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Vanishing of Certain Axially-Symmetric Periodic D-solutions to the Stationary Navier-Stokes Equations

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ABSTRACT OF THE DISSERTATION

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by

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Dr. Qi S. Zhang, Chairperson

One open question in the study of the steady incompressible three-dimensional Navier-Stokes equations is if the only solution with finite Dirichlet integral and vanishing condition at infinity is the trivial solution. Several partial results have been proven by requiring certain integral or decay conditions on the solution. We will explore a certain class of solutions, called axially-symmetric D-solutions, and discuss some results about these solutions. In this thesis, we will prove that certain axially-symmetric periodic D-solutions are identicially zero.

Contents

Chapter 1

Introduction

The **Navier-Stokes equations** (NSE) are a set of equations that describe the movement of viscous fluids. The equations are important to real world applications because they may be used to model a variety of phenomena, including blood flow, water flow in a pipe, weather and more by considering the NSE under suitable boundary and initial conditions and by coupling them with other Partial Differential Equations. Due to the plethora of real-world applications, it is important to further study these equations. There is also significant interest in these equations from a purely mathematical viewpoint. Topics of interest include existence and uniqueness theorems for the solutions, regularity (smoothness) of solutions, growth or decay rate of the solutions, and more. We focus on studying globally bounded solutions.

In Cartesian coordinates, the **time-dependent, incompressible Navier-Stokes equations** are

$$\begin{cases} \partial_t u - \Delta u + (u \cdot \nabla)u + \nabla p = f, \\ \\ \operatorname{div}(u) = 0, \end{cases} \tag{1.0.1}$$

where u is a vector-valued function, p is a scaled-valued function, f is a vector-valued function. In applications, u is the unknown velocity of the fluid, p is the unknown pressure of the fluid, and f is the given external force applied to the fluid. The first equation is the momentum equation while the second equation is the incompressible condition. In this dissertation, we are interested in the homogeneous problem, that is when $f \equiv 0$.

We say that a function u in $L^\infty((-\infty, 0) \times \mathbb{R}^n)$ is a **bounded, ancient weak solution** of (1.0.1) if for all smooth compactly supported functions ϕ we have

$$\int_{-\infty}^{0} \int_{\mathbb{R}^n} u \cdot \nabla \phi dx dt = 0, \tag{1.0.2}$$

and for all smooth compactly supported divergence-free vector fields φ we have that

$$\int_{-\infty}^{0} \int_{\mathbb{R}^n} u \cdot (\partial_t \varphi + \Delta \varphi) dx dt = -\int_{-\infty}^{0} \int_{\mathbb{R}^n} (u \otimes u \colon \nabla \varphi)\, dx dt. \tag{1.0.3}$$

where $a \otimes b = (a^i b^j)$ and $A \colon B = A_{ij} B_{ij}$. We remark that to obtain the weak formulation, we simply multiple the equations by test functions, integrate, then do integration by parts to move all the derivatives to the test function.

We are interested in studying the problem in $\mathbb{R}^3$. Are there bounded, ancient weak solutions to the above problem? If we take $b = b(t) = (e^t, e^t, e^t)$ and define $p(t, x) = -\partial_t b \cdot x = -e^t x_1 - e^t x_2 - e^t x_3$, then b is an $L^\infty(-\infty, 0)$ function and $u(t, x) = b(t)$ is a bounded, ancient weak solution.

Indeed we have we have that

$$\begin{aligned}\partial_t u - \Delta u + (u\cdot\nabla)u + \nabla p &= \partial_t b + \nabla p \\ &= b - b \\ &= 0\end{aligned} \tag{1.0.4}$$

which implies (1.0.3). We also have

$$\begin{aligned}\int_{-\infty}^{0}\int_{\mathbb{R}^3} u\cdot\nabla\phi dxdt &= \int_{-\infty}^{0}\int_{\mathbb{R}^3} b\cdot\nabla\phi dxdt \\ &= \int_{-\infty}^{0} b\cdot\left(\int_{\mathbb{R}^3}\nabla\phi dx\right)dt \\ &= \int_{-\infty}^{0} b\cdot(0,0,0)dt \\ &= 0\end{aligned} \tag{1.0.5}$$

due to the compact support of the test function ϕ. Hence $u = b(t)$ is a bounded, ancient weak solution.

One natural question to ask is whether any ancient bounded weak solution is of this form. This is an open-ended question in full three dimensional space, although much work has been done.

For example, Koch, Nadirashvili, Seregin, and Sverak proved in [KNSS] that if $u = (u^r, u^\theta, u^z)$ is a bounded axi-symmetric weak solution of (1.0.1) with no swirl, meaning $u^\theta = 0$, then $u \equiv 0$. Another result in [KNSS] is that if u is a bounded axi-symmetric weak solution of (1.0.1) in $(-\infty, 0)\times\mathbb{R}^3$ and there is a positive constant C such that

$$|u(t,x)| \leq \frac{C}{\sqrt{x_1^2+x_2^2}}, \tag{1.0.6}$$

then $u \equiv 0$. Similarly, in [LZ2], Lei and Zhang proved that if $r|u^\theta|$ is bounded and the stream function is a BMO function then $u \equiv 0$. Although the axi-symmetric condition simplifies the problem because there are only two partial derivatives in space to consider rather than three partial derivatives, the problem is still open in the three-dimensional axially symmetric case. It is still not known whether all axially symmetric solutions satisfy the above bounds or if all solutions satisfy the swirl-free condition.

Another direction others have taken with this problem is considering time-independent or **stationary, incompressible Navier-Stokes equations**. This means that the solution does not dependent on time so there is no partial derivative in time. Lerary studied the equations and constructed solutions with an extra property in [Le]. The solutions Lerary constructed satisfy:

$$\begin{cases} -\Delta u + (u \cdot \nabla) u + \nabla p = 0, \\ \operatorname{div}(u) = 0, \\ \lim\limits_{|x| \to \infty} u = 0, \\ \displaystyle\int_{\mathbb{R}^3} |\nabla u|^2 dx < +\infty. \end{cases} \tag{1.0.7}$$

Solutions that satisfy the above conditions are called ***D*-solutions.** The D-solutions are called so because the **Dirichlet integral**, that is the L^2 norm of gradient of the the velocity, is finite. Just like in the time-dependent case, we wish to classify solutions to (1.0.7). Certainly the pair $u \equiv 0$ and $p = C$, where C is any constant, will satisfy (1.0.7). Just like in the time-dependent case we ask the question: is the only smooth solution to (1.0.7) the trivial solution $u \equiv 0$? This problem, even if we only assume that the solution is axially-symmetric, remains open in in $\mathbb{R}^3$.

We will note briefly some partial results to the problem in $\mathbb{R}^3$; such results are known as **Liouville Theorems**. If one assumes that u belongs to certain L^p spaces or specific partial derivatives of u belong in L^p spaces, then one can conclude that $u \equiv 0$. For example, Theorem X.9.5 in [Ga] states that if u is a homogeneous D-solution in the domain $D = \mathbb{R}^3$ and u is an $L^{9/2}(\mathbb{R}^3)$ function, then $u = 0$. This result was improved by a log factor in Chae and Wolf [CW]. In the paper [Ch], Chae proved that if Δu isa $L^{6/5}(\mathbb{R}^3)$ function, then $u = 0$.

The problem can also be tackled by assuming that u satisfies decay estimates. One can take a stationary solution and regard it as an ancient solution and use the result in [KNSS] to conclude that $u = 0$ if (1.0.6) holds. In [KTW], Kozono, Terasawa, and Wakasugi showed that if the vorticity $w = \text{curl}(u)$ decays faster than $C/|x|^{5/3}$ at infinity, then homogeneous D-solutions in $\mathbb{R}^3$ is 0. In [ZH], it is shown that if u decays like $C/|x|^{2/3-\varepsilon}$ for any $\varepsilon > 0$ small, then u is 0.

In this dissertation we will prove a Liouville Theorem in not the full space $\mathbb{R}^3$, but in $\mathbb{R}^2 \times [-\pi, \pi]$.

Theorem 1 *Let u be a smooth axially symmetric solution to the problem*

$$\begin{cases} -\Delta u + (u \cdot \nabla)u + \nabla p = 0, \quad in \quad \mathbb{R}^2 \times [-\pi, \pi], \\ \text{div}(u) = 0, \\ u(x_1, x_2, z) = u(x_1, x_2, z + 2\pi), \\ \lim_{|x| \to \infty} u = 0, \end{cases} \tag{1.0.8}$$

such that the Dirichlet integral satisfies the condition: for $0 \le \alpha < 1/5$, we have that for all $R \ge 1$,

$$\int_{-\pi}^{\pi} \int_{|x'| \le R} |\nabla u(x)|^2 dx < R^\alpha < \infty. \tag{1.0.9}$$

Suppose also $\int_{-\pi}^{\pi} u^\theta(\cdot, z) dz = \int_{-\pi}^{\pi} u^z(\cdot, z) dz = 0$. Then $u = 0$.

We note that this is a generalization of the result in [CPZ] because the L^2 norm on the whole space $\mathbb{R}^2 \times [-\pi, \pi]$ of the gradient of u is finite, while here we allow for some growth.

To prove u is zero on the entire domain, we use the Green's function G on $\mathbb{R}^2 \times [-\pi, \pi]$ for functions whose integral on $[-\pi, \pi]$ is zero. By making use of the divergence free condition, we can show that u^r satisfies this requirement. However, we must impose this condition on u^θ and u^z. Unlike the Green's function on the whole space $\mathbb{R}^3$, the Green's function on $\mathbb{R}^2 \times [-\pi, \pi]$ has exponential decay near infinity. This advantage in decay is what allows us to show the u is identically zero.

We first use the integral representations of u^r and u^z in terms of G and w^θ. This gives us an estimate on u^r, u^z in terms of w^θ. Then by using the Brezis-Gallouet inequality and scaling technique, we can obtain a bound on w^r and w^z by the L^∞ norm of u^r and u^z. Then by a different calculation, one can bound u^θ by the L^∞ norm of w^r. Finally, we obtain an estimate on w^θ by using the decay of u. This improves the decay of w^θ which allows us to improve the decay of u^r and u^z. We repeat this process a finite number of times to obtain that the decay of u and w is $r^{-1+2\alpha+\delta}$, where $\delta > 0$ is small.

To obtain the complete decay of r^{-1}, we differentiation (1.0.7) to obtain equations for ∇w. By using the estimates obtained before, we show that ∇w decays roughly like $r^{-3/2+(5/2)\alpha+\delta}$. Using the representations for u^r and u^z, making a calculation for u^θ, and choosing δ small enough, we can show the decay rate of u is r^{-1}. By the results in [CSTY] and [KNSS], we can conclude u is identically zero. We will prove this result in this dissertation.

In Chapter 2, we will discuss and give a derivation of the axially-symmetric Navier-

Stokes equations. We will also derive the vorticity equations, which are a key part of the proof. In Chapter 3, we will derive the Green's function in $\mathbb{R}^2 \times [-\pi, \pi]$ and prove key estimates for the Green's function. Compared to the Green's function in $\mathbb{R}^3$, this Green's function exhibits exponential decay that is crucial to the proof. In Chapter 4, we will prove the main theorem as we outlined above. Finally in Chapter 5 we will discuss future work.

Chapter 2

Axially-Symmetric Navier-Stokes Equation

2.1 Cartesian Coordinates

A vector field $v\colon \mathbb{R}^3 \to \mathbb{R}^3$ can be rewritten in Cartesian coordinates as

$$v(x) = v^1(x)e_1 + v^2(x)e_2 + v^3(x)e_3 \tag{2.1.1}$$

where $x = (x_1, x_2, x_3)$ is a point in $\mathbb{R}^3$, e_1, e_2, e_3 are the standard basis in $\mathbb{R}^3$, and v^1, v^2, v^3 are the component functions. Given a scale-valued function f and vector field $F = (F^1, F^2, F^3)$, we have that the gradient, divergence, curl and Laplacian in Cartesian coordinates are given by

$$\nabla f = (\partial_{x_1} f)e_1 + (\partial_{x_2} f)e_2 + (\partial_{x_3} f)e_3, \tag{2.1.2}$$

$$\mathrm{div} F = \partial_{x_1} F^1 + \partial_{x_2} F^2 + \partial_{x_3} F^3, \tag{2.1.3}$$

$$\mathrm{curl} F = (\partial_{x_2} F^3 - \partial_{x_3} F^2)e_1 - (\partial_{x_1} F^3 - \partial_{x_3} F^1)e_2 + (\partial_{x_1} F^2 - \partial_{x_2} F^1)e_3, \tag{2.1.4}$$

and

$$\Delta f = \partial_{x_1}^2 f + \partial_{x_2}^2 f + \partial_{x_3}^2 f. \tag{2.1.5}$$

Since we are working with a vector field, we will also need the vector laplacian:

$$\Delta F = \Delta F^1 e_1 + \Delta F^2 e_2 + \Delta F^3 e_3. \tag{2.1.6}$$

Essentially we take the Laplacian on each component function.

We note that the Navier-Stokes Equations are actually a system of equations. Component-wise we see that (1.0.7) can be rewritten as

$$\Delta u^i - \sum_{j=1}^{3} u^j \partial_j u^i + \partial_i p = 0. \tag{2.1.7}$$

In addition to the velocity, we will also need to work with another quantity called the vorticity. The vector field $w = \text{curl}(u)$ is called the **vorticity**. By taking the curl of (1.0.7) we have a new set of equations the vorticity satisfies. By direct calculation we have

$$\begin{aligned} -\text{curl}(\Delta u) + \text{curl}[(u \cdot \nabla)u] + \text{curl}(\nabla p) = 0 \\ -\Delta w + \text{curl}[(u \cdot \nabla)u] = 0, \end{aligned} \tag{2.1.8}$$

where we used the fact that the curl of the gradient is zero. To simplify the equation further, we first rewrite the term $(u \cdot \nabla)u$ by noting that

$$(u \cdot \nabla)u = \frac{1}{2}\nabla(u \cdot u) - u \times w. \tag{2.1.9}$$

This follows by using the following vector identity by letting $A = B = u$:

$$\nabla(A \cdot B) = (A \cdot \nabla)B + (B \cdot \nabla)A + A \times (\text{curl}(B)) + B \times (\text{curl}(A)).$$

Hence we have

$$\begin{aligned}\operatorname{curl}[(u\cdot\nabla)u] &= \frac{1}{2}\operatorname{curl}(\nabla(u\cdot u)) - \operatorname{curl}(u\times w)\\ &= \operatorname{curl}(w\times u)\\ &= w(\operatorname{div}(u)) - u(\operatorname{div}(w)) + (u\cdot\nabla)w - (w\cdot\nabla)u\\ &= (u\cdot\nabla)w - (w\cdot\nabla)u,\end{aligned}\tag{2.1.10}$$

where we used the divergence-free condition of u (and consequently w) and the vector identity

$$\operatorname{curl}(A\times B) = A(\operatorname{div}(B)) - B(\operatorname{div}(A)) + (B\cdot\nabla)A - (A\cdot\nabla)B. \tag{2.1.11}$$

Therefore, the vorticity equation for the stationary Navier-Stokes equation is

$$-\Delta w + (u\cdot\nabla)w - (w\cdot\nabla)u = 0. \tag{2.1.12}$$

Although the vorticity equation involves even more partial derivatives of the velocity, the equations no longer involve the pressure.

2.2 Cylindrical Coordinates

Because our solutions are axially-symmetric along the z-axis, it will be useful to convert all the equations into cylindrical coordinates. We can do a change of variables to cylindrical coordinates (r,θ,z) in the following way:

$$r = \sqrt{x_1^2 + x_2^2}, \quad \theta = \arctan\left(\frac{x_2}{x_1}\right), \quad z = x_3, \tag{2.2.1}$$

or equivalently

$$x_1 = r\cos(\theta), \quad x_2 = r\sin(\theta), \quad z = x_3. \tag{2.2.2}$$

Here $r \geq 0$, $0 \leq \theta < 2\pi$, and z is any real number. If $x_1 = 0$, then θ will be either 0, $\frac{\pi}{2}$, or $\frac{3\pi}{2}$ depending on what x_2 is.

Hence a vector field can be represented in the following way:

$$v(x) = v^r(x)e_r + v^\theta(x)e_\theta + v^z(x)e_\theta \tag{2.2.3}$$

where

$$e_r = \left(\frac{x_1}{r}, \frac{x_2}{r}, 0\right) = (\cos(\theta), \sin(\theta), 0), \tag{2.2.4}$$

$$e_\theta = \left(-\frac{x_2}{r}, \frac{x_1}{r}, 0\right) = (-\sin(\theta), \cos(\theta), 0), \tag{2.2.5}$$

and

$$e_z = (0, 0, 1). \tag{2.2.6}$$

Given a scale-valued function f and vector field $F = (F^r, F^\theta, F^z)$, we have that the gradient, divergence, curl, Laplacian, and vector Laplacian in cylindrical coordinates are given by

$$\nabla f = (\partial_r f)e_r + \frac{1}{r}(\partial_\theta f)e_\theta + (\partial_z f)e_z, \tag{2.2.7}$$

$$\mathrm{div} F = \partial_r F^r + \frac{1}{r}F^r + \frac{1}{r}\partial_\theta F^\theta + \partial_z F^z, \tag{2.2.8}$$

$$\mathrm{curl} F = \left(\frac{1}{r}\partial_\theta F^z - \partial_z F^\theta\right) e_r - \left(\partial_r F^z - \partial_z F^r\right) e_\theta + \left(\partial_r F^\theta + \frac{1}{r}F^\theta - \frac{\partial_\theta F^r}{r}\right) e_z, \tag{2.2.9}$$

$$\Delta f = \partial_r^2 f + \frac{1}{r}\partial_r f + \frac{1}{r^2}\partial_\theta^2 f + \partial_z^2 f, \tag{2.2.10}$$

and

$$\Delta F = \left(\Delta F^r - \frac{F^r}{r^2} - \frac{2}{r^2}\partial_\theta F^\theta\right) e_r + \left(\Delta F^\theta - \frac{F^\theta}{r^2} + \frac{2}{r^2}\partial_\theta F^r\right) e_\theta + \Delta F^z e_z. \tag{2.2.11}$$

Compared to Cartesian coordinates, these differential operators are more complex. The reason is because, unlike in Cartesian coordinates, the basis vectors in cylindrical depend on the other variables. In Cartesian coordinates, $\partial_{x_i} e_j$=0 for any i and j. In contrast, in cylindrical coordinates we have that $\partial_\theta e_r = e_\theta$ and $\partial_\theta e_\theta = -e_r$. As a result, the partial derivative hits the basis vectors and produces extra terms. Because the solutions we work with are axially-symmetric, the partial derivatives with respect to θ are zero.

Since we know what the Laplacian, divergence, and gradient are in cylindrical coordinates, we only need to determine how to write the inertia term $(u \cdot \nabla)u$ in cylindrical coordinates. By noting that

$$(u \cdot \nabla) = \left(u^r \partial_r + \frac{u^\theta}{r} \partial_\theta + u^z \partial_z \right) \tag{2.2.12}$$

we have that

$$\begin{aligned}(u \cdot \nabla)u &= (u \cdot \nabla)(u^r e_r + u^\theta e_\theta + u^z e_z) \\ &= \left[(u \cdot \nabla)u^r\right] e_r + \left[(u \cdot \nabla)u^\theta\right] e_\theta + \left[(u \cdot \nabla)u^z\right] e_z \\ &\quad \left[(u \cdot \nabla)e_r\right] u^r + \left[(u \cdot \nabla)e_\theta\right] u^\theta + \left[(u \cdot \nabla)e_z\right] u^z \\ &= \left[(u \cdot \nabla)u^r\right] e_r + \left[(u \cdot \nabla)u^\theta\right] e_\theta + \left[(u \cdot \nabla)u^z\right] e_z + \frac{u^r u^\theta}{r} e_\theta - \frac{(u^\theta)^2}{r} e_r.\end{aligned} \tag{2.2.13}$$

Similarly, we have for the terms $(u \cdot \nabla)w$ and $(w \cdot \nabla)u$ that

$$(u \cdot \nabla)w = \left[(u \cdot \nabla)w^r\right] e_r + \left[(u \cdot \nabla)w^\theta\right] e_\theta + \left[(u \cdot \nabla)w^z\right] e_z + \frac{w^r u^\theta}{r} e_\theta - \frac{w^\theta u^\theta}{r} e_r. \tag{2.2.14}$$

and

$$(w \cdot \nabla)u = \left[(w \cdot \nabla)u^r\right] e_r + \left[(w \cdot \nabla)u^\theta\right] e_\theta + \left[(w \cdot \nabla)u^z\right] e_z + \frac{u^r w^\theta}{r} e_\theta - \frac{w^\theta u^\theta}{r} e_r. \tag{2.2.15}$$

Hence we have the axially symmetric Navier-Stokes equations:

$$\begin{cases} (u^r\partial_r + u^z\partial_z)u^r - \dfrac{(u^\theta)^2}{r} + \partial_r p = \left(\partial_r^2 + \dfrac{1}{r}\partial_r + \partial_z^2 - \dfrac{1}{r^2}\right) u^r \\ (u^r\partial_r + u^z\partial_z)u^\theta + \dfrac{u^r u^\theta}{r} = \left(\partial_r^2 + \dfrac{1}{r}\partial_r + \partial_z^2 - \dfrac{1}{r^2}\right) u^\theta \\ (u^r\partial_r + u^z\partial_z)u^z + \partial_z p = \left(\partial_r^2 + \dfrac{1}{r}\partial_r + \partial_z^2\right) u^z \\ \partial_r u^r + \dfrac{u^r}{r} + \partial_z u^z = 0. \end{cases} \tag{2.2.16}$$

The vorticity equations are as follows:

$$\begin{cases} (u^r\partial_r + u^z\partial_z)w^r - (w^r\partial_r + w^z\partial_z)u^r = \left(\partial_r^2 + \dfrac{1}{r}\partial_r + \partial_z^2 - \dfrac{1}{r^2}\right) w^r \\ (u^r\partial_r + u^z\partial_z)w^\theta - \dfrac{u^r}{r}w^\theta - \dfrac{1}{r}\partial_z(u^\theta)^2 = \left(\partial_r^2 + \dfrac{1}{r}\partial_r + \partial_z^2 - \dfrac{1}{r^2}\right) w^\theta \\ (u^r\partial_r + u^z\partial_z)w^z - (w^r\partial_r + w^z\partial_z)u^r = \left(\partial_r^2 + \dfrac{1}{r}\partial_r + \partial_z^2\right) w^z \end{cases} \tag{2.2.17}$$

where

$$w^r = -\partial_z u^\theta, \quad w^\theta = \partial_z u^r - \partial_r u^z, \quad w^z = \frac{1}{r}\partial_r(r u^\theta). \tag{2.2.18}$$

Chapter 3

Green's Function

Finding solutions to general partial differential equations (PDEs) is an important, but difficult task. Even if one focuses on linear PDEs, there is no general theory that will work for all linear PDEs, regardless if the PDE is homogeneous or inhomogenous. However, in certain cases one may construct a solution to an inhomogeneous PDE using a special kind of solution to the homogeneous problem. Given a linear differential operator $L = L(x)$, a **Green's function** $G = G(x)$ satisfies the following for any point x:

$$LG(x) = \delta(x),$$

where δ is the Dirac delta function. In particular, if we consider $L = \Delta = \partial_{x_1}^2 + \partial_{x_1}^2 + \partial_{x_3}^2$, that is **Laplace's equation** in $\mathbb{R}^3$

$$\Delta G = \delta(x),$$

then it is well-known that the Green's function is $G(x) = \dfrac{1}{4\pi|x|}$.

Now suppose we wish to solve **Poisson's equation**, the inhomogenous Laplace's equation, in $\mathbb{R}^3$. That is, given a function $f\colon \mathbb{R}^3 \to \mathbb{R}$, find a function $u\colon \mathbb{R}^3 \to \mathbb{R}$ that satisfies the PDE

$$\Delta u(x) = f(x).$$

If the given function f is sufficiently well-behaved, we have that $u(x) = \displaystyle\int_{\mathbb{R}^3} f(x)G(x-y)dy$ will be a solution to Poisson's equation. Therefore to find a solution to Poisson's equation in a specific domain, we use the Green's function for the corresponding Laplace's equation and preform a convolution. However, if we consider Poisson's equation or Laplace's equation on a different domain, then the Green's function will change. For a general domain, it is difficult to find explicit formulas for Green's functions directly. An alternative to finding Green's functions for Laplace's equation is constructing them by using a solution to the **heat equation.**

The **heat equation** in $(0, \infty) \times \mathbb{R}^n$ is

$$\partial_t \Gamma - \Delta \Gamma = 0,$$

where $\Gamma = \Gamma(t, x)$ is unknown. For dimension $n \geq 1$ we have that the **fundamental solution of the heat equation** is

$$\Gamma_n(t, x) = \begin{cases} \frac{1}{(4\pi t)^{n/2}} e^{-\frac{|x|^2}{4t}} & t > 0, \quad x \in \mathbb{R}^n \\ 0 & t < 0, \quad x \in \mathbb{R}^n \end{cases}$$

The reason the fundamental solution to the heat equation is important in finding solutions to Poisson's equation is that Green's function for Laplace's equation are related in the following way: for $n = 3$, we have that

$$\begin{aligned}\int_0^\infty \Gamma_3(t,x)dt &= \int_0^\infty \frac{1}{(4\pi t)^{3/2}} e^{-\frac{|x|^2}{4t}} dt \\ &= \int_0^\infty \frac{1}{(4\pi t)^{3/2}} e^{-\left(\frac{|x|}{\sqrt{4t}}\right)^2} dt \\ &= \frac{1}{2\pi^{3/2}|x|} \int_0^\infty e^{-u^2} du \\ &= \frac{1}{4\pi|x|} \\ &= G(x).\end{aligned}$$

where we made the substitution $u(x) = \frac{|x|}{\sqrt{4t}}$. In fact, one can show that for $n \geq 3$,

$$G_n(x) = \int_0^\infty \Gamma_n(t,x)dt$$

where $G_n(x) = \frac{1}{n(n-2)\alpha(n)} \frac{1}{|x|^{n-2}}$ represents the Green's function for Laplace's equation in $\mathbb{R}^n$, $\alpha(n)$ is the measure of the unit ball in $\mathbb{R}^n$, and Γ_n represents the fundamental solution to the heat equation in $\mathbb{R}^n$. It is important to note that this is not the case for $n = 2$, that is to say that

$$G_2(x) = -\frac{1}{2\pi} \log|x| \neq \int_0^\infty \frac{1}{4\pi t} e^{-\frac{|x|^2}{4t}} dt = \int_0^\infty \Gamma_2(t,x)dt,$$

where G_2 is the Green's function for Laplace's equation in $\mathbb{R}^2$. In fact, the integration of the two-dimensional fundamental solution to the heat equation over time is infinite, despite the fact that there is a Green's function for Laplace's equation in dimension two. However, this idea of integrating the fundamental solution will be useful for our goal of studying the stationary Navier-Stokes equation in $\mathbb{R}^2 \times [-\pi, \pi]$.

3.1 The Green function on $\mathbb{R}^2 \times [-\pi, \pi]$ for functions whose integral on $[-\pi, \pi]$ is zero

To find the Green's function on $\mathbb{R}^2 \times [-\pi, \pi]$, we need to find the fundamental solution to the heat equation in $(0, \infty) \times \left(\mathbb{R}^2 \times [-\pi, \pi]\right)$. We might guess that the fundamental solution will be the product of the fundamental solution in $\mathbb{R}^2$ and the fundamental solution in $[-\pi, \pi]$. Indeed this will solve the heat equation on $(0, \infty) \times \left(\mathbb{R}^2 \times [-\pi, \pi]\right)$, but when we try to integrate the fundamental solution over time, we will have a divergent integral just like we did for the two-dimensional heat equation. We can overcome this by "subtracting" off the problematic term to avoid having a divergent integral. However, the price we pay is that we must restrict the class of functions the Green's function will act on.

We will use the following notation for this chapter and beyond. Given $x = (x_1, x_2, x_3) \in \mathbb{R}^3$, we write $x = (x', x_3)$ or $x = (x', z)$, and for $y = (y_1, y_2, y_3) \in \mathbb{R}^3$, we write $y = (y', y_3)$. In this next lemma, we construct the Green's function we will use. We will sometimes write S^1 to represent $[-\pi, \pi]$.

Lemma 2 *Let $\tilde{\Gamma}_1$ and Γ_2 be the standard heat kernel on S^1 and $\mathbb{R}^2$ respectively.*

(a). Let $\Gamma(t; x_3, y_3) = (\tilde{\Gamma}_1 - \frac{1}{2\pi})(t; x_3, y_3)\Gamma_2(t; x', y')$. The function $\Gamma(t; x_3, y_3)$ satisfies the heat equation on $\mathbb{R}^2 \times S^1$:

$$\begin{cases} \Delta_x \Gamma - \partial_t \Gamma = 0 \\ \Gamma|_{t=0} = \delta(x, y) \end{cases} \tag{3.1.1}$$

The last equation means that for any bounded, smooth test function φ with $\displaystyle\int_{-\pi}^{\pi} \varphi(y', y_3) dy_3 = 0$, we have $\displaystyle\lim_{t \to 0^+} \langle \Gamma, \varphi \rangle = \varphi$.

(b). Let $G(x,y) = \int_0^\infty \Gamma(t;x,y)dt$*. Suppose that* f *is a smooth and compactly supported function in* $\mathbb{R}^2 \times S^1$ *such that* $\displaystyle\int_{S^1} f(x', x_3)dx_3 = 0$ *for all* $x' \in \mathbb{R}^2$*. Then*

$$-\Delta(G * f) = f,$$

where $*$ *stands for the usual convolution.*

(c). Let u *and* f *be smooth, bounded functions on* $\mathbb{R}^2 \times S^1$ *such that*

$$-\Delta u = f.$$

Suppose $\displaystyle\int_{S^1} f(x', x_3)dx_3 = 0$ *for all* $x' \in \mathbb{R}^2$ *and* f *is compactly supported. Then for some constant* C*,*

$$u(x) = \int_{\mathbb{R}^2 \times S^1} G(x,y)f(y)dy + C.$$

Thus G *is the Green's function for those* f*.*

Proof. Define $\tilde{\Gamma}_1(t; x_3, y_3) := \frac{1}{2\pi}\big(1 + 2\sum_{m=1}^{\infty} e^{-m^2 t}\cos(m(x_3 - y_3))\big)$, where x_3 and y_3 are in $[-\pi, \pi]$. First for $t \neq 0$ we have that $0 < e^{-t} < 1$ and

$$\left|\sum_{m=1}^{\infty} e^{-m^2 t}\cos(m(x_3 - y_3))\right| \leq \sum_{m=1}^{\infty} e^{-m^2 t} \leq \sum_{m=1}^{\infty} \left(e^{-t}\right)^m < \infty. \tag{3.1.2}$$

This shows that $\tilde{\Gamma}$ is well-defined for $t \neq 0$ and in fact the series converges uniformly by the Weierstrass criterion. Moreover, we can preform term by term differentiation on $\tilde{\Gamma}$ in the variables t, x and y because the respective series converge uniformly and we can interchange summation and differentiation. Hence $\partial_t \tilde{\Gamma}$, $\Delta_{x_3}\tilde{\Gamma}$ are well-defined. We will need to use the fact later that

$$\int_{-\pi}^{\pi} \tilde{\Gamma}_1(t; x_3, 0)dx_3 = \int_{-\pi}^{\pi} \tilde{\Gamma}_1(t; 0, y_3)dy_3 = 1. \tag{3.1.3}$$

This is because for $m \neq 0$, $e^{-m^2t}\int_{-\pi}^{\pi}\cos(my_3)dy_3 = 0$. Therefore, the only non-zero term in the sum is the $m = 0$ term which equals 1. Finally, we wish to note that $\tilde{\Gamma}$ may also be rewritten out as

$$\tilde{\Gamma}_1(t;x_3,y_3) = \frac{1}{2\pi}\sum_{m=-\infty}^{\infty} e^{-m^2t}e^{im(x_3-y_3)}. \tag{3.1.4}$$

We will interchange between these forms as needed. We claim that $\tilde{\Gamma}_1(t;x_3,y_3)$ is the heat kernel on S^1, which means that

$$\begin{cases} \Delta_{x_3}\tilde{\Gamma}_1 - \partial_t\tilde{\Gamma}_1 = 0, \\ \tilde{\Gamma}_1|_{t=0} = \delta(x_3-y_3). \end{cases}$$

Direct computation shows that $\tilde{\Gamma}_1$ satisfies the equation, so we only check that $\tilde{\Gamma}_1|_{t=0} = \delta(x_3-y_3)$. This means we must show that for any test function $\varphi(y_3)$ in $C_0^\infty(\mathbb{R})$,

$$\lim_{t\to 0^+}\langle\tilde{\Gamma}_1(t;x_3,y_3),\varphi(y_3)\rangle - \varphi(x_3) \tag{3.1.5}$$

$$= \lim_{t\to 0^+}\int_{-\pi}^{\pi}\left(1+2\sum_{m=1}^{\infty}e^{-m^2t}\cos(m(x_3-y_3))\right)\varphi(y_3)dy_3 - \varphi(y_3) \tag{3.1.6}$$

$$= 0. \tag{3.1.7}$$

First we note that

$$\sum_{m=1}^{\infty}\int_{-\pi}^{\pi}\left|e^{-m^2t}\cos(m(x_3-y_3))\right|dy_3 = \sum_{m=1}^{\infty}4e^{-m^2t} \leq \sum_{m=1}^{\infty}4\left(e^{-t}\right)^m < \infty. \tag{3.1.8}$$

Hence by Lebesgue theory we are allowed to interchange integration and summation. So we have that

$$\langle\tilde{\Gamma}_1(t;x_3,y_3),\varphi(y_3)\rangle - \varphi(x_3) \tag{3.1.9}$$

$$= \int_{-\pi}^{\pi} \tilde{\Gamma}_1(t;0,y_3)\varphi(x_3-y_3)dy_3 - \varphi(x_3)\int_{-\pi}^{\pi} \tilde{\Gamma}_1(t;0,y_3)dy_3 \tag{3.1.10}$$

$$= \int_{-\pi}^{\pi} \tilde{\Gamma}_1(t;x_3,y_3)\left(\varphi(y_3)-\varphi(x_3)\right)\mathrm{d}y_3 \tag{3.1.11}$$

$$= \sum_{m=-\infty}^{\infty} e^{-m^2 t}\int_{-\pi}^{\pi} e^{-im(x_3-y_3)}\left(\varphi(y_3)-\varphi(x_3)\right)\mathrm{d}y_3 \tag{3.1.12}$$

$$= \lim_{N\to\infty}\sum_{|m|\le N} e^{-m^2 t}\int_{-\pi}^{\pi} e^{-im(x_3-y_3)}\left(\varphi(y_3)-\varphi(x_3)\right)\mathrm{d}y_3. \tag{3.1.13}$$

Next we have

$$\lim_{t\to 0^+}\langle\tilde{\Gamma}_1(t;x_3,y_3),\varphi(y_3)\rangle - \varphi(x_3) \tag{3.1.14}$$

$$= \lim_{t\to 0^+}\lim_{N\to\infty}\sum_{|m|\le N} e^{-m^2 t}\int_{-\pi}^{\pi} e^{-im(x_3-y_3)}\left(\varphi(y_3)-\varphi(x_3)\right)\mathrm{d}y_3 \tag{3.1.15}$$

$$= \lim_{N\to\infty}\lim_{t\to 0^+}\sum_{|m|\le N} e^{-m^2 t}\int_{-\pi}^{\pi} e^{-im(x_3-y_3)}\left(\varphi(y_3)-\varphi(x_3)\right)\mathrm{d}y_3 \tag{3.1.16}$$

$$= \lim_{N\to\infty}\sum_{|m|\le N}\int_{-\pi}^{\pi} e^{-im(x_3-y_3)}\left(\varphi(y_3)-\varphi(x_3)\right)dy_3, \tag{3.1.17}$$

$$= \lim_{N\to\infty}\int_{-\pi}^{\pi}\sum_{|m|\le N} e^{-im(x_3-y_3)}\left(\varphi(y_3)-\varphi(x_3)\right)dy_3, \tag{3.1.18}$$

where we used the Moore-Osgood theorem to interchange the two limits. To proceed further, we note that

$$\sum_{|m|\le N} e^{-im(x_3-y_3)} = \frac{\sin\left((N+1/2)(x_3-y_3)\right)}{\sin\left(\frac{x_3-y_3}{2}\right)}. \tag{3.1.19}$$

We thus have

$$\lim_{t\to 0^+}\langle\tilde{\Gamma}_1(t;x_3,y_3),\varphi(y_3)\rangle - \varphi(x_3) \tag{3.1.20}$$

$$= \lim_{N\to\infty}\int_{-\pi}^{\pi}\frac{\sin\left((N+1/2)(x_3-y_3)\right)}{\sin\left(\frac{x_3-y_3}{2}\right)}\left(\varphi(y_3)-\varphi(x_3)\right)dy_3 \tag{3.1.21}$$

$$= \lim_{N\to\infty}\int_{-\pi}^{\pi}\left(\frac{\varphi(y_3)-\varphi(x_3)}{\sin\left(\frac{x_3-y_3}{2}\right)}\right)\sin\left((N+1/2)(x_3-y_3)\right)dy_3. \tag{3.1.22}$$

Because

$$\lim_{x_3 \to y_3} \frac{\varphi(y_3) - \varphi(x_3)}{\sin\left(\frac{x_3 - y_3}{2}\right)} = 2\varphi'(y_3), \tag{3.1.23}$$

we have that $\frac{\varphi(y_3)-\varphi(x_3)}{\sin\left(\frac{x_3-y_3}{2}\right)}$ is integrable on $[-\pi, \pi]$. Therefore, by the Riemann-Lebesgue Lemma, we obtain

$$\lim_{t\to 0} \langle \tilde{\Gamma}_1(t; x_3, y_3), \varphi(y_3)\rangle - \varphi(x_3) \tag{3.1.24}$$

$$= \lim_{N\to\infty} \int_{-\pi}^{\pi} \left(\frac{\varphi(y_3) - \varphi(x_3)}{\sin\left(\frac{x_3 - y_3}{2}\right)} \right) \sin\left((N + 1/2)(x_3 - y_3)\right) dy_3. \tag{3.1.25}$$

$$= 0, \tag{3.1.26}$$

as desired. This shows that $\tilde{\Gamma}_1(t; x_3, y_3)$ is the heat kernel on $[-\pi, \pi]$.

Now we define $\Gamma_1(t; x_3, y_3) = \tilde{\Gamma}_1(t; x_3, y_3) - \frac{1}{2\pi}$. Then $\Gamma_1(t; x_3, y_3)$ still satisfies the heat equation and is a heat kernel for the test function $\varphi(y_3)$ with $\int_{-\pi}^{\pi} \varphi(y_3)dy_3 = 0$ because

$$\lim_{t\to 0} \langle \Gamma_1(t; x_3, y_3), \varphi(y_3)\rangle - \varphi(x_3) \tag{3.1.27}$$

$$= \lim_{t\to 0} \langle \tilde{\Gamma}_1(t; x_3, y_3), \varphi(y_3)\rangle - \frac{1}{2\pi} \int_{-\pi}^{\pi} \varphi(y_3)dy_3 - \varphi(x_3) \tag{3.1.28}$$

$$= \lim_{t\to 0} \Big(\langle \tilde{\Gamma}_1(t; x_3, y_3), \varphi(y_3)\rangle - \varphi(x_3) \Big) - \frac{1}{2\pi} \int_{-\pi}^{\pi} \varphi(y_3)dy_3 \tag{3.1.29}$$

$$= 0. \tag{3.1.30}$$

We can now construct the heat kernel on $\mathbb{R}^2 \times [-\pi, \pi]$ for bounded smooth functions φ such that

$$\int_{-\pi}^{\pi} \varphi(y', y_3)dy_3 = 0. \tag{3.1.31}$$

Let $\Gamma_2(t; x', y') = \frac{1}{4\pi t} e^{-\frac{|x'-y'|^2}{4t}}$ be the heat kernel on $\mathbb{R}^2$. Define

$$\Gamma(t; x, y) = \Gamma_2(t; x', y')\Gamma_1(t; x_3, y_3).$$

Then we claim that $\Gamma(t; x, y)$ is the heat kernel that satisfies

$$\begin{cases} \Delta_x \Gamma - \partial_t \Gamma = 0 \\ \\ \Gamma|_{t=0} = \delta(x, y) \end{cases} \tag{3.1.32}$$

for bounded, smooth test functions satisfying (3.1.31). The second equation, as before, means that $\lim_{t \to 0^+} \langle \Gamma, \varphi \rangle - \varphi = 0$. First we show that Γ satisfies the heat equation. By direct computation we have

$$\Delta_x \Gamma - \partial_t \Gamma \tag{3.1.33}$$

$$= \Gamma_1 \Delta_{x'} \Gamma_2 + \Gamma_2 \Delta_{x_3} \Gamma_1 - \Gamma_1 \partial_t \Gamma_2 - \Gamma_2 \partial_t \Gamma_1 \tag{3.1.34}$$

$$= \Gamma_1 \left(\Delta_{x'} \Gamma_2 - \partial_t \Gamma_2 \right) + \Gamma_2 \left(\Delta_{x_3} \Gamma_1 - \partial_t \Gamma_1 \right) \tag{3.1.35}$$

$$= 0.$$

Next for any bounded, smooth test function $\varphi(y)$ satisfying (3.1.31), we have

$$\begin{aligned}
&\lim_{t \to 0} \int_{\mathbb{R}^2 \times S^1} \Gamma(t; x, y) \varphi(y) dy \\
&= \lim_{t \to 0} \int_{\mathbb{R}^2} \Gamma_2(t; x', y') \int_{S^1} \Gamma_1(t; x_3, y_3) \varphi(y', y_3) dy_3 dy' \\
&= \lim_{t \to 0} \int_{\mathbb{R}^2} \Gamma_2(t; x', y') \int_{S^1} \tilde{\Gamma}_1(t; x_3, y_3) \varphi(y', y_3) dy_3 dy' \\
&= \lim_{t \to 0} \int_{\mathbb{R}^2} \Gamma_2(t; x', y') \int_{S^1} \tilde{\Gamma}_1(t; x_3, y_3) \left[\varphi(y', y_3) + \varphi(y', x_3) - \varphi(y', x_3) \right] dy_3 dy' \\
&= \lim_{t \to 0} \int_{\mathbb{R}^2} \Gamma_2(t; x', y') \varphi(y', x_3) \int_{S^1} \tilde{\Gamma}(t; x_3, y_3) dy_3 dy' + \lim_{t \to 0} \int_{\mathbb{R}^2} \Gamma_2(t; x', y') \Big[- \varphi(y', x_3) \\
&\qquad + \int_{S^1} \tilde{\Gamma}_1(t; x_3, y_3) \varphi(y', y_3) dy_3 \Big] dy' \\
&= \lim_{t \to 0} \int_{\mathbb{R}^2} \Gamma_2(t; x', y') \varphi(y', x_3) dy' + \lim_{t \to 0} \int_{\mathbb{R}^2} \Gamma_2(t; x', y') \Big[- \varphi(y', x_3) \\
&\qquad + \int_{S^1} \tilde{\Gamma}_1(t; x_3, y_3) \varphi(y', y_3) dy_3 \Big] dy' \\
&= \varphi(x', x_3) + 0 \\
&= \varphi(x).
\end{aligned}$$

We flesh out some of the details of the above computation. First, we claim that the term

$$\int_{\mathbb{R}^2} \Gamma_2(t; x', y') \left[-\varphi(y', x_3) + \int_{S^1} \tilde{\Gamma}_1(t; x_3, y_3)\varphi(y', y_3) dy_3 \right] dy'$$

goes to zero as $t \to 0^+$. This is because since $\lim\limits_{t\to 0} \int_{S^1} \Gamma_1(t; x_3, y_3)\varphi(y', y_3)dy_3 = \varphi(y', x_3)$ for all y', then for some $\delta > 0$ we have that if $0 < t < \delta$, then

$$\left| \int_{S^1} \Gamma_1(t; x_3, y_3)\varphi(y', y_3)dy_3 - \varphi(y', x_3) \right| < \varepsilon.$$

We thus have

$$\left| \int_{\mathbb{R}^2} \Gamma_2(t; x', y') \left[-\varphi(y', x_3) + \int_{S^1} \tilde{\Gamma}_1(t; x_3, y_3)\varphi(y', y_3) dy_3 \right] dy' \right| < \int_{\mathbb{R}^2} \Gamma_2(t; x', y')\varepsilon dy' = \varepsilon.$$

We have thus proven (a), which is that Γ is the heat kernel on $\mathbb{R}^2 \times [-\pi, \pi]$ for functions whose integrals on S^1 are zero. Now we are ready to define the Green's function on $\mathbb{R}^2 \times [-\pi, \pi]$:

$$G(x, y) = \int_0^\infty \Gamma(t; x, y)dt.$$

The integral is finite except when $x = y$. This follows from the exponential decay property of Γ. We can also show that the partial derivatives are defined and we are allowed to interchange differentiation and integration. We will justify all the details in the next lemma.

Let f be a smooth, compactly supported function on $\mathbb{R}^2 \times S^1$, whose integral on S^1 is 0. Then since Γ satisfies the heat equation on $\mathbb{R}^2 \times S^1$, we have that

$$\begin{aligned}
-\int_{\mathbb{R}^2\times S^1} \Delta_x G(x, y)f(y)dy &= \int_{\mathbb{R}^2\times S^1} \left[\int_0^\infty -\Delta_x \Gamma(t; x, y)dt \right] f(y)dy \\
&= -\int_{\mathbb{R}^2\times S^1} \left[\int_0^\infty \partial_t \Gamma(t; x, y)dt \right] f(y)dy \\
&= \int_{\mathbb{R}^2\times S^1} \lim_{t\to 0^+} \Gamma(t; x, y)f(y)dy \\
&= f(x).
\end{aligned} \tag{3.1.36}$$

Hence $G(x,y)$ satisfies the Poisson's equation for any smooth, compactly supported function on $\mathbb{R}^2 \times S^1$ which proves (b). Now let u be any bounded, smooth solution of

$$-\Delta u = f.$$

First, we note that $\int G(x,y)f(y)dy$ is in fact a bounded solution. This follows because in Lemma 3, we will show that

$$|G(x,y)| \leq C\frac{1}{|x-y|}e^{-c_0|x'-y'|}.$$

If $f(y)$ is supported in some ball $B(x_0,R)$, we have that for $|x-x_0| > 2R$ and $|y-x_0| \leq R$ that $\frac{1}{|x-y|} \leq \frac{2}{|x-x_0|}$. Hence

$$\begin{aligned}\left|\int G(x,y)f(y)dy\right| &\leq C\int_{B(x_0,R)} C\frac{1}{|x-y|}e^{-c_0|x'-y'|}|f(y)|dy \\ &\leq C\|f\|_{L^\infty}\int_{B(x_0,R)}\frac{1}{|x-x_0|}dy \\ &\leq C\frac{1}{|x-x_0|}\end{aligned} \tag{3.1.37}$$

which implies that $G(x,y)$ is bounded as $|x|$ becomes unbounded.

We thus have that $u - \int G(x,y)f(y)dy$ is bounded and

$$\Delta[u(x) - \int G(x,y)f(y)dy] = 0.$$

Hence, $u - \int G(x,y)f(y)dy$ is a bounded, harmonic function so by the classical Liouville theorem we have that

$$u = \int G(x,y)f(y)dy + C.$$

This shows that G is the Green function on $\mathbb{R}^2 \times [-\pi,\pi]$ of those functions with its integral on S^1 is zero which proves (c). ■

3.2 Estimates for the Green's Function and its Gradient on $\mathbb{R}^2 \times [-\pi, \pi]$

In the next lemma we prove that that the Green's function is well-defined and in fact satisfies the following estimates.

Lemma 3 *Let $G(x,y)$ be the Green function on $\mathbb{R}^2 \times S^1$ defined above. Then we have the following estimates for some constants $c_0, C_1, C_2 > 0$:*

$$\big|G(x,y)\big| \leq C_1 \frac{1}{|x-y|} e^{-c_0|x'-y'|} \tag{3.2.1}$$

$$\big|\nabla G(x,y)\big| \leq C_2 \frac{1}{|x-y|^2} e^{-c_0|x'-y'|} \tag{3.2.2}$$

with $x' = (x_1, x_2)$ and $y' = (y_1, y_2)$.

Proof. We will first prove that

$$\big|G(x,y)\big| \leq C e^{-\frac{|x'-y'|}{4}} \tag{3.2.3}$$

and

$$\big|\nabla G(x,y)\big| \leq C e^{-\frac{|x'-y'|}{4}} \tag{3.2.4}$$

for the case when $|x'-y'| > 1$. We have

$$\begin{aligned} G(x,y) &= \int_0^\infty (4\pi t)^{-1} e^{-\frac{|x'-y'|^2}{4t}} \frac{1}{\pi} \sum_{m=1}^\infty e^{-m^2 t} \cos(m(x_3-y_3))dt \\ &= \frac{1}{4\pi^2} \sum_{m=1}^\infty \int_0^\infty t^{-1} e^{-\frac{|x'-y'|^2}{4t}} e^{-m^2 t} dt \cos(m(x_3-y_3)) \\ &= \frac{1}{4\pi^2} \sum_{m=1}^\infty I_m \cos(m(x_3-y_3)). \end{aligned} \tag{3.2.5}$$

We will first estimate I_m. By making a change of variables, we see that

$$\begin{aligned} I_m &= \int_0^\infty t^{-1} e^{-\frac{(m|x'-y'|)^2}{4t}} e^{-t} dt \\ &= \Big(\int_0^{\frac{m|x'-y'|}{2}} + \int_{\frac{m|x'-y'|}{2}}^\infty \Big) t^{-1} e^{-\frac{(m|x'-y'|)^2}{4t}} e^{-t} dt \\ &= 2 \int_{\frac{m|x'-y'|}{2}}^\infty t^{-1} e^{-\frac{(m|x'-y'|)^2}{4t}} e^{-t} dt \\ &\leq C \int_{\frac{m|x'-y'|}{2}}^\infty t^{-1} e^{-t} dt \end{aligned} \tag{3.2.6}$$

This follows by first making the change of variables $u = -m^2 t$ so that the $-m^2$ coefficient transfers from $e^{-m^2 t}$ to $e^{-\frac{|x'-y'|^2}{4t}}$. Then we split the integral from 0 to $m|x'-y'|/2$ and $m|x'-y'|/2$ to ∞. We make use of the symmetry of the integrand $t^{-1} e^{-\frac{(m|x'-y'|)^2}{4t}} e^{-t}$ by making a substitution $u = \frac{(m|x'-y'|)^2}{4t}$ to in fact get that the two integrals are equal. Finally, for any $m \geq 1$, $e^{-\frac{(m|x'-y'|)^2}{4t}} \leq 1$. So all we need to do is estimate the integral

$$\int_{\frac{m|x'-y'|}{2}}^\infty t^{-1} e^{-t} dt = E_1 \left(\frac{m|x'-y'|}{2} \right),$$

where $E_1(z)$ is the exponential integral. To get the final estimate, we first note that

$$E_1(x) < e^{-x} \ln \left(1 + \frac{1}{x} \right)$$

for $x > 0$. See [DLMF] for reference. Hence

$$I_m \leq C \ln \left(\frac{2}{m|x'-y'|} + 1 \right) e^{-\frac{m|x'-y'|}{2}}.$$

From (3.2.5) and (3.2.6), we have

$$\begin{aligned} \big| G(x,y) \big| &\leq C \sum_{m=1}^\infty \ln \left(\frac{2}{m|x'-y'|} + 1 \right) e^{-\frac{m|x'-y'|}{2}} \\ &\leq C \ln \left(\frac{2}{|x'-y'|} + 1 \right) e^{-\frac{|x'-y'|}{4}} \\ &\leq C e^{-\frac{|x'-y'|}{4}}. \end{aligned} \tag{3.2.7}$$

This bound is found by first noting that $\ln\left(\frac{2}{m|x'-y'|}+1\right) \leq \ln\left(\frac{2}{|x'-y'|}+1\right) \leq \ln(3)$ for $m \geq 1$ and $|x'-y'| > 1$. So it remains to bound

$$\sum_{m=1}^{\infty} e^{-\frac{m|x'-y'|}{2}} = \sum_{m=1}^{\infty} e^{-\frac{m|x'-y'|}{4}} \left(e^{-\frac{|x'-y'|}{4}}\right)^m \leq e^{-\frac{|x'-y'|}{4}} \sum_{m=1}^{\infty} \left(e^{-\frac{|x'-y'|}{4}}\right)^m.$$

However, $e^{-\frac{|x'-y'|}{4}} < 1$, so we have a geometric series whose sum equals $\frac{1}{1-e^{-\frac{|x'-y'|}{4}}} < \frac{1}{1-e^{-1/4}}$. This proves (3.2.3).

We now prove (3.2.4). We have

$$\begin{aligned}
\left|\partial_{x_3} G(x,y)\right| &= \Big|\frac{1}{4\pi^2} \sum_{m=1}^{\infty} m \int_0^{\infty} t^{-1} e^{-\frac{|x'-y'|^2}{4t}} e^{-m^2 t} dt \sin(m(x_3-y_3))\Big| \\
&\leq C \sum_{m=1}^{\infty} \int_0^{\infty} t^{-\frac{3}{2}} e^{-\frac{|x'-y'|^2}{4t}} e^{-\frac{m^2 t}{2}} dt \qquad (3.2.8) \\
&\leq C \sum_{m=1}^{\infty} m \int_0^{\infty} t^{-\frac{3}{2}} e^{-\frac{(m|x'-y'|)^2}{4t}} e^{-\frac{t}{2}} dt
\end{aligned}$$

We now explain the above calculations. We first make the change of variables $u = m^2 t$ again. Then we note that $t^{-1}e^{-t} \leq t^{-3/2} e^{-\frac{t}{2}}$, for $t > 0$. Equivalently, we break up $e^{-m^2 t}$ as $e^{-\frac{m^2 t}{2}} e^{-\frac{m^2 t}{2}}$ and then we use that

$$m e^{-\frac{m^2 t}{2}} \leq \sup_{m \in [1,\infty)} m e^{-\frac{m^2 t}{2}} = \frac{1}{\sqrt{t}} e^{-\frac{1}{2}}.$$

Hence we obtain

$$m e^{-\frac{m^2 t}{2}} t^{-1} e^{-\frac{|x'-y'|^2}{4t}} e^{-\frac{m^2 t}{2}} \leq t^{-3/2} e^{-\frac{m^2 t}{2}} e^{-\frac{|x'-y'|^2}{4t}},$$

from which we make the change of variables $u = m^2 t$ to obtain the m variable.

Next we estimate the integral

$$m \int_0^{\infty} t^{-\frac{3}{2}} e^{-\frac{(m|x'-y'|)^2}{4t}} e^{-\frac{t}{2}} dt.$$

We first multiply by $e^{-\frac{\sqrt{2}m|x'-y'|}{2}}e^{\frac{\sqrt{2}m|x'-y'|}{2}}$. Then by completing the square, we have that

$$\begin{aligned}
m\int_0^\infty t^{-\frac{3}{2}}e^{-\frac{(m|x'-y'|)^2}{4t}}e^{-\frac{t}{2}}dt &= me^{-\frac{\sqrt{2}m|x'-y'|}{2}}\int_0^\infty t^{-\frac{3}{2}}e^{-\frac{(m|x'-y'|)^2}{4t}+\frac{\sqrt{2}m|x'-y'|}{2}-\frac{t}{2}}dt \\
&= me^{-\frac{\sqrt{2}m|x'-y'|}{2}}\int_0^\infty t^{-\frac{3}{2}}e^{-\frac{1}{4t}((m|x'-y'|)^2-2\sqrt{2}m|x'-y'|t+2t^2)}dt \\
&= me^{-\frac{\sqrt{2}m|x'-y'|}{2}}\int_0^\infty t^{-\frac{3}{2}}e^{-\frac{1}{4t}(m|x'-y'|-\sqrt{2}t)^2}dt \\
&= me^{-\frac{\sqrt{2}m|x'-y'|}{2}}\int_0^\infty t^{-\frac{3}{2}}e^{-\left(\frac{m|x'-y'|-\sqrt{2}t}{2\sqrt{t}}\right)^2}dt. \\
&= me^{-\frac{\sqrt{2}m|x'-y'|}{2}}\frac{2\sqrt{\pi}}{m|x'-y'|} \\
&= e^{-\frac{\sqrt{2}m|x'-y'|}{2}}\frac{2\sqrt{\pi}}{|x'-y'|}.
\end{aligned}$$

We will show that

$$\int_0^\infty t^{-3/2}e^{-\left(\frac{m|x'-y'|-\sqrt{2}t}{2\sqrt{t}}\right)^2}dt = \frac{2\sqrt{\pi}}{m|x'-y'|}. \tag{3.2.9}$$

We first make the substitution $u(t) = \frac{1}{\sqrt{t}}$ to get $-2t^{3/2}du = dt$, then we obtain

$$\int_0^\infty t^{-3/2}e^{-\left(\frac{m|x'-y'|-\sqrt{2}t}{2\sqrt{t}}\right)^2}dt = 2\int_0^\infty e^{-\left(\frac{m|x'-y'|u-\sqrt{2}u^{-1}}{2}\right)^2}du.$$

We make one final substitution: $\theta(u) = m|x'-y'|u$ to obtain

$$2\int_0^\infty e^{-\left(\frac{m|x'-y'|u-\sqrt{2}u^{-1}}{2}\right)^2}du = \frac{2}{m|x'-y'|}\int_0^\infty e^{-\left(\frac{\theta-\sqrt{2}m|x'-y'|\theta^{-1}}{2}\right)^2}d\theta.$$

Now we define

$$F(w) = \int_0^\infty e^{-\left(\frac{\theta-\sqrt{2}w\theta^{-1}}{2}\right)^2}d\theta.$$

We claim that $F(w) = \sqrt{\pi}$ for any choice of $w \geq 0$. To show this, we note that

$$\partial_w F(w) = \frac{\sqrt{2}}{2}\int_0^\infty e^{-\left(\frac{\theta-\sqrt{2}w\theta^{-1}}{2}\right)^2}\left(\theta-\sqrt{2}w\theta^{-1}\right)\theta^{-1}d\theta.$$

The integrand has a singularity at $\theta = 0$, however,

$$\lim_{\theta \to 0} e^{-\left(\frac{\theta - \sqrt{2}w\theta^{-1}}{2}\right)^2} \left(\theta - \sqrt{2}w\theta^{-1}\right)\theta^{-1} = 0,$$

so there is no issue in the integral being infinite. We split the integral into two pieces:

$$\partial_w F(w) = \frac{\sqrt{2}}{2}\left[\int_0^{\sqrt[4]{2}\sqrt{w}} + \int_{\sqrt[4]{2}\sqrt{w}}^{\infty} e^{-\left(\frac{\theta - \sqrt{2}w\theta^{-1}}{2}\right)^2} \left(\theta - \sqrt{2}w\theta^{-1}\right)\theta^{-1} d\theta\right].$$

By making the change of variables $u(\theta) = \sqrt{2}w\theta^{-1}$ so that $du = -\sqrt{2}w\theta^{-2}d\theta$ we get

$$\begin{aligned}&\int_0^{\sqrt[4]{2}\sqrt{w}} e^{-\left(\frac{\theta - \sqrt{2}w\theta^{-1}}{2}\right)^2} \left(\theta - \sqrt{2}w\theta^{-1}\right)\theta^{-1} d\theta \\ &= -\int_{\infty}^{\sqrt[4]{2}\sqrt{w}} e^{-\left(\frac{\sqrt{2}wu^{-1} - u}{2}\right)^2} \left(\sqrt{2}wu^{-1} - u\right) u^{-1} du.\end{aligned}$$

However, we note that by symmetry:

$$\begin{aligned}&-\int_{\infty}^{\sqrt[4]{2}\sqrt{w}} e^{-\left(\frac{\sqrt{2}wu^{-1} - u}{2}\right)^2} \left(\sqrt{2}wu^{-1} - u\right) u^{-1} du \\ &= -\int_{\sqrt[4]{2}\sqrt{w}}^{\infty} e^{-\left(\frac{u - \sqrt{2}wu^{-1}}{2}\right)^2} \left(u - \sqrt{2}wu^{-1}\right) u^{-1} du.\end{aligned}$$

This means that in fact

$$\begin{aligned}&\int_0^{\sqrt[4]{2}\sqrt{w}} e^{-\left(\frac{\sqrt{2}wu^{-1} - u}{2}\right)^2} \left(\sqrt{2}wu^{-1} - u\right) u^{-1} du \\ &= -\int_{\sqrt[4]{2}\sqrt{w}}^{\infty} e^{-\left(\frac{u - \sqrt{2}wu^{-1}}{2}\right)^2} \left(u - \sqrt{2}wu^{-1}\right) u^{-1} du.\end{aligned}$$

This implies that

$$\partial_w F(w) = 0,$$

which means that $F(w)$ is independent of choice of w.

We shall compute $F(0)$:

$$\begin{aligned}\int_0^\infty e^{-\left(\frac{\theta}{2}\right)^2} d\theta &= 2\int_0^\infty e^{-(\xi)^2} d\xi \\ &= 2\frac{\sqrt{\pi}}{2} \\ &= \sqrt{\pi}.\end{aligned}$$

Hence we have proved (3.2.9). We now use (3.2.9) and insert it into (3.2.8) to obtain

$$\begin{aligned}\left|\partial_{x_3} G(x,y)\right| &\leq C\sum_{m=1}^\infty \frac{e^{-\frac{\sqrt{2}m|x'-y'|}{2}}}{|x'-y'|} \\ &\leq C\sum_{m=1}^\infty \frac{e^{-\frac{m|x'-y'|}{2}}}{|x'-y'|} \\ &\leq C\frac{1}{|x'-y'|}e^{-\frac{|x'-y'|}{4}} \\ &\leq Ce^{-\frac{|x'-y'|}{4}}.\end{aligned} \tag{3.2.10}$$

We used the geometric series trick from before and note that $1 < |x'-y'|$. We have finished the estimate for $\partial_{x_3} G(x,y)$. The estimate of $\partial_{y_3} G(x,y)$ is the same as (3.2.10).

We now estimate $\partial_{x',y'} G(x,y)$. From (3.2.5), we have

$$\begin{aligned}\left|\partial_{x',y'} G(x,y)\right| &\leq C\sum_{m=1}^\infty \int_0^\infty t^{-1}\frac{|x'-y'|}{t}e^{-\frac{|x'-y'|^2}{4t}}e^{-m^2 t}dt \\ &\leq C|x'-y'|\sum_{m=1}^\infty m^2\int_0^\infty t^{-2}e^{-\frac{(m|x'-y'|)^2}{4t}}e^{-t}dt,\end{aligned} \tag{3.2.11}$$

where we again make the substitution $u = m^2 t$ from which we get a m^2 in the integral. Next, we split the resulting integral into two parts and integrate separately:

$$\left|\partial_{x',y'} G(x,y)\right| \leq C|x'-y'|\sum_{m=1}^\infty m^2\Big(\int_{\frac{m|x'-y'|}{2}}^\infty + \int_0^{\frac{m|x'-y'|}{2}}\Big)t^{-2}e^{-\frac{(m|x'-y'|)^2}{4t}}e^{-t}dt. \tag{3.2.12}$$

For the first integral, we have that

$$m^2|x'-y'|\int_{\frac{m|x'-y'|}{2}}^{\infty} t^{-2}e^{-\frac{(m|x'-y'|)^2}{4t}}e^{-t}dt \le m^2|x'-y'|\frac{4}{m^2|x'-y'|^2}\int_{\frac{m|x'-y'|}{2}}^{\infty} e^{-t}dt$$
$$= \frac{4}{|x'-y'|}e^{-\frac{m|x'-y'|}{2}}.$$

Here we used the fact that $e^{-\frac{(m|x'-y'|)^2}{4t}} \le 1$ and also $t^{-2} \le \frac{4}{(m|x'-y'|)^2}$ on the interval $[\frac{m|x'-y'|}{2}, \infty)$.

For the second integral we make a change of variables $u = \frac{(m|x'-y'|)^2}{4t}$ so that $du = -\frac{(m|x'-y'|)^2}{4t^2}dt$ and we obtain:

$$m^2|x'-y'|\int_0^{\frac{m|x'-y'|}{2}} t^{-2}e^{-\frac{(m|x'-y'|)^2}{4t}}e^{-t}dt = \frac{4m^2|x'-y'|}{(m|x'-y'|)^2}\int_{\frac{m|x'-y'|}{2}}^{\infty} e^{-\frac{(m|x'-y'|)^2}{4u}}e^{-u}du.$$
$$\le \frac{4}{|x'-y'|}\int_{\frac{m|x'-y'|}{2}}^{\infty} e^{-u}du$$
$$= \frac{4}{|x'-y'|}e^{-\frac{m|x'-y'|}{2}}.$$

Now using our estimates for the two integrals and using the geometric trick again we have that

$$\left|\partial_{x',y'}G(x,y)\right| \le C\frac{1}{|x'-y'|}e^{-\frac{|x'-y'|}{4}}. \tag{3.2.13}$$

This proves (3.2.4).

Now we consider the case when $|x'-y'| \le 1$. First there exists positive constants c_1, c_2 such that

$$|\Gamma(t;x,y)| \le Ct^{-3/2}e^{-c_1\frac{|x-y|^2}{t}}e^{-c_2t}, \tag{3.2.14}$$

and

$$|\nabla\Gamma(t;x,y)| \le Ct^{-2}e^{-c_1\frac{|x-y|^2}{t}}e^{-c_2 t}. \tag{3.2.15}$$

This follows by noting that $|x'-y'| \le 1$,

$$|\Gamma_2(t;x,y)| = Ct^{-1}e^{-\frac{|x'-y'|^2}{4t}},$$

$$|\nabla\Gamma_2(t;x,y)| = Ct^{-2}|x'-y'|e^{\frac{-|x'-y'|^2}{4t}} \le Ct^{-2}e^{\frac{-|x'-y'|^2}{4t}},$$

and by using global estimates for Γ_1 and it's partial derivatives. See Lemma 4.2 in [TZ] for details of the estimate on Γ_1.

We now integrate (3.2.14) to obtain

$$\begin{aligned}
|G(x,y)| &\le C\int_0^\infty t^{-3/2}e^{-c_1\frac{|x-y|^2}{t}}e^{-c_2 t}dt \\
&= Ce^{-2\sqrt{c_1c_2}|x-y|}\int_0^\infty t^{-3/2}e^{-c_1\frac{|x-y|^2}{t}+2\sqrt{c_1c_2}|x-y|-c_2 t}dt \\
&= Ce^{-2\sqrt{c_1c_2}|x-y|}\int_0^\infty t^{-3/2}e^{-\frac{c_1|x-y|^2-2\sqrt{c_1c_2}|x-y|t+c_2t^2}{t}}dt \\
&= Ce^{-2\sqrt{c_1c_2}|x-y|}\int_0^\infty t^{-3/2}e^{-\left(\frac{\sqrt{c_1}|x-y|-\sqrt{c_2}t}{\sqrt{t}}\right)^2}dt \qquad (3.2.16)\\
&= Ce^{-2\sqrt{c_1c_2}|x-y|}\int_0^\infty e^{-\left(\sqrt{c_1}|x-y|u-\sqrt{c_2}u^{-1}\right)^2}du \\
&= C\frac{1}{\sqrt{c_1}|x-y|}e^{-2\sqrt{c_1c_2}|x-y|}\int_0^\infty e^{-\left(\theta-\sqrt{c_1c_2}|x-y|\theta^{-1}\right)}d\theta \\
&= C\frac{1}{\sqrt{c_1}|x-y|}e^{-2\sqrt{c_1c_2}|x-y|},
\end{aligned}$$

where a similar computation like (3.2.9) can be used to show that

$$\int_0^\infty e^{-\left(\theta-\sqrt{c_1c_2}|x-y|\theta^{-1}\right)}d\theta = \frac{\sqrt{\pi}}{2}.$$

A similar computation like (3.2.11) gives us that we can find a constant c_3 so that

$$|\nabla G(x,y)| \le C\frac{1}{|x-y|^2}e^{-c_3|x-y|}. \tag{3.2.17}$$

Hence using the fact that $|x_3 - y_3| \leq 2\pi$ and by combining (3.2.3), (3.2.4),(3.2.16), (3.2.17), we choose a small enough c_0 so that (3.2.1) and (3.2.2) hold for all $|x-y| \neq 0$. ∎

Although the above estimates will suffice for general purposes, we need sharper estimates. In particular, we will be working later with hallowed cylinders. We note that upon integrating the Green's function along the θ variable, we can get the following estimates:

Lemma 4 *Denote $x = (r\cos\theta, r\sin\theta, z)$ and $y = (\rho\cos\phi, \rho\sin\phi, \ell)$. For $|\rho - r| \leq \frac{1}{4}r$,*

$$\int_0^{2\pi} |G(x,y)|d\phi \leq Ce^{-c_0|\rho-r|}\frac{1}{r}\ln\left(2+\frac{r}{|\rho-r|}\right) \tag{3.2.18}$$

$$\int_0^{2\pi} |\nabla G(x,y)|d\phi \leq C\frac{1}{\rho\left(|\rho-r|+|z-\ell|\right)}e^{-c_0|\rho-r|}. \tag{3.2.19}$$

For $\frac{1}{8}r \leq |\rho - r| \leq \frac{1}{4}r$, with $1 \leq r$, we have

$$\int_0^{2\pi} \left(|G(x,y)| + |\nabla G(x,y)|\right)d\phi \leq Ce^{-c_0|\rho-r|}. \tag{3.2.20}$$

Proof. From Lemma 3, we see that

$$\begin{aligned}\int_0^{2\pi} |G(x,y)|d\phi &\leq C\int_0^{2\pi} \frac{1}{|x-y|}e^{-c_0|x'-y'|}d\phi. \\ &\leq Ce^{-c_0|\rho-r|}\int_0^{2\pi}\frac{1}{|x'-y'|}d\phi.\end{aligned} \tag{3.2.21}$$

and

$$\begin{aligned}\int_0^{2\pi} |\nabla\Gamma(x,y)|d\phi &\leq C\int_0^{2\pi} \frac{1}{|x-y|^2}e^{-c_0|x'-y'|}d\phi. \\ &\leq Ce^{-c_0|\rho-r|}\int_0^{2\pi}\frac{1}{|x-y|^2}d\phi.\end{aligned} \tag{3.2.22}$$

Recall that $x' = (r\cos\theta, r\sin\theta), y' = (\rho\cos\phi, \sin\phi)$,

$$|x'-y'| = \sqrt{\rho^2 + r^2 - 2\rho r\cos(\theta-\phi)}, \tag{3.2.23}$$

and

$$|x-y| = \sqrt{\rho^2 + r^2 - 2\rho r\cos(\theta-\phi) + |z-\ell|^2}. \tag{3.2.24}$$

Without loss of generality, we will set $\theta = 0$ because the integrals are independent of choice of θ. So we work with

$$|x' - y'| = \sqrt{\rho^2 + r^2 - 2\rho r\cos(\phi)} = \sqrt{(\rho - r)^2 + 4\rho r \sin^2\left(\frac{\phi}{2}\right)}, \tag{3.2.25}$$

and

$$|x - y| = \sqrt{(\rho - r)^2 + 4\rho r \sin^2\left(\frac{\phi}{2}\right) + |z - \ell|^2}, \tag{3.2.26}$$

where we used the half-angle formula $2\sin^2\left(\frac{\theta}{2}\right) = 1 - \cos(\theta)$.

We will show that

$$\int_0^{2\pi} \frac{1}{|x' - y'|} d\phi \leq C\frac{1}{r}\ln\left(2 + \frac{r}{|\rho - r|}\right), \tag{3.2.27}$$

and

$$\int_0^{2\pi} \frac{1}{|x - y|^2} d\phi \leq C\frac{1}{\rho(|\rho - r| + |z - \ell|)}. \tag{3.2.28}$$

We see that

$$\begin{aligned}
\int_0^{2\pi} \frac{1}{|x' - y'|} d\phi &= 4\int_0^{\pi/2} \frac{1}{\sqrt{(\rho - r)^2 + 4\rho r \sin^2(\phi)}} d\phi \\
&= 4\int_0^{\pi/2} \frac{1}{\sqrt{4\rho r\left(\frac{|\rho - r|^2}{4\rho r} + \sin^2(\phi)\right)}} d\phi \\
&= \frac{2}{\sqrt{\rho r}}\int_0^{\pi/2} \frac{1}{\sqrt{\frac{|\rho - r|^2}{4\rho r} + \sin^2(\phi)}} d\phi \\
&= \frac{2}{\sqrt{\rho r}}\int_0^{\pi/2} \frac{1}{\sqrt{k^2 + \sin^2(\phi)}} d\phi, \\
&= \frac{2}{\sqrt{\rho r}}\left(\int_0^{\pi/4} + \int_{\pi/4}^{\pi/2}\right) \frac{1}{\sqrt{k^2 + \sin^2(\phi)}} d\phi,
\end{aligned} \tag{3.2.29}$$

where $k^2 = \frac{|\rho - r|^2}{4\rho r}$. To estimate the first integral we note that on $[0, \frac{\pi}{4}]$, $Cx \leq \sin(x) \leq x$ where $0 < C < \frac{2\sqrt{2}}{\pi}$ and $\frac{1}{2}(k + \sqrt{C}\phi)^2 \leq k^2 + C\phi^2 \leq (k + \sqrt{C}\phi)^2$.

Hence we have

$$\begin{aligned}
\int_0^{\pi/4} \frac{1}{\sqrt{k^2+\sin^2(\phi)}}d\phi &\le \int_0^{\pi/4} \frac{1}{\sqrt{k^2+C\phi^2}}d\phi \\
&\le \int_0^{\pi/4} \frac{1}{k+\sqrt{C}\phi}d\phi \\
&= \frac{1}{k}\int_0^{\pi/4} \frac{1}{1+\sqrt{C}\frac{\phi}{k}}d\phi \\
&= \int_0^{\frac{\pi}{4k}} \frac{1}{1+\sqrt{C}\phi}d\phi \\
&\le C\ln\left(1+\sqrt{C}\frac{\pi}{4k}\right) \\
&\le C\ln\left(2+\frac{r}{|\rho-r|}\right)
\end{aligned} \tag{3.2.30}$$

We used the fact that since $|\rho - r| \le \frac{1}{4}r$, we have that $\frac{3}{4}r \le \rho \le \frac{5}{4}r$ so ρ is comparable to r and this implies that $k^2 = \frac{|\rho-r|^2}{4\rho r} \le C$.

We also have for second integral that on $[\frac{\pi}{4}, \frac{\pi}{2}]$, $\frac{\sqrt{2}}{2} \le \sin(x) \le 1$, so

$$\begin{aligned}
\int_{\pi/4}^{\pi/2} \frac{1}{\sqrt{k^2+\sin^2(\phi)}}d\phi &\le \int_{\pi/4}^{\pi/2} \frac{1}{\sqrt{k^2+1/2}}d\phi \\
&\le \int_{\pi/4}^{\pi/2} \frac{1}{\sqrt{1/2}}d\phi \\
&= C.
\end{aligned} \tag{3.2.31}$$

Hence by (3.2.30), (3.2.31), and using the fact that ρ is comparable to r, we obtain from (3.2.29) the inequality (3.2.27).

Now we work on (3.2.28). Similar to the above computations for (3.2.27), we see that

$$
\begin{aligned}
\int_0^{2\pi} \frac{1}{|x-y|^2} d\phi` &= C\int_0^{\pi/2} \frac{1}{(\rho-r)^2+|z-\ell|+4\rho r\sin^2(\phi)} d\phi \\
&\le \frac{C}{\rho^2}\int_0^{\pi/2} \frac{1}{\frac{|\rho-r|^2+|z-\ell|^2}{4\rho r}+4\rho r\sin^2(\phi)} \\
&\le \frac{C}{\rho^2}\int_0^{\pi/2} \frac{1}{\kappa^2+\sin^2(\phi)} d\phi \\
&\le \frac{C}{\rho^2}\left(\int_0^{\pi/4} \frac{1}{\kappa^2+\sin^2(\phi)} d\phi + \int_{\pi/4}^{\pi/2} \frac{1}{\kappa^2+\sin^2(\phi)} d\phi\right) \\
&\le \frac{C}{\rho^2}\left(\frac{1}{\kappa}\int_0^{\pi/4k} \frac{1}{1+\phi^2} d\phi + \int_{\pi/4}^{\pi/2} \frac{1}{\kappa^2+1/2} d\phi\right) \\
&\le \frac{C}{\rho^2}\left(\frac{1}{\kappa}\int_0^{\infty} \frac{1}{1+\phi^2} d\phi + \int_{\pi/4}^{\pi/2} \frac{1}{\kappa^2+1/2} d\phi\right) \\
&\le \frac{C}{\rho^2}\left(\frac{1}{\kappa}\right) \\
&\le \frac{C}{\rho(|\rho-r|+|z-\ell|)}.
\end{aligned} \tag{3.2.32}
$$

Finally, substituting (3.2.27) and (3.2.28) into (3.2.21) and (3.2.22) gives us the result. To get (3.2.20), we note that since $1 \le r$ and $\frac{1}{8}r \le |\rho - r| \le \frac{1}{4}r$, we have $\frac{1}{r} < 1$ and $\frac{1}{|\rho-r|} \le \frac{8}{r}$ respectively. Hence from (3.2.18) and (3.2.19),

$$
\begin{aligned}
\int_0^{2\pi} |G(x,y)| d\phi &\le Ce^{-c_0|\rho-r|}\frac{1}{r}\ln\left(2+\frac{r}{|\rho-r|}\right) \\
&\le Ce^{-c_0|\rho-r|}\ln\left(2+\frac{8r}{r}\right) \\
&\le Ce^{-c_0|\rho-r|}\ln(10) \\
&\le Ce^{-c_0|\rho-r|}
\end{aligned} \tag{3.2.33}
$$

and

$$
\begin{aligned}
\int_0^{2\pi} |\nabla G(x,y)| d\phi &\le C\frac{1}{\rho\left(|\rho-r|+|z-\ell|\right)}e^{-c_0|\rho-r|} \\
&\le C\frac{1}{r(|\rho-r|)}e^{-c_0|\rho-r|} \\
&\le Ce^{-c_0|\rho-r|}.
\end{aligned} \tag{3.2.34}
$$

Together these two estimates will give us the result (3.2.20). ■

We remark that compared to the Green's function in full three-dimensions, this Green's function has faster decay because of the exponential term. This will make a critical difference in proving our main result because it will allow for better estimates of the velocity.

Chapter 4

Decay and Vanishing of the Velocity

4.1 Problem Statement

Theorem 5 *Let u be a smooth axially symmetric solution to the problem*

$$\begin{cases} -\Delta u + (u\cdot\nabla)u + \nabla p = 0, \quad in \quad \mathbb{R}^2\times[-\pi,\pi], \\ \operatorname{div}(u) = 0, \\ u(x_1,x_2,z) = u(x_1,x_2,z+2\pi), \\ \lim\limits_{|x|\to\infty} u = 0, \end{cases} \tag{4.1.1}$$

such that the Dirichlet integral satisfies the condition: for $0 \le \alpha < 1/5$, we have that for all $R \ge 1$,

$$\int_{-\pi}^{\pi}\int_{|x'|\le R} |\nabla u(x)|^2 dx < R^\alpha < \infty. \tag{4.1.2}$$

Suppose also $\int_{-\pi}^{\pi} u^\theta(\cdot,z)dz = \int_{-\pi}^{\pi} u^z(\cdot,z)dz = 0$. Then $u = 0$.

Notice that in the theorem there is no requirement that $\int_{-\pi}^{\pi} u^r dz = 0$. This is because one can actually prove this holds without any additional assumptions. First note that by the incompressible condition we have that

$$\partial_r u^r + \frac{u^r}{r} + \partial_z u^z = 0, \tag{4.1.3}$$

which can be rewritten as

$$\frac{1}{r}\partial_r(ru^r) + \partial_z u^z = 0. \tag{4.1.4}$$

We can integrate this equation from $-\pi$ to π along the z variable to obtain

$$\frac{1}{r}\int_{-\pi}^{\pi} \partial_r(ru^r)dz + \int_{-\pi}^{\pi} \partial_z u^z dz = 0. \tag{4.1.5}$$

However, since u is periodic on $[-\pi, \pi]$ we have that

$$\int_{-\pi}^{\pi} \partial_z u^z dz = u^z\Big|_{-\pi}^{\pi} = 0, \tag{4.1.6}$$

which gives us that

$$\int_{-\pi}^{\pi} \partial_r(ru^r)dz = 0. \tag{4.1.7}$$

Therefore, by fundamental theorem of calculus and the above, we have

$$\int_{-\pi}^{\pi} u^r(r,z)dz = \frac{1}{r}\int_0^r \partial_{\tilde{r}}\left(\int_{-\pi}^{\pi} \tilde{r}u^r(\tilde{r},z)dz\right)d\tilde{r} = \frac{1}{r}\int_0^r \int_{-\pi}^{\pi} \partial_{\tilde{r}}\left(\tilde{r}u^r(\tilde{r},z)\right)dzd\tilde{r} = 0. \tag{4.1.8}$$

By differentiation we also have that $\int_{-\pi}^{\pi} \partial_r u^r dz = \partial_r\left(\int_{-\pi}^{\pi} u^r dz\right) = 0.$

4.2 Brezis-Gallouet Inequality

To prove Theorem 5, we will need to get some bounds. One inequality that we need is the $Brezis - Gallouet$ inequality found in [BG]:

Lemma 6 *Let $f \in H^2(\mathcal{O})$ where $\mathcal{O} \subset \mathbb{R}^2$. Then there exists a constant $C_{\mathcal{O}}$, depending only on $\mathcal{O}$, such that*

$$\|f\|_{L^\infty(\mathcal{O})} \leq C_{\mathcal{O}} \|f\|_{H^1(\mathcal{O})} \log^{1/2}\left(e + \frac{\|\Delta f\|_{L^2(\mathcal{O})}}{\|f\|_{H^1(\mathcal{O})}}\right). \tag{4.2.1}$$

We note that the original $Brezis - Gallouet$ inequality is written as:

$$\|f\|_{L^\infty(\mathcal{O})} \leq C_{\mathcal{O}} \|f\|_{H^1(\mathcal{O})} \log^{1/2}\left(e + \frac{\|f\|_{H^2(\mathcal{O})}}{\|f\|_{H^1(\mathcal{O})}}\right).$$

However, by going through the proof in [BG], we see that the norm $\|f\|_{H^2(\mathcal{O})}$ in the log term can be replaced by $\|\Delta f\|_{L^2(\mathcal{O})} + \|f\|_{L^2(\mathcal{O})}$. Moreover,

$$\frac{\|\Delta f\|_{L^2(\mathcal{O})} + \|f\|_{L^2(\mathcal{O})}}{\|f\|_{H^1(\mathcal{O})}} = \frac{\|\Delta f\|_{L^2(\mathcal{O})}}{\|f\|_{H^1(\mathcal{O})}} + \frac{\|f\|_{L^2(\mathcal{O})}}{\|f\|_{H^1(\mathcal{O})}} \leq \frac{\|\Delta f\|_{L^2(\mathcal{O})}}{\|f\|_{H^1(\mathcal{O})}} + \frac{\|f\|_{L^2(\mathcal{O})}}{\|f\|_{L^2(\mathcal{O})}} \leq \frac{\|\Delta f\|_{L^2(\mathcal{O})}}{\|f\|_{H^1(\mathcal{O})}} + e.$$

Hence (4.2.1) is valid. However, for our purposes we will need the following modified B-Z inequality:

$$\|f\|_{L^\infty(\mathcal{O})} \leq C_{\mathcal{O}} \left(\|f\|_{H^1(\mathcal{O})} + 1\right) \log^{1/2}\left(e + \|\Delta f\|_{L^2(\mathcal{O})}\right) \tag{4.2.2}$$

which follows from the following inequality

$$C\|f\|_{H^1(\mathcal{O})} \log^{1/2}\left(e + \frac{\|\Delta f\|_{L^2(\mathcal{O})}}{\|f\|_{H^1(\mathcal{O})}}\right) \leq C\left(\|f\|_{H^1(\mathcal{O})} + 1\right) \log^{1/2}\left(e + \|\Delta f\|_{L^2(\mathcal{O})}\right).$$

The inequality holds because if $\|f\|_{H^1(\mathcal{O})} \geq 1$, then $\frac{\|\Delta f\|_{L^2(\mathcal{O})}}{\|f\|_{H^1(\mathcal{O})}} \leq |\Delta f\|_{L^2(\mathcal{O})}$. If $\|f\|_{H^1(\mathcal{O})} < 1$, then consider the function f defined on $[0,1]$ by

$$f(x) = \begin{cases} 0 & x = 0 \\ \frac{x}{x+1} \frac{\log^{1/2}(e+\frac{A}{x})}{\log^{1/2}(e+A)} & 0 < x \leq 1. \end{cases}$$

As $x \to 0^+$, we see that $f \to 0$. Hence f attains a maximum on $[0,1]$. So $f(x) \leq C$ for some C. If we set $A = \|\Delta f\|_{L^2(\mathcal{O})}$ and regard x as $x = \|f\|_{H^1(\mathcal{O})}$, then

$$\begin{gathered} f(x) \leq C \\ \frac{\|f\|_{H^1(\mathcal{O})}}{\|f\|_{H^1(\mathcal{O})}+1} \frac{\log^{1/2}\left(e + \frac{\|\Delta f\|_{L^2(\mathcal{O})}}{\|f\|_{H^1(\mathcal{O})}}\right)}{\log^{1/2}\left(e + \|\Delta f\|_{L^2(\mathcal{O})}\right)} \leq C \\ \|f\|_{H^1(\mathcal{O})} \log^{1/2}\left(e + \frac{\|\Delta f\|_{L^2(\mathcal{O})}}{\|f\|_{H^1(\mathcal{O})}}\right) \leq C\left(\|f\|_{H^1(\mathcal{O})}+1\right) \log^{1/2}\left(e + \|\Delta f\|_{L^2(\mathcal{O})}\right). \end{gathered} \tag{4.2.3}$$

Hence we obtain (4.2.2).

4.3 First Decay of the Velocitiy and Vorticity

We note that in this thesis C stands for a positive constant that may change from line to line. If C depends on any significant parameter or variable, we will use subscripts to denote such a dependence.

4.3.1 First Decay of w^θ

The first goal is to obtain some decay on w^θ. Pick $x_0 \in \mathbb{R}^3$ and we assume that, without loss of generality, in cylindrical coordinates $x_0 = (r_0, 0, 0)$. Also choose x_0 so that $|x_0'| = \lambda$ is large. The following arguments will work on any point where λ is large. Now scale the velocity and vorticity with respect to the scaling $\tilde{x} = \frac{x}{\lambda}$:

$$\tilde{u}(\tilde{x}) = \lambda u(\lambda \tilde{x}) = \lambda u(x), \tag{4.3.1}$$

$$\tilde{w}(\tilde{x}) = \lambda^2 w(\lambda \tilde{x}) = \lambda^2 w(x). \tag{4.3.2}$$

We note that these are the standard scaling that the velocity and the vorticity of the NSE satisfy.

4.3.1.1 Scaled Computations

In the calculations that follow, we will be working with the scaled functions $\tilde{u}$ and $\tilde{w}$ and the scaled variable $\tilde{x}$. To simplify notation, we will not use the "$\sim$ " throughout the following calculations. We note that the scaled velocity and vorticity still satisfy the NSE and vorticty equations.

Define the domains

$$\mathcal{D}_1 = \left\{ (r, \theta, z) : \frac{1}{2} < r < \frac{3}{2},\ 0 \leq \theta \leq 2\pi, -\frac{\pi}{\lambda} \leq z \leq \frac{\pi}{\lambda} \right\} \tag{4.3.3}$$

and

$$\mathcal{D}_2 = \left\{ (r, \theta, z) : \frac{3}{4} < r < \frac{5}{4},\ 0 \leq \theta \leq 2\pi, -\frac{\pi}{\lambda} \leq z \leq \frac{\pi}{\lambda} \right\}. \tag{4.3.4}$$

Note that $\mathcal{D}_2 \subseteq \mathcal{D}_1$. Let $\psi(y)$ be a cut-off function depending only on r such that $\sup(\psi) \subseteq \mathcal{D}_1$, $\psi = 1$ on $\mathcal{D}_2$, and $\nabla\psi$ is bounded. The first step is to get estimates on $\|\nabla w\|^2_{L^2(\mathcal{D})_2}$. We will do this by testing the vorticity equations with respect to $\psi^2 w^r, \psi^2 w^\theta, \psi^2 w^z$ to obtain the following:

$$-\int_{\mathcal{D}_1} \psi^2 w^r \left(\Delta - \frac{1}{r^2} \right) w^r dy = -\int_{\mathcal{D}_1} \left[(u^r \partial_r + u^z \partial_z) w^r \psi^2 w^r + (w^r \partial_r + w^z \partial_z) u^r \psi^2 w^r \right] dy, \tag{4.3.5}$$

$$-\int_{\mathcal{D}_1} \psi^2 w^\theta \left(\Delta - \frac{1}{r^2}\right) w^\theta dy = -\int_{\mathcal{D}_1} \left[(u^r\partial_r + u^z\partial_z)w^\theta\psi^2 w^\theta + \frac{u^r}{r}(\psi w^\theta)^2 + 2\frac{u^\theta w^r}{r}\psi^2 w^\theta\right] dy, \tag{4.3.6}$$

$$-\int_{\mathcal{D}_1} \psi^2 w^z \Delta w^z dy = -\int_{\mathcal{D}_1} \left[(u^r\partial_r + u^z\partial_z)w^z\psi^2 w^z + (w^r\partial_r + w^z\partial_z)u^z\psi^2 w^z\right] dy. \tag{4.3.7}$$

For equations (4.3.5) through (4.3.7), we will do integration by parts, use the incompressible condition, and use the following identity:

$$-\int \Delta f \psi^2 f dy = \int |\nabla(f\psi)|^2 - f^2|\nabla\psi|^2 dy. \tag{4.3.8}$$

We first work with (4.3.5):

$$\begin{aligned}
&\int_{\mathcal{D}_1} \left(|\nabla(w^r\psi)|^2 + \frac{(w^r\psi)^2}{r^2}\right) dy \\
=& \int_{\mathcal{D}_1} \left((w^r)^2|\nabla\psi|^2 - \frac{1}{2}\psi^2(u^r\partial_r + u^z\partial_z)(w^r)^2 + (w^r\psi)^2\partial_r u^r + w^r w^z\psi^2\partial_z u^r\right) dy \\
=& \int_{\mathcal{D}_1} \Big((w^r)^2|\nabla\psi|^2 + \frac{1}{2}(w^r)^2(u^r\partial_r + u^z\partial_z)\psi^2 - u^r\partial_r(w^r\psi)^2 - (w^r\psi)^2\frac{u^r}{r} \\
&- u^r\partial_z(w^r\psi w^z\psi)\Big) dy \\
=& \int_{\mathcal{D}_1} \Big((w^r)^2|\nabla\psi|^2 + \frac{1}{2}(w^r)^2(u^r\partial_r + u^z\partial_z)\psi^2 - 2u^r w^r\psi\partial_r(w^r\psi) - (w^r\psi)^2\frac{u^r}{r} \\
&- u^r\partial_z(w^r\psi w^z\psi)\Big) dy.
\end{aligned} \tag{4.3.9}$$

We now estimate each integral:

$$\int_{\mathcal{D}_1} (w^r)^2|\nabla\psi|^2 dy \le \|\nabla\psi\|^2_{L^\infty(\mathcal{D}_1)}\|w^r\|^2_{L^2(\mathcal{D}_1)}, \tag{4.3.10}$$

$$\begin{aligned}
\int_{\mathcal{D}_1} (w^r)^2(u^r\partial_r + u^z\partial_z)\psi^2 dy &\le \int_{\mathcal{D}_1} (w^r)^2 \left(\frac{|(u^r)^2}{2} + \frac{|\partial_r(\psi^2)|^2}{2} + \frac{(u^z)^2}{2} + \frac{|\partial_z(\psi^2)|^2}{2}\right) dy \\
&\le C(\|\nabla\psi^2\|^2_{L^\infty(\mathcal{D}_1)} + \|(u^r, u^z)\|^2_{L^\infty(\mathcal{D}_1)})\|w^r\|^2_{L^2(\mathcal{D}_1)} \\
&\le C(1 + \|(u^r, u^z)\|^2_{L^\infty(\mathcal{D}_1)})\|w^r\|^2_{L^2(\mathcal{D}_1)},
\end{aligned} \tag{4.3.11}$$

$$
\begin{aligned}
-2\int_{\mathcal{D}_1} u^r w^r\psi\partial_r(w^r\psi)dy &\le 2\int_{\mathcal{D}_1} 4(u^r w^r\psi)^2 + \frac{|\partial_r(w^r\psi)|^2}{16}dy \\
&\le 8\|u^r\|^2_{L^\infty(\mathcal{D}_1)}\|\psi\|^2_{L^\infty(\mathcal{D}_1)}\|w^r\|^2_{L^2(\mathcal{D}_1)} + \frac{1}{8}\|\nabla(w^r\psi)\|^2_{L^2(\mathcal{D}_1)},
\end{aligned}
\tag{4.3.12}
$$

$$
\begin{aligned}
\int_{\mathcal{D}_1}(w^r\psi)^2\frac{u^r}{r}dy &= \int_{\mathcal{D}_1}(w^r\psi)^2\left(\frac{(u^r)^2}{2}+\frac{1}{2r^2}\right)dy \\
&\le C\|\psi\|^2_{L^\infty(\mathcal{D}_1)}\|u^r\|^2_{L^\infty(\mathcal{D}_1)}\|w^r\|^2_{L^2(\mathcal{D}_1)} + C\|w^r\|^2_{L^2(\mathcal{D}_1)}\|\psi\|^2_{L^\infty(\mathcal{D}_1)}\|r^{-1}\|^2_{L^2(\mathcal{D}_1)} \\
&\le C(1+\|u^r\|^2_{L^\infty(\mathcal{D}_1)})\|w^r\|^2_{L^2(\mathcal{D}_1)},
\end{aligned}
\tag{4.3.13}
$$

$$
\begin{aligned}
\int_{\mathcal{D}_1} u^r\partial_z(w^r\psi w^z\psi)dy &= \int_{\mathcal{D}_1} u^r(\partial_z(w^r\psi)w^z\psi + \partial_z(w^z\psi)w^r\psi)dy \\
&\le \int_{\mathcal{D}_1} 2(u^r w^z\psi)^2 + \frac{(\partial_z(w^r\psi))^2}{8} + 2(u^r w^r\psi)^2 + \frac{(\partial_z(w^z\psi))^2}{8}dy \\
&\le C\|u^r\|^2_{L^\infty(\mathcal{D}_1)}\|\psi\|^2_{L^\infty(\mathcal{D}_1)}\|(w^r,w^z)\|^2_{L^2(\mathcal{D}_1)} \\
&\quad + \frac{1}{8}\left(\|\nabla(w^r\psi)\|^2_{L^2(\mathcal{D}_1)} + \|\nabla(w^z\psi)\|^2_{L^2(\mathcal{D}_1)}\right).
\end{aligned}
\tag{4.3.14}
$$

In the above calculations we used the epsilon Young Inequality: for $a, b \ge 0$ and $\varepsilon > 0$ we have $ab \le \frac{a^2}{2\varepsilon} + \frac{\epsilon b^2}{2}$.

By the above estimates and (4.3.9) we have

$$
\begin{aligned}
\|\nabla(w^r\psi)\|^2_{L^2(\mathcal{D}_1)} &\le C\left(1+\|(u^r,u^z)\|^2_{L^\infty(\mathcal{D}_1)}\right)\|(w^r,w^z)\|^2_{L^2(\mathcal{D}_1)} \\
&\quad + \frac{1}{4}\left(\|\nabla(w^r\psi)\|^2_{L^2(\mathcal{D}_1)} + \|\nabla(w^z\psi)\|^2_{L^2(\mathcal{D}_1)}\right).
\end{aligned}
\tag{4.3.15}
$$

This takes care of the estimates for ∇w^r.

Next we work with (4.3.7):

$$\begin{aligned}
&\int_{\mathcal{D}_1} |\nabla(w^z\psi)|^2 dy \\
&= \int_{\mathcal{D}_1} \left((w^z)^2|\nabla\psi|^2 - \frac{1}{2}\psi^2(u^r\partial_r + u^z\partial_z)(w^z)^2 + w^r w^z \psi^2 \partial_r u^z + (w^z\psi)^2\partial_z u^z \right) dy \\
&= \int_{\mathcal{D}_1} \bigg((w^z)^2|\nabla\psi|^2 + \frac{1}{2}(w^z)^2(u^r\partial_r + u^z\partial_z)\psi^2 - u^z\partial_r(w^r w^z\psi^2) - \frac{u^z}{r} w^r w^z \psi^2 \\
&\quad - 2u^z w^z \psi \partial_z(w^z\psi) \bigg) dy.
\end{aligned} \tag{4.3.16}$$

We again estimate each integral:

$$\int_{\mathcal{D}_1} (w^z)^2|\nabla\psi|^2 dy \leq \|\nabla\psi\|^2_{L^\infty(\mathcal{D}_1)} \|w^z\|^2_{L^2(\mathcal{D}_1)}, \tag{4.3.17}$$

$$\begin{aligned}
\int_{\mathcal{D}_1} (w^z)^2(u^r\partial_r + u^z\partial_z)\psi^2 dy &\leq \int_{\mathcal{D}_1} (w^z)^2 \left(\frac{|(u^r)^2}{2} + \frac{|\partial_r(\psi^2)|^2}{2} + \frac{(u^z)^2}{2} + \frac{|\partial_z(\psi^2)|^2}{2} \right) dy \\
&\leq C(1 + \|(u^r, u^z)\|^2_{L^\infty(\mathcal{D}_1)} \|w^z\|^2_{L^2(\mathcal{D}_1)},
\end{aligned} \tag{4.3.18}$$

$$\begin{aligned}
\int_{\mathcal{D}_1} u^z\partial_r(w^r\psi w^z\psi) dy &= \int_{\mathcal{D}_1} u^z(\partial_r(w^r\psi)w^z\psi + \partial_r(w^z\psi)w^r\psi) dy \\
&\leq \int_{\mathcal{D}_1} 2(u^z w^z\psi)^2 + \frac{(\partial_r(w^r\psi))^2}{8} + 2(u^z w^r\psi)^2 + \frac{(\partial_r(w^z\psi))^2}{8} dy \\
&\leq C\|u^z\|^2_{L^\infty(\mathcal{D}_1)} \|\psi\|^2_{L^\infty(\mathcal{D}_1)} \|(w^r, w^z)\|^2_{L^2(\mathcal{D}_1)} \\
&\quad + \frac{1}{8}\left(\|\nabla(w^r\psi)\|^2_{L^2(\mathcal{D}_1)} + \|\nabla(w^z\psi)\|^2_{L^2(\mathcal{D}_1)} \right),
\end{aligned} \tag{4.3.19}$$

$$\begin{aligned}
\int_{\mathcal{D}_1} \frac{u^z}{r} w^r w^z \psi^2 dy &= \int_{\mathcal{D}_1} \left(\frac{(w^r\psi)^2}{2} + \frac{(w^z\psi)^2}{2} \right) \left(\frac{(u^z)^2}{2} + \frac{1}{2r^2} \right) dy \\
&\leq C(1 + \|u^z\|^2_{L^\infty(\mathcal{D}_1)}) \|(w^r, w^z)\|^2_{L^2(\mathcal{D}_1)},
\end{aligned} \tag{4.3.20}$$

$$\begin{aligned}
-2\int_{\mathcal{D}_1} u^z w^z \psi \partial_z(w^z\psi) dy &\leq 2\int_{\mathcal{D}_1} 4(u^z w^z\psi)^2 + \frac{|\partial_z(w^z\psi)|^2}{16} dy \\
&\leq 8\|u^z\|^2_{L^\infty(\mathcal{D}_1)} \|\psi\|^2_{L^\infty(\mathcal{D}_1)} \|w^z\|^2_{L^2(\mathcal{D}_1)} + \frac{1}{8}\|\nabla(w^z\psi)\|^2_{L^2(\mathcal{D}_1)}.
\end{aligned} \tag{4.3.21}$$

As before we used epsilon Young Inequality. By the above estimates and (4.3.16) we have

$$\begin{aligned}\|\nabla(w^z\psi)\|^2_{L^2(\mathcal{D}_1)} \le C\left(1+\|(u^r,u^z)\|^2_{L^\infty(\mathcal{D}_1)}\right)\|(w^r,w^z)\|^2_{L^2(\mathcal{D}_1)}\\ +\frac{1}{4}\left(\|\nabla(w^r\psi)\|^2_{L^2(\mathcal{D}_1)}+\|\nabla(w^z\psi)\|^2_{L^2(\mathcal{D}_1)}\right).\end{aligned} \tag{4.3.22}$$

Now we combine (4.3.15) and (4.3.22) together, group the $\|\nabla(w^r\psi)\|^2_{L2(\mathcal{D}_1)}+|\nabla(w^z\psi)\|^2_{L2(\mathcal{D}_1)}$ term to the left-hand side of the inequality, then use the fact that $\psi=1$ on $\mathcal{D}_2$ to obtain

$$\|(\nabla w^r,\nabla w^z)\|^2_{L^2(\mathcal{D}_2)} \le C\left(1+\|(u^r,u^z)\|^2_{L^\infty(\mathcal{D}_1)}\right)\|(w^r,w^z)\|^2_{L^2(\mathcal{D}_1)}. \tag{4.3.23}$$

Finally we work with (4.3.6):

$$\begin{aligned}&\int_{\mathcal{D}_1}\Big(|\nabla(w^\theta\psi)|^2+\frac{(w^\theta)^2\psi^2}{r^2}\Big)dy\\ &=\int_{\mathcal{D}_1}\Big((w^\theta)^2|\nabla\psi|^2-\frac{1}{2}\psi^2(u^r\partial_r+u^z\partial_z)(w^\theta)^2-\frac{u^r}{r}(w^\theta\psi)^2-2\frac{u^\theta}{r}w^rw^\theta\psi^2\Big)dy\\ &=\int_{\mathcal{D}_1}\Big((w^\theta)^2|\nabla\psi|^2+\frac{1}{2}(w^\theta)^2(u^r\partial_r+u^z\partial_z)\psi^2-\frac{u^r}{r}(w^\theta\psi)^2-2\frac{u^\theta}{r}w^rw^\theta\psi^2\Big)dy.\end{aligned} \tag{4.3.24}$$

And as before we estimate each integral:

$$\int_{\mathcal{D}_1}(w^\theta)^2|\nabla\psi|^2dy \le \|\nabla\psi\|^2_{L^\infty(\mathcal{D}_1)}\|w^\theta\|^2_{L^2(\mathcal{D}_1)},$$

$$\int_{\mathcal{D}_1}(w^\theta)^2(u^r\partial_r+u^z\partial_z)\psi^2dy \le \|\nabla\psi^2\|_{L^\infty(\mathcal{D}_1)}\|(u^r,u^z)\|_{L^\infty(\mathcal{D}_1)}\|w^\theta\|^2_{L^2(\mathcal{D}_1)},$$

$$\begin{aligned}\int_{\mathcal{D}_1}\frac{u^r}{r}(w^\theta\psi)^2dy &\le \|r^{-1}\|_{L^\infty(\mathcal{D}_1)}\|\psi\|^2_{L^\infty(\mathcal{D}_1)}\|u^r\|_{L^\infty(\mathcal{D}_1)}\|w^\theta\|^2_{L^2(\mathcal{D}_1)}\\ &\le C\|\psi\|^2_{L^\infty(\mathcal{D}_1)}\|u^r\|_{L^\infty(\mathcal{D}_1)}\|w^\theta\|^2_{L^2(\mathcal{D}_1)},\end{aligned}$$

$$\begin{aligned}\int_{\mathcal{D}_1}\frac{u^\theta}{r}w^rw^\theta\psi^2dy &\le \|r^{-1}\|_{L^\infty(\mathcal{D}_1)}\|\psi\|^2_{L^\infty(\mathcal{D}_1)}\|u^\theta\|_{L^\infty(\mathcal{D}_1)}\int_{\mathcal{D}_1}\frac{(w^r)^2}{2}+\frac{(w^\theta)^2}{2}dy\\ &\le C\|\psi\|^2_{L^\infty(\mathcal{D}_1)}\|u^\theta\|_{L^\infty(\mathcal{D}_1)}\|(w^r,w^\theta)\|^2_{L^2(\mathcal{D}_1)}.\end{aligned}$$

By the above estimates and (4.3.24) we have

$$\|\nabla w^\theta\|^2_{L^2(\mathcal{D}_2)} \le C\left(1+\|(u^r,u^\theta,u^z)\|_{L^\infty(\mathcal{D}_1)}\right)\|(w^r,w^\theta)\|^2_{L^2(\mathcal{D}_1)}. \tag{4.3.25}$$

Define the two-dimensional domains

$$\bar{\mathcal{D}}_1 = \left\{ (r,z) : \frac{1}{2} < r < \frac{3}{2}, -\frac{\pi}{\lambda} \le z \le \frac{\pi}{\lambda} \right\} \tag{4.3.26}$$

and

$$\bar{\mathcal{D}}_2 = \left\{ (r,z) : \frac{3}{4} < r < \frac{5}{4}, -\frac{\pi}{\lambda} \le z \le \frac{\pi}{\lambda} \right\}. \tag{4.3.27}$$

Note that since r is bounded in $\bar{\mathcal{D}}_1, \bar{\mathcal{D}}_2, \mathcal{D}_1$, and $\mathcal{D}_2$, we have that

$$\|f\|^2_{L^2(\bar{\mathcal{D}})} = \int_{\bar{\mathcal{D}}} |f|^2 drdz \le C \int_{\bar{\mathcal{D}}} |f|^2 rdrdz \le 2\pi C \int_{\mathcal{D}} |f|^2 rdrd\theta dz = C\|f\|^2_{L^2(\mathcal{D})}, \tag{4.3.28}$$

$$\|\nabla_2 f\|^2_{L^2(\bar{\mathcal{D}})} = \int_{\bar{\mathcal{D}}} |\nabla_2 f|^2 drdz \le C \int_{\bar{\mathcal{D}}} |\nabla_2 f|^2 rdrdz \le 2\pi C \int_{\mathcal{D}} |\nabla f|^2 rdrd\theta dz = C\|\nabla f\|^2_{L^2(\mathcal{D})}, \tag{4.3.29}$$

$$\|\Delta_2 f\|^2_{L^2(\bar{\mathcal{D}})} = \int_{\bar{\mathcal{D}}} |\Delta_2 f|^2 drdz \le C \int_{\bar{\mathcal{D}}} |\Delta_2 f|^2 rdrdz \le 2\pi C \int_{\mathcal{D}} |\Delta f|^2 rdrd\theta dz = C\|\Delta f\|^2_{L^2(\mathcal{D})}, \tag{4.3.30}$$

and

$$\|f\|_{L^\infty(\bar{\mathcal{D}})} = \|f\|_{L^\infty(\mathcal{D})} \tag{4.3.31}$$

for any axially symmetric function f. Then by applying the B-Z inequality and (4.3.25), we have

$$\begin{aligned}
\|w^\theta\|_{L^\infty(\mathcal{D}_2)} &= \|w^\theta\|_{L^\infty(\bar{\mathcal{D}}_2)} \\
&\le C\lambda^{1/2}\big(1 + \|w^\theta\|_{H^1(\bar{\mathcal{D}}_2)}\big) \log^{1/2}\left(e + \|\Delta_2 w^\theta\|_{L^2(\bar{\mathcal{D}}_2)}\right) \\
&= C\lambda^{1/2}\big(1 + \|w^\theta\|_{L^2(\bar{\mathcal{D}}_2)} + \|\nabla_2 w^\theta\|_{L^2(\bar{\mathcal{D}}_2)}\big) \log^{1/2}\left(e + \|\Delta_2 w^\theta\|_{L^2(\bar{\mathcal{D}}_2)}\right) \\
&\le C\lambda^{1/2}\big(1 + \|w^\theta\|_{L^2(\mathcal{D}_2)} + \|\nabla w^\theta\|_{L^2(\mathcal{D}_2)}\big) \log^{1/2}\left(e + C\|\Delta w^\theta\|_{L^2(\mathcal{D}_2)}\right) \\
&\le C\lambda^{1/2}\big(1 + (1 + \|(u^r, u^\theta, u^z)\|^{1/2}_{L^\infty(\mathcal{D}_1)})\|(w^r, w^\theta)\|_{L^2(\mathcal{D}_1)}\big) \\
&\quad \times \log^{1/2}\left(e + C\|\Delta w^\theta\|_{L^2(\mathcal{D}_1)}\right).
\end{aligned} \tag{4.3.32}$$

Similarly, we apply the B-Z inequality and (4.3.23), to obtain

$$
\begin{aligned}
\|(w^r, w^z)\|_{L^\infty(\mathcal{D}_2)} &= \|(w^r, w^z)\|_{L^\infty(\bar{\mathcal{D}}_2)} \\
&\leq C\lambda^{1/2}\big(1 + \|(w^r, w^z)\|_{H^1(\bar{\mathcal{D}}_2)}\big) \log^{1/2}\Big(e + \|\Delta(w^r, w^z)\|_{L^2(\bar{\mathcal{D}}_2)}\Big) \\
&\leq C\lambda^{1/2}\big(1 + (1 + \|(u^r, u^z)\|_{L^\infty(\mathcal{D}_1)})\|(w^r, w^z)\|_{L^2(\mathcal{D}_1)}\big) \\
&\quad \times \log^{1/2}\big(e + C\|\Delta(w^r, w^z)\|_{L^2(\mathcal{D}_1)}\big).
\end{aligned}
\tag{4.3.33}
$$

4.3.1.2 Un-Scaled Computations

Recall that all the calculations above took place in terms of the scaled functions $\tilde{u}, \tilde{w}$ and scaled variable $\tilde{x}$. Therefore, we will scale back to the domains:

$$
\mathcal{D}_{1,\lambda} = \left\{(r, \theta, z) : \frac{1}{2}\lambda < r < \frac{3}{2}\lambda,\ 0 \leq \theta \leq 2\pi,\ -\pi \leq z \leq \pi\right\} \tag{4.3.34}
$$

and

$$
\mathcal{D}_{2,\lambda} = \left\{(r, \theta, z) : \frac{3}{4}\lambda < r < \frac{5}{4}\lambda,\ 0 \leq \theta \leq 2\pi,\ -\pi \leq z \leq \pi\right\}. \tag{4.3.35}
$$

Based on the scaling we will have

$$
\|\tilde{u}\|_{L^\infty(\mathcal{D}_\cdot)} = \lambda\|u\|_{L^\infty(\mathcal{D}_{\cdot,\lambda})}, \tag{4.3.36}
$$

$$
\|\tilde{w}\|_{L^\infty(\mathcal{D}_\cdot)} = \lambda^2\|w\|_{L^\infty(\mathcal{D}_{\cdot,\lambda})}, \tag{4.3.37}
$$

$$
\|\tilde{w}\|_{L^2(\mathcal{D}_\cdot)} = \left(\int_{\mathcal{D}_\cdot} |\lambda^2 w(\lambda\tilde{x})|^2 d\tilde{x}\right)^{1/2} = \left(\int_{\mathcal{D}_{\cdot,\lambda}} \lambda^4 |w(x)| \frac{1}{\lambda^3} dx\right)^{1/2} = \lambda^{1/2}\|w\|_{L^2(\mathcal{D}_{\cdot,\lambda})}, \tag{4.3.38}
$$

$$
\|\nabla\tilde{w}\|_{L^2(\mathcal{D}_\cdot)} = \left(\int_{\mathcal{D}_\cdot} |\lambda^3 \nabla w(\lambda\tilde{x})|^2 d\tilde{x}\right)^{1/2} = \left(\int_{\mathcal{D}_{\cdot,\lambda}} \lambda^6 |\nabla w(x)| \frac{1}{\lambda^3} dx\right)^{1/2} = \lambda^{3/2}\|\nabla w\|_{L^2(\mathcal{D}_{\cdot,\lambda})}, \tag{4.3.39}
$$

and

$$\|\Delta\tilde{w}\|_{L^2(\mathcal{D}_\cdot)} = \left(\int_{\mathcal{D}_\cdot} |\lambda^4 \Delta w(\lambda\tilde{x})|^2 d\tilde{x}\right)^{1/2} = \left(\int_{\mathcal{D}_{\cdot,\lambda}} \lambda^8 |\Delta w(x)| \frac{1}{\lambda^3} dx\right)^{1/2} = \lambda^{5/2}\|\Delta w\|_{L^2(\mathcal{D}_{\cdot,\lambda})}. \tag{4.3.40}$$

Hence scaling back (4.3.23),(4.3.25), (4.3.32) and (4.3.33), we will obtain

$$\begin{aligned}\lambda^{3/2}\|(\nabla w^r, \nabla w^z)\|_{L^2(\mathcal{D}_{2,\lambda})} &\le C\left(1+\lambda\|(u^r,u^z)\|_{L^\infty(\mathcal{D}_{1,\lambda})}\right)\lambda^{1/2}\|(w^r,w^z)\|_{L^2(\mathcal{D}_{1,\lambda})} \\ &\le C\lambda^{3/2}\|(u^r,u^z)\|_{L^\infty(\mathcal{D}_{1,\lambda})}\|(w^r,w^z)\|_{L^2(\mathcal{D}_{1,\lambda})},\end{aligned} \tag{4.3.41}$$

$$\begin{aligned}\lambda^{3/2}\|\nabla w^\theta\|_{L^2(\mathcal{D}_{2,\lambda})} &\le C\left(1+\lambda^{1/2}\|(u^r,u^z)\|^{1/2}_{L^\infty(\mathcal{D}_{1,\lambda})}\right)\lambda^{1/2}\|(w^r,w^\theta)\|_{L^2(\mathcal{D}_{1,\lambda})} \\ &\le C\lambda\|(u^r,u^z)\|_{L^\infty(\mathcal{D}_{1,\lambda})}\|(w^r,w^\theta)\|_{L^2(\mathcal{D}_{1,\lambda})},\end{aligned} \tag{4.3.42}$$

$$\begin{aligned}\lambda^2\|w^\theta\|_{L^\infty(\mathcal{D}_{2,\lambda})} &\le C\lambda^{1/2}\big(1+(1+\lambda^{1/2}\|(u^r,u^\theta,u^z)\|^{1/2}_{L^\infty(\mathcal{D}_{1,\lambda})})\lambda^{1/2}\|(w^r,w^\theta)\|_{L^2(\mathcal{D}_{1,\lambda})}\big) \\ &\quad\times \log^{1/2}\left(e + C\lambda^{5/2}\|\Delta w^\theta\|_{L^2(\mathcal{D}_{1,\lambda})}\right) \\ &\le C\lambda^{1/2}\big(\lambda^{1/2}\|(u^r,u^\theta,u^z)\|^{1/2}_{L^\infty(\mathcal{D}_{1,\lambda})}\lambda^{1/2}\|(w^r,w^\theta)\|_{L^2(\mathcal{D}_{1,\lambda})}\big) \\ &\quad\times \log^{1/2}\left(C\lambda\|\Delta w^\theta\|_{L^2(\mathcal{D}_{1,\lambda})}\right),\end{aligned} \tag{4.3.43}$$

and

$$\begin{aligned}\lambda^2\|(w^r,w^z)\|_{L^\infty(\mathcal{D}_{2,\lambda})} &\le C\lambda^{1/2}\big(1+(1+\lambda\|(u^r,u^z)\|_{L^\infty(\mathcal{D}_{1,\lambda})})\lambda^{1/2}\|(w^r,w^\theta)\|_{L^2(\mathcal{D}_{1,\lambda})}\big) \\ &\quad\times \log^{1/2}\left(e + C\lambda^{5/2}\|\Delta(w^r,w^z)\|_{L^2(\mathcal{D}_{1,\lambda})}\right) \\ &\le C\lambda^{1/2}\big(\lambda\|(u^r,u^z)\|_{L^\infty(\mathcal{D}_{1,\lambda})}\lambda^{1/2}\|(w^r,w^\theta)\|_{L^2(\mathcal{D}_{1,\lambda})}\big) \\ &\quad\times \log^{1/2}\left(C\lambda\|\Delta(w^r,w^z)\|_{L^2(\mathcal{D}_{1,\lambda})}\right).\end{aligned} \tag{4.3.44}$$

Since u is smooth, bounded, and has bounded higher-order derivatives, we have that $\|\Delta(w^r, w^\theta, w^z)\|_{L^2(\mathcal{D}_{1,\lambda})}$ is finite and so we have

$$\|(\nabla w^r, \nabla w^z)\|_{L^2(\mathcal{D}_{2,\lambda})} \leq C\|(u^r, u^z)\|_{L^\infty(\mathcal{D}_{1,\lambda})}\|(w^r, w^z)\|_{L^2(\mathcal{D}_{1,\lambda})}, \tag{4.3.45}$$

$$\|\nabla w^\theta\|_{L^2(\mathcal{D}_{2,\lambda})} \leq C\lambda^{-1/2}\|(u^r, u^z)\|_{L^\infty(\mathcal{D}_{1,\lambda})}\|(w^r, w^\theta)\|_{L^2(\mathcal{D}_{1,\lambda})}, \tag{4.3.46}$$

$$\|w^\theta\|_{L^\infty(\mathcal{D}_{2,\lambda})} \leq C\lambda^{-1/2}\|(u^r, u^\theta, u^z)\|^{1/2}_{L^\infty(\mathcal{D}_{1,\lambda})}\|(w^r, w^\theta)\|_{L^2(\mathcal{D}_{1,\lambda})} \log^{1/2}(\lambda). \tag{4.3.47}$$

and

$$\|(w^r, w^z)\|_{L^\infty(\mathcal{D}_{2,\lambda})} \leq C\|(u^r, u^z)\|_{L^\infty(\mathcal{D}_{1,\lambda})}\|(w^r, w^\theta)\|_{L^2(\mathcal{D}_{1,\lambda})} \log^{1/2}(\lambda). \tag{4.3.48}$$

Now using the fact that $\|(u^r, u^\theta, u^z)\|_{L^\infty(\mathbb{R}^2\times[-\pi,\pi])}$ is finite and (4.1.2), we have our first decay of w^θ:

$$\|w^\theta\|_{L^\infty(\mathcal{D}_{2,\lambda})} \leq C\lambda^{\frac{-1+\alpha}{2}} \log^{1/2}(\lambda). \tag{4.3.49}$$

We recall that although the above computations were done for points that are of the form $x = (r_0, 0, 0)$ where $|x'| = \lambda$ is large, all the computations can be generalized to any point x in $\mathbb{R}^3$ such that $|x'| = \lambda$ is large.

4.3.2 First Decay of u and (w^r, w^z)

Define

$$\mathcal{D}_{3,\lambda} = \left\{(r, \theta, z) : \frac{7}{8}\lambda < r < \frac{9}{8}\lambda,\ 0 \leq \theta \leq 2\pi,\ -\pi \leq z \leq \pi\right\}. \tag{4.3.50}$$

Note that $\mathcal{D}_{3,\lambda} \subseteq \mathcal{D}_{2,\lambda}$. Let $x = (r\cos\theta, r\sin\theta, z)$, where $|x'| = r$ is large. This will be our λ from now on. Since we'll be working with the Green's function, we let $y = (\rho\cos\phi, \rho, \sin\phi, \ell)$ be the variable we integrate with respect with in the representation formula. Let $\psi(x')$

be a cutoff function that is independent of z such that $\sup(\psi) \subseteq \mathcal{D}_{2,r}$, $\psi = 1$ on $\mathcal{D}_{3,r}$, $|\psi| \leq C, |\nabla\psi| \leq \frac{C}{r}$, and $|\nabla^2\psi| \leq \frac{C}{r^2}$.

Note that for any smooth, divergence free vector field f, we have

$$-\Delta(f\psi) = \psi\nabla \times (\nabla \times f) - 2(\nabla\psi \cdot \nabla)f - f\Delta\psi. \tag{4.3.51}$$

In particular, for $b = u^r e_r + u^z e_z$, we have $\nabla \times b = w^\theta e_\theta$ and so we obtain

$$-\Delta(b\psi) = \psi\nabla \times (w^\theta e_\theta) - 2(\nabla\psi \cdot \nabla)b - b\Delta\psi. \tag{4.3.52}$$

The terms on the left-hand side of the equation all have mean zero on $[-\pi, \pi]$ because of our assumption that $\int_{-\pi}^{\pi} u^\theta dz = \int_{-\pi}^{\pi} u^z dz = 0$. Therefore, by the representation formula for Poisson's equation, we have using the Green's function on $\mathbb{R}^2 \times S^1$ that

$$\begin{aligned}(b\psi)(x) = &\int_{S^1}\int_{\mathbb{R}^2} G(x,y)\psi\nabla \times (w^\theta e_\theta)dy - 2\int_{S^1}\int_{\mathbb{R}^2} G(x,y)(\nabla\psi \cdot \nabla)bdy \\ &- \int_{S^1}\int_{\mathbb{R}^2} G(x,y)(\Delta\psi)bdy.\end{aligned} \tag{4.3.53}$$

Component-wise we see that on $\mathcal{D}_{3,r}$

$$\begin{aligned}u^r(x) = &\int_{S^1}\int_{\mathbb{R}^2} G(x,y)\psi\nabla \times (w^\theta e_\theta) \cdot e_r dy - 2\int_{S^1}\int_{\mathbb{R}^2} G(x,y)(\nabla\psi \cdot \nabla)b \cdot e_r dy \\ &- \int_{S^1}\int_{\mathbb{R}^2} G(x,y)(\Delta\psi)b \cdot e_r dy \\ &= \int_{-\pi}^{\pi}\int_0^{2\pi}\int_0^\infty G(x,y)\psi\partial_\ell w^\phi \cos(\phi - \theta)\rho d\rho d\phi d\ell \\ &- 2\int_{-\pi}^{\pi}\int_0^{2\pi}\int_0^\infty G(x,y)\partial_\rho\psi\partial_\rho u^\rho \cos(\phi - \theta)\rho d\rho d\phi d\ell \\ &- \int_{-\pi}^{\pi}\int_0^{2\pi}\int_0^\infty G(x,y)\left(\partial_\rho^2\psi + \frac{1}{\rho}\partial_\rho\psi\right) u^\rho \cos(\phi - \theta)\rho d\rho d\phi d\ell.\end{aligned} \tag{4.3.54}$$

All calculations are justified because $\int_{-\pi}^{\pi} \partial_\ell w^\phi d\ell = w^\phi\big|_{-\pi}^{\pi} = 0$, $\int_{-\pi}^{\pi} \partial_\rho u^\rho d\ell = 0$ and $\int_{-\pi}^{\pi} u^\rho = 0$.

To simplify notation, we will assume $\theta = 0$. This is because our solution is axially-symmetric and all the evaluations of the integrals are independent of θ. Since ψ is independent of ℓ, we can do an integration by parts with respect to ℓ to obtain

$$\begin{aligned}
u^r(x) &= \int_{-\pi}^{\pi}\int_0^{2\pi}\int_0^{\infty} G(x,y)\psi\partial_\ell w^\phi \cos\phi\rho d\rho d\phi d\ell \\
&\quad - 2\int_{-\pi}^{\pi}\int_0^{2\pi}\int_0^{\infty} G(x,y)\partial_\rho\psi\partial_\rho u^\rho \cos\phi\rho d\rho d\phi d\ell \\
&\quad - \int_{-\pi}^{\pi}\int_0^{2\pi}\int_0^{\infty} G(x,y)\left(\partial_\rho^2\psi + \frac{1}{\rho}\partial_\rho\psi\right) u^\rho \cos\phi\rho d\rho d\phi d\ell \\
&= -\int_{-\pi}^{\pi}\int_0^{\infty}\left(\int_0^{2\pi}\partial_\ell G(x,y)\cos(\phi-\theta)d\phi\right)\psi w^\phi \rho d\rho d\ell \qquad (4.3.55)\\
&\quad - 2\int_{-\pi}^{\pi}\int_0^{\infty}\left(\int_0^{2\pi} G(x,y)\cos\phi d\phi\right)\partial_\rho\psi\partial_\rho u^\rho \rho d\rho d\ell \\
&\quad - \int_{-\pi}^{\pi}\int_0^{\infty}\left(\int_0^{2\pi} G(x,y)\cos\phi d\phi\right)\left(\partial_\rho^2\psi + \frac{1}{\rho}\partial_\rho\psi\right) u^\rho \rho d\rho d\ell \\
&= I_1 + I_2 + I_3.
\end{aligned}$$

We will now estimate each integral. We first start with

$$I_1 = \int_{-\pi}^{\pi}\int_0^{\infty}\left(\int_0^{2\pi}\partial_\ell G(x,y)\cos(\phi)d\phi\right)\psi w^\phi \rho d\rho d\ell. \qquad (4.3.56)$$

Because of the cutoff function ϕ, this integral is actually on the region $[3/4r, 5/4r]\times[0,2\pi]\times[-\pi,\pi] = \mathcal{D}_{2,r}$. Because we are working in the region $|\rho - r| \leq \frac{1}{4}r$, we can use the following estimate (3.2.19) for the Green's function from on $\mathbb{R}^2\times[-\pi,\pi]$:

$$\int_0^{2\pi}\partial_\ell G(x,y)d\phi \leq C\frac{e^{-c_0|\rho-r|}}{\rho(|\rho-r|+|z-\ell|)}.$$

Then, using this estimate, the Tonelli theorem to switch the integration order, we have

$$
\begin{aligned}
|I_1| &\leq \|w^\phi\|_{L^\infty(\mathcal{D}_{2,r})}\|\psi\|_{L^\infty}\int_{-\pi}^{\pi}\int_{|\rho-r|\leq\frac{1}{4}r}\frac{e^{-c_0|\rho-r|}}{\rho(|\rho-r|+|z-\ell|)}\rho d\rho d\ell \\
&\leq C|w^\phi\|_{L^\infty(\mathcal{D}_{2,r})}\int_{|\rho-r|\leq\frac{1}{4}r}\int_{-\pi}^{\pi}\frac{1}{|\rho-r|+|z-\ell|}d\ell e^{-c_0|\rho-r|}d\rho \\
&= C\|w^\phi\|_{L^\infty(\mathcal{D}_{2,r})}\int_{|\rho-r|\leq\frac{1}{4}r}\int_{-\pi}^{\pi}\frac{1}{|\rho-r|}\left(\frac{1}{1+\frac{|z-\ell|}{|\rho-r|}}\right)d\ell e^{-c_0|\rho-r|}d\rho \\
&\leq C\|w^\phi\|_{L^\infty(\mathcal{D}_{2,r})}\int_{|\rho-r|\leq\frac{1}{4}r}\ln\left(1+\frac{2\pi}{|\rho-r|}+\frac{\pi^2}{|\rho-r|^2}\right)e^{-c_0|\rho-r|}d\rho \\
&= C\|w^\phi\|_{L^\infty(\mathcal{D}_{2,r})}J.
\end{aligned}
\tag{4.3.57}
$$

We claim that

$$
\begin{aligned}
J &= \int_{|\rho-r|\leq\frac{1}{4}r}\ln\left(1+\frac{2\pi}{|\rho-r|}+\frac{\pi^2}{|\rho-r|^2}\right)e^{-c_0|\rho-r|}d\rho \\
&= \int_{\frac{3}{4}r}^{\frac{5}{4}r}\ln\left(1+\frac{2\pi}{|\rho-r|}+\frac{\pi^2}{|\rho-r|^2}\right)e^{-c_0|\rho-r|}d\rho
\end{aligned}
\tag{4.3.58}
$$

can be controlled by an absolute constant independent of r.

By a change of variables we have

$$
\begin{aligned}
\int_{\frac{3}{4}r}^{r}\ln\left(1+\frac{2\pi}{|\rho-r|}+\frac{\pi^2}{|\rho-r|^2}\right)e^{-c_0|\rho-r|}\,d\rho &= \int_{\frac{3}{4}r}^{r}\ln\left(1+\frac{2\pi}{r-\rho}+\frac{\pi^2}{|r-\rho|^2}\right)e^{-c_0(r-\rho)}\,d\rho \\
&= \int_0^{\frac{1}{4}r}\ln\left(1+\frac{2\pi}{u}+\frac{\pi^2}{u^2}\right)e^{-c_0u}\,\mathrm{d}u,
\end{aligned}
$$

and

$$
\begin{aligned}
\int_{r}^{\frac{5}{4}r}\ln\left(1+\frac{2\pi}{|\rho-r|}+\frac{\pi^2}{|\rho-r|^2}\right)e^{-c_0|\rho-r|}d\rho &= \int_{r}^{\frac{5}{4}r}\ln\left(1+\frac{2\pi}{\rho-r}+\frac{\pi^2}{|\rho-r|^2}\right)e^{-c_0(\rho-r)}d\rho \\
&= \int_0^{\frac{1}{4}r}\ln\left(1+\frac{2\pi}{u}+\frac{\pi^2}{u^2}\right)e^{-c_0u}du,
\end{aligned}
$$

so

$$
J = 2\int_0^{\frac{1}{4}r}\ln\left(1+\frac{2\pi}{u}+\frac{\pi^2}{u^2}\right)e^{-c_0u}du \leq 2\int_0^{\infty}\ln\left(1+\frac{2\pi}{u}+\frac{\pi^2}{u^2}\right)e^{-c_0u}du. \tag{4.3.59}
$$

If $u \in (0,1]$ then $\ln\left(1+\frac{2\pi}{u}+\frac{\pi^2}{u^2}\right)e^{-c_0 u} \le \ln\left(1+\frac{C}{u^2}\right)$ so

$$\begin{aligned}
\int_\varepsilon^1 \ln\left(1+\frac{2\pi}{u}+\frac{\pi^2}{u^2}\right)e^{-c_0 u}du &\le \int_\varepsilon^1 \ln\left(1+\frac{C}{u^2}\right)du \\
&= \ln\left(1+\frac{C}{u^2}\right)u + \frac{2}{\sqrt{C}}\arctan\left(\frac{u}{\sqrt{C}}\right)\Big|_\varepsilon^1 \\
&= \ln(1+C) + \frac{2}{\sqrt{C}}\arctan\left(\frac{1}{\sqrt{C}}\right) - \ln\left(1+\frac{C}{\varepsilon}\right)\varepsilon.
\end{aligned}$$

Therefore,

$$\lim_{\varepsilon\to 0^+}\int_\varepsilon^1 \ln\left(1+\frac{C}{u^2}\right)du = \ln(1+C) + \frac{2}{\sqrt{C}}\arctan\left(\frac{1}{\sqrt{C}}\right), \tag{4.3.60}$$

so by the monotone convergence theorem for sequences

$$\int_0^1 \ln\left(1+\frac{2\pi}{u}+\frac{\pi^2}{u^2}\right)e^{-c_0 u}\,du \le \ln(1+C) + \frac{2}{\sqrt{C}}\arctan\left(\frac{1}{\sqrt{C}}\right). \tag{4.3.61}$$

If $u \in [1,\infty)$, then $\ln\left(1+\frac{2\pi}{u}+\frac{\pi^2}{u^2}\right)e^{-c_0 u} \le \ln\left(1+\frac{C}{u}\right)e^{-c_0 u}$, so

$$\begin{aligned}
\int_1^\infty \ln\left(1+\frac{2\pi}{u}+\frac{\pi^2}{u^2}\right)e^{-c_0 u}du &\le \int_1^\infty \ln\left(1+\frac{C}{u}\right)e^{-c_0 u}du \\
&\le \sum_{N=1}^\infty \ln\left(1+\frac{C}{N}\right)e^{-c_0 N} \qquad (4.3.62) \\
&< \infty,
\end{aligned}$$

where the sum is finite by the ratio test. Hence $J \le C$ and we therefore have

$$|I_1| \le C\|w^\phi\|_{L^\infty(\mathcal{D}_{2,r})} \tag{4.3.63}$$

For $I_2 = \int_{-\pi}^{\pi}\int_0^\infty\left(\int_0^{2\pi} G(x,y)\cos(\phi)d\phi\right)\partial_\rho\psi\partial_\rho u^\rho \rho d\rho d\ell$, we use the fact that $|\psi| \le \frac{C}{r}$, that the integration takes place on the region $\frac{1}{8}r \le |p-r| \le \frac{1}{4}r$ or $(\bar{\mathcal{D}}_{3,r})^c \cap \bar{\mathcal{D}}_{2,r}$

and the estimate (3.2.20) to obtain

$$
\begin{aligned}
|I_2| &\leq \int_{-\pi}^{\pi} \int_{\frac{1}{8}r \leq |\rho - r| \leq \frac{1}{4}r} e^{-c_0|p-r|} |\partial_\rho \psi| |\partial_\rho u^\rho| \rho d\rho d\ell \\
&\leq C \|u^\rho\|_{L^\infty((\bar{\mathcal{D}}_{3,r})^c \cap \bar{\mathcal{D}}_{2,r})} \int_{-\pi}^{\pi} \int_{\frac{1}{8}r \leq |\rho - r| \leq \frac{1}{4}r} e^{-c_0|\rho - r|} d\rho d\ell \\
&= C \int_{\frac{1}{8}r \leq |\rho - r| \leq \frac{1}{4}r} e^{-c_0|\rho - r|} d\rho. \\
&\leq Ce^{-\frac{c_0}{8}r} \\
&\leq Ce^{-\frac{c_0}{8}r}.
\end{aligned}
\tag{4.3.64}
$$

Note that the integration along $\frac{1}{8}r \leq |p - r| \leq \frac{1}{4}r$ can be split into two intervals: $[\frac{3}{4}r, \frac{7}{8}r]$ if $r < \rho$ and $[\frac{9}{8}r, \frac{5}{4}r]$ if $\rho < r$. In either case, both integrals can be bounded by $\dfrac{e^{-\frac{c_0}{8}r}}{c_0}$.

For $I_3 = \displaystyle\int_{-\pi}^{\pi} \int_0^\infty \left(\int_0^{2\pi} G(x,y) \cos(\phi) d\phi \right) \left(\partial_\rho^2 \psi + \frac{1}{\rho} \partial_\rho \psi \right) u^\rho \rho d\rho d\ell$, we have that the integration is on $\frac{1}{8}r \leq |p - r| \leq \frac{1}{4}r$, so we can use the estimate (3.2.20). Also, by properties of the cutoff function, we have $|\partial_\rho \psi| \leq \frac{C}{r}, |\partial_\rho^2 \psi| \leq \frac{C}{r^2}$ and thus for large r

$$
\begin{aligned}
|I_3| &\leq \int_{-\pi}^{\pi} \int_{\frac{1}{8}r \leq |\rho - r| \leq \frac{1}{4}r} e^{-c_0|p-r|} | \left(\partial_\rho^2 \psi + \frac{1}{\rho} \partial_\rho \psi \right) u^\rho \rho d\rho d\ell \\
&\leq C \sup_{\frac{1}{8}r \leq |\rho - r| \leq \frac{1}{4}r \times [-\pi, \pi]} |\partial_\rho u^\rho| \int_{\frac{1}{8}r \leq |\rho - r| \leq \frac{1}{4}r} e^{-c_0|p-r|} \left(\frac{1}{r^2} + \frac{1}{r} \right) \rho d\rho \\
&\leq C \int_{\frac{1}{8}r \leq |\rho - r| \leq \frac{1}{4}r} e^{-c_0|p-r|} d\rho \\
&\leq Ce^{-\frac{c_0}{8}r}.
\end{aligned}
\tag{4.3.65}
$$

Hence by (4.3.63) - (4.3.65), and using (4.3.49) we have for r sufficiently large,

$$
\|u^r\|_{L^\infty(\mathcal{D}_{3,r})} \leq Cr^{\frac{-1+\alpha}{2}} \log^{1/2} r. \tag{4.3.66}
$$

Now we return to (4.3.53) and look at the representation for u^z on $\mathcal{D}_{3,r}$:

$$
\begin{aligned}
u^z &= \int_{-\pi}^{\pi}\int_0^{2\pi}\int_0^{\infty} G(x,y)\psi\partial_\rho(\rho w^\phi)d\rho d\phi d\ell - 2\int_{-\pi}^{\pi}\int_0^{2\pi}\int_0^{\infty} G(x,y)\partial_\rho\psi\partial_\rho u^z \rho d\rho d\phi d\ell \\
&\quad - \int_{-\pi}^{\pi}\int_0^{2\pi}\int_0^{\infty} G(x,y)(\partial_\rho^2\psi + \frac{1}{\rho}\partial_\rho\psi)u^z\rho d\rho d\phi d\ell.
\end{aligned}
$$

As mentioned before, because of the assumption that integration on $[-\pi,\pi]$ of u^z is zero, we have that $\int_{-\pi}^{\pi} w^\theta dz = \int_{-\pi}^{\pi}\partial_z u^r - \partial_r u^z dz = 0$ so the representation is justified. By performing an integration by parts and using the axially symmetric condition we have that

$$
\begin{aligned}
u^z &= \int_{-\pi}^{\pi}\int_0^{2\pi}\int_0^{\infty} G(x,y)\psi\partial_\rho(\rho w^\phi)d\rho d\phi d\ell - 2\int_{-\pi}^{\pi}\int_0^{2\pi}\int_0^{\infty} G(x,y)\partial_\rho\psi\partial_\rho u^z \rho d\rho d\phi d\ell \\
&\quad - \int_{-\pi}^{\pi}\int_0^{2\pi}\int_0^{\infty} G(x,y)(\partial_\rho^2\psi + \frac{1}{\rho}\partial_\rho\psi)u^z\rho d\rho d\phi d\ell \\
&= -\int_{-\pi}^{\pi}\int_0^{\infty}\left(\int_0^{2\pi}\partial_\rho G(x,y)d\phi\right)\psi w^\phi\rho d\rho d\ell - \int_{-\pi}^{\pi}\int_0^{\infty}\left(\int_0^{2\pi} G(x,y)d\phi\right)\partial_\rho\psi w^\phi\rho d\rho d\ell \\
&\quad - 2\int_{-\pi}^{\pi}\int_0^{\infty}\left(\int_0^{2\pi} G(x,y)d\phi\right)\partial_\rho\psi\partial_\rho u^z\rho d\rho d\ell \\
&\quad - \int_{-\pi}^{\pi}\int_0^{\infty}\left(\int_0^{2\pi} G(x,y)d\phi\right)(\partial_\rho^2\psi + \frac{1}{\rho}\partial_\rho\psi)u^z\rho d\rho d\ell \\
&= J_1 + J_2 + J_3 + J_4.
\end{aligned}
$$

We now estimate each integral. For $J_1 = \int_{-\pi}^{\pi}\int_0^{\infty}\left(\int_0^{2\pi}\partial_\rho G(x,y)\,\mathrm{d}\phi\right)\psi w^\phi\rho\mathrm{d}\rho\mathrm{d}\ell$ we do an estimate similar to the estimate we did for I_1 to get

$$
\begin{aligned}
|J_1| &\le C\|\psi\|_{L^\infty}\|w^\phi\|_{L^\infty(\mathcal{D}_{2,r})} \\
&\le C\|w^\phi\|_{L^\infty(\mathcal{D}_{2,\lambda})}.
\end{aligned} \tag{4.3.67}
$$

For $J_2 = \int_{-\pi}^{\pi}\int_0^{\infty}\left(\int_0^{2\pi} G(x,y)\mathrm{d}\phi\right)\partial_\rho\psi w^\phi\rho\mathrm{d}\rho\mathrm{d}\ell$ we do an estimate similar to the estimate we did for I_2 to get

$$
\begin{aligned}
|J_2| &\le C\|\partial_\rho\psi\|_{L^\infty}\|w^\phi\|_{L^\infty(\mathcal{D}_{2,r})}e^{-\frac{c_0}{8}r} \\
&\le \|w^\phi\|_{L^\infty(\mathcal{D}_{2,r})}e^{-\frac{c_0}{8}r}.
\end{aligned} \tag{4.3.68}
$$

For $J_3 = \int_{-\pi}^{\pi}\int_0^{\infty}\left(\int_0^{2\pi} G(x,y)\mathrm{d}\phi\right)\partial_\rho\psi\partial_\rho u^\ell \rho\mathrm{d}\rho\mathrm{d}\ell$ we do an estimate similar to the estimate we did for I_2 to get

$$|J_3| \leq C\|\partial_\rho u^\ell\|_{L^\infty(\mathcal{D}_{2,r})}e^{-\frac{c_0}{8}r}. \tag{4.3.69}$$

For $J_4 = \int_{-\pi}^{\pi}\int_0^{\infty}\left(\int_0^{2\pi} G(x,y)\mathrm{d}\phi\right)\left(\partial_\rho^2\psi + \frac{1}{\rho}\partial_\rho\psi\right)u^\ell\rho\mathrm{d}\rho\mathrm{d}\ell$ we do the same kind of estimate as in I_3 to get

$$|J_4| \leq C\|\partial_\rho\psi\|_{L^\infty}\|u^\ell\|_{L^\infty(\mathcal{D}_{2,r})}e^{-\frac{c_0}{8}r}. \tag{4.3.70}$$

By combining (4.3.67) - (4.3.70) and (4.3.49), we have for r sufficiently large,

$$\|u^z\|_{L^\infty(\mathcal{D}_{3,r})} \leq Cr^{\frac{-1+\alpha}{2}}\log^{1/2} r. \tag{4.3.71}$$

Define

$$\mathcal{D}_{4,\lambda} = \left\{(r,\theta,z) : \frac{15}{16}\lambda < r < \frac{17}{16}\lambda,\ 0 \leq \theta \leq 2\pi,\ -\pi \leq z \leq \pi\right\}. \tag{4.3.72}$$

Note that $\mathcal{D}_{4,\lambda} \subseteq \mathcal{D}_{3,\lambda}$.

By following the same calculation that yielded (4.3.48) on the domains $\mathcal{D}_{3,\lambda}$ and $\mathcal{D}_{4,\lambda}$ instead of the domains $\mathcal{D}_{1,\lambda}$ and $\mathcal{D}_{2,\lambda}$, we can obtain similar estimates on the domains $\mathcal{D}_{3,\lambda}$ and $\mathcal{D}_{4,\lambda}$. If we use the decay of u^r and u^z, then we can substitute (4.3.66) and (4.3.71) and use the D-condition (4.1.2) to obtain the following decay for w^r and w^z:

$$\begin{aligned}
\|(w^r,w^z)\|_{L^\infty(\mathcal{D}_{4,r})} &\leq C\|(u^r,u^z)\|_{L^\infty(\mathcal{D}_{3,r})}\|(w^r,w^\theta)\|_{L^2(\mathcal{D}_{3,r})}\log^{1/2}(r). \\
&\leq Cr^{\frac{-1+\alpha}{2}}r^{\frac{\alpha}{2}}\log^{1/2} r \\
&= Cr^{\frac{-1+2\alpha}{2}}\log^{1/2} r.
\end{aligned} \tag{4.3.73}$$

Now since we have $\int_{-\pi}^{\pi} u^\theta dz = 0$, we have by the mean value theorem that for any r, we can find a z_0 such that $u^\theta(r,z_0) = 0$. And since $\partial_z u^\theta = -w^r$ we have

$$
\begin{aligned}
|u^\theta(r,z)| &= \left| \int_{z_0}^{z} \partial_\ell u^\theta d\ell \right| \\
&\le C|\partial_z u^\theta| \\
&\le C\|w^r\|_{L^\infty(\mathcal{D}_{2,r})} \\
&\le Cr^{\frac{-1+2\alpha}{2}} \log^{1/2} r.
\end{aligned} \tag{4.3.74}
$$

Therefore,

$$
\|(u^r, u^\theta, u^z)\|_{L^\infty(\mathcal{D}_{4,r})} \le Cr^{\frac{-1+2\alpha}{2}} \log^{1/2} r. \tag{4.3.75}
$$

4.4 Almost $1-2\alpha$ Decay by Iteration.

We will repeat the previous process in smaller and smaller domains to improve the decay of u and w.

Let

$$
\mathcal{S}_n = \left\{ (\rho, z) : |\rho - r| \le \frac{r}{2^{n+2}}, -\pi \le z \le \pi \right\}.
$$

Note that $\mathcal{S}_0, \mathcal{S}_1$, and $\mathcal{S}_2$ correspond to $\mathcal{D}_{2,r}, \mathcal{D}_{3,r}$, and $\mathcal{D}_{4,r}$ respectively.

We will iterate in the following way:

1. Use the decay on w^θ to improve the decay of u^r, u^z by means of the Green's function.

2. Use the decay of u^r, u^z to improve the decay of w^r, w^z by means of B-Z inequality.

3. Use the decay of w^r, w^z to improve the decay of u^θ by means of the mean zero condition.

4. Use the improved decay of u^r, u^θ, u^z to improve the decay of w^θ.

5. Repeat.

We need to pay close attention when we do the iteration because every time we use (4.1.2), we obtain a $r^{\alpha/2}$ term. In particular, this occurs whenever we use the B-Z inequality (4.2.2) because we need to use the condition (4.1.2) to bound the L^2 norm of the vorticities.

Specifically, we have the following relationships:

$$\|(u^r, u^z)\|_{L^\infty(\mathcal{S}_{n+1})} \leq C_n \|w^\theta\|_{L^\infty(\mathcal{S}_n)}, \tag{4.4.1}$$

$$\|(w^r, w^z)\|_{L^\infty(\mathcal{S}_{n+2})} \leq C_{n+1} \|(u^r, u^z)\|_{L^\infty(\mathcal{S}_{n+1})} \|(w^r, w^\theta)\|_{L^2(\mathcal{S}_{n+1})} \log^{1/2} r, \tag{4.4.2}$$

$$\|u^\theta\|_{L^\infty(\mathcal{S}_{n+2})} \leq C_{n+2} \|(w^r, w^\theta)\|_{L^\infty(\mathcal{S}_{n+2})}, \tag{4.4.3}$$

$$\|w^\theta\|_{L^\infty(\mathcal{S}_{n+3})} \leq C_{n+3} r^{-1/2} \|(u^r, u^\theta, u^z)\|^{1/2}_{L^\infty(\mathcal{S}_{n+2})} \|(w^r, w^\theta)\|_{L^2(\mathcal{S}_{n+2})} \log^{1/2} r, \tag{4.4.4}$$

where $\lim\limits_{n\to\infty} C_n = \infty$.

We begin the process by remembering that $\|w^\theta\|_{L^\infty(\mathcal{S}_0)} \leq C_0 r^{\frac{-1+\alpha}{2}} \log^{1/2} r$. After iterating n times we get

$$\|w^\theta\|_{L^\infty(\mathcal{S}_{3n})} \leq A_n r^{\frac{-\sum_{i=0}^{n} 2^i + \left(2^n + \sum_{i=1}^{n} 2^i\right)\alpha}{2^{n+1}}} (\log r)^{\frac{2^n + \sum_{i=1}^{n} 2^i}{2^n + 1}} \tag{4.4.5}$$

$$\|(u^r, u^z)\|_{L^\infty(\mathcal{S}_{3n+1})} \leq B_n r^{\frac{-\sum_{i=0}^{n} 2^i + \left(2^n + \sum_{i=1}^{n} 2^i\right)\alpha}{2^{n+1}}} (\log r)^{\frac{2^n + \sum_{i=1}^{n} 2^i}{2^{n+1}}}, \tag{4.4.6}$$

$$\begin{aligned}\|(w^r, w^z)\|_{L^\infty(\mathcal{S}_{3n+2})} &\leq D_n r^{\frac{-\sum_{i=0}^{n} 2^i + \left(2^n + \sum_{i=1}^{n} 2^i\right)\alpha}{2^{n+1}}} (\log r)^{\frac{2^n + \sum_{i=1}^{n} 2^i}{2^{n+1}}} \cdot r^{\alpha/2} (\log r)^{1/2} \\ &= D_n r^{\frac{-\sum_{i=0}^{n} 2^i + \sum_{i=1}^{n+1} 2^i \alpha}{2^{n+1}}} (\log r)^{\frac{\sum_{i=1}^{n+1} 2^i}{2^{n+1}}},\end{aligned} \tag{4.4.7}$$

$$\|u^\theta\|_{L^\infty(\mathcal{S}_{3n+2})} \leq D_n r^{\frac{-\sum_{i=0}^{n} 2^i + \sum_{i=1}^{n+1} 2^i \alpha}{2^{n+1}}} (\log r)^{\frac{\sum_{i=1}^{n+1} 2^i}{2^{n+1}}}, \tag{4.4.8}$$

$$\|(u^r, u^\theta, u^z)\|_{L^\infty(\mathcal{S}_{3n+2})} \leq 2\pi D_n r^{\frac{-\sum_{i=0}^{n} 2^i + \sum_{i=1}^{n+1} 2^i \alpha}{2^{n+1}}} (\log r)^{\frac{\sum_{i=1}^{n+1} 2^i}{2^{n+1}}}, \tag{4.4.9}$$

where $\lim_{n\to\infty} A_n = \lim_{n\to\infty} B_n = \lim_{n\to\infty} D_n = \infty$. Now in the limit we have

$$\begin{aligned} \lim_{n\to\infty} \frac{-\sum_{i=0}^{n} 2^i + \sum_{i=1}^{n+1} 2^i \alpha}{2^{n+1}} &= \lim_{n\to\infty} \frac{-2^{n+1} + 1 + 2^{n+2}\alpha - 2\alpha}{2^{n+1}} \\ &= -1 + 2\alpha + \lim_{n\to\infty} \frac{1-2\alpha}{2^{n+1}} \\ &= -1 + 2\alpha. \end{aligned} \tag{4.4.10}$$

Define

$$\Omega_\delta = \left\{ (\rho, z) : |\rho - r| \leq \frac{\delta^3}{4} r,\ -\pi \leq z \leq \pi \right\}.$$

Then for large r, we can get

$$\|(u^r, u^\theta, u^z)\|_{L^\infty(\Omega_\delta)} \leq C_\delta r^{-1+2\alpha+\delta} \tag{4.4.11}$$

and

$$\|(w^r, w^\theta, w^z)\|_{L^\infty(\Omega_\delta)} \leq C_\delta r^{-1+2\alpha+\delta} \tag{4.4.12}$$

where $\delta > 0$ and $\lim_{\delta\to 0} C_\delta = \infty$.

4.5 Decay and Vanishing of the Velocity

We first start with obtaining first order decay of $|\nabla w|$. Both r and δ we will be chosen later where r will be sufficiently larger and δ will be sufficiently small. For the next step, we also need to iterate the inequalities (4.3.45) and (4.3.46). We can do this after we update u^r, u^θ, u^z on $\mathcal{S}_{3n+2}$ to obtain

$$
\begin{aligned}
\|(\nabla w^r, \nabla w^z)\|_{L^2(\mathcal{S}_{3n+3})} &\le C\|(u^r, u^z)\|_{L^\infty(\mathcal{S}_{3n+2})}\|(w^r, w^z)\|_{L^\infty(\mathcal{S}_{3n+2})} \\
&\le Cr^{\frac{-\sum_{i=0}^{n} 2^i + \sum_{i=1}^{n+1} 2^i\alpha}{2^{n+1}}} (\log r)^{\frac{\sum_{i=1}^{n+1} 2^i}{2^{n+1}}} r^{\frac{\alpha}{2}} \\
&\le Cr^{\frac{-\sum_{i=0}^{n} 2^i + \left(2^n + \sum_{i=1}^{n+1} 2^i\right)\alpha}{2^{n+1}}} (\log r)^{\frac{\sum_{i=1}^{n+1} 2^i}{2^{n+1}}},
\end{aligned} \tag{4.5.1}
$$

and

$$
\begin{aligned}
\|\nabla w^\theta\|_{L^2(\mathcal{S}_{3n+3})} &\le Cr^{-1/2}\|(u^r, u^z)\|_{L^\infty(\mathcal{S}_{3n+2})}\|(w^r, w^\theta)\|_{L^2(\mathcal{S}_{3n+2})} \\
&\le Cr^{\frac{-2^n - \sum_{i=0}^{n} 2^i + \left(2^n + \sum_{i=1}^{n+1} 2^i\right)\alpha}{2^{n+1}}} (\log r)^{\frac{\sum_{i=1}^{n+1} 2^i}{2^{n+1}}} r^{\frac{\alpha}{2}}.
\end{aligned} \tag{4.5.2}
$$

Note that in the limit

$$
\begin{aligned}
\lim_{n\to\infty} \frac{-2^n - \sum_{i=0}^{n} 2^i + \left(2^n + \sum_{i=1}^{n+1} 2^i\right)\alpha}{2^{n+1}} &= \lim_{n\to\infty} \frac{-2^n - 2^{n+1} + 1 + \left(2^n + 2^{n+2} - 2\right)\alpha}{2^{n+1}} \\
&= -\frac{3}{2} + \frac{5}{2}\alpha.
\end{aligned} \tag{4.5.3}
$$

and

$$
\begin{aligned}
\lim_{n\to\infty} \frac{-\sum_{i=0}^{n} 2^i + \left(2^n + \sum_{i=1}^{n+1} 2^i\right)\alpha}{2^{n+1}} &= \lim_{n\to\infty} \frac{-2^{n+1} + 1 + \left(2^n + 2^{n+2} - 2\right)\alpha}{2^{n+1}} \\
&= -1 + \frac{5}{2}\alpha.
\end{aligned} \tag{4.5.4}
$$

Hence for a suitable δ we have

$$\|\nabla(w^r, w^\theta, w^z)\|_{L^2(\Omega_{\delta/2})} \leq C_\delta r^{-1+\frac{5}{2}\alpha+\delta}. \tag{4.5.5}$$

For the following calculations, we will switch to Euclidean coordinates so we will write

$$w = w^1 e_1 + w^2 e_2 + w^3 e_3 = w^r e_r + w^\theta e_\theta + w^z e_z. \tag{4.5.6}$$

Fix a point $x = (r\cos(\theta), r\sin(\theta), z)$ where r is large. Denote

$$B_R = \{y \in \mathbb{R}^3 : |x-y| \leq R\}. \tag{4.5.7}$$

We consider $\mathcal{H}$ the hallowed out cylinder at height z, with inner radius $\left(1 - \dfrac{\delta^3}{4}\right) r$, and outer radius $\left(1 + \dfrac{\delta^3}{4}\right) r$, generated by rotating the rectangle $\Omega_{\delta/2}$ around the curve

$$\left\{(y_1, y_2, y_3) \colon \sqrt{y_1^2 + y_2^2} = r, y_3 = z\right\}. \tag{4.5.8}$$

By dividing the circumference of this curve by the diameter of the ball with radius 1, we can fill up the hallowed out cylinder with $\frac{2\pi r}{2}$ (rounding up to the nearest integer) many disjoint balls of radius 1 centered at x whose union is contained in the hallowed out cylinder. Essentially, these collections of balls will fit inside a torus at height z and radius r which will fit inside $\mathcal{H}$ as long as we choose δ so that $1 \leq \frac{\delta^3}{4} r$. One possible choice is δ satisfying $\left(\frac{4}{r}\right)^{1/3} \leq \delta$. This will also ensure so that the estimates we have for u, w, and ∇w on $\Omega_{\delta/2}$ will also hold on B_1.

Call the collection of balls $\mathcal{B}$. So we have

$$\sum_{B \in \mathcal{B}} \int_{\mathcal{B}} |\nabla w|^2 dx \leq \int_{\mathcal{H}} |\nabla w|^2 dx. \tag{4.5.9}$$

However, since our functions are axially symmetric, we have that for any ball B in the collection $\mathcal{B}$, $\int_{\mathcal{B}} |\nabla w|^2 dx = \int_{B_1} |\nabla w|^2 dx$ and $\int_{\mathcal{H}} |\nabla w|^2 dx = 2\pi \int_{\Omega_{\delta/2}} |\nabla w^2| dx$.

Hence

$$\int_{B_1} |\nabla w|^2 dx \leq \frac{C}{r} \int_{\Omega_{\delta/2}} |\nabla w|^2 dx. \tag{4.5.10}$$

Identical calculation shows

$$\int_{B_1} |\nabla u|^2 dx \leq \frac{C}{r} \int_{\Omega_{\delta/2}} |\nabla u|^2 dx, \tag{4.5.11}$$

as well.

By using (4.5.5) and the D-condition, we have

$$\int_{B_1} |\nabla w|^2 dx \leq \frac{C}{r} \int_{\Omega_{\delta/2}} |\nabla w|^2 dx \leq C_\delta r^{-3+5\alpha+2\delta} \tag{4.5.12}$$

and

$$\int_{B_1} |\nabla u|^2 dx \leq \frac{C}{r} \int_{\Omega_{\delta/2}} |\nabla u|^2 dx \leq C_\delta r^{-1+\alpha}. \tag{4.5.13}$$

The next goal is to get decay estimates on ∇w. In Euclidean coordinates, the vorticity w satisfies

$$-\Delta w = -(u \cdot \nabla) w + (w \cdot \nabla) u. \tag{4.5.14}$$

We will let ∂w represent the partial derivative of w in the x_1, x_2 or x_3 variable. Using (4.5.14) we have

$$-\Delta(\partial w) = -(\partial u \cdot \nabla) w - (u \cdot \nabla) \partial w + (w \cdot \nabla) \partial u + (\partial w \cdot \nabla) u. \tag{4.5.15}$$

Define a cut-off function ϕ that is supported in B_1 and equal to 1 in $B_{1/2}$. Then by the (4.5.15) we have

$$
\begin{aligned}
-\Delta(\phi\partial w) &= -\phi\Delta(\partial w) - 2(\nabla\phi\cdot\nabla)\partial w - \partial w\Delta\phi \\
&= -\phi(\partial u\cdot\nabla)w - \phi(u\cdot\nabla)\partial w + \phi(w\cdot\nabla)\partial u \\
&\quad + \phi(\partial w\cdot\nabla)u - 2(\nabla\phi\cdot\nabla)\partial w - \partial w\Delta\phi.
\end{aligned}
\tag{4.5.16}
$$

By the use of the Green's function in three-dimensions $G(x,y) = \frac{1}{4\pi|x-y|}$, we have that

$$
\begin{aligned}
\partial w(x) = &-\int_{B_1} G(x,y)\phi(\partial u\cdot\nabla)w dy - \int_{B_1} G(x,y)\phi(u\cdot\nabla)\partial w dy + \int_{B_1} G(x,y)\phi(w\cdot\nabla)\partial u dy \\
&+ \int_{B_1} G(x,y)\phi(\partial w\cdot\nabla)u dy - \int_{B_1} G(x,y)2(\nabla\phi\cdot\nabla)\partial w dy - \int_{B_1} G(x,y)\partial w\Delta\phi dy. \\
&= \sum_{n=1}^{6} I_n.
\end{aligned}
\tag{4.5.17}
$$

We now estimate each integral. Note that for any fixed x, we have

$$
\begin{aligned}
\|G(x)\|^2_{L^2(B_1)} &= \frac{1}{16\pi^2}\int_{B_1}\frac{1}{|x-y|^2}dy \\
&= \frac{1}{16\pi^2}\int_0^1\int_{\partial B_r}\frac{1}{r^2}dSdr \\
&= \frac{1}{16\pi^2}\int_0^1\frac{1}{r^2}4\pi r^2 dr \\
&= \frac{1}{4\pi},
\end{aligned}
\tag{4.5.18}
$$

$$
\begin{aligned}
\|G(x)\|_{L^1(B_1)} &= \frac{1}{16\pi^2}\int_{B_1}\frac{1}{|x-y|}dy \\
&= \frac{1}{16\pi^2}\int_0^1\int_{\partial B_r}\frac{1}{r}dSdr \\
&= \frac{1}{16\pi^2}\int_0^1\frac{1}{r}4\pi r^2 dr \\
&= \frac{1}{8\pi},
\end{aligned}
\tag{4.5.19}
$$

$$
\begin{aligned}
\|\nabla G(x)\|_{L^1(B_1)} &\leq \frac{1}{16\pi^2}\int_{B_1}\frac{1}{|x-y|^2}dy \\
&= \frac{1}{16\pi^2}\int_0^1\int_{\partial B_r}\frac{1}{r^2}dSdr \\
&= \frac{1}{16\pi^2}\int_0^1\frac{1}{r^2}4\pi r^2 dr \\
&= \frac{1}{4\pi},
\end{aligned}
\tag{4.5.20}
$$

and

$$
\begin{aligned}
\|\nabla G(x)\|_{L^2(B_1\setminus B_{1/2})} &\leq \frac{1}{16\pi^2}\int_{(B_1\setminus B_{1/2})}\frac{1}{|x-y|^4}dy \\
&= \frac{1}{16\pi^2}\int_{1/2}^1\int_{\partial B_r}\frac{1}{r^4}dSdr \\
&= \frac{1}{16\pi^2}\int_{1/2}^1\frac{1}{r^4}4\pi r^2 dr \\
&= \frac{1}{4\pi}.
\end{aligned}
\tag{4.5.21}
$$

Thus, we have

$$
\begin{aligned}
|I_1| &\leq \|\phi\|_{L^\infty(B_1)}\|\partial u\|_{L^\infty(B_1)}\|G\|_{L^2(B_1)}\|\nabla w\|_{L^2(B_1)} \\
&\leq C\|\nabla w\|_{L^2(B_1)} \\
&\leq C_\delta r^{-3/2+5/2\alpha+\delta},
\end{aligned}
\tag{4.5.22}
$$

where we also use the general boundeness of ∂u and the estimate of w in (4.5.12). Similarly, we have

$$
\begin{aligned}
|I_4| &\leq \|\phi\|_{L^\infty(B_1)}\|(\nabla u)^T\|_{L^\infty(B_1)}\|G\|_{L^2(B_1)}\|\partial w\|_{L^2(B_1)} \\
&\leq C\|\partial w\|_{L^2(B_1)} \\
&\leq C_\delta r^{-3/2+5/2\alpha+\delta},
\end{aligned}
\tag{4.5.23}
$$

and

$$
\begin{aligned}
|I_6| &\leq \|\Delta\phi\|_{L^\infty(B_1)}\|G\|_{L^2(B_1)}\|\partial w\|_{L^2(B_1)} \\
&\leq C\|\partial w\|_{L^2(B_1)} \\
&\leq C_\delta r^{-3/2+5/2\alpha+\delta}.
\end{aligned} \tag{4.5.24}
$$

Next by an integration by parts we have

$$
\begin{aligned}
|I_2| \leq & \left|\int_{B_1} \partial w\cdot(uG\nabla\phi + G\phi\nabla u + \phi(u\cdot\nabla)G)dy\right| \\
&\leq \|\nabla\phi\|_{L^\infty}\|u\|_{L^\infty(B_1)}\|\partial w\|_{L^2(B_1)}\|G\|_{L^2(B_1)} + \|\phi\|_{L^\infty}\|\nabla u\|_{L^\infty(B_1)}\|\partial w\|_{L^2(B_1)}\|G\|_{L^2(B_1)} \\
&\quad + \|\phi\|_{L^\infty(B_1)}\|u\|_{L^\infty(B_1)}\|\partial w\|_{L^\infty(B_1)}\|\nabla G\|_{L^1(B_1)} \\
&\leq C\|\partial w\|_{L^2(B_1)} + C\|\nabla u\|_{L^\infty(B_1)}\|\partial w\|_{L^2(B_1)} + C\|u\|_{L^\infty(B_1)}\|\partial w\|_{L^\infty(B_1)} \\
&\leq C_\delta r^{-3/2+5/2\alpha+\delta} + C_\delta r^{-3/2+5/2\alpha+\delta} + C\|u\|_{L^\infty(B_1)} \\
&\leq C_\delta r^{-3/2+5/2\alpha+\delta} + C_\delta r^{-1+2\alpha+\delta} \\
&\leq C_\delta r^{-1+2\alpha+\delta},
\end{aligned} \tag{4.5.25}
$$

where we used the general boundedness of ∂w and the the decay of u we obtained in (4.4.11).

Next we have

$$
\begin{aligned}
|I_3| \leq & \left|\int_{B_1} \partial u\cdot(w\phi\nabla G + Gw\nabla\phi + G\phi\nabla w)dy\right| \\
&\leq \|\phi\|_{L^\infty}\|\partial u\|_{L^\infty(B_1)}\|w\|_{L^\infty(B_1)}\|\nabla G\|_{L^1(B_1)} + \|\nabla\phi\|_{L^\infty}\|\partial u\|_{L^\infty(B_1)}\|w\|_{L^\infty(B_1)}\|G\|_{L^1(B_1)} \\
&\quad + \|\phi\|_{L^\infty(B_1)}\|\partial u\|_{L^\infty(B_1)}\|\nabla w\|_{L^2(B_1)}\|G\|_{L^2(B_1)} \\
&\leq C\|\partial u\|_{L^\infty(B_1)}\|w\|_{L^\infty(B_1)} + C\|\partial u\|_{L^\infty(B_1)}\|w\|_{L^\infty(B_1)} + C\|\nabla w\|_{L^2(B_1)} \\
&\leq C_\delta r^{-1+2\alpha+\delta} + C_\delta r^{-1+2\alpha+\delta} + C_\delta r^{-3/2+5/2\alpha+\delta} \\
&\leq C_\delta r^{-1+2\alpha+\delta},
\end{aligned} \tag{4.5.26}
$$

where we used the general boundedness of ∂u, the $L^2(B_1)$ estimate of ∇w in (4.5.12) and the decay of w obtained in (4.4.12). Finally, by an integration by parts we have

$$\begin{aligned}
|I_5| &\leq \left| \int_{B_1} (\partial w)^T \cdot (\nabla G \cdot \nabla \phi + G \Delta \phi) dy \right| \\
&\leq \|\nabla \phi\|_{L^\infty(B_1)} \|\nabla w\|_{L^2(B_1)} \|\nabla G\|_{L^2(B_1 \setminus B_{1/2})} + \|\Delta \phi\|_{L^\infty(B_1)} \|\nabla w\|_{L^2(B_1)} \|G\|_{L^2(B_1 \setminus B_{1/2})} \\
&\leq C_\delta r^{-3/2+5/2\alpha+\delta}.
\end{aligned} \tag{4.5.27}$$

Hence by the previous calculations we have that for large r

$$|\nabla w| \leq C_\delta r^{-1+2\alpha+\delta}. \tag{4.5.28}$$

Now we use this to obtain an estimate on $|\nabla u|$. Define a cut-off function ψ that is supported in B_1 and equal to 1 in $B_{1/2}$. Recall that

$$\begin{aligned}
u^r(x) &= \int_{-\pi}^{\pi} \int_0^{2\pi} \int_0^\infty G(x,y) \psi \partial_\ell w^\phi \cos(\phi - \theta) \rho d\rho d\phi d\ell \\
&- 2 \int_{-\pi}^{\pi} \int_0^{2\pi} \int_0^\infty G(x,y) \partial_\rho \psi \partial_\rho u^\rho \cos(\phi - \theta) \rho d\rho d\phi d\ell \\
&- \int_{-\pi}^{\pi} \int_0^{2\pi} \int_0^\infty G(x,y) (\partial_\rho^2 \psi + \frac{1}{\rho} \partial_\rho \psi) u^\rho \cos(\phi - \theta) \rho d\rho d\phi d\ell.
\end{aligned} \tag{4.5.29}$$

If we differentiate this equation we obtain

$$\begin{aligned}
\nabla u^r(x) &= \int_{-\pi}^{\pi} \int_0^{2\pi} \int_0^\infty \nabla G(x,y) \psi \partial_\ell w^\phi \cos(\phi) \rho d\rho d\phi d\ell \\
&- 2 \int_{-\pi}^{\pi} \int_0^{2\pi} \int_0^\infty \nabla G(x,y) \partial_\rho \psi \partial_\rho u^\rho \cos(\phi) \rho d\rho d\phi d\ell \\
&- \int_{-\pi}^{\pi} \int_0^{2\pi} \int_0^\infty \nabla G(x,y) (\partial_\rho^2 \psi + \frac{1}{\rho} \partial_\rho \psi) u^\rho \cos(\phi) \rho d\rho d\phi d\ell \\
&= I_1 + I_2 + I_3.
\end{aligned} \tag{4.5.30}$$

We now estimate each term. Using the estimates on $|\nabla G|$ and using (4.5.28) we obtain

$$
\begin{aligned}
|I_1| &\leq \int_{-\pi}^{\pi}\int_0^{\infty}\left(\int_0^{2\pi}|\nabla G(x,y)|d\phi\right)|\psi||\partial_\ell w^\phi|\rho d\rho d\ell \\
&\leq C\|\nabla w\|_{L^\infty(B_1)}\int_{B_1}|\nabla G|dy \\
&\leq C\|\nabla w\|_{L^\infty(B_1)}\frac{1}{|x-y|^2}e^{-c_0|x'-y'|}dy \\
&\leq C\|\nabla w\|_{L^\infty(B_1)}\frac{1}{|x-y|^2}dy \\
&\leq C\|\nabla w\|_{L^\infty(B_1)} \\
&\leq C_\delta r^{-1+2\alpha+\delta}.
\end{aligned}
\tag{4.5.31}
$$

For the next term we will use integration by parts to move the partial derivative with respect to ρ to the other terms. Then we have

$$
\begin{aligned}
|I_2| &\leq 2\int_{B_1}|\partial_\rho\nabla G(x,y)\partial_\rho\psi u^\rho|\rho d\phi d\rho d\ell + 2\int_{B_1}|\nabla G(x,y)\partial_\rho^2\psi u^\rho|\rho d\phi d\rho d\ell \\
&\quad + 2\int_{B_1}|\nabla G(x,y)\partial_\rho^2\psi u^\rho|d\phi d\rho d\ell \\
&\leq C\|u^\rho\|_{L^\infty(B_1)}\int_{B_1\setminus B_{1/2}}\left[|\partial\rho\nabla G| + |\nabla G|\right]dy \\
&\leq C\|u^\rho\|_{L^\infty(B_1)} \\
&\leq Cr^{-1+2\alpha+\delta}.
\end{aligned}
\tag{4.5.32}
$$

Here we used the decay of u in (4.4.11) and we stress that due to the cutoff function, the singularity at $x = y$ is cut-off so the integration of $\partial_\rho\nabla G$ is justified. Lastly, for I_3, we use the bounds on $|\nabla G|$ and the decay of u in (4.4.11) to obtain

$$
|I_3| \leq Cr^{-1+2\alpha+\delta}. \tag{4.5.33}
$$

Hence we obtain

$$
|\nabla u^r| \leq C_\delta r^{-1+2\alpha+\delta}. \tag{4.5.34}
$$

Next we obtain estimates on $|\nabla u^z|$. By the divergence-free condition,

$$\partial_r u^r + \frac{u^r}{r} + \partial_z u^z = 0 \tag{4.5.35}$$

so using the boundedness of u^r and the bound on $|\nabla u^r|$, we get

$$|\partial_z u^z| \le |\partial_r u^r| + \left|\frac{u^r}{r}\right| \le C r^{-1+2\alpha+\delta}. \tag{4.5.36}$$

By using the fact that $w^\theta = \partial_z u^r - \partial_r u^z$ and the decay of w^θ in (4.4.12), we have

$$|\partial_r u^z| \le |\partial_z u^r| + |w^\theta| \le C r^{-1+2\alpha+\delta}. \tag{4.5.37}$$

Hence we obtain

$$|\nabla u^z| \le C_\delta r^{-1+2\alpha+\delta}. \tag{4.5.38}$$

We finally obtain estimates on $|\nabla u^\theta|$. We use the fact that $w^r = -\partial_z u^\theta$ and $w^z = \partial_r u^\theta + \frac{1}{r} u^\theta$ to obtain

$$|\partial_r u^\theta| \le |w^z| + \left|\frac{u^\theta}{r}\right| \le C r^{-1+2\alpha+\delta}. \tag{4.5.39}$$

and

$$|\partial_z u^\theta| \le |w^r| \le C r^{-1+2\alpha+\delta}. \tag{4.5.40}$$

This yields

$$|\nabla u^\theta| \le C_\delta r^{-1+2\alpha+\delta}, \tag{4.5.41}$$

which together with (4.5.34) and (4.5.38) we get that

$$|\nabla u| \le C_\delta r^{-1+2\alpha+\delta}. \tag{4.5.42}$$

Now that we have a decay estimate on $|\nabla u|$, we can use this to obtain a better estimate on $|\nabla w|$. In (4.5.17), we note that all terms except for I_2 and I_3 decay like $C_\delta r^{-3/2+5/2\alpha+\delta}$. Therefore, we only recalulate I_2 and I_3. By using (4.5.28) we have

$$\begin{aligned}|I_2| &\leq C\|\partial w\|_{L^2(B_1)} + C\|\nabla u\|_{L^\infty(B_1)}\|\partial w\|_{L^2(B_1)} + C\|u\|_{L^\infty(B_1)}\|\partial w\|_{L^\infty(B_1)} \\ &\leq C_\delta r^{-3/2+5/2\alpha+\delta} + C_\delta r^{-3/2+5/2\alpha+\delta} + C\|u\|_{L^\infty(B_1)}\|\partial w\|_{L^\infty(B_1)} \\ &\leq C_\delta r^{-3/2+5/2\alpha+\delta} + C_\delta r^{-2+4\alpha+2\delta} \\ &\leq C_\delta r^{-3/2+5/2\alpha+\delta}.\end{aligned} \tag{4.5.43}$$

Next using (4.5.28) we have

$$\begin{aligned}|I_3| &\leq C\|\partial u\|_{L^\infty(B_1)}\|w\|_{L^\infty(B_1)} + C\|\partial u\|_{L^\infty(B_1)}\|w\|_{L^\infty(B_1)} + C\|\nabla w\|_{L^2(B_1)} \\ &\leq C_\delta r^{-2+4\alpha+2\delta} + +C_\delta r^{-3/2+5/2\alpha+\delta} \\ &\leq C_\delta r^{-3/2+5/2\alpha+\delta}.\end{aligned} \tag{4.5.44}$$

Therefore we have

$$|\nabla w| \leq C_\delta r^{-3/2+5/2\alpha+\delta}. \tag{4.5.45}$$

To finish the proof, we use this decay to improve the decay of u. Let ψ be a cut-off function independent of x_3 and supported in $B(x, r/2)$ such that $\psi = 1$ in $B(x, r/4)$. Then by the Green's function representation in (4.3.55) we have

$$\begin{aligned}u^r(x) = &\int_{-\pi}^{\pi}\int_0^{2\pi}\int_0^\infty G(x,y)\psi\partial_\ell w^\phi \cos\phi\rho d\rho d\phi d\ell \\ &- 2\int_{-\pi}^{\pi}\int_0^{2\pi}\int_0^\infty G(x,y)\partial_\rho\psi\partial_\rho u^\rho \cos\phi\rho d\rho d\phi d\ell \\ &- \int_{-\pi}^{\pi}\int_0^{2\pi}\int_0^\infty G(x,y)(\partial_\rho^2\psi + \frac{1}{\rho}\partial_\rho\psi)u^\rho \cos\phi\rho d\rho d\phi d\ell.\end{aligned} \tag{4.5.46}$$

By recalling the estimates we did following (4.3.55), we see that the last two terms decay exponetially so only the first term needs attention. By using the decay $|\nabla w|$ in (4.5.45) and the decay of G we have that altogether

$$|u^r| \leq C_\delta r^{-3/2+5/2\alpha+\delta} \tag{4.5.47}$$

By differentiating the equation, we have

$$\begin{aligned}
\nabla u^r(x) = &\int_{-\pi}^{\pi}\int_0^{2\pi}\int_0^{\infty} \nabla G(x,y)\psi\partial_\ell w^\phi \cos\phi\rho d\rho d\phi d\ell \\
&- 2\int_{-\pi}^{\pi}\int_0^{2\pi}\int_0^{\infty} \nabla G(x,y)\partial_\rho\psi\partial_\rho u^\rho \cos\phi\rho d\rho d\phi d\ell \\
&- \int_{-\pi}^{\pi}\int_0^{2\pi}\int_0^{\infty} \nabla G(x,y)(\partial_\rho^2\psi + \frac{1}{\rho}\partial_\rho\psi)u^\rho \cos\phi\rho d\rho d\phi d\ell \\
= &I_1 + I_2 + I_3.
\end{aligned} \tag{4.5.48}$$

We now estimate each term. By using the gradient estimate of G we have

$$\begin{aligned}
|I_1| &\leq C\|\partial_\ell w^\phi\|_{L^\infty(B(x,r/2))}\int_{B(x,r/2)} |\nabla G(x,y)|dy \\
&\leq C\|\nabla w^\phi\|_{L^\infty(B(x,r/2))}\int_{B(x,r/2)} \frac{e^{-c_0|x'-y'|}}{|x-y|^2}dy \\
&\leq C\|\nabla w^\phi\|_{L^\infty(B(x,r/2))}\int_{B(x,r/2)} \frac{1}{|x-y|^2}dy \\
&\leq C\|\nabla w^\phi\|_{L^\infty(B(x,r/2))} \\
&\leq C_\delta r^{-3/2+5/2\alpha+\delta}.
\end{aligned} \tag{4.5.49}$$

We also have by $|\nabla\psi| \leq \frac{C}{r}$, the decay of ∇u and ∇G that

$$\begin{aligned}
|I_2| &\leq \frac{C}{r}\|\nabla u\|_{L^\infty(B(x,r/2))}\int_{B(x,r/2)}|\nabla G(x,y)|dy \\
&\leq \frac{C}{r}\|\nabla u\|_{L^\infty(B(x,r/2))}\int_{B(x,r/2)}\frac{e^{-c_0|x'-y'|}}{|x-y|^2}dy \\
&\leq \frac{C}{r}\|\nabla u\|_{L^\infty(B(x,r/2))}\int_{B(x,r/2)}\frac{1}{|x-y|^2}dy \\
&\leq \frac{C}{r}\|\nabla u\|_{L^\infty(B(x,r/2))} \\
&\leq C_\delta r^{-2+2\alpha+\delta}.
\end{aligned} \tag{4.5.50}$$

Similarly we have

$$|I_3| \leq C_\delta r^{-2+2\alpha+\delta}, \tag{4.5.51}$$

which gives us that

$$|\nabla u^r| \leq C_\delta r^{-3/2+5/2\alpha+\delta}. \tag{4.5.52}$$

Hence we have

$$|u^r| + |\nabla u^r| \leq C_\delta r^{-3/2+5/2\alpha+\delta}. \tag{4.5.53}$$

Next by using the divergence free condition and the decay in (4.5.53) we have

$$\begin{aligned}
|\partial_z u^z| &\leq \left|\frac{u^r}{r}\right| + |\partial_r u^r| \\
&\leq C_\delta r^{-3/2+5/2\alpha+\delta}.
\end{aligned} \tag{4.5.54}$$

Since $\partial_z u^\theta = -w^r$, we have

$$|\partial_z u^\theta| \leq C_\delta r^{-3/2+5/2\alpha+\delta}. \tag{4.5.55}$$

Now because of the condition $\int_{-\pi}^{\pi} u^\theta dz = \int_{-\pi}^{\pi} u^z dz = 0$, we know by mean value theorem that for any r, there exists $z_0, z_1 \in [-\pi, \pi]$ such that $u^\theta(r, z_0) = u^z(r, z_1) = 0$. Hence by fundamental theorem of calculus, (4.5.54) and (4.5.55)

$$|u^\theta(r,z)| = \left| \int_{z_0}^{z} \partial_z u^\theta dz \right| \leq 2\pi \|\partial_z u^\theta(r,\cdot)\|_{L^\infty([-\pi,\pi])} \leq C_\delta r^{-3/2+5/2\alpha+\delta}. \tag{4.5.56}$$

and

$$|u^z(r,z)| = \left| \int_{z_1}^{z} \partial_z u^z dz \right| \leq 2\pi \|\partial_z u^z(r,\cdot)\|_{L^\infty([-\pi,\pi])} \leq C_\delta r^{-3/2+5/2\alpha+\delta}. \tag{4.5.57}$$

Hence by (4.5.53), (4.5.56), and (4.5.57) we have that for large r

$$|u(x)| \leq C_\delta r^{-3/2+5/2\alpha+\delta}. \tag{4.5.58}$$

We need both $-3/2 + 5/2\alpha + \delta \leq -1$ and $\left(\frac{4}{r}\right)^{1/3} \leq \delta$. This means we need both a suitably small δ which depends on a suitably large r. By supposing that $\delta = \left(\frac{4}{r}\right)^{1/3}$ and solving the first inequality we see that if we pick $r \geq -\frac{32}{(5\alpha-1)^3} > 0$, then $-3/2 + 5/2\alpha + \delta \leq -1$. Let $r_0 = \frac{-32}{(5\alpha-1)^3}$. If we choose $\delta = \left(\frac{4}{r_0}\right)^{1/3}$, then (4.5.58) will hold for all points x with $|x'| = r \geq r_0$ and the exponent will be less than or equal to -1. This is exactly what we needed to use the results in [KNSS] to obtain that $u \equiv 0$.

Chapter 5

Future Work

A natural question to ask is if the conditions of the theorem can be relaxed so that we do not require that $\int_{-\pi}^{\pi} u^z dz = \int_{-\pi}^{\pi} u^\theta dz = 0$. This condition cannot be removed if we wish to use the method presented in this thesis. However, in [CPZZ], if we assume that

$$\int_{\mathbb{R}^2 \times [-\pi,\pi]} |\nabla u|^2 dx < \infty$$

instead of (4.1.2), then the conditions $\int_{-\pi}^{\pi} u^z dz = \int_{-\pi}^{\pi} u^\theta dz = 0$ are not necessary. This involves a different method that requires we have no growth in the Dirichlet integral.

However, one can still ask if there are uniqueness theorems if we allow there to be a non-zero, divergence free forcing term. This is if u_1, u_2 satisfy (4.1.1) with a forcing term, then under what extra conditions will it be the case that $u_1 = u_2$? This is still an open problem that we will investigate as our future work.

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