(q) = 2p_1 + 2p_2 \cdots + p_{k-2}, \text{ for some } q \text{ and distinct } p_i\}}$
3. $U := \overline{\{[\alpha : C \to \mathbb{P}^1] : \alpha^{-1}(q) = 4p_1 + p_2 \cdots + p_{k-2}, \text{ for some } q \text{ and distinct } p_i\}}$

The loci T and D have codimension 1. The locus U is one component of the intersection $T \cap D$, and U has codimension 2.

Of course, one could consider other ramification behavior, but these three suffice for the applications in this paper. One benefit of these classes is that their push forwards to the moduli space of curves are known to be tautological. We make this precise in the next subsection. Then in the next two subsections, we rewrite the κ-classes and ramification loci in terms of CE classes to show that $[T], [D], [T] \cdot [D]$ and $[U]$ generate $R^*(\mathcal{H}_{k,g})$ as a module over $R^*(\mathcal{M}_g)$ in degrees $k = 4, 5$ respectively.

2.7.1 Push forwards to $\mathcal{M}_g$

To push forward cycles from the Hurwitz stack to $\mathcal{M}_g$, we first need to show that the relevant forgetful maps are proper. Consider the gonality stratification on the moduli space of curves:

$$\mathcal{M}_g^d := \{[C] \in \mathcal{M}_g : C \text{ has a } g_d^1\}.$$

Because we don't require base point freeness in the equation above, we have the inclusions $\mathcal{M}_g^d \subset \mathcal{M}_g^{d+1}$. Because gonality is lower semi-continuous, $\mathcal{M}_g \setminus \mathcal{M}_g^d$ is open for any d. We have the map

$$\beta : \mathcal{H}_{k,g} \to \mathcal{M}_g$$

obtained by forgetting the map to $\mathbb{P}^1$. After removing curves of lower gonality, we obtain a proper map

$$\beta_k : \mathcal{H}_{k,g} \setminus \beta^{-1}(\mathcal{M}_g^{k-1}) \to \mathcal{M}_g \setminus \mathcal{M}_g^{k-1},$$

essentially by the same proof as [BV12, Proposition 2.3].

Remark 2.7.2. If $k = 3$ or 5 and g is sufficiently large, the maps β_k are actually closed embeddings. See [BV12, Proposition 2.3] for the $k = 3$ case. On the other hand, the map β_4 is not injective on points because bielliptic curves admit infinitely many degree 4 maps to $\mathbb{P}^1$.

Because $\mathcal{M}_g \setminus \mathcal{M}_g^k$ is open in $\mathcal{M}_g$, there is a restriction map $A^*(\mathcal{M}_g) \to A^*(\mathcal{M}_g \setminus$

$\mathcal{M}_g^k$).

Definition 2.7.3. The tautological ring $R^*(\mathcal{M}_g \setminus \mathcal{M}_g^k)$ of $\mathcal{M}_g \setminus \mathcal{M}_g^k$ is defined to be the image of the tautological ring $R^*(\mathcal{M}_g)$ under the restriction map $A^*(\mathcal{M}_g) \to A^*(\mathcal{M}_g \setminus \mathcal{M}_g^k)$.

We need the following result of Faber-Pandharipande [FP05], which concerns push forwards of classes of ramification loci quite generally. Let $\mu^1, \ldots, \mu^m$ be m partitions of equal size k and length $\ell(\mu^i)$ that satisfy

$$2g - 2 + 2k = \sum_{i=1}^{m} (d - \ell(\mu^i)).$$

Faber and Pandharipande use the Hurwitz space $\mathcal{H}_g(\mu^1, \ldots, \mu^m)$ that parametrizes morphisms $\alpha : C \to \mathbb{P}^1$ that has marked ramification profiles $\mu^1, \ldots, \mu^m$ over m ordered points of the target and no ramification elsewhere. Two morphisms are equivalent if they are related by composition with an automorphism on $\mathbb{P}^1$. By the Riemann-Hurwitz formula, these are covers of genus g and degree k. They then consider the compactification by admissible covers $\overline{\mathcal{H}}_g(\mu^1, \ldots, \mu^m)$. It admits a natural map to the moduli space of stable curves with marked points by forgetting the map to $\mathbb{P}^1$:

$$\rho : \overline{\mathcal{H}}_g(\mu^1, \ldots, \mu^m) \to \overline{\mathcal{M}}_{g, \sum_{i=1}^m \ell(\mu^i)}.$$

Theorem 2.7.4 (Faber-Pandharipande [FP05])**.** *The pushforwards $\rho_*(\overline{\mathcal{H}}_g(\mu^1, \ldots, \mu^m))$ are tautological classes in $A^*(\overline{\mathcal{M}}_{g, \sum_{i=1}^m \ell(\mu^i)})$.*

We then have the following diagram:

$$\begin{tikzcd}
 & & & \overline{\mathcal{H}}_g(\mu^1,\dots,\mu^m) \arrow[d, "\rho"] \\
 & & & \overline{\mathcal{M}}_{g,\sum_{i=1}^m \ell(\mu^i)} \arrow[d] \\
\mathcal{H}_{k,g} \setminus \beta^{-1}(\mathcal{M}_g^{k-1}) \arrow[r, "\beta_k"] & \mathcal{M}_g \setminus \mathcal{M}_g^{k-1} \arrow[r, hook] & \mathcal{M}_g \arrow[r, hook] & \overline{\mathcal{M}}_g
\end{tikzcd}$$

Because the tautological ring is closed under forgetting marked points and under the pullback from $\overline{\mathcal{M}}_g$ to $\mathcal{M}_g$, it follows that the image of $[\overline{\mathcal{H}}_g(\mu^1,\dots,\mu^m)]$ in $A^*(\mathcal{M}_g \setminus \mathcal{M}_g^{k-1})$ is a tautological class.

Corollary 2.7.5. *Let $k \in \{3,4,5\}$. Then the classes $\beta_{k*}[T]$, $\beta_{k*}[D]$, $\beta_{k*}[U]$, and $\beta_{k*}([T]\cdot[D])$ lie in the tautological ring of $\mathcal{M}_g \setminus \mathcal{M}_g^{k-1}$.*

Proof. We explain the proof in the case $k=5$. The other cases are similar. The image of T, D, and U in $\mathcal{M}_g \setminus \mathcal{M}_g^{k-1}$ are the images of the corresponding spaces considered by Faber-Pandharipande. Indeed, for T, take $\mu_1 = (3,1,1)$ and $\mu_i = (2,1,1,1)$ for all other i. For D, take $\mu_1 = (2,2,1)$ and $\mu_i = (2,1,1,1)$ for all other i. For U, take $\mu_1 = (4,1)$ and $\mu_i = (2,1,1,1)$ for all other i.

One can see that the image of $T \cap D$ under β_k is supported on the image of the following three spaces considered by Faber-Pandharipande:

1. The image of the space with $\mu_1 = (4,1)$ and all other $\mu_i = (2,1,1,1)$
2. The image of the space with $\mu_1 = (3,2)$ and all other $\mu_i = (2,1,1,1)$
3. The image of the space with $\mu_1 = (3,1,1)$ and $\mu_2 = (2,2,1)$

It follows that the pushforward of $[T]\cdot[D]$ is a linear combination of the restrictions of images of the above three spaces. Hence, $\beta_{k*}([T]\cdot[D])$ is also tautological. $\square$

2.7.2 Formulas in degree 4

In this section, we compute formulas for the some of the natural classes on $\mathcal{H}_{4,g}$. We will do the computations in $A^*(\mathcal{H}_{4,g})$ in order to simplify the intersection theory calculation. This simplification is of no consequence to the end results because of the isomorphism $A^*(\mathcal{H}_{4,g}) \cong A^*(\mathcal{H}_{4,g})$.

Deopurkar-Patel [DP18, Proposition 2.8] computed formulas for the classes of T and D in terms of κ_1 and a_1. In [CL21d, Example 3.12] we explained how to write the κ-classes in terms of CE classes, so we obtain the following.

Lemma 2.7.6. *The following identities hold in* $A^1(\mathcal{H}_{4,g})$

$$\kappa_1 = (12g+24)a_1 - 12a_2', \qquad [T] = (24g+60)a_1 - 24a_2', \qquad [D] = (-32g-80)a_1 + 36a_2'.$$

Next, we compute the codimension two class $[U]$. In particular, we will see that $[U]$ is not in the span of products of codimension 1 classes, from which it follows that the classes of $[T], [D], [U]$ generate $R^*(\mathcal{H}_{4,g})$ as a ring.

Lemma 2.7.7. *The class of the quadruple ramification stratum U on $\mathcal{H}_{4,g}$ is*

$$[U] = 36a_1a_2' - (32g+80)a_1^2 + (4g+4)a_2 - (4g+4)b_2.$$

Modulo the relations from Proposition 2.5.4, we have $[U] = 4a_3'$.

Proof. The fibers of a degree 4 cover $\alpha : C \to \mathbb{P}^1$ are given by the base locus of a pencil of conics. A pencil of conics has base locus $4p$ if and only if every element of the pencil is tangent to a given line L and $2L$ is a member of the pencil. Equivalently, $4p$ is the base locus of a pencil of conics if and only if in some choice of local coordinates x, y at p

(U1) All members of the pencil are tangent to the line $y = 0$ at p, i.e. have vanishing coefficient of $x, 1$.

(U2) Some member of the pencil is a multiple of y^2, i.e. has vanishing coefficient of $1, x, y, x^2, xy$.

Note that the base locus of a pencil containing two double lines is not a *curve-linear scheme* (i.e. a subscheme of smooth curve) since it has two dimensional tangent space at the intersection point. Therefore, if If p is a point of quadruple ramification on a smooth curve $C \xrightarrow{\alpha} \mathbb{P}^1$, then the line $L \subset (\mathbb{P}E_\alpha^\vee)_{\alpha^{-1}(\alpha(p))} \cong \mathbb{P}^2$ is unique. That is, there is a unique direction and member of the pencil satisfying (U1) and (U2).

We will use the theory of restricted bundles of principal parts developed in Section 6 to characterize the covers satisfying these conditions. Let $X := \mathbb{P}T_{\mathbb{P}\mathcal{E}^\vee/\mathcal{P}} \times_{\mathcal{P}} \mathbb{P}\mathcal{F}$. The first factor $\mathbb{P}T_{\mathbb{P}\mathcal{E}^\vee/\mathcal{P}}$ keeps track of a "x-direction" and the second factor $\mathbb{P}\mathcal{F}$ keeps track of a particular member of the pencil. We will apply the constructions of Section 6 to the tower

$$X \xrightarrow{a} \mathbb{P}\mathcal{E}^\vee \xrightarrow{\gamma} \mathcal{P}.$$

In particular, pulling back the dual of the tautological sequence on the $\mathbb{P}T_{\mathbb{P}\mathcal{E}^\vee/\mathcal{P}}$ factor, we obtain a filtration on X

$$0 \to \Omega_y \to a^*\Omega_{\mathbb{P}\mathcal{E}^\vee/\mathcal{P}} \to \Omega_x \to 0.$$

Meanwhile, pulling back the dual of the tautological sequence from the $\mathbb{P}\mathcal{F}$ we obtain a quotient

$$a^*\gamma^*\mathcal{F}^\vee \to \mathcal{O}_{\mathbb{P}\mathcal{F}}(1) \to 0$$

Tensoring with $a^*\mathcal{O}_{\mathbb{P}\mathcal{E}^\vee}(2)$, we obtain a filtration of $a^*\mathcal{W} = a^*(\gamma^*\mathcal{F}^\vee \otimes \mathcal{O}_{\mathbb{P}\mathcal{E}^\vee}(2))$:

$$0 \to K \to a^*\mathcal{W} \to \mathcal{O}_{\mathbb{P}\mathcal{F}}(1) \otimes \mathcal{O}_{\mathbb{P}\mathcal{E}^\vee}(2) =: \mathcal{W}' \to 0.$$

To track the data in (U1) and (U2) we define $Q := P^{S \subset S'}_{\mathbb{P}\mathcal{E}^\vee/\mathcal{P}}(\mathcal{W} \to \mathcal{W}')$ where

$S = \{1, x\}$ and $S' = \{1, x, y, x^2, xy\}$. This is represented by the diagram

$$\begin{array}{ccc} \bullet & \bullet & \LEFTcircle \\ \LEFTcircle & \LEFTcircle & \end{array} \tag{2.7.1}$$

There is a natural quotient $a^*P^2_{\mathbb{P}\mathcal{E}^\vee/\mathcal{P}}(\mathcal{W}) \to Q$, corresponding to the picture below.

$$\begin{array}{ccc} \bullet & \bullet & \bullet \\ \bullet & \bullet & \\ \bullet & & \end{array} \longrightarrow \begin{array}{ccc} \bullet & \bullet & \LEFTcircle \\ \LEFTcircle & \LEFTcircle & \\ & & \end{array}$$

As discussed in Section 2.5 the Casnati–Ekedahl theorem determines a global section δ^{univ} of $\mathcal{W}$ whose vanishing is the universal curve. The induced section of Q

$$\mathcal{O}_X \xrightarrow{a^*\delta^{\text{univ}\prime}} a^*P^2_{\mathbb{P}\mathcal{E}^\vee/\mathcal{P}}(\mathcal{W}) \to Q \tag{2.7.2}$$

vanishes at a point of X over p precisely when conditions (U1) and (U2) above are satisfied at p for the corresponding direction and member of the pencil. Let $\widetilde{U}$ be the vanishing locus of the section in (2.7.2).

The map a sends $\widetilde{U}$ one-to-one onto the universal quadruple ramification point. In turn, the universal quadruple ramification point maps generically one-to-one onto U, so

$$[U] = \pi_*\gamma_*a_*[\widetilde{U}].$$

Since all fibers of the map $\widetilde{U} \to U$ are finite we have $\dim \widetilde{U} = \dim U$. Note that X has relative dimension 2 over $\mathbb{P}\mathcal{E}^\vee$, which has relative dimension 3 over $\mathcal{H}_{4,g}$. Thus, we have

$$\operatorname{codim}(\widetilde{U} \subset X) = \operatorname{codim}(U \subset \mathcal{H}_{4,g}) + \text{relative dim of } X/\mathcal{H}_{4,g} = 2 + (2+3) = 7.$$

Meanwhile, $\operatorname{rank}\Omega_x = \operatorname{rank}\Omega_y = \operatorname{rank}\mathcal{W}' = \operatorname{rank}K = 1$. Each dot in the diagram (2.7.1) corresponds to a piece of a filtration of Q. The filled dots $\bullet$ correspond to pieces of rank

2 and half-filled dots ◐ correspond to pieces of rank 1. Hence, rank $Q = 7$. In particular, $\operatorname{codim}(\widetilde{U} \subset X) = \operatorname{rank} Q$, so $[\widetilde{U}] = c_7(Q)$. The top Chern class of Q can be computed using its filtration, and its push forward to $\mathcal{H}_{4,g}$ is computed in Macaulay2 [CL21c], which gives the expressions in the statement of the Lemma. □

In the example below, we provide expressions for some other codimension 2 classes in terms of our preferred generators.

Example 2.7.8. Using the relations provided in the code, we can rewrite c_2 in terms of our preferred generators as

$$c_2 = \frac{3}{g^2+4g+3}a_1^2 - \frac{8}{g^3+6g^2+11g+6}a_3' \tag{2.7.3}$$

Using [CL21d, Example 3.12], we can compute

$$\begin{aligned} \kappa_2 &= a_1 b_2' - 6a_1 a_2' + (6g+6)a_1^2 - (6g-6)a_2 + (g-3)b_2 \qquad (2.7.4)\\ &\quad - (2g^3+6g^2+6g-14)c_2 + 4a_3' \\ &= \frac{44g^2+200g+300}{g^2+4g+3}a_1^2 - \frac{44}{g+1}a_1 a_2' + \frac{2g^3-32g^2+138g-12}{3g^3+18g^2+33g+18}a_3'. \qquad (2.7.5) \end{aligned}$$

Since the coefficient of a_3' is non-zero in (2.7.3) (resp. (2.7.5)), we see that c_2 (resp. κ_2) may be used instead of a_3' as the generator of $R^*(\mathcal{H}_{4,g})$ in codimension 2.

We can now prove Theorem 2.1.10 in when $k = 4$.

Proof of Theorem 2.1.10, $k = 4$. By Lemmas 2.7.6 and 2.7.7 and Theorem 2.1.1, it follows that $[T], [D], [U]$ generate $R^*(\mathcal{H}_{4,g})$. Moreover, $R^i(\mathcal{H}_{4,g}) \to A^i(\mathcal{H}_{4,g}^{\mathrm{nf}})$ is surjective in degrees $i \leq \frac{g+3}{4} - 4$ by Theorem 2.1.1 (2). We have that $A^*(\mathcal{H}_{4,g}^{\mathrm{nf}}) \to A^*(\mathcal{H}_{4,g}^{s})$ is surjective and the ideal generated by T, D, U is in the kernel. Hence, $A^i(\mathcal{H}_{4,g}^{s}) = 0$ for $i \leq \frac{g+3}{4} - 4$. □

Above, we showed that $[T],[D],[U]$ generate $R^*(\mathcal{H}_{4,g})$ as a *ring*. We now show that $[T],[D],[U],[T]\cdot[D]$ generate $R^*(\mathcal{H}_{4,g})$ as a *module* over $\mathbb{Q}[\kappa_1]$.

Lemma 2.7.9. *The following are true:*

1. *$R^1(\mathcal{H}_{4,g})$ is spanned by $[T]$ and $[D]$. Alternatively, it is spanned by $[T]$ and κ_1.*

2. *$R^2(\mathcal{H}_{4,g})$ is spanned by $[T]\kappa_1,[D]\kappa_1,[T]\cdot[D]$ and $[U]$.*

3. *$R^3(\mathcal{H}_{4,g})$ is spanned by $\kappa_1^2[T],\kappa_1^2[D],\kappa_1[U]$.*

4. *$R^4(\mathcal{H}_{4,g})$ is spanned by κ_1^4 and $\kappa_1^2[U]$.*

5. *For $i\geq 5$, $R^i(\mathcal{H}_{4,g})$ is spanned by κ_1^i.*

Proof. (1) By Lemma 2.7.6, any pair of $[T],[D],\kappa_1$ span $R^1(\mathcal{H}_{4,g})$.

(2) By Corollary 2.5.6, we have that $R^2(\mathcal{H}_{4,g})$ is spanned by $\{a_1^2,a_1a_2',a_2'^2,a_3'\}$. Hence, Lemma 2.7.7 shows that $[U]$ and products of codimension 1 classes span $R^2(\mathcal{H}_{4,g})$.

(3) Since a_1,a_2',a_3' generate $R^*(\mathcal{H}_{4,g})$ as a ring, the classes

$$\{a_1^3,a_1^2a_2',a_1a_2'^2,a_2'^3,a_1a_3',a_2'a_3'\}$$

span $R^3(\mathcal{H}_{4,g})$. To show that $\kappa_1^2[T],\kappa_1^2[D]$, and $\kappa_1[U]$ span $R^3(\mathcal{H}_{4,g})$, we first rewrite them in terms of CE classes. It then suffices to see that these three classes, together with the codimension 3 relations r_1,r_2,r_3 of Section 2.5.4, span $\{a_1^3,a_1^2a_2',a_1a_2'^2,a_2'^3,a_1a_3',a_2'a_3'\}$. One way to accomplish this is as follows. By Corollary 2.5.6, $\{a_1a_3',a_2'^3,a_2'a_3'\}$ is a spanning set modulo r_1,r_2,r_3 and one can readily rewrite $\kappa_1^2[T],\kappa_1^2[D]$, and $\kappa_1[U]$ in terms of $\{a_1a_3',a_2'^3,a_2'a_3'\}$ modulo the relations. We record the coefficients of these expressions in a 3×3 matrix. The determinant of this matrix has non-vanishing determinant for all g, so we conclude that $\kappa_1^2[T],\kappa_1^2[D]$, and $\kappa_1[U]$ are also a spanning set modulo the relations. The calculation of the determinant is provided at [CL21c].

(4) The proof is similar to the previous part. By Corollary 2.5.6, $\{a_2'^4, a_3'^2\}$ spans the degree 4 piece of $\mathbb{Q}[a_1, a_2', a_3']/\langle r_1, r_2, r_3, r_4\rangle$. We then write a 2×2 matrix of coefficients that expresses κ_1^4 and $\kappa_1^2[U]$ in terms of $\{a_2'^4, a_3'^2\}$ modulo the relations. We then check that the determinant is non-vanshing.

(5) From a direct calculation provided in the code, we see that κ_1^i is a nonzero multiple of $a_2'^i$ for $5 \leq i \leq 10$. For all $i \geq 11$, a monomial of degree i in the generators a_1, a_2', a_3' can be written as a product of monomials having degrees between 5 and 10, so the claim follows. $\square$

Proof of Theorem 2.1.7, $k = 4$. By Lemma 2.7.9, we see that every class in $R^*(\mathcal{H}_{4,g})$ is expressible as a polynomial in κ_1 times $[T], [D], [T] \cdot [D]$, or $[U]$. By Corollary 2.7.5, the push forwards of $[T], [D], [T] \cdot [D], [U]$ are tautological, so by push-pull, the push forwards of all classes in $R^*(\mathcal{H}_{4,g})$ are tautological on $\mathcal{M}_g \smallsetminus \mathcal{M}_g^3$. $\square$

2.7.3 Formulas in degree 5

As in the previous section, we will perform the calculations on the spaces $\mathcal{H}_{5,g}$ instead of $\mathcal{H}_{5,g}$. As in degree 4, the codimension 1 identities are easily converted from Deopukar-Patel [DP18, Proposition 2.8] and [CL21d, Example 3.12], which computes κ_1 in terms of CE classes.

Lemma 2.7.10. *The following identities hold in $A^1(\mathcal{H}_{5,g})$*

$$\kappa_1 = (12g + 36)a_1 - 12a_2' \qquad [T] = (24g + 84)a_1 - 24a_2' \qquad [D] = -(32g + 112)a_1 + 36a_2'.$$

Using the method explained in [CL21d, Example 3.12], it is not difficult to compute κ_2 in terms of CE classes with our code [CL21c].

Lemma 2.7.11. *The following identities hold in $A^2(\mathcal{H}_{5,g})$*

$$\kappa_2 = (6g^2 + 24g + 40)c_2 - 6a_1^2 + (-7g + 2)a_2 - 7a_1a_2' + (2g + 2)b_2 + 2a_1b_2' + 5a_3' - b_3'.$$

Modulo the relations found in Lemma 2.6.9,

$$\kappa_2 = (30g + 66)a_1^2 + (-21g + 2)a_2 - 21a_1a_2' - (10g^3 + 66g^2 + 104g)c_2.$$

Next, we wish to compute $[U]$ in terms of CE classes, which will require more work and geometric input. Once we have $[U]$ in terms of CE classes, it will not be hard to see that $[T], [D], [U]$ and $[T] \cdot [U]$ generate $R^*(\mathcal{H}_{5,g})$ as a module over $\mathbb{Q}[\kappa_1, \kappa_2]$. However, in contrast with the case $k = 4$, the classes $[T], [D], [U]$ do not generate $R^*(\mathcal{H}_{5,g})$ as a ring, so additional work is needed to prove the vanishing results for $A^i(\mathcal{H}_{5,g}^s)$. We do this by constructing the universal triple ramification point and showing that an additional codimension 2 class needed to generate $R^*(\mathcal{H}_{5,g})$ as a ring is supported on T.

For these last computations, we work with the realization of the universal curve $\mathcal{C} \subset G(2, \mathcal{F})$ as the vanishing locus of a section of a rank 6 vector bundle, as we now describe. On $\pi : \mathcal{P} \to \mathcal{H}_{5,g}$, Casnati's structure theorem in degree 5 determines a universal injection $\eta^{\text{univ}} : \mathcal{E}' \to \wedge^2\mathcal{F}$. Let $\mathcal{Q}$ be the rank 6 cokernel. Let $\mu : G := Gr(2, \mathcal{F}) \to \mathcal{P}$ be the Grassmann bundle. Then $\mathcal{C} \subset G$ is defined by the vanishing of the composition

$$\mathcal{O}_G(-1) := \mathcal{O}_{\mathbb{P}(\wedge^2\mathcal{F})}(-1)|_G \to \mu^*(\wedge^2\mathcal{F}) \to \mu^*\mathcal{Q},$$

which we view as a section σ of $\mu^*\mathcal{Q} \otimes \mathcal{O}_G(1) =: \mathcal{W}$. Studying appropriate principal parts of this section σ of $\mathcal{W}$ on G over $\mathcal{P}$ helps us describe when $\mathcal{C} \to \mathcal{P}$ has a point of higher order ramification.

Precisely, the universal curve has a triple (resp. quadruple) ramification point at $p \in \mathcal{C} \subset G$ if and only if there exists a direction x in $(T_{G/\mathcal{P}})_p$ such that

1. the coefficient of x vanishes in all equations. This implies that the universal curve has a vertical tangent vector in the x direction, and so is ramified at p.

2. Let $y_1, \ldots, y_5$ be the remaining first order coordinates on $(T_{G/\mathcal{P}})_p$. Locally σ cor-

responds to 6 equations on G. Since the universal curve is smooth, when we expand these equations to first order, the coefficients of $y_1, \dots, y_5$ must span a five-dimensional space. That is, on $\mathcal{C}$ each y_i may be solved for as a power series in x with leading term order 2. Moreover, there is also a "distinguished equation" whose first order parts are all zero. This "distinguished equation" will correspond to a particular quotient of $\mathcal{W}$.

3. After substituting for y_i as a power series in x using (2), all equations vanish to order 2 (resp. order 3). This is only a condition on the distinguished equation (the substitutions for y_i were determined so that the other five are identically zero). For order 2 vanishing, this condition is just that the coefficient of x^2 in the distinguished equation is zero. For order 3 vanishing, this will involve expanding through the coefficients of xy_i and x^3.

Note that because $\mathcal{C}$ is smooth over $\mathcal{H}$, the distinguished direction x and distinguished equation of (2) are unique.

Let $X := \mathbb{P}T_{G/\mathcal{P}} \times_{\mathcal{P}} \mathbb{P}\mathcal{W}^\vee$. The first factor keeps track of an "x-direction" and the second factor keeps track of a "distinguished equation" among the equations. We apply the constructions of Section 6 to the tower

$$X \xrightarrow{a} G \xrightarrow{\mu} \mathcal{P}.$$

The pullback to X of the dual of the tautological sequence on $\mathbb{P}T_{G/\mathcal{P}}$ gives a filtration

$$0 \to \Omega_y \to a^*\Omega_{G/\mathcal{P}} \to \Omega_x \to 0.$$

Meanwhile, the pullback of the dual of the tautological sequence on $\mathbb{P}\mathcal{W}^\vee$ gives a quotient

$$a^*\mu^*\mathcal{W} \to \mathcal{O}_{\mathbb{P}\mathcal{W}^\vee}(1) =: \mathcal{W}' \to 0.$$

Let $S = \{1, x\}$ and $S' = \{1, x, y, x^2\}$ and set $M := P^{S \subset S'}_{G/\mathcal{P}}(\mathcal{W} \to \mathcal{W}')$, which is a quotient $a^* P^2_{G/\mathcal{P}}(\mathcal{W})$ corresponding to (2.3.4A), pictured again below. The bundles that appear in the filtration are listed in the corresponding location to the right.

$$\begin{array}{ccc} \bullet & \bullet & \LEFTcircle \\ \LEFTcircle & & \end{array} \qquad \begin{array}{ccc} \mathcal{W} & \mathcal{W} \otimes \Omega_x & \mathcal{W}' \otimes \Omega_x^2 \\ \mathcal{W}' \otimes \Omega_y & & \end{array} \tag{2.7.6}$$

The bundle M measures the values and coefficients of x in the equations, as well as the coefficients of the y_i and x^2 in a distinguished equation. It has rank 18.

A section of $a^* P^2_{G/\mathcal{P}}(\mathcal{W})$ induces a section of M. In particular, the global section σ of $\mathcal{W}$ induces a section σ' of $a^* P^2_{G/\mathcal{P}}(\mathcal{W})$, which then gives a section σ'' of M. We claim that this section σ'' vanishes at some point $\widetilde{p} \in X$ lying over $p \in G$ if and only if conditions (1) – (3) above are satisfied (to order 2) for the distinguished direction and distinguished equation referred to by $\widetilde{p}$. In more detail: the left $\bullet = \mathcal{W}$ corresponds to the condition $p \in \mathcal{C}$; the right $\bullet = \mathcal{W} \otimes \Omega_x$ gives condition (1); the lower $\LEFTcircle = \mathcal{W}' \otimes \Omega_y$ corresponds to condition (2); and and the right $\LEFTcircle = \mathcal{W}' \otimes \Omega_x^2$ corresponds to condition (3).

Hence, the vanishing locus $\widetilde{T}$ of this induced section of M maps isomorphically to the universal triple ramification point. A computation similar to the one in Lemma 2.7.7 shows that this vanishing occurs in the expected codimension, so $[\widetilde{T}] = c_{18}(M)$. The composition from $\widetilde{T} \to \mathcal{H}_{5,g}$ is generically one-to-one onto its image, so we obtain an equality of classes

$$[T] = \pi_* \mu_* a_* [\widetilde{T}].$$

This pushforward can be computed using a computer, and agrees with Lemma 2.7.10.

The universal quadruple ramification point is cut out inside $\widetilde{T}$ by one more condition: namely, after replacing each y_i with its power series in x as in (2), the coefficient of x^3 in the distinguished equation must vanish.

Since y_i is of order 2 in x, only the terms xy_i can contribute to the coefficient of x^3. We already know that the coefficients of $1, y_1, \ldots, y_5, x, x^2$ vanish in the distinguished equation (corresponding to the shape (2.7.6)). We therefore wish to study the expansion of the distinguished equation through its coefficients of $xy_1, \ldots, xy_5$ and x^3. This will correspond to two new dots (represented below in red). Let $S'' = \{1, x, y, x^2, xy, y^2, x^3\}$. The part of the Taylor expansion we need corresponds to the bundle $N := P^{S \subset S''}_{G/\mathcal{P}}(\mathcal{W} \to \mathcal{W}')$ from (2.3.4C), pictured below. The bundles in the filtration are listed in the corresponding location on the right.

$$\begin{array}{cccc} \bullet & \bullet & \color{blue}{\LEFTcircle} & \color{red}{\LEFTcircle} \\ \color{blue}{\LEFTcircle} & \color{red}{\LEFTcircle} & & \end{array} \qquad \begin{array}{cccc} \mathcal{W} & \mathcal{W} \otimes \Omega_x & \color{blue}{\mathcal{W}' \otimes \Omega_x^2} & \color{red}{\mathcal{W}' \otimes \Omega_x^3} \\ \color{blue}{\mathcal{W}' \otimes \Omega_y} & \color{red}{\mathcal{W}' \otimes \Omega_x \otimes \Omega_y} & & \end{array}$$

Let $N^{\color{red}{\LEFTcircle}} \subset N$ be the kernel of $N \to M$. Visually, $N^{\color{red}{\LEFTcircle}}$ is subbundle corresponding to the right-most partially filled circles, which is filtered by $\mathcal{W}' \otimes \Omega_x \otimes \Omega_y$ and $\mathcal{W}' \otimes \Omega_x^3$. By the definition of $\widetilde{T}$, on $\widetilde{T} \subset X$, the section of N induced by σ factors through $N^{\color{red}{\LEFTcircle}}$. We call this section $\sigma^{\color{red}{\LEFTcircle}}$.

To get a quadruple point, it needs to be the case that when we sub in the power series of the y_i's in terms of x into the distinguished equation, the coefficient of x^3 vanishes. This is the same as saying that the expansion of the distinguished equation lies in the span of "x times" the $\{y, x^2\}$ parts of the other equations. This will correspond to vanishing of evaluation in a rank 1 quotient of $N^{\color{red}{\LEFTcircle}}$ that we define below. This quotient will be isomorphic to $\mathcal{W}' \otimes \Omega_x^3$.

Remark 2.7.12. The vanishing order filtration on $N^{\color{red}{\LEFTcircle}}$ provides a *sub*bundle $\mathcal{W}' \otimes \Omega_x^3 \subset N^{\color{red}{\LEFTcircle}}$. The construction of our desired *quotient* $N^{\color{red}{\LEFTcircle}} \to \mathcal{W}' \otimes \Omega_x^3$ on $\widetilde{T}$ will crucially use the fact that the subschemes in the fibers of $\mathcal{C} \to \mathcal{P}$ are curve-linear (in particular, have 1 dimensional tangent space). This is equivalent to the statement in (2) that the other y_i's may be solved for as power series in x.

To make this precise, let V be the kernel of $P_{G/P}^{\{1,x,y,x^2\}}(\mathcal{O}) \to \mathcal{O}$, which comes equipped with a filtration

$$0 \to \Omega_x^2 \to V \to a^*\Omega_{G/P} \to 0.$$

The bundle V is like the tangent bundle but "with a bit of second order information in the distinguished direction." Considering the triple point inside G referred to by each point of $\widetilde{T}$ determines a rank 2 quotient Q_{trip} of V on $\widetilde{T}$ that fits in a diagram

$$\begin{tikzcd}
0 \arrow[r] & \Omega_x^2 \arrow[r] & V \arrow[r] \arrow[d] & a^*\Omega_{G/P} \arrow[r] \arrow[d] & 0 \\
& & Q_{\text{trip}} \arrow[r] & \Omega_x. &
\end{tikzcd}$$

Just as having a distinguished quotient of $a^*\Omega_{G/P}$ allowed us to refine bundles of principal parts in Section 2.3.2, so too does having this rank 2 quotient of V. Let L be the kernel of $Q_{\text{trip}} \to \Omega_x$, so L corresponds to the second order data along a triple ramification point. The map from upper left to lower right, $\Omega_x^2 \to L$, is non-vanishing because the square of the first order coordinate is non-zero on the triple point (this uses curve-linearity), so $L \cong \Omega_x^2$. Equivalently, the quotient $V \to Q_{\text{trip}}$ does factor through $a^*\Omega_{G/\mathcal{P}}$ on any fiber (which would mean the fiber through p had two-dimensional tangent space). Now, $\ker(V \to \Omega_x)$ corresponds to the $\{y, x^2\}$ parts of our expansions. Similarly, $\ker(V \to \Omega_x) \otimes \Omega_x$ corresponds to the $\{xy, x^3\}$ parts. Tensoring $\ker(V \to \Omega_x) \to L$ with $\mathcal{W}' \otimes \Omega_x$, we get the desired quotient

$$N^{\LEFTcircle} = \mathcal{W}' \otimes \Omega_x \otimes \ker(V \to \Omega_x) \to \mathcal{W}' \otimes \Omega_x \otimes L \cong \mathcal{W}' \otimes \Omega_x^3.$$

The evaluation of $\delta^{\LEFTcircle}$ in this quotient is zero precisely when condition (3) above is satisfied to order 3.

Hence, the universal quadruple ramification point is determined by the vanishing of a section of a line bundle $\mathcal{W}' \otimes \Omega_x^3$ on $\widetilde{T}$. In particular,

$$[U] = \pi_* \mu_* a_* ([\widetilde{T}] \cdot c_1(\mathcal{W}' \otimes \Omega_x^3)),$$

which we computed in Macaulay.

Lemma 2.7.13. *The class of the ramification locus U on $\mathcal{H}_{5,g}$ is*

$$[U] = (12g+48)a_1^2 - (4g+16)b_2 - (4g^3+48g^2+192g+256)c_2 - 4a_1b_2' + 4b_3'.$$

Modulo the relations from Lemma 2.6.9,

$$[U] = \frac{156g+468}{5}a_1^2 - \frac{108g+216}{5}a_2 - \frac{108}{5}a_1a_2' - \frac{52g^3+468g^2+1352g+1248}{5}c_2.$$

We now give additive generators for $R^*(\mathcal{H}_{5,g})$.

Lemma 2.7.14. *Suppose $g \geq 2$. Then,*

1. *$R^1(\mathcal{H}_{5,g})$ is spanned by $[T]$ and $[D]$. Alternately, it is spanned by $[T]$ and κ_1.*
2. *$R^2(\mathcal{H}_{5,g})$ is spanned by $[T]\kappa_1$, $[D]\kappa_1$, $[T]\cdot[D]$, $[U]$, κ_2.*
3. *$R^3(\mathcal{H}_{5,g})$ is spanned by $[T]\kappa_1^2, [D]\kappa_1^2, [T]\cdot[D]\kappa_1, [U]\kappa_1, [T]\kappa_2, [D]\kappa_2$.*
4. *$R^4(\mathcal{H}_{5,g})$ is spanned by $[T]\kappa_1^3, \kappa_1^4, [T]\kappa_1\kappa_2, [T]\cdot[D]\kappa_2, \kappa_2^2, \kappa_1^2\kappa_2, [U]\kappa_2$.*
5. *$R^5(\mathcal{H}_{5,g})$ is spanned by $[T]\kappa_1^4, [T]\kappa_2^2, \kappa_1^5, \kappa_1\kappa_2^2$.*
6. *$R^6(\mathcal{H}_{5,g})$ is spanned by $[T]\kappa_1^5, \kappa_1^6, \kappa_1^4\kappa_2$.*
7. *$R^i(\mathcal{H}_{5,g})$ is spanned by $[T]\kappa_1^{i-1}, \kappa_1^i$ for $i \geq 7$.*

Proof. Using Lemmas 2.7.10, 2.7.11, and 2.7.13, we can write down expressions for each class in the statement of the Lemma in terms of Casnati–Ekedahl classes. Modulo our relations in Section 2.6.9, Macaulay gives a formula for these classes in terms of the spanning sets of Corollary 2.6.10.

For each i, we can then write down a matrix whose entries are the coefficients of the expression for the classes in the statement of the lemma in terms of the CE spanning set. We then check if the determinant of the matrix of coefficients, which is a polynomial in g, has no positive integer roots. For example, in codimension 1, we have that $\{a_1, a_2'\}$ is a spanning set, and we have

$$[T] = (24g + 84)a_1 - 24a_2' \qquad [D] = -(32g + 112)a_1 + 36a_2'.$$

The matrix of coefficients

$$\begin{pmatrix} 24g + 84 & -24 \\ -32g - 112 & 36 \end{pmatrix}$$

has determinant $96g + 336$, which has no integer roots, so $[T]$ and $[D]$ span $R^1(\mathcal{H}_{5,g})$. A similar calculation shows that $[T]$ and κ_1 span $R^1(\mathcal{H}_{5,g})$. For $2 \leq i \leq 6$, we repeat the process, and the determinants are calculated at [CL21c]. None of them has roots at any integer $g \geq 2$.

When $i \geq 7$, we use an argument similar to Section 2.6.4. For $7 \leq i \leq 14$, we check that $[T]\kappa_1^{i-1}$ and κ_1^i span, by showing that the matrix of coefficients to express these in terms of $a_1 a_2'^{i-1}$ and $a_2'^i$ is invertible. Because it $R^*(\mathcal{H}_{5,g})$ is generated in degrees 1 and 2, for $i \geq 15$, every monomial class in $R^*(\mathcal{H}_{5,g})$ is expressible as a product of two monomials, both of degree at least 7. Then the product of two such monomials is in the span of $\kappa_1^i, \kappa_1^{i-1}[T]$ and $\kappa_1^{i-2}[T]^2 = \kappa_1^{i-7}(\kappa_1^5[T]^2)$. The last monomial is already in the span of the first two because $R^7(\mathcal{H}_{5,g})$ is spanned by $\kappa_1^7, \kappa_1^6[T]$. The last part (7) now follows. □

As a consequence, we finish the proofs of Theorem 2.1.7 and Theorem 2.1.10.

Proof of Theorem 2.1.7, $k = 5$. By Lemma 2.7.14, we see that every class in $R^*(\mathcal{H}_{5,g})$ is expressible as a polynomial in the kappa classes times $[T], [D]$, or $[U]$. By Corollary 2.7.5, the push forwards of $[T], [D], [U]$ are tautological, so by push-pull, the push forwards of all classes in $R^*(\mathcal{H}_{5,g})$ are tautological on $\mathcal{M}_g$. $\square$

Proof of Theorem 2.1.10, $k = 5$. For i in the range of the statement, we have $A^i(\mathcal{H}_{5,g}) = R^i(\mathcal{H}_{5,g})$. Thus, it suffices to produce generators for $R^*(\mathcal{H}_{5,g})$ as a ring that are supported on T and D. We know from Theorem 2.1.1 (3) that $R^*(\mathcal{H}_{5,g})$ is generated by two classes in degree 1 and two classes in degree 2. The classes $[T]$ and $[D]$ generate $R^1(\mathcal{H}_{5,g})$. Then, we computed $\pi_*(\mu_* a_*([\widetilde{T}]) \cdot z)$, which is supported on T, in the code [CL21c]. The result is that

$$\pi_*(\mu_* a_*([\widetilde{T}]) \cdot z) = (3g^2 + 24g + 48)c_2 - 3a_1^2 - 3a_2 + 3b_2.$$

Modulo the relations from Lemma 2.6.6, this class is given by

$$\pi_*(\mu_* a_*([\widetilde{T}]) = 12a_1^2 - 24a_2 - (12g^2 + 84g - 144)c_2. \tag{2.7.7}$$

Using Lemma 2.7.13, we see that $\pi_*(\mu_* a_*([\widetilde{T}]) \cdot z)$ and $[U]$ are independent modulo products of codimension 1 classes. Since $R^*(\mathcal{H}_{5,g})$ is generated in codimension 1 and 2, we conclude that $R^*(\mathcal{H}_{5,g})$ is generated by $[T], [D], [U]$ and the class in (2.7.7), which are all supported on T and D. $\square$

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Chapter 2 is coauthored with Larson, Hannah. The dissertation author was co-primary investigator and author of this paper.

Chapter 3

Integral Picard groups of low-degree Hurwitz spaces

3.1 Introduction

Let $\mathscr{H}_{k,g}$ be the Hurwitz stack parametrizing degree k covers by genus g curves of $\mathbb{P}^1$, up to automorphisms of the target $\mathbb{P}^1$. Let $\mathscr{H}^s_{k,g} \subseteq \mathscr{H}_{k,g}$ be the open substack parametrizing simply branched covers. The Hurwitz space Picard rank conjecture posits that $\mathrm{Pic}(\mathscr{H}^s_{k,g}) \otimes \mathbb{Q} = 0$. For $k \geq 4$, the complement of $\mathscr{H}^s_{k,g} \subset \mathscr{H}_{k,g}$ consists of two irreducible divisors: D, parametrizing covers with two points of ramification in the same fiber (pictured left) and T, parametrizing covers with a point of triple ramification (pictured right).

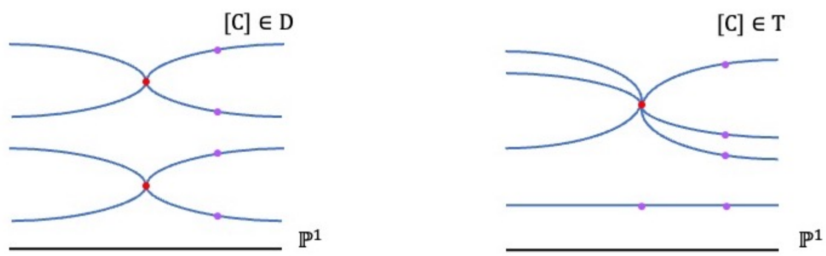

Figure 3.1. Components of the complement of $\mathscr{H}^s_{k,g} \subset \mathscr{H}_{k,g}$

In [DP15], Deopurkar–Patel prove that for $k \geq 4$, the classes of T and D are linearly independent in $\mathrm{Pic}(\mathscr{H}_{k,g}) \otimes \mathbb{Q}$. Note that D is empty when $k = 3$. For $k \geq 4$, the Picard rank conjecture is then equivalent to $\mathrm{Pic}(\mathscr{H}_{k,g}) \otimes \mathbb{Q} \cong \mathbb{Q}^{\oplus 2}$. The conjecture is known by work of Stankova-Frenkel and Deopurkar–Patel for $k \leq 5$ [SF00, DP15], and for $k > g - 1$ by work of Mullane [Mul20].

In this paper, we will study $\mathrm{Pic}(\mathscr{H}_{k,g})$ and $\mathrm{Pic}(\mathscr{H}^s_{k,g})$ with *integral* coefficients. Much less is known in this case: Arsie–Vistoli [AV04] computed $\mathrm{Pic}(\mathscr{H}_{2,g})$, and Bolognesi–Vistoli [BV12] computed $\mathrm{Pic}(\mathscr{H}_{3,g})$. In both cases, there are torsion classes depending on the genus. Our main theorem shows that this torsion phenomenon does not extend to $k = 4, 5$ when $g \geq 3$.

Theorem 3.1.1. *For $g \geq 2$, the integral Picard groups of the Hurwitz stacks are as follows.*

1. *We have*

$$\mathrm{Pic}(\mathscr{H}_{4,g}) = \begin{cases} \mathbb{Z} \oplus \mathbb{Z}/10\mathbb{Z} & \text{if } g = 2 \\ \mathbb{Z} \oplus \mathbb{Z} & \text{if } g \geq 3. \end{cases}$$

2. *We have*

$$\mathrm{Pic}(\mathscr{H}_{5,g}) = \begin{cases} \mathbb{Z} \oplus \mathbb{Z}/10\mathbb{Z} & \text{if } g = 2 \\ \mathbb{Z} \oplus \mathbb{Z} & \text{if } g \geq 3. \end{cases}$$

We also provide explicit line bundles generating $\mathrm{Pic}(\mathscr{H}_{k,g})$ (Sections 3.4.1, 3.5.3, and 3.6.3). One of those line bundles is the determinant of the Hodge bundle, which is pulled back from the moduli space of curves. The torsion in the case $g = 2$ arises from the torsion in the Picard group of the moduli space of genus 2 curves, as the first Chern class λ of the Hodge bundle is 10-torsion when $g = 2$ [Vis98].

Remark 3.1.2. This 10-torsion phenomenon is also present when $k = 3$, namely we shall prove $\mathrm{Pic}(\mathscr{H}_{3,2}) = \mathbb{Z}/10\mathbb{Z}$, correcting the $g = 2$ case of [BV12].

The classes of the divisors T and D have been previously computed [DP15, CL21a]. Using these computations, we determine the integral Picard groups of the simply branched Hurwitz spaces. They are, of course, torsion.

Corollary 3.1.3. *For $g \geq 2$, the integral Picard groups of the simply branched Hurwitz stacks are as follows:*

1. *We have*

$$\mathrm{Pic}(\mathscr{H}^s_{3,g}) = \begin{cases} \mathbb{Z}/2\mathbb{Z} & \text{if } g = 2 \\ \mathbb{Z}/(4g+6)\mathbb{Z} \oplus \mathbb{Z}/3\mathbb{Z} & \text{if } g \geq 3 \text{ odd} \\ \mathbb{Z}/(8g+12)\mathbb{Z} \oplus \mathbb{Z}/3\mathbb{Z} & \text{if } g \geq 3 \text{ even} \end{cases}$$

2. We have

$$\operatorname{Pic}(\mathscr{H}^s_{4,g}) = \begin{cases} \mathbb{Z}/18\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} & \text{if } g = 2 \\ \mathbb{Z}/(8g+20)\mathbb{Z} \oplus \mathbb{Z}/12\mathbb{Z} & \text{if } g \geq 3 \text{ odd} \\ \mathbb{Z}/(4g+10)\mathbb{Z} \oplus \mathbb{Z}/12\mathbb{Z} & \text{if } g \geq 3 \text{ even.} \end{cases}$$

3. We have

$$\operatorname{Pic}(\mathscr{H}^s_{5,g}) = \begin{cases} \mathbb{Z}/44\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} & \text{if } g = 2 \\ \mathbb{Z}/(4g+14)\mathbb{Z} \oplus \mathbb{Z}/12\mathbb{Z} & \text{if } g \geq 3 \text{ odd} \\ \mathbb{Z}/(8g+28)\mathbb{Z} \oplus \mathbb{Z}/12\mathbb{Z} & \text{if } g \geq 3 \text{ even.} \end{cases}$$

The paper is structured as follows. In Section 3.2, we define the stacks $\mathscr{H}_{k,g}$ and their closely related counterparts $\mathcal{H}_{k,g}$. When $k \leq 5$, these stacks have a nice relationship with stacks parametrizing pairs of vector bundles on $\mathbb{P}^1$-fibrations, respectively pairs of vector bundles on $\mathbb{P}^1$-bundles (the former may not have a relative degree 1 line bundle; this distinction is important for results with integral coefficients). We construct these stacks of vector bundles on $\mathbb{P}^1$-fibrations, respectively $\mathbb{P}^1$-bundles, in Section 3.3, compute their integral Picard groups and explain how they are related to each other. In Section 3.4, we calculate $\operatorname{Pic}(\mathscr{H}_{3,g})$, which is originally due to Bolognesi–Vistoli, in a different way. This new perspective is used to prove Corollary 3.1.3(1). Along the way, we also obtain results in genus 2 that will be useful in Sections 3.5 and 3.6. In Section 3.5, we prove Theorem 3.1.1(1) and from it Corollary 3.1.3(2). In Section 3.6, we prove Theorem 3.1.1(2) and Corollary 3.1.3(3).

3.2 Hurwitz Stacks

We say a morphism $P \to S$ is a $\mathbb{P}^1$-fibration if it is a flat, proper, finitely presented morphism of schemes whose geometric fibers are isomorphic to $\mathbb{P}^1$. We define the unparametrized Hurwitz stack $\mathscr{H}_{k,g}$ of degree k, genus g covers of $\mathbb{P}^1$ to be the stack whose objects over a scheme S are of the form $(C \to P \to S)$ where $P \to S$ is a $\mathbb{P}^1$-fibration, $C \to P$ is a finite, flat, finitely presented morphism of constant degree k, and the composition $C \to S$ is smooth with geometrically connected fibers. We do not impose the condition that our covers $C \to \mathbb{P}^1$ be simply branched. In the case $k = 3$, $\mathscr{H}_{3,g}$ is the stack $\mathcal{T}_g$ from [BV12].

The parametrized Hurwitz scheme $\mathscr{H}^{\dagger}_{k,g}$ is defined similarly, except $P \to S$ is replaced by $\mathbb{P}^1_S$. Therefore, the unparametrized Hurwitz stack is the PGL_2 quotient of the parametrized Hurwitz scheme. There is also a natural action of SL_2 on $\mathscr{H}^{\dagger}_{k,g}$ (via $\mathrm{SL}_2 \subset \mathrm{GL}_2 \to \mathrm{PGL}_2$).

We shall use script font $\mathscr{H}_{k,g} := [\mathscr{H}^{\dagger}_{k,g} / \mathrm{PGL}_2]$ for the PGL_2 quotient,
and caligraphic font $\mathcal{H}_{k,g} := [\mathscr{H}^{\dagger}_{k,g} / \mathrm{SL}_2]$ for the SL_2 quotient.

Explicitly, the SL_2 quotient $\mathcal{H}_{k,g}$ is the stack whose objects over a scheme S are families $(C \to P \to S)$ where $P = \mathbb{P}V \to S$ is the projectivization of a rank 2 vector bundle V with trivial determinant, $C \to P$ is a finite flat finitely presented morphism of constant degree k, and the composition $C \to S$ has smooth fibers of genus g. The benefit of working with $\mathcal{H}_{k,g}$ is that the SL_2 quotient is equipped with a universal $\mathbb{P}^1$-bundle $\mathcal{P} \to \mathcal{H}_{k,g}$ that has a relative degree one line bundle $\mathcal{O}_{\mathcal{P}}(1)$ (a $\mathbb{P}^1$-fibration does not).

The Hurwitz stacks come with universal diagrams

$$\begin{tikzcd} \mathscr{C} \arrow[r, "\alpha"] \arrow[dr, "f"'] & \mathscr{P} \arrow[d, "\pi"] \\ & \mathscr{H}_{k,g}, \end{tikzcd} \tag{3.2.1}$$

where $\mathscr{C} \to \mathscr{H}_{k,g}$ is the universal curve, $\mathscr{C} \to \mathscr{P}$ is the universal degree k cover, and $\mathscr{P} \to \mathscr{H}_{k,g}$ is the universal $\mathbb{P}^1$-fibration. One can also form the analogous diagram for $\mathcal{H}_{k,g}$. We set

$$\lambda := c_1(f_*\omega_f),$$

which is pulled back from the moduli space of curves $\mathcal{M}_g$.

3.3 Stacks of vector bundles on $\mathbb{P}^1$

In this section, we discuss these stacks of vector bundles on $\mathbb{P}^1$-fibrations and $\mathbb{P}^1$-bundles, and compute their Picard groups.

Definition 3.3.1. Let r, d be nonnegative integers.

1. The objects of $\mathcal{V}_{r,d}$ are pairs $(P \to S, E)$ where $P \to S$ is a $\mathbb{P}^1$-fibration over a k-scheme S and E is a locally free sheaf of rank r on P whose restriction to each of the fibers of $P \to S$ is globally generated of degree d. A morphism between objects $(P \to S, E)$ and $(P' \to S', E')$ is a Cartesian diagram

$$\begin{array}{ccc} P' & \xrightarrow{F} & P \\ \downarrow & & \downarrow \\ S' & \longrightarrow & S \end{array}$$

together with an isomorphism $\phi : F^*E \to E'$.

2. The objects of $\mathcal{V}_{r,d}$ are triples (S, V, E) where S is a k-scheme, V is a rank 2 vector bundle on S with trivial determinant, and E is a rank r vector bundle on $\mathbb{P}V$ whose restrictions to the fibers of $\mathbb{P}V \to S$ are globally generated of degree d. A morphism between objects (S, V, E) and (S', V', E') is a Cartesian diagram

$$\begin{array}{ccc} \mathbb{P}V' & \xrightarrow{F} & \mathbb{P}V \\ \downarrow & & \downarrow \\ S' & \longrightarrow & S \end{array}$$

together with an isomorphism $\phi : F^*E \to E'$.

Bolognesi–Vistoli [BV12] gave a presentation for $\mathcal{V}_{r,d}$ as a quotient stack, which we briefly summarize here. Let $M_{r,d}$ be the affine space that represents the functor which sends a scheme S to the set of matrices of size $(r+d)\times d$ with entries in $H^0(\mathbb{P}^1_S, \mathcal{O}_{\mathbb{P}^1_S}(1))$. We can identify such a matrix with the associated map

$$\mathcal{O}_{\mathbb{P}^1_S}(-1)^d \to \mathcal{O}_{\mathbb{P}^1_S}^{r+d}.$$

Let $\Omega_{r,d} \subset M_{r,d}$ denote the open subscheme parametrizing injective maps with locally free cokernel. The group GL_d acts on $M_{r,d}$ by multiplication on the left, GL_{r+d} acts by multiplication on the right, and GL_2 acts by change of coordinates on $H^0(\mathbb{P}^1_S, \mathcal{O}_{\mathbb{P}^1_S}(1))$. These actions commute with each other and leave $\Omega_{r,d}$ invariant, and hence $\mathrm{GL}_d \times \mathrm{GL}_{r+d} \times \mathrm{GL}_2$ acts on $\Omega_{r,d}$. There is a copy of $\mathbb{G}_m$ inside of $\mathrm{GL}_d \times \mathrm{GL}_{r+d} \times \mathrm{GL}_2$ embedded by $t \mapsto (t\,\mathrm{Id}_d, \mathrm{Id}_{r+d}, t^{-1}\,\mathrm{Id}_2)$. The image T acts trivially on $M_{r,d}$ and so we can define an action of the quotient

$$\Gamma_{r,d} := \mathrm{GL}_d \times \mathrm{GL}_{r+d} \times \mathrm{GL}_2 / T$$

on $\Omega_{r,d}$. There is an exact sequence

$$1 \to \mathrm{GL}_d \times \mathrm{GL}_{r+d} \to \Gamma_{r,d} \to \mathrm{PGL}_2 \to 1,$$

where the map $\Gamma_{r,d} \to \mathrm{PGL}_2$ is induced by the projection of $\mathrm{GL}_d \times \mathrm{GL}_{r+d} \times \mathrm{GL}_2 \to \mathrm{GL}_2$.

Theorem 3.3.2 (Bolognesi–Vistoli [BV12], Theorem 4.4)**.** *There is an isomorphism of fibered categories*

$$\mathcal{V}_{r,d} \cong [\Omega_{r,d}/\Gamma_{r,d}.]$$

A slight modification of the argument in Bolognesi–Vistoli gives a quotient structure for $\mathcal{V}_{r,d}$, which we have utilized in our previous work [CL21d, CL21a].

Proposition 3.3.3. *There is an isomorphism of fibered categories*

$$\mathcal{V}_{r,d} \cong [\Omega_{r,d}/\,\mathrm{GL}_d \times \mathrm{GL}_{r+d} \times \mathrm{SL}_2].$$

Proof. The proof is the same as in [BV12, Theorem 4.4], except that instead of taking $P \to S$ a $\mathbb{P}^1$-fibration in the definition of the various stacks, we take $P = \mathbb{P}V \to S$ where V is a rank 2 vector bundle with trivial determinant. $\square$

To parametrize the linear algebraic data associated to a low degree cover of $\mathbb{P}^1$, we will need to construct stacks parametrizing pairs of vector bundles on $\mathbb{P}^1$. These stacks are products of the form $\mathcal{V}_{r,d} \times_{\mathrm{BSL}_2} \mathcal{V}_{s,e}$, which parametrize a pair of vector bundles on the same $\mathbb{P}^1$-bundle, or $\mathcal{V}_{r,d} \times_{\mathrm{BPGL}_2} \mathcal{V}_{s,e}$, which parametrize a pair of vector bundles on the same $\mathbb{P}^1$-fibration.

Let $G_{r,d,s,e} := \mathrm{GL}_d \times \mathrm{GL}_{r+d} \times \mathrm{GL}_e \times \mathrm{GL}_{s+e}$. The group $G_{r,d,s,e} \times \mathrm{SL}_2$ acts on $M_{r,d}$ via the projection $G_{r,d,s,e} \times \mathrm{SL}_2 \to \mathrm{GL}_d \times \mathrm{GL}_{r+d} \times \mathrm{SL}_2$; and similarly on $M_{s,e}$ via the projection $G_{r,d,s,e} \times \mathrm{SL}_2 \to \mathrm{GL}_e \times \mathrm{GL}_{s+e} \times \mathrm{SL}_2$. By Proposition 3.3.3, we have

$$\mathcal{V}_{r,d} \times_{\mathrm{BSL}_2} \mathcal{V}_{s,e} = [\Omega_{r,d} \times \Omega_{s,e} / G_{r,d,s,e} \times \mathrm{SL}_2]. \tag{3.3.1}$$

Let T_d and T_{r+d} denote the universal vector bundles on BGL_d and BGL_{r+d}; similarly, let S_e and S_{s+e} be the universal vector bundles on BGL_e and BGL_{s+e}. The integral Chow ring of $\mathrm{B}(G_{r,d,s,e} \times \mathrm{SL}_2)$ is the free $\mathbb{Z}$-algebra on the Chern classes of $T_d, T_{r+d}, S_e, S_{s+e}$, together with the universal second Chern class c_2 on BSL_2. Let us denote these classes by

$$\begin{aligned} t_i = c_i(T_d) \qquad &\text{and} \qquad u_i = c_i(T_{r+d}) \\ v_i = c_i(S_e) \qquad &\text{and} \qquad w_i = c_i(S_{s+e}). \end{aligned}$$

Since $\Omega_{r,d} \times \Omega_{s,e}$ is open inside the affine space $M_{r,d} \times M_{s,e}$, the excision and homotopy properties imply

$$\mathrm{Pic}(\mathcal{V}_{r,d} \times_{\mathrm{BSL}_2} \mathcal{V}_{s,e}) \text{ is generated by the restrictions of } t_1, u_1, v_1, w_1. \tag{3.3.2}$$

We now identify the restrictions of the tautological bundles T_d and T_{d+r} in terms of the universal rank r, degree d vector bundle on $\mathbb{P}^1$. Let $\pi : \mathcal{P} \to \mathcal{V}_{r,d}$ be the universal $\mathbb{P}^1$-bundle. We write $z := c_1(\mathcal{O}_{\mathcal{P}}(1)) \in A^1(\mathcal{P})$. We have $c_2 = c_2(\pi_*\mathcal{O}_{\mathcal{P}}(1)) \in A^2(\mathcal{V}_{r,d})$, the universal second Chern class, pulled back via the natural map $\mathcal{V}_{r,d} \to \mathrm{BSL}_2$). Note that $c_1(\pi_*\mathcal{O}_{\mathcal{P}}(1)) = 0$, so by the projective bundle formula

$$A^*(\mathcal{P}) = A^*(\mathcal{V}_{r,d})[z]/(z^2 + \pi^*c_2).$$

Let $\mathcal{E}$ be the universal rank r, degree d vector bundle on $\mathcal{P}$. The Chern classes of $\mathcal{E}$ may thus be written as

$$c_i(\mathcal{E}) = \pi^*a_i + (\pi^*a_i')z \qquad \text{where} \qquad a_i \in A^i(\mathcal{V}_{r,d}), \quad a_i' \in A^{i-1}(\mathcal{V}_{r,d}).$$

Note that $a_1' = d$. Let $\gamma : \mathcal{V}_{r,d} \times_{\mathrm{BSL}_2} \mathcal{V}_{s,e} \to B(G_{r,d,s,e} \times \mathrm{SL}_2)$ be the structure map. Then by [Lar21b, Lemma 3.2] (noting that $\det(\pi_*\mathcal{O}_{\mathcal{P}}(1))$ is trivial), we have

$$\gamma^*T_d = \pi_*\mathcal{E}(-1) \qquad \text{and} \qquad \gamma^*T_{r+d} = \pi_*\mathcal{E}. \tag{3.3.3}$$

Since $R^1\pi_*\mathcal{E}(-1)$ and $R^1\pi_*\mathcal{E}$ are zero, Grothendieck–Riemann–Roch says that the Chern characters of $\pi_*\mathcal{E}(-1)$ and $\pi_*\mathcal{E}$ are push forwards by π of polynomials in the $c_i(\mathcal{E})$ and z. The push forward of such a polynomial is a polynomial in the a_i, a_i' and c_2. In particular, the restrictions of t_1 and u_1 to $\mathrm{Pic}(\mathcal{V}_{r,d})$ are linear combinations of a_1 and a_2'. We calculate this explicitly in the following example.

Example 3.3.4 (First Chern classes)**.** Let $T_\pi = \mathcal{O}_{\mathcal{P}}(2)$ denote the relative tangent bundle of $\pi : \mathcal{P} \to \mathcal{V}_{r,d}$, so the the relative Todd class is $\mathrm{Td}_\pi = 1 + \frac{1}{2}c_1(T_\pi) + \ldots = 1 + z + \ldots$.

Using Equation (1.4.3), and then Grothedieck–Riemann–Roch, we have that on $\mathcal{V}_{r,d}$,

$$\begin{aligned}
t_1 &= c_1(\pi_*\mathcal{E}(-1)) = \mathrm{ch}_1(\pi_*\mathcal{E}(-1)) = [\pi_*(\mathrm{ch}(\mathcal{E}) \cdot \mathrm{ch}(\mathcal{O}_{\mathcal{P}}(-1)) \cdot \mathrm{Td}_\pi)]_1 \\
&= [\pi_*(\mathrm{ch}(\mathcal{E}) \cdot (1-z) \cdot (1+z))]_1 = \pi_*(\mathrm{ch}_2(\mathcal{E})) = \pi_*\left(\frac{1}{2}c_1(\mathcal{E})^2 - c_2(\mathcal{E})\right) \\
&= da_1 - a_2' \\
u_1 &= c_1(\pi_*\mathcal{E}) = \mathrm{ch}_1(\pi_*\mathcal{E}) = [\pi_*(\mathrm{ch}(\mathcal{E}) \cdot \mathrm{Td}_\pi)]_1 = [\pi_*(\mathrm{ch}(\mathcal{E}) \cdot (1+z))]_1 \\
&= \pi_*(\mathrm{ch}_2(\mathcal{E}) + \mathrm{ch}_1(\mathcal{E})z) = (da_1 - a_2') + a_1 \\
&= (d+1)a_1 - a_2'.
\end{aligned}$$

It follows that $a_1 = u_1 - t_1$ and $a_2' = du_1 - (d+1)t_1$.

In Equation (1.4.1), we described $\mathcal{V}_{r,d} \times_{\mathrm{BSL}_2} \mathcal{V}_{s,e}$ as a quotient. To similarly understand the moduli space of pairs of vector bundles on a $\mathbb{P}^1$-*fibration*, we need the "pair" version of $\Gamma_{r,d}$ and of Theorem 3.3.2. Precisely, let us define $\Gamma_{r,d,s,e}$ to be the quotient of $G_{r,d,s,e} \times \mathrm{GL}_2$ by $t \mapsto (t\,\mathrm{Id}_d, \mathrm{Id}_{r+d}, t\,\mathrm{Id}_e, I_{s+e}, t^{-1}\,\mathrm{Id}_2)$. Then, we have

$$\mathcal{V}_{r,d} \times_{\mathrm{BPGL}_2} \mathcal{V}_{s,e} = [\Omega_{r,d} \times \Omega_{s,e}/\Gamma_{r,d,s,e}].$$

Considering the commutative diagram

$$\begin{tikzcd}
1 \arrow[r] & \mu_2 \arrow[r] \arrow[d] & G_{r,d,s,e} \times \mathrm{SL}_2 \arrow[r] \arrow[d] & \bullet \arrow[r] \arrow[d, "\sim"] & 1 \\
1 \arrow[r] & \mathbb{G}_m \arrow[r] \arrow[d, "(-)^2"'] & G_{r,d,s,e} \times \mathrm{GL}_2 \arrow[r] \arrow[d, "\det"] & \Gamma_{r,d,s,e} \arrow[r] & 1 \\
& \mathbb{G}_m \arrow[r, "\mathrm{id}"'] & \mathbb{G}_m & &
\end{tikzcd}$$

we see by the snake lemma that $\Gamma_{r,d,s,e}$ is a μ_2 quotient of $G_{r,d,s,e} \times \mathrm{SL}_2$.

Let us assume $r, s > 1$, so that the complement of $\Omega_{r,d} \times \Omega_{s,e} \subset M_{r,d} \times M_{s,e}$ has codimension at least 2. In particular, by the excision and homotopy properties, we have natural identifications

$$\mathrm{Pic}(\mathcal{V}_{r,d} \times_{\mathrm{BSL}_2} \mathcal{V}_{s,e}) = \mathrm{Pic}(\mathrm{B}(G_{r,d,s,e} \times \mathrm{SL}_2)),$$

and

$$\mathrm{Pic}(\mathcal{V}_{r,d} \times_{\mathrm{BPGL}_2} \mathcal{V}_{s,e}) = \mathrm{Pic}(\mathrm{B}\Gamma_{r,d,s,e}).$$

The group $\mathrm{Pic}(\mathrm{B}(G_{r,d,s,e} \times \mathrm{SL}_2))$ is the free $\mathbb{Z}$-module generated by t_1, u_1, v_1, w_1 (see (1.4.2)). Using Example 3.3.4, we see that the classes a_1, a_2', b_1, b_2' also freely generate $\mathrm{Pic}(\mathcal{V}_{r,d} \times_{\mathrm{BSL}_2} \mathcal{V}_{s,e})$.

Lemma 3.3.5. *The natural map $\mathcal{V}_{r,d} \times_{\mathrm{BSL}_2} \mathcal{V}_{s,e} \to \mathcal{V}_{r,d} \times_{\mathrm{BPGL}_2} \mathcal{V}_{s,e}$ induces an inclusion*

$$\mathrm{Pic}(\mathcal{V}_{r,d} \times_{\mathrm{BPGL}_2} \mathcal{V}_{s,e}) \hookrightarrow \mathrm{Pic}(\mathcal{V}_{r,d} \times_{\mathrm{BSL}_2} \mathcal{V}_{s,e}),$$

whose image is the subgroup generated by

$$\begin{cases} t_1, u_1, v_1, w_1 & \textit{if } d, e \textit{ both even} \\ 2t_1, u_1, v_1, w_1 & \textit{if } d \textit{ odd and } e \textit{ even} \\ t_1, u_1, 2v_1, w_1 & \textit{if } d \textit{ even and } e \textit{ odd} \\ t_1 - v_1, 2t_1, u_1, w_1 & \textit{if } d, e \textit{ both odd,} \end{cases} \tag{3.3.4}$$

or equivalently by

$$\begin{cases} a_1, a_2', b_1, b_2' & \textit{if } d, e \textit{ both even} \\ 2a_1, a_2', b_1, b_2' & \textit{if } d \textit{ odd and } e \textit{ even} \\ a_1, a_2', 2b_1, b_2' & \textit{if } d \textit{ even and } e \textit{ odd} \\ a_1 - b_1, 2a_1, a_2', b_2' & \textit{if } d, e \textit{ both odd.} \end{cases} \tag{3.3.5}$$

Proof. Recall that $\mathrm{Pic}(\mathrm{B}G)$ is naturally identified with the character group of G because it is identified with Mumford's functorial Picard group. The exact sequence of groups

$$0 \to \mu_2 \to G_{r,d,s,e} \times \mathrm{SL}_2 \to \Gamma_{r,d,s,e} \to 0$$

induces a left exact sequence

$$0 \to \mathrm{Pic}(\mathrm{B}\Gamma_{r,d,s,e}) \to \mathrm{Pic}(\mathrm{B}(G_{r,d,s,e} \times \mathrm{SL}_2)) \to \mathrm{Pic}(\mathrm{B}\mu_2). \tag{3.3.6}$$

The Picard group $\mathrm{Pic}(\mathrm{B}\mu_2)$ is isomorphic to $\mathbb{Z}/2\mathbb{Z}$. Let h be a generator of $\mathrm{Pic}(\mathrm{B}\mu_2)$. Recall that the map $\mu_2 \to G_{r,d,s,e} \times \mathrm{SL}_2$ sends -1 to $(-\mathrm{Id}_d, \mathrm{Id}_{r+d}, -\mathrm{Id}_e, \mathrm{Id}_{s+e}, -\mathrm{Id}_2)$. The generator $t_1 \in \mathrm{Pic}(\mathrm{B}(G_{r,d,s,e} \times \mathrm{SL}_2))$ corresponds to the determinant of the rank d matrix. Thus, the right-hand map in (3.3.6) sends t_1 to dh. Similarly, u_1 and w_1 are sent to zero, and v_1 to eh. The kernel is thus the subgroup generated by the classes listed in (3.3.4).

The translation between (3.3.4) and (3.3.5) follows from Example 3.3.4. We explain the case d, e both odd, the other cases being similar but simpler. Since d and e are both odd, the following change of basis matrix has integer coefficients

$$\begin{pmatrix} t_1 - v_1 \\ 2t_1 \\ u_1 \\ w_1 \end{pmatrix} = \begin{pmatrix} e & \frac{d-e}{2} & -1 & 1 \\ 0 & d & -2 & 0 \\ 0 & \frac{d+1}{2} & -1 & 0 \\ -(e+1) & \frac{e+1}{2} & 0 & -1 \end{pmatrix} \begin{pmatrix} a_1 - b_1 \\ 2a_1 \\ a_2' \\ b_2' \end{pmatrix}.$$

The determinant of the 4×4 matrix above is 1 so the entries of the two column vectors generate the same subgroup with $\mathbb{Z}$-coefficients. $\square$

Lemma 3.3.6. *Let $X' \subset X$ be an open substack. Given a smooth map $f : Y \to X$, let $Y' \subset Y$ be the preimage of X'. If $\mathrm{Pic}(X) \to \mathrm{Pic}(Y)$ is injective, then $\mathrm{Pic}(X') \to \mathrm{Pic}(Y')$ is injective.*

Proof. It suffices to treat the case when $X' \subset X$ is the complement of an irreducible divisor D. (Removing any component of codimension two or more from X does not change $\mathrm{Pic}(X)$; if X' is the complement of a reducible divisor, then we just apply the irreducible case to each component in turn.)

Because $f : Y \to X$ is smooth, $Y' \subset Y$ is the complement of the irreducible divisor $f^{-1}(D)$, which has class $f^*[D]$. Let $\langle [D] \rangle$ denote the subgroup of $\mathrm{Pic}(X)$ generated by the fundamental class of D and similarly for $\langle f^*[D] \rangle$ inside $\mathrm{Pic}(Y)$. We therefore have a diagram of exact sequences where the left vertical map is surjective.

$$\begin{tikzcd}
0 \arrow[r] & \langle [D] \rangle \arrow[r] \arrow[d] & \mathrm{Pic}(X) \arrow[r] \arrow[d, hook] & \mathrm{Pic}(X') \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \langle f^*[D] \rangle \arrow[r] & \mathrm{Pic}(Y) \arrow[r] & \mathrm{Pic}(Y') \arrow[r] & 0.
\end{tikzcd}$$

The result now follows from the snake lemma. $\square$

We shall be applying Lemma 3.3.6 in the context of a smooth map $Y \to X$ which is induced by a base change $\mathrm{BSL}_2 \to \mathrm{BPGL}_2$. The basic idea is that injectivity of a certain map of Picard groups will allow us to argue that our previous calculations determining relations in $\mathrm{Pic}(\mathcal{H}_{k,g})$ actually hold in $\mathrm{Pic}(\mathscr{H}_{k,g})$ with the "same formulas." We then just need to understand which classes are integral, which will be deduced from Lemma 3.3.5.

3.3.1 Construction of the base stacks

To keep track of the integrality conditions arising from Lemma 3.3.5, we shall find it useful to use the quantity

$$\epsilon := \epsilon_{k,g} = \begin{cases} 1 & \text{if } g+k-1 \text{ is even} \\ 2 & \text{if } g+k-1 \text{ is odd,} \end{cases} \tag{3.3.7}$$

which keeps track of the parity of the degree of certain vector bundles on $\mathbb{P}^1$ we associate to degree k, genus g covers. We shall often drop the subscript when k and g are understood.

Below, we introduce the "base stacks" $\mathscr{B}_{k,g}$ that parametrize the bundles on $\mathbb{P}^1$ we shall associate to degree k, genus g covers in Sections 3.4, 3.5, and 3.6.

Degree 3

We set $\mathscr{B}_{3,g} := \mathcal{V}_{2,g+2}$. Then, by Lemma 3.3.5, we have

$$\operatorname{Pic}(\mathscr{B}_{3,g}) = \mathbb{Z}(\epsilon a_1) \oplus \mathbb{Z}a_2', \tag{3.3.8}$$

where ϵ is as in (3.3.7).

Degree 4

First let us recall a standard, but very useful fact.

Lemma 3.3.7. *Suppose that $X \to Y$ is a $\mathbb{G}_m$-torsor with associated line bundle $\mathcal{L}$. Then,*

$$\operatorname{Pic}(X) \cong \operatorname{Pic}(Y)/\langle c_1(\mathcal{L})\rangle.$$

Proof. Under the correspondence between $\mathbb{G}_m$-torsors and line bundles, X is the complement of the zero section in the total space of the line bundle $\mathcal{L} \to Y$. We thus have the exact sequence

$$\operatorname{Pic}(Y) \to \operatorname{Pic}(\mathcal{L}) \to \operatorname{Pic}(X) \to 0.$$

After identifying $\operatorname{Pic}(\mathcal{L})$ with $\operatorname{Pic}(Y)$, the first map in the sequence is given by multiplication by the first Chern class $c_1(\mathcal{L})$, and the result follows. $\square$

Let $\pi : \mathscr{P} \to \mathcal{V}_{3,g+3} \times_{\mathrm{BPGL}_2} \mathcal{V}_{2,g+3}$ be the universal $\mathbb{P}^1$-fibration, equipped with universal bundles $\mathscr{E}$ of rank 3 and $\mathscr{F}$ of rank 2. Define $\mathscr{B}_{4,g}$ be the $\mathbb{G}_m$-torsor over $\mathcal{V}_{3,g+3} \times_{\mathrm{BPGL}_2} \mathcal{V}_{2,g+3}$ associated to $\pi_*(\det \mathscr{E} \otimes \det \mathscr{F}^\vee)$. This push forward is a line bundle by cohomology and base change, and has class $c_1(\pi_*(\det \mathscr{E} \otimes \det \mathscr{F}^\vee)) = a_1 - b_1 \in$

$\mathrm{Pic}(\mathcal{V}_{3,g+3} \times_{\mathrm{BPGL}_2} \mathcal{V}_{2,g+3})$. The stack $\mathscr{B}_{4,g}$ parametrizes pairs of vector bundles (E, F) on $\mathbb{P}^1$-fibrations together with an isomorphism of their determinants.

Combining Lemma 3.3.7 and Lemma 3.3.5 (where we are necessarily either in the first or last case of (3.3.5)), we find

$$\mathrm{Pic}(\mathscr{B}_{4,g}) = \mathbb{Z}(\epsilon a_1) \oplus \mathbb{Z}a_2' \oplus \mathbb{Z}b_2', \tag{3.3.9}$$

where again, ϵ is as in (3.3.7).

Degree 5

Let $\pi : \mathscr{P} \to \mathcal{V}_{4,g+4} \times_{\mathrm{BPGL}_2} \mathcal{V}_{5,2g+8}$ be the universal $\mathbb{P}^1$-fibration, equipped with universal bundles $\mathscr{E}$ of rank 4, degree $g+4$, and $\mathscr{F}$ of rank 5 degree $2g+8$. Define $\mathscr{B}_{5,g}$ to be the $\mathbb{G}_m$-torsor over $\mathcal{V}_{4,g+4} \times_{\mathrm{BPGL}_2} \mathcal{V}_{5,2g+8}$ associated to the bundle $\pi_*(\det \mathscr{E}^{\otimes 2} \otimes \det \mathscr{F}^\vee)$, which is a line bundle by cohomology and base change. It has first Chern class $2a_1 - b_1$. The stack $\mathscr{B}_{5,g}$ parametrizes pairs of vector bundles (E, F) on a $\mathbb{P}^1$-fibration together with an isomorphism between $(\det E)^{\otimes 2}$ and $\det F$. By Lemmas 3.3.5 and 3.3.7,

$$\mathrm{Pic}(\mathscr{B}_{5,g}) = \mathbb{Z}(\epsilon a_1) \oplus \mathbb{Z}a_2' \oplus \mathbb{Z}b_2'. \tag{3.3.10}$$

3.4 Trigonal

The Picard group of $\mathscr{H}_{3,g}$ was computed by Bolognesi–Vistoli [BV12, Theorem 1.1] when $g \geq 3$. As a warm-up for $k = 4, 5$, we give a slightly different proof of their result; we also treat the case $g = 2$, and compute $\mathrm{Pic}(\mathscr{H}_{3,2})$.

Let us recall the linear algebraic data associated to a degree 3 cover, as developed by Miranda [Mir85], and later Casnati–Ekedahl [CE96]. For more details in our context see also [BV12] and [CL21d, Section 3.1]. Given a degree 3 cover $\alpha : C \to \mathbb{P}^1$, we associate a rank 2 vector bundle $E_\alpha := (\alpha_* \mathcal{O}_C / \mathcal{O}_{\mathbb{P}^1})^\vee$ on $\mathbb{P}^1$. The cover naturally factors through an embedding $C \subset \mathbb{P}E_\alpha^\vee \to \mathbb{P}^1$ and $C \subset \mathbb{P}E_\alpha^\vee$ is defined as the zero locus of a section

$$\eta_\alpha \in H^0(\mathbb{P}^1, \det E_\alpha^\vee \otimes \operatorname{Sym}^3 E_\alpha) \cong H^0(\mathbb{P}E_\alpha, \det E_\alpha^\vee \otimes \mathcal{O}_{\mathbb{P}E_\alpha}(3)).$$

The association of α with E_α defines a map $\mathscr{H}_{3,g} \to \mathscr{B}_{3,g} := \mathscr{V}_{2,g+2}$. Let $\pi : \mathscr{P} \to \mathscr{B}_{3,g}$ be the universal $\mathbb{P}^1$-fibration, equipped with universal rank 2 bundle $\mathscr{E}$. We define $\mathscr{B}'_{3,g}$ to be the locus where $\det \mathscr{E}^\vee \otimes \operatorname{Sym}^3 \mathscr{E}$ is globally generated on the fibers of π. Equivalently, $\mathscr{B}'_{3,g}$ is the locus where $\mathscr{E}$ has splitting type (e_1, e_2) for $e_1 \leq e_2$ and $2e_1 - e_2 \geq 0$ on fibers of π.

Let $\mathscr{X}'_{3,g}$ be the total space of the vector bundle $\pi_*(\det \mathscr{E}^\vee \otimes \operatorname{Sym}^3 \mathscr{E})|_{\mathscr{B}'_{3,g}}$, which is locally free by the theorem on cohomology and base change, on $\mathscr{B}'_{3,g}$. Arguing exactly as in [CL21d, Lemma 5.1], one sees that the association of $\alpha : C \to \mathbb{P}^1$ with (E_α, η_α) defines an open embedding of $\mathscr{H}_{3,g}$ into $\mathscr{X}'_{3,g}$. Let $\mathscr{D}_{3,g} := \mathscr{X}'_{3,g} \smallsetminus \mathscr{H}_{3,g}$ be the closed complement.

At this point, we have described stacks and morphisms

$$\mathscr{D}_{3,g} \to \mathscr{X}'_{3,g} \to \mathscr{B}'_{3,g} \to \mathscr{B}_{3,g} = \mathscr{V}_{2,g+3} \to \mathrm{BPGL}_2,$$

Base changing by $\mathrm{BSL}_2 \to \mathrm{BPGL}_2$, we obtain the stacks and morphisms studied in [CL21a, Section 4.2].

$$\Delta_{3,g} \to \mathcal{X}'_{3,g} \to \mathcal{B}'_{3,g} \to \mathcal{B}_{3,g} = \mathcal{V}_{2,g+2} \to \mathrm{BSL}_2 .$$

In [CL21a, Equation 4.5], we showed that $\Delta_{3,g}$ is irreducible with fundamental class given by

$$[\Delta_{3,g}] = (8g+12)a_1 - 9a_2' \in \operatorname{Pic}(\mathcal{X}'_{3,g}) \cong \operatorname{Pic}(\mathcal{B}'_{3,g}).$$

Recall that $\operatorname{Pic}(\mathcal{V}_{2,g+2}) \to \operatorname{Pic}(\mathcal{V}_{2,g+2})$ is injective (Lemma 3.3.5). Applying Lemma 3.3.6, we also see that $\operatorname{Pic}(\mathscr{B}'_{3,g}) \to \operatorname{Pic}(\mathcal{B}'_{3,g})$ is injective, and then so too is $\operatorname{Pic}(\mathscr{X}'_{3,g}) \to$

$\mathrm{Pic}(\mathcal{X}'_{3,g})$. Since $\mathscr{D}_{3,g}$ pulls back to $\Delta_{3,g}$, it follows that $\mathscr{D}_{3,g}$ is irreducible of class

$$[\mathscr{D}_{3,g}] = \frac{8g+12}{\epsilon}(\epsilon a_1) - 9a_2' \in \mathrm{Pic}(\mathscr{X}'_{3,g}) \cong \mathrm{Pic}(\mathscr{B}'_{3,g}).$$

where ϵ is as in (3.3.7). (Recall ϵ is always 1 or 2 so the coefficient above is an integer.)

Now, by excision, we have

$$\mathrm{Pic}(\mathscr{H}_{3,g}) = \frac{\mathrm{Pic}(\mathscr{X}'_{3,g})}{\langle[\mathscr{D}_{3,g}]\rangle} = \frac{\mathrm{Pic}(\mathscr{B}'_{3,g})}{\langle[\mathscr{D}_{3,g}]\rangle}.$$

When $g > 2$, the complement of $\mathscr{B}'_{3,g} \subset \mathscr{B}_{3,g}$ has codimension at least 2 by [CL21a, p. 16]. Hence, $\mathrm{Pic}(\mathscr{B}'_{3,g}) = \mathrm{Pic}(\mathscr{B}_{3,g}) = \mathbb{Z}(\epsilon a_1) \oplus \mathbb{Z}a_2'$. Therefore, for $g > 2$, we have

$$\mathrm{Pic}(\mathscr{H}_{3,g}) = \frac{\mathbb{Z}(\epsilon a_1) \oplus \mathbb{Z}a_2'}{\langle(\frac{8g+12}{\epsilon})a_1 - 9a_2'\rangle} \cong \begin{cases} \mathbb{Z} & \text{if } g \neq 0 \pmod 3 \text{ and } g \neq 2 \\ \mathbb{Z} \oplus \mathbb{Z}/3\mathbb{Z} & \text{if } g = 0 \pmod 3 \text{ and } g \neq 3 \pmod 9 \\ \mathbb{Z} \oplus \mathbb{Z}/9\mathbb{Z} & \text{if } g = 3 \pmod 9. \end{cases}$$

Meahwhile, when $g = 2$, the complement of $\mathscr{B}'_{3,g} \subset \mathscr{B}_{3,g}$ is an irreducible divisor corresponding to the locus where the universal bundle $\mathscr{E}$ has splitting type $(1,3)$ on the fibers of $\mathscr{P} \to \mathscr{B}_{3,g}$. As found in the proof of [CL21d, Lemma 4.3], the (pullback to $\mathrm{Pic}(\mathcal{B}_{3,g})$ of the) class of this splitting locus is

$$s_{1,3} = a_2' - 2a_1 \in \mathrm{Pic}(\mathscr{B}_{3,g}) \subseteq \mathrm{Pic}(\mathcal{B}_{3,g}). \tag{3.4.1}$$

Hence, noting that $\epsilon = 1$ when $g = 2$, we have

$$\mathrm{Pic}(\mathscr{H}_{3,2}) = \frac{\mathbb{Z}a_1 \oplus \mathbb{Z}a_2'}{\langle 28a_1 - 9a_2', a_2' - 2a_1\rangle} \cong \mathbb{Z}/10\mathbb{Z}.$$

3.4.1 Generating line bundles

One natural class on $\mathscr{H}_{3,g}$ is $\lambda := c_1(f_*\omega_f)$ (which is pulled back from $\mathcal{M}_g$). Using Example 3.3.4, we compute that the pullback of λ to $\mathcal{H}_{3,2}$ is

$$\lambda = c_1(f_*\omega_f) = c_1(\pi_*(\alpha_*\omega_\alpha) \otimes \omega_\pi) = c_1(\pi_*\mathcal{E}(-2)) = (g+1)a_1 - a_2'.$$

(See (3.2.1) for definitions of the maps f, α, π). Note that when g is odd, the coefficient of a_1 is even, so using Lemma 3.3.5, this class lies in the subgroup $\mathrm{Pic}(\mathscr{H}_{3,g})$, as it must. In the case of genus 2, we have $\lambda = a_1 + s_{1,3}$ (where $s_{1,3}$ is the relation in (3.4.1)), so λ generates $\mathrm{Pic}(\mathscr{H}_{3,2})$. This is not surprising: in [Vis98], Vistoli computed the integral Chow ring of the stack $\mathcal{M}_2$ and found in particular that $\mathrm{Pic}(\mathcal{M}_2) = \mathbb{Z}/10\mathbb{Z}$, generated by λ. This means that the pullback map $\mathrm{Pic}(\mathcal{M}_2) \to \mathrm{Pic}(\mathscr{H}_{3,2})$ is an isomorphism.

We note here a corollary of this fact for use in the later sections of the paper. Let $\mathscr{P}ic^k$ be the universal Picard stack over $\mathcal{M}_2$. Over a scheme S, its objects are families of smooth curves $\mathcal{C} \to S$ of genus 2 together with a line bundle $\mathcal{L}$ of relative degree k on the fibers. The group $\mathbb{G}_m$ injects into the automorphism group of every object by scaling the line bundle. One can form the so-called $\mathbb{G}_m$-rigidifcation of $\mathscr{P}ic^k$, which is a stack $\mathscr{P}^k$ such that $\mathscr{P}ic^k \to \mathscr{P}^k$ is a $\mathbb{G}_m$-banded gerbe.

Corollary 3.4.1. *Let $\mathscr{P}^k \to \mathcal{M}_2$ be as above. Then the pullback map $\mathrm{Pic}(\mathcal{M}_2) \to \mathrm{Pic}(\mathscr{P}^k)$ is injective.*

Proof. There are natural isomorphisms $\mathscr{P}^k \cong \mathscr{P}^{k+2}$ (given by tensoring with the canonical), so it suffices to prove the claim for $k = 2$ and $k = 3$. When $k = 2$, the canonical line bundle gives a section $\mathcal{M}_2 \to \mathscr{P}^k$, so the pullback map must be injective.

Now consider the case $k = 3$. Every degree 3 line bundle on a genus 2 curve has 2 sections. Therefore, $\mathscr{H}_{3,2}$ is naturally an open substack inside $\mathscr{P}^3$. (It is the complement of the universal curve $\mathscr{C} \hookrightarrow \mathscr{P}^3$ embedded by summing each point with a canonical divisor.)

The isomorphism $\mathrm{Pic}(\mathcal{M}_2) \to \mathrm{Pic}(\mathscr{H}_{3,2})$ factors through $\mathrm{Pic}(\mathcal{M}_2) \to \mathrm{Pic}(\mathscr{P}^3)$, so the latter must also be injective. □

For $g \neq 2, 5$, however, λ cannot be used as a generator of $\mathrm{Pic}(\mathscr{H}_{3,g})$. (This follows from the fact that $\det \begin{pmatrix} \frac{g+1}{\epsilon} & -1 \\ \frac{8g+12}{\epsilon} & -9 \end{pmatrix} = \frac{g-3}{\epsilon}$ is not a unit, unless $g = 2$ or 5.)

To describe line bundles generating $\mathrm{Pic}(\mathscr{H}_{3,g})$, let $\mathscr{E}$ be the universal rank 2 bundle on $\pi : \mathscr{P} \to \mathscr{H}_{3,g}$ (recall $\mathscr{E} = (\alpha_* \mathcal{O}_{\mathscr{C}} / \mathcal{O}_{\mathscr{P}})^\vee$.) For $g > 2$, we can generate the free part of $\mathrm{Pic}(\mathscr{H}_{3,g})$ by

$$\mathscr{L} = \begin{cases} \pi_* \left(\det \mathscr{E} \otimes \omega_\pi^{\otimes (g+2)/2} \right) & \text{if } g \text{ even} \\ \\ \pi_* \left((\det \mathscr{E})^{\otimes 2} \otimes \omega_\pi^{\otimes (g+2)} \right) & \text{if } g \text{ odd} \end{cases} \qquad \text{which has} \qquad c_1(\mathscr{L}_1) = \epsilon a_1.$$

When $g = 0 \pmod 3$ and $g \neq 3 \pmod 9$, the torsion subgroup is generated by

$$\frac{8g+12}{3} a_1 - 3a_2' = 3\lambda - \left(\frac{g-3}{3\epsilon} \right) c_1(\mathscr{L}).$$

When $g = 3 \pmod 9$, the torsion subgroup is generated by

$$\frac{8g+12}{9} a_1 - a_2' = \lambda - \left(\frac{g-3}{9\epsilon} \right) c_1(\mathscr{L}).$$

3.4.2 Simple branching

Let $T \subset \mathscr{H}_{3,g}$ be the divisor of covers with a point of triple ramification, as defined in the introduction (see Figure 3.1). In [DP18, Proposition 2.8], Deopurkar–Patel compute the class of T. In terms of our generators, we have

$$T = (24g + 36) a_1 - 24 a_2'.$$

Proof of Corollary 3.1.3(1). Using excision, for $g \geq 3$, we have

$$\operatorname{Pic}(\mathscr{H}_{3,g}^s) = \operatorname{Pic}(\mathscr{H}_{3,g})/\langle T\rangle = \frac{\mathbb{Z}(\epsilon a_1) \oplus \mathbb{Z}a_2'}{\langle \left(\frac{8g+12}{\epsilon}\right)\epsilon a_1 - 9a_2', \left(\frac{24g+36}{\epsilon}\right)\epsilon a_1 - 24a_2'\rangle}$$

Now observe that

$$\begin{pmatrix} -8 & 3 \\ -3 & 1 \end{pmatrix} \begin{pmatrix} (8g+12)/\epsilon & -9 \\ (24g+36)/\epsilon & -24 \end{pmatrix} = \begin{pmatrix} (8g+12)/\epsilon & 0 \\ 0 & 3 \end{pmatrix}. \tag{3.4.2}$$

The matrix on the left of (3.4.2) is invertible over $\mathbb{Z}$. Thus, $\operatorname{Pic}(\mathscr{H}_{3,g}^s)$ is the sum of two cyclic groups with orders given by the diagonal entries of the matrix on the right of (3.4.2). This completes the proof for $g \geq 3$.

Finally, for $g = 2$, we already know $a_2' = 2a_1$ by (3.4.1) and that $10a_1 = 0$ in $\operatorname{Pic}(\mathscr{H}_{3,g})$. When we remove T, this creates one additional relation $0 = T = 84a_1 - 24a_2' = 36a_1$ in $\operatorname{Pic}(\mathscr{H}_{3,g}^s)$. We have $\gcd(10, 36) = 2$, so $\operatorname{Pic}(\mathscr{H}_{3,2}^s) = \mathbb{Z}/2\mathbb{Z}$. $\square$

3.5 Tetragonal

We begin by briefly recalling the linear algebraic data associated to a degree 4 cover, as developed by Casnati–Ekedahl [CE96]. For more details in our context, see [CL21d, Section 3.2]. Given a degree 4 cover $\alpha : C \to \mathbb{P}^1$, we associate two vector bundles on $\mathbb{P}^1$:

$$E_\alpha := (\alpha_*\mathcal{O}_C/\mathcal{O}_{\mathbb{P}^1})^\vee = \ker(\alpha_*\omega_\alpha \to \mathcal{O}_{\mathbb{P}^1}) \qquad \text{and} \qquad F_\alpha := \ker(\operatorname{Sym}^2 E_\alpha \to \alpha_*\omega_\alpha^{\otimes 2}).$$

The first is rank 3 and the second is rank 2. If C has genus g, then both bundles have degree $g + 3$. Geometrically, the curve C is embedded in $\gamma : \mathbb{P}E_\alpha^\vee \to \mathbb{P}^1$ as the zero locus of a section

$$\delta_\alpha \in H^0(\mathbb{P}E_\alpha^\vee, \mathcal{O}_{\mathbb{P}E_\alpha^\vee}(2) \otimes \gamma^* F_\alpha^\vee).$$

In each fiber of γ, the four points are the base locus of a pencil of conics parametrized by F_α. We can also think of δ_α as a section of a bundle on $\mathbb{P}^1$ through the natural isomorphism

$$H^0(\mathbb{P}E_\alpha^\vee, \mathcal{O}_{\mathbb{P}E_\alpha^\vee}(2) \otimes \gamma^* F_\alpha^\vee) \cong H^0(\mathbb{P}^1, \operatorname{Sym}^2 E_\alpha \otimes F_\alpha^\vee).$$

The cover α also determines an isomorphism $\phi_\alpha : \det E_\alpha \cong \det F_\alpha$ (see [CL21d, Section 3.2])

The association of $\alpha : C \to \mathbb{P}^1$ with the triple $(E_\alpha, F_\alpha, \phi_\alpha)$ gives rise to a map of $\mathscr{H}_{4,g}$ to the base stack $\mathscr{B}_{4,g}$ defined in 3.3.1. Unlike in the degree 3 case, the map $\mathscr{H}_{4,g} \to \mathscr{B}_{4,g}$ does *not* factor through a vector bundle over $\mathscr{B}_{4,g}$. Nevertheless, we shall define an open substack $\mathscr{H}'_{4,g}$ that does admit such a nice description. The key fact, to be established in Lemma 3.5.1, is that the complement of $\mathscr{H}'_{4,g} \subset \mathscr{H}_{4,g}$ has codimension at least 2 for all $g \neq 3$. Thus, it will suffice to compute $\operatorname{Pic}(\mathscr{H}'_{4,g})$.

3.5.1 The open $\mathscr{H}'_{4,g}$

First define $\mathscr{B}'_{4,g} := \mathscr{B}_{4,g} \smallsetminus R^1\pi_*(\mathscr{F}^\vee \otimes \operatorname{Sym}^2 \mathscr{E})$. We define $\mathscr{H}'_{4,g} \subset \mathscr{H}_{4,g}$ to be the base change of the map $\mathscr{H}_{4,g} \to \mathscr{B}_{4,g}$ along the open embedding $\mathscr{B}'_{4,g} \hookrightarrow \mathscr{B}_{4,g}$. The key property of $\mathscr{H}'_{4,g}$ is that the map $\mathscr{H}'_{4,g} \to \mathscr{B}'_{4,g}$ factors through an open inclusion in the total space of a vector bundle (by an argument identical to [CL21d, Lemma 5.3]):

$$\mathscr{H}'_{4,g} \hookrightarrow \mathscr{X}'_{4,g} := \pi_*(\mathscr{F}^\vee \otimes \operatorname{Sym}^2 \mathscr{E})|_{\mathscr{B}'_{4,g}}. \tag{3.5.1}$$

As promised, we now show that the complement of $\mathscr{H}'_{4,g} \subset \mathscr{H}_{4,g}$ has codimension at least 2 (except when $g = 3$). This essentially follows from earlier work of Deopurkar–Patel.

Lemma 3.5.1. *If $g \neq 3$, every component of the complement of $\mathscr{H}'_{4,g} \subset \mathscr{H}_{4,g}$ has codimension at least* 2*. Hence, there is an isomorphism* $\operatorname{Pic}(\mathscr{H}_{4,g}) \cong \operatorname{Pic}(\mathscr{H}'_{4,g})$.

Proof. Following the notation of [DP15], let $M(E, F) \subset \mathscr{H}_{4,g}$ denote the locus of covers α

with $E_\alpha \cong E$ and $F_\alpha \cong F$. The complement of $\mathscr{H}'_{4,g} \subset \mathscr{H}_{4,g}$ is the union of $M(E,F)$ such that

$$h^1(\mathbb{P}^1, F^\vee \otimes \operatorname{Sym}^2 E) > 0. \tag{3.5.2}$$

If $E = \mathcal{O}(e_1) \oplus \mathcal{O}(e_2) \oplus \mathcal{O}(e_3)$ with $e_1 \leq e_2 \leq e_3$ and $F = \mathcal{O}(f_1) \oplus \mathcal{O}(f_2)$ with $f_1 \leq f_2$, then (3.5.2) is equivalent to $2e_1 - f_1 \leq -2$.

Next, let E_{gen} and F_{gen} denote the balanced bundles of rank 3 and 2 and degree $g+3$. First note that $h^1(F_{\text{gen}}^\vee \otimes \operatorname{Sym}^2 E_{\text{gen}}) = 0$: this is equivalent to $2\lfloor \frac{g+3}{3} \rfloor - \lceil \frac{g+3}{2} \rceil \geq -1$. This says that $M(E_{\text{gen}}, F_{\text{gen}}) \subseteq \mathscr{H}'_{4,g}$. Hence, any divisorial component of $\mathscr{H}_{4,g} \smallsetminus \mathscr{H}'_{4,g}$ is contained in a divisorial component of $\mathscr{H}_{4,g} \smallsetminus M(E_{\text{gen}}, F_{\text{gen}})$.

Next, let us define bundles that are "one-off" from balanced

$$\begin{aligned} F_1 &:= \mathcal{O}(n-1) \oplus \mathcal{O}(n+1) && \text{if } n = \frac{g+3}{2} \text{ is an integer} \\ E_1 &:= \mathcal{O}(m-1) \oplus \mathcal{O}(m) \oplus \mathcal{O}(m+1) && \text{if } m = \frac{g+3}{3} \text{ is an integer.} \end{aligned}$$

In [DP15, p. 20], Deopurkar–Patel enumerate the divisorial components of $\mathscr{H}_{4,g} \smallsetminus M(E_{\text{gen}}, F_{\text{gen}})$ and show that, for $g \neq 3$, they are always of the form

$$\overline{M(E_{\text{gen}}, F_1)} \qquad \text{if } 2 \mid g+3 \tag{3.5.3}$$

$$\overline{M(E_1, F_{\text{gen}})} \qquad \text{if } 3 \mid g+3. \tag{3.5.4}$$

Note that we are using the irreducibility of M and CE in [DP15, Propositions 4.5 and 4.7] to write these divisors as the closures above. Meanwhile, when $g = 3$, the stratum

$$\overline{M(E_1, F_1)} = \overline{M(\mathcal{O}(1) \oplus \mathcal{O}(2) \oplus \mathcal{O}(3), \mathcal{O}(2) \oplus \mathcal{O}(4))} \tag{3.5.5}$$

is also a divisor. This divisor does not lie in $\mathscr{H}'_{4,g}$.

One readily checks that

$$h^1(F_1^\vee \otimes \operatorname{Sym}^2 E_{\mathrm{gen}}) = h^1(F_{\mathrm{gen}}^\vee \otimes \operatorname{Sym}^2 E_1) = 0. \tag{3.5.6}$$

Hence, when they are defined, $\mathscr{H}'_{4,g}$ contains each of

$$M(E_{\mathrm{gen}}, F_{\mathrm{gen}}), \quad M(E_{\mathrm{gen}}, F_1), \quad M(E_1, F_{\mathrm{gen}}),$$

and, for $g \neq 3$, all other possible $M(E,F)$ have codimension at least 2. □

Remark 3.5.2. Each $M(E,F)$ can be constructed directly as a global quotient, giving rise to a bound on $\operatorname{Pic}(M(E,F))$. Deopurkar–Patel use their enumeration of the components of $\mathcal{H}_{4,g} \smallsetminus M(E_{\mathrm{gen}}, F_{\mathrm{gen}})$ to count ranks and prove the Picard rank conjecture for $k \leq 5$.

The new innovation in our work is that we have built a larger open $\mathscr{H}'_{4,g}$ which contains several $M(E,F)$ and, in particular, is not missing any divisorial components (when $g \neq 3$). Hence, we see $\operatorname{Pic}(\mathscr{H}_{4,g}) = \operatorname{Pic}(\mathscr{H}'_{4,g})$, and we can compute the later *integrally* using excision on $\mathscr{H}'_{4,g} \subset \mathscr{X}'_{4,g}$.

Lemma 3.5.3. *When $g = 3$, the complement of $\mathscr{H}'_{4,3} \subset \mathscr{H}_{4,3}$ is an irreducible divisor whose class lies in the subgroup generated by ϵa_1 and a'_2 (these classes are defined on all of $\mathscr{H}_{4,3}$ via pull back along $\operatorname{Pic}(\mathscr{B}_{4,3}) \to \operatorname{Pic}(\mathscr{H}_{4,g})$.)*

Proof. Continuing the notation of the previous lemma, first note that if

$$M(\mathcal{O}(1) \oplus \mathcal{O}(2) \oplus \mathcal{O}(3), \mathcal{O}(f_1) \oplus \mathcal{O}(f_2)) \neq \varnothing$$

then by [CL21d, Proposition 5.6(2)], we have $f_1 \leq 2$ and $f_2 \leq 4$ and $f_1 + f_2 = 6$, hence $f_1 = 2$ and $f_2 = 4$. The divisor in (3.5.5) can therefore be viewed as the locus were the universal $\mathscr{E}$ (pulled back along $\mathscr{H}_{4,3} \to \mathscr{B}_{4,3}$) has splitting type $(1,2,3)$ on fibers of the universal $\mathbb{P}^1$-fibration. As a splitting locus for $\mathscr{E}$, this divisor occurs in the expected codimension, so its fundamental class is determined by the universal splitting loci formulas of [Lar21c]. In particular, it can be expressed in terms of the classes $\epsilon a_1, a_2'$. $\square$

3.5.2 Excision

Recall the inclusion of (3.5.1) and let $\mathscr{D}_{4,g} := \mathscr{X}_{4,g}' \smallsetminus \mathscr{H}_{4,g}'$ be the complement. By excision, we have a series of surjections (the middle map is an isomorphism because $\mathscr{X}_{4,g}'$ is a vector bundle over $\mathscr{B}_{4,g}'$):

$$\operatorname{Pic}(\mathscr{B}_{4,g}) \to \operatorname{Pic}(\mathscr{B}_{4,g}') \cong \operatorname{Pic}(\mathscr{X}_{4,g}') \to \operatorname{Pic}(\mathscr{H}_{4,g}'). \tag{3.5.7}$$

Moreover, the fundamental class $[\mathscr{D}_{4,g}] \in \operatorname{Pic}(\mathscr{X}_{4,g}')$ lies in the kernel of the last map in (3.5.7) (and it generates the kernel when $\mathscr{D}_{4,g}$ is irreducible.)

At this point we have defined a sequence of morphisms

$$\mathscr{D}_{4,g} \to \mathscr{X}_{4,g}' \to \mathscr{B}_{4,g}' \to \mathscr{B}_{4,g} \to \mathscr{V}_{3,g+3} \times_{\mathrm{BPGL}_2} \mathscr{V}_{2,g+3} \to \mathrm{BPGL}_2\,. \tag{3.5.8}$$

Lemma 3.5.4. *For $g \geq 2$, some combination of components of $\mathscr{D}_{4,g}$ has class*

$$\frac{8g+20}{\epsilon}(\epsilon a_1) - 8a_2' - b_2' \in \operatorname{Pic}(\mathscr{X}_{4,g}') \cong \operatorname{Pic}(\mathscr{B}_{4,g}').$$

In particular, an integral relation holds in $\operatorname{Pic}(\mathscr{H}_{4,g}')$ expressing b_2' in terms of ϵa_1 and a_2'.

Proof. Base changing (3.5.8) by $\mathrm{BSL}_2 \to \mathrm{BPGL}_2$, we obtain the stacks and morphisms considered in [CL21d, Section 5.2] (below $\Delta_{4,g}'$ is the complement of the open inclusion $\mathcal{H}_{4,g}' \hookrightarrow \mathcal{X}_{4,g}'$ of [CL21d, Lemma 5.3]):

$$\Delta_{4,g}' \to \mathcal{X}_{4,g}' \to \mathcal{B}_{4,g}' \to \mathcal{B}_{4,g} \to \mathcal{V}_{3,g+3} \times_{\mathrm{BSL}_2} \mathcal{V}_{2,g+3} \to \mathrm{BSL}_2\,.$$

We claim some combination of components of $\Delta'_{4,g}$ has class

$$(8g+20)a_1 - 8a'_2 - b'_2 \in \mathrm{Pic}(\mathcal{X}'_{4,g}) \cong \mathrm{Pic}(\mathcal{B}_{4,g}).$$

This will establish the lemma since the map $\mathrm{Pic}(\mathscr{X}'_{4,g}) \to \mathrm{Pic}(\mathcal{X}'_{4,g})$ sends the class of a component of $\mathscr{D}_{4,g}$ to the class of the corresponding component of $\Delta'_{4,g}$. Because $\mathrm{Pic}(\mathscr{X}'_{4,g}) \to \mathrm{Pic}(\mathcal{X}'_{4,g})$ is injective (by Lemma 3.3.6), classes are represented by the same formulas in either group.

By [CL21a, Lemma 5.2 and Equation 5.7], we know that $(8g+20)a_1 - 8a'_2 - b'_2 = 0$ in $\mathrm{Pic}(\mathcal{H}_{4,g})$, so this relation must also hold on the open substack $\mathcal{H}'_{4,g} \subset \mathcal{H}_{4,g}$. But, we also know $\mathcal{H}'_{4,g} \cong \mathcal{X}'_{4,g} \smallsetminus \Delta'_{4,g}$. By excision, every relation among a_1, a'_2, b'_2 restricted to $\mathrm{Pic}(\mathcal{H}'_{4,g})$ comes from a class supported on $\Delta'_{4,g} \subset \mathcal{X}'_{4,g}$. That is, some combination of components of $\Delta'_{4,g}$ has class $(8g+20)a_1 - 8a'_2 - b'_2 = 0$. The corresponding combination of components of $\mathscr{D}_{4,g}$ will have the same class. $\square$

Remark 3.5.5. In fact, the fundamental class of $\mathscr{D}_{4,g}$ has the class displayed in Lemma 3.5.4. For a more conceptual explanation, we sketch the following argument. Recall that in [CL21a, Equation 5.7], we computed the restriction of $[\Delta'_{4,g}]$ to a slightly smaller open $\mathcal{X}^\circ_{4,g} \subset \mathcal{X}'_{4,g}$ via principal parts bundle techniques. For g sufficiently large, the complement of $\mathcal{X}^\circ_{4,g} \subset \mathcal{X}'_{4,g}$ has codimension at least 2, so the codimesnion 1 calculation holds on all of $\mathcal{X}'_{4,g}$. That is $[\Delta'_{4,g}] = (8g+20)a_1 - 8a'_2 - b'_2$ and so $[\mathscr{D}_{4,g}]$ also has this class.

But even for smaller g, we claim the formula for $[\Delta_{4,g}]$ in [CL21a, Equation 5.7] holds on all of $\mathcal{X}'_{4,g}$. Although the principal parts map of [CL21a, Equation 5.4] need not be surjective over all of $\mathcal{X}'_{4,g}$ (so the vanishing locus $\widetilde{\Delta}_{4,g}$ of the principal parts bundle map need not be a vector bundle) the calculation of the fundamental class holds so long as $\widetilde{\Delta}_{4,g}$ has the correct codimension. One can verify this by stratifying the base by loci where the rank drops and checking that the strata where the rank drops by δ have codimension greater than δ. This also establishes irreducibility of $\Delta'_{4,g}$ and $\mathscr{D}_{4,g}$ when $g \geq 4$. However, this fact is not actually necessary for our argument.

Proof of Theorem 3.1.1(1) for $g \geq 4$. By excision, we know that

$$\mathrm{Pic}(\mathscr{H}'_{4,g}) \cong \frac{\mathrm{Pic}(\mathscr{X}'_{4,g})}{\text{classes supported on } \mathscr{D}_{4,g}}.$$

Meanwhile, $\mathrm{Pic}(\mathscr{X}'_{4,g})$ is a quotient of $\mathrm{Pic}(\mathscr{B}_{4,g}) = \mathbb{Z}(\epsilon a_1) \oplus \mathbb{Z}a'_2 \oplus \mathbb{Z}b'_2$ (see (3.3.9)). Hence, using Lemma 3.5.4, we see $\mathrm{Pic}(\mathscr{H}'_{4,g})$ is a quotient of

$$\frac{\mathbb{Z}(\epsilon a_1) \oplus \mathbb{Z}a'_2 \oplus \mathbb{Z}b'_2}{\langle \frac{8g+20}{\epsilon}(\epsilon a_1) - 8a'_2 - b'_2 \rangle} \cong \mathbb{Z}(\epsilon a_1) \oplus \mathbb{Z}a'_2.$$

Recall that in Lemma 3.5.1, we showed $\mathrm{Pic}(\mathscr{H}_{4,g}) = \mathrm{Pic}(\mathscr{H}'_{4,g})$ for $g \geq 4$. By [DP15, Proposition 2.15], this group has rank at least 2. Therefore, $\mathrm{Pic}(\mathscr{H}_{4,g}) = \mathbb{Z} \oplus \mathbb{Z}$, since any quotient would have smaller rank. □

Genus 3

When $g = 3$, we require a different argument, as Lemma 3.5.3 tells us that $\mathscr{H}'_{4,3} \subset \mathscr{H}_{4,3}$ is the complement of a divisor.

Proof of Theorem 3.1.1(1) when $g = 3$. By Lemma 3.5.3, the kernel of the restriction map

$$\mathrm{Pic}(\mathscr{H}_{4,3}) \to \mathrm{Pic}(\mathscr{H}'_{4,3})$$

lies in the subgroup $\langle a_1, a'_2 \rangle$. We know that $\mathrm{Pic}(\mathscr{H}'_{4,g})$ is generated by the classes a_1, a'_2, b'_2, so it follows that $\mathrm{Pic}(\mathscr{H}_{4,3})$ is also generated by these 3 classes.

Next, we claim that b'_2 is integrally expressible in terms of a_1, a'_2. By Lemma 3.5.4, we know b'_2 is expressible in terms of a_1, a'_2 in $\mathrm{Pic}(\mathscr{H}'_{4,3})$. But, the kernel of $\mathrm{Pic}(\mathscr{H}_{4,3}) \to \mathrm{Pic}(\mathscr{H}'_{4,3})$ lies in $\langle a_1, a'_2 \rangle$, so b'_2 must also be expressible in terms of a_1, a'_2 in $\mathrm{Pic}(\mathscr{H}_{4,3})$.

It follows that $\operatorname{Pic}(\mathscr{H}_{4,3})$ is a quotient of $\mathbb{Z}a_1 \oplus \mathbb{Z}a_2'$. However, by [DP15], we know that $\operatorname{Pic}(\mathscr{H}_{4,3})$ has rank 2. Any further quotient would have lower rank, so we are done. □

Genus 2

The proof of [DP15, Proposition 2.15] (showing $\operatorname{Pic}(\mathscr{H}_{4,g})$ has rank 2) does not go through when $g = 2$ because Deopurkar–Patel's test family B_3 has curves with disconnecting nodes, so it does not lie in their $\widetilde{\mathcal{H}}_{4,2}^{ns}$. However, their proof does establish that the rank of $\operatorname{Pic}(\mathscr{H}_{4,2})$ is at least 1. This, together with Lemma 3.4.1, provides enough information to determine the Picard group.

Proof of Theorem 3.1.1(1) when $g = 2$. We have already established that $\operatorname{Pic}(\mathscr{H}_{4,2})$ is generated by $2a_1$ and a_2'. Using Example 3.3.4, we compute that (see (3.2.1) for definitions of the maps f, α, π)

$$\lambda = c_1(f_*\omega_f) = c_1(\pi_*(\alpha_*\omega_\alpha) \otimes \omega_\pi) = c_1(\pi_*\mathcal{E}(-2)) = 4a_1 - a_2' = 2(2a_1) - a_2'.$$

From this we see that λ and $2a_1$ are generators for $\operatorname{Pic}(\mathscr{H}_{4,2})$. Since λ is the generator of $\operatorname{Pic}(\mathcal{M}_2) \cong \mathbb{Z}/10\mathbb{Z}$, we see that $\operatorname{Pic}(\mathscr{H}_{4,2})$ is a quotient of $\mathbb{Z} \oplus \mathbb{Z}/10$.

By the discussion at the start of this section, we know $\operatorname{Pic}(\mathscr{H}_{4,2})$ has rank at least 1. Thus, it remains to prove that $\operatorname{Pic}(\mathcal{M}_2) \to \operatorname{Pic}(\mathscr{H}_{4,2})$ is injective.

Let $\mathscr{P}^k$ be the universal Picard variety over $\mathcal{M}_2$ as in Section 3.4.1 (the $\mathbb{G}_m$-rigidification of the universal Picard stack $\mathscr{P}ic^k$). As in [Moc95, Section 6], the natural map $\mathscr{H}_{4,2} \to \mathscr{P}^4$ factors through a Grassmann fibration. For this, recall that every degree 4 line bundle on a genus 2 curve has a 3-dimensional space of sections. Let $\mathscr{G} \to \mathscr{P}^4$ be the Grassmann fibration parametrizing two-dimensional subspaces of the space of global sections of a degree 4 line bundle. Then $\mathscr{H}_{4,2}$ sits naturally as an open substack $\mathscr{G}$. Its complement $Z = \mathscr{G} \smallsetminus \mathscr{H}_{4,2}$ is the locus of pencils with a base point. Note that Z has 1-dimensional irreducible fibers over $\mathscr{P}^4$, so Z is irreducible. Since Z meets each fiber of $\mathscr{G} \to \mathscr{P}^4$, it is not equivalent to the pullback of a divisor on $\mathscr{P}^4$. In particular, the map $\mathrm{Pic}(\mathscr{P}^4) \to \mathrm{Pic}(\mathscr{H}_{4,2})$ must be injective. Using Lemma 3.4.1, we conclude that $\mathrm{Pic}(\mathcal{M}_2) \to \mathrm{Pic}(\mathscr{H}_{4,2})$ is also injective, completing the proof. □

Remark 3.5.6. Geometrically, the fact that $\mathrm{Pic}(\mathscr{H}_{4,2})$ has rank 1 can be explained by the fact that $\Delta'_{4,2}$ is reducible. Thus, its components give rise to further relations beyond just its fundamental class.

3.5.3 Generating line bundles

Let $\epsilon = 1$ if g is odd and $\epsilon = 2$ if g is even. We have shown that $\mathrm{Pic}(\mathscr{H}_{4,g})$ is generated by ϵa_1 and a_2', or equivalently by ϵa_1 and $\lambda := (g+2)a_1 - a_2'$. Let $\pi : \mathscr{P} \to \mathscr{H}_{4,g}$ be the universal $\mathbb{P}^1$-fibration and $\mathscr{E}$ the universal rank 3, degree $g+3$ vector bundle on $\mathscr{P}$. Recall that ω_π has relative degree -2. Line bundles generating $\mathrm{Pic}(\mathscr{H}_{4,g})$ are given by

$$\mathscr{L}_1 = \begin{cases} \pi_*\left(\det \mathscr{E} \otimes \omega_\pi^{\otimes(g+3)/2}\right) & \text{if } g \text{ odd} \\ \\ \pi_*\left((\det \mathscr{E})^{\otimes 2} \otimes \omega_\pi^{\otimes(g+3)}\right) & \text{if } g \text{ even} \end{cases} \qquad \text{which has} \qquad c_1(\mathscr{L}_1) = \epsilon a_1$$

and

$$\mathscr{L}_2 = \det f_*(\omega_f) = \det \pi_*(\mathscr{E} \otimes \omega_\pi) \qquad \text{which has} \qquad c_1(\mathscr{L}_2) = \lambda = (g+2)a_1 - a_2'.$$

3.5.4 Simple branching

Let T and D be the divisors in $\mathscr{H}_{4,g}$ as in the introduction (see Figure 3.1). In [CL21a, Lemma 7.6], we wrote T and D (pulled back to $\mathrm{Pic}(\mathcal{H}_{4,g})$) in terms of our generators a_1 and a_2':

$$T = (24g+60)a_1 - 24a_2' \qquad D = -(32g+80)a_1 + 36a_2'.$$

Note that the coefficient of a_1 is a multiple of ϵ, as it must be, since these classes are defined in $\mathrm{Pic}(\mathscr{H}_{4,g}) \subseteq \mathrm{Pic}(\mathcal{H}_{4,g})$.

Proof of Corollary 3.1.3(2). By excision, we have $\mathrm{Pic}(\mathscr{H}_{4,g}^s) = \mathrm{Pic}(\mathscr{H}_{4,g})/\langle T, D\rangle$. Row operations over $\mathbb{Z}$ diagonalize the change of basis matrix for $\epsilon a_1, a_2'$ to T, D, namely we have

$$\begin{pmatrix} 3 & 2 \\ 4 & 3 \end{pmatrix} \begin{pmatrix} (24g+60)/\epsilon & -24 \\ (-32g-80)/\epsilon & 36 \end{pmatrix} = \begin{pmatrix} (8g+20)/\epsilon & 0 \\ 0 & 12 \end{pmatrix},$$

where the matrix on the left is invertible over $\mathbb{Z}$. By its definition in (3.3.7), we have $\epsilon = 1$ when g is odd and $\epsilon = 2$ when g is even, so the corollary follows for $g \geq 3$.

Finally, when $g = 2$, the above tells us $18(2a_1) = 0$ and $12a_2' = 0$, and we have the additional relation

$$0 = 10\lambda = 10(4a_1 - a_2') \qquad \Rightarrow \qquad 0 = 2(2a_1 + a_2').$$

Using generators $2a_1 + a_2'$ and $2a_1$, we see that they generate cyclic groups of order 2 and 18 respectively. $\square$

3.6 Pentagonal

We begin by recalling the linear algebraic data associated to degree 5 covers, as developed by Casnati [Cas96]. For more details in our context, see [CL21d, Section 3.3]. To a degree 5, cover $\alpha : C \to \mathbb{P}^1$, we again associate two vector bundles on $\mathbb{P}^1$:

$$E_\alpha := (\alpha_* \mathcal{O}_C / \mathcal{O}_{\mathbb{P}^1})^\vee = \ker(\alpha_* \omega_\alpha \to \mathcal{O}_{\mathbb{P}^1}) \qquad \text{and} \qquad F_\alpha := \ker(\mathrm{Sym}^2 E_\alpha \to \alpha_* \omega_\alpha^{\otimes 2}).$$

If C has genus g, then E_α has degree $g+4$, and rank 4, while F_α has degree $2g+8$ and rank 5. Geometrically, the curve C is embedded in $\gamma : \mathbb{P}(E_\alpha^\vee \otimes \det E_\alpha) \to \mathbb{P}^1$, which further maps to $\mathbb{P}(\wedge^2 F_\alpha)$ via an associated section

$$\eta_\alpha \in H^0(\mathbb{P}^1, \mathcal{H}om(E_\alpha^\vee \otimes \det E_\alpha, \wedge^2 F_\alpha)).$$

The curve C is obtained as the intersection of the image of $\mathbb{P}(E_\alpha^\vee \otimes \det E_\alpha)$ with the Grassmann bundle $G(2, F_\alpha) \subset \mathbb{P}(\wedge^2 F_\alpha)$. The cover α also determines an isomorphism $\phi_\alpha : \det E_\alpha^{\otimes 2} \to \det F_\alpha$, [CL21d, p. 10].

The association of $\alpha : C \to \mathbb{P}^1$ with the triple $(E_\alpha, F_\alpha, \phi_\alpha)$ gives rise to a map $\mathscr{H}_{5,g} \to \mathscr{B}_{5,g}$, for the base stack $\mathscr{B}_{5,g}$ defined in 3.3.1. Just like in the degree 4 case, the map $\mathscr{H}_{5,g} \to \mathscr{B}_{5,g}$ does not factor through a vector bundle over $\mathscr{B}_{5,g}$, but an open substack $\mathscr{H}'_{5,g}$ does. When $g \neq 3$, we will show that the complement of $\mathscr{H}'_{5,g}$ in $\mathscr{H}_{5,g}$ has codimension at least 2, so it will suffice to compute $\mathrm{Pic}(\mathscr{H}_{5,g})$. We will then deal with the $g = 3$ case separately.

3.6.1 The open substack $\mathscr{H}'_{5,g}$

First, define $\mathscr{B}'_{5,g} := \mathscr{B}_{5,g} \smallsetminus R^1\pi_*(\mathcal{H}om(\mathscr{E}^\vee \otimes \det \mathscr{E}, \wedge^2 \mathscr{F}))$. Let $\mathscr{H}'_{5,g}$ be the base change of $\mathscr{H}_{5,g} \to \mathscr{B}_{5,g}$ along the open embedding $\mathscr{B}'_{5,g} \hookrightarrow \mathscr{B}_{5,g}$. Arguing exactly as in [CL21d, Lemma 5.3], the morphism $\mathscr{H}'_{5,g} \to \mathscr{B}'_{5,g}$ factors through the total space of a vector bundle over $\mathscr{B}'_{5,g}$:

$$\mathscr{H}'_{5,g} \hookrightarrow \mathscr{X}'_{5,g} := \pi_*(\mathcal{H}om(\mathscr{E}^\vee \otimes \det \mathscr{E}, \wedge^2 \mathscr{F}))|_{\mathscr{B}'_{5,g}}.$$

Lemma 3.6.1. *Suppose $g \neq 3$. Then, every component of the complement of $\mathscr{H}'_{5,g} \subset \mathscr{H}_{5,g}$ has codimension at least 2. In particular,* $\mathrm{Pic}(\mathscr{H}_{5,g}) = \mathrm{Pic}(\mathscr{H}'_{5,g})$

Proof. Following the notation of [DP15], let $M(E,F) \subset \mathscr{H}_{5,g}$ denote the locus of covers α with $E_\alpha \cong E$ and $F_\alpha \cong F$. The complement of $\mathscr{H}'_{5,g} \subset \mathscr{H}_{5,g}$ is the union of $M(E,F)$ such that

$$h^1(E \otimes \det E^\vee \otimes \wedge^2 F^\vee) > 0. \tag{3.6.1}$$

If $E = \mathcal{O}(e_1) \oplus \cdots \oplus \mathcal{O}(e_4)$ with $e_1 \leq \cdots \leq e_4$ and $F = \mathcal{O}(f_1) \oplus \cdots \oplus \mathcal{O}(f_5)$ with $f_1 \leq \cdots \leq f_5$, then (3.6.1) is equivalent to $e_1 + f_1 + f_2 - (g+4) \leq -2$.

Next, let E_{gen}, respectively F_{gen}, denote the balanced bundle of rank 4 and degree $g+4$, respectively rank 5 and degree $2g+8$. First note that $h^1(E_{\text{gen}} \otimes \det E_{\text{gen}}^\vee \otimes \wedge^2 F_{\text{gen}}^\vee) = 0$, as in this case, $e_1 + f_1 + f_2 - (g+4) \geq \lfloor \frac{g+4}{4} \rfloor + 2\lfloor \frac{2(g+4)}{5} \rfloor - (g+4) \geq -1$ (except when $g = 3$, in which case $f_2 = \lceil \frac{2(g+4)}{5} \rceil$, so we still have $e_1 + f_1 + f_2 \geq -1$.) This says that $M(E_{\text{gen}}, F_{\text{gen}}) \subseteq \mathscr{H}'_{5,g}$. Hence, any divisorial component of $\mathscr{H}_{5,g} \smallsetminus \mathscr{H}'_{5,g}$ is contained in a divisorial component of $\mathscr{H}_{5,g} \smallsetminus M(E_{\text{gen}}, F_{\text{gen}})$.

Again, let us define bundles that are "one-off" from balanced

$$F_1 := \mathcal{O}(n-1) \oplus \oplus \mathcal{O}(n)^{\oplus 3} \mathcal{O}(n+1) \qquad \text{if } n = \frac{2g+8}{5} \text{ is an integer}$$
$$E_1 := \mathcal{O}(m-1) \oplus \mathcal{O}(m)^{\oplus 2} \oplus \mathcal{O}(m+1) \qquad \text{if } m = \frac{g+4}{4} \text{ is an integer.}$$

In [DP15, p. 25], Deopurkar–Patel enumerate the divisorial components of $\mathscr{H}_{5,g} \smallsetminus M(E_{\text{gen}}, F_{\text{gen}})$ and show that, for $g \neq 3$, they are always of the form

$$\overline{M(E_{\text{gen}}, F_1)} \qquad \text{if } 5 \mid 2g+8$$
$$\overline{M(E_1, F_{\text{gen}})} \qquad \text{if } 4 \mid g+4.$$

Note that we are using the irreducibility of M and CE in [DP15, Propositions 5.1 and 5.2] to write these divisors as the closures above. One readily checks that

$$h^1(E_{\text{gen}} \otimes \det E_{\text{gen}}^\vee \otimes \wedge^2 F_1) = h^1(E_1 \otimes \det E_1^\vee \otimes \wedge^2 F_{\text{gen}}) = 0$$

Hence, when they are defined, $\mathscr{H}'_{5,g}$ contains each of

$$M(E_{\text{gen}}, F_{\text{gen}}), \quad M(E_{\text{gen}}, F_1), \quad M(E_1, F_{\text{gen}}),$$

and all other possible $M(E,F)$ have codimension at least 2. $\square$

Lemma 3.6.2. *When $g=3$, the complement of $\mathscr{H}'_{5,3} \subset \mathscr{H}_{5,3}$ is an irreducible divisor whose class lies in the subgroup generated by $2a_1$ and a'_2.*

Proof. The divisorial component of the complement of $\mathscr{H}'_{5,3}$ inside $\mathscr{H}_{5,3}$ is the locus

$$\overline{M(E_1, F_1)} = \overline{M(\mathcal{O}(1) \oplus \mathcal{O}(2)^{\oplus 3}, \mathcal{O}(2)^{\oplus 2} \oplus \mathcal{O}(3)^{\oplus 2} \oplus \mathcal{O}(4))}.$$

By [DP15, Proposition 5.2], this locus is precisely the preimage of the hyperelliptic locus under the natural morphism $\mathscr{H}_{5,3} \to \mathcal{M}_3$. By [DL20], the hyperelliptic locus in $\mathcal{M}_3$ has class 9λ. The class $\lambda = c_1(f_*\omega_f) = c_1(\pi_*\mathscr{E} \otimes \omega_\pi) \in \operatorname{Pic}(\mathscr{H}_{5,3})$ is pulled back from $\operatorname{Pic}(\mathscr{B}_{5,3})$. Since $\operatorname{Pic}(\mathscr{B}_{5,3})$ includes into $\operatorname{Pic}(\mathcal{B}_{5,3})$, we can determine this class via a calculation on the SL_2 quotient, as in Example 3.3.4:

$$[\overline{M(E_1, F_1)}] = 9\lambda = 9c_1(f_*\omega_f) = 9c_1(\pi_*\mathcal{E}(-2)) = 54a_1 - 9a'_2 = 27(2a_1) - 9a'_2,$$

which is in the span of $2a_1$ and a'_2. $\square$

3.6.2 Excision

We proceed similarly to the $k=4$ case. Let $\mathscr{D}_{5,g} \subset \mathscr{X}'_{5,g} \smallsetminus \mathscr{H}'_{5,g}$. There is a series of surjections

$$\operatorname{Pic}(\mathscr{B}_{5,g}) \to \operatorname{Pic}(\mathscr{B}'_{5,g}) \cong \operatorname{Pic}(\mathscr{X}'_{5,g}) \to \operatorname{Pic}(\mathscr{H}'_{5,g}). \tag{3.6.2}$$

The middle map is an isomorphism because $\mathscr{X}'_{5,g} \to \mathscr{B}'_{5,g}$ is a vector bundle. We have defined a sequence of morphisms

$$\mathscr{D}_{5,g} \to \mathscr{X}'_{5,g} \to \mathscr{B}'_{5,g} \to \mathscr{B}_{5,g} \to \mathscr{V}_{4,g+4} \times_{\mathrm{BPGL}_2} \mathscr{V}_{5,2g+8} \to \mathrm{BPGL}_2\,. \tag{3.6.3}$$

Lemma 3.6.3. *For $g \geq 2$, some combination of the components of $\mathscr{D}_{5,g}$ has class*

$$\frac{(10g+36)}{\epsilon}(\epsilon a_1) - 7a'_2 - b'_2 \in \mathrm{Pic}(\mathscr{X}'_{5,g}) \cong \mathrm{Pic}(\mathscr{B}'_{5,g}).$$

Proof. Base changing (3.6.3) by $\mathrm{BSL}_2 \to \mathrm{BPGL}_2$, we obtain the stacks and morphisms considered in [CL21d, Section 5.3] (below $\Delta'_{5,g}$ is the complement of the open inclusion $\mathcal{H}'_{5,g} \hookrightarrow \mathcal{X}'_{5,g}$ of [CL21d, Lemma 5.11]):

$$\Delta'_{5,g} \to \mathcal{X}'_{5,g} \to \mathcal{B}'_{5,g} \to \mathcal{B}_{5,g} \to \mathcal{V}_{4,g+4} \times_{\mathrm{BSL}_2} \mathcal{V}_{5,2g+8} \to \mathrm{BSL}_2\,.$$

By [CL21a, Lemma 6.6 and Equation 6.19], the relation $\frac{(10g+36)}{\epsilon}(\epsilon a_1) - 7a'_2 - b'_2 = 0$ holds in $\mathrm{Pic}(\mathcal{H}_{5,g})$, so it must also hold on the open substack $\mathcal{H}'_{5,g}$. Since $\mathcal{H}'_{5,g} = \mathcal{X}'_{5,g} \smallsetminus \Delta'_{5,g}$, any relation among a_1, a'_2, b'_2 on $\mathcal{H}'_{5,g}$ must come from a class supported on $\Delta'_{5,g}$. Therefore, some combination of components of $\Delta'_{5,g}$ has class $(10g+36)a_1 - 7a'_2 - b'_2$. The corresponding combination of components on $\mathscr{D}_{5,g}$ has the same class. $\square$

Proof of Theorem 1.1(3) when $g \geq 4$. By Lemma 3.6.1, $\mathrm{Pic}(\mathscr{H}_{5,g}) \cong \mathrm{Pic}(\mathscr{H}'_{5,g})$ for $g \geq 4$. We have that $\mathrm{Pic}(\mathscr{H}'_{5,g})$ is a quotient of $\mathrm{Pic}(\mathscr{X}'_{5,g}) \cong \mathrm{Pic}(\mathscr{B}'_{5,g})$ by classes supported on $\mathscr{D}_{5,g}$. By Lemma 3.6.3, one such class is $(10g+36)a_1 - 7a'_2 - b'_2$. Therefore, $\mathrm{Pic}(\mathscr{H}'_{5,g})$ is a quotient of

$$\frac{\mathbb{Z}(\epsilon a_1) \oplus \mathbb{Z}a'_2 \oplus \mathbb{Z}b'_2}{\langle \frac{10g+36}{\epsilon}(\epsilon a_1) - 7a'_2 - b'_2 \rangle} \cong \mathbb{Z}(\epsilon a_1) \oplus \mathbb{Z}a'_2.$$

By [DP15, Proposition 2.15], the rank of $\mathrm{Pic}(\mathscr{H}_{5,g}) \otimes \mathbb{Q}$ is at least 2, so we must have that $\mathrm{Pic}(\mathscr{H}_{5,g}) \cong \mathbb{Z}(\epsilon a_1) \oplus \mathbb{Z}a'_2$. $\square$

Genus 3

As in the $k = 4$ case, when $g = 3$, we require a different argument because Lemma 3.6.2 tells us that $\mathscr{H}'_{5,3} \subset \mathscr{H}_{5,3}$ is the complement of a divisor.

Proof of Theorem 3.1.1(2) when $g = 3$. By Lemma 3.6.2, the kernel of the restriction map

$$\mathrm{Pic}(\mathscr{H}_{5,3}) \to \mathrm{Pic}(\mathscr{H}'_{5,3})$$

lies in the subgroup $\langle 2a_1, a'_2 \rangle$. We know that $\mathrm{Pic}(\mathscr{H}'_{5,g})$ is generated by the classes a_1, a'_2, b'_2, so it follows that $\mathrm{Pic}(\mathscr{H}_{5,3})$ is also generated by these 3 classes.

Next, we claim that b'_2 is integrally expressible in terms of a_1, a'_2. By Lemma 3.6.3, we know b'_2 is expressible in terms of a_1, a'_2 in $\mathrm{Pic}(\mathscr{H}'_{5,3})$. But, the kernel of $\mathrm{Pic}(\mathscr{H}_{5,3}) \to \mathrm{Pic}(\mathscr{H}'_{5,3})$ lies in $\langle a_1, a'_2 \rangle$, so b'_2 must also be expressible in terms of a_1, a'_2 in $\mathrm{Pic}(\mathscr{H}_{5,3})$.

It follows that $\mathrm{Pic}(\mathscr{H}_{5,3})$ is a quotient of $\mathbb{Z}a_1 \oplus \mathbb{Z}a'_2$. However, by [DP15], we know that $\mathrm{Pic}(\mathscr{H}_{5,3}) \otimes \mathbb{Q}$ has rank 2, so $\mathrm{Pic}(\mathscr{H}_{5,3}) \cong \mathbb{Z} \oplus \mathbb{Z}$. □

Genus 2

As in Section 3.5.2, the argument of Deopurkar–Patel [DP15, Proposition 2.15] in genus 2 establishes that the rank of $\mathrm{Pic}(\mathscr{H}_{5,2})$ is at least 1.

Proof of Theorem 3.1.1(2) when $g = 2$. We have already established that a_1, a'_2 are generators for $\mathrm{Pic}(\mathscr{H}_{5,2})$. We compute directly $\lambda = 5a_1 - a'_2$ as in Section 3.5.2, from which we see λ and a_1 are generators for $\mathrm{Pic}(\mathscr{H}_{5,2})$. Arguing as in Section 3.5.2, it suffices to show that the map $\mathrm{Pic}(\mathcal{M}_2) \to \mathrm{Pic}(\mathscr{H}_{5,2})$ is injective.

Every degree 5 line bundle on a genus 2 curve has a 4-dimensional space of sections. The Hurwitz space $\mathscr{H}_{5,2}$ then sits naturally as an open inside the Grassmann fibration $\mathscr{G} \to \mathscr{P}^5$ parametrizing 2-dimensional subspaces of the space of global sections of a degree

5 line bundle. The complement of $\mathscr{H}_{5,2} \subset \mathscr{G}$ is the locus of pencils with a base point, which we again see is irreducible and not equivalent to the pullback of a divisor on $\mathscr{P}^5$. In particular, the map $\mathrm{Pic}(\mathscr{P}^5) \to \mathrm{Pic}(\mathscr{H}_{5,2})$ must be injective. Applying Lemma 3.4.1, we conclude that $\mathrm{Pic}(\mathcal{M}_2) \to \mathrm{Pic}(\mathscr{H}_{5,2})$ is also injective, completing the proof. $\square$

3.6.3 Generating line bundles

Line bundles generating $\mathrm{Pic}(\mathscr{H}_{5,g})$ are given by

$$\mathscr{L}_1 = \begin{cases} \pi_*\left(\det \mathscr{E} \otimes \omega_\pi^{\otimes(g+4)/2}\right) & \text{if } g \text{ even} \\ \\ \pi_*\left((\det \mathscr{E})^{\otimes 2} \otimes \omega_\pi^{\otimes(g+4)}\right) & \text{if } g \text{ odd} \end{cases} \qquad \text{which has} \qquad c_1(\mathscr{L}_1) = \epsilon a_1$$

and

$$\mathscr{L}_2 = \det f_*(\omega_f) = \det \pi_*(\mathscr{E} \otimes \omega_\pi) \qquad \text{which has} \qquad c_1(\mathscr{L}_2) = \lambda = (g+3)a_1 - a_2'.$$

3.6.4 Simple branching

Let T and D be as in Figure 3.1. In [CL21a, Lemma 7.10], we wrote the classes of T and D in terms of our generators a_1 and a_2':

$$T = (24g + 84)a_1 - 24a_2' \qquad D = -(32g + 112)a_1 + 36a_2'.$$

Note that the coefficient of a_1 is a multiple of ϵ, as it must be because these classes are defined in $\mathrm{Pic}(\mathscr{H}_{5,g}) \subseteq \mathrm{Pic}(\mathcal{H}_{5,g})$.

Proof of Corollary 3.1.3(3). By excision $\mathrm{Pic}(\mathscr{H}_{5,g}^s) = \mathrm{Pic}(\mathscr{H}_{5,g})/\langle T, D\rangle$. Again, row operations over $\mathbb{Z}$ diagonalize the change of basis matrix for $\epsilon a_1, a_2'$ to T, D:

$$\begin{pmatrix} 3 & 2 \\ 4 & 3 \end{pmatrix} \begin{pmatrix} (24g+84)/\epsilon & -24 \\ (-32g-112)/\epsilon & 36 \end{pmatrix} = \begin{pmatrix} (8g+28)/\epsilon & 0 \\ 0 & 12 \end{pmatrix}.$$

For $g \geq 3$, these are the only relations, so $\mathrm{Pic}(\mathscr{H}^s_{5,g})$ is the sum of two cyclic groups of orders equal to the diagonal entries above.

In genus 2, the above gives $44a_1 = 0$ and $12a_2' = 0$, and we have the additional relation

$$0 = 10\lambda = 10(5a_1 - a_2') \qquad \Rightarrow \qquad 0 = 2(3a_1 + a_2')$$

The generators a_1 and $3a_1 + a_2'$ generate cyclic groups of order 44 and 2 respectively. $\square$

This chapter, in full, has been submitted for publication. It is is coauthored with Larson, Hannah. The dissertation author was co-primary investigator and author of this paper.

Chapter 4

The Chow rings of the moduli spaces of curves of genus 7, 8, and 9

4.1 Introduction

In his landmark paper [Mum83], Mumford introduced the Chow ring of the moduli space $\mathcal{M}_g$ of genus g curves. Since then, much progress has been made on the determination of $A^*(\mathcal{M}_g)$ in low genus, which we summarize below.

- ($g = 2$) Mumford [Mum83] in 1983, determined $A^*(\overline{\mathcal{M}}_2)$ with rational coefficients. Vistoli [Vis98] in 1998, determined $A^*(\mathcal{M}_2)$ with integral coefficients, E. Larson [Lar21a] in 2020, determined $A^*(\overline{\mathcal{M}}_2)$ with integral coefficients.
- ($g = 3$) Faber [Fab90a] in 1990, determined $A^*(\overline{\mathcal{M}}_3)$ with rational coefficients, Di Lorenzo–Fulghesu–Vistoli [LFV20] in 2020, determined the integral Chow ring of the locus of smooth plane quartics.
- ($g = 4$) Faber [Fab90b] in 1990, determined $A^*(\mathcal{M}_4)$ with rational coefficients.
- ($g = 5$) Izadi [Iza95] in 1995, determined $A^*(\mathcal{M}_5)$ with rational coefficients.
- ($g = 6$) Penev–Vakil [PV15b] in 2015, determined $A^*(\mathcal{M}_6)$ with rational coefficients.

In each of the above cases, the rational Chow ring of $\mathcal{M}_g$ is equal to the *tautological subring* $R^*(\mathcal{M}_g) \subseteq A^*(\mathcal{M}_g)$, a subring generated by certain natural classes which we now define. Let $f : \mathcal{C} \to \mathcal{M}_g$ be the universal curve. The tautological subring is the subring of $A^*(\mathcal{M}_g)$ generated by the *kappa classes*, $\kappa_i := f_*(c_1(\omega_f)^{i+1})$.

In this paper, we tackle the next open cases of genus $7, 8$, and 9 using the new machinery of tautological classes on the Hurwitz space [CL21d, CL21a]. We prove that the rational Chow rings of $\mathcal{M}_7, \mathcal{M}_8$, and $\mathcal{M}_9$ are all generated by tautological classes, and thereby determine these Chow rings using work of Faber [Fab99]. In addition to our theorems in genus $7, 8$, and 9, our techniques give new and much simpler proofs of the genus 5 and 6 cases (see Section 4.4.3). In particular, in genus 6, we establish that all classes supported on the bielliptic locus are tautological, which was not fully explained in [PV15b].

Theorem 4.1.1. *The Chow ring of the moduli space of genus* 7 *curves is generated by tautological classes. Hence,*

$$A^*(\mathcal{M}_7) \cong \mathbb{Q}[\kappa_1, \kappa_2]/I_7,$$

where I_7 *is the ideal generated by the classes*

$$\begin{cases} 2423\kappa_1^2\kappa_2 - 52632\kappa_2^2 \\ 1152000\kappa_2^2 - 2423\kappa_1^4 \\ 16000\kappa_1^3\kappa_2 - 731\kappa_1^4. \end{cases}$$

The computation of the tautological ring is originally due to Faber [Fab99]. We used the Sage [S$^+$20] package admcycles [DSvZ20] and a program of Pixton [Pix20] to obtain the above presentation and those below.

Theorem 4.1.2. *The Chow ring of the moduli space of genus* 8 *curves is generated by tautological classes. Hence,*

$$A^*(\mathcal{M}_8) = \mathbb{Q}[\kappa_1, \kappa_2]/I_8,$$

where I_8 *is the ideal generated by the classes*

$$\begin{cases} 714894336\kappa_2^2 + 55211328\kappa_1^2\kappa_2 - 1058587\kappa_1^4 \\ 62208000\kappa_1\kappa_2^2 - 95287\kappa_1^5 \\ 144000\kappa_1^3\kappa_2 - 5617\kappa_1^5. \end{cases}$$

Remark 4.1.3. The authors would like to point out contemporaneous work of Maxwell da Paixão de Jesus Santos, which, using different techniques, makes significant progress towards showing $A^*(\mathcal{M}_8)$ is tautological (it is proved that non-tautological classes must be supported on the bielliptic locus).

Theorem 4.1.4. *The Chow ring of the moduli space of genus* 9 *curves is generated by tautological classes. Hence,*

$$A^*(\mathcal{M}_9) = \mathbb{Q}[\kappa_1, \kappa_2, \kappa_3]/I_9,$$

where I_9 *is the ideal generated by the classes*

$$\begin{cases} 5195\kappa_1^4 + 3644694\kappa_1\kappa_3 + 749412\kappa_2^2 - 265788\kappa_1^2\kappa_2 \\ 33859814400\kappa_2\kappa_3 - 95311440\kappa_1^3\kappa_2 + 2288539\kappa_1^5 \\ 19151377\kappa_1^5 + 16929907200\kappa_1\kappa_2^2 - 114345520\kappa_1^3\kappa_2 \\ 1422489600\kappa_3^2 - 983\kappa_1^6 \\ 1185408000\kappa_2^3 - 47543\kappa_1^6. \end{cases}$$

Remark 4.1.5. Despite their complicated looking presentations, the tautological rings above have many nice properties. Faber proved that they are Gorenstein rings with socle in degree $g-2$ [Fab99]. He also points out that $g=9$ is the first case in which the tautological ring is not a complete intersection ring. Several different methods of producing relations among tautological classes in arbitrary genus have found only the Faber–Zagier relations, which may suggest that the Gorenstein property only occurs in low genus cases (see [Pan18] for a discussion).

Remark 4.1.6. An interesting consequence of Theorems 4.1.1, 4.1.2, and 4.1.4 is that for $g = 7, 8, 9$, the cycle class map $A^*(\mathcal{M}_g) \to H^{2*}(\mathcal{M}_g, \mathbb{Q})$ is injective. It is unknown whether this holds in general, and it could even fail quite dramatically: when g is large, it is unknown whether $A^*(\mathcal{M}_g)$ is finite or infinite dimensional as a $\mathbb{Q}$-vector space, whereas cohomology is finitely generated for any algebraic variety. On the moduli space of stable curves, Pikaart [Pik95, Corollary 4.7] has shown that $H^{33}(\overline{\mathcal{M}}_g, \mathbb{Q}) \neq 0$ for g sufficiently large. It then follows from work of Jannsen [Jan94, Theorem 3.6] that the map $A^*(\overline{\mathcal{M}}_g) \to H^{2*}(\overline{\mathcal{M}}_g, \mathbb{Q})$ is not injective for g sufficiently large.

In our previous work about tautological classes on the Hurwitz space [CL21d,CL21a], degree four covers $C \to \mathbb{P}^1$ that factor through a lower genus curve presented a major difficulty. The primary example of this issue is when C is bielliptic, where degree four covers $C \to \mathbb{P}^1$ arise from the double cover $C \to E$ and any double cover $E \to \mathbb{P}^1$, where E is an elliptic curve. A main challenge of this paper is therefore to prove that classes supported on the bielliptic locus of $\mathcal{M}_g$ for $g \leq 9$ are tautological. Indeed, the bielliptic locus is the source of the first known example of a *nontautological* algebraic class on $\mathcal{M}_g$: in [vZ18], van Zelm proves that the fundamental class of the locus of bielliptic curves $\mathcal{B}_{12} \subset \mathcal{M}_{12}$ is nontautological. The techniques we develop for the bielliptic locus in genus $g \leq 9$ also extend to genus 10.

Theorem 4.1.7. *The fundamental class of the bielliptic locus $\mathcal{B}_{10} \subset \mathcal{M}_{10}$ is tautological (hence equal to zero).*

4.1.1 Overview of the proof

Our basic approach is to use the stratification of $\mathcal{M}_g$ by *gonality*, the minimal degree of a map $C \to \mathbb{P}^1$. Precisely, let us define

$$\mathcal{M}_g^k := \{[C] \in \mathcal{M}_g : C \text{ has a } g^1_k\},$$

which is the locus of curves of gonality less than or equal to k. For $g = 7, 8$, a general curve of genus g has gonality 5, so our stratification takes the form

$$\mathcal{M}_g^2 \subset \mathcal{M}_g^3 \subset \mathcal{M}_g^4 \subset \mathcal{M}_g^5 = \mathcal{M}_g.$$

In genus 9, a general curve has gonality 6, so we have one more stratum

$$\mathcal{M}_9^2 \subset \mathcal{M}_9^3 \subset \mathcal{M}_9^4 \subset \mathcal{M}_9^5 \subset \mathcal{M}_9^6 = \mathcal{M}_9.$$

It suffices to show for each k that all classes supported on $\mathcal{M}_g^k$ are tautological up to classes supported on $\mathcal{M}_g^{k-1}$. In other words, we must show that every class in $A^*(\mathcal{M}_g^k \smallsetminus \mathcal{M}_g^{k-1})$ pushes forward to a class in $A^*(\mathcal{M}_g \smallsetminus \mathcal{M}_g^{k-1})$ that is the restriction of a tautological class on $\mathcal{M}_g$.

We shall call a class on $A^*(\mathcal{M}_g \smallsetminus \mathcal{M}_g^{k-1})$ tautological if it is the restriction of a tautological class from $\mathcal{M}_g$. As k increases, each stratum $\mathcal{M}_g^k \smallsetminus \mathcal{M}_g^{k-1}$ becomes more complicated. Our main contribution is a better understanding of the strata for $k = 4, 5$, which was the main stumbling block in extending previous work on low genus curves. We now explain our process in more detail, starting with the curves of lowest gonality and working upwards.

(1) An easy start

Faber [Fab99] showed that the fundamental class of any Brill–Noether locus of the expected dimension is tautological. In particular the fundamental class of $\mathcal{M}_g^k$ is tautological. It is well known that $A^*(\mathcal{M}_g^2) \cong \mathbb{Q}$ for all g. By a result of Patel-Vakil [PV15a], $A^*(\mathcal{M}_g^3 \smallsetminus \mathcal{M}_g^2)$ is generated by the restriction of κ_1 for all $g \neq 3$. Using Faber's result and the push-pull formula this establishes for $g \neq 3$ that

$$\text{all classes supported on } \mathcal{M}_g^3 \text{ are tautological.} \tag{4.1.1}$$

(See Remark 4.2.5 for an alternative argument when $g = 3$.) Note that (4.1.1) already establishes that $A^*(\mathcal{M}_g) = R^*(\mathcal{M}_g)$ for $g \leq 4$. More generally, using the push-pull formula and Faber's result, if the Chow ring of each locally closed stratum $A^*(\mathcal{M}_g^k \smallsetminus \mathcal{M}_g^{k-1})$ were generated by the restrictions of tautological classes for all k, we would be done. However, this is not the case for $k > 3$.

(2) Why it must get harder

For $k = 4, 5$, the Chow ring of $\mathcal{M}_g^k \smallsetminus \mathcal{M}_g^{k-1}$ is *not* in general generated by restrictions of tautological classes. By considering curves of bidegree $(4, 4)$ on $\mathbb{P}^1 \times \mathbb{P}^1$, one can show that when $g \geq 8$, the map $\beta : \mathcal{H}_{4,g} \to \mathcal{M}_g^4$ is an isomorphism away from loci of codimension 2 in both spaces. Therefore, the Picard rank conjecture, proved by Deopurkar–Patel [DP15] for $k = 4, 5$, shows that

$$\dim A^1(\mathcal{M}_g^4 \smallsetminus \mathcal{M}_g^3) = \dim A^1(\mathcal{H}_{4,g}) = 2.$$

Hence, the first Chow group of the locally closed stratum $\mathcal{M}_g^4 \smallsetminus \mathcal{M}_g^3$ cannot be generated by the restriction of κ_1. The analogous result holds for $k = 5$ when $g \geq 10$. Furthermore, it is known that there exist classes supported on $\mathcal{M}_g^4$ that are not tautological in some genera: van Zelm [vZ18] has shown that the fundamental class of the bielliptic locus $\mathcal{B}_{12} \subset \mathcal{M}_{12}$ is *not* a tautological class. This makes the tetragonal locus (Section 4.4) one of the most interesting parts, and it will, of course, require some special observations about genus 7, 8, and 9 curves. (In Section 4.4.3, we also explain how to prove $A^*(\mathcal{M}_g) = R^*(\mathcal{M}_g)$ for $g = 5$ and 6 using our techniques.) In Section 4.4.7, we discuss why our techniques cannot access the bielliptic locus when $g \geq 11$. In the case $g = 10$, we prove Theorem 4.1.7: the class of the bielliptic locus $\mathcal{B}_{10} \subset \mathcal{M}_{10}$ is tautological.

(3) Using the Hurwitz space

Our approach is to study the Chow rings of the Hurwitz stacks $\mathcal{H}_{4,g}$ and $\mathcal{H}_{5,g}$ parametrizing degree 4 and 5 covers, respectively, of the projective line. Let $\beta : \mathcal{H}_{k,g} \to \mathcal{M}_g$ denote the forgetful map. The induced map $\mathcal{H}_{k,g} \smallsetminus \beta^{-1}(\mathcal{M}_g^{k-1}) \to \mathcal{M}_g^k \smallsetminus \mathcal{M}_g^{k-1}$ is proper and surjective, and thus induces a surjection on rational Chow groups. In [CL21a], we showed that for $k = 4, 5$, classes in the *tautological ring of* $\mathcal{H}_{k,g} \smallsetminus \beta^{-1}(\mathcal{M}_g^{k-1})$ (see Definition 4.2.2) push forward to tautological classes on $\mathcal{M}_g \smallsetminus \mathcal{M}_g^{k-1}$ (see Theorem 4.2.4). This is a useful (and non-trivial) tool because there are tautological classes on the Hurwitz space which are *not* pullbacks of tautological classes on $\mathcal{M}_g$. Thus, we wish to show that the Chow rings of $\mathcal{H}_{k,g} \smallsetminus \beta^{-1}(\mathcal{M}_g^{k-1})$ are tautological for $k = 4, 5$ and $g = 7, 8, 9$. We succeed in proving this in each of these cases except $k = 4, g = 9$, where some additional special arguments are used. These arguments are carried out in Section 4.4 for $k = 4$ and Section 4.5 for $k = 5$.

To accomplish this, we further stratify $\mathcal{H}_{k,g}$. Given a cover $\alpha : C \to \mathbb{P}^1$, we define $E_\alpha := (\alpha_* \mathcal{O}_C / \mathcal{O}_{\mathbb{P}^1})^\vee$ and $F_\alpha := \ker(\operatorname{Sym}^2 E_\alpha \to \alpha_* \omega_\alpha^{\otimes 2})$, which are vector bundles on $\mathbb{P}^1$. See Section 4.2.1 for an elaboration of the properties of E_α and F_α. We then stratify $\mathcal{H}_{k,g}$ by the pair of splitting types of E_α and F_α. Each of these "pair splitting loci" has a nice description as a quotient stack (Lemmas 4.3.10 and 4.3.11). As a starting point, our previous work [CL21a] shows that the Chow ring of a union Ψ of the several largest strata is generated by tautological classes (Proposition 4.2.9). This result allows us to narrow down the possible sources of non-tautological classes: they all occur on the complement of Ψ. Some "bad" pair splitting loci Σ_i remain outside of Ψ and not inside $\beta^{-1}(\mathcal{M}_g^{k-1})$. These bad Σ_i are the main focus of this paper. Part of the difficulty of these strata is that they all occur in the "unexpected (pair) codimension" in the sense of Deopurkar–Patel [DP15, Remark 4.2].

(4) The key coincidence and work to be done.

Using universal degeneracy formulas from [Lar21c], we show that the fundamental class of a single splitting locus (i.e. where *one* of the two vector bundles has a given splitting type) is tautological if it occurs in the "expected codimension." Perhaps the most surprising part of the proof is the following coincidence (when $(k, g) \neq (4, 9)$): after excising strata contained in $\beta^{-1}(\mathcal{M}_g^{k-1})$, every bad Σ_i can be realized as a single splitting locus, and that single splitting locus occurs in the expected codimension (proofs of Lemmas 4.4.8, 4.4.9, 4.5.7, 4.5.9). Hence, the fundamental class of the closure of each bad Σ_i is tautological in $A^*(\mathcal{H}_{k,g} \smallsetminus \beta^{-1}(\mathcal{M}_g^{k-1}))$. That these fundamental classes are tautological is a "coincidence of small numbers." It in fact fails for $k = 4$, $g = 12$ by the result of van Zelm [vZ18].

We then study the Chow rings of the locally closed strata Σ_i. Using the description of Σ_i as a quotient, we show that the Chow ring of each stratum $A^*(\Sigma_i \smallsetminus \beta^{-1}(\mathcal{M}_g^{k-1}))$ is generated by restrictions of tautological classes on $\mathcal{H}_{k,g}$ (Sections 4.4.2 and 4.5.2). This last step requires a geometric understanding of the equations that define C inside the associated scroll $\mathbb{P}E_\alpha^\vee$, and when a collection of equations of this type fail to define a smooth curve or produce a curve of gonality less than k. These ideas do extend to arbitrary genus, unlike the results in the previous paragraph concerning fundamental classes. We state them as broadly as possible for arbitrary genera as they may be of future use.

(5) The further work in genus 9

As the genus increases, the luck with fundamental classes starts to run out and more subtle arguments are required. In the case $g = 9, k = 4$, we encounter two bad pair strata Σ_i that occur in unexpected codimension and cannot be realized as a single splitting locus. Although we do not compute their classes on $\mathcal{H}_{4,9}$, we still manage to show that their push forwards to $\mathcal{M}_9 \smallsetminus \mathcal{M}_9^3$ are tautological. For example, one of these problem strata corresponds to the locus of plane sextics with one double point (Lemma 4.4.14). The class of this locus is tautological on $\mathcal{M}_9 \smallsetminus \mathcal{M}_9^3$ because it is a Brill–Noether locus of the expected codimension. We then show that the Chow ring of this stratum is generated by the pullback of κ_1 and κ_2, which is a stronger statement than being generated by restrictions of tautological classes on $\mathcal{H}_{4,9}$. By the push-pull formula, the push forward of every class supported on this stratum is tautological on $\mathcal{M}_9 \smallsetminus \mathcal{M}_9^3$ (though we remain unsure if they are tautological upstairs on $\mathcal{H}_{4,9}$). We deal with the other problem stratum by showing that its union with the bielliptic locus $\beta^{-1}(\mathcal{B}_9)$ has tautological fundamental class on $\mathcal{H}_{4,9} \smallsetminus \beta^{-1}(\mathcal{M}_9^3)$ and a trick explained in Figure 4.5.

In genus 9, we must also deal with curves of gonality 6. The approach we take to these curves in Section 4.6 is quite different from the approach taken to curves of gonality 5 and below because there is no uniform description for degree 6 covers in terms of associated vector bundles. Instead, using results of Mukai [Muk10], we realize $\mathcal{M}_9 \smallsetminus \mathcal{M}_9^5$ (up to a μ_2 gerbe) as a global quotient of an open subvariety of a Grassmannian by Sp_6. The tautological subbundle on the resulting Grassmann bundle over BSp_6 is the Hodge bundle (up to possibly twisting by a line bundle, see Lemma 4.6.9). It then remains to see that Chern classes of the rank 6 vector bundle $\mathcal{V}$ associated to this quotient are tautological. We see this by proving that the rank 21 vector bundle $\mathrm{Sym}^2 \mathcal{V}$ is the bundle of 21 quadrics that cut out a canonical genus 9 curve (Lemma 4.6.10). From this, it follows that the Chern classes of $\mathrm{Sym}^2 \mathcal{V}$ are tautological, and, using the splitting principle, the Chern classes of $\mathcal{V}$ are seen to be tautological as well.

4.1.2 Notations and conventions.

All schemes in this paper are taken over a fixed algebraically closed field of characteristic 0 or $p > 5$. All Chow rings are taken with rational coefficients. We use the subspace convention for projective bundles and Grassmann bundles.

4.2 Hurwitz Schemes and the Tautological Ring

In order to study the loci $\mathcal{M}_g^4 \smallsetminus \mathcal{M}_g^3$ and $\mathcal{M}_g^5 \smallsetminus \mathcal{M}_g^4$, we will study the Hurwitz stacks $\mathcal{H}_{4,g}$ and $\mathcal{H}_{5,g}$ parametrizing degree 4 and 5 covers, respectively, of the projective line.

Definition 4.2.1. The unparametrized Hurwitz stack $\mathcal{H}_{k,g}$ is the stack whose objects over a scheme S are of the form $(C \to P \to S)$ where $P \to S$ is a $\mathbb{P}^1$-fibration, $C \to P$ is a finite flat finitely presented morphism of constant degree k, and the composition $C \to S$ is a family of smooth genus g curves.

The Hurwitz stack $\mathcal{H}_{k,g}$ admits a universal diagram

$$\begin{tikzcd} \mathcal{C} \arrow[r, "\alpha"] \arrow[dr, "f"'] & \mathcal{P} \arrow[d, "\pi"] \\ & \mathcal{H}_{k,g} \end{tikzcd}$$

The universal diagram furnishes several natural classes in the Chow ring of $\mathcal{H}_{k,g}$.

Definition 4.2.2. The tautological ring $R^*(\mathcal{H}_{k,g})$ is the subring of $A^*(\mathcal{H}_{k,g})$ generated by classes of the form

$$f_*(c_1(\omega_f)^i \cdot \alpha^* c_1(\omega_\pi)^j).$$

If $U \subseteq \mathcal{H}_{k,g}$ is an open substack of $\mathcal{H}_{k,g}$, we define the tautological ring of the open $R^*(U)$ to be the image of the tautological ring under the restriction map

$$A^*(\mathcal{H}_{k,g}) \to A^*(U).$$

Remark 4.2.3 (A note on the SL_2 quotient)**.** The Hurwitz stack $\mathcal{H}_{k,g}$ is the PGL_2 quotient of the parametrized Hurwitz scheme $\mathcal{H}^\dagger_{k,g}$. One can also take the quotient of $\mathcal{H}^\dagger_{k,g}$ by SL_2. The map $[\mathcal{H}^\dagger_{k,g}/\mathrm{SL}_2] \to [\mathcal{H}^\dagger_{k,g}/\mathrm{PGL}_2] = \mathcal{H}_{k,g}$ is a μ_2-banded gerbe. It is a general fact that, *with rational coefficients*, the pullback map along any gerbe banded by a finite group induces an isomorphism on Chow rings [PV15b, Section 2.3]. In particular, $A^*(\mathcal{H}_{k,g}) \cong A^*([\mathcal{H}^\dagger_{k,g}/\mathrm{SL}_2])$. The benefit of the SL_2 quotient is that the pullback of the universal $\mathbb{P}^1$-fibration to the SL_2 quotient is a $\mathbb{P}^1$ bundle, i.e. it is equipped with a line bundle of relative degree 1. Since we work with rational coefficients throughout, we do not distinguish the PGL_2 and SL_2 quotients and freely assume that $\mathcal{P}$ is equipped with a line bundle $\mathcal{O}_{\mathcal{P}}(1)$ of relative degree 1. The push forward $\pi_*\mathcal{O}_{\mathcal{P}}(1)$ is the pullback of the universal rank 2 vector bundle on BSL_2.

By forgetting the map $C \to P$, we obtain a morphism

$$\beta : \mathcal{H}_{k,g} \to \mathcal{M}_g.$$

Let

$$\beta' : \mathcal{H}_{k,g} \smallsetminus \beta^{-1}(\mathcal{M}_g^{k-1}) \to \mathcal{M}_g \smallsetminus \mathcal{M}_g^{k-1}$$

be the restriction of β to $\mathcal{H}_{k,g} \smallsetminus \beta^{-1}(\mathcal{M}_g^{k-1})$. The map β' is proper (see e.g. [BV12, Proposition 2.3]). In [CL21a, Theorem 1.7], we showed the following result relating the tautological rings of the relevant Hurwitz stacks and $\mathcal{M}_g$.

Theorem 4.2.4. *Let $k = 4, 5$. The map β' is proper, so the induced push forward map*

$$\beta'_* : A^*(\mathcal{H}_{k,g} \smallsetminus \beta^{-1}(\mathcal{M}_g^{k-1})) \to A^*(\mathcal{M}_g^k \smallsetminus \mathcal{M}_g^{k-1})$$

is surjective. Moreover, $\beta'_*(R^*(\mathcal{H}_{k,g} \smallsetminus \beta^{-1}(\mathcal{M}_g^{k-1}))) \subseteq R^*(\mathcal{M}_g \smallsetminus \mathcal{M}_g^{k-1})$.

Remark 4.2.5. In the case $k = 3$, it is also true that $\beta'_*(R^*(\mathcal{H}_{3,g} \smallsetminus \beta^{-1}(\mathcal{M}_g^2))) \subset R^*(\mathcal{M}_g \smallsetminus \mathcal{M}_g^2)$. For $g \neq 3$, this follows from work of Patel–Vakil [PV15b] which shows $A^*(\mathcal{H}_{3,g}) = R^*(\mathcal{H}_{3,g})$ is generated by $\beta^*\kappa_1$. In genus 3, it turns out $\beta^*\kappa_1 = 0$, so we instead prove the claim as follows. (The following argument does not presuppose $A^*(\mathcal{M}_3) = R^*(\mathcal{M}_3)$ and therefore provides a new proof of this fact in line with our approach.) Let $T \in A^1(\mathcal{H}_{3,3})$ be the class of the locus of covers with a point of triple ramification. By [DP15, Proposition 2.15], we have $A^1(\mathcal{H}_{3,3}) = R^1(\mathcal{H}_{3,3}) = \mathbb{Q} \cdot T$. By [CL21a, Theorem 1.1 (1)], we have $A^i(\mathcal{H}_{3,3}) = 0$ for all $i \geq 2$ and $R^*(\mathcal{H}_{3,3}) = A^*(\mathcal{H}_{3,3})$. By [CL21a, Corollary 7.5], $\beta'_*(T)$ is tautological. Hence, the push forwards of all classes from $\mathcal{H}_{3,3}$ are tautological on $\mathcal{M}_3 \smallsetminus \mathcal{M}_3^2$. This argument is representative of the ideas that were used to prove Theorem

4.2.4 in [CL21a, Theorem 1.7].

In light of Theorem 4.2.4, in order to prove Theorems 4.1.1 and 4.1.2, it suffices to show that $A^*(\mathcal{H}_{4,g} \smallsetminus \beta^{-1}(\mathcal{M}_g^3))$ and $A^*(\mathcal{H}_{5,g} \smallsetminus \beta^{-1}(\mathcal{M}_g^4))$ are generated by tautological classes for $g = 7, 8$. We will prove this in Sections 4.4 and 4.5. More work is required when $g = 9$.

4.2.1 The Casnati–Ekedahl structure theorem

Here, we recall the Casnati–Ekedahl structure theorems for finite Gorenstein covers. The structure theorems furnish distinguished tautological classes, which we call the Casnati–Ekedahl classes, abbreviated CE classes.

We begin with the most general statement, which holds for covers of every degree. Given a degree k cover $\alpha : X \to Y$ where Y is integral, one obtains an exact sequence

$$0 \to \mathcal{O}_Y \to \alpha_*\mathcal{O}_X \to E_\alpha^\vee \to 0, \tag{4.2.1}$$

where E_α is a vector bundle of rank $k-1$. When α is Gorenstein, $\alpha_*\mathcal{O}_X \cong (\alpha_*\omega_\alpha)^\vee$. Pulling back and using adjunction, we therefore obtain a map

$$\omega_\alpha^\vee \to (\alpha^*\alpha_*\omega_\alpha)^\vee \to \alpha^* E_\alpha^\vee,$$

which induces a map $X \to \mathbb{P}E_\alpha^\vee$ that factors $\alpha : X \to Y$. The Casnati–Ekedahl structure theorem gives a resolution of the ideal sheaf of X inside of $\mathbb{P}E_\alpha^\vee$ [CE96].

Theorem 4.2.6 (Casnati–Ekedahl [CE96])**.** *Let X and Y be schemes, Y integral and let $\alpha : X \to Y$ be a Gorenstein cover of degree $k \geq 3$. There exists a unique $\mathbb{P}^{k-2}$-bundle $\gamma : \mathbb{P} \to Y$ and an embedding $i : X \hookrightarrow \mathbb{P}$ such that $\alpha = \gamma \circ i$ and $X_y := \alpha^{-1}(y) \subset \gamma^{-1}(y) \cong \mathbb{P}^{k-2}$ is a nondegenerate arithmetically Gorenstein subscheme for each $y \in Y$. Moreover, the following properties hold.*

1. $\mathbb{P} \cong \mathbb{P}E_\alpha^\vee$ *where* $E_\alpha^\vee := \operatorname{coker}(\mathcal{O}_Y \to \alpha_*\mathcal{O}_X)$*, and* $i^*\mathcal{O}_{\mathbb{P}}(1) \cong \omega_\alpha$*.*

2. There is a unique up to unique isomorphism exact sequence of locally free $\mathcal{O}_{\mathbb{P}}$ *sheaves*

$$0 \to \gamma^* F_{k-2}(-k) \to \gamma^* F_{k-3}(-k+2) \to \cdots \to \gamma^* F_1(-2) \to \mathcal{O}_{\mathbb{P}} \to \mathcal{O}_X \to 0. \quad (4.2.2)$$

where F_i *is locally free on* Y*. The restriction of the exact sequence above to a fiber gives a minimal free resolution of* $X_y := \alpha^{-1}(y)$*. Moreover the resolution is self-dual, so there is a canonical isomorphism* $\mathcal{H}om_{\mathcal{O}_{\mathbb{P}}}(F_i, F_{k-2}) \cong F_{k-2-i}$*.*

3. The ranks of the F_i *are*

$$\operatorname{rank} F_i = \frac{i(k-2-i)}{k-1}\binom{k}{i+1}.$$

4. There is a canonical isomorphism $F_{k-2} \cong \det E_\alpha$*.*

In the cases $k = 4, 5$, self-duality of the resolution determines all of the bundles F_i in terms of E_α and $F_\alpha := F_1$ and tensor products and determinants thereof. Twisting (1.3.3) by $\mathcal{O}_{\mathbb{P}}(2)$ and pushing forward, we see that $F_\alpha = \ker(\operatorname{Sym}^2 E_\alpha \to \alpha_*\omega_\alpha^{\otimes 2})$. We shall use this notation throughout.

Applying this to the universal cover $\alpha : \mathcal{C} \to \mathcal{P}$ over $\mathcal{H}_{k,g}$, we obtain vector bundles $\mathcal{E} := E_\alpha$ and $\mathcal{F} := F_\alpha$ on $\mathcal{P}$. The bundle $\mathcal{E}$ is sometimes called the "universal Tschirnhausen bundle" and has degree $g + k - 1$ on the fibers of $\pi : \mathcal{P} \to \mathcal{H}_{k,g}$ (see e.g. [CL21d, Example 3.1]). Next, let $z := -\frac{1}{2}c_1(\omega_\pi) = c_1(\mathcal{O}_{\mathcal{P}}(1))$. For $i = 1, \ldots, k-1$, we define classes $a_i \in A^i(\mathcal{H}_{k,g})$ and $a_i' \in A^{i-1}(\mathcal{H}_{k,g})$ by the formula

$$a_i := \pi_*(z \cdot c_i(\mathcal{E})), \quad a_i' := \pi_*(c_i(\mathcal{E})) \qquad \Rightarrow \qquad c_i(\mathcal{E}) = \pi^* a_i + \pi^* a_i' z. \qquad (4.2.3)$$

Similarly, we define

$$b_i := \pi_*(z \cdot c_i(\mathcal{F})), \quad b_i' := \pi_*(c_i(\mathcal{F})) \qquad \Rightarrow \qquad c_i(\mathcal{F}) = \pi^* b_i + \pi^* b_i' z. \tag{4.2.4}$$

Finally, we set $c_2 := c_2(\pi_* \mathcal{O}_{\mathcal{P}}(1)) \in A^*(\mathcal{H}_{k,g})$, so $z^2 = -\pi^* c_2 \in A^2(\mathcal{P})$.

Definition 4.2.7. We define $a_i, a_i', b_i, b_i', c_2$ to be the *Casnati–Ekedahl (CE) classes.*

Note that the CE classes generate all π_*'s of polynomials in z and the Chern classes of $\mathcal{E}$ and $\mathcal{F}$. In [CL21d, Theorem 3.10], we proved that the CE classes (together with some suitable generalizations when $k > 5$) are generators for the tautological ring.

Lemma 4.2.8. *The Casnati–Ekedahl classes lie in the tautological ring $R^*(\mathcal{H}_{k,g})$. Conversely, when $k = 4, 5$, every tautological class is a polynomial in the above Casnati–Ekedahl classes.*

Furthermore, we proved in [CL21d, Lemmas 5.3 and 5.11] that the CE classes are generators for the entire Chow ring of a certain open substack of $\mathcal{H}_{k,g}$ when $k = 4, 5$.

Proposition 4.2.9. *Let $g \geq 2$ be an integer. Then the following hold:*

1. *The Chow ring of $\Psi = \mathcal{H}_{4,g} \smallsetminus \operatorname{Supp} R^1\pi_*(\mathcal{F}^\vee \otimes \operatorname{Sym}^2 \mathcal{E})$ is generated by the restrictions of CE classes.*

2. *The Chow ring of $\Psi = \mathcal{H}_{5,g} \smallsetminus \operatorname{Supp} R^1\pi_*(\wedge^2 \mathcal{F} \otimes \mathcal{E} \otimes \det \mathcal{E}^\vee)$ is generated by the restrictions of CE classes.*

Remark 4.2.10. Combining Theorem 4.2.4, Lemma 4.2.8, and Proposition 4.2.9, if we knew that $\operatorname{Supp} R^1\pi_*(\mathcal{F}^\vee \otimes \operatorname{Sym}^2 \mathcal{E})$ were contained in $\beta^{-1}(\mathcal{M}_g^3)$ and $\operatorname{Supp} R^1\pi_*(\wedge^2 \mathcal{F} \otimes \mathcal{E} \otimes \det \mathcal{E}^\vee)$ were contained in $\beta^{-1}(\mathcal{M}_g^4)$, we would be done. However, as we shall see, this is not the case (except when $k = 5, g = 7$, which seems mostly a coincidence).

4.3 Splitting Loci

Every vector bundle E on $\mathbb{P}^1$ splits as a direct sum of line bundles, $E \cong \mathcal{O}(e_1) \oplus \cdots \oplus \mathcal{O}(e_r)$. We call the tuple of integers $\vec{e} = (e_1, \ldots, e_r)$ with $e_1 \leq \cdots \leq e_r$ the *splitting type* of E and abbreviate the corresponding sum of line bundles by $\mathcal{O}(\vec{e}) := \mathcal{O}(e_1) \oplus \cdots \oplus \mathcal{O}(e_r)$.

If E is a vector bundle on a $\mathbb{P}^1$ bundle $\pi : P \to B$, then the base B is stratified by locally closed subvarieties called *splitting loci*

$$\Sigma_{\vec{e}}(E) := \{b \in B : E|_{\pi^{-1}(b)} \cong \mathcal{O}(\vec{e})\}.$$

The above equation describes splitting loci set-theoretically. Below, we give a moduli-theoretic interpretation. Though not necessary here, equations giving a subscheme structure to $\Sigma_{\vec{e}}(E) \subset B$ in terms of Fitting supports can be found in [Lar21c, Section 2].

Suppose W is a rank 2 vector bundle with trivial determinant. We say that a vector bundle E on $\pi : \mathbb{P}W \to B$ is a *family of vector bundles of splitting type* $\vec{e}$ if B admits a cover U_i so that:

- there exist isomorphisms $\psi_i : W|_{U_i} \cong \mathbb{A}^2 \times U_i$ (and therefore $\pi^{-1}(U_i) \cong U_i \times \mathbb{P}^1$).
- there exist isomorphisms $\phi_i : E|_{\pi^{-1}(U_i)} \cong q_i^* \mathcal{O}(\vec{e})$, where $q_i : \pi^{-1}(U_i) \cong U_i \times \mathbb{P}^1 \to \mathbb{P}^1$ is the composition of the isomorphism above with the second projection.

This gives rise to gluing data on the overlaps which satisfy a cocycle condition on the triple overlaps. The data of the vector bundle W is equivalent to the data of a principal SL_2 bundle. A family of vector bundles of splitting type $\vec{e}$, is equivalent to the data of:

- transition functions for W over $U_i \cap U_j$, i.e. maps $\psi_{ij} : U_i \cap U_j \to \mathrm{SL}_2$ satisfying the cocycle condition $\psi_{ik} = \psi_{ij} \circ \psi_{jk}$ on $U_i \cap U_j \cap U_k$
- transition functions for E over $U_i \cap U_j$, i.e. maps $\phi_{ij} : U_i \cap U_j \to \mathrm{Aut}(\mathcal{O}(\vec{e}))$ such that when restricted to the triple overlap $U_i \cap U_j \cap U_k$ we have $\phi_{ik} = \phi_{ij} \circ (\psi_{ij} \cdot \phi_{jk})$

where ψ_{ij} acts on ϕ_{jk} by change of coordinates (made precise below).

The action of SL_2 on $\mathrm{Aut}(\mathcal{O}(\vec{e}))$ that arises above can be described concretely as follows. We have

$$\mathrm{Aut}(\vec{e}) := \mathrm{Aut}(\mathcal{O}(\vec{e})) \subset H^0(\mathbb{P}^1, \mathcal{E}nd(\mathcal{O}(\vec{e})) = \bigoplus_{i,j} H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(e_j - e_i)).$$

We let SL_2 act on a factor $H^0(\mathbb{P}^1, \mathcal{O}(e_j - e_i))$ via the $(e_j - e_i)$th symmetric power of the standard representation (if $e_j - e_i < 0$ then this cohomology group is 0). The cocycle conditions above are described by multiplication in the semidirect product $\mathrm{SL}_2 \ltimes \mathrm{Aut}(\vec{e})$.

By this discussion, a family of vector bundles of splitting type $\vec{e}$ over B determines a principal $\mathrm{SL}_2 \ltimes \mathrm{Aut}(\vec{e})$ bundle on B and vice versa. In other words, the universal $\vec{e}$ splitting locus is the classifying stack $B(\mathrm{SL}_2 \ltimes \mathrm{Aut}(\vec{e}))$. Let us write $\pi : \mathcal{P} \to B(\mathrm{SL}_2 \ltimes \mathrm{Aut}(\vec{e}))$ for the universal $\mathbb{P}^1$ bundle (which is pulled back from BSL_2), and let $\mathcal{V}(\vec{e})$ denote the universal vector bundle of splitting type $\vec{e}$ on $\mathcal{P}$.

Suppose that $\vec{e} = (e_1, \ldots, e_r)$ consists of distinct degrees $d_1 < \cdots < d_s$ and that d_i occurs with multiplicity n_i. Then, we have

$$\mathrm{Aut}(\vec{e}) = \prod_{i=1}^{s} \mathrm{GL}_{n_i} \ltimes \prod_{i<j} H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(d_j - d_i))^{\oplus(n_i n_j)}.$$

Elements of $\mathrm{Aut}(\vec{e})$ can be represented by block upper triangular matrices where the off diagonal entries are polynomials of the specified degrees on $\mathbb{P}^1$.

The SL_2 action is trivial on the block diagonal matrices (the product of GL_{n_i} subgroup). It follows that

$$\mathrm{SL}_2 \ltimes \mathrm{Aut}(\vec{e}) \cong \left(\mathrm{SL}_2 \times \prod_{i=1}^{s} \mathrm{GL}_{n_i}\right) \ltimes \prod_{i<j} H^0(\mathbb{P}^1, \mathcal{O}(d_j - d_i))^{\oplus n_i n_j}. \tag{4.3.1}$$

Hence, we have a map $\mathrm{SL}_2 \ltimes \mathrm{Aut}(\vec{e}) \to \prod_{i=1}^{s} \mathrm{GL}_{n_i}$. Let $\mathcal{N}_i$ on $B(\mathrm{SL}_2 \ltimes \mathrm{Aut}(\vec{e}))$ be the pullback of the tautological rank n_i vector bundle from the BGL_{n_i} factor. The Harder-Narasimhan filtration on the restriction of $\mathcal{V}(\vec{e})$ to each fiber of $\mathcal{P} \to B$ induces a filtration of $\mathcal{V}(\vec{e})$ where the successive quotients are $(\pi^*\mathcal{N}_i)(d_i)$. We call this the *HN filtration* of $\mathcal{V}(\vec{e})$ and we call the bundles $\mathcal{N}_i$ on $B(\mathrm{SL}_2 \ltimes \mathrm{Aut}(\vec{e}))$ the *HN bundles for* $\mathcal{V}(\vec{e})$.

Meanwhile, we also have an inclusion $\mathrm{SL}_2 \times \prod_{i=1}^s \mathrm{GL}_{n_i} \to \mathrm{SL}_2 \ltimes \mathrm{Aut}(\vec{e})$. This induces a map $\varphi : \mathrm{BSL}_2 \times \prod_{i=1}^s \mathrm{BGL}_{n_i} \to B(\mathrm{SL}_2 \ltimes \mathrm{Aut}(\vec{e}))$, which by (4.3.1) is an affine bundle. The pullback $\varphi^*\mathcal{N}_i$ is again the tautological rank n_i bundle coming from the BGL_{n_i} factor. We have the fiber diagram

$$\begin{tikzcd}
\mathcal{P}' \arrow[r, "\varphi'"] \arrow[d, "\pi'"'] & \mathcal{P} \arrow[d, "\pi"] \\
\mathrm{BSL}_2 \times \prod_{i=1}^s \mathrm{BGL}_{n_i} \arrow[r, "\varphi"'] & B(\mathrm{SL}_2 \ltimes \mathrm{Aut}(\vec{e})).
\end{tikzcd}$$

We note in passing that the pullback $\varphi'^*\mathcal{V}(\vec{e})$ on $\mathcal{P}'$ actually splits as a direct sum

$$\varphi'^*\mathcal{V}(\vec{e}) \cong \bigoplus_{i=1}^{s} \varphi'^*(\pi^*\mathcal{N}_i)(d_i).$$

Since φ is a vector bundle map, it induces an isomorphism on Chow. This establishes the following.

Lemma 4.3.1. *The Chow ring of $B(\mathrm{SL}_2 \ltimes \mathrm{Aut}(\vec{e}))$ is the free $\mathbb{Z}$ algebra on the universal c_2 pulled back from BSL_2 and the Chern classes of the HN bundles $\mathcal{N}_1, \dots, \mathcal{N}_s$.*

Remark 4.3.2. The statement above holds with $\mathbb{Z}$-coefficients. We will only use it, however, with $\mathbb{Q}$-coefficients.

The above argument works just as well for a pair of splitting types.

Lemma 4.3.3. *The Chow ring of $B(\mathrm{SL}_2 \ltimes (\mathrm{Aut}(\vec{e}) \times \mathrm{Aut}(\vec{f})))$ is the free $\mathbb{Z}$ algebra on the universal c_2 pulled back from BSL_2, the Chern classes of the HN bundles for $\mathcal{V}(\vec{e})$, and Chern classes of the HN bundles for $\mathcal{V}(\vec{f})$.*

From our description of the universal $\vec{e}$ splitting locus, one sees that its codimension in the moduli stack of vector bundles on $\mathbb{P}^1$ bundles is $h^1(\mathbb{P}^1, \mathcal{E}nd(\mathcal{O}(\vec{e})))$. Given a family of vector bundles on $\mathbb{P}^1$ bundles with splitting type $\vec{e}$, we say that $h^1(\mathbb{P}^1, \mathcal{E}nd(\mathcal{O}(\vec{e})))$ is the *expected codimension*. It follows that, if non-empty, the codimension of a splitting locus is always at most the expected codimension. There is a partial ordering on splitting types defined by $\vec{e}' \leq \vec{e}$ if $e'_1 + \ldots + e'_j \leq e_1 + \ldots + e_j$ for all j. On the moduli space of vector bundles on $\mathbb{P}^1$ bundles, the $\vec{e}'$ splitting locus is in the closure of the $\vec{e}$ splitting locus if and only if $\vec{e}' \leq \vec{e}$. (Of course, this need not be the case in every family, as we shall see.) Since codimension can only decrease under pullback, this implies the following fact

$$\text{Every component of } \bigcup_{\vec{e}' \leq \vec{e}} \Sigma_{\vec{e}}(E) \text{ has at most the expected codimension.} \tag{4.3.2}$$

We note that the union $\bigcup_{\vec{e}' \leq \vec{e}} \Sigma_{\vec{e}}(E)$ may not be the closure of $\Sigma_{\vec{e}}(E)$, but it is always closed in the base.

Definition 4.3.4 (Stratifications)**.** Throughout this paper a *stratification* of B shall mean a disjoint union $B = \bigsqcup_{S \in \mathcal{S}} S$ into locally closed subvarieties (or substacks) equipped with a partial ordering $S' \leq S$ such that for each $S \in \mathcal{S}$, the union $\bigcup_{S' \leq S} S'$ is closed in B.

Example 4.3.5 (Warning)**.** Our notion of stratification is weaker than some in the literature. For example, say B is the union of the two coordinate axes $B = V(xy) \subset \operatorname{Spec} k[x, y] \cong \mathbb{A}^2$. Then $\mathcal{S} = \{V(y), V(x) \smallsetminus (0, 0)\}$ is a stratification of B with partial order $V(y) \leq V(x) \smallsetminus (0, 0)$. We represent this partial order diagramatically as pictured on the right.

We make use of the following key result from [Lar21c].

Theorem 4.3.6 (Theorem 1.2 of [Lar21c])**.** *Let E be a vector bundle on a $\mathbb{P}^1$ bundle $\pi : P \to B$. Suppose that $\Sigma_{\vec{e}}(E)$ occurs in the expected codimension. Then, modulo classes supported on $\Sigma_{\vec{e}'}(E)$ for $\vec{e}' < \vec{e}$, the fundamental class of the closure of $\Sigma_{\vec{e}}(E)$ is given by*

Figure 4.1. Example of a stratification

a universal formula in terms of the Chern classes of $\pi_*\mathcal{O}_P(1), \pi_*E(m-1)$ *and* $\pi_*E(m)$ *for some* m *suitably large.*

Applying this to the universal CE bundles, we obtain the following.

Lemma 4.3.7. *Let* $\mathcal{E}$ *and* $\mathcal{F}$ *be the universal CE bundles on* $\mathcal{P} \to \mathcal{H}_{k,g}$*. If* $\Sigma_{\vec{e}}(\mathcal{E})$ *occurs in the expected codimension, then, modulo classes supported on* $\Sigma_{\vec{e}\,'}(\mathcal{E})$ *for* $\vec{e}\,' < \vec{e}$ *the fundamental class of its closure is expressible in terms of CE classes. The analogous statement holds for the classes of* $\Sigma_{\vec{f}}(\mathcal{F})$*.*

Proof. Recall that the class $c_2 = c_2(\pi_*\mathcal{O}_{\mathcal{P}}(1))$ is a CE class by definition (and $c_1(\mathcal{O}_{\mathcal{P}}(1)) = 0$ because we are working over BSL_2). For m suitably large, we have $R^1\pi_*\mathcal{E}(m) = 0$ and $R^1\pi_*\mathcal{E}(m-1) = 0$, so by Grothendieck–Riemann–Roch, the Chern classes of $\pi_*\mathcal{E}(m)$ and $\pi_*\mathcal{E}(m-1)$ are polynomials in the CE classes. Similarly, the Chern classes of $\pi_*\mathcal{F}(m)$ and $\pi_*\mathcal{F}(m-1)$ are polynomials in the CE classes for m suitably large. The result now follows from Theorem 4.3.6. $\square$

We shall need a slight variant of the above lemma concerning particular unions of splitting loci (some of which will be allowed to occur in the wrong codimension). Let us define

$$\Sigma_{(n,*,\ldots,*)}(E) := \bigcup_{e_1=n} \Sigma_{\vec{e}}(E)$$

In [Lar21c, Lemma 5.1], it was shown that if the above union occurs in its expected codimension, equal to $\deg(\vec{e})+1-(n+1)r$, then — modulo classes supported on $\Sigma_{(n',*,\ldots,*)}(E)$ for $n' < n$ — its fundamental class can be computed with the Porteous formula. In particular, it is expressible in terms of the Chern classes of $\pi_*E(m-1), \pi_*E(m)$ and $\pi_*\mathcal{O}_P(1)$ for m suitably large. Arguing as in Lemma 4.3.7, we obtain the following result.

Lemma 4.3.8. *Suppose that every component of the union $\Sigma_{(n,*,\ldots,*)}(\mathcal{E})$ occurs in codimension $\deg(\vec{e})+1-(n+1)r$. Then, modulo classes supported on $\Sigma_{(n',*,\ldots,*)}(\mathcal{E})$ for $n' < n$, the fundamental class of the closure of $\Sigma_{(n,*,\ldots,*)}(\mathcal{E})$ is expressible in terms of CE classes.*

This is useful to us as illustrated in the following example.

Example 4.3.9. The expected codimension for splitting type $(2,4,6)$ is 5, but suppose that $\Sigma_{(2,4,6)}(\mathcal{E})$ occurs in codimension 4. Suppose $\Sigma_{(2,5,5)}(\mathcal{E})$ also occurs in codimension 4 and $\Sigma_{(2,3,7)}(\mathcal{E})$ and $\Sigma_{(2,2,8)}(\mathcal{E})$ are empty. Then, we have $\Sigma_{(2,*,*)}(\mathcal{E}) = \Sigma_{(2,4,6)}(\mathcal{E}) \cup \Sigma_{(2,5,5)}(\mathcal{E})$, and every component occurs occurs in codimension $4 = 13-9 = \deg(\vec{e})+1-(2+1)(3)$. Thus, the above lemma shows that the fundamental class of the union $\Sigma_{(2,4,6)}(\mathcal{E}) \cup \Sigma_{(2,5,5)}(\mathcal{E})$ is expressible in terms of CE classes (modulo classes supported on $\Sigma_{(n',*,*)}(\mathcal{E})$ for $n' < 2$).

4.3.1 Pair splitting loci on $\mathcal{H}_{4,g}$

Let $\mathcal{E}$ and $\mathcal{F}$ be the universal CE bundles on $\mathcal{P}$, the universal $\mathbb{P}^1$-bundle on $\mathcal{H}_{4,g}$. Let $\vec{e}$ be a splitting type of rank 3 and degree $g+3$, and let $\vec{f}$ be a splitting type of rank 2 and degree $g+3$. Each splitting locus of the form $\Sigma := \Sigma_{\vec{e}}(\mathcal{E}) \cap \Sigma_{\vec{f}}(\mathcal{F})$ has a concrete description as a quotient stack. This description seems well-known in the literature, but with slightly different presentations (see for example [DP15, p. 20], [CE96, Theorem 4.4], and [CDC02, Section 3]). Here, we outline our preferred way of thinking about this quotient, following our set-up in [CL21d, Section 3].

The vector space

$$\Phi : H^0(\mathbb{P}^1, \mathcal{O}(\vec{f})^\vee \otimes \operatorname{Sym}^2 \mathcal{O}(\vec{e})) \xrightarrow{\sim} H^0(\mathbb{P}\mathcal{O}(\vec{e})^\vee, \gamma^*\mathcal{O}(\vec{f})^\vee \otimes \mathcal{O}_{\mathbb{P}\mathcal{O}(\vec{e})^\vee}(2))$$

parametrizes pencils of relative quadrics on the $\mathbb{P}^1$ bundle $\mathbb{P}\mathcal{O}(\vec{e})^\vee$. Let

$$U \subset H^0(\mathbb{P}^1, \mathcal{O}(\vec{f})^\vee \otimes \operatorname{Sym}^2 \mathcal{O}(\vec{e}))$$

be the open subset of sections η whose vanishing locus $V(\Phi(\eta)) \subset \mathbb{P}\mathcal{O}(\vec{e})^\vee$ defines a smooth, irreducible quadruple cover of $\mathbb{P}^1$. Considering its Hilbert polynomial, one can show that such a cover will have genus g. It turns out — essentially from the Casnati–Ekedahl structure theorem — that all degree 4, genus g covers $\alpha : C \to \mathbb{P}^1$ with $E_\alpha \cong \mathcal{O}(\vec{e})$ and $F_\alpha \cong \mathcal{O}(\vec{f})$ arise in this way. We make this precise below.

There is a natural action of $\mathrm{SL}_2 \ltimes (\mathrm{Aut}(\vec{e}) \times \mathrm{Aut}(\vec{f}))$ on U. Since $\vec{e}$ and $\vec{f}$ are the same degree, we have $\det \mathcal{O}(\vec{e}) \otimes \det \mathcal{O}(\vec{f})^\vee \cong \mathcal{O}_{\mathbb{P}^1}$, so $\mathrm{SL}_2 \ltimes (\mathrm{Aut}(\vec{e}) \times \mathrm{Aut}(\vec{f}))$ also acts on a copy of $\mathbb{G}_m \subset H^0(\mathbb{P}^1, \det \mathcal{O}(\vec{e}) \otimes \det \mathcal{O}(\vec{f})^\vee)$. Our discussion will be simplified slightly by considering also the framed Hurwitz space $\rho : \mathcal{H}_{k,g}^\dagger \to \mathcal{H}_{k,g}$ (see Remark 4.2.3). Let us write $\Sigma^\dagger := \rho^{-1}(\Sigma)$, so $\Sigma = [\Sigma^\dagger / \mathrm{SL}_2]$. This allows us to think about the quotient in two steps.

Lemma 4.3.10. *We have* $\Sigma^\dagger \cong [(U \times \mathbb{G}_m)/\operatorname{Aut}(\vec{e}) \times \operatorname{Aut}(\vec{f})]$*, and therefore*

$$\Sigma = [\Sigma^\dagger / \mathrm{SL}_2] \cong [(U \times \mathbb{G}_m)/\, \mathrm{SL}_2 \ltimes (\operatorname{Aut}(\vec{e}) \times \operatorname{Aut}(\vec{f}))].$$

Proof. We shall prove the statement for the framed stratum $\Sigma^\dagger$, from which the second statement follows. By definition, $\Sigma^\dagger \subset \mathcal{H}_{4,g}$ parametrizes covers $\alpha : C \to \mathbb{P}^1$ such that $E_\alpha \cong \mathcal{O}(\vec{e})$ and $F_\alpha \cong \mathcal{O}(\vec{e})$. As a fiber category, the objects of $\Sigma^\dagger(S)$ for a scheme S are degree 4 covers $\alpha : C \to \mathbb{P}^1 \times S$ such that

1. E_α on $\mathbb{P}^1 \times S$ is a family of vector bundles of splitting type $\vec{e}$
2. F_α on $\mathbb{P}^1 \times S$ is a family of vector bundles on splitting type $\vec{f}$.
3. $C \to S$ is a family of smooth genus g curves.

The morphisms in $\Sigma^\dagger(S)$ are isomorphisms of covers over $\mathbb{P}^1 \times S$. The category $\Sigma^\dagger(S)$ is the subcategory of $\mathrm{Quad}(\mathbb{P}^1 \times S)$ from [CL21d, Section 3.2] where we impose the additional conditions on the splitting types in (1) and (2), and the smoothness in condition (3).

Let $G := \mathrm{Aut}(\vec{e}) \times \mathrm{Aut}(\vec{f})$. As explained in the start of this section, a principal $\mathrm{Aut}(\vec{e})$ bundle is equivalent to a family of vector bundles of splitting type $\vec{e}$ on $\mathbb{P}^1 \times S$. Via this identification, $[(U \times \mathbb{G}_m)/G](S)$ is the category whose objects are tuples (E, F, ϕ, η) where

(1′) E on $\mathbb{P}^1 \times S$ is a family of vector bundles of splitting type $\vec{e}$

(2′) F on $\mathbb{P}^1 \times S$ is a family of vector bundles of splitting type $\vec{f}$

(3′) ϕ is an isomorphism $\det E \cong \det F$

(4′) η is a global section of $F^\vee \otimes \mathrm{Sym}^2 E$ such that $V(\eta) \subset \mathbb{P}E^\vee \to \mathbb{P}^1 \times S$ is a degree 4 cover over S, and the composition $V(\eta) \to S$ is a family of smooth curves.

An arrow $(E_1, F_1, \phi_1, \eta_1)$ to $(E_2, F_2, \phi_2, \eta_2)$ is a pair of isomorphisms $\xi : E_1 \to E_2$, and $\psi : F_1 \to F_2$, such that the following diagrams commute

$$\begin{tikzcd} F_1 \arrow[r, "\eta_1"] \arrow[d, "\psi"'] & \mathrm{Sym}^2 E_1 \arrow[d, "\mathrm{Sym}^2 \xi"] \\ F_2 \arrow[r, "\eta_2"] & \mathrm{Sym}^2 E_2 \end{tikzcd} \qquad \begin{tikzcd} \det F_1 \arrow[r, "\phi_1"] \arrow[d, "\det \psi"'] & \det E_1 \arrow[d, "\det \xi"] \\ \det F_2 \arrow[r, "\phi_2"] & \det E_2. \end{tikzcd}$$

Thus, the category $[(U \times \mathbb{G}_m)/G](S)$ is the subcategory of $\mathrm{Quad}'(\mathbb{P}^1 \times S)$ from [CL21d, Section 3.2] where we impose the additional conditions on the splitting types in (1′) and (2′) and the smoothness in condition (4′).

There is a natural map $[(U \times \mathbb{G}_m)/G] \to \Sigma^\dagger$ that sends a tuple (E, F, ϕ, η) over S to the degree 4 cover $V(\Phi(\eta)) \subset \mathbb{P}E^\vee \to \mathbb{P}^1 \times S$. Theorem 3.6 of [CL21d] showed that a corresponding map $\mathrm{Quad}'(\mathbb{P}^1 \times S) \to \mathrm{Quad}(\mathbb{P}^1 \times S)$ is an equivalence of categories. The argument there restricts to give an equivalence of the subcategories $[(U \times \mathbb{G}_m)/G](S)$ and $\Sigma^\dagger(S)$. $\square$

It follows from Lemma 4.3.10 that the $\vec{e}, \vec{f}$ splitting locus is irreducible of codimension

$$h^1(\mathbb{P}^1, \mathcal{E}nd(\mathcal{O}(\vec{e}))) + h^1(\mathbb{P}^1, \mathcal{E}nd(\mathcal{O}(\vec{f}))) - h^1(\mathbb{P}^1, \mathcal{O}(\vec{f})^\vee \otimes \operatorname{Sym}^2 \mathcal{O}(\vec{e})) \tag{4.3.3}$$

inside $\mathcal{H}_{4,g}$ (see also [DP15, Remark 4.2]). In light of Proposition 4.2.9, we are primarily concerned with the $\vec{e}, \vec{f}$ splitting loci for which $h^1(\mathbb{P}^1, \mathcal{O}(\vec{f})^\vee \otimes \operatorname{Sym}^2 \mathcal{O}(\vec{e})) \neq 0$, equivalently $2e_1 - f_2 \leq -2$. By (4.3.3) these are the pair splitting loci whose codimension is *not* the sum of the expected codimensions for $\vec{e}$ and $\vec{f}$.

4.3.2 Pair splitting loci on $\mathcal{H}_{5,g}$

Let $\mathcal{E}$ and $\mathcal{F}$ be the universal CE bundles on $\pi : \mathcal{P} \to \mathcal{H}_{5,g}$. Let $\vec{e}$ be a splitting type of rank 4 and degree $g + 4$, and let $\vec{f}$ be a splitting type of rank 5 and degree $2(g + 4)$. Similar to the previous subsection, we describe each splitting locus of the form $\Sigma := \Sigma_{\vec{e}}(\mathcal{E}) \cap \Sigma_{\vec{f}}(\mathcal{F})$ as a quotient stack. Again this description is well-known, though in varying language (see for example [DP15, p. 24], [Cas96, Theorem 3.8]). We give a presentation following our set up in [CL21d, Section 3].

In degree 5, the relevant space of section is

$$\begin{aligned}\Phi : H^0(&\mathbb{P}^1, \mathcal{O}(\vec{e}) \otimes \mathcal{O}(-g-4) \otimes \wedge^2 \mathcal{O}(\vec{f})) \\ &\xrightarrow{\sim} H^0(\mathbb{P}\mathcal{O}(\vec{e})^\vee, \mathcal{O}_{\mathbb{P}\mathcal{O}(\vec{e})^\vee}(1) \otimes \gamma^*(\mathcal{O}_{\mathbb{P}^1}(-g-4) \otimes \wedge^2 \mathcal{O}(\vec{f})))\end{aligned}$$

Sections of the right-hand side are represented by 5×5 skew-symmetric matrices M of linear forms on $\mathbb{P}\mathcal{O}(\vec{e})^\vee$. Given such a matrix M, we write $D(M) \subset \mathbb{P}\mathcal{O}(\vec{e})^\vee$ to mean the subscheme defined by the 4×4 Pfaffians of M (see Section 4.5 for explicit equations in coordinates). These Pfaffians correspond to the equations of the Grassmann bundle $G(2, \mathcal{O}(\vec{f})) \subset \mathbb{P}(\wedge^2 \mathcal{O}(\vec{f}))$ under its relative Plücker embedding, as we now explain. A section

$$\eta \in H := H^0(\mathbb{P}^1, \mathcal{O}(\vec{e}) \otimes \mathcal{O}(-g-4) \otimes \wedge^2 \mathcal{O}(\vec{f}))$$

can be viewed as a linear map $\eta : \mathcal{O}(\vec{e})^\vee \otimes \mathcal{O}(g+4) \to \wedge^2\mathcal{O}(\vec{f})$. If this map is injective with locally free cokernel, then $D(\Phi(\eta)) \subset \mathbb{P}\mathcal{O}(\vec{e})^\vee$ is the intersection of $\eta(\mathbb{P}\mathcal{O}(\vec{e})^\vee)$ with $G(2, \mathcal{O}(\vec{f})) \subset \mathbb{P}(\wedge^2\mathcal{O}(\vec{f}))$. The Grassmann bundle $G(2, \mathcal{O}(\vec{f})) \subset \mathbb{P}(\wedge^2\mathcal{O}(\vec{f}))$ has degree 5 and codimension 3 in each fiber over $\mathbb{P}^1$, so one expects this intersection to be a degree 5 cover of $\mathbb{P}^1$.

Let $U \subset H$ be the open subvariety of sections η such that $D(\Phi(\eta))$ is a smooth, irreducible degree 5 cover of $\mathbb{P}^1$. Considering its Hilbert polynomial, one can show that such a cover will have genus g. It turns out — essentially from the Casnati–Ekedahl structure theorem and further work of Casnati [Cas96] — that all degree 5, genus g smooth covers $\alpha : C \to \mathbb{P}^1$ with $E_\alpha \cong \mathcal{O}(\vec{e})$ and $F_\alpha \cong \mathcal{O}(\vec{f})$ arise in this way.

Precisely, there is a natural action of $\mathrm{SL}_2 \ltimes (\mathrm{Aut}(\vec{e}) \times \mathrm{Aut}(\vec{f}))$ on U. Since $\deg(\vec{f}) = 2\deg(\vec{e})$, we have $\det \mathcal{O}(\vec{e})^{\otimes 2} \otimes \det \mathcal{O}(\vec{f})^\vee \cong \mathcal{O}_{\mathbb{P}^1}$, so $\mathrm{SL}_2 \ltimes (\mathrm{Aut}(\vec{e}) \times \mathrm{Aut}(\vec{f}))$ acts on a copy of $\mathbb{G}_m \subset H^0(\mathbb{P}^1, \det \mathcal{O}(\vec{e})^{\otimes 2} \otimes \det \mathcal{O}(\vec{f})^\vee)$. As in the previous subsection, we will consider the quotient in two steps. Let $\rho : \mathcal{H}^\dagger_{5,g} \to \mathcal{H}_{5,g}$ be the parametrized Hurwitz space and set $\Sigma^\dagger := \rho^{-1}(\Sigma)$ so $\Sigma = [\Sigma^\dagger / \mathrm{SL}_2]$.

Lemma 4.3.11. *We have $\Sigma^\dagger \cong [(U \times \mathbb{G}_m)/\mathrm{Aut}(\vec{e}) \times \mathrm{Aut}(\vec{f})]$. Therefore,*

$$\Sigma = [\Sigma^\dagger / \mathrm{SL}_2] \cong [(U \times \mathbb{G}_m)/\mathrm{SL}_2 \ltimes (\mathrm{Aut}(\vec{e}) \times \mathrm{Aut}(\vec{f}))].$$

Proof. The proof is very similar to Lemma 4.3.10. There is a map

$$[(U \times \mathbb{G}_m)/\mathrm{Aut}(\vec{e}) \times \mathrm{Aut}(\vec{f})] \to \Sigma^\dagger$$

that comes from sending a section $\eta \in U$ to the associated cover $D(\Phi(\eta)) \to \mathbb{P}^1$. The categories $\Sigma^\dagger(S)$ and $[(U \times \mathbb{G}_m)/\mathrm{Aut}(\mathcal{O}(\vec{e})) \times \mathrm{Aut}(\mathcal{O}(\vec{f}))](S)$ are readily seen to be subcategories of $\mathrm{Pent}(\mathbb{P}^1 \times S)$ and $\mathrm{Pent}'(\mathbb{P}^1 \times S)$ respectively, defined in [CL21d, Section 3.3]; these two subcategories are seen to be equivalent under the equivalence given in [CL21d, Theorem 3.8]. $\square$

It follows from Lemma 4.3.11 that the $\vec{e}, \vec{f}$ splitting locus is irreducible of codimension

$$h^1(\mathbb{P}^1, \mathcal{E}nd(\mathcal{O}(\vec{e}))) + h^1(\mathbb{P}^1, \mathcal{E}nd(\mathcal{O}(\vec{f}))) - h^1(\mathbb{P}^1, \mathcal{O}(\vec{e}) \otimes \mathcal{O}(-g-4) \otimes \wedge^2 \mathcal{O}(\vec{f})) \quad (4.3.4)$$

inside $\mathcal{H}_{5,g}$. In light of Proposition 4.2.9, our primary interest will be in strata where the last term $h^1(\mathbb{P}^1, \mathcal{O}(\vec{e}) \otimes \mathcal{O}(-g-4) \otimes \wedge^2 \mathcal{O}(\vec{f})) \neq 0$, or equivalently $e_1 + f_1 + f_2 - (g+4) \leq -2$.

4.4 The Tetragonal Locus

In this section, we study the stratification of $\mathcal{H}_{4,g}$ by the pair splitting loci of the CE bundles $\mathcal{E}$ and $\mathcal{F}$. Given a degree 4, genus g cover $\alpha : C \to \mathbb{P}^1$, we let $E = E_\alpha$ and $F = F_\alpha$ be the associated vector bundles as in Section 4.2.1. Since they are vector bundles on $\mathbb{P}^1$, the bundles E and F split.

$$E = \mathcal{O}(e_1) \oplus \mathcal{O}(e_2) \oplus \mathcal{O}(e_3) \qquad\qquad e_1 \leq e_2 \leq e_3$$

and

$$F = \mathcal{O}(f_1) \oplus \mathcal{O}(f_2) \qquad\qquad f_1 \leq f_2.$$

In this section, we use the roman font, E and F, to denote vector bundles of a fixed splitting type. By slight abuse of notation, we sometimes write $E = \vec{e}$ to mean $E \cong \mathcal{O}(\vec{e})$.

When C is not hyperelliptic, the splitting type of E can be interpreted geometrically as follows: under the canonical embedding, the fibers of α span a 2-plane. The union of these two planes is called the *associated* 3-*fold scroll.* The embedding $C \subset \mathbb{P}E^\vee$ given by the Casnati–Ekedahl theorem is constructed so that $\mathcal{O}_{\mathbb{P}E^\vee}(1)|_C = \omega_\alpha = \omega_C \otimes \alpha^*\omega_{\mathbb{P}^1}^\vee$. Let $\gamma : \mathbb{P}E^\vee \to \mathbb{P}^1$ be the structure map. Then, the associated scroll is the image of $\mathbb{P}E^\vee \to \mathbb{P}^{g-1}$ via the line bundle $\mathcal{O}_{\mathbb{P}E^\vee}(1) \otimes \gamma^*\omega_{\mathbb{P}^1}$ on $\mathbb{P}E^\vee$.

Meanwhile, the bundle F parametrizes the pencil of relative quadrics that define $C \subset \mathbb{P}E^\vee$. If X, Y, Z are relative coordinates on $\mathbb{P}E^\vee$ corresponding to a splitting, then the pencil is generated by

$$p = p_{1,1}X^2 + p_{1,2}XY + p_{2,2}Y^2 + p_{1,3}XZ + p_{2,3}YZ + p_{3,3}Z^2 \tag{4.4.1}$$

$$q = q_{1,1}X^2 + q_{1,2}XY + q_{2,2}Y^2 + q_{1,3}XZ + q_{2,3}YZ + q_{3,3}, Z^2, \tag{4.4.2}$$

where $p_{i,j}$ and $q_{i,j}$ are polynomials on $\mathbb{P}^1$ of degrees

$$\deg(p_{i,j}) = e_i + e_j - f_1 \qquad \text{and} \deg(q_{i,j}) = e_i + e_j - f_2.$$

For a stratum to be non-empty, $\vec{e}$ and $\vec{f}$ must satisfy certain constraints, which we collect below. Considering the defining sequence (4.2.1) of E_α, we see that $\deg(E) = -\deg(\alpha_*\mathcal{O}_C) = -\chi(\alpha_*\mathcal{O}_C) + 4 = g + 3$. By [CE96, Theorem 4.4], one must have $\det E \cong \det F$, so

$$e_1 + e_2 + e_3 = f_1 + f_2 = g + 3. \tag{4.4.3}$$

For a cover to be irreducible, we must have $1 = h^0(C, \mathcal{O}_C) = h^0(\mathbb{P}^1, E^\vee) + 1$. This implies $e_1 \geq 1$. An upper bound on the largest part was given in [DP15, Proposition 2.6]:

$$e_1 \geq 1 \qquad \text{and} \qquad e_3 \leq \frac{g+3}{2}. \tag{4.4.4}$$

It is well-known (see e.g. [Sch86, p. 127]) that

$$e_1 = 1 \text{ if and only if } C \text{ is hyperelliptic,} \tag{4.4.5}$$

in which case α factors as $C \xrightarrow{h} \mathbb{P}^1 \xrightarrow{i} \mathbb{P}^1$, where $h : C \to \mathbb{P}^1$ is the hyperelliptic map and $i : \mathbb{P}^1 \to \mathbb{P}^1$ is a degree 2 cover.

We now turn to the geometry of the quadrics that cut out C. If $p_{1,1} = 0$ and $q_{1,1} = 0$, then $V(p,q)$ contains the section $Y = Z = 0$. Thus,

$$p_{1,1} \text{ and } q_{1,1} \text{ cannot both be } 0 \qquad \Rightarrow \qquad 2e_1 \geq f_1. \tag{4.4.6}$$

If $q_{1,1} = q_{1,2} = q_{2,2} = 0$, then the quadric q is divisible by Z. That is, $V(q)$ is reducible, so C, being irreducible, must lie in one component. Then fibers of $C \to \mathbb{P}^1$ would then each span a line under the canonical embedding, giving C a g^2_4, which is impossible when $g > 3$.

$$q_{1,1}, q_{1,2} \text{ and } q_{2,2} \text{ cannot all be } 0 \qquad \Rightarrow \qquad 2e_2 \geq f_2. \tag{4.4.7}$$

On the other hand, if $q_{1,1} = q_{1,2} = q_{1,3} = 0$, then $V(q)$ is singular all along the section $Y = Z = 0$. Therefore, in order for C to be smooth, no other quadric in the pencil can vanish at any point along the section $Y = Z = 0$:

$$\text{if } q_{1,1} = q_{1,2} = q_{1,3} = 0, \text{ then } p_{1,1} \text{ must be non-vanishing on } \mathbb{P}^1. \tag{4.4.8}$$

In terms of splitting types this implies,

$$\text{if } f_2 > e_1 + e_3, \text{ then } f_1 = 2e_1. \tag{4.4.9}$$

Let us write $\Psi := \mathcal{H}_{4,g} \smallsetminus \operatorname{Supp} R^1\pi_*(\mathcal{F}^\vee \otimes \operatorname{Sym}^2 \mathcal{E})$. By Proposition 4.2.9, we know that $A^*(\Psi)$ is generated by tautological classes. The complement of Ψ is the union of splitting loci which satisfy $2e_1 - f_2 \leq -2$. We will therefore need some results concerning the Chow rings of locally closed strata $\Sigma_{\vec{e}}(\mathcal{E}) \cap \Sigma_{\vec{f}}(\mathcal{F})$ for such $\vec{e}, \vec{f}$, which we prove in Section 4.4.2. In Sections 4.4.4, 4.4.5, 4.4.6, we specialize to the cases $g = 7, 8, 9$ respectively.

4.4.1 Strategy

Our basic strategy will be as follows:

1. Use conditions (4.4.3) – (4.4.9) to determine the allowed pairs of splitting types $\vec{e}, \vec{f}$. The partial order on splitting types of Section 4.3 induces a partial order on pairs of splitting types by $(\vec{e}', \vec{f}') \leq (\vec{e}, \vec{f})$ if $\vec{e}' \leq \vec{e}$ and $\vec{f}' \leq \vec{f}$.

2. Starting with strata at the bottom of our $\leq$ order and working upwards, show that for each stratum outside of Ψ, at least one of the following is satisfied:

   (a) the stratum is contained in $\beta^{-1}(\mathcal{M}_g^3)$

   (b) its fundamental class in $\mathcal{H}_{4,g} \smallsetminus \beta^{-1}(\mathcal{M}_g^3)$ is tautological (modulo classes supported on strata below it in the partial order) *and* the Chow ring of the locally closed stratum is generated by the restrictions of CE classes.

   (c) the push forward of its fundamental class to $\mathcal{M}_g \smallsetminus \mathcal{M}_g^3$ is tautological *and* the Chow ring of the locally closed stratum is generated by the restrictions of κ_1, κ_2.

Case (c) will only be needed in genus 9; thus, in genus 7 and 8, we will actually establish that $A^*(\mathcal{H}_{4,g} \smallsetminus \beta^{-1}(\mathcal{M}_g^3))$ is generated by CE classes.

Remark 4.4.1. We note the "trade-off" between choices (b) and (c) above. If a class is tautological on $\mathcal{H}_{4,g} \smallsetminus \beta^{-1}(\mathcal{M}_g^3)$, then its push forward to $\mathcal{M}_g \smallsetminus \mathcal{M}_g^3$ is tautological by Theorem 4.2.4. On the other hand, κ_1 and κ_2 are CE classes, but need not generate all CE classes. Therefore, in (b) if we prove the stronger statement about the fundamental class, we only need the weaker statement about the Chow ring; in (c) if we only prove the weaker condition about the fundamental class, we need the stronger statement about the Chow ring.

4.4.2 Chow rings of locally closed strata outside Ψ

In Lemma 4.3.10, each $\vec{e}, \vec{f}$ splitting locus $\Sigma = \Sigma_{\vec{e}}(\mathcal{E}) \cap \Sigma_{\vec{f}}(\mathcal{F})$ was described as a quotient of the form $[(U \times \mathbb{G}_m)/G]$, where $G := \mathrm{SL}_2 \ltimes (\mathrm{Aut}(\vec{e}) \times \mathrm{Aut}(\vec{f}))$. The quotient $[(U \times \mathbb{G}_m)/G]$ is a $\mathbb{G}_m$ bundle over $[U/G]$, and $U \subset H := H^0(\mathbb{P}^1, \mathcal{O}(\vec{f})^\vee \otimes \mathrm{Sym}^2 \mathcal{O}(\vec{e}))$ is an open subvariety of affine space. Hence, there is a series of surjections

$$A^*(BG) \twoheadrightarrow A^*([U/G]) \twoheadrightarrow A^*(\Sigma), \tag{4.4.10}$$

We gave generators for $A^*(BG)$ in Lemma 4.3.3. To show that $A^*(\Sigma)$ is generated by CE classes, it will suffice to show that the images of these generators under (4.4.10) can be written in terms of CE classes. Similarly, to show the stronger statement that $A^*(\Sigma)$ is generated by κ_1 and κ_2, we must show that the images of the generators of $A^*(BG)$ under (4.4.10) are all expressible in terms of κ_1 and κ_2. We first consider the case when $\vec{e}$ has a repeated part.

Lemma 4.4.2. *Let Σ be the $\vec{e}, \vec{f}$ splitting locus and suppose $e_1 < e_2 = e_3$ and $f_1 < f_2$. Then, $A^*(\Sigma)$ is generated by the restrictions of CE classes.*

Proof. Set $G := \mathrm{SL}_2 \ltimes (\mathrm{Aut}(\vec{e}) \times \mathrm{Aut}(\vec{f}))$ and let $\pi : \mathcal{P} \to BG$ be the $\mathbb{P}^1$ bundle pulled back from BSL_2. Let L of rank 1 and R of rank 2 be the HN bundles for $\vec{e}$ so that we have a filtration

$$0 \to (\pi^* R)(e_2) \to \mathcal{V}(\vec{e}) \to (\pi^* L)(e_1) \to 0. \tag{4.4.11}$$

Similarly, let M and N be the rank 1 HN bundles for $\vec{f}$ so that we have a filtration

$$0 \to (\pi^* N)(f_2) \to \mathcal{V}(\vec{f}) \to (\pi^* M)(f_1) \to 0. \tag{4.4.12}$$

Let $r_i = c_i(R)$, and $\ell = c_1(L), m = c_1(M)$ and $n = c_1(N)$. Let c_2 be the second Chern class pulled back from BSL_2. By Lemma 4.3.3, the classes r_1, r_2, ℓ, m, n and c_2 freely generate $A^*(BG)$. Setting $z = c_1(\mathcal{O}_{\mathcal{P}}(1))$, and using the splitting principle with (4.4.11). we obtain the identities

$$c_1(\mathcal{V}(\vec{e})) = c_1(R(e_2)) + c_1(L(e_1)) = r_1 + 2e_2 z + \ell + e_1 z = (r_1 + \ell) + (g+3)z. \quad (4.4.13)$$

Recalling that $z^2 = -c_2$ on $\mathcal{P}$, and using the splittng principle we also have

$$c_2(\mathcal{V}(\vec{e})) = (2e_2\ell + (e_1 + e_2)r_1)z - (2e_1e_2 + e_2^2)c_2 + \ell r_1 + r_2. \quad (4.4.14)$$

Similarly, using the splitting principle on (4.4.12), we obtain the identities

$$c_1(\mathcal{V}(\vec{f})) = c_1(M(f_1)) + c_1(N(f_2)) = m + f_1 z + n + f_2 z = (m+n) + (g+3)z, \quad (4.4.15)$$
$$c_2(\mathcal{V}(\vec{f})) = (f_2 m + f_1 n)z - f_2 f_1 c_2 + mn. \quad (4.4.16)$$

By slight abuse of notation, let us denote the images of r_1, r_2, ℓ, m, n and c_2 under the map (4.4.10) by the same letters. (The pullback of c_2 is the CE class c_2, as both are pulled back from BSL_2.) These classes are generators for $A^*(\Sigma)$. By (4.4.13), we have $a_1 = r_1 + \ell$. By (4.4.14), we have $a_2' = e_1 r_1 + 2e_2\ell$. We have $e_1 < 2e_1$, so the classes r_1 and ℓ are expressible in terms of a_1 and a_2'. Next, (4.4.14) shows $a_2 = r_2 + r_1\ell - (2e_1e_2 + e_2^2)c_2$, so r_2 is also expressible in terms of CE classes. Finally, $b_1 = m + n$ by (4.4.15) and $b_2' = f_2 m + f_1 n$ by (4.4.16), so m and n are expressible in terms of b_1 and b_2' because $f_1 < f_2$. Hence, the CE classes generate $A^*(\Sigma)$. □

Now we consider the case when all parts of $\vec{e}$ are distinct. The proof follows a similar set up, but requires that we also make use of some relations among the generators of $A^*(BG)$ when pulled back to $A^*([U/G])$, i.e. that the first map v^* in (4.4.10) has a kernel. The classes in the kernel come from considering the complement of $U \subset H := H^0(\mathbb{P}^1, \mathcal{O}(\vec{f})^\vee \otimes \operatorname{Sym}^2 \mathcal{O}(\vec{e}))$, which corresponds equations whose vanishing locus in $\mathbb{P}E^\vee$ fails to be a smooth, irreducible curve. The second map in (4.4.10) also has a kernel. By Lemma 4.3.10, we have that $\Sigma \to [U/G]$ is the $\mathbb{G}_m$ bundle associated to the line bundle $\pi_*(\det \mathcal{V}(\vec{e}) \otimes \det \mathcal{V}(\vec{f})^\vee)$. Thus, by a theorem of Vistoli, the kernel of $A^*([U/G]) \to A^*(\Sigma)$ is generated by $c_1(\pi_*(\det \mathcal{V}(\vec{e}) \otimes \det \mathcal{V}(\vec{f})^\vee)) = a_1 - b_1$.

Lemma 4.4.3. *Let Σ be the $\vec{e}, \vec{f}$ splitting locus and suppose $e_1 < e_2 < e_3$ and $f_1 < f_2$ and $2e_1 < f_2$. Then the following are true:*

1. *If $2e_1 = f_1$, then $A^*(\Sigma)$ is generated by the restrictions of CE classes.*

2. *If $2e_1 = f_1$, and $e_1 + e_2 < 2e_2 = f_2$, then $A^*(\Sigma)$ is generated by κ_1 and κ_2.*

3. (i) *If $2e_1 > f_1$, and $e_1 + e_2 < e_1 + e_3 = 2e_2 = f_2$, then $A^*(\Sigma)$ is generated by restrictions of CE classes.*

   (ii) *Furthermore, if we also have $g \neq 9 - f_1$, then $A^*(\Sigma)$ is generated by κ_1 and κ_2.*

Proof. Set $G = \mathrm{SL}_2 \ltimes (\mathrm{Aut}(\vec{e}) \times \mathrm{Aut}(\vec{f}))$ and let $\pi : \mathcal{P} \to BG$ be the $\mathbb{P}^1$ bundle pulled back from BSL_2 as before. Let L, S, T be the rank 1 HN bundles on BG for $\vec{e}$, so that $\mathcal{V}(\vec{e})$ is filtered by $(\pi^*L)(e_1), (\pi^*S)(e_2)$, and $(\pi^*T)(e_3)$. Similarly, let M and N be the rank 1 HN bundles on BG for $\vec{f}$ so that $\mathcal{V}(\vec{f})$ is filtered by $(\pi^*M)(f_1)$ and $(\pi^*N)(f_2)$. Let $s = c_1(S), t = c_1(T), \ell = c_1(L), m = c_1(M)$ and $n = c_1(N)$. By Lemma 4.3.3, the classes s, t, ℓ, m, n and c_2 freely generate $A^*(BG)$.

Using the splitting principle (and omitting π pullbacks) as in Lemma 4.4.2, we have

$$c_1(\mathcal{V}(\vec{e})) = c_1(L(e_1)) + c_1(S(e_2)) + c_1(T(e_3)) = (\ell + s + t) + (g + 3)z.$$

Recalling that $z^2 = -c_2$ on $\mathcal{P}$, we also have

$$c_2(\mathcal{V}(\vec{e})) = ((e_2 + e_3)\ell + (e_1 + e_3)t + (e_1 + e_2)s)z - (e_1e_2 + e_1e_3 + e_2e_3)c_2 + \ell(t + s) + ts$$

Thus, we have

$$a_1 = \ell + s + t \qquad \text{and} \qquad a_2' = (e_2 + e_3)\ell + (e_1 + e_3)s + (e_1 + e_2)t. \tag{4.4.17}$$

Similarly, using the splitting principle, the Chern classes of $\mathcal{V}(\vec{f})$ satisfy the same identities as in (4.4.15) and (4.4.16), so

$$b_1 = m + n \qquad \text{and} \qquad b_2' = f_2 m + f_1 n. \tag{4.4.18}$$

Notice that $A^*(BG)$ has 5 generators in codimension 1, but there are only 4 codimension 1 CE classes (namely a_1, a_2', b_1, b_2'). Thus, to have any hope of the CE classes generating $A^*(\Sigma)$, the first map (4.4.10) must have some kernel in codimension 1. In each of the cases below, we describe one, or two such relations.

(1) Assume that $2e_1 = f_1$. Corresponding to our filtration of $\mathcal{V}(\vec{e})$, there is a quotient $\operatorname{Sym}^2 \mathcal{V}(\vec{e}) \to (\pi^* L)^{\otimes 2}(2e_1)$. Tensoring with $\mathcal{V}(\vec{f})^\vee$, we obtain a quotient

$$\mathcal{V}(\vec{f})^\vee \otimes \operatorname{Sym}^2 \mathcal{V}(\vec{e}) \to \mathcal{V}(\vec{f})^\vee \otimes (\pi^* L)^{\otimes 2}(2e_1). \tag{4.4.19}$$

Using our filtration of $\mathcal{V}(\vec{f})^\vee$, we see that the right hand term above is filtered by the line bundles $\pi^*(N^\vee \otimes L^{\otimes 2})(2e_1 - f_2)$ and $\pi^*(M^\vee \otimes L^{\otimes 2})(2e_1 - f_1) = \pi^*(M^\vee \otimes L^{\otimes 2})$. Noting that $2e_1 - f_2 < 0$, cohomology and base change then shows that the push forward of the right hand term of (4.4.19) is $\pi_*(\mathcal{V}(\vec{f})^\vee \otimes (\pi^* L)^{\otimes 2}(2e_1)) \cong M^\vee \otimes L^{\otimes 2}$. Applying π_* to (4.4.19), we therefore obtain a surjection

$$\pi_*(\mathcal{V}(\vec{f})^\vee \otimes \operatorname{Sym}^2 \mathcal{V}(\vec{e})) \to M^\vee \otimes L^{\otimes 2}.$$

The total space of $\pi_*(\mathcal{V}(\vec{f})^\vee \otimes \operatorname{Sym}^2 \mathcal{V}(\vec{e}))$ is simply $[H^0(\mathbb{P}^1, \mathcal{O}(\vec{f})^\vee \otimes \operatorname{Sym}^2 \mathcal{O}(\vec{e}))/G]$. In the notation of (4.4.1), the above surjection corresponds to projection onto the coefficient

$p_{1,1}$. Since $2e_1 - f_2 < 0$, the coefficient $q_{1,1} = 0$. Thus by (4.4.6), we must have $p_{1,1} \neq 0$. That is, U lies in the complement of the kernel of this projection. Put differently, writing $v : [U/G] \to BG$ for the map to the base, the line bundle $v^*(M^\vee \otimes L^{\otimes 2})$ has a non-vanishing section on $[U/G]$. Hence, we have the relation

$$0 = v^*(2\ell - m). \tag{4.4.20}$$

We collect (4.4.17), (4.4.18) and (4.4.20) into a 5×5 matrix equation in $A^1(\Sigma)$:

$$\begin{pmatrix} a_1 \\ a_2' \\ b_1 \\ b_2' \\ 0 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 & 0 & 0 \\ e_2+e_3 & e_1+e_3 & e_1+e_2 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & f_2 & f_1 \\ 2 & 0 & 0 & -1 & 0 \end{pmatrix} \begin{pmatrix} \ell \\ s \\ t \\ m \\ n \end{pmatrix}.$$

The matrix of coefficients above has determinant $2(e_3 - e_2)(f_2 - f_1)$. Since $e_2 \neq e_3$ and $f_1 \neq f_2$, the matrix is invertible, so the images of the classes ℓ, s, t, m, n are expressible in terms of the CE classes a_1, a_2', b_1, a_2'. The images of ℓ, s, t, m, n and c_2 under (4.4.10) generate $A^*(\Sigma)$. Hence, $A^*(\Sigma)$ is generated by CE classes.

(2) Suppose further that $2e_1 = f_1$, and $e_1+e_2 < 2e_2 = f_2$. Because $2e_1 < e_1+e_2 < f_2$, both $q_{1,1}$ and $q_{1,2}$ are zero. By (4.4.7), $q_{2,2}$ must be nonzero. Since $2e_2 = f_2$, the coefficient $q_{2,2}$ is degree 0, so its non-vanishing is a codimension 1 condition. Using an argument similar to the above, this gives rise to a non-vanishing global section of $v^*(S^{\otimes 2} \otimes N^\vee)$ on $[U/G]$. Hence, we obtain the relation $v^*(2s - n) = 0$. As in (1) we still have the relation $v^*(2\ell - m) = 0$. Meanwhile, we also know of some relations among CE classes that hold in $A^1(\mathcal{H}_{4,g})$. First off, we have $0 = a_1 - b_1$, which corresponds to $\Sigma \to [U/G]$ being a $\mathbb{G}_m$ bundle associated to a line bundle with first Chern class $a_1 - b_1$. Second, we have $0 = (8g + 20)a_1 - 8a_2' - b_2'$, by [CL21a, Equation 5.7], corresponding to the

fundamental class of $\Delta := H^0(\mathbb{P}^1, \mathcal{O}(\vec{f})^\vee \otimes \operatorname{Sym}^2 \mathcal{O}(\vec{e})) \smallsetminus U$. (Our other relations above $2\ell - m = 2s - n = 0$ came from certain components of Δ). Finally, by [CL21a, Lemma 7.6], we have $\kappa_1 = (12g+24)a_1 - 12a_2'$. Using (4.4.17) and (4.4.18) to rewrite these in terms of s, t, ℓ, m, n (and that $e_1 + e_2 = e_3 = f_1 + f_2 = g + 3$), we lay out a 5×5 matrix summarizing these relations that hold in $A^1(\Sigma)$:

$$\begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ \kappa_1 \end{pmatrix} = \begin{pmatrix} 2 & 0 & 0 & 0 & -1 \\ 0 & 0 & 2 & -1 & 0 \\ 1 & 1 & 1 & -1 & -1 \\ 8e_2 + 44 & 8e_3 + 44 & 8e_1 + 44 & -f_2 & -f_1 \\ 12e_2 + 60 & 12e_3 + 60 & 12e_1 + 60 & 0 & 0 \end{pmatrix} \begin{pmatrix} s \\ t \\ \ell \\ m \\ n \end{pmatrix}. \tag{4.4.21}$$

Taking into account $f_2 = 2e_2$ and $f_1 = 2e_1$, the determinant of the above matrix is equal to $48(g+5)(e_2 - e_1) \neq 0$, so κ_1 is a generator for $A^1(\Sigma)$.

Next, we want to show that the entire ring $A^*(\Sigma)$ is generated by κ_1 and κ_2. Because $A^*(\Sigma)$ is generated in codimension 1 and 2, it suffices to show that $A^2(\Sigma)$ is generated by κ_2, together with products of codimension 1 classes. This in turn follows if we can write c_2 in terms of products of codimension 1 classes and κ_2. Such an identity in fact holds in $A^2(\mathcal{H}_{4,g})$, as implied by [CL21a, Example 7.8]. Precisely, combining [CL21a, Equations (7.3) and (7.5)], we see

$$c_2 = -24(2g^3 - 32g^2 + 138g - 12)\kappa_2 + \text{products of codimension 1 classes} \in A^2(\mathcal{H}_{4,g}). \tag{4.4.22}$$

(3) Now we assume that $2e_1 > f_1$ and $e_1 + e_2 < e_1 + e_3 = 2e_2 = f_2$. Since $e_1 + e_2 < f_2$, we have $q_{1,1} = q_{1,2} = 0$, so by (4.4.7), we must have $q_{2,2} \neq 0$. Since $\deg(q_{2,2}) = 2e_2 - f_2 = 0$, this is also a codimension 1 condition. The coefficient $q_{2,2}$ is corresponds to a non-zero section of $v^*(S^{\otimes 2} \otimes N^\vee)$, so we obtain the relation $v^*(2s - n) = 0$ as in (2). Collecting

(4.4.17), (4.4.18), and the relation $v^*(2s-n)=0$ in a matrix equation, we have

$$\begin{pmatrix} a_1 \\ a_2' \\ b_1 \\ b_2' \\ 0 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 & 0 & 0 \\ e_2+e_3 & e_1+e_3 & e_1+e_2 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & f_2 & f_1 \\ 0 & 2 & 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} \ell \\ s \\ t \\ m \\ n \end{pmatrix}.$$

The determinant of the above matrix is $-2(e_3-e_1)(f_2-f_1)$. Because $e_3>e_1$ and $f_2>f_1$, this determinant does not vanish, and so the codimension 1 generators are expressible in terms of CE classes. This completes the proof of (3)(i).

To show (3)(ii), we will need to produce more relations. Because $2e_1>f_1$, the coefficient $q_{1,1}$ is a polynomial of positive degree on $\mathbb{P}^1$, in particular it must vanish somewhere. Thus, by (4.4.8), one of $q_{1,1},q_{1,2},q_{1,3}$ must be nonzero. However, we know $q_{1,1}=q_{1,2}=0$, so we must have $q_{1,3}\neq 0$. Again, $\deg(q_{1,3})=e_1+e_3-f_2=0$, so this is a codimension 1 condition. This coefficient of $q_{1,3}$ gives a non-zero section of $v^*(L\otimes T\otimes N^\vee)$, on $[U/G]$, so we obtain the relation $v^*(\ell+t-n)=0$. Now, we can just replace the second row of the matrix in (4.4.21) (the top row and bottom three are still valid relations), to get an equation in $A^1(\Sigma)$:

$$\begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ \kappa_1 \end{pmatrix} = \begin{pmatrix} 2 & 0 & 0 & 0 & -1 \\ 0 & 1 & 1 & 0 & -1 \\ 1 & 1 & 1 & -1 & -1 \\ 8e_2+44 & 8e_3+44 & 8e_1+44 & -f_2 & -f_1 \\ 12e_2+60 & 12e_3+60 & 12e_1+60 & 0 & 0 \end{pmatrix} \begin{pmatrix} s \\ t \\ \ell \\ m \\ n \end{pmatrix}. \tag{4.4.23}$$

The determinant of the above matrix is $-12(f_1+g-9)(e_3-e_1)$, which is non-zero given the hypotheses in the lemma. Thus κ_1 generates $A^1(\Sigma)$. By (4.4.22), we see that the codimension 2 generator c_2 is expressible in terms of κ_2 and κ_1^2. Hence, $A^*(\Sigma)$ is generated by κ_1 and κ_2, as desired. $\square$

4.4.3 Genus 5 and 6

As a warm-up, we will explain how the argument works in genus 5 and 6, thus giving new proofs of the results of Izadi [Iza95] and Penev-Vakil [PV15b], who proved that the Chow ring is equal to the tautological ring in genus 5 and 6, respectively.

Using (4.4.3)–(4.4.9), we have the following allowed pairs of splitting types in genus 5. We label a stratum with a Ψ_i if it is contained within Ψ (see Proposition 4.2.9), equivalently if $2e_1 - f_2 \geq -1$:

(Ψ_0) $E = (2, 3, 3), F = (4, 4)$.

(Ψ_1) $E = (2, 3, 3), F = (3, 5)$.

(Ψ_2) $E = (2, 2, 4), F = (3, 5)$.

(Z) $E = (1, 3, 4), F = (2, 6)$.

Proposition 4.4.4. *The Chow ring $A^*(\mathcal{H}_{4,5} \smallsetminus \beta^{-1}(\mathcal{M}_5^3))$ is generated by tautological classes. Hence, $A^*(\mathcal{M}_5)$ is tautological.*

Proof. By (4.4.5), the stratum Z consists of entirely hyperelliptic curves. Hence, $\mathcal{H}_{4,5} \smallsetminus \beta^{-1}(\mathcal{M}_5^3)$ is contained in $\mathcal{H}_{4,5} \smallsetminus Z = \Psi_0 \cup \Psi_1 \cup \Psi_2 = \Psi$. In particular, by Proposition 4.2.9, we see $A^*(\mathcal{H}_{4,5} \smallsetminus \beta^{-1}(\mathcal{M}_5^3))$ is generated by tautological classes. By Theorem 4.2.4, it follows that $A^*(\mathcal{M}_5 \smallsetminus \mathcal{M}_5^3)$ is generated by tautological classes. Classes supported on $\mathcal{M}_5^3$ are known to be tautological (4.1.1), so we conclude that $A^*(\mathcal{M}_5)$ is tautological. $\square$

The genus 6 case is similar. By (4.4.3)–(4.4.9), we have the following pairs of splitting types:

(Ψ_0) $E = (3, 3, 3), F = (4, 5)$, generic stratum.

(Ψ_1) $E = (2,3,4), F = (4,5)$, codimension 1, with E unbalanced.

(Ψ_2) $E = (3,3,3), F = (3,6)$, codimension 2, with F unbalanced.

(Σ_3) $E = (2,3,4), F = (3,6)$, codimension 2, with E and F unbalanced.

(Z) $E = (1,4,4), F = (2,7)$, codimension 2.

We first identify the curves of lower gonality

Lemma 4.4.5. *We have $\beta^{-1}(\mathcal{M}_6^3) = Z \cup \Psi_2$*

Proof. By (4.4.5), we already know $Z = \beta^{-1}(\mathcal{M}_6^2)$, so must show that $\Psi_2 = \beta^{-1}(\mathcal{M}_6^3 \smallsetminus \mathcal{M}_6^2)$. We first show $\Psi_2 \subseteq \beta^{-1}(\mathcal{M}_6^3 \smallsetminus \mathcal{M}_6^2)$. On Ψ_2, we have $\mathbb{P}E^\vee \cong \mathbb{P}^1 \times \mathbb{P}^2$. Since $\deg(q_{i,j}) = 0$ and $\deg(p_{i,j}) = 3$ for all i, j, the projection onto the $\mathbb{P}^2$ factor realizes C as a degree 3 cover of a conic in $\mathbb{P}^2$. To show the reverse inclusion, suppose $\sigma : C \to \mathbb{P}^1$ is a trigonal curve that also admits a degree 4 map $\alpha : C \to \mathbb{P}^1$. Then $(\alpha, \sigma) : C \to \mathbb{P}^1 \times \mathbb{P}^1$ is birational onto its image, which is a curve of bidegree $(3,4)$. By the genus formula, the genus of the image is 6, so $(\alpha, \sigma) : C \to \mathbb{P}^1 \times \mathbb{P}^1$ is an embedding. Composing with the degree 2 Veronese on the second factor, we obtain a map $C \hookrightarrow \mathbb{P}^1 \times \mathbb{P}^1 \hookrightarrow \mathbb{P}^1 \times \mathbb{P}^2$ which is an embedding of C in a $\mathbb{P}^2$ bundle satisfying the properties of $\mathbb{P}$ in Theorem 4.2.6. By its uniqueness, we see that $\mathbb{P}E_\alpha \cong \mathbb{P}^1 \times \mathbb{P}^2$, i.e. $E_\alpha = (3,3,3)$. Meanwhile, the bundle F_α corresponds to the quadrics vanishing on $C \subset \mathbb{P}E_\alpha^\vee \cong \mathbb{P}^1 \times \mathbb{P}^2$. The curve C lies on a quadric whose equation is pulled back from the $\mathbb{P}^2$ factor. Writing this quadric in the form (4.4.2), we see that $\deg(q_{i,j}) = 0$, so $f_2 = 6$. Hence, $F_\alpha = (3,6)$. □

Proposition 4.4.6. *The Chow ring $A^*(\mathcal{H}_{4,6} \smallsetminus \beta^{-1}(\mathcal{M}_6^3))$ is generated by tautological classes. Hence, $A^*(\mathcal{M}_6)$ is tautological.*

Proof. Working on the complement of $\beta^{-1}(\mathcal{M}_6^3) = Z \cup \Psi_2$, we observe that Σ_3 is the $(3,6)$ splitting locus for $\mathcal{F}$, i.e. $\Sigma_3 = \Sigma_{(3,6)}(\mathcal{F})$. Moreover,

$$\operatorname{codim} \Sigma_3 = 2 = h^1(\mathbb{P}^1, \mathcal{E}nd(\mathcal{O}(3,6))).$$

Thus, by Lemma 4.3.7, the fundamental class of $\Sigma_3 \subset \mathcal{H}_{4,6} \smallsetminus \beta^{-1}(\mathcal{M}_6^3)$ is expressible in terms of CE classes. By Lemma 4.4.3 (3) (i), we see $A^*(\Sigma_3)$ is generated by restrictions of CE classes. Hence, using the push-pull formula, every class supported on $\Sigma_3 \subset \mathcal{H}_{4,6} \smallsetminus \beta^{-1}(\mathcal{M}_6^3)$ is tautological.

Meanwhile, $\mathcal{H}_{4,6} \smallsetminus (\beta^{-1}(\mathcal{M}_6^3) \cup \Sigma_3) = \Psi_0 \cup \Psi_1 = \Psi$ is the open subset considered in Proposition 4.2.9. Hence, $A^*(\mathcal{H}_{4,6} \smallsetminus (\beta^{-1}(\mathcal{M}_6^3) \cup \Sigma_3))$ is generated by tautological classes. By excision and the first paragraph of the proof, all of $A^*(\mathcal{H}_{4,6} \smallsetminus \beta^{-1}(\mathcal{M}_6^3))$ is tautological. By Theorem 4.2.4, $A^*(\mathcal{M}_6 \smallsetminus \mathcal{M}_6^3)$ is generated by tautological classes. Combined with (4.1.1), we obtain that $A^*(\mathcal{M}_6)$ is tautological. $\square$

Remark 4.4.7. (1) We note that the stratum Σ_3 consists of plane quintic curves. Indeed, on Σ_3, we have $p_{1,1} = 0$ and $\deg(q_{1,1}) = 1$ so the curve meets the line $Y = Z = 0$ in $\mathbb{P}E^\vee$ in one point, say $\nu \in C$. The canonical line bundle on C is the restriction of $\mathcal{O}_{\mathbb{P}E^\vee}(1) \otimes \omega_{\mathbb{P}^1}$, which contracts the line $Y = Z = 0$ in the map $\mathbb{P}E^\vee \to \mathbb{P}^5$. Thus, ν is contained in each of the planes spanned by the image of a fiber of α under the canonical embedding. Hence, by geometric Riemann–Roch, the g^1_4 plus ν is a g^2_5. The locus of genus 6 curves possessing a g^2_5 is codimension 3 in $\mathcal{M}_6$, but this stratum has codimension 2 in $\mathcal{H}_{4,6}$ because projection from any point on a plane quintic gives a g^1_4.

(2) It turns out Σ_3 in genus 6 is the only case where Lemma 4.4.3 (3)(i) holds but $g = 9 - f_1$. The fact that $\Sigma_3 \to \mathcal{M}_6$ has positive-dimensional fibers seems to provide some geometric intuition for this exception where we fail to obtain the stronger statement in (3)(ii).

4.4.4 Genus 7

Using (4.4.3) – (4.4.9), the allowed splitting types in genus 7 are as follows. We label a stratum with a Ψ_i if it is contained within Ψ (see Proposition 4.2.9), equivalently if $2e_1 - f_2 \geq -1$.

(Ψ_0) $E = (3, 3, 4), F = (5, 5)$: generic stratum; associated scroll is smooth.

(Ψ_1) $E = (3,3,4), F = (4,6)$: associated scroll is smooth, F unbalanced.

(Σ_2) $E = (2,4,4), F = (4,6)$: associated scroll is a cone over $\mathbb{P}^1 \times \mathbb{P}^1$ (which is embedded in a hyperplane in $\mathbb{P}^6$ via $\mathcal{O}(2,1)$). General bielliptics live in here as a proper closed subvariety, described in [CDC02, Theorem 2.3].

(Σ_3) $E = (2,3,5), F = (4,6)$: the associated scroll is a cone over the Hirzebruch surface $\mathbb{F}_2$ (embedded via $\mathcal{O}(1)$ on $\mathbb{P}(\mathcal{O}(1) \oplus \mathcal{O}(3))$. The "special bielliptics" live in here as a proper closed subvariety, described in [CDC02, Theorem 2.3].

(Z) $E = (1,4,5), F = (2,8)$: stratum of hyperelliptic curves (see (4.4.5)).

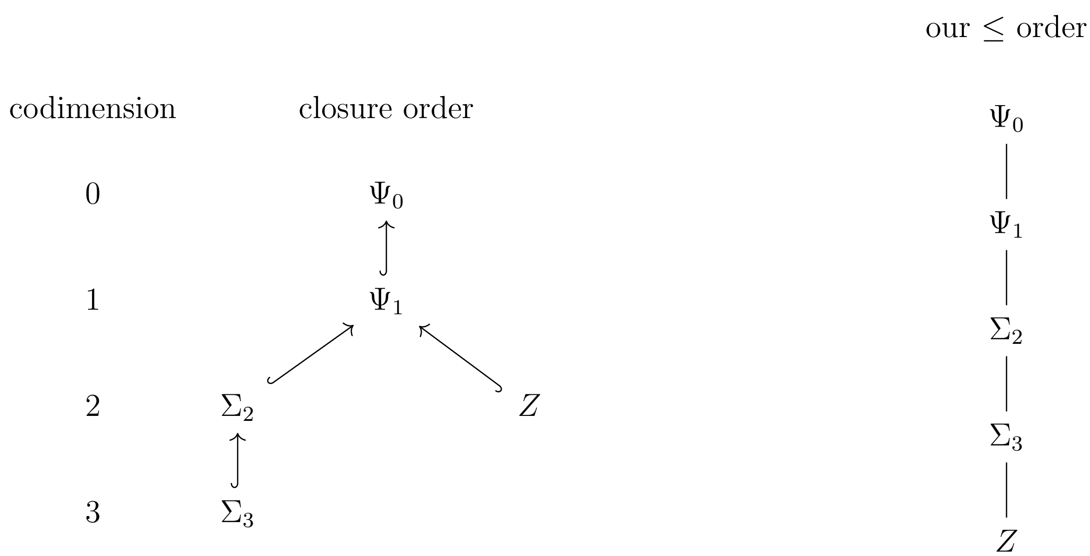

Figure 4.2. Two partial orders on the genus 7 strata

The table on the left of Figure 4.2 lists the codimensions of strata (computed with (4.3.3)). It also indicates the partial order of which strata lie in the closure of others, which can be seen by considering (4.3.2). This should be contrasted with the our partial ordering $\leq$, which is pictured on the right.

Lemma 4.4.8. *The Chow ring $A^*(\mathcal{H}_{4,7} \smallsetminus \beta^{-1}(\mathcal{M}_7^3))$ is generated by CE classes. Hence, all classes supported on $\mathcal{M}_7^4$ are tautological on $\mathcal{M}_7$.*

Proof. We implement Strategy 4.4.1, starting at the bottom of the partial ordering. By (4.4.5), we have

$$Z = \beta^{-1}(\mathcal{M}_7^2) = \beta^{-1}(\mathcal{M}_7^3).$$

The second equality follows because a genus 7 curve cannot possess maps to $\mathbb{P}^1$ of degrees 3 and 4, otherwise it would map to $\mathbb{P}^1 \times \mathbb{P}^1$ with image a curve of bidegree $(3, 4)$, which has genus 6.

Next, we claim that, modulo classes supported on Z, the fundamental class of Σ_3 is expressible in terms of CE classes. To see this, observe that $\Sigma_3 = \Sigma_{(2,3,5)}(\mathcal{E})$. Moreover,

$$\operatorname{codim} \Sigma_3 = 3 = h^1(\mathbb{P}^1, \mathcal{E}nd(\mathcal{O}(2, 3, 5))).$$

Thus, the claim follows from Lemma 4.3.7. By Lemma 4.4.3 (1), we see $A^*(\Sigma_3)$ is generated by the restrictions of CE classes, so by the push-pull formula, every class supported on on $\Sigma_3 \subset \mathcal{H}_{4,7} \smallsetminus Z$ is expressible in terms of CE classes, i.e. is tautological.

Similarly, modulo classes supported on Z and Σ_3, we claim that the fundamental class of Σ_2 is expressible in terms of CE classes. To see this, observe that $\Sigma_2 = \Sigma_{(2,4,4)}(\mathcal{E})$ and

$$\operatorname{codim} \Sigma_2 = 2 = h^1(\mathbb{P}^1, \mathcal{E}nd(\mathcal{O}(2, 4, 4))).$$

Thus, the claim again follows from Lemma 4.3.7. By Lemma 4.4.3 (1), we see $A^*(\Sigma_2)$ is generated by restrictions of CE classes. Using the push-pull formula, along with the previous paragraph, we see that every class supported on $\Sigma_2 \cup \Sigma_3 \subset \mathcal{H}_{4,7} \smallsetminus Z$ is tautological.

By Proposition 4.2.9, we know $A^*(\Psi)$ is generated by tautological classes. Putting this together with the above, we find that $A^*(\Psi \cup \Sigma_2 \cup \Sigma_3) = A^*(\mathcal{H}_{4,7} \smallsetminus \beta^{-1}(\mathcal{M}_7^3))$ is generated by tautological classes. Applying Theorem 4.2.4, every class supported on $\mathcal{M}_7^4 \smallsetminus \mathcal{M}_7^3$ is tautological in $\mathcal{M}_7 \smallsetminus \mathcal{M}_7^3$. classes supported on $\mathcal{M}_7^3$ are known to be tautological (see (4.1.1)), so the result follows. $\square$

4.4.5 Genus 8

Using (4.4.3) – (4.4.9), there are five allowed splitting types for the CE bundles, which give rise to the following stratification of $\mathcal{H}_{4,8}$. Again, we label a stratum with a Ψ_i if it is contained within Ψ, equivalently if $2e_1 - f_2 \geq -1$.

(Ψ_0) $E = (3,4,4), F = (5,6)$: generic stratum; the associated scroll is smooth.

(Ψ_1) $E = (3,4,4), F = (4,7)$: associated scroll is smooth, F unbalanced.

(Ψ_2) $E = (3,3,5), F = (5,6)$: associated scroll is smooth, E unbalanced.

(Σ_3) $E = (2,4,5), F = (4,7)$: associated scroll is a cone over $\mathbb{F}_1$. Bielliptic curves are a proper closed subvariety here, see [CDC02, Theorem 2.3].

(Z) $E = (1,5,5), F = (2,9)$: stratum of hyperelliptic curves (see (4.4.5)).

The table on the left of Figure 4.3 lists the codimension of strata (see (4.3.3)) and indicates which strata are in the closure of others, which can be seen by considering (4.3.2). This should be contrasted with our partial ordering $\leq$, which is pictured on the right.

codimension closure order

$$\begin{tikzcd}
0 & & \Psi_0 & \\
1 & & & \\
2 & \Psi_2 \arrow[ruu, hook] & \Psi_1 \arrow[uu, hook] & Z \arrow[luu, hook] \\
3 & \Sigma_3 \arrow[u, hook] \arrow[ru, hook] & &
\end{tikzcd}$$

our $\leq$ order

$$\begin{tikzcd}[arrows=dash]
& \Psi_0 & \\
\Psi_1 \arrow[ru] & & \Psi_2 \arrow[lu] \\
& \Sigma_3 \arrow[lu] \arrow[ru] & \\
& Z \arrow[u] &
\end{tikzcd}$$

Figure 4.3. Two partial orders on the genus 8 strata

Lemma 4.4.9. *The Chow ring $A^*(\mathcal{H}_{4,8} \smallsetminus \beta^{-1}(\mathcal{M}_8^3))$ is generated by CE classes. Hence, all classes supported on $\mathcal{M}_8^4$ are tautological on $\mathcal{M}_8$.*

Proof. The proof is very similar to Lemma 4.4.8. The lowest stratum is again the hyperelliptic locus: $Z = \beta^{-1}(\mathcal{M}_8^2) = \beta^{-1}(\mathcal{M}_8^3)$. Then, we notice that Σ_3 is equal to $\Sigma_{(2,4,5)}(\mathcal{E})$ and

$$\operatorname{codim} \Sigma_3 = 3 = h^1(\mathbb{P}^1, \mathcal{E}nd(\mathcal{O}(2,4,5))).$$

Thus, by Lemma 4.3.7, the fundamental class of Σ_3 inside $\mathcal{H}_{4,8} \smallsetminus \beta^{-1}(\mathcal{M}_8^3)$ is tautological. By Lemma 4.4.3 (1), the Chow ring of the locally closed stratum $A^*(\Sigma_3)$ is generated by the restrictions of CE classes. By the push-pull formula, every class supported on $\Sigma_3 \subset \mathcal{H}_{4,8} \smallsetminus \beta^{-1}(\mathcal{M}_8^3)$ is tautological.

Meanwhile, Proposition 4.2.9 shows that $A^*(\Psi) = A^*(\Psi_0 \cup \Psi_1 \cup \Psi_2)$ is generated by tautological classes. Putting this together with the previous paragraph, all of $A^*(\mathcal{H}_{4,8} \smallsetminus \beta^{-1}(\mathcal{M}_8^3))$ is generated by tautological classes. □

4.4.6 Genus 9

Using (4.4.3) – (4.4.9), we find that the allowed splitting types in genus 9 are as follows. Again, we label a stratum Ψ_i if $2e_1 - f_1 \geq -1$.

(Ψ_0) $E = (4,4,4), F = (6,6)$: the general stratum, the associated scroll is $\mathbb{P}^2 \times \mathbb{P}^1$.

(Ψ_1) $E = (4,4,4), F = (5,7)$: codimension 1, with F unbalanced.

(Ψ_2) $E = (3,4,5), F = (6,6)$: codimension 1, with E unbalanced.

(Ψ_3) $E = (3,4,5), F = (5,7)$: codimension 2, both E and F unbalanced.

(Ψ_4) $E = (4,4,4), F = (4,8)$: codimension 3, such curves have bidegree $(4,4)$ on $\mathbb{P}^1 \times \mathbb{P}^1$.

(Ψ_5) $E = (3,3,6), F = (6,6)$: codimension 4.

(Σ_6) $E = (3,4,5), F = (4,8)$: codimension 3, such curves posess a g^2_6 (Lemma 4.4.14).

(Σ_7) $E = (2,5,5), F = (4,8)$: codimension 4, all such covers factor through a degree 2 cover of an elliptic curve (see Lemma 4.4.11).

(Σ_8) $E = (2,4,6), F = (4,8)$: codimension 4, the "special bielliptics" live here as a proper closed locus of codimension 1 (see Figure 4.5).

(Z) $E = (1,5,6), F = (2,10)$: codimension 2, the hyperelliptic curves (see (4.4.5)).

In genus 9, it is less clear which strata lie in the closure of others. However, for our purposes, all we need is our $\leq$ order, pictured in Figure 4.4 below.

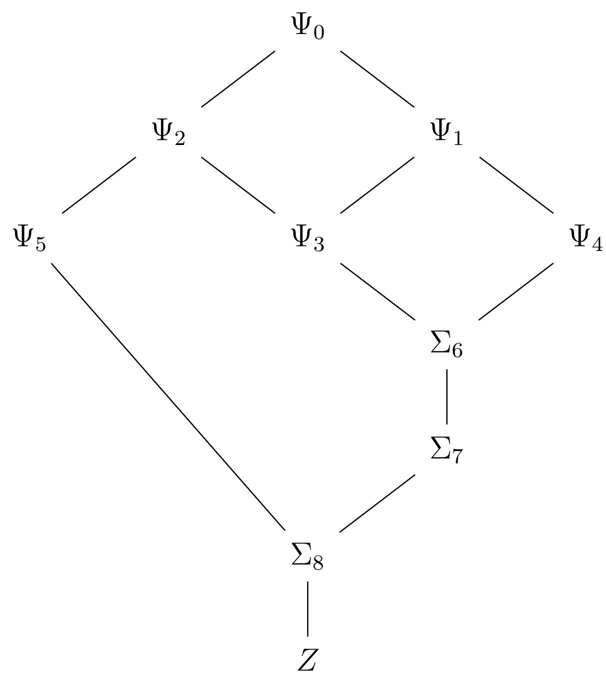

Figure 4.4. Our $\leq$ order in genus 9

Note that Σ_7 and Σ_8 have the same dimension, so Σ_8 is not contained in the closure of Σ_7 (see Example 4.3.5 for a baby case of this phenomenon).

Remark 4.4.10. The two "problem strata" mentioned in the introduction are Σ_8 and Σ_6. Note that these are more "interesting" nodes in the diagram above: they live directly below two different strata (i.e. there are two lines coming out the tops of these nodes).

The key to our argument is a good geometric understanding of Σ_8, Σ_7 and Σ_6. (We already know that Z consists of hyperelliptic curves, so it is not a concern.)

We first describe the bielliptic locus, making use of the explicit description due to Casnati–Del Centina [CDC02] for curves of genus $6 \leq g \leq 9$. Let $\mathcal{B}_9 \subset \mathcal{M}_9$ denote the locus of curves C which are double covers of an elliptic curve E. By the Castelnuovo–Severi inequality, $C \to E$ is the unique degree 2 map of C to an elliptic curve. By Riemann–Hurwitz, $\mathcal{B}_9$ is irreducible of dimension 16. Also by the Castelnuovo–Severi inequality, every degree 4 map $C \to \mathbb{P}^1$ from a bielliptic curve C factors through the map $C \to E$. That is, $\beta^{-1}(\mathcal{B}_9) \subset \mathcal{H}_{4,9}$ consists of maps of the form $C \to E \to \mathbb{P}^1$. Hence, $\beta^{-1}(\mathcal{B}_9) \subset \mathcal{H}_{4,9}$ is irreducible of dimension 17 and $\beta^{-1}(\mathcal{B}_9) \to \mathcal{B}_9$ has 1-dimensional fibers. Recall that by Riemann-Hurwitz, $\dim \mathcal{H}_{4,9} = 21$.

Lemma 4.4.11. *Every curve in stratum Σ_7 is bielliptic. Moreover, every degree 4 cover that factors through an elliptic curve lives in the closure of Σ_7, i.e. $\overline{\Sigma}_7 = \beta^{-1}(\mathcal{B}_9)$.*

Proof. On Σ_7, we have $E = (2, 5, 5)$ and $F = (4, 8)$. Thus, for degree reasons, we have $q_{1,2} = q_{1,3} = 0$, so the conditions [CDC02, Theorem 2.3 (general case)] are automatically satisfied. This says that $\Sigma_7 \subset \beta^{-1}(\mathcal{B}_9)$. Meanwhile, Σ_7 is irreducible of codimension 4, hence dimension 17. Since $\beta^{-1}(\mathcal{B}_9)$ is closed and irreducible of dimension 17, we must have $\overline{\Sigma}_7 = \beta^{-1}(\mathcal{B}_9)$. □

Theorem 2.3 of [CDC02] shows that $\beta^{-1}(\mathcal{B}_9)$ meets precisely one other stratum, Σ_8, in codimension 1 inside Σ_8. Casnati-Del Centina call the intersection $\overline{\Sigma}_7 \cap \Sigma_8$ the *special biellíptics* (pictured in purple in Figure 4.5). The special bielliptics $C \xrightarrow{\varphi} E \xrightarrow{\sigma} \mathbb{P}^1$ are characterized by the property that the branch locus of φ is linearly equivalent to $\sigma^*\mathcal{O}_{\mathbb{P}^1}(8)$ on E. Given a bielliptic curve $C \to E$, one may always choose a map $E \to \mathbb{P}^1$ so that the composition $C \to E \to \mathbb{P}^1$ is special. Hence, $\beta(\overline{\Sigma}_7) \subset \beta(\Sigma_8)$.

Recall that (see Figure 4.4), the stratum Σ_8 is closed in $\mathcal{H}_{4,9} \smallsetminus Z = \mathcal{H}_{4,9} \smallsetminus \beta^{-1}(\mathcal{M}_9^3)$.

Lemma 4.4.12. *The push forward $\beta_*(\Sigma_8)$ is tautological on $\mathcal{M}_9 \smallsetminus \mathcal{M}_9^3$. Furthermore, the push forward of any class supported on $\Sigma_8 \cup \Sigma_7$ is tautological on $\mathcal{M}_9 \smallsetminus \mathcal{M}_9^3$.*

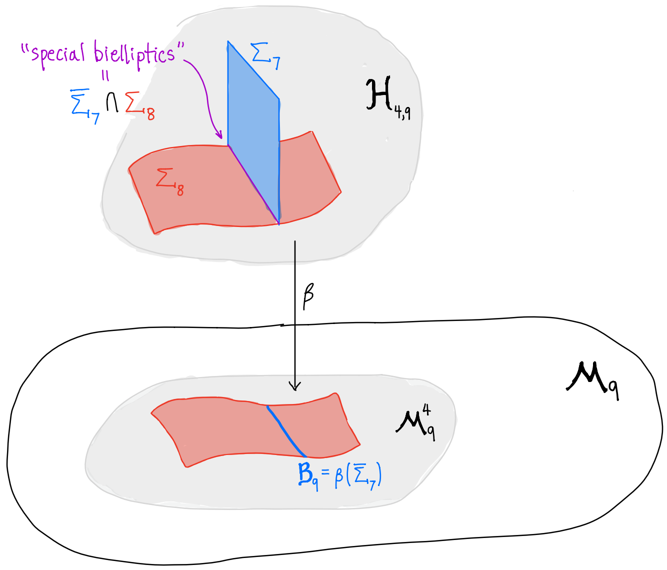

Figure 4.5. The map β contracts Σ_7 and $\beta(\overline{\Sigma}_7) \subset \beta(\Sigma_8)$

Proof. We first observe that $\Sigma_{(2,*,*)}(\mathcal{E}) = \Sigma_7 \cup \Sigma_8$ and has pure codimension 4, which is the expected codimension. By Lemma 4.3.8 (see also Example 4.3.9), the fundamental class of $\Sigma_{(2,*,*)}(\mathcal{E})$ is tautological in $\mathcal{H}_{4,9} \smallsetminus Z = \mathcal{H}_{4,9} \smallsetminus \beta^{-1}(\mathcal{M}_9^3)$. Hence, by Theorem 4.2.4, we have

$$\beta_*[\Sigma_{(2,*,*)}(\mathcal{E})] = \beta_*[\Sigma_8] + \beta_*[\overline{\Sigma}_7]$$

is tautological on $\mathcal{M}_9 \smallsetminus \mathcal{M}_9^3$. But, by Lemma 4.4.11, $\overline{\Sigma}_7 = \beta^{-1}(\mathcal{B}_9)$ maps to $\mathcal{M}_9$ with 1-dimensional fibers (pictured in blue in Figure 4.5). Hence, $\beta_*[\overline{\Sigma}_7] = 0$, so $\beta_*[\Sigma_8]$ is tautological on $\mathcal{M}_9 \smallsetminus \mathcal{M}_9^3$.

By Lemma 4.4.3 (2), we know that $A^*(\Sigma_8)$ is generated by the pullbacks of κ_1 and κ_2. Hence, using the push-pull formula, the push forward of every class supported on Σ_8 is tautological. Since we are working with rational coefficients, the pushforward map from $A^*(\Sigma_8)$ to $A^*(\beta(\Sigma_8))$ is surjective. Hence, every class supported on $\beta(\Sigma_8)$ is tautological on $\mathcal{M}_9 \smallsetminus \mathcal{M}_9^3$. The last sentence now follows because $\beta(\Sigma_8 \cup \Sigma_7) = \beta(\Sigma_8)$. $\square$

Example 4.4.13 (Regarding the class of $\mathcal{B}_9$)**.** To further explicate the second paragraph of the above proof, we explain why the fundamental class of $\mathcal{B}_9 \subset \mathcal{M}_9 \smallsetminus \mathcal{M}_9^3$ is tautological.

(Once known to be tautological, this class actually must be 0 by a result of Looijenga [Loo95].) Let $S = \overline{\Sigma}_7 \cap \Sigma_8 \subset \Sigma_8$ be the locus of special bielliptics (pictured in purple in Figure 4.5). We know that S maps finitely and surjectively onto $\mathcal{B}_9$. Thus, the fundamental class of $\mathcal{B}_9 \subset \mathcal{M}_9 \smallsetminus \mathcal{M}_9^3$ is a multiple of $\beta_*[S]$. By Lemma 4.4.3 (2), the class of S inside $A^*(\Sigma_8)$ is a multiple of $\beta^*\kappa_1$, so by the push-pull formula, $\beta_*[S]$ is a multiple of $\kappa_1 \cdot \beta_*[\Sigma_8]$.

Continuing up the partial order, we turn next to Σ_6.

Lemma 4.4.14. *Every curve in Σ_6 possesses a g^2_6 which is birational onto its image.*

Proof. On Σ_6, we have $2e_1 - f_2 < 0$ and $2e_1 - f_2 = 2$. Therefore, $C = V(p, q)$ meets the line $V(Y, Z) \subset \mathbb{P}E^\vee$ in 2 points (counted with multiplicity), say $p + q$. The line $V(Y, Z) \subset \mathbb{P}E^\vee$ is dual to the canonical quotient $E \to \mathcal{O}(3)$. This line is sent to a line with degree 1 under the map $\mathbb{P}E^\vee \to \mathbb{P}^8$ that factors the canonical embedding (which is given by $\mathcal{O}_{\mathbb{P}E^\vee}(1) \otimes \gamma^*\omega_{\mathbb{P}^1}$).

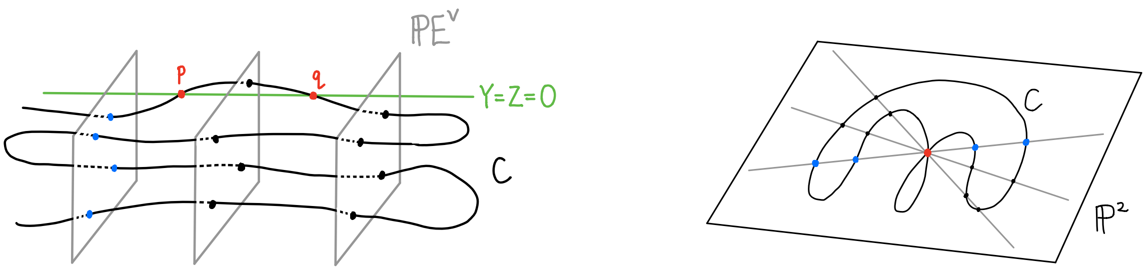

Figure 4.6. Curves in Σ_6 possess a g^2_6

As pictured on the left of Figure 4.6, the line spanned by p, q meets each plane spanned by the fibers of the g^1_4. Taking a fiber of the g^1_4 plus p and q, we obtain 6 points whose span under the canonical is 3-dimensional. By Geometric Riemann–Roch, these six points constitute a g^2_6 (pictured on the right of Figure 4.6).

A g^2_6 is either (1) birational onto its image (2) a double cover of a degree 3 plane curve or (3) a degree 3 cover of a conic. A genus 9 curve cannot have maps to $\mathbb{P}^1$ of degrees 3 and 4 (if so it would map birationally to a curve of bidegree $(3, 4)$ on $\mathbb{P}^1 \times \mathbb{P}^1$, which has genus 6), so we are not in case (3). Meanwhile, we have already established that everything in case (2) is contained in $\Sigma_8 \cup \Sigma_7$, which is disjoint from Σ_6. Thus, we must be in case (1). $\square$

Let $\mathrm{PS} \subset \mathcal{M}_9$ denote the locus of plane sextics, i.e. curves of genus 9 with a g^2_6 that is birational onto its image. Let $\Delta^\circ \subset H^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(6))$ be the locally closed subvariety of degree 6 equations on $\mathbb{P}^2$ whose vanishing locus has exactly one double point. Such curves have geometric genus 9 and $\Delta^\circ / \mathrm{GL}_3$ maps surjectively onto $\mathrm{PS} \subset \mathcal{M}_9$. In particular, PS is irreducible and

$$\dim \mathrm{PS} \leq h^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(6)) - 1 - \dim \mathrm{GL}_3 = 28 - 1 - 9 = 18. \tag{4.4.24}$$

Lemma 4.4.15. *The push forward $\beta_*[\overline{\Sigma}_6]$ is tautological on $\mathcal{M}_9 \smallsetminus \mathcal{M}^3_9$. Hence, the push forward of any class supported on $\Sigma_8 \cup \Sigma_7 \cup \Sigma_6$ is tautological on $\mathcal{M}_9 \smallsetminus \mathcal{M}^3_9$.*

Proof. If $\Sigma_6 \to \mathcal{M}_9$ has positive dimensional fibers, then $\beta_*[\overline{\Sigma}_6] = 0$, which is tautological. So let us assume $\Sigma_6 \to \mathcal{M}_9$ is generically finite onto its image, in which case $\beta_*[\overline{\Sigma}_6]$ is a multiple of $[\beta(\overline{\Sigma}_6)]$. By 4.4.14, we have $\beta(\overline{\Sigma}_6) \subset \overline{\mathrm{PS}}$ (where the closure of PS is taken in $\mathcal{M}_9 \smallsetminus \mathcal{M}^3_9$). It follows that $\dim \mathrm{PS} \geq \dim \Sigma_6 = 18$. By (4.4.24), we conclude that $\dim \mathrm{PS} = 18$ and such curves possess finitely many g^2_6's. Now, both $\beta(\overline{\Sigma}_6)$ and $\overline{\mathrm{PS}}$ are irreducible of dimension 18, so they must be equal. Therefore, we wish to show that $[\overline{\mathrm{PS}}]$ is tautological on $\mathcal{M}_9 \smallsetminus \mathcal{M}^3_9$. We know by Lemma 4.4.12 that all classes supported on $\mathcal{B}_9$ are tautological, so it suffices to work on the further open $\mathcal{M}_9 \smallsetminus (\mathcal{M}^3_9 \cup \mathcal{B}_9)$.

Let $\rho(g, r, d) := g - (r+1)(g - d + r)$ be the Brill–Noether number. In particular, $\rho(9, 2, 6) = -6$, so the plane sextics are "expected" to occur in codimension 6 on $\mathcal{M}_9$. On the open $\mathcal{M}_9 \smallsetminus (\mathcal{M}^3_9 \cup \mathcal{B}_9)$, the locus of curves that possess a g^2_6 is PS, which has dimension $18 = \dim \mathcal{M}_9 + \rho(9, 2, 6)$, so it is a Brill–Noether locus of the expected dimension. (Notice we need to work on the complement of $\mathcal{M}^3_9 \cup \mathcal{B}_9$, as curves of gonality less than or equal to 3 possess a g^2_6, but $\dim \mathcal{M}^3_9 = 19$ is too large. In the end, this is not an issue because we already know all classes supported on $\mathcal{M}^3_9$ are tautological.) Moreover, each curve in PS possesses finitely many g^2_6's (this is where it is important we also work on the complement of $\mathcal{B}_9$). We can therefore apply Faber's argument as in [Fab99, p. 15-16] on the open $\mathcal{M}_9 \smallsetminus (\mathcal{M}^3_9 \cup \mathcal{B}_9)$ to see that [PS] is tautological in $A^*(\mathcal{M}_9 \smallsetminus (\mathcal{M}^3_9 \cup \mathcal{B}_9))$. Since all classes supported on $\mathcal{B}_9$ have already been shown to be tautological, $[\overline{\mathrm{PS}}]$ is tautological in $A^*(\mathcal{M}_9 \smallsetminus \mathcal{M}^3_9)$ too.

By Lemma 4.4.3 (3)(ii), we know that $A^*(\Sigma_6)$ is generated by the pullback of κ_1 and κ_2. The second claim now follows by the push-pull formula and Lemma 4.4.12. □

We now complete the goal of this subsection.

Lemma 4.4.16. *All classes supported on $\mathcal{M}^4_9$ are tautological on $\mathcal{M}_9$.*

Proof. By Proposition 4.2.9, the Chow ring of $\Psi = \Psi_0 \cup \cdots \cup \Psi_5$ is generated by CE classes. Combining this with Theorem 4.2.4 and Lemma 4.4.15, we see that every class in $\mathcal{H}_{4,9} \smallsetminus \beta^{-1}(\mathcal{M}^3_9)$ pushes forward to a tautological class on $\mathcal{M}_9 \smallsetminus \mathcal{M}^3_9$. Such push forwards span all classes supported on $\mathcal{M}^4_9 \smallsetminus \mathcal{M}^3_9 \subset \mathcal{M}_9 \smallsetminus \mathcal{M}^3_9$. Finally, all classes supported on $\mathcal{M}^3_9$ are known to be tautological (see (4.1.1)). □

4.4.7 The end of our luck: bielliptics in genus 10 and beyond

We point out here one last coincidence in genus 10, which allows us to see that the bielliptic locus on $\mathcal{M}_{10}$ is tautological. We then explain why these coincidences that drive our technique do not continue into higher genus.

For $g \geq 10$, the bielliptics completely fill the strata they occupy. Let $h = \lfloor \frac{g}{2} \rfloor$. By [CDC02, Proposition 2.1] and the sentence following it, for $g \geq 10$, we have

$$\beta^{-1}(\mathcal{B}_g) = \begin{cases} \Sigma_{(2,h,h+1)}(\mathcal{E}) \cap \Sigma_{(4,g-1)}(\mathcal{F}) & \text{if } g \text{ even} \\ [\Sigma_{(2,h+1,h+1)}(\mathcal{E}) \cap \Sigma_{(4,g-1)}(\mathcal{F})] \cup [\Sigma_{(2,h,h+2)}(\mathcal{E}) \cap \Sigma_{(4,g-1)}(\mathcal{F})] & \text{if } g \text{ odd.} \end{cases} \tag{4.4.25}$$

Using similar techniques as in genus $7, 8, 9$ we establish that the fundamental class of the bielliptic locus $\mathcal{B}_{10} \subset \mathcal{M}_{10}$ is tautological.

Proof of Theorem 4.1.7. Using (4.4.3) – (4.4.9) in genus 10, one sees that $F = (4, 9)$ occurs only with $E = (2, 5, 6)$. Let Σ be this splitting locus. By (4.4.25), we have $\Sigma = \beta^{-1}(\mathcal{B}_{10})$. Meanwhile, $\operatorname{codim} \Sigma = 4$ is the expected codimension for $\Sigma_{(4,9)}(\mathcal{F})$, so by Lemma 4.3.7, we see that $[\Sigma] = [\Sigma_{(4,9)}(\mathcal{F})]$ is tautological (modulo classes supported on $\beta^{-1}(\mathcal{M}_{10}^3)$). By Lemma 4.4.3 (1), we know $A^*(\Sigma)$ is generated by restrictions of CE classes. By Theorem 4.2.4, the push forward of every class supported on Σ is tautological on $\mathcal{M}_{10} \smallsetminus \mathcal{M}_{10}^3$. Since we are working with rational coefficients, the push forward map on Chow groups from Σ to $\beta(\Sigma)$ is surjective. In particular, $[\beta(\Sigma)] = [\mathcal{B}_{10}]$ is tautological. The vanishing of $[\mathcal{B}_{10}]$ then follows from a theorem of Looijenga [Loo95], which says that the tautological ring vanishes in codimension $d > g - 2$. $\square$

The codimension of $\beta^{-1}(\mathcal{B}_g) \subset \mathcal{H}_{4,g}$ is always 4. However, for $g \geq 11$, neither $\Sigma_{(4,g-1)}(\mathcal{F})$ nor $\Sigma_{(2,*,*)}(\mathcal{E})$ has expected codimension 4. Thus, there is no way to realize the strata in (4.4.25) as splitting loci of the expected dimension. As an example, in genus 12, the biellipitics have $E = (2, 6, 7)$ and $F = (4, 11)$. In this case,

$$h^1(\mathbb{P}^1, \mathcal{E}nd(\mathcal{O}(2,6,7))) = 7 \qquad \text{and} \qquad h^1(\mathbb{P}^1, \mathcal{E}nd(\mathcal{O}(4,11))) = 6,$$

so neither expected codimension is 4. In fact, in genus 12, we claim van Zelm's result [vZ18] that $[\mathcal{B}_{12}]$ is non-tautological on $\mathcal{M}_{12}$ implies $[\beta^{-1}(\mathcal{B}_{12})]$ is non-tautological on $\mathcal{H}_{4,12}$. By Lemma 4.4.3 (1), we know that $\beta^{-1}(\mathcal{B}_{12})$ is generated by the restrictions of CE classes. Therefore, if $[\beta^{-1}(\mathcal{B}_{12})]$ were tautological, using the push-pull formula and Theorem 4.2.4, we would see that all classes supported on $\mathcal{B}_{12}$ were tautological, which is a contradiction.

4.5 The Pentagonal Locus

In this section, we show that $A^*(\mathcal{H}_{5,g} \smallsetminus \beta^{-1}(\mathcal{M}_g^4))$ is generated by tautological classes for $g = 7, 8, 9$. Given a degree 5, genus g cover $\alpha : C \to \mathbb{P}^1$, let $E = E_\alpha$ and $F = F_\alpha$ be the associated vector bundles on $\mathbb{P}^1$ as in Section 4.2.1. Let $\gamma : \mathbb{P}E^\vee \to \mathbb{P}^1$. The line bundle $\mathcal{O}_{\mathbb{P}E^\vee}(1) \otimes \gamma^*\omega_{\mathbb{P}^1}$ defines a map $\mathbb{P}E^\vee \to \mathbb{P}^{g-1}$ such that the composition $C \subset \mathbb{P}E^\vee \to \mathbb{P}^{g-1}$ is the canonical embedding. The bundles E and F split

$$E = \mathcal{O}(e_1) \oplus \mathcal{O}(e_2) \oplus \mathcal{O}(e_3) \oplus \mathcal{O}(e_4) \qquad e_1 \leq e_2 \leq e_3 \leq e_4$$

and

$$F = \mathcal{O}(f_1) \oplus \mathcal{O}(f_2) \oplus \mathcal{O}(f_3) \oplus \mathcal{O}(f_4) \oplus \mathcal{O}(f_5) \qquad f_1 \leq f_2 \leq f_3 \leq f_4 \leq f_5.$$

As in the degree 4 case, the splitting types of E and F give a stratification of $\mathcal{H}_{5,g}$. This stratification was studied by Schreyer [Sch86] when $g = 7, 8$, and Sagraloff [Sag05] when $g = 9$. (The translation between our notation and Schreyer's is that $a_i = f_i - 4$; the splitting type of E determines the type of Schreyer's determinantal surface Y.)

The condition to be inside $\Psi = \mathcal{H}_{5,g} \smallsetminus \mathrm{Supp}(R^1\pi_*\mathcal{E} \otimes \det \mathcal{E}^\vee \otimes \wedge^2\mathcal{F})$ is that

$$e_1 + f_1 + f_2 - (g+4) \geq -1 \qquad \iff \qquad e_1 + f_1 + f_2 \geq \begin{cases} 10 & \text{if } g = 7 \\ 11 & \text{if } g = 8. \\ 12 & \text{if } g = 9. \end{cases} \tag{4.5.1}$$

Just as in the degree 4 case, there are several constraints on the splitting types. We collect some of these constraints below. Using these constraints, we recover the stratifications found by Schreyer [Sch86] in genus 7 and 8 and Sagraloff [Sag05] in genus 9.

To begin, we know that $\deg(E) = g + 4$ and $\det E^{\otimes 2} \cong \det F$, so we have

$$e_1 + e_2 + e_3 + e_4 = g + 4, \tag{4.5.2}$$

$$f_1 + f_2 + f_3 + f_4 + f_5 = 2g + 8. \tag{4.5.3}$$

By [DP15, Proposition 2.6], we have

$$\frac{g+4}{10} \leq e_1 \leq \frac{g+4}{4} \qquad \text{and} \qquad e_4 \leq \frac{2g+8}{5}. \tag{4.5.4}$$

Because F is a subbundle of $\operatorname{Sym}^2 E$,

$$f_5 \leq 2e_4. \tag{4.5.5}$$

Note that equations (4.5.2)–(4.5.5) always reduce us to a finite list of allowed splitting types.

Next, we introduce some notation. Every section in $H := H^0(E \otimes \det E^\vee \otimes \wedge^2 F)$ can be represented by a skew symmetric matrix

$$M = \begin{pmatrix} 0 & L_{1,2} & L_{1,3} & L_{1,4} & L_{1,5} \\ -L_{1,2} & 0 & L_{2,3} & L_{2,4} & L_{2,5} \\ -L_{1,3} & -L_{2,3} & 0 & L_{3,4} & L_{3,5} \\ -L_{1,4} & -L_{2,4} & -L_{3,4} & 0 & L_{4,5} \\ -L_{1,5} & -L_{2,5} & -L_{3,5} & -L_{4,5} & 0 \end{pmatrix}, \tag{4.5.6}$$

where $L_{i,j} \in H^0(\mathcal{O}(f_i + f_j) \otimes \mathcal{O}(\vec{e}) \otimes \mathcal{O}(-g-4))$. The equations defining $D(\Phi(\eta)) \subset \mathbb{P}E^\vee$

are the 5 Pfaffian quadrics listed below:

$$Q_1 = L_{2,5}L_{3,4} - L_{2,4}L_{3,5} + L_{2,3}L_{4,5}$$
$$Q_2 = L_{1,5}L_{3,4} - L_{1,4}L_{3,5} + L_{1,3}L_{4,5}$$
$$Q_3 = L_{1,5}L_{2,4} - L_{1,4}L_{2,5} + L_{1,2}L_{4,5}$$
$$Q_4 = L_{1,5}L_{2,3} - L_{1,3}L_{2,5} + L_{1,2}L_{3,5}$$
$$Q_5 = L_{1,4}L_{2,3} - L_{1,3}L_{2,4} + L_{1,2}L_{3,4}.$$

Corresponding to the splitting of $E = \mathcal{O}(\vec{e})$, we can take coordinates $X_1, \ldots, X_4$ on $\mathbb{P}E^\vee$. The $L_{i,j}$ are linear homogeneous polynomials in the X_k whose coefficients are elements of $H^0(\mathbb{P}^1, \mathcal{O}(f_i + f_j + e_k - g - 4))$. We write these as:

$$L_{i,j} = a_{i,j}X_1 + b_{i,j}X_2 + c_{i,j}X_3 + d_{i,j}X_4.$$

If $L_{1,2}$ and $L_{1,3}$ were identically zero, then Q_5 would be reducible, which is impossible because C is irreducible. Therefore, we must have

$$f_1 + f_3 + e_4 \geq g + 4 \tag{4.5.7}$$

If X_4 divides $L_{1,2}, L_{1,3}$, and $L_{1,4}$, then Q_5 is reducible. This will occur if $a_{1,4}, b_{1,4}, c_{1,4}$ all identically vanish. In order for $a_{1,4}, b_{1,4}$, and $c_{1,4}$ to not all identically vanish, we must have

$$f_1 + f_4 + e_3 \geq g + 4. \tag{4.5.8}$$

Similarly, if X_4 divides $L_{1,2}$, $L_{1,3}$ and $L_{2,3}$, then X_4 divides Q_5 and Q_5 is reducible. To prevent this, we must have

$$f_2 + f_3 + e_3 \geq g + 4. \tag{4.5.9}$$

Note that the curve C cannot contain the section defined by $X_2 = X_3 = X_4 = 0$. Otherwise, it would be reducible. Therefore, at least one of the Q_i must have a nonzero coefficient of X_1^2. If the coefficient of X_1 is zero in $L_{1,2}, \ldots, L_{1,5}$ and $L_{2,3}, L_{2,4}$, and $L_{2,5}$, then the coefficient of X_1^2 vanishes for all Q_i. Therefore, we must have that

$$f_2 + f_5 + e_1 \geq g + 4. \tag{4.5.10}$$

Similarly, we note that we must have

$$f_3 + f_4 + e_1 \geq g + 4. \tag{4.5.11}$$

Indeed, if not, then the coefficient of X_1 vanishes for all $L_{i,j}$ except possibly $j = 5$. It follows that none of the quadrics have an X_1^2 term in them, and thus they contain the section $X_2 = X_3 = X_4 = 0$, so C would be reducible.

If all $a_{1,j} = b_{1,j} = 0$, then the quadrics $Q_2, \ldots, Q_5$ all vanish on $V(X_3, X_4)$. The remaining equation Q_1 then cuts out a divisor on the surface $V(X_3, X_4)$, so either C is reducible or is entirely contained in $V(X_3, X_4)$. But this is impossible because then in the canonical embedding would send five points on C to a common line, which means C has a g^3_5. Projection from a point gives a g^2_5, so C would have genus at most 6. To prevent this,

$$f_1 + f_5 + e_2 \geq g + 4. \tag{4.5.12}$$

Another bad thing is if $L_{1,2}, L_{1,3}, L_{2,3}, L_{1,4}, L_{2,4}$ are all zero on $V(X_3, X_4)$. In this case, the restriction of the quadrics to $V(X_3, X_4)$ is $Q_1 = L_{2,5}L_{3,4}, Q_2 = L_{1,5}L_{3,4}, Q_3 = Q_4 = Q_5 = 0$, so $V(Q_1)$ and $V(Q_2)$ share 1-dimensional component inside $V(X_3, X_4)$. To prevent this, we need

$$f_2 + f_4 + e_2 \geq g + 4. \tag{4.5.13}$$

Next, we note some conditions that imply C has a special linear series. If $a_{1,5} = 0$ and $a_{2,3} = 0$, then the curve meets the line $V(X_2, X_3, X_4)$ along $V(a_{2,5}a_{3,4} - a_{3,5}a_{2,4})$. The degree of $V(a_{2,5}a_{3,4} - a_{3,5}a_{2,4})$ is $\deg(a_{2,5}) + \deg(a_{3,4}) = 2e_1 - f_1$. If $e_1 = 2$, the line $V(X_2, X_3, X_4)$ is contracted in the map $\mathbb{P}E^\vee \to \mathbb{P}^{g-1}$. If $2e_1 - f_1 = 1$, then C meets $V(X_2, X_3, X_4)$ in a point p. Under the canonical, this point p lies in the span of the five points in a fiber so p plus the g^1_5 is a g^2_6 on C. This yields the condition

$$\text{if } e_1 = 2 \text{ and } e_1 + f_1 + f_5, e_1 + f_2 + f_3 < g + 4 \text{ and } 2e_1 - f_1 = 1, \text{ then } C \text{ has a } g^2_6. \tag{4.5.14}$$

As another source of special linear series, Schreyer shows [Sch86, p. 136] that if $L_{1,2} = 0$, then C lies on a certain determinantal surface, which is birational to a Hirzebruch surface $\mathbb{F}_k := \mathbb{P}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(k))$ for $k = f_2 - f_1$. Schreyer determines determines the class of the image of C on this Hirzebruch surface in [Sch86, Theorem 5.7]. In the case $k = 0$, we have $\mathbb{F}_0 \cong \mathbb{P}^1 \times \mathbb{P}^1$ and projection onto the other factor determines another pencil on C. Similarly, if $k = 1$, then $\mathbb{F}_1$ admits a map to $\mathbb{P}^2$ and we obtain a g^2_d. The degree d of these special linear series is given by intersecting Schreyer's class C' with the $\mathcal{O}_{\mathbb{F}_k}(1)$ (which Schreyer calls A). This calculation is summarized nicely by Sagraloff in [Sag05, p. 65] (to translate our splitting types, $f_i = a_{5-i} + 4$):

$$\text{if } f_2 - f_1 = k \text{ and } L_{1,2} = 0, \text{ then } C \text{ possesses a } g^{1+k}_{f_1}. \tag{4.5.15}$$

The final condition we note concerns the situation when $e_1 + f_2 + f_4 - (g + 4) < 0$. In this case, $a_{1,2} = a_{1,3} = a_{2,3} = a_{1,4} = a_{2,4} = 0$. Restricting the five quadrics $Q_1, \ldots, Q_5$ to the line $Z = V(X_2, X_3, X_4)$ we obtain

$$Q_1|_Z = a_{2,5}a_{3,4}X_1^2, \qquad Q_2|_Z = a_{1,5}a_{3,4}X_1^2, \qquad Q_3|_Z = Q_4|_Z = Q_5|_Z = 0.$$

In particular, if $a_{3,4} = 0$, then $C = D(\Phi(\eta))$ would contain the line Z, but such a curve would be reducible. Therefore,

$$\text{if } e_1 + f_2 + f_4 < g + 4, \text{ then } a_{3,4} \neq 0. \tag{4.5.16}$$

4.5.1 Strategy

The strategy for the pentagonal locus is the same as, or even simpler than, the strategy for the tetragonal locus.

1. Use conditions (4.5.2)–(4.5.13) to determine the allowed pairs of splitting types $\vec{e}, \vec{f}$. The partial order on splitting types of Section 4.3 induces a partial order on pairs of splitting types by $(\vec{e}\,', \vec{f}') \leq (\vec{e}, \vec{f})$ if $\vec{e}\,' \leq \vec{e}$ and $\vec{f}' \leq \vec{f}$.

2. Starting with strata at the bottom of the partial ordering and working upwards, show that for each stratum outside of Ψ at least one of the following is satisfied:

   (a) the stratum is contained in $\beta^{-1}(\mathcal{M}_g^4)$.

   (b) its fundamental class in $\mathcal{H}_{5,g} \smallsetminus \beta^{-1}(\mathcal{M}_g^4)$ is tautological (modulo classes supported on strata below it in the partial order) *and* the Chow ring of the locally closed stratum $\Sigma' := \Sigma \smallsetminus \beta^{-1}(\mathcal{M}_g^4)$ is generated by the restrictions of CE classes.

This will establish that $A^*(\mathcal{H}_{5,g} \smallsetminus \beta^{-1}(\mathcal{M}_g^4))$ is generated by CE classes when $g = 7, 8, 9$. In Section 4.5.2, we show that the Chow rings of certain Σ' are generated by restrictions of CE classes. Then, in Sections 4.5.3, 4.5.4, 4.5.5, we treat the cases $g = 7, 8, 9$ respectively.

4.5.2 Chow rings of locally closed strata outside Ψ

In Lemma 4.3.11, we described each pair splitting locus as a quotient $\Sigma = [(U \times \mathbb{G}_m)/G]$ where $G = \mathrm{SL}_2 \ltimes (\mathrm{Aut}(\vec{e}) \times \mathrm{Aut}(\vec{f}))$, and $U \subset H := H^0(E \otimes \det E^\vee \otimes \wedge^2 F)$ was the open subvariety of sections η so that the Pfaffian locus $D(\Phi(\eta)) \subseteq \mathbb{P}E^\vee$ is a smooth, irreducible curve. Let $U' \subset U$ be the further open where $C = D(\Phi(\eta))$ does not possess a g^1_d for $d < 5$, so $\Sigma' := \Sigma \smallsetminus \beta^{-1}(\mathcal{M}^4_g) = [(U' \times \mathbb{G}_m)/G]$. We have a series of surjections

$$A^*(BG) \twoheadrightarrow A^*([U'/G]) \twoheadrightarrow A^*(\Sigma'), \tag{4.5.17}$$

The first map is induced by pullback from the structure map $v : [U'/G] \to BG$. It will suffice to show that the images of generators on the left are expressible in terms of CE classes. To see this, we will need to know about the relations that come from the complement of $U' \subset H$.

We consider several different "shapes" of splitting types that occur for pentagonal strata.

Lemma 4.5.1. *Let $\vec{e} = (e_1, e_2, e_3, e_4)$ and $\vec{f} = (f_1, f_2, f_3, f_4, f_5)$ satisfy the following conditions:*

1. *$e_1 < e_2 = e_3 < e_4$,*
2. *$f_1 = f_2 < f_3 = f_4 < f_5$,*
3. *$e_4 + f_1 + f_2 = g + 4$,*
4. *$e_1 + f_3 + f_4 = g + 4$,*
5. *$-e_1 f_1 + e_2 f_1 - e_2 f_3 + e_4 f_3 + e_1 f_5 - e_4 f_5 \neq 0$.*

Let Σ denote the corresponding stratum with splitting types $\vec{e}$ and $\vec{f}$. Then $A^*(\Sigma')$ is generated by the restrictions of CE classes.

Proof. Set $G = \mathrm{SL}_2 \ltimes (\mathrm{Aut}(\vec{e}) \times \mathrm{Aut}(\vec{f}))$ and let $\pi : \mathcal{P} \to BG$ be the $\mathbb{P}^1$ bundle pulled back from BSL_2. The first part of the HN filtration for $\mathcal{V}(\vec{e})$ is

$$0 \to \pi^* L(e_4) \to \mathcal{V}(\vec{e}) \to Q_1 \to 0, \tag{4.5.18}$$

where L is rank 1. The next part is

$$0 \to \pi^* R(e_2) \to Q_1 \to \pi^* T(e_1) \to 0, \tag{4.5.19}$$

where R is rank 2 and T is rank 1. Similarly, we have the HN filtration for $\mathcal{V}(\vec{f})$:

$$0 \to \pi^* S(f_5) \to \mathcal{V}(\vec{f}) \to W_1 \to 0, \tag{4.5.20}$$

and then

$$0 \to \pi^* M(f_3) \to W_1 \to \pi^* N(f_1) \to 0. \tag{4.5.21}$$

The bundle S is of rank 1, and M and N are of rank 2. We denote the Chern classes of an HN bundle by the corresponding lowercase letter, with subscripts $i = 1, 2$ when the HN bundle has rank 2. In particular, the right hand column vector in (4.5.22) below consists of the first Chern classes of the HN bundles. From (4.5.18)–(4.5.21), an application of the splitting principle gives the following expressions:

$$\begin{pmatrix} a_1 \\ a_2' \\ b_1 \\ b_2' \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 & 0 & 0 & 0 \\ e_1 + 2e_2 & e_1 + e_2 + e_4 & 2e_2 + e_4 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 2f_3 + 2f_1 & f_5 + f_3 + 2f_1 & f_5 + 2f_3 + f_1 \end{pmatrix} \begin{pmatrix} \ell \\ r_1 \\ t \\ s \\ m_1 \\ n_1 \end{pmatrix}. \tag{4.5.22}$$

Next, we show that the geometry of the curves in this stratum imposes some relations among $\ell, r_1, t, s, m_1, n_1$. We will show that modulo these relations, the Chern classes of the HN bundles are all expressible in terms of CE classes, finishing the proof. Note that BG has six generators in codimension 1 (namely $\ell, r_1, t, s, m_1, n_1$) but there are only four CE classes in codimension 1, so we will need to show $A^1(BG) \to A^1([U'/G])$ has a kernel.

The two sources of relations are from the conditions (4.5.15) and (4.5.16). From (4.5.15), we see that if $L_{1,2} = 0$, then C possesses a g^1_4. For degree reasons, we see that in this stratum

$$L_{1,2} = d_{1,2}X_4.$$

Therefore, the vanishing of $L_{1,2}$ is equivalent to the vanishing of $d_{1,2}$. This is a codimension 1 condition because $\deg(d_{1,2}) = f_1 + f_2 + e_4 - (g+4) = 0$.

Corresponding to the filtration on $\mathcal{V}(\vec{f})$, there is a quotient map

$$\wedge^2\mathcal{V}(\vec{f}) \to \det(\pi^*N(f_1)).$$

Tensoring by $\mathcal{V}(\vec{e}) \otimes \det\mathcal{V}(\vec{e})^\vee$, we obtain a surjection

$$\wedge^2\mathcal{V}(\vec{f}) \otimes \mathcal{V}(\vec{e}) \otimes \det\mathcal{V}(\vec{e})^\vee \to \det(\pi^*N(f_1)) \otimes \mathcal{V}(\vec{e}) \otimes \det\mathcal{V}(\vec{e})^\vee.$$

From the isomorphism

$$\det\mathcal{V}(\vec{e})^\vee \cong \pi^*(\det R^\vee \otimes T^\vee \otimes L^\vee)(-g-4),$$

we obtain a surjection

$$\wedge^2\mathcal{V}(\vec{f}) \otimes \mathcal{V}(\vec{e}) \otimes \det\mathcal{V}(\vec{e})^\vee \to \mathcal{V}(\vec{e}) \otimes \pi^*(\det N \otimes \det R^\vee \otimes \det T^\vee \otimes L^\vee)(2f_1 - g - 4). \quad (4.5.23)$$

Because $2f_1 + e_i - g - 4 < 0$ for $1 \leq i \leq 3$ and $2f_1 + e_4 - g - 4 = 0$, cohomology and base change implies that the push forward of (4.5.23) is given by

$$\begin{aligned} \pi_*(\wedge^2\mathcal{V}(\vec{f}) \otimes \mathcal{V}(\vec{e}) \otimes \det\mathcal{V}(\vec{e})^\vee) &\to \pi_*(\pi^*(\det N \otimes \det R^\vee \otimes \det T^\vee)(2f_1 + e_4 - g - 4)) \\ &\cong \det N \otimes \det R^\vee \otimes \det T^\vee. \end{aligned}$$

The left-hand side above is the total space of $[H/G]$. The above map corresponds to projection of H onto the coefficient $d_{1,2}$. Let $v : [U'/G] \to BG$ be the structure map, so $v^* : A^*(BG) \to A^*([U'/G])$ is the first map in (4.5.17). Since $d_{1,2}$ is non-vanishing on U', the pullback $v^*(\det N \otimes \det R^\vee \otimes \det T^\vee)$ admits a non-vanishing section on $[U'/G]$. In particular we obtain the relation

$$v^*(n_1 - r_1 - t) = 0. \tag{4.5.24}$$

Next, we turn to the condition (4.5.16). We want to write down a similar map that picks out the coefficient $a_{3,4}$. From the filtration on $\mathcal{V}(\vec{e})$, we have a surjection

$$\mathcal{V}(\vec{e}) \to \pi^* T(e_1).$$

By tensoring with $\det \mathcal{V}(\vec{e})^\vee$, we obtain a surjection

$$\mathcal{V}(\vec{e}) \otimes \det \mathcal{V}(\vec{e})^\vee \to \pi^*(\det R^\vee \otimes L^\vee)(e_1 - g - 4).$$

Next, we note that we have a surjection $\wedge^2 \mathcal{V}(\vec{f}) \to \wedge^2 W_1$, and so we obtain a surjection

$$\wedge^2 \mathcal{V}(\vec{f}) \otimes \mathcal{V}(\vec{e}) \otimes \det \mathcal{V}(\vec{e})^\vee \to \wedge^2 W_1 \otimes \pi^*(\det R^\vee \otimes L^\vee)(e_1 - g - 4). \tag{4.5.25}$$

There is a filtration of $\wedge^2 W_1$ with subquotients $\wedge^2(\pi^* N(f_1))$, $\pi^* N(f_1) \otimes \pi^* M(f_3)$ and $\wedge^2(\pi^* M(f_3))$. Because $2f_1 + e_1 - g - 4 < 0$, $f_1 + f_3 + e_1 - g - 4 < 0$, and $2f_3 + e_1 - g - 4 = 0$, the π push forward of (4.5.25) is given by

$$\pi_*(\wedge^2 \mathcal{V}(\vec{f}) \otimes \mathcal{V}(\vec{e}) \otimes \det \mathcal{V}(\vec{e})^\vee) \to \det M \otimes \det R^\vee \otimes \det L^\vee.$$

This map corresponds to projecting onto $a_{3,4}$. Since $a_{3,4} \neq 0$ on U', we obtain the relation

$$v^*(m_1 - r_1 - \ell) = 0. \tag{4.5.26}$$

Augmenting matrix (4.5.22) by the relations (4.5.26) and (4.5.24), we obtain the matrix

$$\begin{pmatrix} 0 \\ 0 \\ a_1 \\ a_2' \\ b_1 \\ b_2' \end{pmatrix} = \begin{pmatrix} 0 & -1 & -1 & 0 & 0 & 1 \\ -1 & -1 & 0 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 \\ e_1 + 2e_2 & e_1 + e_2 + e_4 & 2e_2 + e_4 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 2f_3 + 2f_1 & f_5 + f_3 + 2f_1 & f_5 + 2f_3 + f_1 \end{pmatrix} \begin{pmatrix} \ell \\ r_1 \\ t \\ s \\ m_1 \\ n_1 \end{pmatrix}. \tag{4.5.27}$$

The determinant of the above 6×6 matrix is the quantity in part (5) of the statement of the lemma. By assumption, this determinant does not vanish, so the classes $\ell, r_1, t, s, m_1, n_1$ are expressible in terms of CE classes.

Besides products of codimension 1 generators, BG has four codimension 2 generators: c_2, m_2, n_2, r_2. By definition, c_2 is a CE class, so we just need to show that m_2, n_2, and r_2 are expressible in terms of CE classes. Using the splitting principle on (4.5.18) and (4.5.19), we obtain the following expression for a_2:

$$a_2 = r_2 + r_1(\ell + t) + \ell t - (2e_1e_2 + e_2^2 + e_1e_4 + 2e_2e_4)c_2.$$

Therefore, r_2 is expressible in terms of CE classes. Similarly, from (4.5.20) and (4.5.21), we have the following expressions for b_2 and b_3':

$$b_2 = s(m_1 + n_1) + m_1 n_1 + m_2 + n_2 - (2f_5 f_3 + f_3^2 + 2f_5 f_1 + 4f_3 f_1 + f_1^2)c_2,$$
$$b_3' = (f_3 + 2f_1)sm_1 + (2f_3 + f_1)sn_1 + (f_5 + f_3 + f_1)m_1 n_1 + (f_5 + 2f_1)m_2 + (f_5 + 2f_3)n_2$$
$$- (f_5 f_3^2 + 4f_5 f_3 f_1 + 2f_3^2 f_1 + f_5 f_1^2 + 2f_3 f_1^2)c_2.$$

Since $f_1 \neq f_3$, we see that m_2 and n_2 are expressible in terms of CE classes. □

Lemma 4.5.2. *Let $\vec{e} = (e_1, e_2, e_3, e_4)$ and $\vec{f} = (f_1, f_2, f_3, f_4, f_5)$ satisfy the following conditions:*

1. $e_1 < e_2 < e_3 = e_4$,
2. $f_1 < f_2 = f_3 < f_4 = f_5$,
3. $e_1 + f_2 + f_5 = g + 4$,
4. $e_3 + f_1 + f_2 = g + 4$,
5. $2e_2 f_1 - 2e_3 f_1 - e_1 f_2 - 3e_2 f_2 + 4e_3 f_2 + e_1 f_4 + e_2 f_4 - 2e_3 f_4 \neq 0$.

Let Σ denote the corresponding stratum with splitting types $\vec{e}$ and $\vec{f}$. Then $A^*(\Sigma')$ is generated by the restrictions of CE classes.

Proof. Set $G = \mathrm{SL}_2 \ltimes (\mathrm{Aut}(\vec{e}) \times \mathrm{Aut}(\vec{f}))$ and let $\pi : \mathcal{P} \to BG$ be the $\mathbb{P}^1$ bundle pulled back from BSL_2. The first part of the HN filtration for $\mathcal{V}(\vec{e})$ is

$$0 \to \pi^* R(e_3) \to \mathcal{V}(\vec{e}) \to \pi^* W_1 \to 0 \tag{4.5.28}$$

and then

$$0 \to \pi^* S(e_2) \to W_1 \to \pi^* L(e_1) \to 0, \tag{4.5.29}$$

where R is of rank 2 and S and L are of rank 1. Similarly, for $\mathcal{V}(\vec{f})$ we have

$$0 \to \pi^* M(f_4) \to \mathcal{V}(\vec{f}) \to W_2 \to 0 \tag{4.5.30}$$

and

$$0 \to \pi^* N(f_2) \to W_2 \to T(f_1) \to 0, \tag{4.5.31}$$

where T is rank 1, and M and N are rank 2. As usual, we denote the Chern classes of an HN bundle by the corresponding lowercase letter, with subscripts $i = 1, 2$ when the HN bundle has rank 2. Using the splitting principle and the definitions of CE classes, we obtain the following expressions:

$$\begin{pmatrix} a_1 \\ a_2' \\ b_1 \\ b_2' \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 & 0 & 0 & 0 \\ e_2 + 2e_3 & e_1 + e_2 + e_3 & e_1 + 2e_3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 2f_4 + 2f_2 & f_4 + 2f_2 + f_1 & 2f_4 + f_2 + f_1 \end{pmatrix} \begin{pmatrix} \ell \\ r_1 \\ s \\ t \\ m_1 \\ n_1 \end{pmatrix}. \tag{4.5.32}$$

There are 6 generators for $A^1(BG)$, but only 4 codimension 1 CE classes. Therefore, we will need to study the kernel of $A^1(BG) \to A^1([U'/G])$. Let $Z = V(X_2, X_3, X_4) \subset \mathbb{P}E^\vee$. For degree reasons, when we restrict the quadrics to Z, we see that they all vanish except for possibly $Q_1|_Z$, which takes the form

$$Q_1|_Z = (a_{2,5}a_{3,4} - a_{2,4}a_{3,5})X_1^2.$$

The coefficient of X_1^2 is of degree 0 on $\mathbb{P}^1$. Note that if it vanishes, the curve becomes reducible, so the vanishing of this coefficient should impose a codimension 1 relation. To find this relation, we need to find a way of picking out the quadric Q_1.

Recall that the quadrics Q_i cutting out the curve are obtained from the Pfaffians of the skew-symmetric matrix (4.5.6). Using the canonical identification $\wedge^4 F \cong F^\vee \otimes \det F$, these 5 Pfaffians correspond to a global section of the rank 5 vector bundle $\mathcal{O}_{\mathbb{P}E}(2) \otimes (\det E)^{\otimes 2} \otimes F^\vee \otimes \det F$ on $\mathbb{P}E^\vee$. Equivalently, this is a global section of $(\operatorname{Sym}^2 E^\vee) \otimes (\det E)^{\otimes 2} \otimes F^\vee \otimes \det F$ on $\mathbb{P}^1$. Working on $\pi : \mathcal{P} \to BG$, the Pfaffians thus correspond to a section of $\operatorname{Sym}^2 \mathcal{V}(\vec{e}) \otimes \det \mathcal{V}(\vec{e})^{\vee\otimes 2} \otimes \mathcal{V}(\vec{f})^\vee \otimes \det \mathcal{V}(\vec{f})$. From the HN filtration on $\mathcal{V}(\vec{e})$, there is a quotient

$$\operatorname{Sym}^2 \mathcal{V}(\vec{e}) \otimes \det \mathcal{V}(\vec{e})^{\vee\otimes 2} \otimes \mathcal{V}(\vec{f})^\vee \otimes \det \mathcal{V}(\vec{f}) \to \pi^* L^{\otimes 2}(2e_1) \otimes \det \mathcal{V}(\vec{e})^{\vee\otimes 2} \otimes \mathcal{V}(\vec{f})^\vee \otimes \det \mathcal{V}(\vec{f}), \tag{4.5.33}$$

corresponding to the X_1^2 parts of the Pfaffians. Note that we have

$$\det \mathcal{V}(\vec{e})^{\vee\otimes 2} \cong \pi^*(\det R^{\vee\otimes 2} \otimes S^{\vee\otimes 2} \otimes L^{\vee\otimes 2})(-4e_3 - 2e_2 - 2e_1)$$

and

$$\det \mathcal{V}(\vec{f}) = \pi^*(\det M \otimes \det N \otimes T)(f_1 + 2f_2 + 2f_4).$$

By the assumptions on the splitting type $\vec{f}$ and cohomology and base change, the π push forward of (4.5.33) is

$$\pi_*(\operatorname{Sym}^2 \mathcal{V}(\vec{e}) \otimes \det \mathcal{V}(\vec{e})^{\vee\otimes 2} \otimes \mathcal{V}(\vec{f})^\vee \otimes \det \mathcal{V}(\vec{f})) \to \det M \otimes \det N \otimes \det R^{\vee\otimes 2} \otimes S^{\vee\otimes 2}.$$

This quotient map corresponds to evaluating the coefficient of X_1^2 in Q_1. The line bundle on the right thus admits a non-vanishing section when pulled back to $[U'/G]$. This gives a relation:

$$v^*(m_1 + n_1 - 2s - 2r_1) = 0 \in A^1([U'/G]). \tag{4.5.34}$$

We need one more codimension 1 relation. Note that if $L_{1,2}$ and $L_{1,3}$ are linearly dependent then, after change of basis (within the $\mathcal{O}(f_2) \oplus \mathcal{O}(f_3)$ part of F), we can assume

$L_{1,2} = 0$. Then (4.5.15) shows that the resulting curve would have a g^2_4, which is impossible. Therefore, $L_{1,2}$ and $L_{1,3}$ must be linearly independent. Conditions (1), (2) and (4) imply that the degrees of $a_{1,2}, a_{1,3}, b_{1,2}, b_{1,3}$ are negative, so $L_{1,2}$ and $L_{1,3}$ are dependent if and only if

$$c_{1,2}d_{1,3} - c_{1,3}d_{1,2} = 0.$$

Below, we construct a morphism of vector bundles whose vanishing locus is the locus where $L_{1,2}$ and $L_{1,3}$ become dependent. From the HN filtration, we have a series of surjections

$$\wedge^2\mathcal{V}(\vec{f}) \to \wedge^2 W_2 \to \pi^*(N \otimes T)(f_1 + f_2).$$

Tensoring with $\mathcal{V}(\vec{e}) \otimes \det \mathcal{V}(\vec{e})^\vee$ and pushing forward, we have

$$\pi_*(\wedge^2\mathcal{V}(\vec{f}) \otimes \mathcal{V}(\vec{e}) \otimes \det \mathcal{V}(\vec{e})^\vee) \to \pi_*(\pi^*N \otimes \pi^*T \otimes \mathcal{V}(\vec{e}) \otimes \det \mathcal{V}(\vec{e})^\vee)(f_1 + f_2).$$

Because $f_1 + f_2 - g - 4 + e_i < 0$ for $i < 3$ and $f_1 + f_2 - g - 4 + e_3 = 0$, by cohomology and base change, the above map is

$$\pi_*(\wedge^2\mathcal{V}(\vec{f}) \otimes \mathcal{V}(\vec{e}) \otimes \det \mathcal{V}(\vec{e})^\vee) \to \det R^\vee \otimes S^\vee \otimes L^\vee \otimes R \otimes N \otimes T.$$

This map corresponds to projection onto the tuple of coefficients $(c_{1,2}, d_{1,2}, c_{1,3}, d_{1,3})$. Note that $R \otimes \det R^\vee \cong R^\vee$, so we can identify the section we obtained from the above map as a morphism

$$S \otimes L \otimes R \to N \otimes T.$$

Taking the determinant of this morphism, we have

$$S^{\otimes 2} \otimes L^{\otimes 2} \otimes \det R \to \det N \otimes T^{\otimes 2},$$

and if this determinant morphism vanishes, then $L_{1,2}$ and $L_{1,3}$ are dependent. Therefore, we obtain the relation

$$v^*(-n_1 - 2t + r_1 + 2s + 2\ell) = 0. \tag{4.5.35}$$

Augmenting matrix (4.5.32) by the relations (4.5.34) and (4.5.35), we the following matrix

$$\begin{pmatrix} 0 \\ 0 \\ a_1 \\ a_2' \\ b_1 \\ b_2' \end{pmatrix} = \begin{pmatrix} 0 & -2 & -2 & 0 & 1 & 1 \\ 2 & 1 & 2 & -2 & 0 & -1 \\ 1 & 1 & 1 & 0 & 0 & 0 \\ e_2 + 2e_3 & e_1 + e_2 + e_3 & e_1 + 2e_3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 2f_4 + 2f_2 & f_4 + 2f_2 + f_1 & 2f_4 + f_2 + f_1 \end{pmatrix} \begin{pmatrix} \ell \\ r_1 \\ s \\ t \\ m_1 \\ n_1 \end{pmatrix}. \tag{4.5.36}$$

The determinant of the above 6×6 matrix is the quantity in part (5) of the statement of the lemma, which does not vanish by assumption. Hence, $\ell, r_1, s, t, m_1, n_1 \in A^1([U'/G])$ are expressible in terms of CE classes.

In addition to the products of codimension 1 generators, $A^2(BG)$ has four codimension 2 generators: c_2, r_2, n_2, m_2. By definition c_2 is a CE class, so it remains to show that r_2, n_2 and m_2 are expressible in terms of CE classes. From the HN filtrations and the splitting principle, we obtain the following expression for a_2:

$$a_2 = -(e_1e_2 + 2e_1e_3 + 2e_2e_3 + e_3^2)c_2 + \ell r_1 + \ell s + r_1 s + r_2,$$

from which it follows that r_2 is expressible in terms of CE classes. Similarly, we obtain the following expressions for b_2 and b_3':

$$b_2 = -(f_4^2 + 4f_4f_2 + f_2^2 + 2f_4f_1 + 2f_2f_1)c_2 + tm_1 + tn_1 + m_1n_1 + m_2 + n_2$$

$$b_3' = -(2f_4^2f_2 + 2f_4f_2^2 + f_4^2f_1 + 4f_4f_2f_1 + f_2^2f_1)c_2 + (f_4 + 2f_2)tm_1 + (2f_4 + f_2)tn_1 +$$
$$(f_4 + f_2 + f_1)m_1n_1 + (2f_2 + f_1)m_2 + (2f_4 + f_1)n_2.$$

Because $f_2 \neq f_4$, we see that m_2 and n_2 are expressible in terms of CE classes. □

We consider one more pair of shapes of splitting types.

Lemma 4.5.3. *Let $\vec{e} = (e_1, e_2, e_3, e_4)$ and $\vec{f} = (f_1, f_2, f_3, f_4, f_5)$ satisfy the following conditions:*

1. $e_1 < e_2 = e_3 < e_4$,
2. $f_1 < f_2 = f_3 < f_4 = f_5$,
3. $e_1 + f_2 + f_5 = g + 4$,
4. $e_2 + f_1 + f_4 = g + 4$,
5. $-2e_2f_1 + 2e_4f_1 + e_1f_2 - 2e_2f_2 + e_4f_2 - e_1f_4 + 4e_2f_4 - 3e_4f_4 \neq 0$.

Let Σ be the $\vec{e}, \vec{f}$ splitting locus. Then $A^*(\Sigma')$ is generated by the restrictions of CE classes.

Proof. Set $G = \mathrm{SL}_2 \ltimes (\mathrm{Aut}(\vec{e}) \times \mathrm{Aut}(\vec{f}))$ and let $\pi : \mathcal{P} \to BG$ be the $\mathbb{P}^1$ bundle pulled back from BSL_2. The HN filtration for $\mathcal{V}(\vec{f})$ is given by

$$0 \to \pi^*S(e_4) \to \mathcal{V}(\vec{e}) \to W_1 \to 0$$

and

$$0 \to \pi^*R(e_2) \to W_1 \to \pi^*L(e_1) \to 0.$$

The filtration for $\mathcal{V}(\vec{f})$ is the same as in the previous Lemma 4.5.2. As a result, we have the following expressions for the Casnati–Ekedahl classes in terms of the generators of the Chow ring of this stratum:

$$\begin{pmatrix} a_1 \\ a_2' \\ b_1 \\ b_2' \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 & 0 & 0 & 0 \\ 2e_2+e_4 & e_1+e_2+e_4 & e_1+2e_2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 2f_4+2f_2 & f_4+2f_2+f_1 & 2f_4+f_2+f_1 \end{pmatrix} \begin{pmatrix} \ell \\ r_1 \\ s \\ t \\ m_1 \\ n_1 \end{pmatrix}. \tag{4.5.37}$$

As in the previous lemma, there are 6 codimension 1 generators for $A^*(BG)$, but only 4 codimension 1 CE classes. We will need to show $A^1(BG) \to A^1([U'/G])$ has a kernel, meaning we have relations between the generators. The first relation is quite similar to the first relation from the previous Lemma 4.5.2. Not all of the quadrics cutting out the curve can vanish on $Z = V(X_2, X_3, X_4)$. We see that upon restriction to Z all of the quadrics vanish, except for possibly Q_1, which is given by

$$Q_1|_Z = (a_{2,5}a_{3,4} - a_{2,4}a_{3,5})X_1^2.$$

As in Lemma 4.5.2, there is a quotient map

$$\mathrm{Sym}^2\,\mathcal{V}(\vec{e})\otimes\det\mathcal{V}(\vec{e})^{\vee\otimes 2}\otimes\mathcal{V}(\vec{f})^{\vee}\otimes\det\mathcal{V}(\vec{f}) \to \pi^*L^{\otimes 2}(2e_1)\otimes\det\mathcal{V}(\vec{e})^{\vee\otimes 2}\otimes\mathcal{V}(\vec{f})^{\vee}\otimes\det\mathcal{V}(\vec{f}),$$

which corresponds to the coefficients of X_1^2 in the Pfaffians. Note that we have

$$\det\mathcal{V}(\vec{e})^{\vee\otimes 2} \cong \pi^*(L^{\vee\otimes 2}\otimes S^{\vee\otimes 2}\otimes R^{\vee\otimes 2})(-2e_1-2e_4-4e_2)$$

and

$$\det \mathcal{V}(\vec{f}) = \pi^*(\det M \otimes \det N \otimes T)(f_1 + 2f_2 + 2f_4).$$

From cohomology and base change and the filtration on $\mathcal{V}(\vec{f})$, we see that the π push forward of this map is

$$\pi_*(\mathrm{Sym}^2 \mathcal{V}(\vec{e}) \otimes \det \mathcal{V}(\vec{e})^{\vee\otimes 2} \otimes \mathcal{V}(\vec{f})^\vee \otimes \det \mathcal{V}(\vec{f})) \to \det M \otimes \det N \otimes S^{\vee\otimes 2} \otimes R^{\vee\otimes 2}.$$

This quotient map corresponds to the coefficient of X_1^2 in Q_1. The non-vanishing of this coefficient means that the pullback along $v : [U'/G] \to BG$ of the line bundle on the right has a non-vanishing section. This gives us the relation

$$v^*(m_1 + n_1 - 2r_1 - 2s) = 0 \in A^1([U'/G]). \tag{4.5.38}$$

The next relation comes from considering the equations for the curve when restricted to $V(X_4)$. For degree reasons, $L_{1,2}$ and $L_{1,3}$ vanish when restricted to $V(X_4)$. Suppose that $L_{1,4}$ and $L_{1,5}$ are dependent. Then, the quadrics $Q_2, \ldots, Q_5$ all vanish along $V(L_{1,4}, X_4) = V(L_{1,5}, X_4)$. It follows that $V(Q_1, L_{1,4}, X_4)$ is contained in the curve. However, $\mathbb{P}E^\vee$ has dimension 4, so the locus $V(Q_1, L_{1,4}, X_4)$ has dimension at least 1. This means that C would be contained in $V(X_4)$, which is impossible. Therefore, the restrictions of $L_{1,4}$ and $L_{1,5}$ to $V(X_4)$ must be independent. Because $e_1 + f_1 + f_5 - g - 4 < 0$, we have

$$L_{1,4}|_{V(X_4)} = b_{1,4}X_2 + c_{1,4}X_3 \qquad \text{and} \qquad L_{1,5}|_{V(X_4)} = b_{1,5}X_2 + c_{1,5}X_3.$$

Therefore, $L_{1,4}|_{V(X_4)}$ and $L_{1,5}|_{V(X_4)}$ are dependent if and only if

$$b_{1,4}c_{1,5} - b_{1,5}c_{1,4} = 0.$$

As in Lemma 4.5.2, we construct a morphism of vector bundles whose vanishing locus is the locus where $L_{1,4}|_{V(X_4)}$ and $L_{1,5}|_{V(X_4)}$ are dependent. From the HN filtration and the corresponding filtration on $\wedge^2\mathcal{V}(\vec{f})$, we have a surjection

$$\wedge^2\mathcal{V}(\vec{f}) \to K, \tag{4.5.39}$$

where K is a vector bundle admitting a filtration

$$0 \to \pi^*M(f_4) \otimes \pi^*T(f_1) \to K \to \wedge^2 W_2 \to 0.$$

Tensoring (4.5.39) with the map $\mathcal{V}(\vec{e}) \otimes \det \mathcal{V}(\vec{e})^\vee \to W_1 \otimes \det \mathcal{V}(\vec{e})^\vee$, we obtain a surjection

$$\wedge^2\mathcal{V}(\vec{f}) \otimes \mathcal{V}(\vec{e}) \otimes \det \mathcal{V}(\vec{e})^\vee \to K \otimes W_1 \otimes \det \mathcal{V}(\vec{e})^\vee. \tag{4.5.40}$$

By cohomology and base change and the assumptions on the splitting types $\vec{e}$ and $\vec{f}$, the π push forward of (4.5.40) is given by

$$\pi_*(\wedge^2\mathcal{V}(\vec{f}) \otimes \mathcal{V}(\vec{e}) \otimes \det \mathcal{V}(\vec{e})^\vee) \to R \otimes \det R^\vee \otimes L^\vee \otimes S^\vee \otimes M \otimes T.$$

This map corresponds to projection onto the tuple of coefficients $(b_{1,4}, c_{1,4}, b_{1,5}, c_{1,5})$. Since R has rank 2, we have $R \otimes \det R^\vee \cong R^\vee$. The section obtained from the above map can be identified with a morphism

$$R \otimes L \otimes S \to M \otimes T.$$

The associated determinant morphism

$$\det R \otimes L^{\otimes 2} \otimes S^{\otimes 2} \to \det M \otimes T^{\otimes 2}$$

vanishes precisely when $b_{1,4}c_{1,5} - b_{1,5}c_{1,4} = 0$. Since this quantity is non-vanishing on $[U'/G]$, we obtain the relation

$$v^*(2t + m_1 - 2s - 2\ell - r_1) = 0. \tag{4.5.41}$$

We augment the matrix (4.5.37) by the relations (4.5.38), (4.5.41) to obtain

$$\begin{pmatrix} 0 \\ 0 \\ a_1 \\ a_2' \\ b_1 \\ b_2' \end{pmatrix} = \begin{pmatrix} 0 & -2 & -2 & 0 & 1 & 1 \\ -2 & -1 & -2 & 2 & 1 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 \\ 2e_2 + e_4 & e_1 + e_2 + e_4 & e_1 + 2e_2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 2f_4 + 2f_2 & f_4 + 2f_2 + f_1 & 2f_4 + f_2 + f_1 \end{pmatrix} \begin{pmatrix} \ell \\ r_1 \\ s \\ t \\ m_1 \\ n_1 \end{pmatrix}. \tag{4.5.42}$$

The determinant of this matrix is the quantity in part (5) of the statement of the lemma. It is non-vanishing by assumption, so on $[U'/G]$, the classes $\ell, r_1, s, t, m_1, n_1$ are expressible in terms of the CE classes.

Besides products of codimension 1 classes, $A^*(BG)$ has 4 codimension 2 generators: c_2, r_2, m_2, n_2. Using the splitting principle and the HN filtrations, we obtain the following expression for a_2:

$$a_2 = -(2e_1e_2 + e_2^2 + e_1e_4 + 2e_2e_4)c_2 + \ell r_1 + \ell s + r_1 s + r_2.$$

It follows that r_2 is expressible in terms of CE classes. Similarly, we obtain the following expressions for b_2 and b_3'.

$$b_2 = -(f_4^2 + 4f_4f_2 + f_2^2 + 2f_4f_1 + 2f_2f_1)c_2 + tm_1 + tn_1 + m_1n_1 + m_2 + n_2$$

$$b_3' = -(2f_4^2f_2 + 2f_4f_2^2 + f_4^2f_1 + 4f_4f_2f_1 + f_2^2f_1)c_2 + (f_4 + 2f_2)tm_1 + (2f_4 + f_2)tn_1 +$$
$$(f_4 + f_2 + f_1)m_1n_1 + (2f_2 + f_1)m_2 + (2f_4 + f_1)n_2.$$

Because $f_4 \neq f_2$, both m_2 and n_2 are expressible in terms of CE classes. □

4.5.3 Genus 7

Applying the constraints in (4.5.2) – (4.5.13), one obtains a stratification of $\mathcal{H}_{5,7}$ based on the allowable splitting types of $\mathcal{E}$ and $\mathcal{F}$. This stratification was obtained by Schreyer [Sch86, p. 133], and we translate it here into our notation. The claimed special linear series (which are also listed in Schreyer's table) can be seen from (4.5.14) and (4.5.15).

Lemma 4.5.4 (Schreyer)**.** *Let $g = 7$. There are 5 allowed pairs of splitting types for the bundles E and F. They give rise to the following stratification of $\mathcal{H}_{5,7}$:*

(Ψ_0) $E = (2,3,3,3), F = (4,4,4,5,5)$: the general stratum.

(Z_1) $E = (2,2,3,4), F = (4,4,4,5,5)$: such curves possess a g^1_4.

(Z_2) $E = (2,3,3,3), F = (3,4,5,5,5)$: such curves possess a g^2_6.

(Z_3) $E = (2,2,3,4), F = (3,4,4,5,6)$: such curves possess a g^2_6.

(Z_4) $E = (2,3,3,3), F = (3,3,5,5,6)$: such curves possess a g^1_3.

As our labeling suggests, by a happy coincidence, all strata outside of the "good open" Ψ actually lie inside $\beta^{-1}(\mathcal{M}^4_7)$.

Corollary 4.5.5. *The Chow ring of $\mathcal{H}_{5,7} \smallsetminus \beta^{-1}(\mathcal{M}^4_7)$ is generated by the restrictions of tautological classes. Hence, all classes supported on $\mathcal{M}_8 \smallsetminus \mathcal{M}^4_8$ are tautological.*

Proof. It this case, we have $\Psi = \Psi_0$. Applying Proposition 4.2.9, it suffices to show that all other strata Z_i are contained in $\beta^{-1}(\mathcal{M}^4_7)$. This follows immediately for Z_1 and Z_4. Suppose that a 7 curve C possesses a g^2_6. Degree 6 plane curves have arithmetic genus 10. If the g^2_6 sends C birationally onto its image, then the image must have a double point (or worse). Projection from such a point gives a g^1_4 (or g^1_k for $k < 4$). Otherwise, the g^2_6 sends C with degree three onto a conic (so C has a g^1_3) or with degree two onto a cubic. Every cubic admits a degree 2 map to $\mathbb{P}^1$ (by projecting from a point) so composing these two degree 2 maps, we see that C has a g^1_4. Thus, Z_2 and Z_3 are also contained in $\beta^{-1}(\mathcal{M}^4_7)$. □

Combining Lemma 4.5.5 with Lemma 4.4.8 completes the proof of Theorem 4.1.1. There is still some work to do in genus 8 and 9.

4.5.4 Genus 8

The constraints (4.5.2)–(4.5.13) from the beginning of the section give a stratification of $\mathcal{H}_{5,8}$, which was first observed by Schreyer [Sch86, p. 133]. The claimed linear series can be seen from (4.5.14) and (4.5.15). The codimensions of strata are determined by (4.3.4).

Lemma 4.5.6 (Schreyer)**.** *Let $g = 8$. There are 7 allowed pairs of splitting types for the bundles E and F. They give rise to the following stratification of $\mathcal{H}_{5,8}$:*

(Ψ_0) $E = (3, 3, 3, 3), F = (4, 5, 5, 5, 5)$*: the general stratum.*

(Ψ_1) $E = (2, 3, 3, 4), F = (4, 5, 5, 5, 5)$*: codimension* 1*.*

(Σ_2) $E = (2, 3, 3, 4), F = (4, 4, 5, 5, 6)$*: codimension* 2*.*

(Z_3) $E = (3, 3, 3, 3), F = (4, 4, 5, 5, 6)$*: such curves possess a g^1_4.*

(Z_4) $E = (2, 2, 4, 4), F = (4, 4, 4, 6, 6)$*: such curves possess a g^1_4.*

(Z_5) $E = (2,3,3,4), F = (3,4,5,6,6)$: such curves possess a g^2_6.

(Z_6) $E = (3,3,3,3), F = (3,3,6,6,6)$: such curves possess a g^1_3.

This time there is a stratum, Σ_2, which lives outside Ψ and not inside $\beta^{-1}(\mathcal{M}^4_8)$. Nevertheless, using arguments similar to Lemmas 4.4.8 and 4.4.9, we have the following.

Lemma 4.5.7. *The Chow ring $A^*(\mathcal{H}_{5,8} \smallsetminus \beta^{-1}(\mathcal{M}^4_8))$ is generated by CE classes. Hence, all classes supported on $\mathcal{M}_8 \smallsetminus \mathcal{M}^4_8$ are tautological.*

Proof. By a similar argument to the proof of Corollary 4.5.5, every genus 8 curve possessing a g^2_6 also possesses a g^1_k for $k \leq 4$. In particular, we see that $Z = Z_3 \cup Z_4 \cup Z_5 \cup Z_6$ is contained in $\beta^{-1}(\mathcal{M}^4_8)$. Next, on $\mathcal{H}_{5,8} \smallsetminus Z$, we have $\Sigma_2 = \Sigma_{(4,4,5,5,6)}(\mathcal{F})$. Hence, we have

$$\operatorname{codim} \Sigma_2 = 2 = h^1(\mathbb{P}^1, \mathcal{E}nd(\mathcal{O}(4,4,5,5,6))).$$

Thus, by Lemma 4.3.6, the fundamental class of Σ_2 inside $\mathcal{H}_{5,8} \smallsetminus \beta^{-1}(\mathcal{M}^4_8)$ is tautological. By Lemma 4.5.1, $A^*(\Sigma'_2)$ is generated by tautological classes. It then follows from the push-pull formula that every class supported on $\Sigma'_2 \subset \mathcal{H}_{5,8} \smallsetminus \beta^{-1}(\mathcal{M}^4_8)$ is tautological. By Proposition 4.5.1, we know $A^*(\Psi) = A^*(\Psi_0 \cup \Psi_1)$ is generated by tautological classes. It follows that all of $A^*(\mathcal{H}_{5,8} \smallsetminus \beta^{-1}(\mathcal{M}^4_8))$ is generated by tautological classes. $\square$

Combining Lemma 4.5.7 with Lemma 4.4.9 completes the proof of Theorem 4.1.2. The rest of the paper will deal with the case $g = 9$.

4.5.5 Genus 9

There is a similar stratificaion in genus 9, which was given by Sagraloff [Sag05]. Below, we translate Sagraloff's notation into ours. The stratification can be obtained from the conditions (4.5.2)–(4.5.13), and the claimed linear series can be seen from (4.5.14) and (4.5.15). The codimensions of strata are determined by (4.3.4).

Lemma 4.5.8 (Sagraloff). *Let $g = 9$. There are 7 allowed pairs of splitting types for the bundles E and F. They give rise to the following stratification of $\mathcal{H}_{5,9}$:*

(Ψ_0) $E = (3, 3, 3, 4), F = (5, 5, 5, 5, 6)$*: the general stratum.*

(Ψ_1) $E = (3, 3, 3, 4),\ F = (4, 5, 5, 6, 6)$*: codimension* 2.

(Σ_2) $E = (2, 3, 4, 4),\ F = (4, 5, 5, 6, 6)$*: codimension* 2.

(Σ_3) $E = (2, 3, 3, 5),\ F = (4, 5, 5, 6, 6)$*: codimension* 4.

(Z_4) $E = (3, 3, 3, 4),\ F = (4, 4, 6, 6, 6)$*: such curves possess a* g^1_4.

(Z_5) $E = (2, 3, 4, 4),\ F = (4, 4, 5, 6, 7)$*: such curves possess a* g^1_4.

(Z_6) $E = (2, 3, 4, 4),\ F = (3, 4, 6, 6, 7)$*: such curves possess a* g^2_6.

Lemma 4.5.9. *The Chow ring $A^*(\mathcal{H}_{5,9} \smallsetminus \beta^{-1}(\mathcal{M}^4_9))$ is generated by tautological classes. Hence, all classes supported on $\mathcal{M}^5_9$ are tautological.*

Proof. First, we see that $Z = Z_4 \cup Z_5 \cup Z_6$ is contained in $\beta^{-1}(\mathcal{M}^4_9)$. Then, note that on $\mathcal{H}_{5,9} \smallsetminus Z$, we have that $\Sigma_3 = \Sigma_{(2,3,3,5)}(\mathcal{E})$. Moreover, we see that

$$\operatorname{codim} \Sigma_3 = 4 = h^1(\mathbb{P}^1, \mathcal{E}nd(\mathcal{O}(2, 3, 3, 5))).$$

By Lemma 4.3.6, it follows that the fundamental class of Σ_3 is tautological. By Lemma 4.5.3, we see $A^*(\Sigma'_3)$ is generated by tautological classes, so by the push-pull formula, every class supported on $\Sigma'_3 \subset \mathcal{H}_{5,9} \smallsetminus \beta^{-1}(\mathcal{M}^4_9)$ is tautological.

Similarly, on $\mathcal{H}_{5,9} \smallsetminus \beta^{-1}(\mathcal{M}^4_9)$, we have $\Sigma_2 = \Sigma_{(2,3,4,4)}(\mathcal{E})$, and

$$\operatorname{codim} \Sigma_2 = 2 = h^1(\mathbb{P}^1, \mathcal{E}nd(\mathcal{O}(2, 3, 4, 4))).$$

Applying Lemma 4.3.6, the fundamental class of Σ_2 is tautological. Applying Lemma 4.5.2, we see that every class supported on $\Sigma'_2 \subset \mathcal{H}_{5,9} \smallsetminus \beta^{-1}(\mathcal{M}^4_9)$ is expressible in terms of tautological classes. By Proposition 4.5.1, $A^*(\Psi_0 \cup \Psi_1)$ is generated by tautological classes. Therefore, $A^*(\mathcal{H}_{5,9} \smallsetminus \beta^{-1}(\mathcal{M}_9^4))$ is generated by tautological classes. □

4.6 The General Genus 9 Curve

Mukai [Muk10] completely characterized canonical curves of genus 9 without a g^1_5 as linear sections of a symplectic Grassmannian. We briefly recall his construction here. Let V be a six-dimensional vector space equipped with a symplectic form σ. The *symplectic Grassmannian* $Sp(3, V) \subset G(3, V)$ parametrizes three-dimensional *symplectic subspaces* $U \subset V$, i.e. subspaces such that $\sigma|_U = 0$. The Grassmannian $G(3, V)$ embeds in $\mathbb{P}(\wedge^3 V) \cong \mathbb{P}^{19}$ via the Plücker embedding. Contracting with the symplectic form gives a map

$$\sigma^\sharp : \wedge^3 V \to V,$$

and the symplectic Grassmannian is the intersection of $G(3, V)$ with $\mathbb{P}(\ker \sigma^\sharp) \subset \mathbb{P}(\wedge^3 V)$. Note that the subspace $\mathbb{P}(\ker \sigma^\sharp) \subset \mathbb{P}(\wedge^3 V)$ corresponds to subspace of symmetric matrices in Mukai's description of the Plücker embedding [Muk10, p. 1544].

Recall that we use the subspace convention for Grassmannians and projective spaces. For example, given a globally generated rank 3 vector bundle E on C, the evaluation map $H^0(E) \to E$ determines a map $C \to G(3, H^0(E)^\vee)$ by considering the dual $E^\vee \to H^0(E)^\vee$. Similarly, the canonical embedding sends a curve $C \hookrightarrow \mathbb{P}(H^0(\omega_C)^\vee)$. The following is an amalgamation of Mukai's Theorems A, B, and C of [Muk10].

Theorem 4.6.1. *Suppose C is a smooth curve of genus 9 with no g^1_5. Then there is a unique rank 3 vector bundle E on C with the following properties:*

1. $\det E \cong \omega_C$.

2. $h^0(C, E) = 6$.

3. E is globally generated and for every 3-dimensional subspace $U \subset H^0(E)$, the evaluation homomorphism $U \otimes \mathcal{O}_C \to E$ is injective or everywhere of rank 2.

The bundle E induces a morphism $C \to G(3, H^0(E)^\vee)$ whose image is contained in the symplectic Grassmannian $Sp(3, H^0(E)^\vee) \subset G(3, H^0(E)^\vee)$. The curve C in its canonical embedding is obtained by intersecting $Sp(3, H^0(E)^\vee) \subset \mathbb{P}(\ker \sigma^\sharp) \cong \mathbb{P}^{13}$ with an eight dimensional linear subspace $\mathbb{P}^8 \subset \mathbb{P}^{13}$. Such a linear space is unique up to the action of PSp_6, the subgroup of PGL_6 fixing the one dimensional space spanned by a symplectic form.

Moreover, a canonical curve C of genus 9 is the intersection $\mathbb{P}^8 \cap Sp(3,6)$ if and only if C has no g^1_5.

Let $\Delta \subset G(9,14) = \mathbb{G}(8,13)$ be the locus of linear subspaces whose intersection with $Sp(3,6) \subset \mathbb{P}(\ker \sigma^\sharp) = \mathbb{P}^{13}$ is not a smooth genus 9 curve. The above theorem provides a morphism

$$\phi : [G(9,14) \smallsetminus \Delta / \mathrm{PSp}_6] \to \mathcal{M}_9 \smallsetminus \mathcal{M}^5_9. \tag{4.6.1}$$

We wish to show that ϕ is an isomorphism. The basic idea of our proof is modeled after [PV15b, Theorem 5.7]. In particular, we make use of the following standard lemma, whose proof we include for completeness.

Lemma 4.6.2. *Let $f : \mathcal{X} \to \mathcal{Y}$ be a separated morphism of connected smooth Deligne–Mumford stacks that are of finite type over a field. Suppose that*

1. *the characteristic of the ground field is zero,*

2. *f induces an isomorphism on stabilizer groups of geometric points, and*

3. *f induces a bijection on geometric points.*

Then f is an isomorphism.

Proof. Let $a : Y \to \mathcal{Y}$ be a connected, smooth cover by a smooth scheme Y. Let $X := \mathcal{X} \times_{\mathcal{Y}} Y$ be the fiber product, so we have a diagram

$$\begin{tikzcd} X \arrow[r, "g"] \arrow[d, "b"'] & Y \arrow[d, "a"] \\ \mathcal{X} \arrow[r, "f"'] & \mathcal{Y}. \end{tikzcd}$$

Suppose $x : \operatorname{Spec} k \to X$ is a geometric point. The stabilizer group G_x of x is equal to the fiber product of stabilizer groups $G_{b(x)} \times_{G_{f(b(x))}} G_{g(x)}$. But Y is a scheme, so $G_{g(x)}$ is trivial. Hence, $G_x = \ker(G_{b(x)} \to G_{f(b(x))})$, which is trivial by hypothesis. By [Con07, Theorem 2.2.5], it follows that X is an algebraic space. Further, the map $f' : X \to Y$ is quasi-finite and separated so by [Sta21, Tag 03XX], we know X is a scheme. Because f induces an isomorphism on stabilizer groups of geometric points, the map $f' : X \to Y$ is a bijection on geometric points. Because a is smooth and $\mathcal{X}$ is smooth and connected, we know X is also smooth and connected. Working in characteristic zero, the map f' is generically smooth, hence birational. Now, Zariski's Main Theorem shows that $f' : X \to Y$ is an isomorphism. □

Lemma 4.6.3. *Suppose the characteristic of the ground field is zero. The map ϕ induces an isomorphism on stabilizer groups of geometric points.*

Proof. In characteristic zero, finite group schemes are smooth, so it suffices to show the map induces a bijection on the finite stabilizer groups. Suppose $x : \operatorname{Spec} k \to [G(9, 14) \smallsetminus \Delta / \operatorname{PSp}_6]$ is a geometric point. Such a point is the data of (V, σ, W) where V is a six-dimensional vector space, σ is a symplectic form remembered up to scaling and $W \subset \ker \sigma^\sharp \subset \wedge^3 V$ is a 9 dimensional subspace. The stabilizer group of x is the subgroup of elements $\gamma \in \operatorname{PSp}_6 \subset \operatorname{PGL}_6$ that send $W \subset \ker \sigma^\sharp \subset \wedge^3 V$ into itself. The image $\phi(x)$ is the genus 9 curve

$$C := \mathbb{P}W \cap Sp(3, V) \subset \mathbb{P}(\ker \sigma^\sharp) \subset \mathbb{P}(\wedge^3 V).$$

The automorphism group of $\phi(x)$ is the automorphism group of C.

To see that ϕ induces an injection on these stabilizer groups, suppose $\gamma \in \mathrm{PSp}_6$ induces the identity on C. Let $E^\vee \to V$ be the restriction of the tautological sequence on $Sp(3, V)$ to C. By [Muk10, Section 4], the bundle E is the unique rank 3 bundle of Mukai's Theorem 4.6.1 and $E^\vee \to V$ is dual to the evaluation map $H^0(E) \to E$. Let $\widetilde{\gamma} \in \mathrm{GSp}_6 \subset \mathrm{GL}_6$ be a lift of γ. Since γ induces the identity on $C \subset Sp(3, V)$, there must exist an automorphism ϵ of E on C so that the diagram below commutes

$$\begin{tikzcd} E^\vee \arrow[r] \arrow[d, "\epsilon^\vee"'] & V \arrow[d, "\widetilde{\gamma}"] \\ E^\vee \arrow[r] & V. \end{tikzcd} \tag{4.6.2}$$

Above, the horizontal maps are the same (restricted from the tautological sequence on $Sp(3, V)$) In [Muk10, Proposition 3.5(3)], Mukai showed that the only automorphisms of E are scalars, so ϵ is a scalar. For (4.6.2) to commute, the map $\widetilde{\gamma}$ must be the dual of the effect of ϵ on global sections. Hence, $\widetilde{\gamma}$ is also a scalar, so γ is the identity.

To see that ϕ induces a surjection on stabilizer groups, suppose we have an automorphism $i : C \to C$. We need to find an element $\gamma \in \mathrm{PSp}_6(S)$ that induces i on $C \subset \mathbb{P}(\ker \sigma^\sharp) \subset \mathbb{P}(\wedge^3 V)$. Let E on C be the restriction of the tautological bundle on $Sp(3, V)$, which is the Mukai bundle. The tautological surjection $V \to E$ induces an isomorphism $V \cong H^0(E)$. The pullback i^*E has the properties of Theorem 4.6.1, so Mukai's uniqueness tells us $i^*E \cong E$, and moreover, this isomorphism is unique up to scaling [Muk10, Proposition 3.5(3)]. Now i^* gives rise to an automorphism

$$\gamma^\vee : V^\vee \cong H^0(E) \xrightarrow{i^*} H^0(i^*E) \cong H^0(E) \cong V^\vee,$$

which is well-defined up to scaling, and preserves the symplectic form up to scaling. By construction, the dual of this element, $\gamma \in \mathrm{PSp}_6$ induces the automorphism $i : C \to C$. $\square$

Lemma 4.6.4. *The quotient* $[G(9, 14) \smallsetminus \Delta / \mathrm{PSp}_6]$ *is separated.*

Proof. By Mukai [Muk10, Lemma 4.1], if the intersection $\mathbb{P}^8 \cap Sp(3,6)$ is smooth of the expected dimension 1, then it is a genus 9 curve. Thus, Δ is the locus of linear spaces whose intersection with $Sp(3,6)$ has a point with tangent space of dimension 2 or more. Considering the incidence correspondence

$$\{(p, \Lambda) \in Sp(3,6) \times G(9,14) : \dim(\mathbb{P}\Lambda \cap \mathbb{T}_p Sp(3,6)) \geq 2\},$$

one sees that Δ is an irreducible divisor. Let $L = \mathcal{O}(\Delta)$ be the corresponding ample line bundle on $G(9,14)$.

Let V be a 6-dimensional vector space equipped with a symplectic form σ. The group $G := \mathrm{PSp}_6$ acts on $G(9,14) = G(9, \ker \sigma^\sharp)$ via the 14-dimensional representation $\ker \sigma^\sharp$. Let $X = G(9,14) \smallsetminus \Delta$. We claim that the orbit of every point in X is closed in X. Indeed suppose x' is in the closure of the orbit of $x \in X$. The orbit of x corresponds to a constant family of a curve $[C] \in \mathcal{M}_9 \smallsetminus \mathcal{M}_9^5$. If $x' \in X$ is in the closure of the orbit of x, the intersection of the corresponding linear space with $Sp(3,6)$ is a smooth curve C' in the closure of the constant family of C, so $C' = C$. By Mukai's Theorem 4.6.1, x' is in the orbit of x. Because the orbits are closed, X is contained in the stable locus of the action of G on $G(9,14)$ with respect to L (see [MFK94, Definition 1.7(c)]).

By Lemma 4.6.3, the stabilizers of G acting on X are all finite. Therefore, [Edi13, Theorem 4.18] shows that the action of G on X is proper. By [Edi13, Proposition 4.17] the quotient stack $[X/G]$ is separated. $\square$

Corollary 4.6.5. *Assume the characteristic of the ground field is* 0. *The map* ϕ *in* (4.6.1) *is an isomorphism.*

Proof. Mukai's Theorem 4.6.1 says that ϕ induces a bijection on geometric points. By Lemma 4.6.4, the source of ϕ is separated, and hence the map ϕ is separated. By Lemma 4.6.3, we know ϕ induces an isomorphism on stabilizer groups of geometric points. Thus, ϕ is an isomorphism by Lemma 4.6.2. $\square$

Remark 4.6.6. In positive characteristic, the only way ϕ can fail to be an isomorphism is if ϕ induces a purely inseparable extension of function fields. However, even if ϕ is a purely inseparable extension of degree d, the maps ϕ^* and $\frac{1}{d}\phi_*$ are mutually inverse. Therefore, we still have an isomorphism of Chow rings $A^*([G(9,14) \smallsetminus \Delta / \mathrm{PSp}_6]) \cong A^*(\mathcal{M}_9 \smallsetminus \mathcal{M}_9^5)$, which is actually all we need for our purposes.

Our task is now to compute generators for the Chow ring of $[G(9,14) \smallsetminus \Delta / \mathrm{PSp}_6]$ and show that they are tautological. First, note that there is an exact sequence

$$1 \to \mu_2 \to \mathrm{Sp}_6 \to \mathrm{PSp}_6 \to 0.$$

It follows that

$$[G(9,14) \smallsetminus \Delta / \mathrm{Sp}_6] \to [G(9,14) \smallsetminus \Delta / \mathrm{PSp}_6]$$

is a μ_2-banded gerbe. Hence, the two stacks have isomorphic Chow rings (with $\mathbb{Q}$-coefficients), so we may work with the Sp_6 quotient instead. The stack $[G(9,14) \smallsetminus \Delta / \mathrm{Sp}_6]$ is an open substack of a Grassmann bundle over BSp_6. Therefore, its Chow ring is generated by the Chern classes of the tautological subbundle $\mathcal{S}$ of the Grassmann bundle together with the (pullbacks of) generators of the Chow ring of BSp_6. Totaro [Tot99, Section 15] computed the Chow ring of BSp_{2n}.

Proposition 4.6.7 (Totaro)**.** *The Chow ring $A^*(\mathrm{BSp}_{2n})$ is isomorphic to $\mathbb{Z}[c_2, c_4, \ldots, c_{2n}]$ where c_{2i} are the classes of the standard representation (induced via $\mathrm{Sp}_{2n} \hookrightarrow \mathrm{SL}_{2n}$).*

As a result, we obtain generators for the Chow ring of $\mathcal{M}_9 \smallsetminus \mathcal{M}_9^5$.

Lemma 4.6.8. *The Chow ring of $\mathcal{M}_9 \smallsetminus \mathcal{M}_9^5$ is generated by the Chern classes of $\mathcal{S}$ and the Chern classes $c_2(\mathcal{V}), c_4(\mathcal{V}), c_6(\mathcal{V})$, where $\mathcal{V}$ is the standard representation of Sp_6.*

First, we deal with the Chern classes $c_i(\mathcal{S})$. Let $f : \mathcal{C} \to \mathcal{M}_9 \smallsetminus \mathcal{M}_9^5$ be the universal curve.

Lemma 4.6.9. *The Chern classes of $\mathcal{S}$ are tautological.*

Proof. By Mukai's theorem, the projectivization of dual of the tautological subbundle $\mathbb{P}\mathcal{S}^\vee$ is identified with projectivization of the Hodge bundle $\mathbb{P}(f_*\omega_f)$. Therefore, $\mathcal{S} \cong (f_*\omega_f)^\vee \otimes \mathcal{L}$ where $\mathcal{L}$ is some line bundle on $\mathcal{M}_9 \smallsetminus \mathcal{M}_9^5$. By a theorem of Harer [Har85] in characteristic 0 and Moriwaki [Mor01] in characteristic p, $\mathrm{Pic}(\mathcal{M}_g)$ and hence $\mathrm{Pic}(\mathcal{M}_9 \smallsetminus \mathcal{M}_9^5)$ is generated by $c_1(f_*\omega_f)$. It follows from the splitting principle that the Chern classes of $\mathcal{S}$ are tautological. $\square$

Next we deal with the Chern classes $c_i(\mathcal{V})$. Writing $f : \mathcal{C} \to \mathcal{M}_9 \smallsetminus \mathcal{M}_9^5$ for the universal curve, let $\mathcal{I}_2(\mathcal{C})$ be defined by the exact sequence

$$0 \to \mathcal{I}_2(\mathcal{C}) \to \mathrm{Sym}^2(f_*\omega_f) \to f_*(\omega_f^{\otimes 2}) \to 0.$$

The bundle $\mathcal{I}_2(\mathcal{C})$ is a rank 21 bundle parametrizing the quadrics vanishing on the curve under its canonical embedding. By Petri's theorem, a nontrigonal canonical curve of genus 9 is exactly the common zero locus of these 21 quadrics.

Lemma 4.6.10. *The bundle $\mathcal{I}_2(\mathcal{C})$ is isomorphic to* $\mathrm{Sym}^2 \mathcal{V}$.

Proof. Because a canonical curve of genus 9 with no g^1_5 is a linear section of the symplectic Grassmannian, we see that we can identify the space of quadrics vanishing on the canonical curve with the restriction to $\mathbb{P}^8$ of the space of quadrics defining the symplectic Grassmannian $Sp(3, V) \subset \mathbb{P}(\ker \sigma^\sharp) \cong \mathbb{P}^{13}$. The symplectic Grassmannian is the zero locus of 21 quadrics in $\mathbb{P}^{13}$, see [Muk10, Equation 0.1]. That is, $\mathcal{I}_2(\mathcal{C})$ is the corresponding 21-dimensional representation of Sp_6. Following Mukai's notation on p. 1544, let V be a six-dimensional vector space with a symplectic form and choose a decomposition $V \cong U_0 \oplus U_\infty$ for two symplectic subspaces U_0, U_∞. Then we identify the representations in [Muk10, Equation 0.1] as follows. The first equation (representing 6 quadrics) lives in a space of symmetric 3×3 matrices $\mathrm{Sym}_3\, k$ corresponding to $\mathrm{Sym}^2 U_0$, the second equation (representing another 6 quadrics) lives in $\operatorname{Sym}_3 k \cong \operatorname{Sym}^2 U_\infty$ and the third equation (representing 9 quadrics) lives in a space of 3×3 matrices, $\mathrm{Mat}_3 \cong U_0 \otimes U_\infty$. Together, we recognize $(\operatorname{Sym}^2 U_0) \oplus (\operatorname{Sym}^2 U_\infty) \oplus (U_0 \otimes U_\infty)$ as $\operatorname{Sym}^2 V$, which is also isomorphic to the adjoint representation of Sp_6. □

Corollary 4.6.11. *The Chern classes $c_2(\mathcal{V}), c_4(\mathcal{V}), c_6(\mathcal{V})$ are all tautological.*

Proof. By the previous Lemma, there is an exact sequence

$$0 \to \operatorname{Sym}^2 \mathcal{V} \to \operatorname{Sym}^2(f_*\omega_f) \to f_*(\omega_f^{\otimes 2}) \to 0.$$

By the splitting principle and Grothendieck–Riemann–Roch, the Chern classes of $\operatorname{Sym}^2 f_*\omega_f$ and $f_*(\omega_f^{\otimes 2})$ are tautological. Hence, the Chern classes of $\operatorname{Sym}^2 \mathcal{V}$ are tautological. By the splitting principle and the fact that the odd Chern classes of $\mathcal{V}$ vanish, we have

$$c(\operatorname{Sym}^2 \mathcal{V}) = 1 + 8c_2(\mathcal{V}) + [22c_2(\mathcal{V})^2 + 14c_4(\mathcal{V})] + [28c_2(\mathcal{V})^3 + 54c_2(\mathcal{V})c_4(\mathcal{V}) + 38c_6(\mathcal{V})] + \dots.$$

It follows that $c_2(\mathcal{V}), c_4(\mathcal{V}), c_6(\mathcal{V})$ are tautological. □

By Lemmas 4.6.8 and 4.6.9 and Corollary 4.6.11, we conclude that $A^*(\mathcal{M}_9 \smallsetminus \mathcal{M}_9^5)$ is tautological. Combining this with Lemma 4.5.9 completes the proof of Theorem 4.1.4.

This chapter, in full, has been submitted for publication. It is coauthored with Larson, Hannah. The dissertation author was co-primary investigator and author of this paper.

Chapter 5

The Chow rings of moduli spaces of elliptic surfaces over $\mathbb{P}^1$

5.1 Introduction

Given a smooth stack X that is the solution to a moduli problem, there are often natural algebraic cycles called tautological classes in $A^*(X)$, the Chow ring of X with rational coefficients. For example, when $X = \mathcal{M}_g$, the moduli space of smooth curves of genus g, there is the tautological subring $R^*(\mathcal{M}_g) \subset A^*(\mathcal{M}_g)$ generated by the κ-classes. Faber [Fab99] gave a series of conjectures on the structure of $R^*(\mathcal{M}_g)$, which assert that $R^*(\mathcal{M}_g)$ behaves like the algebraic cohomology ring of a smooth projective variety of dimension $g-2$, even though $\mathcal{M}_g$ is neither projective nor of dimension $g-2$. Looijenga [Loo95] proved that $R^i(\mathcal{M}_g) = 0$ for $i > g-2$ and that $R^{g-2}(\mathcal{M}_g) \cong \mathbb{Q}$, settling one of Faber's conjectures. Looijenga's theorem gives a new proof of Diaz's result [Dia84] that the maximal dimension of a complete subvariety of $\mathcal{M}_g$ is $g-2$. Faber further conjectured that $R^*(\mathcal{M}_g)$ should be a Gorenstein ring with socle in codimension $g-2$, meaning that the intersection product is a perfect pairing

$$R^i(\mathcal{M}_g) \times R^{g-2-i}(\mathcal{M}_g) \to R^{g-2}(\mathcal{M}_g) \cong \mathbb{Q}.$$

Faber [Fab99] and Faber–Zagier proved this conjecture for $g \leq 23$ by producing relations in the tautological ring and showing computationally that the resulting quotient is Gorenstein.

Recently, there has been significant interest in the tautological rings $R^*(\mathcal{F}_\Lambda)$ of the moduli spaces $\mathcal{F}_\Lambda$ of lattice polarized K3 surfaces [MP13, MOP17, PY20, BLMM17, BL19]. In [MOP17], the tautological rings are defined as the subrings of $A^*(\mathcal{F}_\Lambda)$ generated by the fundamental classes of Noether–Lefschetz loci together with push forwards of κ-classes from all Noether-Lefschetz loci. There are natural analogues of Faber's conjectures for $R^*(\mathcal{F}_\Lambda)$.[1]

Conjecture 5.1.1 (Oprea–Pandharipande)**.** *Let* $d = \dim \mathcal{F}_\Lambda$*.*

[1] We learned about these analogues from a lecture given by Rahul Pandharipande in the algebraic geometry seminar at UCSD and from a course on K3 surfaces given by Dragos Oprea.

1. *For $i > d-2$, $R^i(\mathcal{F}_\Lambda) = 0$.*

2. *There is an isomorphism $R^{d-2}(\mathcal{F}_\Lambda) \cong \mathbb{Q}$.*

The primary evidence for part (1) of this conjecture is a theorem of Petersen [Pet19, Theorem 2.2], which says that the image $RH^{2*}(\mathcal{F}_\Lambda)$ of $R^*(\mathcal{F}_\Lambda)$ in cohomology under the cycle class map vanishes above cohomology degree $2(d-2)$. If Conjecture 5.1.1 holds, then one can further ask for the analogue of Faber's Gorenstein conjecture: is there a perfect pairing

$$R^i(\mathcal{F}_\Lambda) \times R^{d-2-i}(\mathcal{F}_\Lambda) \to R^{d-2}(\mathcal{F}_\Lambda) \cong \mathbb{Q}?$$

In this paper, we study the Chow rings of moduli spaces E_N of elliptic surfaces Y fibered over $\mathbb{P}^1$ with section $s : \mathbb{P}^1 \to Y$ and fundamental invariant N (see Section 2 for definitions). The main result is that natural analogues of Faber's vanishing and Gorenstein conjectures hold for the entire Chow ring $A^*(E_N)$ for each $N \geq 2$.

Theorem 5.1.2. *Let $N \geq 2$ be an integer.*

1. *The Chow ring has the form*

$$A^*(E_N) = \mathbb{Q}[a_1, c_2]/I_N$$

   where $a_1 \in A^1(E_N), c_2 \in A^2(E_N)$, and I_N is the ideal generated by the two relations from Proposition 5.3.4.

2. *The Poincaré polynomial collecting dimensions of the Chow groups is given by*

$$\begin{aligned} p_N(t) &= \sum \dim A^i(E_N) t^i \\ &= 1 + t + 2t^2 + 2t^3 + 3t^4 + 3t^5 + 4t^6 + 4t^7 + 5t^8 + \\ &\quad + 4t^9 + 4t^{10} + 3t^{11} + 3t^{12} + 2t^{13} + 2t^{14} + t^{15} + t^{16}. \end{aligned}$$

3. *The Chow ring $A^*(E_N)$ is Gorenstein with socle in codimension* 16.

We also have similar partial results for Poincaré polynomial for the cohomology ring when $N = 2$ that will appear in future work.

A notable property is that the dimensions of the Chow groups are independent of N. In particular, the Chow groups $A^i(E_N)$ are only nonzero in codimension $0 \leq i \leq 16$, despite the fact that the dimensions of the moduli spaces E_N go to infinity with N. Moreover, the ring structure depends in a simple and explicit way on N coming from the relations in Proposition 5.3.4. As a consequence of Theorem 5.1.2, we obtain an analogue of Diaz's theorem [Dia84] on the maximal dimension of a complete subvariety of $\mathcal{M}_g$. In our case, the bound is independent of N.

Corollary 5.1.3. *Let $N \geq 2$ be an integer. The maximal dimension of a complete subvariety of E_N is* 16*.*

When $N = 2$, the corresponding elliptic surfaces are K3 surfaces polarized by a hyperbolic lattice U with intersection matrix

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

We show that the generators a_1 and c_2 of $A^*(E_2)$ have natural interpretations as tautological classes in $R^*(\mathcal{F}_U)$, where $\mathcal{F}_U$ is the moduli space of U-polarized K3 surfaces.

Theorem 5.1.4. *Under the identification of $A^*(E_2)$ with $A^*(\mathcal{F}_U)$, the classes a_1 and c_2 lie in $R^*(\mathcal{F}_U)$. Therefore, $A^*(\mathcal{F}_U) = R^*(\mathcal{F}_U)$ is a Gorenstein ring with socle in codimension* 16*. In particular, Conjecture 5.1.1 is true for $\Lambda = U$, the hyperbolic lattice.*

The paper is structured as follows. In Section 5.2, we collect the necessary background on elliptic surfaces, the closely related notion of Weierstrass fibrations, and their moduli. In Section 5.3, we prove Theorem 5.1.2 and Corollary 5.1.3. In Section 5.4, we explore the case $N = 2$ and prove Theorem 5.1.4. We also compute relations among the codimension 1 κ-classes.

Notations and Conventions

1. Schemes are over a fixed algebraically closed field k of characteristic not 2 or 3. All stacks are fibered over the category of schemes over k.

2. We denote the Chow ring of a space X with *rational coefficients* by $A^*(X)$.

3. We use the subspace (classical) convention for projective bundles.

5.2 Elliptic Surfaces and Weierstrass Fibrations

In this section, we collect the necessary background information on elliptic surfaces and Weierstrass fibrations following Miranda [Mir81]. The main objects of interest in this paper will be moduli spaces of minimal elliptic surfaces over $\mathbb{P}^1$ with section.

Definition 5.2.1. A minimal elliptic surface over $\mathbb{P}^1$ with section consists of the following data:

1. a smooth projective surface Y,

2. a proper morphism $\pi : Y \to \mathbb{P}^1$ whose general fiber is a smooth connected curve of genus 1 and such that none of the fibers contain any (-1)-curves,

3. a section $s : \mathbb{P}^1 \to Y$ of π.

Remark 5.2.2. Note that the minimality condition is different from the usual one given in the birational geometry of surfaces. There can be (-1)-curves on the surface Y, but they must not lie in the fibers of p.

We will study moduli spaces of minimal elliptic surfaces by studying the closely related notion of Weierstrass fibrations.

Definition 5.2.3. A Weierstrass fibration over $\mathbb{P}^1$ consists of the following data:

1. a projective surface X,
2. a flat proper morphism $p : X \to \mathbb{P}^1$ such that every fiber is an irreducible curve of arithmetic genus 1 and the general fiber is smooth,
3. a section $s : \mathbb{P}^1 \to X$ of p whose image does not intersect the singular points of any fiber.

Weierstrass fibrations $X \to \mathbb{P}^1$ have a natural invariant associated to them that governs aspects of the geometry of X and the associated moduli spaces.

Definition 5.2.4. Let $p : X \to \mathbb{P}^1$ be a Weierstrass fibration.

1. The fundamental line bundle associated to $p : X \to \mathbb{P}^1$ is the line bundle

$$\mathbb{L} = (R^1 p_* \mathcal{O}_X)^\vee.$$

2. The fundamental invariant associated to $p : X \to \mathbb{P}^1$ is the integer

$$N = \deg \mathbb{L}.$$

Because $\mathbb{L}$ is a line bundle on $\mathbb{P}^1$, it is of the form $\mathcal{O}(N)$ where N is the fundamental invariant. By [Mir81, Corollary 2.4], the fundamental invariant is always nonnegative.

There is a one-to-one correspondence between minimal elliptic surfaces with section and Weierstrass fibrations with at worst rational double points as singularities. Given a minimal elliptic surface $\pi : Y \to \mathbb{P}^1$, we obtain a Weierstrass fibration with at worst rational double points $p : X \to \mathbb{P}^1$ by contracting any rational components in the fibers that do not meet the section. Conversely, given a Weierstrass fibration $p : X \to \mathbb{P}^1$ with at worst rational double points as singularities, resolving the singularities and blowing down (-1)-curves in the fibers yields a minimal elliptic surface $\pi : Y \to \mathbb{P}^1$. We say that Y contracts to X and X resolves to Y. Weierstrass fibrations have a representation as divisors on a $\mathbb{P}^2$-bundle over $\mathbb{P}^1$, which Miranda [Mir81] used to construct coarse moduli spaces for Weierstrass fibrations, and hence elliptic surfaces, using Geometric Invariant Theory.

Lemma 5.2.5 ((Corollary 2.5 of [Mir81]))**.** *Let $\pi : Y \to \mathbb{P}^1$ be a minimal elliptic surface with section contracting to a Weierstrass fibration $p : X \to \mathbb{P}^1$ with fundamental invariant N. Then X is isomorphic to the closed subscheme of $\mathbb{P}(\mathcal{O} \oplus \mathcal{O}(2N) \oplus \mathcal{O}(3N))$ defined by*

$$y^2z = x^3 + Axz^2 + Bz^3.$$

where $A \in H^0(\mathbb{P}^1, \mathcal{O}(4N))$ and $B \in H^0(\mathbb{P}^1, \mathcal{O}(6N))$. Moreover,

1. *$4A^3 + 27B^2$ is not identically zero. If it vanishes at $q \in \mathbb{P}^1$, the fiber of X over q is singular.*
2. *For every $q \in \mathbb{P}^1$, $v_q(A) \leq 3$ or $v_q(B) \leq 5$, where v_q is the order of vanishing at q.*

Set $V_{4N} := H^0(\mathbb{P}^1, \mathcal{O}(4N))$ and $V_{6N} := H^0(\mathbb{P}^1, \mathcal{O}(6N))$. Let $T_N \subset V_{4N} \oplus V_{6N}$ denote the open subspace satisfying conditions (1) and (2) from Lemma 5.2.5. The following is [Mir81, Corollary 2.8].

Corollary 5.2.6. *The set of isomorphism classes of minimal elliptic surfaces $\pi : Y \to \mathbb{P}^1$ with $\deg R^1p_*\mathcal{O}_X = -N$ and with fixed section (equivalently, Weierstrass fibrations with only rational double points) is in $1-1$ correspondence with the set of orbits of $\mathrm{SL}_2 \times \mathbb{G}_m$ on T_N.*

In order to give the set of orbits a geometric structure, Miranda analyzes the stability of the action of $\mathrm{SL}_2 \times \mathbb{G}_m$ on T_N.

Proposition 5.2.7. *Let $(A, B) \in V_{4N} \oplus V_{6N}$ be a pair of forms.*

1. *The point corresponding to (A, B) is not semistable if and only if there is a point $q \in \mathbb{P}^1$ such that*

$$v_q(A) > 2N \text{ and } v_q(B) > 3N.$$

2. *The point corresponding to (A, B) is not stable if and only if there is a point $q \in \mathbb{P}^1$ such that*

$$v_q(A) \geq 2N \text{ and } v_q(B) \geq 3N.$$

From Lemma 5.2.5 and Proposition 5.2.7, we see that as long as $N \geq 2$, points in T_N are stable, and thus $E_N := T_N // \mathrm{SL}_2 \times \mathbb{G}_m$ is a coarse moduli space for Weierstrass fibrations with fundamental invariant N. In particular, the natural morphism

$$\mathcal{E}_N := [T_N / \mathrm{SL}_2 \times \mathbb{G}_m] \to E_N$$

from the quotient stack to the GIT quotient is a coarse moduli space morphism.

In Section 4, it will be useful for us to work on a stack $\mathcal{W}_N$ of Weierstrass fibrations with fundamental invariant N, not just the coarse moduli space constructed by Miranda. This stack is not the stack $\mathcal{E}_N$ defined above, but it is closely related as we will now explain. The stack $\mathcal{W}_N$ was recently defined in work of Park–Schmitt [PS21], and we will briefly recall their construction.

Definition 5.2.8. Let S be a scheme. A family of Weierstrass fibrations over S is given by the data

$$\mathcal{X} \xrightarrow{p} \mathcal{P} \xrightarrow{\gamma} S, \mathcal{P} \xrightarrow{s} \mathcal{X}$$

where

1. γ is a smooth, proper morphism locally of finite type, with geometric fibers isomorphic to $\mathbb{P}^1$,

2. p is a proper map with section s,

3. the fibers $(\mathcal{X}_t \to \mathcal{P}_t, \mathcal{P}_t \to \mathcal{X}_t)$ on geometric points $t \in S$ are Weierstrass fibrations.

Park–Schmitt [PS21] define $\mathcal{W}$ to be the moduli stack whose objects over S are families of Weierstrass fibrations over S with morphisms over $T \to S$ given by fiber diagrams. The stack $\mathcal{W}_N$ is the open and closed substack parametrizing Weierstrass fibrations with fundamental invariant N. Finally, we consider the open substacks $\mathcal{W}_{\min,N} \subset \mathcal{W}_N$ of Weierstrass fibrations satisfying the two conditions from Lemma 5.2.5. These stacks parametrize the Weierstrass fibrations with fundamental invariant N that resolve to minimal elliptic surfaces. By [PS21, Theorem 1.2], the stacks $\mathcal{W}_{\min,N}$ are smooth, separated Deligne-Mumford stacks for $N \geq 2$, and by [PS21, Theorem 1.4], E_N is a coarse moduli space for $\mathcal{W}_{\min,N}$

We now have three spaces of interest: $\mathcal{E}_N$, $\mathcal{W}_{\min,N}$ and E_N. We want to compare their Chow rings.

Proposition 5.2.9. *The Chow rings of $\mathcal{E}_N$, $\mathcal{W}_{\min,N}$ and E_N are isomorphic.*

Proof. The space E_N is a coarse moduli space for both stacks $\mathcal{E}_N$ and $\mathcal{W}_{\min,n}$. Therefore, since we are using rational coefficients, all three Chow rings are isomorphic by a result of Vistoli [Vis89b, Proposition 6.1]. □

Remark 5.2.10. The difference between the stacks $\mathcal{W}_{\min,N}$ and $\mathcal{E}_N$ is that $\mathcal{E}_N$ is a μ_2-banded gerbe over $\mathcal{W}_{\min,N}$. The gerbe structure arises from the map $\mathrm{BSL}_2 \to \mathrm{BPGL}_2$.

5.3 Computing the Chow ring

By Proposition 5.2.9, it suffices to compute $A^*(\mathcal{E}_N)$ in order to prove Theorem 5.1.2. Let $\Delta_N \subset V_{4N} \oplus V_{6N}$ denote the complement of T_N. We have the excision exact sequence

$$A_*([\Delta_N/\operatorname{SL}_2 \times \mathbb{G}_m]) \to A^*([V_{4N} \oplus V_{6N}/\operatorname{SL}_2 \times \mathbb{G}_m]) \to A^*(\mathcal{E}_N) \to 0. \tag{5.3.1}$$

We want to study the image of $A_*([\Delta_N/\operatorname{SL}_2 \times \mathbb{G}_m])$ in $A^*([V_{4N} \oplus V_{6N}/\operatorname{SL}_2 \times \mathbb{G}_m])$.

We begin with background information on the stack $[V_{4N} \oplus V_{6N} / \operatorname{SL}_2 \times \mathbb{G}_m]$. The stack BSL_2 is the classifying stack for rank 2 vector bundles with trivial first Chern class. Let $\mathcal{V}$ denote the universal rank 2 vector bundle with trivial first Chern class over BSL_2. Set $c_2 := c_2(\mathcal{V})$. Similarly, the stack $\mathrm{B}\mathbb{G}_\mathrm{m}$ is the classifying stack for line bundles. Let $\mathcal{M}$ denote the universal line bundle over $\mathrm{B}\mathbb{G}_\mathrm{m}$. Set $a_1 := c_1(\mathcal{M})$. By abuse of notation, we will not distinguish between $\mathcal{V}$, $\mathcal{M}$, c_2, and a_1 and their pullbacks to the product $\operatorname{BSL}_2 \times \mathrm{B}\mathbb{G}_\mathrm{m}$ under the natural projection maps. We will interpret the stack $\operatorname{BSL}_2 \times \mathrm{B}\mathbb{G}_\mathrm{m}$ as the stack of line bundles of relative degree N on $\mathbb{P}^1$-bundles as in [Lar21b] as follows. Consider the universal $\mathbb{P}^1$-bundle

$$\gamma : \mathbb{P}(\mathcal{V}) \to \operatorname{BSL}_2 \times \mathrm{B}\mathbb{G}_\mathrm{m}\,.$$

Fix $N \geq 0$ and set $\mathcal{L} := \gamma^* \mathcal{M}(N)$, the universal relative degree N line bundle on $\mathbb{P}(\mathcal{V})$.

Lemma 5.3.1.

1. *The stack* $[V_{4N} \oplus V_{6N} / \operatorname{SL}_2 \times \mathbb{G}_m]$ *is the total space of the vector bundle* $\gamma_*(\mathcal{L}^{\otimes 4} \oplus \mathcal{L}^{\otimes 6})$ *on* $\operatorname{BSL}_2 \times \mathrm{B}\mathbb{G}_\mathrm{m}$.

2. *There is an isomorphism of graded rings*

$$A^*([V_{4N} \oplus V_{6N} / \operatorname{SL}_2 \times \mathbb{G}_m]) \cong \mathbb{Q}[a_1, c_2],$$

with a_1 *in degree* 1 *and* c_2 *in degree* 2.

Proof. Part (1) follows from cohomology and base change. Indeed, the fibers of $\gamma_*(\mathcal{L}^{\otimes 4} \oplus \mathcal{L}^{\otimes 6})$ are canonically identified with $V_{4N} \oplus V_{6N}$, and the higher cohomology vanishes. For part (2), we note that by part (1) and the homotopy property for Chow rings, there is an isomorphism

$$A^*([V_{4N} \oplus V_{6N} / \operatorname{SL}_2 \times \mathbb{G}_m]) \cong A^*(\operatorname{BSL}_2 \times \mathrm{B}\mathbb{G}_\mathrm{m}).$$

A standard calculation in equivariant intersection theory [Tot99, Section 15] shows that

$$A^*(\mathrm{BSL}_2 \times \mathrm{B}\mathbb{G}_\mathrm{m}) \cong \mathbb{Q}[a_1, c_2]$$

as graded rings. □

5.3.1 Computing the ideal of relations

By Lemma 5.3.1, the exact sequence (5.3.1) can be rewritten as

$$A_*([\Delta_N / \mathrm{SL}_2 \times \mathbb{G}_m]) \to \mathbb{Q}[a_1, c_2] \to A^*(\mathcal{E}_N) \to 0. \tag{5.3.2}$$

It follows that $A^*(\mathcal{E}_N)$, and hence $A^*(E_N)$, is a quotient of $\mathbb{Q}[a_1, c_2]$ by the ideal I_N generated by the image of $A_*([\Delta_N / \mathrm{SL}_2 \times \mathbb{G}_m])$.

Lemma 5.2.5 tells us exactly when a pair $(A, B) \in V_{4N} \oplus V_{6N}$ is contained in Δ_N. We write $\Delta_N = \Delta_N^1 \cup \Delta_N^2$, where Δ_N^1 parametrizes the pairs of forms (A, B) such that $4A^3 + 27B^2$ is identically zero (corresponding to Lemma 5.2.5 part (1)), and Δ_N^2 parametrizing pairs of forms (A, B) such that $v_q(A) \geq 4$ or $v_q(B) \geq 6$ for some point $p \in \mathbb{P}^1$ (corresponding to Lemma 5.2.5 part (2)). First, we will determine the relations obtained from excising the pairs $(A, B) \in \Delta_N^2$. To do so, we need to introduce bundles of principal parts. We will follow the treatment in [EH16].

Let $b : Y \to Z$ be a smooth proper morphism. Let $\Delta_{Y/Z} \subset Y \times_Z Y$ be the relative diagonal. With p_1 and p_2 the projection maps, we obtain the following commutative diagram:

$$\begin{tikzcd}
\Delta_{Y/Z} \arrow[dr] \arrow[drr, bend left] \arrow[ddr, bend right] & & \\
& Y \times_Z Y \arrow[r, "p_1"] \arrow[d, "p_2"'] & Y \arrow[d, "b"] \\
& Y \arrow[r, "b"'] & Z.
\end{tikzcd}$$

Definition 5.3.2. Let $\mathcal{F}$ be a vector bundle on Y and let $\mathcal{I}_{\Delta_{Y/Z}}$ denote the ideal sheaf of the diagonal in $Y \times_Z Y$. The bundle of relative m^{th} order principal parts $P_b^m(\mathcal{V})$ is defined as

$$P_b^m(\mathcal{F}) = p_{2*}(p_1^*\mathcal{F} \otimes \mathcal{O}_{Y \times_Z Y}/\mathcal{I}_{\Delta_{Y/Z}}^{m+1}).$$

The following explains all the basic properties of bundles of principal parts that we need.

Proposition 5.3.3 ((Theorem 11.2 in [EH16]))**.** *With notation as above,*

1. *There is an isomorphism* $b^*b_*\mathcal{F} \xrightarrow{\sim} p_{2*}p_1^*\mathcal{F}$.

2. *The quotient map* $p_1^*\mathcal{F} \to p_1^*\mathcal{F} \otimes \mathcal{O}_{Y \times_Z Y}/\mathcal{I}_{\Delta_{Y/Z}}^{m+1}$ *pushes forward to a map*

$$b^*b_*\mathcal{F} \cong p_{2*}p_1^*\mathcal{F} \to P_b^m(\mathcal{F}),$$

   which, fiber by fiber, associates to a global section δ *of* $\mathcal{F}$ *a section* δ' *whose value at* $z \in Z$ *is the restriction of* δ *to an* m^{th} *order neighborhood of* z *in the fiber* $b^{-1}b(z)$.

3. $P_b^0(\mathcal{F}) = \mathcal{F}$. *For* $m > 1$, *the filtration of the fibers* $P_b^m(\mathcal{F})_y$ *by order of vanishing at* y *gives a filtration of* $P_b^m(\mathcal{F})$ *by subbundles that are kernels of the natural surjections* $P_b^m(\mathcal{F}) \to P_b^k(\mathcal{F})$ *for* $k < m$. *The graded pieces of the filtration are identified by the exact sequences*

$$0 \to \mathcal{F} \otimes \operatorname{Sym}^m(\Omega_{Y/Z}) \to P_b^m(\mathcal{F}) \to P_b^{m-1}(\mathcal{F}) \to 0.$$

By (2) of Proposition 5.3.3, there is a morphism

$$\psi : \gamma^*\gamma_*(\mathcal{L}^{\otimes 4} \oplus \mathcal{L}^{\otimes 6}) \to P_\gamma^3(\mathcal{L}^{\otimes 4}) \oplus P_\gamma^5(\mathcal{L}^{\otimes 6})$$

which, along points in the $\mathbb{P}^1$ fibers, sends A (respectively, B) to a third (respectively, fifth) order neighborhood. The kernel of this map therefore parametrizes the triples (A, B, q) such that $v_q(A) \geq 4$ and $v_q(B) \geq 6$. Looking fiber-by-fiber, one sees that the map ψ is surjective. Therefore, the kernel K of ψ is a vector bundle. We obtain the following commutative diagram where ϕ, ϕ' and ϕ'' are vector bundle morphisms.

$$\begin{tikzcd} K \arrow[r, hook, "i"] \arrow[dr, "\phi''"'] & \gamma^*\gamma_*(\mathcal{L}^{\otimes 4} \oplus \mathcal{L}^{\otimes 6}) \arrow[r, "\gamma'"] \arrow[d, "\phi'"'] & \gamma_*(\mathcal{L}^{\otimes 4} \oplus \mathcal{L}^{\otimes 6}) \arrow[d, "\phi"] \\ & \mathbb{P}(\mathcal{V}) \arrow[r, "\gamma"] & \mathrm{BSL}_2 \times \mathrm{B}\mathbb{G}_{\mathrm{m}} . \end{tikzcd} \tag{5.3.3}$$

By construction, K maps properly and surjectively onto $[\Delta^2_N / \operatorname{SL}_2 \times \mathbb{G}_m]$ under the identification of $\gamma_*(\mathcal{L}^{\otimes 4} \oplus \mathcal{L}^{\otimes 6})$ with $[V_{4N} \oplus V_{6N} / \operatorname{SL}_2 \times \mathbb{G}_m]$ from Lemma 5.3.1. Consequently, the images of the push forward maps

$$\gamma'_* i_* : A_*(K) \to A^*(\gamma_*(\mathcal{L}^{\otimes 4} \oplus \mathcal{L}^{\otimes 6})) = A^*([V_{4N} \oplus V_{6N} / \operatorname{SL}_2 \times \mathbb{G}_m])$$

and

$$A_*([\Delta^2_N / \operatorname{SL}_2 \times \mathbb{G}_m]) \to A^*([V_{4N} \oplus V_{6N} / \operatorname{SL}_2 \times \mathbb{G}_m])$$

are the same.

Proposition 5.3.4. *Let z denote the hyperplane class of $\mathbb{P}(\mathcal{V})$. The image of the push forward map $\gamma'_* i_* : A^*(K) \to A^*(\gamma_*(\mathcal{L}^{\otimes 4} \oplus \mathcal{L}^{\otimes 6}))$ is the ideal generated by the two classes*

1. *$\phi^*\gamma_*(c_{top}(P^3_\gamma(\mathcal{L}^{\otimes 4}) \oplus P^5_\gamma(\mathcal{L}^{\otimes 6})))$, and*

2. *$\phi^*\gamma_*(c_{top}(P^3_\gamma(\mathcal{L}^{\otimes 4}) \oplus P^5_\gamma(\mathcal{L}^{\otimes 6})) \cdot z)$.*

Proof. Let $\alpha \in A^*(K)$. Then because K is a vector bundle over $\mathbb{P}(\mathcal{V})$, we see that $\alpha = \phi''^*(\beta)$ for some class $\beta \in A^*(\mathbb{P}(\mathcal{V}))$, so we have

$$\alpha = \phi''^*(\beta) = i^*\phi'^*(\beta).$$

Pushing forward, we obtain

$$\gamma'_* i_* \alpha = \gamma'_* i_* i^* \phi'^*(\beta) = \gamma'_*([K] \cdot \phi'^* \beta).$$

Because K is the kernel of the vector bundle morphism

$$\psi : \gamma^* \gamma_* (\mathcal{L}^{\otimes 4} \oplus \mathcal{L}^{\otimes 6}) \to P^3_\gamma(\mathcal{L}^{\otimes 4}) \oplus P^5_\gamma(\mathcal{L}^{\otimes 6}),$$

the fundamental class $[K]$ is given by $\phi'^*(c_{\text{top}}(P^3_\gamma(\mathcal{L}^{\otimes 4}) \oplus P^5_\gamma(\mathcal{L}^{\otimes 6})))$. Because the square in the commutative diagram (5.3.3) is Cartesian, $\gamma'_* \phi'^* = \phi^* \gamma_*$, so

$$\gamma'_* i_* \alpha = \phi^* \gamma_* (c_{\text{top}}(P^3_\gamma(\mathcal{L}^{\otimes 4}) \oplus P^5_\gamma(\mathcal{L}^{\otimes 6})) \cdot \beta).$$

Because $\mathbb{P}(\mathcal{V})$ is a projective bundle, β can be written as

$$\beta = \gamma^* \beta_1 + \gamma^* \beta_2 z,$$

where β_1 and β_2 are classes in $A^*(\mathrm{BSL}_2 \times \mathrm{B}\mathbb{G}_{\mathrm{m}})$. The statement of the proposition follows. $\square$

Lemma 5.3.5. *The codimension of Δ^1_N in $V_{4N} \oplus V_{6N}$ is $8N+1$.*

Proof. Let t be an affine coordinate on $\mathbb{P}^1$. Then we can factor $A(t)$ and $B(t)$ into linear factors as

$$A(t) = a \prod_{i=1}^{4N} (t - c_i) \textit{ and } B(t) = b \prod_{i=1}^{6N} (t - d_i).$$

Because $4A^3 + 27B^2$ is identically zero, we have the equation

$$4a^3 \prod_{i=1}^{4N} (t - c_i)^3 = -27b^2 \prod_{i=1}^{6N} (t - d_i)^2.$$

By comparing the orders of vanishing of each side, we see that $A(t) = aG(t)^2$ and $B(t) = bG(t)^3$, where G is a polynomial of degree $2N$ and $4a^3 + 27b^2 = 0$. It follows that the codimension of Δ_N^1 is given by

$$\dim(V_{4N} \oplus V_{6N}) - \dim V_{2N} = 10N + 2 - 2N - 1 = 8N + 1.$$

$\square$

We can now complete the proof of Theorem 5.1.2.

Proof of Theorem 5.1.2. By a calculation in Macaulay2 [GS], the graded ring $\mathbb{Q}[a_1, c_2]/I_N$ vanishes in degree 17 and higher, where I_N is the ideal generated by the relations from Proposition 5.3.4. We have the excision exact sequence

$$A_*([\Delta_N^1/\operatorname{SL}_2 \times \mathbb{G}_m]) \to \mathbb{Q}[a_1, c_2]/I_N \to A^*(\mathcal{E}_N) \to 0.$$

By Lemma 5.3.5, the image of

$$A_*([\Delta_N^1/\operatorname{SL}_2 \times \mathbb{G}_m]) \to \mathbb{Q}[a_1, c_2]/I_N$$

lies in codimension 17 or higher, so it is identically zero. Therefore,

$$\mathbb{Q}[a_1, c_2]/I_N \cong A^*(\mathcal{E}_N).$$

This completes the proof of Theorem 5.1.2 part (1). Parts (2) and (3) are consequences of part (1) together with a computation in Macaulay2 [GS] that computes the Hilbert Series of the ring $\mathbb{Q}[a_1, c_2]/I_N$ and verifies that the intersection pairing is perfect. $\square$

Proof of Corollary 5.1.3. Miranda's construction of E_N by geometric invariant theory [Mir81] shows that E_N is a quasi-projective variety. It thus admits an ample line bundle

L. If S is a complete subvariety of dimension d, then, because L is ample,

$$c_1(L)^d \cdot S > 0.$$

Hence, $c_1(L)^d$ is numerically nonzero. By Theorem 5.1.2, it follows that $d \leq 16$. $\square$

5.4 The Tautological Ring

5.4.1 Stacks of lattice polarized K3 surfaces

Let $\Lambda \subset U^{\oplus 3} \oplus E_8(-1)^{\oplus 2}$ be a fixed rank r primitive sublattice with signature $(1, r-1)$, and let $v_1, \ldots, v_r$ be an integral basis of Λ. A Λ-polarization on a K3 surface X is a primitive embedding

$$j : \Lambda \hookrightarrow \mathrm{Pic}(X)$$

such that

1. The lattices $H^2(X, \mathbb{Z})$ and $U^{\oplus 3} \oplus E_8(-1)^{\oplus 2}$ are isomorphic via an isometry restricting to the identity on Λ, where we view Λ as sitting inside $H^2(X, \mathbb{Z})$ via $\Lambda \hookrightarrow \mathrm{Pic}(X) \hookrightarrow H^2(X, \mathbb{Z})$.

2. The image of j contains the class of a quasi-polarization.

Beauville [Bea04] constructed moduli stacks $\mathcal{F}_\Lambda$ of Λ-polarized K3 surfaces, and showed that they are smooth Deligne–Mumford stacks of dimension $20 - r$. Using the surjectivity of the period map, one can construct coarse moduli spaces F_Λ for $\mathcal{F}_\Lambda$ [Dol96].

We think of the stacks $\mathcal{F}_\Lambda$ as parametrizing families of K3 surfaces

$$\pi : X \to S$$

together with r line bundles $H_1, \ldots, H_r$ on X corresponding to the basis $v_1, \ldots, v_r$ of Λ, well-defined up to pullbacks from $\mathrm{Pic}(S)$. Technically, these bundles exist only étale locally, as they are defined as sections of the sheaf $\operatorname{Pic}_{X/S}$, which is the étale sheafification of the presheaf on the category of schemes over S

$$T \mapsto \operatorname{Pic}(X_T)/\operatorname{Pic}(T).$$

We will generally suppress this detail, but we will remark when it is important. There are forgetful morphisms

$$\mathcal{F}_{\Lambda'} \hookrightarrow \mathcal{F}_{\Lambda}$$

for any lattice $\Lambda \subset \Lambda'$. We call the subvarieties $\mathcal{F}_{\Lambda'}$ *Noether-Lefschetz loci* of $\mathcal{F}_{\Lambda}$.

5.4.2 The tautological ring of $\mathcal{F}_\Lambda$

The stack $\mathcal{F}_\Lambda$ comes equipped with a universal K3 surface

$$\pi_\Lambda : \mathcal{X}_\Lambda \to \mathcal{F}_\Lambda.$$

and universal bundles $\mathcal{H}_1, \dots \mathcal{H}_r$, well-defined up to pullbacks from $\mathcal{F}_\Lambda$. Let $\mathcal{T}_{\pi_\Lambda}$ denote the relative tangent bundle. Following [MOP17], we define the κ-classes

$$\kappa^{\Lambda}_{a_1,\dots,a_r,b} := \pi_{\Lambda *}\left(c_1(\mathcal{H}_1)^{a_1} \cdots c_1(\mathcal{H}_r)^{a_r} \cdot c_2(\mathcal{T}_{\pi_\Lambda})^b\right).$$

Definition 5.4.1. The tautological ring $R^*(\mathcal{F}_\Lambda)$ is the subring of $A^*(\mathcal{F}_\Lambda)$ generated by pushforwards from the Noether–Lefschetz loci of all κ-classes.

By [Bor99] or [FR20], the Hodge class $\lambda := c_1(\pi_{\Lambda *}\omega_{\pi_\Lambda})$ lies in the tautological ring $R^*(\mathcal{F}_\Lambda)$ for all Λ, as it is supported on Noether–Lefschetz divisors.

5.4.3 Moduli of elliptic K3 surfaces and Weierstrass fibrations

Let $p : X \to \mathbb{P}^1$ be a minimal elliptic surface over $\mathbb{P}^1$ with fundamental invariant 2. Then X is a K3 surface, and the class of the fiber f and section σ form a primitively embedded lattice $U \subset \mathrm{Pic}(X)$ equivalent to a hyperbolic lattice, whose image contains a quasi-polarization $\sigma + 2f$. Conversely, given a K3 surface X, a primitive embedding of a hyperbolic lattice $U \hookrightarrow \mathrm{Pic}(X)$ whose image contains a quasi-polarization allows one to define a morphism $p : X \to \mathbb{P}^1$ with section $s : \mathbb{P}^1 \to X$ with fundamental invariant 2 [CD07, Theorem 2.3]. Because of this, we call the stack $\mathcal{F}_U$ the stack parametrizing elliptic K3 surfaces with section. By [OO21, Theorem 7.9], the coarse moduli space F_U is isomorphic to E_2. By the discussion in subsection 5.4.1, $\mathcal{F}_U$ comes equipped with a universal K3 surface and two universal line bundles

$$\pi_U : \mathcal{X}_U \to \mathcal{F}_U, \quad \mathcal{O}(f) \to \mathcal{X}_U, \quad \mathcal{O}(\sigma) \to \mathcal{X}_U.$$

The intersection matrix of $\mathcal{O}(\sigma)$ and $\mathcal{O}(f)$ is

$$\begin{bmatrix} \mathcal{O}(\sigma)^2 & \mathcal{O}(\sigma)\cdot\mathcal{O}(f) \\ \mathcal{O}(\sigma)\cdot\mathcal{O}(f) & \mathcal{O}(f)^2 \end{bmatrix} = \begin{bmatrix} -2 & 1 \\ 1 & 0 \end{bmatrix},$$

which can be obtained by a change of basis from the usual intersection matrix for a hyperbolic lattice U:

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

We prefer to take $\mathcal{O}(f)$ and $\mathcal{O}(\sigma)$ as our basis because of their geometric meaning. Recall that the stack $\mathcal{W}_{\min}$ parametrizes families of Weierstrass fibrations resolving to minimal elliptic surfaces. We will construct a morphism

$$G : \mathcal{F}_U \to \mathcal{W}_{\min},$$

which is a relative version of the morphism sending an elliptic K3 surface to its associated Weierstrass fibration. Let $\pi : X \to S$ be a family of U-polarized K3 surfaces, equipped with bundles $\mathcal{O}(f)$ and $\mathcal{O}(\sigma)$ on X, up to an étale cover of S. The surjection

$$\pi^*\pi_*\mathcal{O}(f) \to \mathcal{O}(f)$$

defines a morphism

$$p : X \to \mathbb{P}(\pi_*\mathcal{O}(f)^\vee)$$

over S. The relative effective Cartier divisor associated to $\mathcal{O}(\sigma)$ allows us to define a section s of p. The surjection

$$p^*p_*\mathcal{O}(3\sigma) \to \mathcal{O}(3\sigma)$$

defines a morphism $i : X \to \mathbb{P}(p_*\mathcal{O}(3\sigma)^\vee)$. Let Y denote the image of X under i. Then Y is a family of Weierstrass fibrations over S. This construction defines the morphism

$$G : \mathcal{F}_U \to \mathcal{W}_{\min}.$$

Remark 5.4.2. We note that in constructing Y, we chose line bundles $\mathcal{O}(f)$ and $\mathcal{O}(\sigma)$. Technically, we could only do so étale locally. The projective bundle $\mathbb{P}(\pi_*\mathcal{O}(f)^\vee) \to S$ will only descend to a smooth proper morphism, locally of finite type, with geometric fibers isomorphic to $\mathbb{P}^1$: it will not necessarily be the projectivization of a vector bundle on S. Second, even once we pass to an étale cover, $\mathcal{O}(f)$ and $\mathcal{O}(\sigma)$ are only defined up to pullbacks from $\mathrm{Pic}(S)$. If we made different choices for $\mathcal{O}(f)$ and $\mathcal{O}(\sigma)$ the resulting Weierstrass fibration would be canonically isomorphic to the original one because for any vector bundle $\mathcal{E}$ and line bundle $\mathcal{L}$, $\mathbb{P}(\mathcal{E} \otimes \mathcal{L})$ is canonically isomorphic to $\mathbb{P}(\mathcal{E})$.

Consider the following Cartesian diagram, which defines the stack $\widetilde{\mathcal{F}}_U$.

$$\begin{tikzcd} \widetilde{\mathcal{F}}_U \arrow[r, "G'"] \arrow[d] & \mathcal{E}_2 \arrow[d] \\ \mathcal{F}_U \arrow[r, "G"] & \mathcal{W}_{\min} \end{tikzcd}$$

The vertical morphisms are μ_2-banded gerbes. In fact, we can explicitly describe the functor of points for $\widetilde{\mathcal{F}}_U$. A morphism from a scheme S to $\widetilde{\mathcal{F}}_U$ is a family

$$(\pi : X \to S, \mathcal{O}(f), \mathcal{O}(\sigma), \mathcal{N})$$

where $(\pi : X \to S, \mathcal{O}(f), \mathcal{O}(\sigma))$ is a family of U-polarized K3 surfaces and $\mathcal{N}$ is a line bundle on S such that

$$\mathcal{N}^{\otimes 2} \cong \det \pi_* \mathcal{O}(f).$$

Recall that $\mathcal{E}_2$ has a universal rank 2 vector bundle with trivial first Chern class $\mathcal{V}$ and a universal line bundle $\mathcal{M}$. By construction of the map G and its base change G', we have that

$$G'^* \mathcal{V} = \pi_* \mathcal{O}(f)^\vee \otimes \mathcal{N},$$

where $\mathcal{N}$ is the universal square root of $\det \pi_* \mathcal{O}(f)$. We will abuse notation and denote the universal K3 surface on $\mathcal{F}_U$ and $\widetilde{\mathcal{F}}_U$ both by π.

Lemma 5.4.3. *The class $c_2(\pi_* \mathcal{O}(f)^\vee \otimes \mathcal{N})$ on $\widetilde{\mathcal{F}}_U$ is the pullback of a tautological class on $\mathcal{F}_U$.*

Proof. Note that

$$\begin{aligned}
c_2(\pi_*\mathcal{O}(f)^\vee \otimes \mathcal{N}) &= c_1(\mathcal{N})^2 + c_1(\pi_*\mathcal{O}(f)^\vee)c_1(\mathcal{N}) + c_2(\pi_*\mathcal{O}(f)^\vee) \\
&= \frac{1}{4}c_1(\det \pi_*\mathcal{O}(f))^2 - \frac{1}{2}c_1(\pi_*\mathcal{O}(f))c_1(\det \pi_*\mathcal{O}(f)) + c_2(\pi_*\mathcal{O}(f)) \\
&= -\frac{1}{4}c_1(\pi_*\mathcal{O}(f))^2 + c_2(\pi_*\mathcal{O}(f)).
\end{aligned}$$

It thus suffices to show that the Chern classes of $\pi_*\mathcal{O}(f)$ are tautological. By Grothendieck–Riemann–Roch, we have

$$\mathrm{ch}(\pi_!\mathcal{O}(f)) = \pi_*(\mathrm{ch}(\mathcal{O}(f)) \cdot \mathrm{td}(T_\pi)).$$

By definition, the classes on the right hand side are tautological. We note that

$$\pi_!\mathcal{O}(f) = \pi_*\mathcal{O}(f)$$

because π is a relative K3 surface. By comparing degree 1 parts of both sides, we see that $c_1(\pi_*\mathcal{O}(f))$ is tautological. By comparing degree 2 parts, we see that $c_2(\pi_*\mathcal{O}(f))$ is tautological. □

Proof of Theorem 5.1.4. Each of the stacks $\mathcal{E}_2$, $\mathcal{W}_{\min}$, $\mathcal{F}_U$, and $\widetilde{\mathcal{F}}_U$ has the same coarse moduli space E_2. They thus all have isomorphic Chow rings, and proper push forward $A_*(Z) \to A_*(E_2)$ is an isomorphism of Chow groups, where Z is any of the four stacks above [Vis89b, Proposition 6.1]. By Theorem 5.1.2, $A^1(E_2)$ is generated by the push forward of a_1. By [Pet19, Theorem 2.1] or the proof of [vdGK05, Corollary 4.2], the tautological class λ is nonvanishing on $\mathcal{F}_U$. It follows that $A^1(\mathcal{F}_U)$ is generated by λ, so $A^1(\mathcal{F}_U) = R^1(\mathcal{F}_U)$. By Theorem 5.1.2, $A^2(E_2)$ is generated by the push forwards of a_1^2 and c_2. By Lemma 5.4.3, the class c_2 pulls back to a class in $A^2(\widetilde{\mathcal{F}}_U)$ that is the pullback of a tautological class from $A^2(\mathcal{F}_U)$. It follows that $A^2(\mathcal{F}_U) = R^2(\mathcal{F}_U)$, as the images of a_1^2

and c_2 in $A^2(E_2)$ can both be obtained by pushing forward tautological classes from $\mathcal{F}_U$. Therefore, $A^*(\mathcal{F}_U) = R^*(\mathcal{F}_U)$. The fact that $A^*(\mathcal{F}_U) = R^*(\mathcal{F}_U)$ is Gorenstein with socle in codimension 16 follows from Theorem 5.1.2. $\square$

5.4.4 Codimension one classes

By Theorems 5.1.2 and 5.1.4, $A^1(\mathcal{F}_U)$ is of rank one and the Hodge class λ is a generator. It is natural to ask how to represent κ-classes explicitly in terms of the Hodge class λ.

Proposition 5.4.4. *The following four linear combinations of κ-classes are independent of the choice of universal line bundles. Moreover, they are all multiples of the Hodge class λ.*

$$\kappa_{3,0,0} + \frac{1}{4}\kappa_{1,0,1} = \frac{7}{2}\lambda, \quad 3\kappa_{2,1,0} - \frac{1}{4}\kappa_{1,0,1} + \frac{1}{4}\kappa_{0,1,1} = \frac{1}{2}\lambda,$$

$$3\kappa_{1,2,0} - \frac{1}{4}\kappa_{0,1,1} = -3\lambda, \quad \kappa_{0,3,0} = 0.$$

where $\kappa_{i,j,k} := \pi_*\left(c_1(\mathcal{O}(\sigma))^i \cdot c_1(\mathcal{O}(f))^j \cdot c_2(\mathcal{T}_\pi)^k\right)$.

Proof. A direct computation shows the above four κ combinations are invariant under $f \mapsto f + \pi^*(l)$ and $\sigma \mapsto \sigma + \pi^*(l')$ for any $l, l' \in A^1(\mathcal{F}_U)$.

By Theorem 5.1.2, we know $A^1(\mathcal{F}_U)$ is of rank one, so it is sufficient to check the identities by computing their intersection numbers with a suitable test curve:

$$\iota : C \to \mathcal{F}_U.$$

To construct the curve, we use the resolved version of the STU model in [KMPS10]. The STU model is a smooth Calabi-Yau 3-fold, endowed with a map:

$$\pi^{STU} : X^{STU} \to \mathbb{P}^1.$$

It arises as an anti-canonical section of a toric 4-fold Y. The fan datum for Y can be found in [KMPS10, Section 1.3]. We use their notation. There are 10 primitive rays $\{\rho_i; 1 \leq i \leq 10\}$, and the corresponding divisors are denoted as $D_i \in \mathrm{Pic}(Y)$. The anti-canonical class is:

$$-K_Y = \sum_{i=1}^{10} D_i.$$

The general fiber of π^{STU} is a smooth elliptic K3 surface, but there are 528 singular fibers [KMPS10, Proposition 1], each of which has exactly one ordinary double point singularity. Let $\epsilon : C \to \mathbb{P}^1$ be a double cover branched along the 528 points corresponding to the singular fibers. The pullback of X^{STU} by ϵ has double point singularities, and by resolving them we obtain the resolved STU model:

$$\widetilde{\pi}^{STU} : \widetilde{X}^{STU} \to C.$$

All fibers of $\widetilde{\pi}^{STU}$ are smooth elliptic K3 surfaces. Moreover the toric divisors $D_5, D_3 \in \mathrm{Pic}(Y)$ restrict to the universal section and fiber for $\widetilde{\pi}^{STU}$. The family $\widetilde{\pi}^{STU}$ defines a curve in the moduli space $\mathcal{F}_U$:

$$\iota : C \to \mathcal{F}_U.$$

The intersection number $\iota^*(\lambda)$ is computed in [KMPS10, Proposition 2]:

$$\iota^*(\lambda) = 4E_4(q)E_6(q)[0] = 4,$$

where E_4 and E_6 are Eisenstein series, and we take the coefficient of q^0.

For the κ-classes, it suffices to perform the computation over the non-resolved STU model. Since the tautological classes we consider are all invariant, we may assume the universal line bundles on $\mathcal{F}_U$ pull back to the toric divisors D_5, D_3. For $\kappa_{3,0,0}$, we have:

$$\iota^*(\kappa_{3,0,0}) = 2 \cdot \pi_*^{STU}\left(D_5^3 \cdot \sum_{i=1}^{10} D_i\right),$$

where the factor of 2 comes from the double cover ϵ. Using toric geometry, all monomials of the form $D_i \cdot D_j \cdot D_k \cdot D_l$ can be explicitly determined. We obtain:

$$\iota^*(\kappa_{3,0,0}) = 16.$$

Other intersection numbers can be computed analogously. We record the final answers:

$$\iota^*(\kappa_{3,0,0}) = 16 \quad \iota^*(\kappa_{1,0,1}) = -8 \quad \iota^*(\kappa_{2,1,0}) = -4$$

$$\iota^*(\kappa_{0,1,1}) = 48 \quad \iota^*(\kappa_{1,2,0}) = 0 \quad \iota^*(\kappa_{0,3,0}) = 0.$$

The four identities in the proposition then follow immediately. $\square$

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