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Authors

Al Majid, A

Allezy, A

Dufour, R

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Metric Damping of MDOF Systems in High Transient Motion

A. Al Majid

e-mail: ahmad.almajid@insa-lyon.fr

A. Allezy

e-mail: arnaud.allezy@insa-lyon.fr

R. Dufour

e-mail: regis.dufour@insa-lyon.fr

Laboratoire de Dynamique des Machines et des Structures,
UMR CNRS 5006, INSA de Lyon,
69621 Villeurbanne, France

This paper deals with damping due to transient motion in the case of multi-degree-of-freedom (MDOF) system. The main aim of this research is to make the method presented by the authors in a previous paper available for MDOF systems. An method based on relativity concepts is developed in order to identify and evaluate a metric damping due to time-varying forcing frequency. An additional dimension for each degree of freedom (DOF) is introduced. The variational problem of the metric of a Riemannian space gives the geodesic equations, i.e., equations of motion that, after time integration carried out with several types of numerical schemes, permit one to predict the forced transient response of a 3-DOF system. The proposed metric approach makes the experimental results correspond with the simulated results. [DOI: 10.1115/1.2128638]

1 Introduction

The response of mechanical systems is greatly influenced by damping, which has a dissipative effect. Broadly speaking, damping in mechanical systems remains delicate and complicated to model [1,2]. It especially depends on the type of excitation, see for example, [3–5]. In [6,7], Al Majid and Dufour compared the measured and predicted responses of a single-degree-of-freedom (SDOF) system subjected to a forcing frequency varying considerably through time and showed that using a classical model is inefficient, particularly after the resonance phenomenon. Inspired by Einstein's general relativity principle [8–14], they presented an original method for evaluating the dissipative effect due to the transient phenomenon. This method, based on the use of an additional dimension in a pseudo-Riemannian space (a non-Euclidian space) is used to predict the response which, compared to the measured response of a SDOF system, permitted the validation of the proposed model or metric model.

The aim of this paper is to improve and extend the method proposed in [7] to multi-degree-of-freedom (MDOF) systems. Sections 2 and 3 are devoted to the MDOF metric model and the classical model. In Sec. 3, an experimental 3-DOF spring-pendulum system is used for testing the metric and classical models. The metric and classical responses are predicted using several numerical schemes and compared to the measured response.

2 MDOF Metric Model

2.1 Dimension and Metric of the Space. Let a discrete mechanical system be subjected to an external force $F = F_0 e^{i \phi(t)}$ that has a time-varying forcing frequency. It is assumed that the forcing frequency is the only parameter that has an effect on space. The extension of a metric method to MDOF systems is performed by using the modal method. If $\mathbf{x} = (x^1, x^2, \dots, x^n)$ is the real variable vector of the MDOF system studied, the interval between two successive positions is expressed in tensor notation by an element of infinitesimal length ds given by Eq. (1), where $g_{\mu\nu}$ is the metric tensor

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu, \quad (\mu, \nu) \in [1, n]^2 \quad (1)$$

Let $\mathbf{q} = (q^1, q^2, \dots, q^n)$ be the modal variable vector defined by Eq. (2), where the squared matrix Φ is the full modal matrix

$$\mathbf{x} = \Phi \mathbf{q} \quad (2)$$

Then, assuming the mode shapes are uncoupled, each modal equation obtained can be treated independently. In order to take into account the influence of the forcing frequency on space, the frame of reference is equipped with several additional modal coordinates. The variable q^0 , representing time, is defined by Eq. (3) where a is a high speed, t is the time and $i^2 = -1$

$$q^0 = iat \quad (3)$$

Assuming there is only one influential parameter, one additional variable is devoted to each DOF of the MDOF system as defined in Eq. (4), where ρ_μ and θ^μ are, respectively, the constant radius and the angle of rotation

$$q^{-\mu} = i\rho_\mu \theta^\mu, \quad \frac{dq^{-\mu}}{ds} = i\rho_\mu \dot{\theta}^\mu, \quad \mu \in [1, n] \quad (4)$$

Consequently, $\dot{\theta}^\mu$ corresponds to the frequency of the motion of the DOF μ .

2.2 Metric Determination. In the case of a flat space, such as the Minkowski space, the metric tensor is $\delta_{\mu\nu}$ and the Riemann tensor $R^\alpha_{\lambda\mu\nu}$ is nil. Let the motion have a small effect on the space time and slightly curve it. Thus, the Ricci tensor is nil

$$R_{\mu\nu} = 0 \quad (5)$$

This expresses the maximum independence between the coordinates. In general relativity, this corresponds to the Einstein field equations in free space.

The Ricci tensor is calculated by Eq. (6) where $\Gamma^k_{\mu\nu}$ is the symbol of Christoffel of the second kind, defined by Eq. (7),

$$R_{\mu\nu} = \frac{\partial \Gamma^\sigma_{\mu\nu}}{\partial x^\sigma} - \frac{\partial \Gamma^\sigma_{\mu\sigma}}{\partial x^\nu} + \Gamma^\lambda_{\sigma\nu} \Gamma^\sigma_{\mu\lambda} - \Gamma^\lambda_{\sigma\lambda} \Gamma^\sigma_{\mu\nu} \quad (6)$$

$$\Gamma^k_{\mu\nu} = \frac{1}{2} g^{ik} \left(\frac{\partial g_{\mu l}}{\partial x^\nu} + \frac{\partial g_{\nu l}}{\partial x^\mu} - \frac{\partial g_{\mu\nu}}{\partial x^l} \right) \quad (7)$$

The metric tensor $g_{\mu\nu}$ can be expressed by the diagonal form presented in

$$g_{\mu\nu} = \delta_{\mu\nu} + \varepsilon \gamma_{\mu\nu}(q^{-1}, q^{-2}, \dots, q^{-n}) \quad (8)$$

The term $\varepsilon \gamma_{\mu\nu}(q^{-1}, q^{-2}, \dots, q^{-n})$, is a small perturbation metric

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tensor. In order to obtain a corresponding form between geodesic and Lagrange equations, it is necessary to introduce the potential energy $U(q^0, q^1, q^2, \dots, q^n)$, as indicated in [7], and therefore to replace $g_{00} = 1 + \varepsilon \gamma_{00}(q^{-1})$ by the expression given by

$$g_{00} = 1 + \varepsilon \gamma_{00}(q^{-1}, q^{-2}, \dots, q^{-n}) + 2H(q^{-1}, q^{-2}, \dots, q^{-n}) \frac{U(q^0, q^1, q^2, \dots, q^n)}{a^2} \quad (9)$$

The metric tensor obtained is then obtained as presented as follows:

$$\mathbf{g} = \begin{pmatrix} 1 + \varepsilon \gamma_{-n-n} & \cdots & 0 & 0 & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 1 + \varepsilon \gamma_{-1-1} & 0 & 0 & \cdots & 0 \\ 0 & \cdots & 0 & 1 + \varepsilon \gamma_{00} + 2H \frac{U}{a^2} & 0 & \cdots & 0 \\ 0 & \cdots & 0 & 0 & 1 + \varepsilon \gamma_{11} & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & 0 & 0 & \cdots & 1 + \varepsilon \gamma_{nn} \end{pmatrix} \quad (10)$$

To fulfill the condition presented in Eq. (5), $\gamma_{\mu\nu}$ and H are expressed as functions of one unknown function h , which can be determined by solving Einstein's equations as explained in [7]. A nil Ricci tensor forces the Einstein tensor to be nil

$$G_{ij} = R_{ij} - \frac{1}{2} g_{ij} R = 0 \quad (11)$$

R is the Ricci scalar (the trace of the Ricci tensor). Using this method, the coefficients of the metric tensor $g_{\mu\nu}$ are obtained and the geodesic equations give the equations of the motion. As the modal method is used, the results of [7] for SDOF systems give, for each modal DOF, the set of coupled equations presented as follows:

$$\ddot{q}^j + 2\alpha_j \dot{q}^j + \Omega_j^2 q^j = \frac{1}{m_j} f(t) \quad (12)$$

$$\ddot{\theta}^j - 2\frac{\alpha_j}{\rho_j^2} \left(\frac{1}{2} \Omega_j^2 (q^j)^2 - \frac{1}{m_j} f_j(t) q^j - \frac{1}{2} (\dot{q}^j)^2 \right) = 0 \quad (13)$$

Associated with the DOF j , m_j , α_j , and $f_j(t)$ are the modal mass, viscous damping factor, and force, respectively; and Ω_j is the natural frequency. If $\ddot{\theta}^j = 0$, the damping is constant and Eq. (12) is reduced to the classical model equation.

2.3 State-Space Formulation. The state-space form of Eqs. (12) and (13) yields the system given by

$$\dot{\mathbf{X}} = \mathbf{A}\mathbf{X} + \mathbf{B}\mathbf{U}$$

$$\mathbf{Y} = \mathbf{C}\mathbf{X} + \mathbf{D}\mathbf{U} \quad (14)$$

The different terms of the system presented in Eq. (14) are given by

$$\mathbf{X} = \begin{bmatrix} \mathbf{q} \\ \dot{\mathbf{q}} \\ \boldsymbol{\theta} \\ \dot{\boldsymbol{\theta}} \end{bmatrix}, \quad \mathbf{U} = \begin{bmatrix} f(t) \\ \mathbf{q} f(t) \\ \mathbf{q}^2 \\ \dot{\mathbf{q}}^2 \\ \dot{\mathbf{q}} \dot{\boldsymbol{\theta}} \end{bmatrix}, \quad (15)$$

$$\mathbf{A} = \begin{bmatrix} \mathbf{0} & \mathbf{Id} & \mathbf{0} & \mathbf{0} \\ -\mathbf{M}_{\text{mod}}^{-1} \mathbf{K}_{\text{mod}} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{Id} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix}_{4n \times 4n}$$

$$\mathbf{B} = \begin{bmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{M}_{\text{mod}}^{-1} \mathbf{V}_{\text{mod}} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & -\frac{2\alpha_j}{\rho_j^2} \mathbf{M}_{\text{mod}}^{-1} \mathbf{V}_{\text{mod}} & \frac{\alpha_j}{\rho_j^2} \mathbf{M}_{\text{mod}}^{-1} \mathbf{K}_{\text{mod}} & -\frac{\alpha_j}{\rho_j^2} \mathbf{0} & \mathbf{0} \end{bmatrix}_{4n \times (1+4n)} \quad (16)$$

$$\mathbf{C} = \mathbf{Id}_{4n \times 4n}, \quad \mathbf{D} = \begin{bmatrix} \mathbf{0} \\ \mathbf{M}_{\text{mod}}^{-1} \mathbf{V}_{\text{mod}} \end{bmatrix}, \quad \boldsymbol{\alpha} = \begin{bmatrix} \alpha_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \alpha_n \end{bmatrix} \quad (17)$$

$\mathbf{M}_{\text{mod}}$ and $\mathbf{K}_{\text{mod}}$ are, respectively, the modal mass and stiffness modal matrices. Solving Eqs. (12) and (13) requires the conditions presented as follows:

$$t = 0 \Rightarrow \theta^j = 0, \quad q^j = 0, \quad \dot{q}^j = 0 \quad (18)$$

$$t = t_j \Rightarrow \dot{\theta}^j = \Omega_j \quad (19)$$

where t_j is the time when $\omega = \Omega_j$. Constants ρ_j remain unknown but they can be assumed to be very small when comparing them to the unity by virtue of the least-action principle [7].

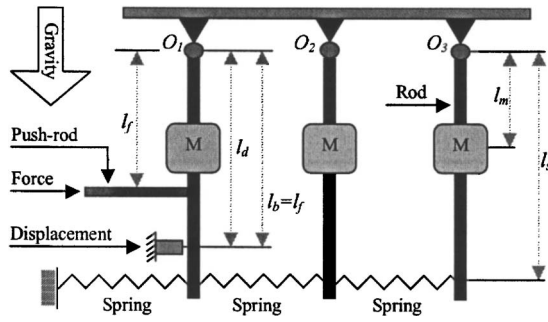


Fig. 1 Diagram of the experimental setup

3 Classical MDOF Model

3.1 Equation of Motion. The motion is governed by the set of linear differential equations given by

$$\mathbf{M}\ddot{\mathbf{x}} + \mathbf{K}\mathbf{x} = \mathbf{V}f(t) \quad (20)$$

where $\mathbf{M}$ is the mass matrix and $\mathbf{K}$ is the stiffness matrix. Moreover $\mathbf{x}$ and $\mathbf{V}f(t)$ are the displacement and force vectors, respectively. Expanding the solution on a modal base requires the introduction of the variable change (Eq. (2)) in the kinetic and strain energies and in the virtual work of the external force. This yields the modal equations presented as follows:

$${}^t\Phi\mathbf{M}\Phi\ddot{\mathbf{q}} + {}^t\Phi\mathbf{K}\Phi\mathbf{q} = {}^t\Phi\mathbf{V}f(t) \Leftrightarrow \mathbf{M}_{\text{mod}}\ddot{\mathbf{q}} + \mathbf{K}_{\text{mod}}\mathbf{q} = \mathbf{V}_{\text{mod}}f(t) \quad (21)$$

The modal form given by Eq. (21) permits the introduction of the modal diagonal damping matrix $\mathbf{C}_{\text{mod}}$, which coefficients are expressed from the measured modal viscous damping coefficients using a steady-state transfer function,

$$\mathbf{M}_{\text{mod}}\ddot{\mathbf{q}} + \mathbf{C}_{\text{mod}}\dot{\mathbf{q}} + \mathbf{K}_{\text{mod}}\mathbf{q} = \mathbf{V}_{\text{mod}}f(t) \quad (22)$$

3.2 State-Space Formulation. The state-space equations given by Eq. (14) are used with the terms defined as follows:

$$\mathbf{A} = \begin{bmatrix} \mathbf{0} & \mathbf{Id} \\ -\mathbf{M}_{\text{mod}}^{-1}\mathbf{K}_{\text{mod}} & -\mathbf{M}_{\text{mod}}^{-1}\mathbf{C}_{\text{mod}} \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} \mathbf{0} \\ -\mathbf{M}_{\text{mod}}^{-1}\mathbf{V}_{\text{mod}} \end{bmatrix}, \quad (23)$$

$$\mathbf{C} = \mathbf{Id}, \quad \mathbf{D} = \begin{bmatrix} \mathbf{0} \\ -\mathbf{M}_{\text{mod}}^{-1}\mathbf{V}_{\text{mod}} \end{bmatrix}, \quad \mathbf{X} = \begin{bmatrix} \mathbf{q} \\ \dot{\mathbf{q}} \end{bmatrix}, \quad \mathbf{U} = f(t)$$

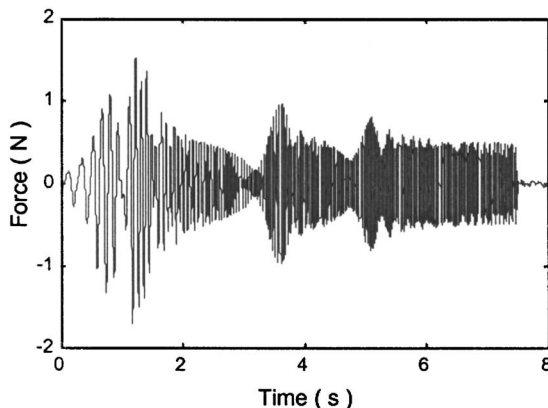


Fig. 2 Excitation force (0–50 Hz)

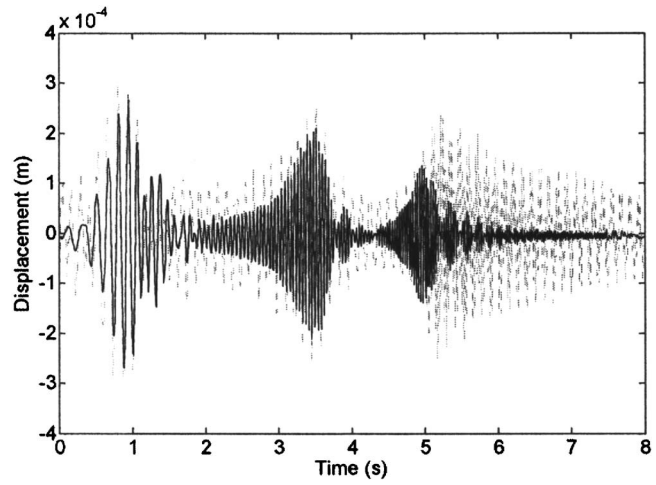


Fig. 3 Predicted response with a classical model (ODE113, dotted line) and measured response (solid line)

4 Experimental Validation

4.1 Experimental Setup. The mechanical system is composed of three in-plane compound and identical pendulums hinged by ball bearings at points O_1 , O_2 , and O_3 , respectively, and connected to each other by identical springs. The position of the three masses M can be modified in order to adjust the natural frequencies of the system. In the frequency range considered, this system can be modeled by three DOF (see Fig. 1).

An electrodynamic shaker permits a force to be applied, measured by a piezoelectric sensor, onto the first subsystem (i.e., first DOF) whose displacement is measured by a proximity probe. A signal dynamic analyzer supplies a periodic chirp source and records the time history of both the force transmitted and the displacement. The measured response plotted in what follows is obtained by using the transmitted force, shown in Fig. 2, which has a forcing frequency that varies from 0 to 50 Hz. This curve-fitted force and the data given in Table 1 in the Appendix A, permit the prediction of the displacement response performed in what follows using the classical and metric models. The three measured natural frequencies are given by

$$v_1 = 8.26 \text{ Hz}, \quad v_2 = 22.96 \text{ Hz}, \quad v_3 = 33.60 \text{ Hz} \quad (24)$$

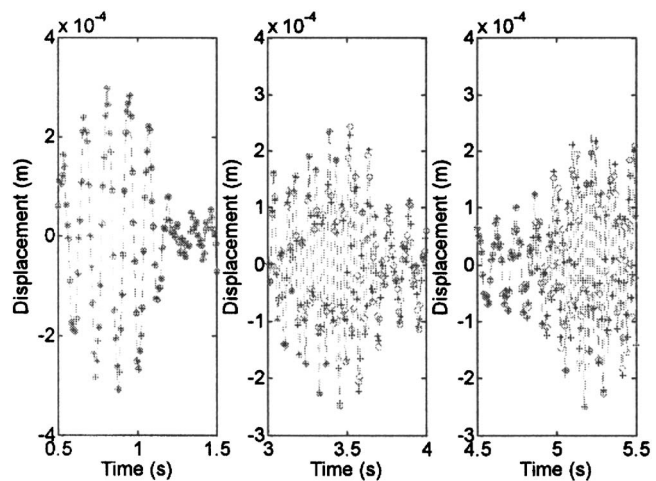


Fig. 4 Classical model. Zooms of the predicted response of three different algorithms: ODE23 (---), ODE45 (+++), ODE113 (ooo)

4.2 Time-Integration Schemes. MATLAB-SIMULINK software, which is used for solving the state-space equations given by Eq. (14), provides a panel of different integration schemes called ODE (ordinary differential equations) to solve initial value problems. In this case, different algorithms are tested to solve the state-space equations. They are differentiated by the problem type, their order of accuracy, and their ease of use [15,16]. It can be noted that the ODE23, ODE45, and ODE113 algorithms are based on the Runge-Kutta method (more or less modified) and that solving the metric model equations by ODE23 requires six times as much memory as the ODE45. Moreover, a scheme combining Runge-Kutta and Newton-Raphson methods is developed. The maximum time step is 0.0002 s. Constant or adaptive time steps have no significant influence on the results.

4.3 Prediction With the Classical Model. The characteristics given in Table 1 in the Appendix allow defining the parameters of the matrices and vectors given in Eq. (23),

$$\begin{aligned} \mathbf{x} &= (x^1 \ x^2 \ x^3), \quad \mathbf{V} = (l_d l_d \ 0 \ 0), \\ \Phi &= \begin{pmatrix} 0.36 & 0.80 & -0.61 \\ 0.64 & 0.31 & 0.82 \\ 0.80 & -0.66 & -0.40 \end{pmatrix}, \quad {}^t\Phi\mathbf{V} = \begin{pmatrix} 0.35 \\ 0.77 \\ -0.59 \end{pmatrix} \\ \mathbf{M}_{\text{mod}} &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \mathbf{K}_{\text{mod}} = \begin{pmatrix} \Omega_1^2 & 0 & 0 \\ 0 & \Omega_2^2 & 0 \\ 0 & 0 & \Omega_3^2 \end{pmatrix}, \\ \mathbf{C}_{\text{mod}} &= \begin{pmatrix} 2\alpha_1\Omega_1 & 0 & 0 \\ 0 & 2\alpha_2\Omega_2 & 0 \\ 0 & 0 & 2\alpha_3\Omega_3 \end{pmatrix} \end{aligned} \quad (25)$$

$$(26)$$

The classical model described by Eqs. (14) and (23) is used to predict the displacement response of the first DOF. Even though different numerical schemes are used, there is a considerable deviation between the predicted and measured responses, especially after each resonance phenomenon (see Fig. 3). The different algorithms tested give identical predicted responses. This is shown by Fig. 4, where the magnified responses predicted by numerical schemes ODE23, 45, and 113 are plotted around the three resonance phenomena. No improvement is obtained even though the nonlinear equations are taken into account.

4.4 Prediction With the Metric Model. Updating the ρ_j finally gives the values $\rho_1 = 3.3 \times 10^{-6}$, $\rho_2 = 6.4 \times 10^{-6}$, and $\rho_3 = 2.1 \times 10^{-6}$ with which simulation is carried out. It should be noted that a variation $\pm 1 \times 10^{-7}$ of ρ_j does not significantly modify the results. The time history of the predicted and measured responses are compared in Fig. 5. The three zooms performed around the three resonance phenomena and plotted in Fig. 6 show that there is only a slight deviation after the third resonance phenomenon. The metric model is proved satisfactory. Figure 7 shows the metric and classical modal damping coefficients (extracted from the experimental transfer function by the bandwidth technique). As can be seen, damping varies around each resonance phenomenon. Finally, the time history of the three variables θ^μ are compared in Fig. 8, with the linear evolution of $\Omega_j t$ corresponding to the classical damping $2\alpha_j\Omega_j$.

5 Conclusion

Predicting the response of a mechanical system in high transient motion using a classical numerical approach could be unsatisfactory. The metric method proposed is based on the concept of general relativity and the introduction of additional axes (an axis per parameter influencing space for each DOF). If the excitation frequency is considered as the influencing parameter, one additional axis is assigned to each DOF.

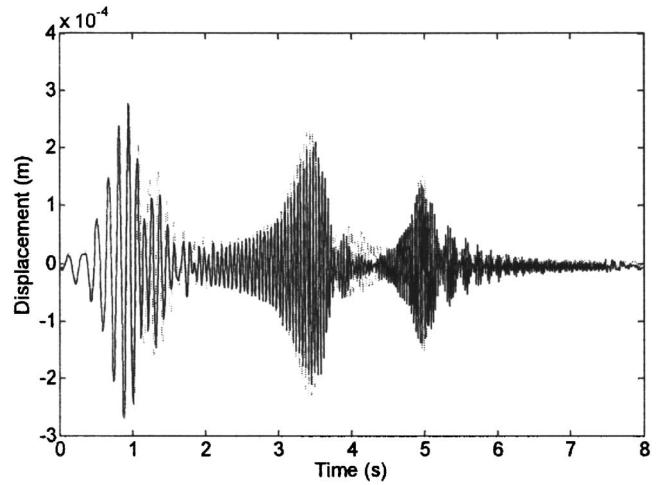


Fig. 5 Displacement measured (solid line) and predicted with the metric model (dotted line)

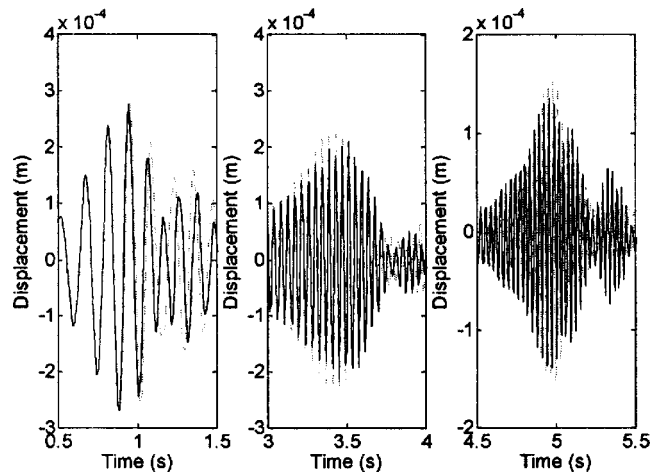


Fig. 6 Zooms of displacement measured (solid line) and predicted with the metric model (dotted line)

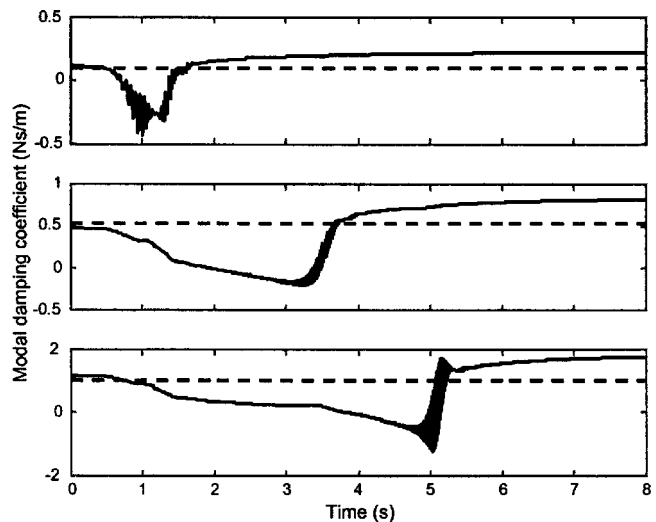


Fig. 7 Modal damping coefficients predicted with the metric (solid line) and classical (dotted line) models

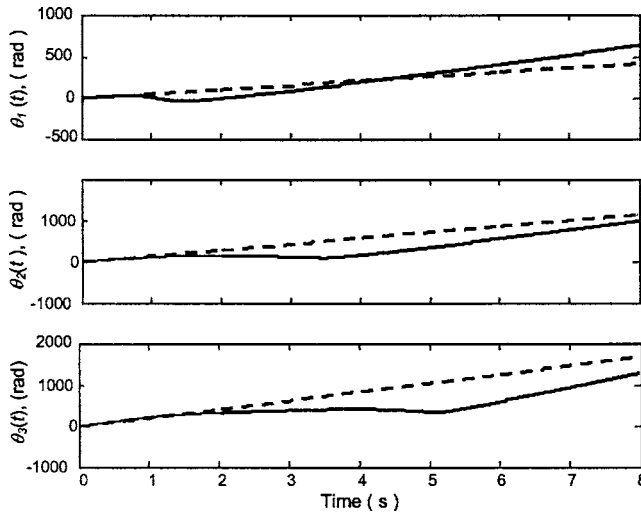


Fig. 8 Progression of $\theta(t)$ of the metric model (solid line) and Ωt of the classic model (dotted line)

Table 1 Characteristics of the 3-DOF system

Definition	Variable	Value
Location of the proximity probe	l_d	0.374 m
Length of the beam	l_t	0.400 m
Location of mass M	l_m	0.327 m
Location of the push-rod	l_b	0.362 m
Location of the force	l_f	0.362 m
Location of the springs	l_s	0.390 m
Length of the push-rod	L_b	0.240 m
Radius of the axis of rotation	r_a	0.008 m
Radius of the mass	r_m	0.025 m
Mass M	m	0.860 kg
Mass of the push-rod	m_b	0.040 kg
Mass of the spring	m_s	0.080 kg
Mass of the beam	m_t	0.333 kg
Mass of the axis of rotation	m_a	0.120 kg
Stiffness of the three springs	$k_1 = k_2 = k_3$	10,500 N.m ⁻¹
Modal damping factor	α_1	0.001
Modal damping factor	α_2	0.002
Modal damping factor	α_3	0.002
Gravity acceleration	g	9.81 m.s ⁻²

The steps of the proposed method are linked as follows:

- defining a frame of reference—a Riemannian space containing additional axes
- establishing the metric
- determining the terms of the metric assuming that the time space is slightly curved (i.e., Ricci tensor nil)
- defining the interval between two successive positions
- minimizing the interval to establish the geodesic equation, i.e., equations of motion
- solving the coupled equations with the state-space form, provided with boundary conditions
- updating the unknown factors ρ_j

Using this metric method makes the experimental results correspond with the simulated results.

Acknowledgment

The authors would like to thank Michel Mizony of the University of Lyon I, for his advice, and the *Région Rhône Alpes* council for its support.

Appendix A: Parameters of Experimental Setup

The parameters of the experimental setup are presented in Table 1.

Appendix B: Matrices of Classical Model

$$\mathbf{M} = \begin{pmatrix} \frac{I_1 + 2m_s l_s^2/3}{l_d^2} & \frac{1}{6} \frac{m_s l_s^2}{l_d^2} & 0 \\ \frac{1}{6} \frac{m_s l_s^2}{l_d^2} & \frac{I_2 + 2m_s l_s^2/3}{l_d^2} & \frac{1}{6} \frac{m_s^2 l_s^2}{l_d^2} \\ 0 & \frac{1}{6} \frac{m_s^2 l_s^2}{l_d^2} & \frac{I_3 + 2m_s l_s^2/3}{l_d^2} \end{pmatrix} \quad (\text{B1})$$

$$\mathbf{K} = \begin{pmatrix} \frac{Q_1 g + k_1 l_s^2 + k_2 l_s^2}{l_d^2} & -\frac{k_2 l_s^2}{l_d^2} & 0 \\ -\frac{k_2 l_s^2}{l_d^2} & \frac{Q_2 g + k_2 l_s^2 + k_3 l_s^2}{l_d^2} & -\frac{k_3 l_s^2}{l_d^2} \\ 0 & -\frac{k_3 l_s^2}{l_d^2} & \frac{Q_3 g + k_3 l_s^2}{l_d^2} \end{pmatrix} \quad (\text{B2})$$

$$I_1 = \frac{1}{3} m_t l_t^2 + m \left(\frac{1}{4} r_m^2 + l_m^2 \right) + \frac{1}{2} m_a r_a^2 + m_b l_b^2 \quad (\text{B3})$$

$$I_2 = I_3 = \frac{1}{3} m_t l_t^2 + m \left(\frac{1}{4} r_m^2 + l_m^2 \right) + \frac{1}{2} m_a r_a^2 \quad (\text{B4})$$

$$Q_1 = m l_m + \frac{1}{2} m_t l_t^2 + m_s l_s + m_b \sqrt{l_b^2 + L_b^2} \quad (\text{B5})$$

$$Q_2 = Q_3 = m l_m + \frac{1}{2} m_t l_t^2 + m_s l_s \quad (\text{B6})$$

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