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Efficient high-speed cornering motions based on continuously-variable feedrates. II. Implementation and performance analysis

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Abstract A novel high-speed cornering strategy for piecewise-linear motions is proposed, based on the G^2 continuous Pythagorean-hodograph (PH) rounding segments and continuously-variable feedrates described in Part I of this two-part paper. This strategy employs an acceleration-limited approach to feedrate scheduling, including smooth deceleration and acceleration profiles along linear segments entering and leaving the rounded segments. A G-code part program parsing software package has been developed, that automatically identifies toolpath corners and inserts appropriately-sized PH quintic corner rounding segments, with associated feedrate suppression ratios. The method has been tested on a three-axis CNC mill with an open-architecture controller incorporating real-time interpolators that realize smoothly varying feedrates along the PH corner curves. Tests on a representative selection of toolpaths yield substantial savings (up to 40 %) in execution times, compared to the traditional “full stop” strategy for unmodified sharp corners.

Keywords Toolpath corner rounding · Pythagorean-hodograph curves · CNC machines · Continuously-variable feedrate · Real-time interpolator algorithm

1 Introduction

In the companion paper [5] of this two-part study, a family of novel real-time interpolator algorithms were described, for the execution of continuously-variable feedrates along G^2 continuous Pythagorean-hodograph (PH) quintic corner segments. These corner curves and feedrate functions form the basis of a comprehensive sharp-corner traversal strategy. The navigation of corners remains an important challenge in many motion control applications, such as traditional machining, laser and water jet cutting, additive manufacturing, and industrial automation. In such applications, the execution time and path accuracy are of particular concern during cornering motions.

Complete details of the PH curve corner rounding constructions, and real-time interpolator algorithms for smooth feedrate suppression along them, may be found in [5]. The present study develops these basic utilities into a cohesive strategy for traversing piecewise-linear part programs that contain many sharp corners, and thoroughly investigates its practical performance. It is of interest to identify significant advantages or shortcomings (if any) of the three cornering feedrate functions proposed in [5], as well as the efficiency improvements that the proposed cornering strategy offers over the traditional “full stop” approach to negotiating sharp corners.

A sharp corner cannot be executed at a non-zero speed, since this implies discontinuities in velocity and acceleration—consequently, significant path errors will be incurred unless the machine comes to a complete stop at each corner, which incurs a significant increase in the overall path execution time. Consequently, the problem of effi-

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ciently accommodating toolpath corners has been studied by several authors [1, 2, 10, 13, 14, 16].

Rounding the corner by a smooth curve segment is one method to control acceleration in cornering. Another is to suppress the feedrate (speed) along the rounded corner. The strategy proposed here combines corner rounding and continuous feedrate modulation to allow relatively fast corner traversal while maintaining accelerations within acceptable bounds. The present focus is on execution of corners with significant turning angles between extended linear segments in planar toolpaths, rather than on smoothing the numerous shallow angles between successive short G01 linear path segments, commonly employed as approximations to free-form curvilinear toolpaths.

A software package has been developed to parse G-code part programs, and identify sharp corners between successive G01 linear path segments. Such linear segments are truncated at distance L before and after the corner, and a PH quintic corner rounding curve is inserted between them, that meets the truncated segments with G^2 —i.e., position, tangent, and curvature—continuity. The dimension L is chosen so that the deviation δ of the rounded corner from the exact corner does not exceed a prescribed tolerance ϵ .

A variable feedrate is associated with each corner curve, specified by two key parameters—the initial/final feedrate V_0 and the feedrate suppression factor f , which defines the minimum (mid-point) cornering feedrate $V_{\min}$ as a fraction of V_0 . Three forms of feedrate variation are considered herein: (i) the *parameter-dependent feedrate* is specified as a quintic polynomial in the curve parameter; (ii) the *curvature-dependent feedrate* employs an inverse linear expression in the curvature; and (iii) the *hybrid feedrate* makes use of a parameter-dependent modulation of the curvature in case (ii), to ensure tangential acceleration continuity. The special algebraic structure of the PH curves [3] allows a closed-form reduction of the interpolation integral for each of these feedrate variations, providing very accurate and efficient real-time interpolator algorithms. Further details on the PH quintic corner curves and associated cornering feedrate functions may be found in [5].

It is desired to implement these feedrate types into a real-time controller as part of a broader strategy that encompasses automated corner rounding and acceleration-limited selection of the feedrate parameters V_0 and f . The problem of corner rounding in conjunction with adaptive feedrate scheduling has received relatively little attention, compared to individual treatments of these two topics. Prior work on corner rounding with PH curves (including some studies that employ variable feedrates) was briefly summarized in [5]. Industrial machine tool controllers must also provide strategies to deal with the kinematic difficulties of

sharp corners. These may be deceleration to zero (or very low) feedrate into the corner, with matching acceleration out of the corner, or more complex strategies that involve actual path rounding. For example, the G641 G-code command found in Siemens controllers executes a modal corner rounding to a specified rounding radius [15]. More traditionally, there are the modal G61 and non-modal G09 exact stop functions available in many controllers that impose a complete stop at sharp corners.

The plan for the remainder of this paper is as follows. Section 2 describes a global kinematic strategy for the execution of piecewise-linear paths with rounded corners, including feedrate planning and acceleration analysis. The algorithms and software used to solve for the PH corner geometry and set the continuously-variable feedrate functions are explained in Section 3. Section 4 then covers the implementation of this strategy on a three-axis milling machine governed by an open-architecture controller, with discussion of interpolation methods and some new G-code commands. Results from the execution of several piecewise-linear toolpaths with rounded corners at high feedrates are described in Section 5. Finally, Section 6 presents some concluding remarks about the effectiveness of the strategy, and ideas for further developments.

2 Kinematic analysis

The primary performance criteria of interest are execution times and motion accuracy, including feedrate smoothness and accuracy. The construction of a *globally* smooth geometry and feedrate solution along toolpath segments and transitions between them is the most direct way to improve these performance criteria. This section formulates and analyzes the global kinematic strategy employed in the high-speed cornering methodology of this paper.

For a given feedrate $V(\xi)$ on a plane parametric curve $\mathbf{r}(\xi) = (x(\xi), y(\xi))$, the acceleration $\mathbf{a}(\xi)$ can be expressed [5] as¹

$$\mathbf{a}(\xi) = \frac{V(\xi)V'(\xi)}{\sigma(\xi)} \mathbf{t}(\xi) - \kappa(\xi)V^2(\xi) \mathbf{n}(\xi), \quad (1)$$

where $\sigma(\xi) = |\mathbf{r}'(\xi)|$ is the parametric speed of curve, $\mathbf{t}(\xi)$ and $\mathbf{n}(\xi)$ are the unit tangent and normal, and $\kappa(\xi)$ is the curvature. The tangential (or feed) acceleration depends only on the feedrate and the curve parameterization, but not the curve geometry. The normal (or centripetal) acceleration depends on the path geometry, through the curvature, and the feedrate.

¹Primes denote derivatives with respect to the curve parameter ξ .

CNC machines typically employ independently-actuated axes, with the acceleration limit for each axis determined by its inertia and motor torque capacity. In the case of a PH curve, the problem of computing the individual axis acceleration extrema corresponding to expression (1) can be reduced [5] to a polynomial root-solving problem. For simplicity, however, the strategy adopted here is to specify the feedrate function $V(\xi)$ so as to ensure that the *magnitude* of the acceleration $a = |\mathbf{a}|$ does not exceed a prescribed bound a_{lim} —this may be regarded as applying an *a fortiori* bound on the accelerations of the individual axes, with a built-in “safety factor” of $\sqrt{2}$.

It is desirable to maintain acceleration continuity throughout the entire toolpath execution. This ensures that physically unrealistic commands, such as an instantaneous change in motor torque, are not implied by the motion. Acceleration continuity also allows for robust implementation of acceleration and jerk-limited feedrate specifications. For instance, jerk-limited feedrates have been described in [8, 12], although these studies consider only the tangential acceleration component. The kinematic strategy adopted in this study employs an *acceleration-limited* approach to determine transitional feedrates and corner feedrate parameters (see Section 3).

From (1), we see that the tangential acceleration is continuous when $V(\xi)$ and its derivative are continuous, while the normal acceleration is continuous when $\kappa(\xi)$ and $V(\xi)$ are continuous. In the specific context of corner rounding of piecewise-linear paths with G^2 PH quintic segments, curvature continuity is guaranteed, since they satisfy $\kappa(0) = \kappa(1) = 0$ at their junctions with the linear segments. Satisfying continuity of $V'(\xi)$ is somewhat more involved.

Three cornering feedrate functions $V(\xi)$ were proposed in [5], together with real-time interpolator algorithms based upon analytic reductions of the interpolation integral. They depend on two simple parameters²—the corner entry/exit feedrate V_0 , and the ratio $f = V_{\text{min}}/V_0$ that specifies the minimum (mid-point) feedrate as a fraction of V_0 . The *parameter-dependent* feedrate

$$V(\xi) = V_0 [1 - 16(1 - f)(1 - \xi)^2 \xi^2]$$

satisfies the tangential acceleration continuity conditions $V(0) = V(1) = V_0$ and $V'(0) = V'(1) = 0$, but has no intuitive geometrical meaning. The *curvature-dependent* feedrate

$$V(\xi) = \frac{V_0}{\rho \kappa(\xi) + 1} \quad \text{where} \quad \rho = \frac{1 - f}{f} r_{\text{min}},$$

²It is assumed that the linear segments enter and leave a corner with feedrate V_0 and zero derivative of feedrate (as is the case for a constant feedrate).

and $r_{\text{min}} = 1/\kappa(\frac{1}{2})$ is the extremum radius of curvature, is defined purely in terms of the corner curve geometry. However, $V'(0)$ and $V'(1)$ are non-zero for this feedrate. The *hybrid* feedrate

$$V(\xi) = \frac{V_0}{4\rho(1 - \xi)\xi\kappa(\xi) + 1}$$

remedies this problem by introducing a parameter-dependent modulation of the curvature. Complete details on these cornering feedrate functions may be found in [5]. Although the curvature-dependent feedrate is not precisely continuous in tangential acceleration, the other feedrates achieve this at the expense of larger intermediate feed accelerations. Therefore, a primary goal of the present study is to investigate in detail the comparative performance of these cornering feedrate functions for a variety of test curves.

The acceleration (1) along a corner curve can be expressed analytically [5] by substituting one of the above feedrate functions $V(\xi)$ and its derivative $V'(\xi)$. By further differentiation, to determine $V''(\xi)$ and $\kappa'(\xi)$, an analytical expression for the jerk (i.e., time derivative of acceleration) can be obtained, but this becomes rather involved and will not be pursued here.

In general, it is necessary to choose the two feedrate parameters V_0 and f independently for each corner (see Section 3), in order to obtain the best results. For example, the desire to minimize the total execution time argues for the use of high feedrates along the interiors of extended linear segments, while adjusting to a lower “safe” entry/exit feedrate V_0 for each corner. In general, this necessitates dedicating a portion of the linear segments entering and leaving each corner curve to a smooth feedrate transition, between a given interior value and a desired corner entry/exit value V_0 .

Polynomial time-dependent feedrate functions are especially well suited [18] to the task of achieving feedrate transitions while maintaining a desired order of continuity. A quintic “S-shaped” feedrate profile, for example, can smoothly transition between given initial and final feedrates V_i and V_f with C^2 continuity (along a linear segment, this provides jerk continuity). In terms of the normalized time $\tau = t/T$ (where T is the total transition time), the quintic feedrate profile can be expressed [18] in Bernstein form as

$$V(\tau) = \sum_{k=0}^5 V_k \binom{5}{k} (1 - \tau)^{n-k} \tau^k, \quad \tau \in [0, 1], \quad (2)$$

with Bernstein coefficients

$$V_0 = V_1 = V_2 = V_i, \quad V_3 = V_4 = V_5 = V_f.$$

These coefficients completely define the feedrate profile in the τ domain, but the total traversal time T remains a

Fig. 1 Three-phase feedrate variations along linear segments. *Upper left* initial acceleration from V_i to V_l , coasting at V_l , and final deceleration to V_f . *Upper right* since $V_i = V_l$, the acceleration phase is missing. *Lower left* since $V_f = V_l$, the deceleration phase is missing. *Lower right* if the segment is too short, the feedrate V_l must be reduced and the coasting phase is absent

free parameter. Its value determines the speed of the transition and the acceleration it incurs. Differentiating (2), the maximum acceleration magnitude (at $\tau = \frac{1}{2}$) is found to be

$$a_{\max} = \frac{15 |V_f - V_i|}{8 T}, \quad (3)$$

and this can be expressed in terms of transition path length S , rather than the traversal time T , as

$$a_{\max} = \frac{15 |V_f^2 - V_i^2|}{16 S}. \quad (4)$$

The expressions (3) and (4) allow determination of the duration T and extent S of an acceleration-limited transition between feedrates V_i and V_f along a linear segment, when the C^2 quintic feedrate profile (2) is used.

In the proposed cornering strategy, the rounded corners use one of the feedrate functions described above, while the linear segments use a constant feedrate V_l bridged by S-shaped quintic profiles to suitable corner entry/exit feedrates. The feedrate V_l is not acceleration-limited (since a constant speed along a linear segment incurs no acceleration)—it represents the maximum allowable feedrate for a given machine or manufacturing process.

Along a single linear segment, the most general case³ involves an initial acceleration from a lower feedrate V_i to V_l upon exiting a corner, maintenance of feedrate V_l for a certain extent, and a final deceleration from V_l to V_f upon entering another corner. However, as illustrated in Fig. 1, one or more of these three phases may be absent in certain situations. For example, if the feedrate differences $V_l - V_i$ and $V_l - V_f$ are large, and the linear segment length S is small, a three-phase profile that satisfies the acceleration limit $a_{\lim}$ cannot be accommodated. In such cases, V_l must be reduced so that the coasting (constant feedrate) phase is

of zero extent, i.e., there is only an acceleration phase and a deceleration phase to and from the feedrate value given by

$$V_l = \sqrt{\frac{16 S a_{\lim} + 15(V_i^2 + V_f^2)}{30}}.$$

Figure 2 illustrates the entire acceleration-limited feedrate profile⁴ for the execution of a single right-angle corner, starting and ending at rest, with a fast linear coasting speed $V_l = 800$ ipm (inches-per-minute). Section 3 below describes how the PH quintic corner curve is constructed to satisfy a desired tolerance ϵ , and the methodology for selecting the feedrate parameters V_0 and f to guarantee that the maximum acceleration on the corner curve does not exceed $a_{\lim}$. It is apparent from the acceleration plot that the overall goal of acceleration continuity, with satisfaction of the acceleration limit $a_{\lim}$, is achieved. Each of the three path segments attains the acceleration bound $a_{\lim} = 250 \text{ in/s}^2$, including the “S-shaped” deceleration and acceleration phases on the linear segments. Note that Fig. 2 shows the true acceleration magnitude, including both the tangential component (due to variation of V) and normal component (due to the path curvature κ).

The kinematic strategy aims to balance considerations of motion smoothness, time optimality, and mathematical complexity. Section 5 below presents results from real-time encoder data of the method implemented on a three-axis milling machine, including paths of considerable intricacy executed at high feedrates. To assess the performance of the method, we use as a comparison reference the usual practice of employing unmodified piecewise-linear paths with a full stop at each corner. To ensure a direct and fair comparison, the full stop method also uses the three-phase feedrate profile on each linear path segment: this is achieved by simply

³In the present context, V_l is always greater than or equal to V_i and V_f .

⁴In this figure, and all subsequent figures, the black and red colors represent linear and corner path segments, respectively.

Fig. 2 Upper a 90° corner (red), rounded to $\epsilon = 0.1$ in. Lower left the three-phase feedrate variation along linear segments (black) with $V_l = 800$ ipm, and parameter-dependent corner feedrate function with

$V_0 = 745$ ipm, $f = 0.487$. Lower right variation of acceleration magnitude a , with $a_{lim} = 250$ in/s²

Fig. 3 The unrounded path geometry corresponding to Fig. 2, executed using the full stop method. A three-phase feedrate profile is used along each segment, starting and ending at rest, with $V_i = V_f = 0$ and $V_l = 800$ ipm

Fig. 4 Simplified logic flowchart for the automated corner rounding and feedrate assignment of traditional piecewise-linear G-code part programs. The converted part program is expressed in the G05 format [4] for PH curves

choosing $V_i = V_f = 0$ (V_i is the same for both methods). Figure 3 shows the path geometry and kinematic behavior analogous to Fig. 2 in the case of the full stop strategy. A theoretical time savings is already apparent in this single-corner example, and will be more fully explored based on experimental data in Section 5.

3 Corner rounding and feedrate algorithms

A manual solution of the corner rounding and feedrate assignment problem becomes impractical for motion paths containing more than a few corners. Consequently, a software package was developed to automate this process for paths described in the standard G-code format (ISO 6983). It incorporates algorithms for both the geometrical PH quintic corner rounding process, and the acceleration-limited assignment of the parameters V_0 and f in the corner feedrate functions. The three-phase acceleration-limited feedrate profile is also applied along the linear path segments, as necessary (Fig. 1).

Figure 4 illustrates the logical flow of the software. It begins by parsing the G-code part program line-by-line, storing information such as modal type, feedrate, and position data for each block. The “rounding algorithm” module detects sharp corners in the original G-code path, characterized by two consecutive G01 segments between non-collinear points $\mathbf{q}_0, \mathbf{q}_1, \mathbf{q}_2$ from which the turning angle θ may be determined. Let

$$\mathbf{v}_i = \frac{\mathbf{q}_1 - \mathbf{q}_0}{|\mathbf{q}_1 - \mathbf{q}_0|} \quad \text{and} \quad \mathbf{v}_o = \frac{\mathbf{q}_2 - \mathbf{q}_1}{|\mathbf{q}_2 - \mathbf{q}_1|}$$

be unit vectors along the incoming and outgoing segments. These segments are replaced by truncated versions from $\mathbf{q}_0$ to $\mathbf{q}_1 - L\mathbf{v}_i$ and $\mathbf{q}_1 + L\mathbf{v}_o$ to $\mathbf{q}_2$, where L is the corner curve side length. The PH quintic corner curve $\mathbf{r}(\xi)$ is specified [5] by the three points $\mathbf{p}_i = \mathbf{q}_1 - L\mathbf{v}_i$, $\mathbf{p}_c = \mathbf{q}_1$, $\mathbf{p}_o = \mathbf{q}_1 + L\mathbf{v}_o$, from which its Bézier control points $\mathbf{p}_0, \dots, \mathbf{p}_5$ are determined [5] so it meets the truncated G01 segments with G^2 continuity at $\xi = 0$ and $\xi = 1$.

The side length L is used [5] as a parameter to ensure that the maximum deviation δ of the corner curve from the orig-

inal sharp corner point $\mathbf{p}_c$ is equal to a specified tolerance ϵ . By symmetry, the maximum deviation occurs at the curve midpoint $\mathbf{r}(\frac{1}{2})$, and L can be expressed [5] in terms of δ as

$$L(\delta) = \frac{8\delta(6\cos\frac{1}{2}\theta + 1)}{(3\cos\frac{1}{2}\theta + 8)|\sin\frac{1}{2}\theta|}. \quad (5)$$

It is also of interest to relate the choice of L to the common practice of using circular arcs of radius R for corner rounding, in which case the maximum deviation from the sharp corner is

$$\delta = \frac{R(1 - \cos\frac{1}{2}\theta)}{\cos\frac{1}{2}\theta}. \quad (6)$$

Using this expression in (5) allows a comparison of the PH quintic and circular arc corner rounding curves, under the constraint of equal maximum deviation from the corner point. The PH corner curve can also be made to conform to the arc length and chord length of the circular arc. However, these are less useful comparisons because the circular arc is only G^1 continuous with the linear segments, and the G^2 continuity of the PH quintic corner curve causes it to have extended initial and final low-curvature portions, that result in a greater arc length and chord length than that of the circular arc for a given extremum deviation or curvature. Using the results of [5], the L values that result from equating the arc length and chord length of the PH quintic and circular arc corner curves may be expressed as

$$L_S = \frac{R|\theta|(6\cos\frac{1}{2}\theta + 1)}{2(6 + \cos\frac{1}{2}\theta)\cos\frac{1}{2}\theta} \quad \text{and} \quad L_C = R|\tan\frac{1}{2}\theta|.$$

Once L has been determined, the Bézier control points of the PH quintic corner curve can be expressed using the results in [5] as

$$\mathbf{p}_0 = \mathbf{q}_1 - L\mathbf{v}_i, \quad \mathbf{p}_1 = \mathbf{p}_2 = \mathbf{p}_0 + \ell\mathbf{v}_i,$$

$$\mathbf{p}_5 = \mathbf{q}_1 + L\mathbf{v}_0, \quad \mathbf{p}_3 = \mathbf{p}_4 = \mathbf{p}_5 - \ell\mathbf{v}_0,$$

where

$$\ell = \frac{6L\cos\frac{1}{2}\theta}{6\cos\frac{1}{2}\theta + 1}.$$

Examples of toolpaths rounded by the algorithm are shown in Section 5.

Once the corner rounding has been accomplished, the feedrate parameters for each corner segment must be assigned (namely, the entry/exit feedrate V_0 and feedrate suppression factor $f = V_{\min}/V_0$). The goal is to choose values so as to minimize the corner traversal time T , while satisfying the prescribed bound $a_{\lim}$ on the acceleration magnitude. This is the most computationally demanding aspect of the overall software architecture illustrated in Fig. 4, and is performed as an off-line pre-processing task.

Fig. 5 Surface $a_{\max}(V_0, f)$ for the corner curve in Fig. 2 executed with the parameter-dependent feedrate, showing the contour $a_{\max} = 250\text{in/sec}^2$

The greatest acceleration magnitude incurred in executing a given corner curve with a specified feedrate function is a bivariate function $a_{\max}(V_0, f)$ of V_0 and f . Figure 5 illustrates the surface specified by this function for the execution of the corner curve in Fig. 2, utilizing the parameter-dependent feedrate function. It is seen that $a_{\max}$ increases sharply with both V_0 and f . Note that, for given V_0 and f , both the tangential and normal acceleration components in (1) contribute to $|\mathbf{a}(\xi)|$, so a numerical iteration on the curve parameter ξ is required to find its maximum value $a_{\max}$.

The constraint $a_{\max}(V_0, f) = a_{\lim}$ specifies a contour on the surface, that identifies V_0, f combinations for which the acceleration bound $a_{\lim}$ is attained at some instant during the corner traversal—Fig. 5 shows such a contour for the value $a_{\lim} = 250\text{in/s}^2$ used in the kinematic analysis of Fig. 2.

For each feedrate type, the corner traversal time is known [5] as a bivariate function $T(V_0, f)$, and the contour $a_{\max}(V_0, f) = a_{\lim}$ can be mapped to a (non-planar) curve $C_{\lim}$ on the surface defined by $T(V_0, f)$ —as illustrated in Fig. 6 for the example in Fig. 5. Figure 6 also shows the projections of $C_{\lim}$ onto the (V_0, T) and (f, T) planes. Two important conclusions, that hold for any corner curve geometry under the constraint $a_{\max}(V_0, f) = a_{\lim}$, can be inferred. First, the traversal time T possesses a well-defined minimum. Second, there may be multiple f values corresponding to a single V_0 value along $C_{\lim}$, but there is only one V_0 value for each f . Hence, only f and T are in one-to-one correspondence along $C_{\lim}$. These facts permit the use of robust numerical methods, with guaranteed convergence, on the minimum-time solution for the feedrate parameters V_0 and f .

As indicated in Fig. 4, the “corner feedrate algorithm” module of the corner rounding software determines the

Fig. 6 Upper: the curve C_{lim} on the corner curve traversal time surface $T(V_0, f)$ —corresponding to the contour $a_{max}(V_0, f) = a_{lim}$ on the surface in Fig. 5. Lower: the projections of C_{lim} onto the (V_0, T) and (f, T) planes

feedrate parameters using a nested series of numerical iterations. Imposing the requirement $a_{max}(V_0, f) = a_{lim}$ and the additional boundary condition $V_0 \leq V_l$, the time function $T(V_0, f)$ is minimized along the acceleration-limited constraint curve C_{lim} . The highest-level iteration is on the parameter f , since it is one-to-one with T . Inside the f iteration, further iterations to find the corresponding V_0 and ξ values to ensure that $a_{max}(V_0, f) = a_{lim}$ are required. The final outcome is the optimal combination of V_0 and f values

that minimize the corner traversal time T along the corner, subject to acceleration constraint $|a(\xi)| \leq a_{lim}$.

For linear path segments between rounded corners, specification of the three-phase feedrate profile requires the exit feedrate V_i of the previous block, the global linear feedrate V_l , and entry feedrate V_f of the following block (V_i and V_f are equal to the V_0 values for the preceding and succeeding corners). Determination of the three-phase feedrate profile requires the use of a rolling three-block buffer, so that G-

Fig. 7 Left three-axis mill governed by OpenCNC controller. Right display showing the G05 part program that corresponds to the toolpath in Fig. 2

code commands are not output to the modified part program before their geometry and feedrate profiles are finalized.

The time-optimal setting of the cornering feedrate function parameters, combined with the three-phase feedrate profiles along linear segments, offers a simple kinematic basis for efficient execution of piecewise-linear paths with rounded corners, that avoids the complicated mathematical formulations and algorithmic implementations incurred by the formal “bang-bang” solutions [1, 6, 17, 19] to time-optimal motion problems under prescribed acceleration bounds. The proposed strategy yields significant time savings, while globally satisfying the specified acceleration bounds over the entire path.

4 Real-time implementation

The high-speed corner rounding strategy was implemented on a three-axis milling machine (see Fig. 7), governed by the MDSI OpenCNC open-architecture software controller that allows incorporation of custom real-time interpolator and motion control algorithms. The controller runs on an “off-the-shelf” PC with a modest 500 MHz processor, and has a position sampling frequency of 1024 Hz, corresponding to a sampling interval $\Delta t \approx 0.001$ s.

Figure 8 outlines the controller logic for the high-speed cornering strategy. It is necessary to communicate to the machine both the toolpath geometry (a combination of lin-

Fig. 8 Simplified controller logic flowchart for real-time implementation of the high-speed corner rounding strategy, including feedrate adjustment to ensure an exact integer number of interpolation steps within each block

ear segments and PH quintic corner curves) and feedrate information that defines the corner feedrate profiles and three-phase feedrate profiles for the linear segments (see Fig. 1). This can be achieved through the use of G-code commands. A G-code part program consists of a sequence of lines or “blocks” containing one or more “words” that comprise an address character paired with a numerical value [15]. The “interpreter” module of the controller software in Fig. 8 parses the G-code part program and stores the motion instructions and associated numerical values.

A G-code structure has been established [4] to analytically specify PH curve toolpaths and associated feedrate functions to CNC machines. The G05 code, unused in traditional part programming, was adopted for this purpose. This G05 structure is an evolving convention that is open to extensions and modifications, such as in recent work on surface machining [9]. In addition to specifying PH curve geometry in terms of a target point and the coefficients of the hodograph pre-image polynomials $\alpha(\xi)$ and $\beta(\xi)$, the G05 format allows assignment of several different feedrate variations through the F word, with associated numerical values specified through the U, V, W words. Building on this, Table 1 summarizes the address characters adopted in the high-speed corner rounding strategy. F indicates the type of feedrate profile for each block, and also whether it is a linear segment or a corner curve, since a linear segment has the three-phase F7 feedrate designation. For the corner curves, the curvature-dependent, parameter-dependent, and hybrid feedrate forms are designated by F8, F9, F10. The toolpath in Fig. 2, for example, is specified by the following G-code part program⁵ when using the parameter-dependent (F9) feedrate type and $V_l = 800$ ipm:

G05 F7 U0.00 V745.07 W800.00 X0.000 Y3.413

G05 F9 V0.486 X0.586 Y4.000 A1.088 B0.000 C1.539 P1.088 Q0.000 R0.000

G05 F7 V0.00 X4.000

The “block look-ahead” controller module performs certain preparatory computations for each block, before execution of the real-time interpolator. These can be used to slightly modify certain feedrate parameters for better execution in the controller, and to minimize the computational load during interpolation (in the case of the PH quintic cornering feedrate functions, for example, by pre-computation of several constants arising in the interpolation function $F(\xi)$ —see [5]). The corner geometry data is also constant during execution, and can be calculated and stored beforehand.

⁵For brevity, numerical data is shown here to three decimal places. In practice, a much higher precision should be employed for the A, B, C and P, Q, R coefficients. Also, modal values such as feedrates are not repeated if they do not change from one block to the next.

Table 1 Address characters for the high-speed corner rounding strategy

Address	Interpretation
G05	Preparatory function
F	Feedrate type
F7	Three-phase linear
F8	Curvature-dependent
F9	Parameter-dependent
F10	Hybrid feedrate
U, V, W	Feedrate parameters
U	V_i for F7; V_0 for F8, F9, F10
V	V_l for F7; f for F8, F9, F10
W	V_f for F7
X, Y, Z	End-point coordinates
A, B, C	$\alpha(\xi)$ Bernstein coefficients
P, Q, R	$\beta(\xi)$ Bernstein coefficients

For optimum motion smoothness, particularly in transitioning between blocks, it is necessary to consider the digital nature of the control algorithm. For a given total segment execution time T and a sampling interval Δt the ratio $T/\Delta t$ does not, in general, correspond to a whole number of intervals. To avoid terminal position and feedrate errors incurred by the fact that reaching a segment end-point requires only a fraction of a sampling interval, the nominal feedrate V is perturbed slightly to the value defined by

$$\tilde{V} = \frac{VT}{N\Delta t}, \quad (7)$$

where N is defined using the nearest-integer function⁶ as

$$N = \lfloor T/\Delta t \rfloor. \quad (8)$$

This strategy is used for all the feedrate types employed in high-speed cornering, to ensure smooth transitions between segments. Expressions for the total cornering time T , corresponding to each of the corner feedrate types, can be found in [5]. The total execution time for a linear path segment using the three-phase feedrate profile can be expressed as

$$T = \frac{S_l}{V_l} + \frac{15}{8} \frac{|V_l - V_i| + |V_l - V_f|}{a_{\text{lim}}},$$

where S_l is the length of the middle (coasting) phase of the feedrate—for a linear segment of total length S , this can be expressed as

$$S_l = S - \frac{15}{16} \frac{|V_l^2 - V_i^2| + |V_l^2 - V_f^2|}{a_{\text{lim}}}.$$

⁶The floor and ceiling functions $\lfloor T/\Delta t \rfloor$ and $\lceil T/\Delta t \rceil$ could also be used instead.

Fig. 9 The velocity and acceleration profiles corresponding to the path in Fig. 2, with (*left*) and without (*right*) the feedrate adjustment strategy

It should be noted that using the perturbed feedrate (7) will incur a small velocity discontinuity at the segment junctures, and the motion is no longer precisely acceleration-limited. In practice, however, these effects are usually of small magnitude compared with the consequence of not using the adjusted feedrate (7). Using (8), the velocity discontinuity $\Delta V = \tilde{V} - V$ satisfies

$$\frac{|\Delta V|}{V} \lesssim \frac{1}{2N}.$$

For the results presented in Section 5, this value is almost always $\lesssim 0.5\%$. Figure 9 compares velocity and acceleration profiles, obtained from the axis encoder data, in executing the path in Fig. 2 with and without the feedrate adjustment. It is apparent that, at the junctures of the path segments, there are significant disturbances in the velocity and acceleration when the feedrate adjustment method is not used, compared to when it is enforced. In fact, the acceleration disturbances incurred when $T/\Delta t$ is not an integer substantially exceed the specified acceleration limit $a_{\text{lim}} = 250 \text{ in/s}^2$. This offers strong empirical evidence for the benefits of the feedrate adjustment strategy.

Analytic reductions of the interpolation integrals for the variable feedrate functions described in Section 2 yield highly accurate and efficient real-time interpolator algorithms for the corner curves, based on computing the unique real root of a monotone function $F(\xi)$ on $\xi \in [0, 1]$ by the Newton–Raphson method [5]. The real-time interpolator algorithm for the “S-shaped” feedrate profiles along linear segments may be found in [18]. Applying the feedrate adjustment for an integer number of sampling intervals

on each segment, the final step yields the parameter value $\xi = 1.0$ within machine precision.

5 Results

Based on the implementation described in Section 4, the high-speed cornering approach was run on several test paths specified by G05 part programs, with real-time axis encoder data recorded in memory for off-line analysis. In a previous study [11], the three-axis milling machine (Fig. 7) used for these experiments was found to have conservative acceleration limits on the X and Y axis drives of $a_{\text{lim}} = 250 \text{ in/s}^2$, and this value has been adopted in the present experiments. The results presented below are intended to elucidate any practical differences between the cornering feedrate functions described in Section 2, and to quantify their performance compared with the customary “full stop” method for executing the original sharp-corner toolpaths.

Consider first the corner of Fig. 2 again. Figure 10 shows the measured contour (normal) error,⁷ feedrate, and acceleration magnitude for execution of this path with the *curvature-dependent* feedrate (results with the other feedrate types are similar, and for brevity are not shown). The feedrate and acceleration plots are quite similar to those shown on the left of Fig. 9 for the *parameter-dependent* feedrate case, although at a somewhat lower V_0 (626.5 versus

⁷The contour error is the deviation from the specified path, in the normal direction at the closest path point.

Table 2 Performance results for a 90° corner with $\epsilon = 0.1$ in, $V_l = 800$ ipm

	Parameter dependent	Curvature dependent	Hybrid	Full stop
T_{total} (s)	0.723	0.737	0.746	0.801
ΔT_{total} (s)	0.078	0.064	0.055	–
ΔT_{total} (%)	9.8	7.9	6.8	–
$\Delta T_{\text{total}}/n$ (s)	0.078	0.064	0.055	–
S_{total} (in)	7.888	7.888	7.888	8.000
ΔS_{total} (%)	1.4	1.4	1.4	–
Max contour error (in)	0.0015	0.0014	0.0014	0.0002

745.1 ipm). A key point of interest is whether the tangential acceleration discontinuity upon entry to and exit from the corner curve is of any practical significance for the curvature-dependent feedrate. As is evident from the velocity and acceleration plots in these two figures (and many other examples that were studied), any effect is lost within the measurement noise and the small discontinuity induced by the feedrate adjustment scheme.

Some key performance results for the single 90° corner path, using the three cornering feedrate functions and full stop method, are listed in Table 2. T_{total} and S_{total} are the total execution time and path length, respectively, ΔT_{total} and ΔS_{total} are their differences relative to the full stop reference case (indicated positive for a reduction), and n is the

number of toolpath corners. The rounded corner, in conjunction with the continuously-variable feedrate functions, provide a significant % improvement in T_{total} , considering that it arises from a single corner (the time along the linear toolpath segments is essentially identical for the rounded-corner and full stop methods, since they employ the same three-phase feedrate profiles). Although the improvement is partly due to the fact that the rounded corner yields a smaller S_{total} , the time savings is mainly attributable to the variable feedrate along the rounded corner, since the % reduction in T_{total} greatly exceeds that in S_{total} .

Figures 11, 12, and 13 and Tables 3, 4, and 5 present results for three more toolpaths of increased complexity using the variable-feedrate types. Note that the corner

Fig. 10 Execution of a single 90° corner rounded to a tolerance $\epsilon = 0.1$ in using the curvature-dependent feedrate function with feedrate $V_l = 800$ ipm

Fig. 11 Execution of the rounded logarithmic spiral approximation with $\epsilon = 0.02$ in, employing the parameter-dependent feedrate and $V_l = 800$ ipm

Fig. 12 A sheet metal part rounded to equivalent circular arc deviation $R = 0.15$ in, executed using the hybrid feedrate function with $V_l = 800$ ipm

Fig. 13 Fourth-order Hilbert curve approximation, rounded to $\epsilon = 0.02$ in and traversed with the parameter-dependent feedrate using $V_l = 440$ ipm

curves typically exhibit long initial/final portions of low curvature, making the corners appear larger than is suggested by their extremum deviation δ .

Figure 11 shows performance results for a piecewise-linear approximation to a logarithmic spiral (polar equation $r(\phi) = a \exp(b\phi)$), with successively sharper turning angles ranging from $\theta = 21.8^\circ$ to 132.8° . As seen in Fig. 11, the shortened linear path segments between consecutive corners forces the feedrate V_l along them to be suppressed below 800 ipm, to ensure that the acceleration limit a_{lim} is

satisfied on both the rounded corners and the linear path segments. The corner rounding strategy yields reductions in the path execution time ranging from about 24 to 29 %, depending on the feedrate function employed (see Table 3). Note also that a contour error of $\lesssim 0.001$ in is maintained throughout the entire path, despite the rather violent nature of the motion as the path spirals inward toward the limit point.

Figure 12 shows an example of a “realistic” toolpath, for a sheet metal part. In this case, the corner tolerance was

Table 3 Performance results for the logarithmic spiral approximation shown in Fig. 11, with tolerance $\epsilon = 0.02$ in, $n = 31$ corners, and $V_l = 800$ ipm

	Parameter dependent	Curvature dependent	Hybrid	Full stop
T_{total} (s)	5.432	5.720	5.826	7.626
ΔT_{total} (s)	2.194	1.906	1.800	—
ΔT_{total} (%)	28.8	25.0	23.6	—
$\Delta T_{\text{total}}/n$ (s)	0.071	0.061	0.058	—
S_{total} (in)	56.156	56.156	56.156	56.618
ΔS_{total} (%)	0.8	0.8	0.8	—
Max normal error (in)	0.0013	0.0013	0.0014	0.0003

Table 4 Performance results for a sheet metal part with tolerance equivalent to a circular fillet radius $R = 0.15$ in, $n = 31$ corners, and $V_l = 800$ ipm

	Parameter dependent	Curvature dependent	Hybrid	Full stop
T_{total} (s)	3.701	4.062	4.114	6.173
ΔT_{total} (s)	2.472	2.111	2.059	–
ΔT_{total} (%)	40.0	34.2	33.3	–
$\Delta T_{\text{total}}/n$ (s)	0.080	0.068	0.066	–
S_{total} (in)	33.162	33.162	33.162	37.027
ΔS_{total} (%)	10.4	10.4	10.4	–
Max normal error (in)	0.0014	0.0012	0.0013	0.0003

specified in terms of an equivalent circular arc fillet radius $R = 0.15$ in, so δ is a function of θ as specified in (6). As seen in Table 4, the corner rounding strategy achieves reductions in total execution time ranging from about 33 to 40 % in this case.

Finally, Fig. 13 shows an application of the corner rounding strategy to a fourth-order approximation of the *Hilbert curve* [7]—a fractal “space-filling” curve, constructed within a 6×6 inch square. As seen in the figure, this involves an intricate sequence of right-angle corners between linear segments. Because of the short lengths of these segments, the reduced feedrate value $V_l = 440$ ipm was employed in this case. Performance results for this example are listed in Table 5—the reduction in total execution time ranges between about 25 and 32 %, depending on the choice of cornering feedrate function. A very tight contour error of $\lesssim 0.001$ in is maintained throughout the path, despite the intricacy of the motion.

Several conclusions can be inferred from the experimental data. First, the parameter-dependent feedrate systematically yields the greatest reduction in the total path execution time under the acceleration constraint $a_{\max}(V_0, f) = a_{\text{lim}}$, while the hybrid feedrate yields the smallest reduction and the curvature-dependent feedrate is intermediate between them. For paths with many sharp corners, however, all three cornering feedrate functions offer substantial time reductions over the full stop strategy required to exactly negotiate a sharp corner. The maximum measured contour errors are quite uniform among the three cornering feedrate functions, and the different toolpaths employed in the experiments. This is consistent with the expected correlation of normal path error with normal acceleration, whose peak value has been limited to the universal value a_{lim} through the feedrate assignment algorithm.

Although it has no intuitive geometrical interpretation, the experimental results support a recommendation of the parameter-dependent feedrate as the preferred choice for

high-speed cornering. Compared to the curvature-dependent feedrate (which exhibits a simple correlation between feedrate and path curvature), it offers faster execution times and tangential acceleration continuity⁸ upon entering and leaving the corner curves. The hybrid feedrate, which is essentially a parameter-dependent modification of the curvature-dependent feedrate, appears to be the least-favored choice.

The empirical results support the principles guiding the kinematic design philosophy presented in Section 2. Despite aggressive feedrates and frequent acceleration limitation, the machine accurately executes intricate toolpaths while faithfully maintaining the prescribed feedrate profile and observing the acceleration magnitude limit $a_{\text{lim}} = 250 \text{ in/s}^2$. Substantial reductions in total path execution times are realized, compared to the traditional full stop method—on a realistic sheet metal part contour, the parameter-dependent feedrate achieved 40 % time savings with a 0.0014 in maximum normal error and corners equivalent to circular arcs of radius $R = 0.15$ in.

The full stop method has two notable advantages—it executes the exact path geometry without any corner rounding and has, in practical terms, zero normal error (since it never incurs any normal path acceleration). However, in many application contexts, a tolerance-based rounding of sharp corners is not only acceptable, but actually a desired goal in its own right—especially in high-speed 2D applications such as laser cutting. As previously noted, the acceleration-limited kinematic design strategy naturally yields a maximum contour error that converges to a constant value. With an appropriate choice of a_{lim} , it is possible to keep this error within very reasonable bounds.

⁸However, as previously noted, the tangential acceleration discontinuities associated with the curvature-dependent feedrate are relatively insignificant.

Table 5 Performance results for the fourth-order approximation to a Hilbert curve in a 6×6 in square, with $\epsilon = 0.02$ in., $n = 204$, and $V_l = 440$ ipm

	Parameter dependent	Curvature dependent	Hybrid	Full stop
T_{total} (s)	18.767	20.021	20.817	27.573
ΔT_{total} (s)	8.807	7.553	6.756	–
ΔT_{total} (%)	31.9	27.4	24.5	–
$\Delta T_{\text{total}}/n$ (s)	0.043	0.037	0.033	–
S_{total} (in)	97.436	97.436	97.436	102.000
ΔS_{total} (%)	4.5	4.5	4.5	–
Max normal error (in)	0.0012	0.0010	0.0009	0.0003

6 Conclusion

Traditional feedrate planning employs constant segment feedrates connected with acceleration/deceleration profiles. For piecewise-linear paths with many sharp corners, this incurs relatively long overall execution times. The present study has shown that using G^2 PH quintic rounded corner curves, combined with continuously-variable cornering feedrate functions, can offer substantial efficiency improvements in high-speed execution of complicated toolpaths. The empirical performance of three different cornering feedrate functions—each of which admits analytic reduction of the interpolation integral—has been evaluated and compared, and a clear overall recommendation for the preferred choice among them emerges from the experimental results.

A key aspect of the proposed methodology is the use of an acceleration-limited strategy in the assignment of segment feedrate parameters. Currently, an off-line algorithm is employed for this purpose, but with a fast processor and block look-ahead strategy, it is conceivable that this aspect of the motion planning can also be performed in real time. Feedrate planning governed by specified acceleration limits yields smoother path executions, and consistent global suppression of contour error along the entire toolpath.

The experiments indicate that the corner-rounding strategy is perfectly amenable to a practical implementation on machines with open-architecture controllers. The advantages over the “full stop” method for exact traversal of sharp corners have been clearly demonstrated, and similar benefits over the simple strategy of using constant-feedrate circular arcs as rounding elements (which necessarily incur acceleration discontinuities) will also hold.

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