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ERRATA

TO: All recipients of UCRL-11407

FROM: Technical Information Division

Subject: UCRL-11407, "The Effect of Collisions on Ion Cyclotron Waves," by David Larry Sachs, May 1, 1964.

Please make the following corrections on subject report.

Page 4, line 13: delete "dielectric constant"; insert "index of refraction".

Page 37, line 4 from bottom: delete "refracting"; insert "refraction".

Page 67, line 14: "$G^-(k)$ in the..." should be "$G^-(k)$ is the...".

Page 70, equation (30a): "$\int_{C_{23}}$" should be "$\int_{C_{12}}$".

Page 80, line 7 from bottom: "equation" should be "equations".

Page 85, line 11: "function" should be "functions".

Page 101, top equation: "$\frac{|ka_i|}{\nu_i}$" should be "$\left|\frac{ka_i}{\nu_i}\right|^2$".

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David Larry Sachs

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CONTENTS

THE EFFECT OF COLLISIONS ON ION CYCLOTRON WAVES*

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May 1, 1964

ABSTRACT

The behavior of a transverse electromagnetic wave propagating in the direction of a uniform magnetic field in a fully ionized plasma is examined. Linearized kinetic equations with collision terms of the Krook-Bhatnager-Gross type extended by Liboff to include interspecies collisional effects are used in the solution of a spatial boundary value problem. The region of ion cyclotron resonance is closely investigated, and the transition of the dispersion relation from the low-temperature, collision-dominated regime to the high-temperature regime is observed. It is found that moments of the equations are adequate in the collision-dominated regime, but the kinetic equation for the ions must be used at higher temperatures. At these higher temperatures a complete solution of the problem requires numerical work near the source plane. Far from the source, explicit solutions for the fields can be written.

I. INTRODUCTION

This paper is concerned with the propagation of plane transverse electromagnetic waves in plasma immersed in a uniform magnetic field. The term transverse signifies that the vector quantities associated with the wave are perpendicular to the direction of propagation of the wave. The term plane signifies that the perturbed quantities associated with the wave are uniform in the plane perpendicular to the direction of propagation. Thus the divergence of all vector quantities associated with the wave vanishes.

In order to determine the characteristics of these waves in a plasma, one must have the constitutive relation $\vec{J}(\vec{E})$, where $\vec{J}$ is the average current density produced in the plasma and $\vec{E}$ is the average electric field in the plasma. $\vec{J}$ and $\vec{E}$ are functions of position and time which are averages over dimensions of space and time small compared to the macroscopic dimensions of the system such as the length and period of the wave and large compared to the microscopic dimensions of the system, such as the interparticle spacing. The problem is then determined by the four Maxwell equations:

$$\vec{\nabla} \times \vec{B} = \frac{4\pi}{c} \vec{J}(\vec{E}) + \frac{1}{c}\frac{\partial \vec{E}}{\partial t} ;$$

$$\vec{\nabla} \times \vec{E} = -\frac{1}{c}\frac{\partial \vec{B}}{\partial t} ;$$

$$\vec{\nabla} \cdot \vec{E} = 0 ;$$

$$\vec{\nabla} \cdot \vec{B} = 0 ;$$

along with appropriate boundary conditions. Under the assumption that the macroscopic properties of the unperturbed plasma are uniform in space and constant in time, Fourier analysis of the wave field reduces the problem to the examination of waves with the simple spatial-temporal dependence, $e^{i(kz-\omega t)}$.

In addition, as the following sections will show, there are waves which do not have the simple dependence shown above. These waves are quantitatively unimportant except in special circumstances which will be explained in the later sections. They arise when the perturbation of the plasma results from a disturbance at a boundary.

Ignoring these new waves for the time being, a linear theory suitable for small amplitude waves places the constitutive relation into the simple form $\vec{J} = \overleftrightarrow{\sigma}(\omega,k) \cdot \vec{E}$ where the conductivity tensor, $\overleftrightarrow{\sigma}$, is dependent on the frequency and wave-number as well as the macroscopic properties of the unperturbed plasma.

If one places the direction of a uniform externally produced magnetic field along the z axis of a right-handed coordinate system and considers a transverse plane wave propagating in the z direction, the conductivity tensor assumes the form:

$$\overleftrightarrow{\sigma} = \begin{pmatrix} \sigma_{xx} & \sigma_{xy} & 0 \\ \sigma_{yx} & \sigma_{yy} & 0 \\ 0 & 0 & \sigma_{zz} \end{pmatrix}$$

where $\sigma_{xx} = \sigma_{yy} = \sigma_1$

and $\sigma_{xy} = -\sigma_{yx} = \sigma_2$.

$\sigma_{xz} = \sigma_{yz} = 0$ which shows that there is no coupling between transverse and longitudinal waves. σ_{zz} will not be of importance for these transverse plane waves.

The off-diagonal elements which result from the presence of the external magnetic field give rise to a Faraday effect, a difference in the phase velocity of right and left-handed circularly polarized waves.

A right-handed circularly polarized wave has an electric vector of the form $\vec{E}_+ = (\hat{x} + i\hat{y})\, e^{i(kz-\omega t)}$. Operation with the conductivity tensor shows this field to give rise to a current density vector of the form $\vec{J} = \vec{J}_+ = (\sigma_1 + i\sigma_2)(\hat{x} + i\hat{y})e^{i(kz-\omega t)} = \sigma_+ \vec{E}_+$.

The vectors rotate in the same sense in which the electrons of the plasma gyrate about the uniform magnetic field.

The electric vector of the left-handed wave has the form $\vec{E}_- = (\hat{x} - i\hat{y})\, e^{i(kz-\omega t)}$ which gives rise to $\vec{J}_- = (\sigma_1 - i\sigma_2)\vec{E}_- = \sigma_- \vec{E}_-$. These vectors rotate in the sense in which the positive ions gyrate.

The problem is thus reduced to the determination of σ_+ and σ_-. With these known, the Maxwell equations furnish separate dispersion relations for $\vec{E}_+$ and $\vec{E}_-$. One obtains the wave equation, $\left[k^2 - (\omega^2/c^2) - 4\pi i\omega\sigma_\pm/c^2\right]\vec{E}_\pm = 0$ which leads to the dispersion relation,

$$k_\pm^{\,2} = (\omega^2/c^2) + 4\pi\, i\, \omega\, \sigma_\pm/c^2 .$$

This relation is often put in the form $k_\pm^{\,2} = \frac{\omega^2}{c^2}\, n_\pm^{\,2}(\omega, k_\pm)$ where $n_\pm = \sqrt{1 + \frac{4\pi i}{\omega}\, \sigma_\pm(\omega, k_\pm)}$ is the dielectric constant.

The left-handed wave is of interest to the controlled fusion programs. At frequencies close to the ion cyclotron frequency, this wave becomes damped and gives its energy to random motion of the plasma, i.e. heats it. Stix[1,2,3] has investigated this wave in high temperature plasma where collisions are infrequent; Engelhardt,[4] in low temperature plasma where the thermal effects such as viscosity are unimportant.

This paper will compare various methods of obtaining $\overleftrightarrow{\sigma}(\omega, k)$ and will discuss their merits and shortcomings. Finally a complete boundary value problem will be done using a kinetic equation with phenomenological relaxation terms. Criteria are determined for the adequacy of simpler methods.

The result of this study will be a continuous observation of the properties of the waves from low temperature where collisions are important and thermal effects unimportant through intermediate temperatures where the thermal properties of the plasma become important to high temperature where collisions are infrequent and thermal effects are responsible for the damping of the wave. This thermal damping (called cyclotron damping by Stix[2]) is analogous to the Landau damping which occurs for longitudinal waves.

II. COLLISIONLESS COLD PLASMA

The earliest work[5] in the determination of $\overleftrightarrow{\sigma}$ neglected two important considerations. These are the correlations of the particles of the plasma with one another (commonly called collisions), and the effects of the thermal motion of the particles.

Neglecting the thermal motion of the particles, the conductivity is independent of k. That is, the current density at a point $\vec{x}$ is a function of the electric field at the point $\vec{x}$ alone for a linear theory. If collisions are also neglected, the equation of motion of the plasma is the same as the equation of motion for a particle of the plasma,

$$m_j \frac{d\vec{v}_j}{dt} = q_j \left(\vec{E} + \frac{\vec{v}_j}{c} \times \vec{B}_0 \right) ,$$

where $\vec{B}_0$ is the uniform magnetic field and the subscript j denotes the type of particle.

Letting $j = i$ for ions and $j = e$ for electrons, we have

$$\vec{J} = q_i \, n_i \, \vec{v}_i + q_e \, n_e \, \vec{v}_e = e n_0 (\vec{v}_i - \vec{v}_e)$$

since $q_e = -q_i = -e$ (assuming singly charged ions) and

$n_i = n_e = n_0$ (assuming the unperturbed plasma to be neutral).

Astrom[5] solves these equations to obtain

$$n^2_{\pm} = \frac{k_{\pm}^2 c^2}{\omega^2} = 1 - \frac{\omega_p^2}{(\omega \pm \omega_{ic})(\omega \mp \omega_{ec})}$$

where $\omega_p^2 = 4\pi n_0 e^2\left(\frac{1}{m_i} + \frac{1}{m_e}\right)$, the sum of the squares of the particle plasma frequencies;

$$\omega_{ic} = \frac{eB_0}{m_i c}, \qquad \text{the ion gyro frequency;}$$

$$\omega_{ec} = \frac{eB_0}{m_e c} \qquad \text{the electron gyro frequency.}$$

Figure 1 is a sketch of the form of $k_{\pm}(\omega)$ for the two polarizations. At extremely low $\omega(\omega \ll \omega_{ic})$ both k_+ and k_- are given by $k = \omega/v$, where $v^2 = A^2 c^2/(A^2 + c^2)$ and

$$A^2 = \frac{c^2 \omega_{ic} \omega_{ec}}{\omega_p^2} = \frac{B_0^2}{4\pi n_0(m_i + m_e)},$$ the square of the Alfven speed. At extremely high $\omega(\omega \gg \omega_p, \omega_{ec})$ both k_+ and k_- are given by $k = \omega/c$, that is, the effects of the plasma are negligible. The intermediate ω behavior shows a resonance for k_- at ω_{ic} and a cutoff at ω_1 and a resonance for k_+ at ω_{ec} with a cutoff at ω_2 where

$$\omega_{\substack{2\\1}} = \frac{1}{2}\left[\sqrt{(\omega_{ic} + \omega_{ec})^2 + 4\omega_p^2} \pm (\omega_{ec} - \omega_{ic})\right].$$

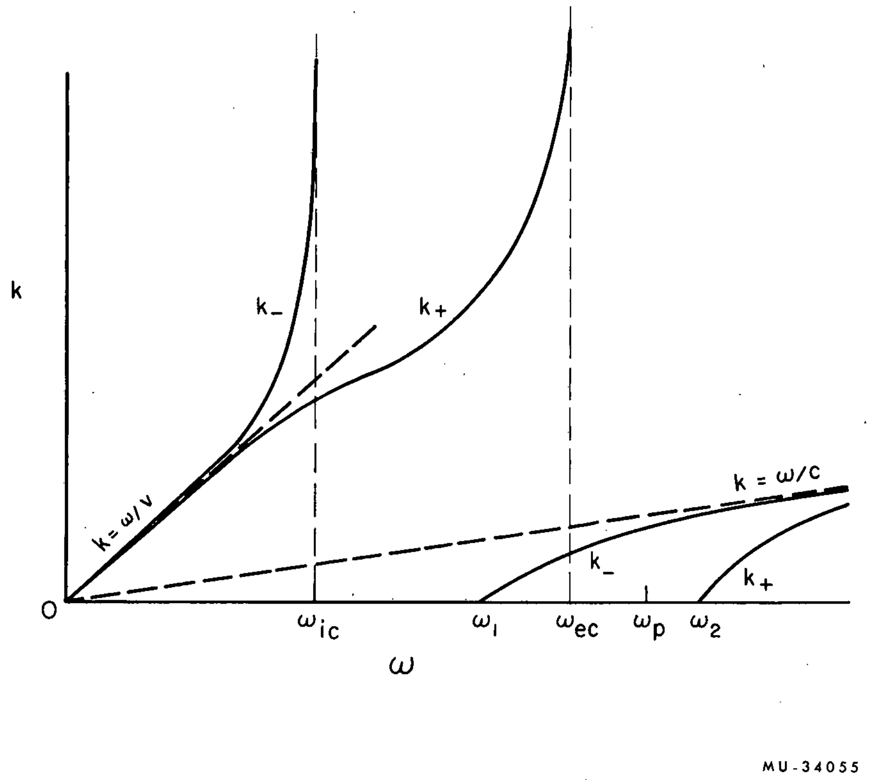

Fig. 1. Dispersion relation for transverse waves in cold collisionless plasma.

Between ω_{ic} and ω_1 , k_- is pure imaginary, decreasing in magnitude from infinity at ω_{ic} to zero at ω_1 . Similar behavior results for k_+ between ω_{ec} and ω_2 . Rather than continue the repetitious remarks about k_+, we will henceforth concentrate on k_- , the ion cyclotron wave.

The behavior at and near the resonance frequency, ω_{ic}, determined by the above model is never correct. The model shows k to be purely real for $\omega < \omega_{ic}$ and approaching infinity as $\omega \to \omega_{ic}$ signifying an undamped wave whose length approaches zero. The behavior for $\omega > \omega_{ic}$ signifies an evanescent disturbance whose penetration length increases from zero at ω_{ic} to infinity at $\omega = \omega_1$.(An evanescent disturbance does not propagate energy. Thus a pure imaginary k signifies perfect reflectance of the medium.)

The actual behavior of k in the resonance region differs greatly from that described above because at least one of the two neglected effects becomes important. For any temperature of the plasma, the collisional effects will be important at high densities while the thermal effects will be important at low densities where collisions are negligible.

In either case, the results show k to be finite and complex in the resonance region, signifying a wave that is highly damped for a stable plasma.

The next step then, is to incorporate the thermal and collisional effects into the plasma model. This may be done using equations governing moments of the appropriate kinetic equation. A more difficult but more general method is to use the kinetic equation itself. Both methods will be used and the results compared.

III. COLLISIONAL AND THERMAL EFFECTS

A. The Kinetic Equation

A kinetic equation is of the form

$$\left(\frac{\partial}{\partial t} + \vec{v} \cdot \vec{\nabla} + \frac{\vec{F}}{m} \cdot \vec{\nabla}_{\vec{v}}\right) f = \left.\frac{\partial f}{\partial t}\right)_{coll.}$$

where $f(\vec{x}, \vec{v}, t)$, the distribution function of a class of particles, corresponds to the number of particles at position $\vec{x}$ with velocity $\vec{v}$ at time t per unit volume of "phase" space, $d\,\vec{x}\,d\,\vec{v}$. The left side of the equation represents the conservation of particles in phase space. The term $\left.\frac{\partial f}{\partial t}\right)_{coll}$ corresponds to the change in f per unit time caused by particle interactions which cannot be accounted for by the force term on the left side, $\vec{F}$, which is assumed constant over the volume $d\,\vec{x}\,d\,\vec{v}$.

This term represents the collisional effects. When this term has the particular form derived by Boltzmann, the equation is correctly called the Boltzmann equation. If the form of this term is unspecified, the equation is in general, called a kinetic equation. The thermal effects are represented by the term on the left side, $\vec{v} \cdot \vec{\nabla} f$ which corresponds to the change in f per unit time due to the streaming of particles of velocity $\vec{v}$ from neighboring regions of configuration space into the volume $d\,\vec{x}$.

The macroscopic quantities of interest are moments of f:

The density,

$$n(\vec{x}, t) = \int d\vec{v}\, f(\vec{x}, \vec{v}, t) \;;$$

the velocity,

$$\vec{u}(\vec{x}, t) = \frac{\int d\vec{v}\, \vec{v}\, f(\vec{x}, \vec{v}, t)}{n(\vec{x}, t)} \,.$$

Defining the random velocity

$$\vec{w} = \vec{v} - \vec{u}(\vec{x}, t) \,,$$

the pressure tensor is given by

$$\overleftrightarrow{P} = \int m\, \vec{w}\, \vec{w}\, f\, d\vec{v} \;;$$

the heat flow tensor,

$$\overleftrightarrow{Q} = \int m\, \vec{w}\, \vec{w}\, \vec{w}\, f\, d\vec{v} \;;$$

and so on. Equations for these macroscopic quantities are obtained by taking appropriate moments of the kinetic equation. Since we are interested in a linear treatment, we may immediately simplify the form of the kinetic equation.

Let $f_j = f_j^{\,0} + f_j^{\,1}$ where the Maxwell distribution function for the unperturbed particles of type j is

$$f_j^{\,0} = \frac{n_0}{\pi^{3/2} a_j^{\,3}} e^{-v^2/a_j^{\,2}}$$

with $a_j = \sqrt{\frac{2T_0}{m_j}}$, the most probable speed of particle j where T_0 is the temperature of the unperturbed plasma in energy units. The linearized kinetic equation is then

$$\left(\frac{\partial}{\partial t} + \vec{v}\cdot\vec{\nabla} + \omega_{jc}\,\vec{v}\times\hat{z}\cdot\vec{\nabla}_{\vec{v}}\right) f_j^{\,1} + \frac{q_j}{m_j}\,\vec{E}\cdot\vec{\nabla}_v f_j^{\,0} = \left.\frac{\partial f_j}{\partial t}\right)_{coll.} .$$

The corresponding moment equations are:

$$\frac{\partial n}{\partial t} + n_0\,\vec{\nabla}\cdot\vec{u}_j = \int d\,\vec{v}\left.\frac{\partial f_j}{\partial t}\right)_{coll} = 0 \quad ; \qquad (1)$$

$$n_0\, m_j \frac{\partial \vec{u}_j}{\partial t} + \vec{\nabla}\cdot\overleftrightarrow{P}_j + \omega_{jc}\,\hat{z}\times n_0\, m_j\,\vec{u}_j$$

$$= q_j\, n_0\,\vec{E} + \int d\,\vec{v}\, m_j\,\vec{v}\left.\frac{\partial f_j}{\partial t}\right)_{coll} \quad ; \qquad (2)$$

$$\frac{\partial \overleftrightarrow{P}_j}{\partial t} + \vec{\nabla}\cdot\overleftrightarrow{Q}_j + P_0\left[\vec{\nabla}\cdot\vec{u}_j\,\overleftrightarrow{I} + \vec{\nabla}\,\vec{u}_j + (\vec{\nabla}\,\vec{u}_j)^T\right]$$

$$+\, \omega_{jc}\left(\hat{z}\times\overleftrightarrow{P}_j - \overleftrightarrow{P}_j\times\hat{z}\right) = \int d\,\vec{v}\, m_j\,\vec{v}\,\vec{v}\left.\frac{\partial f_j}{\partial t}\right)_{coll} \qquad (3)$$

and so on where $P_0 = n_0 T_0$ and $(A)^T$ signifies the transpose of the matrix A; $\overleftrightarrow{I}$ is the unit tensor. In Eq. (3), it was sufficient to use the term, $\vec{v}\vec{v}$, rather than $\vec{w}\vec{w}$ to determine the perturbed moment, $\overleftrightarrow{P}$.

Disregarding the collision terms for the moment, we see that Eq. (2) is of primary interest for obtaining $\vec{J}\,(\vec{E})$ since $\vec{J} = n_0 \sum_j q_j \vec{u}_j$. However, the thermal effect term, $\vec{v} \cdot \vec{\nabla} f$, of the kinetic equation has coupled the next higher moment into the equation of interest through the term, $\vec{\nabla} \cdot \overleftrightarrow{P}_j$.

Upon contemplating Eq. (3) for $\overleftrightarrow{P}_j$ it is found that the thermal effect term has again coupled the next higher moment into the equation through the term $\vec{\nabla} \cdot \overleftrightarrow{Q}_j$ and so on. Thus, the system of moment equations is not closed.

In order to close the set, the thermal effect term must be smaller than the other terms in the kinetic equation. If this is not true, then the moment equations are of no value since the set cannot be closed. Let us assume that the thermal term is small and close the set by ignoring the term $\vec{\nabla} \cdot \overleftrightarrow{P}_j$ in Eq (2). That is, we neglect the thermal effects entirely. Now Eq. (2) will be sufficient for the determination of $\vec{J}\,(\vec{E})$ when the collision term is given. As for Eq. (1), the right-hand side is zero because the number density of particles is conserved by collisions. Since we are interested in transverse plane waves, $\vec{\nabla} \cdot \vec{u}_j = 0$. Thus Eq. (1) simply tells us that the density of particles is unperturbed by the wave.

The determination of the form of the collision term of the kinetic equation is a major and still incomplete problem of plasma physics. This will be discussed more fully later on in this section.

At this point, however, we are concerned with the term,

$$\int d\vec{v}\, m_j\, \vec{v}\, \frac{\partial f_j}{\partial t}\bigg)_{coll}$$

of Eq. (2). This is the total momentum transferred to particles of type j per unit volume per unit time by collisions with other particles. Spitzer[6] states that a reasonable assumption here is that the net momentum exchanged by collisions between two types of particles should be proportional to their relative flow velocity. The above momentum transfer term is thus replaced by $n_0\, m_{ie}\, \nu_1(\vec{u}_i - \vec{u}_e)$ for the case of momentum transfer to electrons by ions where m_{ie} is the reduced mass. Numerical work by Spitzer using the Boltzmann equation leads to the following estimate for the momentum transfer collision frequency:[6]

$$\nu_1 = \frac{3.7\, n_0\, \ell n\, \Lambda}{T_0^{3/2}}\ \sec^{-1}$$

where $$\Lambda = \frac{1.24 \cdot 10^4\, T_0^{3/2}}{\sqrt{n_0}}\ . \qquad (3a)$$

The dimension of n_0 is cm^{-3} and the dimension of T_0 is degrees Kelvin. This number is obtained from Spitzer's value for the plasma resistivity, η, by comparing his definition of η to our definition of ν_1. This collision frequency is applicable to the case of high magnetic field, $\omega_{ec} \gg \nu_1$, for relative velocities perpendicular to the magnetic field, which is our case of interest. For relative velocities parallel to the field or if $\omega_{ec} \ll \nu_1$, about half the above value is correct.

With these simplifications, the factor n_0 is cancelled and Eq. (2) is now

$$m_e \frac{\partial \vec{u}_e}{\partial t} - \omega_{ec} \hat{z} \times m_e \vec{u}_e = - e \vec{E} + m_{ie} \nu_1 (\vec{u}_i - \vec{u}_e)$$

for the electrons and

$$m_i \frac{\partial \vec{u}_i}{\partial t} + \omega_{ic} \hat{z} \times m_i \vec{u}_i = e \vec{E} - m_{ie} \nu_1 (\vec{u}_i - \vec{u}_e)$$

for the ions. These equations are identical to those previously mentioned in the work of Åstrom except for the addition of the damping term. This addition adds no difficulty to the problem. Using again the relation, $\vec{J} = en_0(\vec{u}_i - \vec{u}_e)$ and Maxwell's equations one easily obtains the new index of refraction,

$$n_{\pm}^{2} = \frac{k_{\pm}^{2} c^{2}}{\omega^{2}} = 1 - \frac{\omega_p^{2}}{(\omega \pm \omega_{ic})(\omega \mp \omega_{ec}) + i\,\omega\,\nu_1} . \quad (4)$$

With this model, k remains finite in the resonance region and for $\omega = \omega_{ic}$, we find that the real and imaginary parts of k are comparable, signifying a highly damped wave. This follows from Eq. (4) when $\omega_p^{2} \gg \omega_c \nu_1$ which is always true in our cases of interest. Discussion will always be limited to plasmas where ω_p is much larger than all other frequencies of interest.

Fig. 2 is a plot of the real and imaginary parts of k as a function of ω for the left-handed wave around its resonance region, ω_{ic}. The right-handed wave will have analogous behavior in the vicinity of its resonance at ω_{ec}.

Fig. 2 shows that Im k is negligible for $\omega \ll \omega_{ic}$ and Re k is negligible for $\omega \gg \omega_{ic}$ in agreement with the collisionless theory of Åstrom.

The next question concerns the influence of the thermal term. We can approximate the thermal effects by keeping the term $\vec{\nabla} \cdot \overleftrightarrow{P}$ in Eq. (2) and ignoring the term $\vec{\nabla} \cdot \overleftrightarrow{Q}$ in Eq. (3). We then will have to replace the collision term on the right-hand side of Eq. (3) by a reasonable and workable form as we did in the case of

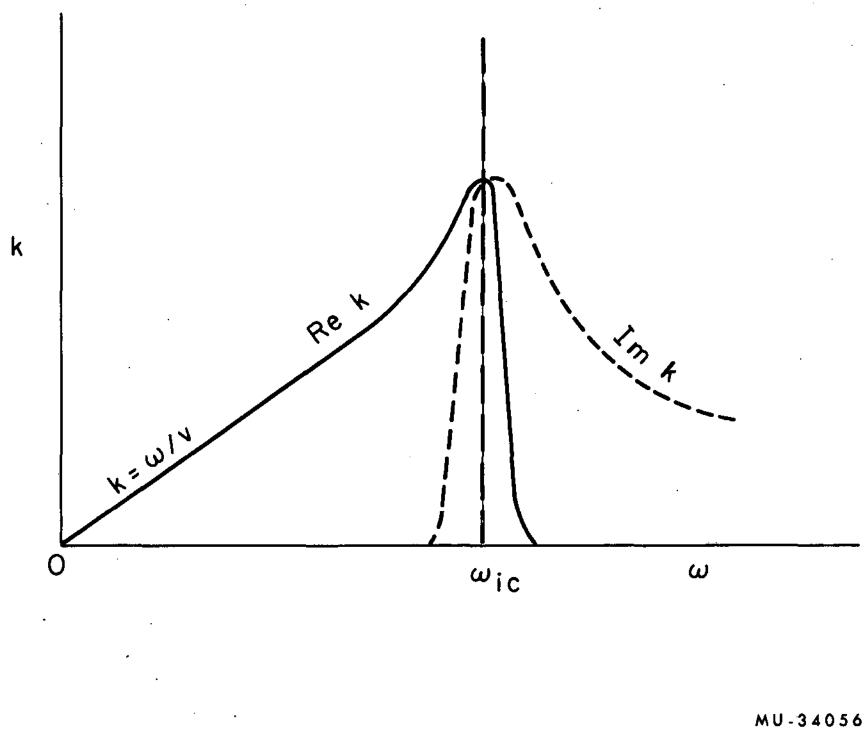

Fig. 2. Dispersion relation for ion cyclotron wave in cold plasma. —— Re k; – – – Im k.

Eq. (2) .

Kaufman[7] in his investigation of viscosity uses the form,

$$\int d\vec{v}\, m \vec{v}\vec{v} \frac{\partial f}{\partial t}\Big)_{coll} = -\nu\,(\overleftrightarrow{P} - P\overleftrightarrow{I})$$

where $P = 1/3$ Trace $\overleftrightarrow{P}$. Therefore, we can define $\overleftrightarrow{\Pi} = \overleftrightarrow{P} - P\overleftrightarrow{I}$ as the traceless part of the stress tensor. Equation (3) is then split into the two equations

$$\frac{\partial P}{\partial t} + 5/3\, P_0 \vec{\nabla} \cdot \vec{u} = 0 \qquad (5)$$

and

$$\frac{\partial \overleftrightarrow{\Pi}}{\partial t} + P_0\left[\vec{\nabla}\vec{u} + (\vec{\nabla}\vec{u})^T - 2/3\, \vec{\nabla}\cdot\vec{u}\,\overleftrightarrow{I}\right] + \omega_c(\hat{z}\times\overleftrightarrow{\Pi} - \overleftrightarrow{\Pi}\times\hat{z}) = -\nu\,\overleftrightarrow{\Pi}. \qquad (6)$$

The intuitive idea is that the collisions of a species of particles with themselves will cause the pressure tensor to approach a scalar pressure $P\overleftrightarrow{I}$. That is, the velocity distribution of the particles becomes isotropic through the collisions, causing $\overleftrightarrow{\Pi}$ to relax to zero. If one then chooses the self collision frequency of Spitzer[6] for ν one has $\nu_e = 1.03\ \nu_1$ for electrons where ν_1 is given in Eq.(3a). For ions, one has $\nu_i = \sqrt{\frac{m_e}{m_i}}\ \nu_e$.

Since $\vec{\nabla} \cdot \vec{u} = 0$ Eq. (5) shows that $P = 0$. That is, a transverse wave does not perturb the pressure of the plasma. Since $P = n_0 T + n T_0$ (where quantities without the subscript, zero, denote first order perturbations), we have $T = 0$ since $P = n = 0$. That is, the transverse wave does not perturb the temperature of the plasma. Thus the thermal effects are essentially viscous effects. Equation (6) will be used later to obtain $\overleftrightarrow{\Pi}$ in terms of $\vec{\nabla} \vec{u}$. Substitution into the momentum Eq. (2), will then be used to produce the first thermal correction to the conductivity. The next question would regard the importance of the second correction, the $\vec{\nabla} \cdot \overleftrightarrow{Q}$ term. We would desire an estimate of its importance in order to know when it can safely be ignored.

Thus intuitive forms would be necessary for the collisional terms of the higher moment equations. Our intuition, however, is lacking for the higher moments. A more reasonable approach is to use intuition to determine the form of the collision term in the kinetic equation itself and then the form of the collision terms of all the moment equations would simply follow from this term.

There exist more precise expressions for this collision term than the one we shall use. For example, the Fokker-Planck collision term[8,9] is an integro-differential expresssion which is a good approximation for $\omega \gg \omega_p$ and $\Lambda \gg 1$. These conditions are satisfied in our region of interest. However, the mathematical complexity of the expression necessitates an expansion in small parameters.

One such expansion technique is the Chapman-Enskog method,[10] which is used to obtain the transport coefficients of the plasma.[11,12] Here the small parameters are the time and space derivatives of the distribution function. These are considered small compared to the collision term and the force term arising from the presence of the homogeneous unperturbed magnetic field.

This expansion cannot be used in our regions of interest since the time and space derivative terms will be of the same order of magnitude as the collision and magnetic field terms. With no alternative technique at our disposal, the more precise expressions cannot be used.

We use collision forms developed by Liboff,[13] which are extensions of the Krook, Bhatnager and Gross[14] model. According to this model, the collision form is a term which would cause the distribution function to relax to a local Maxwellian,

$$f_{LM} = \frac{n_0 + n(\vec{r},\ t)}{\left[\frac{2\pi}{m}(T_0 + T(\vec{r},t))\right]^{3/2}} \exp\left\{\frac{-m\left[\vec{v} - \vec{u}(\vec{r},t)\right]^2}{2\left[T_0 + T(\vec{r},t)\right]}\right\},$$

in the absence of external forces. Liboff includes terms corresponding to the tendency of interactions between the ions and electrons to reduce differences in their average velocities and temperatures. His expression for the ions is

$$\left.\frac{\partial f_i}{\partial t}\right)_{coll} = -\nu_i\left\{f_i^{\,1} - f_i^{\,0}\left[\frac{n_i}{n_0} + \frac{2}{a_i^{\,2}}(\vec{u}_i \cdot \vec{v}) + \frac{T_i}{T_0}\left(\frac{v^2}{a_i^{\,2}} - \frac{3}{2}\right)\right]\right\}$$

$$- \nu_1 \frac{m_{ie}}{T_0} f_i^{\,0}\, \vec{v} \cdot (\vec{u}_i - \vec{u}_e) - \frac{\nu_2 f_i^{\,0}}{T_0}\left(T_i - T_e\right)\left(\frac{v^2}{a_i^{\,2}} - \frac{3}{2}\right).$$

The term in curly brackets in the collision expression is simply $f_{total} - f_{IM}$ where f_{IM} is expressed in terms of f^0 by means of a Taylor expansion in the perturbed quantities.

To obtain the collision term for the electrons, simply interchange the subscripts "i" and "e". Henceforth, the electron equation will be omitted. ν_i, the ion collision frequency, represents the rate at which the ion distribution function approaches a local Maxwellian. ν_e has the analogous meaning for electrons. The magnitudes of ν_i and ν_e are those previously given, the self-collision frequencies of Spitzer. ν_1, the momentum transfer collision frequency, represents the rate at which the difference of the average velocities of the two species, $\vec{u}_i - \vec{u}_e$, approaches zero. Its magnitude is given by Eq. (3a). ν_2, the energy transfer collision frequency, represents the rate at which the difference of the temperatures of the two species, $T_i - T_e$ approaches zero. ν_2 is smaller than ν_1 by about the mass ratio of electrons to ions. We will not need its value in our calculations since the perturbations in temperature are zero.

B. The Moment Equations Method

Having the full linearized kinetic equation, the necessary moment equations will now be derived. We first derive expressions for the pertinent moments of f^0 .

$$\int f^0 \, d\vec{v} = n_0 ,$$

$$\int \vec{v} \, f^0 \, d\vec{v} = 0 ,$$

$$\int m \vec{v} \vec{v} f^0 \, d\vec{v} = P_0 \overleftrightarrow{I} = n_0 T_0 \overleftrightarrow{I} ,$$

$$\int m \vec{v} \vec{v} \vec{v} f^0 \, d\vec{v} = 0 ,$$

and $$\int m \vec{v} \vec{v} \vec{v} \vec{v} f^0 \, d\vec{v} = \frac{n_0 T_0^2}{m} \overleftrightarrow{\overleftrightarrow{M}}$$

where

$$M_{\ell jki} = \delta_{ij} \delta_{k\ell} + \delta_{jk} \delta_{i\ell} + \delta_{j\ell} \delta_{ik} .$$

In the linear theory, the perturbation in the moments of interest are given by

$$n = \int f^1 \, d\vec{v} ;$$

$$n_0 \vec{u} = \int f^1 \vec{v} \, d\vec{v} ;$$

$$P = \int \frac{mv^2}{3} f^1 \, d\vec{v} ;$$

$$\overleftrightarrow{\Pi} = \int m(\vec{v} \vec{v} - \frac{v^2}{3} \overleftrightarrow{I}) f^1 \, d\vec{v} ;$$

$$\overleftrightarrow{\overrightarrow{Q}} = \int d\vec{v} (f^0 + f^1) \, m(\vec{v} - \vec{u})(\vec{v} - \vec{u})(\vec{v} - \vec{u}) .$$

Therefore

$$Q_{ijk} = \int d\vec{v}\, f^1\, mv_i\, v_j\, v_k - P_0(u_i\, \delta_{jk} + u_j\, \delta_{ik} + u_k\, \delta_{ij})$$

in the linear theory. It is apparent then, that the perturbation in $\overleftrightarrow{Q}$ may be obtained by using the form

$$Q_{ijk} = \int d\vec{v}\, f^1 \left[mv_i\, v_j\, v_k - T_0(v_i\, \delta_{jk} + v_j\, \delta_{ik} + v_k\, \delta_{ij}) \right] .$$

The heat flow vector,

$$\vec{q} = \int d\vec{v}(f^0 + f^1)\, \frac{m}{2}\, (\vec{v} - \vec{u})^2 (\vec{v} - \vec{u}) ,$$

is then obtained by a contraction of $\overleftrightarrow{Q}$, that is

$$q_i = \frac{1}{2}\, Q_{ijj} .$$

Before obtaining the moment equations we substitute the collision term into our linearized kinetic equation to obtain the form we will henceforth use. Since $\vec{\nabla}_{\vec{v}}\, f^0 = \frac{-2\,\vec{v}}{a^2}\, f^0$, we have

$$\left(\frac{\partial}{\partial t} + \vec{v} \cdot \vec{\nabla} + \omega_{ci}\, \vec{v} \times \hat{z} \cdot \vec{\nabla}_{\vec{v}} + \nu_i\right) f_i^{\,1} = f_i^{\,0}\left\{\frac{2e}{m_i\, a_i^{\,2}}\, \vec{E} \cdot \vec{v} \right.$$

$$+ \nu_i\left[\frac{n_i}{n_0} + \frac{2}{a_i^{\,2}}\,(\vec{u}_i \cdot \vec{v}) + \frac{T_i}{T_0}\left(\frac{v^2}{a_i^{\,2}} - \frac{3}{2}\right)\right]$$

$$\left. - \frac{\nu_1\, m_{ie}}{T_0}\, \vec{v} \cdot (\vec{u}_i - \vec{u}_e) - \frac{\nu_2}{T_0}\,(T_i - T_e)\left(\frac{v^2}{a_i^{\,2}} - \frac{3}{2}\right)\right\}. \quad (7)$$

In the operations that follow, we integrate by parts to obtain

$$\int v_i\, v_j \cdots v_n\, \vec{v} \times \hat{z} \cdot \vec{\nabla}_{\vec{v}}\, f^1\, d\vec{v} = -\int f^1\, \vec{\nabla}_{\vec{v}} \cdot \left[\vec{v} \times \hat{z}\; v_i v_j \cdots v_n\right] d\vec{v}$$

$$= \int f^1\, d\vec{v}\; \hat{z} \times \vec{v} \cdot \vec{\nabla}_{\vec{v}}\left[v_i v_j \cdots v_n\right]$$

$$= \int f^1\, d\vec{v}\left[(\hat{z} \times \vec{v})_i\; v_j v_k \cdots v_n + v_i (\hat{z} \times \vec{v})_j\; v_k \cdots v_n + \cdots + v_i v_j v_k \cdots (\hat{z} \times \vec{v})_n\right].$$

This equality can be written in tensor form as

$$\int \vec{v}\, \vec{v} \cdots \vec{v}\,(\vec{v} \times \hat{z} \cdot \vec{\nabla}_{\vec{v}}\, f^1)\, d\vec{v} = \hat{z} \otimes \int \vec{v}\, \vec{v} \cdots \vec{v}\, f^1\, d\vec{v}$$

where

$$(\hat{z} \otimes \vec{v}\vec{v} \cdots \vec{v})_{ijk \cdots n} \equiv (\hat{z} \times \vec{v})_i v_j v_k \cdots v_n + v_i (\hat{z} \times \vec{v})_j v_k \cdots v_n + \cdots + v_i v_j \cdots (\hat{z} \times \vec{v})_n .$$

The derivation of the moment equation is simplified by the use of the properties of the Hermite polynomials defined by Grad.[15] These are defined as follows:

$$\mathcal{H}_{ij}^{n}{}_{\cdots n} = \left(\sqrt{\frac{m}{T_0}}\, v_i - \sqrt{\frac{T_0}{m}}\frac{\partial}{\partial v_i}\right)\left(\sqrt{\frac{m}{T_0}}\, v_j - \sqrt{\frac{T_0}{m}}\frac{\partial}{\partial v_j}\right)\cdots\left(\sqrt{\frac{m}{T_0}}\, v_n - \sqrt{\frac{T_0}{m}}\frac{\partial}{\partial v_n}\right) 1 \; ; \tag{8}$$

so

$$\mathcal{H}^0 = 1 \; ;$$
$$\mathcal{H}_i^1 = \sqrt{\frac{m}{T_0}}\, v_i \; ;$$
$$\mathcal{H}_{ij}^2 = \frac{m}{T_0}\, v_i v_j - \delta_{ij} \; ;$$
$$\mathcal{H}_{ijk}^3 = \left(\frac{m}{T_0}\right)^{3/2} v_i v_j v_k - \sqrt{\frac{m}{T_0}}\left(v_i\,\delta_{jk} + v_j\,\delta_{ik} + v_k\,\delta_{ij}\right) ;$$

and so on. These have the following useful property:

$$\int f^0\, d\vec{v}\, \mathcal{H}^n_{ij\cdots n}\;\; \mathcal{H}^m_{\ell m\cdots o} = 0 \text{ if } n \neq m .$$

Since the velocity dependence of the right-hand side of Eq. (7) is composed of the terms $f^0\mathcal{H}^0$, $f^0\mathcal{H}_i^1$ and $f^0\mathcal{H}_{jj}^2$, operating on the equation with $\int \mathcal{H}^n\, d\vec{v}$ is simplified somewhat since these terms do not contribute to the higher moment equations. We also notice that

$$\int \mathcal{H}^n f^0 \, d\vec{v} = 0 \text{ for } n \neq 1$$

and

$$\int \mathcal{H}^0 f^1 \, d\vec{v} = n ;$$

$$\sqrt{\frac{T_0}{m}} \int \mathcal{H}_i^{\,1} f^1 d\vec{v} = n_0 \, u_i ;$$

$$T_0 \int \mathcal{H}_{ij}^{\,2} f^1 d\vec{v} = P_{ij} - n \, T_0 \, \delta_{ij} = \Pi_{ij} + n_0 \, T \, \delta_{ij} ;$$

$$\frac{T_0^{\,3/2}}{m^{1/2}} \int \mathcal{H}_{ijk}^{\,3} f^1 d\vec{v} = Q_{ijk} .$$

Each higher polynomial introduces new information. It introduces the next perturbed moment.

Equation (8) discloses the following useful property of these polynomials:

$$\int \mathcal{H}^n_{jk\cdots n} \, \vec{v} \cdot \vec{\nabla} f^1 \, d\vec{v} = \frac{\partial}{\partial x_i} \int v_i \, \mathcal{H}^n_{jk\cdots n} \, f^1 d\vec{v}$$

$$= \frac{\partial}{\partial x_i} \left[\sqrt{\frac{T_0}{m}} \int \mathcal{H}^{n+1}_{ijk\cdots n} \, f^1 d\vec{v} + \frac{T_0}{m} \int \frac{\partial}{\partial v_i} \, \mathcal{H}^n_{jk\cdots n} \, f^1 \, d\vec{v} \right] .$$

Another property of the polynomials is

$$\frac{\partial}{\partial v_i} \mathcal{H}^n_{jk\cdots mn} = \sqrt{\frac{m}{T_0}} \left[\delta_{ij} \mathcal{H}^{n-1}_{k\ell\cdots mn} + \delta_{ik} \mathcal{H}^{n-1}_{j\ell\cdots mn} + \cdots + \delta_{in} \mathcal{H}^{n-1}_{jk\cdots m}\right].$$

The final result is

$$\int \mathcal{H}^n_{jk\cdots mn} \vec{v}\cdot\vec{\nabla} f^1 \, d\vec{v} = \sqrt{\frac{T_0}{m}} \left\{ \frac{\partial}{\partial x_i} \int \mathcal{H}^{n+1}_{ijk\cdots mn} f^1 \, d\vec{v} + \frac{\partial}{\partial x_j} \int \mathcal{H}^{n-1}_{k\ell\cdots mn} f^1 d\vec{v} + \frac{\partial}{\partial x_k} \int \mathcal{H}^{n-1}_{j\ell\cdots mn} f^1 \, d\vec{v} + \cdots + \frac{\partial}{\partial x_n} \int \mathcal{H}^{n-1}_{jk\cdots m} f^1 d\vec{v} \right\}$$

The $\vec{v} \cdot \vec{\nabla} f^1$ term of the kinetic equation couples both the next higher and the next lower moment into the equation governing the moment corresponding to $\mathcal{H}^n$.

These results are now used to obtain the first three moment equations. We operate on Eq. (7) with $\int \mathcal{H}^n d\vec{v}$ for $n = 0,1,$ and 2 to obtain

$$\frac{\partial n_1}{\partial t} + n_0 \vec{\nabla} \cdot \vec{u}_1 = 0 \quad , \qquad (9)$$

$$n_0 m_i \frac{\partial \vec{u}_i}{\partial t} + \vec{\nabla} \cdot (\overleftrightarrow{\Pi}_i + n_0 T_i \overleftrightarrow{I}) + T_0 \vec{\nabla} n_i + \omega_{ic} \hat{z} \times n_0 m_i \vec{u}_i$$

$$= e n_0 \vec{E} - \nu_1 m_{ie} n_0 (\vec{u}_i - \vec{u}_e) \tag{10}$$

$$\frac{\partial}{\partial t}(\overleftrightarrow{\Pi}_i + n_0 T_i \overleftrightarrow{I}) + \vec{\nabla} \cdot \overleftrightarrow{Q}_i + P_0(\vec{\nabla} \vec{u}_i + (\vec{\nabla} \vec{u}_i)^T) + \omega_{ic} \hat{z} \otimes \overleftrightarrow{\Pi}_i$$

$$= - \nu_i \overleftrightarrow{\Pi}_i - \nu_2 n_0 (T_i - T_e) \overleftrightarrow{I} \ . \tag{11}$$

Equation (9) is the expected result. Equation (10) agrees with Eq. (2) when the intuitive collision form is used in that equation. The term, $\vec{\nabla} \cdot \overleftrightarrow{P}$, $= \vec{\nabla} \cdot \overleftrightarrow{\Pi} + n_0 \vec{\nabla} T + T_0 \vec{\nabla} n$ since $P = n_0 T + T_0 n$. If we multiply Eq. (9) by T_0 and add it to one third of the trace of Eq. (11) we obtain

$$\frac{\partial P_i}{\partial t} + \frac{5}{3} P_0 \vec{\nabla} \cdot \vec{u}_i = -\nu_2 n_0 (T_i - T_e) \ .$$

This is the extension of Eq. (5) to include the relaxation of the temperature difference of the two species of particles. The traceless part of Eq. (11) is

$$\frac{\partial \overleftrightarrow{\Pi}_i}{\partial t} + \vec{\nabla} \cdot \overleftrightarrow{Q}_i - \frac{2}{3} \overleftrightarrow{I} \vec{\nabla} \cdot \vec{q}_i + P_0 (\vec{\nabla}\, \vec{u}_i + (\vec{\nabla}\, \vec{u}_i)^T - \frac{2}{3} \overleftrightarrow{I}\, \vec{\nabla} \cdot \vec{u}_i) + \omega_{ic}\, \hat{z} \otimes \overleftrightarrow{\Pi}_i + \nu_i \overleftrightarrow{\Pi}_i = 0. \tag{11a}$$

which agrees with Eq. (6), except the next moment is retained here.

Operating on Eq. (7) with $\int \mathcal{H}^3 d\vec{v}$, we obtain

$$\left[\left(\frac{\partial}{\partial t} + \nu_i + \omega_{ic}\, \hat{z} \otimes\right) \overleftrightarrow{Q}\right]_{ijk} + \frac{\partial}{\partial x_\ell} R_{\ell ijk} + \frac{T_0}{m} \left[\frac{\partial}{\partial x_i} (\Pi_{jk} + n_0\, T\, \delta_{jk}) + \frac{\partial}{\partial x_j} (\Pi_{ik} + n_0\, T\, \delta_{ik}) + \frac{\partial}{\partial x_k} (\Pi_{ij} + n_0\, T\, \delta_{ij})\right] = 0 \tag{11b}$$

where

$$R_{\ell ijk} = \int f^1\, d\vec{v}\, \frac{T_0^{\,2}}{m}\, \mathcal{H}^4_{\ell ijk} \quad .$$

Now the usefulness of the Hermite polynomials is apparent. If M^n is the moment introduced by $\mathcal{H}^n$, for $n \geq 3$, the corresponding moment equation is

$$\mathcal{L}\, M^n + \vec{\nabla} \cdot M^{n+1} + \frac{T_0}{m} \vec{\mathcal{K}}\ \ M^{n-1} = 0 \tag{11c}$$

where

$$\mathcal{L} = \left(\frac{\partial}{\partial t} + \nu_i + \omega_{ci}\,\hat{z}\otimes\right)$$

and

$$\left(\vec{\mathcal{K}}\, M^{n-1}\right)_{ijk\cdots mn} = \frac{\partial}{\partial x_i} M^{n-1}_{jk\cdots mn} + \frac{\partial}{\partial x_j} M^{n-1}_{ik\cdots mn} + \cdots + \frac{\partial}{\partial x_n} M^{n-1}_{ijk\cdots m} .$$

Because of the orthogonality of the Hermite polynomials, none of the terms on the right side of Eq. (7) are present. The truncation of the series of moment equations can now be formally illustrated treating the operators $\vec{\nabla}, \mathcal{L}$ and $\vec{\mathcal{K}}$ as numbers. One ignores the moment M^{n+1} and obtains

$$M^n = \frac{-T_0}{m} \frac{\vec{\mathcal{K}}}{\mathcal{L}} M^{n-1} .$$

The result is used in the next lower moment equation to obtain

$$M^{n-1} = \frac{T_0}{m} \frac{1}{\mathcal{L}} \vec{\nabla} \cdot \frac{1}{\mathcal{L}} \vec{\mathcal{K}}\, M^{n-1} - \frac{T_0}{m} \frac{\vec{\mathcal{K}}}{\mathcal{L}} M^{n-2}$$

or

$$M^{n-1} = \frac{\frac{-T_0 \vec{\mathcal{K}}}{m\mathcal{L}} M^{n-2}}{1 - \frac{T_0}{m}\frac{1}{\mathcal{L}}\vec{\vec{\nabla}}\cdot\frac{1}{\mathcal{L}}\vec{K}} .$$

Let

$$\epsilon = \frac{T_0}{m}\frac{1}{\mathcal{L}}\vec{\vec{\nabla}}\cdot\frac{1}{\mathcal{L}}\vec{\mathcal{K}} .$$

The continuation of the process leads to the formal expression

$$\vec{\vec{Q}} = \frac{\frac{T_0\vec{\mathcal{K}}}{m\mathcal{L}}(\overleftrightarrow{\Pi} + n_0\, T\, \overleftrightarrow{I})}{1 - \cfrac{\epsilon}{1 - \cfrac{\epsilon}{1 - \epsilon \cdots \cfrac{\epsilon}{1-\epsilon}}}} .$$

where the number of terms kept in the denominator depends on the order of the neglected moment. We now have an equation for $\vec{\vec{Q}}$ in terms of $(\overleftrightarrow{\Pi} + n_0\, T\, \overleftrightarrow{I})$ which along with Eqs. (9), (10) and (11) and Maxwell's equations constitute a closed set of equations.

Thus the truncation of the system of moment equations is equivalent to an expansion in ϵ. If the parameter ϵ is not small, the truncation is incorrect.

We estimate ϵ for the transverse plane waves by ignoring $\overleftrightarrow{\overleftrightarrow{R}}$

in Eq. (11b). The contraction of this equation produces

$$\left(\frac{\partial}{\partial t} + \omega_{ic}\, \hat{z} \times + \nu_i\right)\vec{q}_i + \frac{T_0}{m}\left[\vec{\nabla} \cdot \overleftrightarrow{\Pi}_i + \frac{5}{2} n_0 \vec{\nabla} T_i\right] = 0 \, . \quad (12)$$

For transverse waves with the form $e^{i(kz - \omega t)}$, u_z and E_z are zero. Equation (9) and the trace of Eq. (11) show $n = T = 0$. Equation (11) then shows the only components of $\overleftrightarrow{\Pi}$ produced by the $\vec{\nabla}\vec{u}$ term are $\Pi_{xz} = \Pi_{zx}$ and $\Pi_{yz} = \Pi_{zy}$. Equation (11b) without $\overleftrightarrow{R}$ then shows the only components of $\overleftrightarrow{Q}$ produced by the $\vec{\nabla}\overleftrightarrow{\Pi}$ term are $Q_{xzz} = Q_{zxz} = Q_{zzx}$ and $Q_{yzz} = Q_{zyz} = Q_{zzy}$. Since $Q_{xzz} = 2q_x$ and $Q_{yzz} = 2q_y$, Eq. (12) is sufficient to determine $\overleftrightarrow{Q}$. Equation (12) may be written

$$\left(\frac{\partial}{\partial t} + \nu_i\right) q_x - \omega_{ic}\, q_y = \frac{-T_0}{m_i}\frac{\partial}{\partial z}\, \Pi_{zx}$$

$$\left(\frac{\partial}{\partial t} + \nu_i\right) q_y + \omega_{ic}\, q_x = \frac{-T_0}{m_i}\frac{\partial}{\partial z}\, \Pi_{zy} \, .$$

Since the left-handed wave has the form $\vec{q} = q_-(\hat{x} - i\hat{y})e^{i(kz-\omega t)}$, $\vec{\Pi}_z = \Pi_{z-}(\hat{x} - i\hat{y})\, e^{i(kz-\omega t)}$, and so on, these two equations are combined to yield $(\omega - \omega_{ic} + i\nu_i)q_- \equiv \omega'_{i-}\, q_- = \frac{T_0 k}{m_i}\Pi_{z-}$. (12a).

In the same manner, Eq. (11a) yields the equation

$$\omega_{i-}' \Pi_{z-} - 2kq_- = P_0 k u_{i-} .$$

Substitution for q_- using Eq. (12a) yields

$$\left[\omega_{i-}' - \frac{2T_0}{m_i} \frac{k^2}{\omega_{i-}'}\right] \Pi_{z-} = P_0 k u_{i-}$$

or

$$\omega_{i-}'(1 - \epsilon_{i-})\Pi_{z-} = P_0 k u_- \qquad (12b)$$

where $\epsilon_{i-} = \frac{2T_0}{m_i} \frac{k^2}{(\omega_{i-}')^2} = \frac{k^2 a_i^2}{(\omega_{i-}')^2}$ is the small parameter.

It is a measure of the importance of the inclusion of the heat flow tensor in the viscosity equation (11a). The criterion then, for neglecting the heat flow and therefore all higher moments is

$$|\epsilon_{i-}| = \left| \frac{k^2 a_i^2}{(\omega - \omega_{ic} + i\nu_i)^2} \right| \ll 1 .$$

This is satisfied for ω far from ω_{ic} since $\omega_{ic} \gg k a_i$.

That is, the ion Larmour radius, $\frac{a_i}{\omega_{ic}}$, is smaller than the wavelength, $\frac{1}{k}$, for our cases of interest. In the resonance region, the ion collisions become important. The wave length must now be larger than the mean free path for ion collisions, that is, $\frac{1}{k} > \frac{a_i}{\nu_i}$. When this criterion is satisfied, the heat flow may be neglected. However, the viscosity may still not be negligible. We shall now show that the criterion for neglecting ion viscosity is more stringent in the resonance region.

For the left-handed wave, Eq. (10) has the form

$$m_i\left[\omega - \omega_{ic} + i\,\frac{m_{ie}}{m_i}\,\nu_1\right] u_{i-} - \frac{k}{n_0}\,\Pi_{z-} - i\, m_{ie}\,\nu_1\, u_{e-} = ieE_- \; . \tag{12c}$$

Substitution of Eq. (12b) into Eq.(12c) produces

$$m_i\left[\omega - \omega_{ic} + i\,\frac{m_{ie}\nu_1}{m_i} - \frac{k^2\,T_0}{m_i\omega_{i-}'(1 - \epsilon_{i-})}\right] u_{i-} - i m_{ie}\nu_1 u_{e-} = ieE_- \; . \tag{13}$$

Neglecting ϵ_{i-} in Eq. (13) and assuming $\omega \approx \omega_{ic}$ we have

$$\left[i\,\frac{m_{ie}}{m_i}\,\nu_1 + \frac{ik^2T_0}{m_i\,\nu_i}\right] m_i\, u_{i-} = i\, m_{ie}\,\nu_1\, u_{e-} + ieE_- \; . \tag{13a}$$

For contrast we rewrite Eq. (12b) for $\omega \approx \omega_{ic}$.

$$\left[i\,\nu_i + \frac{2i\,k^2\,T_0}{m_i\,\nu_i} \right] \Pi_{z-} = P_0\,k\,u_{i-} \qquad (13b)$$

Now $\frac{m_{ie}}{m_i}\nu_1$, the ion electron momentum transfer collision frequency, is smaller than ν_i, the ion-ion collision frequency.

$$\frac{m_{ie}}{m_i}\nu_1 \approx \sqrt{\frac{m_e}{m_i}}\;\nu_i \,.$$

Therefore the ratio of the viscous to the collision term in Eq. (13a) is

$$\epsilon_{i-}{}' = \frac{k^2\,T_0}{\nu_i\,m_{ie}\,\nu_1} \approx \frac{1}{2}\sqrt{\frac{m_i}{m_e}}\;\frac{k^2\,a_i^{\,2}}{\nu_i^{\,2}} = \frac{1}{2}\sqrt{\frac{m_i}{m_e}}\left|\epsilon_{i-}(\omega = \omega_{ic})\right| .$$

The ratio of the heat flow to the collision term in Eq. (13b) is

$$\left|\epsilon_{i-}(\omega = \omega_{ic})\right| = \frac{2k^2\,T_0}{m_i\,\nu_i^{\,2}} = \frac{k^2\,a_i^{\,2}}{\nu_i^{\,2}} \,.$$

Therefore for cases where $|\epsilon_{i-}(\omega = \omega_{ic})| \ll 1$ and heat flow is negligible we may find $\epsilon_{i-}' \gtrsim 1$ indicating that ion viscosity is not negligible and may significantly alter the results of the cold plasma theory.

This is not true for the electrons. If we consider the right-handed wave, the corresponding equation for the electron motion is

$$m_e \left[\omega - \omega_{ec} + i \frac{m_{ie}}{m_e} \nu_1 - \frac{k^2 T_0}{m_e \omega_{e+}'(1 + \epsilon_{e+})} \right] u_{e+} - i m_{ie} \nu_1 u_{i+} = -ieE_+$$

where

$$\omega_{e+}' = \omega - \omega_{ec} + i \nu_e$$

and

$$\epsilon_{e+} = \frac{k^2 a_e^2}{(\omega_{e+}')^2} .$$

When $\omega \approx \omega_{ec}$ the criterion for neglecting heat flow and the other moments is $\left| \frac{k^2 a_e^2}{\nu_e^2} \right| \ll 1$. The criterion for neglecting viscosity is

$$\frac{k^2 a_e^2}{\nu_e \left(\frac{m_{ie}}{m_e} \nu_1 \right)} \ll 1 .$$

Since $\frac{m_{ie}}{m_e} \nu_1 \approx \nu_e$ the criteria are identical. The effect of electron-electron collisions is comparable to that of electron-ion collisions. Thus for the electron resonance, viscosity will have a small effect unless $\epsilon_{e+} \gtrsim 1$ in which case, the moment treatment is invalid.

We will be concerned with the ion resonance, which occurs for the left-hand wave. In this case, the equation for the electron motion is

$$m_e\left[\omega + \omega_{ec} + \frac{i\ m_{ie}\ \nu_1}{m_e} - \frac{k^2\ T_0}{m_e \omega_{e-}{}'(1 - \epsilon_{e-})}\right]u_{e-} - i m_{ie} \nu_1 u_{i-} = -ieE_-$$

where

$$\omega_{e-}{}' = \omega + \omega_{ec} + i\ \nu_e$$

and

$$\epsilon_{e-} = \frac{k^2\ a_e{}^2}{(\omega_{e-}{}')^2} .$$

In the region of interest, where the thermal effects may be important, $\omega \approx \omega_{ic}$. Therefore $|\omega_{e-}{}'| \approx |\omega_{ce} + i\ \nu_e| \gg |\nu_e|$ in the region of interest and $|\epsilon_{e-}| \ll |\frac{k^2 a_e{}^2}{\nu_e{}^2}| = |\frac{k^2 a_i{}^2}{\nu_i{}^2}| = |\epsilon_{i-}|$.

Therefore, if the moment expansion is valid for the ions, the thermal effects of the electrons are negligible. The two equations are then

$$m_i\left[\omega - \omega_{ic} + i\frac{m_{ie}}{m_i}\nu_1 - \frac{k^2\, T_0}{m_i(\omega - \omega_{ic} + i\, \nu_i)}\right] u_{i-} - i m_{ie}\nu_1 u_{e-} = i\, e\, E_-$$

and

$$-i\, m_{ie}\, \nu_1\, u_{i-} + m_e\left[\omega + \omega_{ec} + i\frac{m_{ie}\nu_1}{m_e}\right] u_{e-} = -\, i\, e\, E_-$$

resulting in

$$n_-^{\,2} = \frac{k_-^{\,2}\, c^2}{\omega^2}$$

$$= 1 - \frac{\omega_p^{\,2}\left[1 - \dfrac{k_-^2\, T_0}{(m_i + m_e)\omega(\omega - \omega_{ic} + i\, \nu_i)}\right]}{\left\{(\omega-\omega_{ic})(\omega+\omega_{ec})+i\nu_1\omega\right\}\left\{1 - \dfrac{k_-^{\,2} T_0\left[m_e(\omega + \omega_{ec}) + i\, m_{ie}\, \nu_1\right]}{m_i m_e\left[\omega-\omega_{ic}+i\nu_i\right]\left[(\omega-\omega_{ic})(\omega+\omega_{ec})+i\nu_1\omega\right]}\right\}} \qquad (14)$$

exhibiting the lowest order thermal correction which causes the index of refracting to be dependent on k in addition to ω. The dispersion relation, (14), now yields four solutions $k(\omega)$, whereas without the thermal term, there are only the usual two solutions, $k = +k_0(\omega)$, corresponding to a damped wave propagating in the positive z direction and $k = -k_0(\omega)$, corresponding to the same wave propagating in the negative z direction. The solution, $k_0(\omega)$, was plotted in Fig. 2 from Eq. (4), which is the result of setting the terms containing T_0 equal to zero in Eq. (14). The two extra solutions introduced by the ion viscosity require that two more spatial boundary conditions be specified for the total solution to be determined. Prior to the introduction of ion viscosity, the two conditions

$$B_-(z = 0) = 1 \tag{15}$$

and

$$\lim_{z\to\infty} B_- = 0 \tag{16}$$

are sufficient to determine the solution

$$B_-(z) = e^{ik_0(\omega)z} \tag{17}$$

where $B_-(z)$ is the amplitude of the wave magnetic field. Henceforth we will consider the solution for the wave magnetic field, $\vec{B} = B_-(\hat{y} + i\hat{x})e^{-i\omega t}$ rather than the electric field in order to simplify comparison with the following section. Because Eq. (14) is even in k, one of the two extra solutions will have $\text{Im } k < 0$. This may be discarded by condition (16). We are then left with

$$B_-(z) = A_1 e^{ik_1(\omega)z} + A_2 e^{ik_2(\omega)z}$$

where $\mathrm{Im}\, k_1$, $\mathrm{Im}\, k_2 > 0$. Condition (15) determines $A_2 = 1 - A_1$. One more boundary condition is required for A_1. If we take the plane, $z = 0$, to be a boundary of the plasma, we are essentially considering the problem of the transmission of a transverse left-handed circularly polarized wave incident on a semi-infinite plasma in a constant magnetic field which is perpendicular to the plane boundary. We choose the condition that the particles of the plasma reflect specularly from the boundary. There is therefore no parallel stress on the wall by the plasma. That is, $\vec{\Pi}_z$ is zero at the wall. This condition then determines A_1 and the wave form in the plasma is determined.

If we define

$$\delta^2 = \frac{m_i m_e\left[(\omega - \omega_{ic})(\omega + \omega_{ec}) + i\,\nu_1\,\omega\right]\left[\omega - \omega_{ic} + i\,\nu_i\right]}{T_0\left[m_e(\omega + \omega_{ec}) + i\,m_{ie}\,\nu_1\right]},$$

the wave magnetic field is

$$B_-(z) = \frac{1}{k_2^{\,2} - k_1^{\,2}}\left[(\delta^2 - k_1^{\,2})e^{ik_1 z} - (\delta^2 - k_2^{\,2})e^{ik_2 z}\right]. \tag{18}$$

That is, we find

$$A_1 = \frac{\delta^2 - k_1^2}{k_2^2 - k_1^2}$$

and

$$A_2 = 1 - A_1 = \frac{(\delta^2 - k_2^2)}{k_1^2 - k_2^2} .$$

In order to display the relative effects of collisions and viscosity, we choose values of density and magnetic field representative of a wave experiment[16] conducted at the Lawrence Radiation Laboratory, Berkeley. These are

$$n_0 = 3.5 \cdot 10^{14} \text{ cm}^{-3}$$

and

$$B_0 = 1.09 \cdot 10^4 \text{ gauss.}$$

We then have the following values for a Deuterium plasma:

$$\omega_p = 1.06 \cdot 10^{12} \text{ sec}^{-1} ;$$
$$\omega_{ec} = 1.92 \cdot 10^{11} \text{ sec}^{-1} ;$$
$$\omega_{ic} = 5.24 \cdot 10^{7} \text{ sec}^{-1} ;$$

If we choose $T_0 = 2 \cdot 10^4$ °K, we have $\nu_1 = 3.45 \cdot 10^9$ sec^{-1} and $\nu_i = 5.9 \cdot 10^7$ sec.$^{-1}$.

Figure 3 is a plot of the trajectories of k_1 and k_2 in the complex k plane for this case; k_0 in the figure is the result obtained when viscosity is neglected. It is a polar plot of Fig. 2. Let $\Omega = \frac{\omega}{\omega_{ic}}$. The quantities k_0 and k_1 are plotted for Ω values between $\Omega = 0.5$ and $\Omega = 2.0$, while k_2 is plotted for $0.95 \leqslant \Omega \leqslant 1.05$. Beyond this region, k_2 becomes too large to neglect heat flow and higher moments. That is,

$$\left| \frac{k_2^2 \, a_i^2}{(\omega - \omega_{ic} + i\,\nu_i)^2} \right|$$

becomes comparable to unity nullifying the moment approach. Where this is so, the coefficient of the k_2 wave, A_2, becomes negligibly small. ($|A_2|$ is plotted as a function of Ω in the lower left section of Fig. 3.) Therefore, the behavior of k_2 is not known where it is not needed. At $\Omega = 1.015$, however, $|A_2| = 0.35$. At $T_0 = 2 \cdot 10^4$; therefore, the viscosity is at the threshold of importance.

So for $0.95 \leqslant \Omega \leqslant 1.05$, the expression (18) is necessary. Beyond this region, expression (17) will suffice.

For order of magnitude estimates of damping, the viscosity may be ignored and expression (17) used.

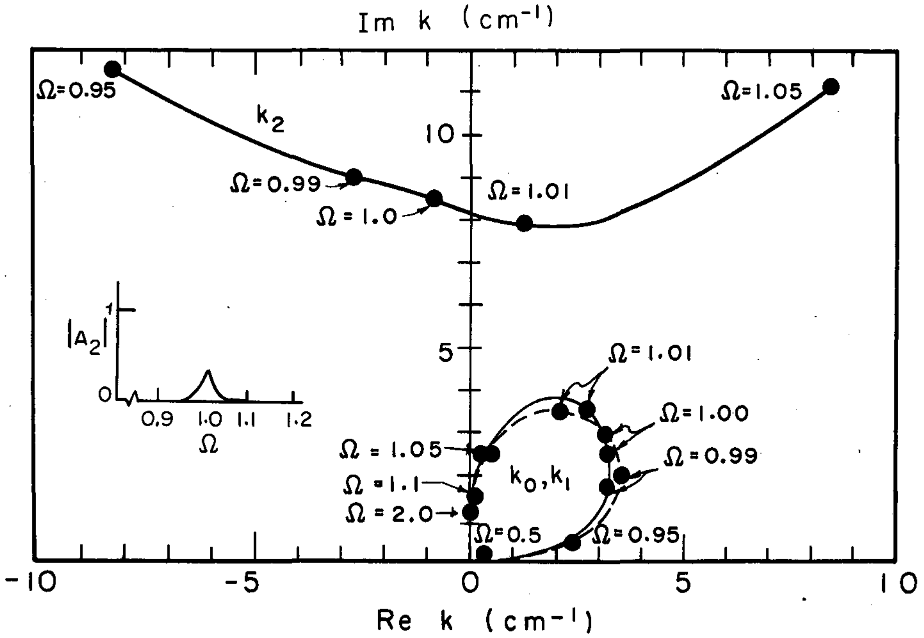

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Fig. 3. Trajectories of k_1 and k_2. $T_0 = 2 \times 10^4$ °K.
—— k_1, k_2; - - - k_0.

At lower temperatures, it is found that the k_2 wave may be entirely ignored. The coefficient, A_1 , remains essentially unity, $k_1 \approx k_0$ and k_2 recedes to infinity corresponding to zero damping length.

Since the viscosity is most important when $\omega \approx \omega_{ic}$ we use this point to derive the criterion for neglecting viscosity. As before, we have the ratio of the viscous to the collision term,

$$\frac{k^2 T_0}{\nu_i m_{ie} \nu_1} \ll 1 .$$

We choose for k^2 the value which is obtained from the non-viscous dispersion relation, (4), when $\omega \approx \omega_{ic}$. We have

$$k_0^2(\omega_{ic}) = \frac{\omega_{ic}^2}{c^2} + \frac{i \omega_{ic} \omega_p^2}{\nu_1 c^2} .$$

The first term on the right, which corresponds to the displacement current may be ignored in our region of interest where

$$\frac{\omega_p^2}{\omega_{ic} \nu_1} \gg 1 .$$

Substitution produces the criterion

$$\frac{\omega_{ic}\,\omega_p^2}{\nu_1\,c^2}\;\frac{T_0}{\nu_i\,\nu_1\,m_{ie}} = \frac{3\cdot 10^3\,B_0\,T_0^{11/2}}{n_0^2\,(\ell n\,\Lambda)^3} \ll 1\;.$$

Inserting the value for parameters pertaining to Fig. 3, we obtain the value 0.28 < 1. Fig. 3 then represents a case where the criterion is barely satisfied. Noticing that the criterion is heavily temperature dependent, we examine the case when $T_0 = 3\cdot 10^4\,{}^\circ K$. We then obtain the value $2.1 > 1$. Now the viscosity must be kept. Figure 4 is a plot of the trajectories of k_1 and k_2 for this case. The trajectory of k_0 is included for comparison and $|A_2|$ is plotted as before. The coefficient $|A_1|$ is approximately $1-|A_2|$ and is therefore not plotted in Figs. 3 and 4.

Now it is found that for $0.9 \leq \Omega \leq 1.1$, expression (18) is necessary. It is also found that the wave k_1 no longer identifies with k_0. $k_1 \approx k_0$ at $\Omega = 0.9$ but $k_2 \approx k_0$ at $\Omega = 1.1$. A_2 varies from negligibly small values near $\Omega = 0.9$ to nearly unity at $\Omega = 1.05$. A_1 and A_2 are about equal at $\Omega = 1.01$. Now, no order of magnitude estimates can be made for the damping at resonance by considering one wave alone. The disturbance is expressed in terms of two wave forms which are not independent in this geometry. That is, the two wave forms cannot be independently excited. This is because of the necessity of maintaining $\vec{\Pi}_z$ zero at the boundary.

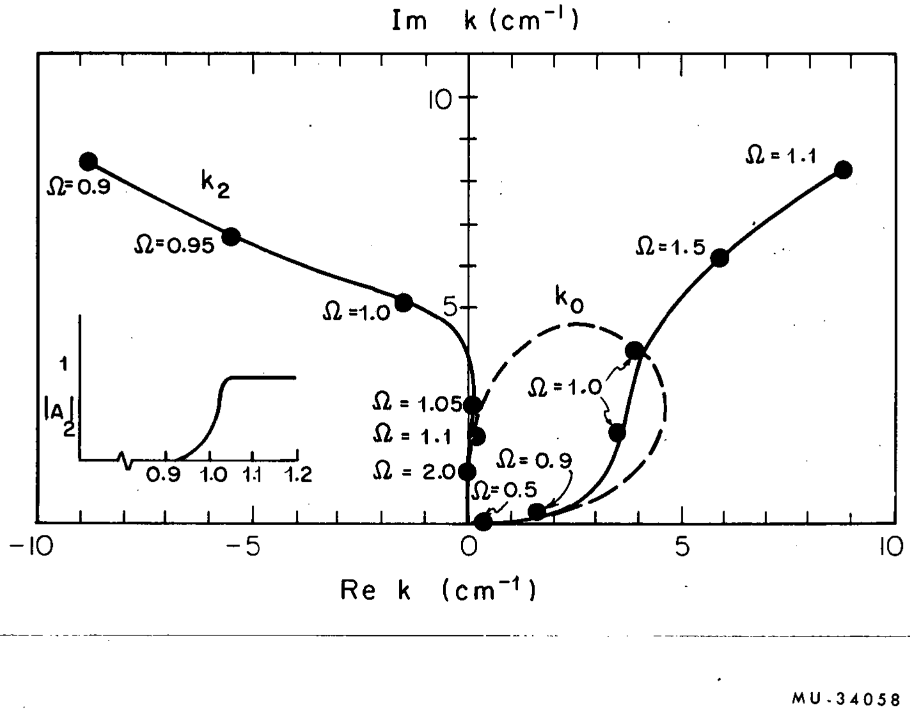

Fig. 4. Trajectories of k_1 and k_2. $T_0 = 3 \times 10^4$ °K. —— k_1, k_2; – – – k_0.

We illustrate the wave form of $B_-(z)$ at resonance on Fig. 5 for the temperature,

$$T = 3 \cdot 10^4 \ {}^\circ K .$$

The upper plot is Real $(A_1 e^{ik_1z})$, the contribution of k_1 to the total solution. The middle plot is Real $(A_2 e^{ik_2z})$. The lower plot is Real $(A_1 e^{ik_1z} + A_2 e^{ik_2z})$, the complete solution as given by Eq. (18). For comparison, we include Real (e^{ik_0z}), the solution without viscosity. Since

$$\vec{B} = B_-(z)(\hat{y} + i\,\hat{x})e^{-i\omega t} ,$$

we are plotting the component of $\vec{B}$ in the direction $(\hat{y}\cos\omega t + \hat{x}\sin\omega t)$ at time t as a function of z. Referring to the lower plot, we find that both waves are severely damped but the wave that includes the viscosity effects does not decrease as abruptly as the other wave. The viscosity acts to reduce the shear caused by the spatial variation of the wave field.

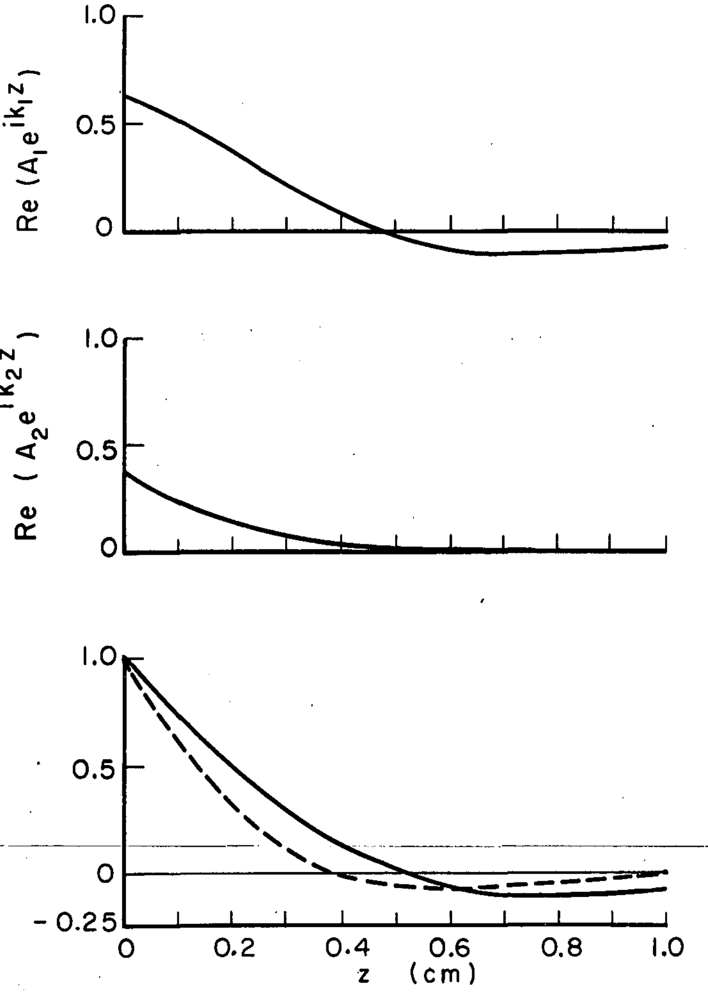

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Fig. 5. Wave forms of B at resonance. $T_0 = 3 \times 10^4$ °K. For lowest drawing only, —— Re $(A_1 e^{ik_1 z} + A_2 e^{ik_2 z})$; – – – Re $e^{ik_0 z}$.

The question arises as to the effect of heat flow on these solutions. Will a third wave form arise? Including heat flow necessitates solving a cubic equation for $k^2(\omega)$. Since heat flow is unimportant unless $\epsilon_{i-} \gtrapprox 1$, in which case the moment expansion is invalid, nothing is gained by its inclusion. Instead, the moment expansion is abandoned. In the next section, the problem is done using the kinetic equation (7) itself. This method is correct at all temperatures where classical non-relativistic physics is adequate. This method is necessary when $\epsilon_{i-} \gtrapprox 1$. Solution of Eq. (14) shows that $\epsilon_{i-} \approx 1$ in the resonance region for $T_0 = 10^5$ °K. Therefore, for temperatures of this order and higher and the previously mentioned values of density and magnetic field, the kinetic equation method must be used.

No simple criterion exists for the determination of ϵ_{i-} in general. A coarse criterion is obtained by again using $k_0^{\,2}(\omega_{ic})$ in the expression for ϵ_{i-}. At $\omega = \omega_{ic}$, we have

$$\epsilon_{i-}^{\,0} = \frac{\omega_{ic}\,\omega_p^{\,2}}{\nu_1\,c^2}\,\frac{a_i^{\,2}}{\nu_i^{\,2}} = \frac{10^2\,B_0\,T_0^{\,11/2}}{n_0^{\,2}(\ell n\,\Lambda)^3}\,.$$

$T_0 = 10^5$ corresponds to $\epsilon_{i-}^{\,0} = 28$. The coarse criterion is too pessimistic. Fig. 4 shows that $|k_1(\omega_{ic})| < |k_0(\omega_{ic})|$. Lacking a simple expression for the pertinent values of k_1 or k_2, $\epsilon_{i-}^{\,0} < 1$ will be considered a sufficient condition for using the moment equations and neglecting heat flow. If $\epsilon_{i-}^{0} > 1$ but not by much, it would be worthwhile to neglect heat flow and check the value of ϵ_{i-} pertaining to the solutions k_1 and k_2 where they are important.

C. The Kinetic Equation Method

The starting point is Eq. (7) for the ions and the corresponding equation for electrons. Choosing cylindrical coordinates for the velocity vector, the magnetic field term simplifies. For

$$\vec{v} = v_\perp \cos \phi \, \hat{x} + v_\perp \sin \phi \, \hat{y} + v_z \, \hat{z}$$

we have

$$\vec{v} \times \hat{z} \cdot \vec{\nabla}_{\vec{v}} f = -\frac{\partial f}{\partial \phi} .$$

The macroscopic vectors, $\vec{E}$ and $\vec{u}$, appearing on the right side of Eq. (7) are put in the form

$$\vec{E} = E_+(\hat{x} + i \, \hat{y}) + E_-(\hat{x} - i \, \hat{y}) + E_z \, \hat{z} .$$

Assuming no spatial variation in the x-y plane, we have

$$\left(\frac{\partial}{\partial t} + v_z \frac{\partial}{\partial z} - \omega_{ic} \frac{\partial}{\partial \phi} + \nu_i\right) f_i^{\,1} = D_+ e^{i\phi} + D_- e^{-i\phi} + D_z \quad (19)$$

where

$$D_{\pm} = \frac{2v_{\perp} f_i^{\,0}}{a_i^{\,2}} \left[\frac{eE_{\pm}}{m_i} + (\nu_i - \frac{m_{ie}}{m_i}\ \nu_1)u_{i\pm} + \nu_1 \frac{m_{ie}}{m_i}\ u_{e\pm}\right]$$

and

$$D_z = f_i^{\,0}\left\{\frac{2v_z}{a_i^{\,2}}\left[\frac{eE_z}{m_i} + (\nu_i - \frac{m_{ie}}{m_i}\ \nu_1)u_{iz} + \nu_1 \frac{m_{ie}}{m_i} u_{ez}\right] + \nu_i \frac{n_i}{n_0} + \frac{\left[(v^2/a_i^2) - 3/2\right]}{T_0}\left(\nu_i\ T_i - \nu_2\left[T_i - T_e\right]\right)\right\} .$$

A similar equation results for the electrons.

The form of the right side of Eq. (19) suggests a separation of $f_i^{\,1}$ into the form

$$f_i^{\,1} = f_{i+}(z,\ t,\ v_{\perp},\ v_z)e^{i\phi} + f_{i-}(z,\ t,\ v_{\perp},\ v_z)e^{-i\phi} + f_{i0}(z, t, v_{\perp},\ v_z)\ .$$

Equating coefficients of like exponentials in Eq. (19) produces

$$\left(\frac{\partial}{\partial t} + v_z \frac{\partial}{\partial z} + \nu_i \mp i\ \omega_{ic}\right) f_{i\ \pm1} = D_{\pm} \tag{20a}$$

and

$$\left(\frac{\partial}{\partial t} + v_z \frac{\partial}{\partial z} + \nu_i\right) f_{i0} = D_z \quad . \tag{20b}$$

To complete the equations, expressions for $\vec{E}$, $\vec{u}$, n and T are needed:

$$n = \int f^1 \, d\vec{v} = 2\pi \iint v_\perp dv_\perp \, dv_z \, f_0 \; ;$$

since $T = \dfrac{P_0 - nT_0}{n_0}$,

$$T = \frac{2T_0}{3n_0} \int \left(\frac{v^2}{a^2} - 3/2\right) f^1 \, d\vec{v} = \frac{4\pi \, T_0}{3n_0} \iint v_\perp \, dv_\perp \, dv_z \, f_0 \left(\frac{v^2}{a^2} - 3/2\right);$$

$$n_0 \, u_z = \int v_z \, f^1 \, d\vec{v} = 2\pi \iint v_\perp \, dv_\perp \, v_z \, dv_z \, f_0 \; ;$$

$$n_0 \, u_\pm = \int \frac{v_\perp}{2} \, e^{\mp i\phi} \, f^1 \, d\vec{v} = \pi \iint v_\perp^2 \, dv_\perp \, dv_z \, f_{\pm 1} \quad .$$

Maxwell's equations furnish $\vec{E}$ in terms of $\vec{u}$.

$$\left(\frac{\partial^2}{\partial z^2} - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right) E_\pm = \frac{4\pi}{c^2}\frac{\partial J_\pm}{\partial t} = \frac{4\pi \, en_0}{c^2} \frac{\partial}{\partial t} \left(u_{i\pm} - u_{e\pm}\right) \; .$$

For E_z , the equations

$$\frac{\partial E_z}{\partial z} = 4\pi e(n_i - n_e)$$

and

$$n_0 \frac{\partial u_z}{\partial z} + \frac{\partial n}{\partial t} = 0$$

are sufficient.

The equations uncouple into three sets of equations for the three independent sets of quantities:

$$f_{+1} , E_+, u_+ ;$$

$$f_{-1} , E_-, u_- ;$$

$$f_0 , E_z, u_z, n, T .$$

The first set corresponds to the electron cyclotron wave. This wave has been studied by Shafranov[17] neglecting collisions and the effect of ion motion. Platzman and Buchsbaum[18] extended his work to include collisions but considered the unperturbed distribution function to be of the form

$$f^0 = \frac{N}{(v_z^2 + \alpha^2)^2}$$

rather than Maxwellian for simplicity in the numerical work. The quantities N and α are normalization factors chosen to give rise to a specified density and temperature. The collision form used by Platzman and Buchsbaum is simply

$$\left.\frac{\partial f_e}{\partial t}\right)_{coll} = -\nu f_e^1 .$$

The neglect of ion motion reduces the Liboff collision term to

$$\left.\frac{\partial f_e}{\partial t}\right)_{coll} = -\nu_e f_e^1 + \frac{2f_e^0}{a_e^2}(\vec{u}_e - \vec{v})\left(\nu_e - \nu_1 \frac{m_{ie}}{m_e}\right)$$

since $n_e = T_e = 0$ for the transverse electron cyclotron wave. Since $\nu_e \approx \nu_1 \frac{m_{ie}}{m_e}$, the second term on the right-hand side is negligible and we see that the collision form used by Platzman and Buchsbaum is adequate for their case of interest. However, their form is completely inadequate for a treatment of the ion cyclotron wave because of the importance of the electron motion and the fact that

$$\nu_i \neq \nu_1 \frac{m_{ie}}{m_i} .$$

The Liboff expression is necessary for an adequate treatment of a two-species plasma when both species are perturbed. The third set corresponds to the longitudinal wave. This problem was first considered by Landau[19] neglecting collisions and ion motion and has since been the subject of many papers. We are interested in the second set which corresponds to the ion cyclotron wave. Dropping the minus sign subscripts, we have the following set of equations:

$$\left(\frac{\partial}{\partial t} + v_z \frac{\partial}{\partial z} + \nu_i + i\,\omega_{ic}\right) f_i = \frac{2f_i^{\,0}\, v_\perp}{a_i^{\,2}} \left[\frac{e}{m_i} E + \left(\nu_i - \frac{m_{ie}}{m_i}\nu_1\right) u_i + \frac{m_{ie}}{m_i}\nu_1 u_e\right];$$

$$\left(\frac{\partial}{\partial t} + v_z \frac{\partial}{\partial z} + \nu_e - i\,\omega_{ec}\right) f_e = \frac{2f_e^{\,0}\, v_\perp}{a_e^{\,2}} \left[\frac{-eE}{m_e} + \left(\nu_e - \frac{m_{ie}}{m_e}\nu_1\right) u_e + \frac{m_{ie}}{m_e}\nu_1 u_i\right];$$

$$n_0\, u_{i,e} = \pi \int\!\!\int v_\perp^{\,2}\, dv_\perp\, dv_z\, f_{i,e} \;;$$

$$\left(\frac{\partial^2}{\partial z^2} - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right) E = \frac{4\pi\, e n_0}{c^2}\, \frac{\partial}{\partial t}\left(u_i - u_e\right) .$$

Assuming a time dependence of the form $e^{-i\omega t}$ with ω real we consider the problem of a wave propagating in a plasma which fills the semi-infinite space $z > 0$. There is a uniform magnetic field in the plasma perpendicular to the plane boundary $z = 0$. The magnetic field of the wave is given as a boundary condition at $z = 0$. Because of the existence of damping there is no disturbance at $z = \infty$. For any perturbed quantity, $P(z, t) = P(z)\, e^{-i\omega t}$, we have the mathematical condition

$$P(z) \neq 0 \qquad \text{for } z \geqslant 0$$
$$P(z) = 0 \qquad \text{for } z < 0$$

since we are interested in the determination of the disturbance in the region $z > 0$ in terms of its value at $z = 0$. The appropriate transform is the one-sided Fourier transform which is identical in theory to the Laplace transform.[20] Define

$$P(k) = \frac{1}{2\pi} \int_0^\infty P(z)\, e^{-ikz}\, dz .$$

The inverse transform is

$$P(z) = \int_{-\infty - i\gamma}^{\infty - i\gamma} dk\, e^{ikz}\, P(k)$$

where γ is chosen so that the contour in the k plane is below all singularities of the integrand. This insures that $P(k)$ exists and $P(z) = 0$ for $z < 0$. The requirement that no disturbance exist in the limit of infinite distance from the boundary means that $P(k)$ is regular in the lower half k plane including the real k axis because of collisional damping. We therefore may take γ to be zero.

Taking the transform of the equations, we have

$$\left(-k^2 + \frac{\omega^2}{c^2}\right)E = \frac{-i\,\omega\,4\pi\,e\,n_0}{c^2}\left(u_i - u_e\right) + \frac{i\,k\,E_b}{2\pi} + \frac{E_b{}'}{2\pi}, \qquad (21a)$$

$$(-i\omega + ikv_z + i\omega_{ic} + \nu_i)f_i = \frac{2f_i{}^0 v_\perp}{a_i{}^2}\left[\frac{e}{m_i}E + \left(\nu_i - \frac{m_{ie}}{m_i}\nu_1\right)u_i + \frac{m_{ie}\nu_1}{m_i}u_e\right] + \frac{v_z f_{ib}}{2\pi}, \qquad (21b)$$

$$(-i\omega + ikv_z - i\omega_{ec} + \nu_e)f_e = \frac{2f_e{}^0 v_\perp}{a_e{}^2}\left[\frac{-e}{m_e}E + \left(\nu_e - \frac{m_{ie}}{m_e}\nu_1\right)u_e + \frac{m_{ie}\nu_1}{m_e}u_i\right] + \frac{v_z f_{eb}}{2\pi}, \qquad (21c)$$

and again

$$n_0\, u_{i,e} = \pi \iint v_\perp{}^2\, dv_\perp\, dv_z\, f_{i,e}\,.$$

Now all perturbed quantities are the Fourier transforms and are functions of k. The quantities with subscript b are the boundary values.

$$f_b = f(z = 0)$$

$$E_b = E(z = 0)$$

and

$$E_b{}' = \lim_{z\to 0}\frac{\partial E(z)}{\partial z} = E'(z = 0)\,.$$

Solving for u_i, we have

$$u_i = \frac{\pi i}{n_0}\int v_\perp{}^2 dv_\perp dv_z \frac{\left\{\frac{2f_i{}^0 v_\perp}{a_i{}^2}\left[\frac{eE}{m_i} + \left(\nu_i - \frac{m_{ie}}{m_i}\nu_1\right)u_i + \frac{m_{ie}}{m_i}\nu_1\ u_e\right] + \frac{v_z\ f_{ib}}{2\pi}\right\}}{\omega - \omega_{ic} + i\ \nu_i - kv_z}$$

$$= -iG_i\left[\frac{eE}{m_i} + \left(\nu_i - \frac{m_{ie}}{m_i}\nu_1\right)u_i + \frac{m_{ie}}{m_i}\nu_1\ u_e\right] + \bar{u}_i$$

where

$$G_i = \frac{-2\pi}{n_0\ a_i{}^2}\int \frac{dv_z\ v_\perp{}^3\ dv_\perp\ f_i{}^0}{\omega - \omega_{ic} + i\nu_i - kv_z}$$

and

$$\bar{u}_i = \frac{i}{2n_0}\int \frac{dv_z\ dv_\perp\ v_\perp{}^2\ v_z\ f_{ib}}{\omega - \omega_{ic} + i\nu_i - kv_z}\ . \tag{21d}$$

Similar results obtain for u_e. Solving for $u_i - u_e$, we find

$$u_i - u_e = \frac{\left(\bar{u}_i - i\frac{G_i eE}{m_i}\right)\left(1 + i\nu_e G_e\right) - \left(\bar{u}_e + i\frac{G_e eE}{m_e}\right)\left(1 + i\ \nu_i\ G_i\right)}{\left(1+i\nu_i G_i\right)\left(1+i\nu_e G_e\right) - i\nu_1 m_{ie}\left[\frac{G_i}{m_i}\left(1+i\nu_e G_e\right) + \frac{G_e}{m_e}\left(1+i\nu_i G_i\right)\right]}\ . \tag{21e}$$

Now

$$G_i = \frac{-2}{\sqrt{\pi}\, a_i^5} \int_{-\infty}^{\infty} dv_z \int_0^{\infty} dv_\perp \frac{v_\perp^3 \; e^{-(v_z^2 + v_\perp^2)/a_i^2}}{\omega - \omega_{ic} + i\nu_i - kv_z} .$$

The $v_\perp$ integration is

$$\int_0^{\infty} v_\perp^3 \, dv_\perp \, e^{-v_\perp^2/a_i^2} = \frac{a_i^4}{2} .$$

Hence

$$G_i = \frac{-1}{\sqrt{\pi}\, a_i} \int_{-\infty}^{\infty} \frac{dv_z \; e^{-v_z^2/a_i^2}}{\omega - \omega_{ic} + i\nu_i - kv_z} .$$

Let $t = v_z/a_i$. Then

$$G_i = \frac{1}{ka_i} \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} \frac{dt \; e^{-t^2}}{t - \Phi_i}$$

where

$$\Phi_i = \frac{\omega - \omega_{ic} + i\,\nu_i}{k\, a_i} .$$

In the definition of the transforms, we have k real. Therefore, if $k > 0$, then $\mathrm{Im}\,\bar{\phi}_i > 0$ and the integral in the t plane is defined. Upon evaluating the inverse transformations later, $G(k)$ will be analytically continued off the real k axis. For the positive real k axis, this analytic continuation is effected by moving the path of integration of the above integral in the t plane so as to always be below the pole at $t = \bar{\phi}_i$. When $k < 0$, then $\mathrm{Im}\,\bar{\phi}_i < 0$ and the reverse definition obtains. That is, the path of integration must now remain above the pole at $t = \bar{\phi}_i$ in the analytic continuation of $G(k)$ from the negative real axis. The t plane contours are illustrated in Fig. 6.

Although $G(k)$ has different definitions depending on the sign of k, it is continuous at $k = 0$ and $k = \infty$.

$$\lim_{k\to 0^-} G_i(k) = \lim_{k\to 0^+} G_i(k) = G_i(0) = \frac{-1}{(\omega - \omega_{ic} + i\,\nu_i)} .$$

$$\lim_{k\to\infty^+} G_i(k) = \lim_{k\to\infty^-} G_i(k) = G_i(\infty) = 0 .$$

The integral defined this way for $k > 0$ is called $Z(\bar{\phi})$, the Plasma Dispersion Function and is tabulated in Fried and Conte.[21] So with the above definition, set

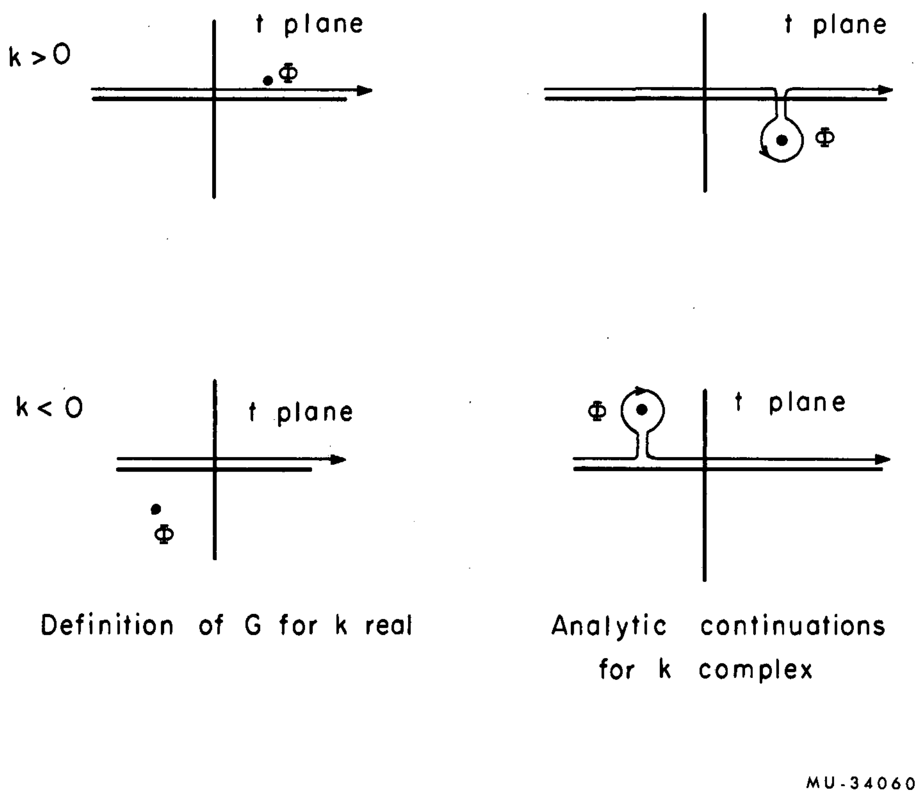

Fig. 6. Relation of contour and pole in t plane.

$$\frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} \frac{dt\, e^{-t^2}}{t - \phi} = Z(\phi)$$

when the contour is below the pole. We then have

$$G_i = \frac{1}{ka_i} Z(\phi_i)$$

where

$$\phi_i = \frac{\omega - \omega_{ic} + i\,\nu_i}{ka_i}$$

and similarly,

$$G_e = \frac{1}{ka_e} Z(\phi_e)$$

where

$$\phi_e = \frac{\omega + \omega_{ec} + i\nu_e}{ka_e}$$

for $k > 0$. For $k < 0$, the integral is not that tabulated in Fried and Conte but is easily expressed in terms of $Z(\phi)$. If we define

$$\frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} \frac{dt\ e^{-t^2}}{t - \phi} = Z^-(\phi)$$

when the contour is above the pole, we see from Fig. 6 that

$$Z^-(\phi) = Z(\phi) - 2i\sqrt{\pi}\ e^{-\phi^2} .$$

For $k < 0$ we then have

$$G_i = \frac{1}{ka_i} Z^-(\phi_i)$$

and

$$G_e = \frac{1}{ka_e} Z^-(\phi_e) .$$

Since

$$J = en_0(u_i - u_e)$$

from Eq. (21e) we have

$$J = \frac{-i\, n_0\, e^2}{D} E \left[\frac{G_i}{m_i} \left(1 + i\, \nu_e\, G_e \right) + \frac{G_e}{m_e} \left(1 + i\, \nu_i\, G_i \right) \right]$$

$$+ \frac{e\, n_0}{D} \left[\bar{u}_i \left(1 + i\, \nu_e\, G_e \right) - \bar{u}_e \left(1 + i\, \nu_i\, G_i \right) \right]$$

where

$$D = (1 + i\nu_i\, G_i)(1 + i\nu_e\, G_e) - i\nu_1\, m_{ie} \left[\frac{G_i}{m_i}(1 + i\nu_e G_e) + \frac{G_e}{m_e}(1 + i\nu_i G_i) \right] .$$

We thus have

$$J(k) = \sigma(k)\, E(k) + \bar{J}(k, f_{ib}, f_{eb}) .$$

The term, $\bar{J}$, represents the effects of the boundary. If this term were not present, it is easily shown that the conductivity kernel would be a function of relative distance alone. That is

$$J(z) = \int dz' \;\; \sigma(z - z')\, E\,(z') . \qquad (22)$$

Shavranov[17] in his work on the electron cyclotron wave, believed (22) to be equivalent to the assumption of specular reflection of particles at the boundary. That is, he believed the specular reflection condition,

$$f_b(v_z) = f_b(-v_z) \; , \tag{22a}$$

will lead to

$$\int_{-\infty}^{\infty} dk \, e^{ikz} \; \overline{J}(k, f_{ib}, f_{eb}) = 0 \; . \tag{22b}$$

We now test this hypothesis. We evaluate $\overline{u}_i$ from Eq. (21d). The specular reflection condition (22a) shows that the integrand of (21d) is even in v_z .

$$\overline{u}_i = \frac{i}{2n_0} \int \frac{dv_z \, dv_\perp \, v_\perp^2 \, v_z \, (\omega - \omega_{ic} + i \, \nu_i + kv_z) f_{ib}}{(\omega - \omega_{ic} + i \, \nu_i)^2 - k^2 \, v_z^2}$$

$$= \frac{ik}{2n_0} \int \frac{dv_z \, v_z^2 \, dv_\perp \, v_\perp^2 \, f_{ib}}{(\omega - \omega_{ic} + i\nu_i)^2 - k^2 v_z^2} \; .$$

Thus

$$\overline{J} = \frac{iek}{2\,D} \int dv_z v_z^2 dv_\perp v_\perp^2 \left[\frac{f_{ib}(1 + i\nu_e \; G_e)}{(\omega - \omega_{ic} + i\nu_i)^2 - k^2 v_z^2} - \frac{f_{eb}(1 + i\nu_i \; G_i)}{(\omega + \omega_{ec} + i\nu_e)^2 - k^2 v_z^2} \right] \; . \tag{22c}$$

This term is not necessarily zero. It is therefore not generally true that the assumption of specular reflection at the boundary gives rise to Eq. (22). That is, this assumption does not necessarily remove the effects of the boundary.

We have shown, however, that if $G(k)$ is an even function of k, then $\overline{J}(k)$ is an odd function of k. This fact does not cause Eq. (22b) to be satisfied, but it will be of use to us as will be seen later. We assume specular reflection as a boundary condition of f and therefore use Eq. (22c) for $\overline{J}$.

Substituting the expression for $J(k)$ into Eq. (21a) we obtain an expression for $E(k)$ in terms of the boundary conditions.

$$E(k) = \frac{\dfrac{-i\,k\,E_b - E_b'}{2\pi} + \dfrac{4\pi\, i\omega}{c^2}\,\overline{J}(k)}{k^2 - (\omega^2/c^2)\,n^2(k)}$$

where

$$n^2(k) = 1 + \frac{4\pi i}{\omega}\,\sigma(k)$$

$$= 1 + \frac{\omega_{pi}^2}{\omega D}\,G_i(1 + i\nu_e G_e) + \frac{\omega_{pe}^2}{\omega D}\,G_e(1 + i\,\nu_i\,G_i)\ .$$

Finally, $E(z)$ is given by the inverse transform

$$E(z) = \int_{-\infty}^{+\infty} \frac{dk e^{ikz}\left[\frac{-ikE_b - E_b'}{2\pi} + \frac{4\pi i \omega}{c^2} \bar{J}(k)\right]}{k^2 - \frac{\omega^2}{c^2} n^2(k)} \quad . \tag{23}$$

It appears that in addition to E_b and E_b', $\bar{J}(k)$ must be given to specify the solution. The expression for $\bar{J}(k)$ to be used is not known since it depends on the unknown quantities, f_{ib} and f_{eb}. Shafranov essentially ignored $\bar{J}(k)$ in his work by his assumption, Eq. (22). He abandoned the semi-infinite problem for the problem of finding the field in an infinite medium excited by a surface current in the plane, $z = 0$. Then the tangential component of the wave magnetic field is an odd function of z and discontinuous at $z = 0$. In the problem of interest, the wave magnetic field has only a tangential component which is simply related to the spatial derivative of the electric field by one of the Maxwell equations. If we write

$$\vec{B} = B(\hat{y} + i\,\hat{x})e^{-i\omega t} \quad ,$$

we have

$$\vec{\nabla} \times E(\hat{x} + i\,\hat{y})e^{-i\omega t} = \frac{-1}{c}\frac{\partial}{\partial t} B(\hat{y} + i\,\hat{x})e^{-i\omega t} \; .$$

Hence,

$$B = \frac{-ic}{\omega} \frac{\partial E}{\partial z} . \tag{24}$$

Shafranov prescribed the fictitious surface current in terms of the boundary value of B in such a way that the resulting electric field had a spatial derivative that corresponded to E_b' at the boundary. His resulting expression expressed in our form was

$$E(z) = \frac{-1}{\pi} \int_{-\infty}^{\infty} \frac{dk\, e^{ikz} E_b'}{k^2 - \frac{\omega^2}{c^2} n^2(k)} .$$

The identical expression will now be obtained in our case without the use of a fictitious current or relation (22).

We assume that the value of E_b' is known. This is one boundary condition. The second boundary condition is $E(z = \infty) = 0$. This second condition is equivalent to the requirement that $E(k)$ be regular in the lower half plane. With k on the real axis, the requirement is equivalent to the statement that $E(z) = 0$ for $z < 0$. This, of course, was part of the original definition of $E(z)$.

Returning to the definition of G, we see that $G(k) = G(-k)$.

$$G_i = \frac{1}{\sqrt{\pi}a_i}\int_{-\infty}^{\infty}\frac{dt\ e^{-t^2}\left[kt + \frac{\omega-\omega_{ic}+i\nu_i}{a_i}\right]}{k^2t^2 - \left(\frac{\omega - \omega_{ic} + i\nu_i}{a_i}\right)^2} = \frac{(\omega-\omega_{ic}+i\nu_i)}{\sqrt{\pi}\ a_i^2}\int_{-\infty}^{\infty}\frac{dt\ e^{-t^2}}{k^2t^2-\left(\frac{\omega-\omega_{ic}+i\nu_i}{a_i}\right)^2} .$$

The part of G_i that is odd in k is also odd in t and is therefore zero. Therefore, by inspection of their definitions,

$$n^2(k) = n^2(-k)$$

and

$$\overline{J}(k) = -\overline{J}(-k) .$$

Using these properties of $n^2(k)$ and $\overline{J}(k)$ and evaluating Eq. (23) for $z < 0$, we have

$$0 = \int_{-\infty}^{\infty}\frac{dke^{-ik|z|}\left[\frac{-ikE_b - E_b'}{2\pi} - \frac{4\pi\ i\omega}{c^2}\ \overline{J}(-k)\right]}{k^2 - \frac{\omega^2}{c^2}\ n^2(-k)} .$$

Replacing the dummy variable k by -k we have

$$0 = \int_{-\infty}^{\infty} \frac{dke^{ik|z|}\left[\frac{ikE_b - E_b'}{2\pi} - \frac{4\pi\ i\omega}{c^2}\ \bar{J}(k)\right]}{k^2 - \frac{\omega^2}{c^2}\ n^2\ (k)} .$$

Therefore

$$\int_{-\infty}^{\infty} \frac{dke^{ik|z|}\left[\frac{ik\ E_b}{2\pi} - \frac{4\pi\ i\omega}{c^2}\ \bar{J}(k)\right]}{k^2 - \frac{\omega^2}{c^2}\ n^2\ (k)} = \int_{-\infty}^{\infty} \frac{\frac{E_b'}{2\pi}\ dke^{ik|z|}}{k^2 - \frac{\omega^2}{c^2}\ n^2(k)} .$$

Now for $z > 0$, $|z| = z$ and

$$E(z) = \int_{-\infty}^{\infty} \frac{dke^{ik|z|}\left[\frac{-ikE_b}{2\pi} + \frac{4\pi i\omega}{c^2}\bar{J}(k)\ \frac{-E_b'}{2\pi}\right]}{k^2 - \frac{\omega^2}{c^2}\ n^2(k)} = \int_{-\infty}^{\infty} \frac{dke^{ik|z|}\left[\frac{-E_b'}{2\pi} - \frac{E_b'}{2\pi}\right]}{k^2 - \frac{\omega^2}{c^2}\ n^2(k)} .$$

The result is

$$E(z) = \frac{-E_b'}{\pi} \int_{-\infty}^{\infty} \frac{dk e^{ikz}}{k^2 - \frac{\omega^2}{c^2} n^2(k)} \qquad \text{for } z \geq 0$$

$$E(z) = 0 \qquad \text{for } z < 0 \qquad (26)$$

which agrees with the result of Shafranov.

We have essentially chosen the boundary values, E_b and $\bar{J}(f_{ib}, f_{eb})$ so as to eliminate the solutions that grow rather than damp with z.

We are interested in the wave magnetic field. Using Eq. (24) and (26), we obtain the equations

$$B(z) = \frac{B_b}{\pi i} \int_{-\infty}^{\infty} \frac{k dk e^{ikz}}{k^2 - \frac{\omega^2}{c^2} n^2(k)} \qquad \text{for } z \geq 0$$

$$B(z) = 0 \qquad \text{for } z < 0 \qquad (27)$$

To obtain $B(z)$ by contour integration, we must analytically continue $n^2(k)$ off the real k axis. Now $G(k)$ for $k > 0$ agrees with the function described in Fried and Conte.[21] Call this $G^+(k)$. $G(k)$ for $k < 0$ is defined with the pole in the t-plane below the contour. Call this $G^-(k)$.

$$G^+(k) = G^-(-k) \tag{28a}$$

and

$$G^+(k) = G^-(k) + \frac{2i\sqrt{\pi}}{ka} e^{-\phi^2} \tag{28b}$$

from the definition of $Z^-(\phi)$.

Since the integrand of Eq. (27) contains both G_i and G_e, there are two branch cuts in the upper half k plane: one separating the functions G_i^+ and G_i^-; the other separating the functions G_e^+ and G_e^-. These cuts separate the upper k plane into three regions. Both cuts extend from $k = 0$ to $k = \infty$. $G^+(k)$ is the analytic continuation of $G(k)$ from the positive real k axis into the complex k plane. $G^-(k)$ in the analytic continuation of $G(k)$ from the negative real k axis into the complex k plane. To maintain $G(k)$ single valued in the k plane, we must cut the k plane along some path between $k = 0$ and $k = \infty$.

We therefore see that each cut in the k plane separates the region of the k plane where we use $G^+(k)$ from the region of the k plane where we use $G^-(k)$. That is, on one side of the cut, k is such that the pole in the t plane is above the contour and on the other side, k is such that the pole in the t plane is below the contour.

Since $G(k = \infty) = 0$, $n^2(k = \infty) = 1$ and we find that the integrand in Eq. (27) vanishes on a semicircle at infinite k, because of the e^{ikz} term. We may therefore add this semicircle to our original contour of integration, the real k axis, without changing the result. We then shrink the resultant closed contour to as small an area as possible, being careful not to cross any singularities of the integrand or the branch cuts. A typical situation is shown in Fig. 7 . The determination of the positions of singularities and cuts will be explained later.

Denoting the three regions by the numbers1, 2 and 3, we have three corresponding different functions $n_1^2(k)$, $n_2^2(k)$ and $n_3^2(k)$ in the integrand. Corresponding to Fig. 7, we have

$$n_1^2 = n^2(G_i^+, G_e^-),$$

$$n_2^2 = n^2(G_i^-, G_e^-),$$

and

$$n_3^2 = n^2(G_i^+, G_e^+) .$$

Assume that there are N_1 singularities of the function

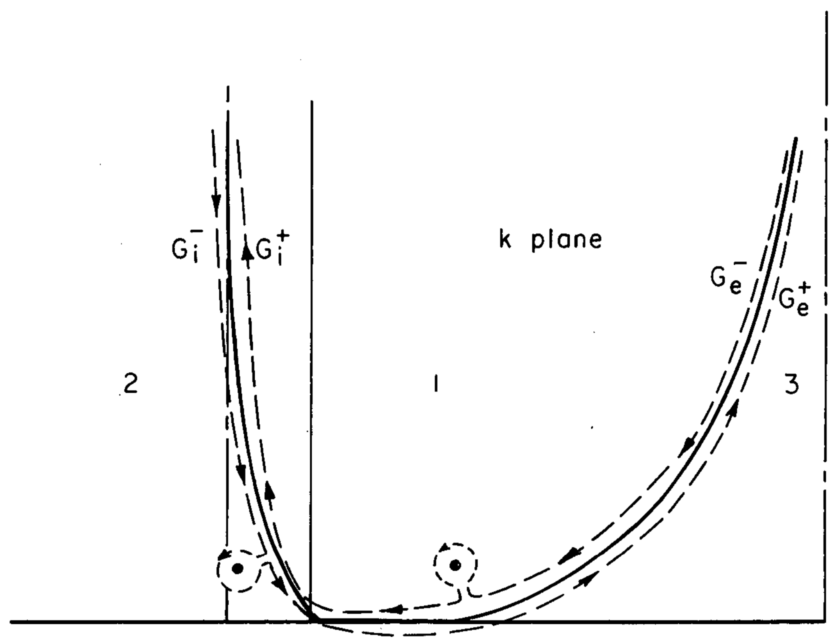

Fig. 7. Integration contour in k plane.

$$H_1(k) = \frac{ke^{ikz}}{k^2 - \frac{\omega^2}{c^2} n_1^{\,2}(k)} \tag{29}$$

in region 1, N_2 singularities of $H_2(k)$ in region 2 and N_3 singularities of $H_3(k)$ in region 3 where H_2 and H_3 are defined as in Eq. (29) in terms of $n_2^{\,2}(k)$ and $n_3^{\,2}(k)$. Let

$$B_1 = 2B_b \sum_{\alpha=1}^{N_1} \left[\text{Res } H_1(k) \; ; \; K_\alpha \right]$$

where

$$\left[\text{Res } H_1(k) \; ; \; k_\alpha \right] = \frac{k_\alpha e^{ik_\alpha z}}{\left[\frac{d}{dk} \left(k^2 - \frac{\omega^2}{c^2} n_1^{\,2}(k) \right) \right]_{k=k_\alpha}}$$

when k_α is a simple zero of the denominator of $H_1(k)$. B_2 and B_3 have similar definitions for regions 2 and 3 . The general solution is then

$$B(z) = B_1 + B_2 + B_3 + B_{12} + B_{31} \; .$$

The contribution B_{12} arises from the integral along the branch cut separating regions 1 and 2 . It is expressible in terms of the difference of $H_1(k)$ and $H_2(k)$. We then have

$$B_{12} = \frac{B_b}{\pi i} \int_{C_{12}} \frac{kdke^{ikz} \frac{\omega^2}{c^2} \left(n_1^2(k) - n_2^2(k)\right)}{\left(k^2 - \frac{\omega^2}{c^2} n_1^2(k)\right)\left(k^2 - \frac{\omega^2}{c^2} n_2^2(k)\right)}$$

where the contour C_{12} is along the cut between regions 1 and 2 from $k = 0$ to $k = \infty$. B_{31} has a similar definition. Upon evaluating $n_1^2 - n_2^2$ we have the following integral for B_{12} .

$$B_{12} = \frac{B_b \omega_{pi}^2 \omega}{\pi \, ic^2} \int_{C_{23}} \frac{kdke^{ikz}(1+i\nu_e G_e^-)^2 (G_i^+ - G_i^-)}{D(G_i^+, G_e^-)\left(k^2 - \frac{\omega^2}{c^2} n_1^2(k)\right) D(G_i^-, G_e^-)\left(k^2 - \frac{\omega^2}{c^2} n_2^2(k)\right)} \quad (30a)$$

where as before,

$$D(G_i, G_e) = (1+i\nu_i G_i)(1+i\nu_e G_e) - i\nu_1 m_{ie}\left[\frac{G_i}{m_i}\left(1+i\nu_e G_e\right) + \frac{G_e}{m_e}\left(1+i\nu_i G_i\right)\right]. \quad (30b)$$

From (28b), the dominant k dependence of the integrand at small and large k comes from

$$e^{\left[ikz - \left(\frac{\omega-\omega_{ic}+i\nu_i}{ka_i}\right)^2\right]}.$$

The factor, e^{ikz}, varies rapidly at large k while

$$e^{-\left(\frac{\omega-\omega_{ic}+i\nu_i}{ka_i}\right)^2}$$

which comes from $(G_i^+ - G_i^-)$ varies rapidly at small k. The integral is put into the form

$$B_{12} = \int_{C_{12}} g_{12}(k)\, dk\, e^{\left[ikz - \left(\frac{\omega-\omega_{ic}+i\nu_i}{ka_i}\right)^2\right]} \qquad (31)$$

where the part of the integrand that is a relatively weak function of k is

$$g_{12}(k) = \frac{2B_b\, \omega_{pi}^2 \omega (1 + i\nu_e G_e^-)^2}{\sqrt{\pi}\, c^2\, a_i\, D(G_i^+, G_e)\left(k^2 - \frac{\omega^2}{c^2} n_1^2(k)\right) D(G_i^-, G_e^-)\left(k^2 - \frac{\omega^2}{c^2} n_2^2(k)\right)}.$$

Since the exponential part of the integrand drops sharply to zero at both end points of the contour, the integral can be approximated by the method of steepest descents.[20] This procedure is outlined in Appendix I. The result is

$$B_{12} = \sqrt{\pi}\ \frac{dk}{d\sigma}\ e^{f(k_{si})}$$

where

$$\frac{dk}{d\sigma} = \left[\frac{-f''(k_{si})}{2}\right]^{-1/2}$$

$$f(k) = ikz - \left(\frac{\omega - \omega_{ic} + i\,\nu_i}{ka_i}\right)^2 + \ln g(k)$$

and k_{si}, the saddle point, is determined by the equation

$$f'(k_{si}) = iz + \frac{2}{k_{si}}\left(\frac{\omega - \omega_{ic} + i\nu_i}{k_{si}\ a_i}\right)^2 + \frac{g'(k_{si})}{g(k_{si})} = 0\ .$$

The steepest descent approximation requires that the contour, C_{12}, be along a specified path from $k = 0$ through the saddle point, k_{si} to $k = \infty$. Thus the position of the branch cut in the k plane is chosen so that the resulting integral for B_{12} (Eq. (31)) can be evaluated by the method of steepest descent. Since $g(k)$ is relatively slowly varying, an approximation to k_{si} is obtained by ignoring the term

$$\frac{g'(k_{si})}{g(k_{si})}$$

to obtain

$$k_{si} = \left[\frac{2i(\omega - \omega_{ic} + i\,\nu_i)}{a_i^2 z}\right]^{1/3} . \qquad (32)$$

Again ignoring the dependence of $g(k)$ we obtain

$$f''(k_{si}) = \frac{-6}{k_{si}^2}\left(\frac{\omega - \omega_{ic} + i\nu_i}{k_{si}a_i}\right)^2$$

and

$$f'''(k_{si}) = \frac{24}{k_{si}^3}\left(\frac{\omega - \omega_{ic} + i\nu_i}{k_{si}a_i}\right)^3 .$$

Therefore, the criterion of validity of the steepest descent approximation,

$$\left|\frac{f'''(k_{si})}{[f''(k_{si})]^{3/2}}\right| \ll 1 ,$$

leads to the requirement

$$\frac{4}{\sqrt{6}} \left| \frac{k_{si}\ a_i}{\omega - \omega_{ic} + i\ \nu_i} \right| \ll 1 . \qquad (33)$$

Using Eq. (32), it is found that the method of steepest descent is adequate at large distances from the boundary. The criterion is

$$z \gg \frac{8.7\ a_i}{|\omega - \omega_{ic} + i\ \nu_i|} .$$

The term, $\frac{dk}{d\sigma}$, is approximated by the formula

$$\frac{dk}{d\sigma} = \frac{a_i\ k_{si}^2}{\sqrt{3}\ (\omega - \omega_{ic} + i\ \nu_i)} .$$

We also find

$$e^{f(k_{si})} = \exp\left\{-3\left[\frac{z(\omega-\omega_{ic}+i\nu_i)}{2ia_i}\right]^{2/3}\right\} g(k_{si}) = \exp\left[-3\left(\frac{\omega - \omega_{ic} + i\ \nu_i}{k_{si}\ a_i}\right)^2\right] g(k_{si}) .$$

The result is then

$$B_{12}=\frac{2B_b\omega_{pi}{}^2\omega k_{si}{}^2\exp\left[-3\left(\frac{\omega-\omega_{ic}+i\nu_i}{k_{si}\;a_i}\right)^2\right]}{\sqrt{3}\;\;c^2(\omega-\omega_{ic}+i\;\nu_i)}\left[\frac{\left(1+i\;\nu_e\;G_e{}^-\right)^2}{D(G_i{}^+,G_e{}^-)\left(k^2-\frac{\omega^2}{c^2}n_1{}^2\right)D(G_i{}^-,G_e{}^-)\left(k^2-\frac{\omega^2}{c^2}\;n_2{}^2\right)}\right]_{k=k_{si}}.$$

A similar result is obtained for B_{31} .

$$B_{31}=\frac{2B_b\omega_{pe}{}^2\omega k_{se}{}^2 e^{-3\left(\frac{\omega+\omega_{ec}+i\nu_e}{k_{se}a_e}\right)^2}}{\sqrt{3}\;\;c^2(\omega+\omega_{ec}+i\;\nu_e)}\left[\frac{(1+i\;\nu_i\;G_i{}^+)^2}{D(G_i{}^+,G_e{}^-)\left(k^2-\frac{\omega^2}{c^2}n_1{}^2\right)D(G_i{}^+,G_e{}^+)\left(k^2-\frac{\omega^2}{c^2}n_3{}^2\right)}\right]_{k=k_{se}}$$

where

$$k_{se}=\left[\frac{2i(\omega+\omega_{ec}+i\;\nu_e)^2}{a_e{}^2\;z}\right]^{1/3}.$$

These results are correct when the requirement (33) and its analog for the electrons are satisfied. The exponential parts of the expression for B_{31} and B_{12} are then very small. The term, B_{31} , for the electron branch cut is much smaller than the term, B_{12} , for the ion branch cut for frequencies satisfying the criterion

$$\omega < \sqrt{\omega_{ic}\, \omega_{ec}} \quad .$$

This criterion is satisfied by the frequencies in our range of interest. B_{31} may therefore be neglected. The electron thermal effects which occur in B_{12} and n^2 are also negligible for the frequencies and wavelengths of interest. This can be explicitly shown by consideration of the function G_e^- which contains these effects.

$$G_e^- = \frac{1}{ka_e \sqrt{\pi}} \int_{-\infty}^{\infty} \frac{e^{-t^2} dt}{t - \mathcal{J}_e}$$

where

$$\mathcal{J}_e = \frac{\omega + \omega_{ec} + i\, \nu_e}{ka_e} \quad .$$

The integral is defined for $\mathcal{J}_e$ below the integration path in the t plane. Since the wave numbers of interest are such that

$\mathrm{Im}\,\phi_e < 0$, the integral has the asymptotic expansion for large argument

$$G_e^- \simeq \frac{-1}{\omega+\omega_{ec}+i\nu_e}\left[1 + \frac{1}{2\phi_e^2} + \frac{3}{4\phi_e^4} + \frac{3\cdot 5}{8\phi_e^6} + \cdots\right] . \qquad (35)$$

This expansion differs from that found in Fried and Conte[21] which is applicable to G_e^+. It is easily obtained from their expansion and the use of Eq. (28b). Since the wavelengths of interest are such that

$$\frac{1}{\epsilon_{e-}} = |\phi_e|^2 \gg 1 ,$$

the first term of the expansion is kept and the rest discarded. This criterion is identical to that used in the section on the moment equations to neglect the electron viscosity and higher moments. Replacing G_e^- by the first term in the asymptotic expansion and neglecting the electron branch cut is equivalent to using the moment equation for the electron flow velocity and neglecting electron viscosity and the higher moments. Since

$$B_{31} \propto e^{-3\phi_e^2}$$

where

$$\mathrm{Re}\ \mathscr{J}_e^{\,2} > 0 ,$$

the contribution of the electron branch cut is not expandable in terms of $\frac{1}{\mathscr{J}_e}$ and is therefore unobtainable from the truncated moment equations.

With the electron thermal effects ignored we have

$$n^2(k)=1+\frac{\frac{\omega_{pi}^2}{\omega}(\omega+\omega_{ec})G_i-\frac{\omega_{pe}^2}{\omega}(1+i\,\nu_i\,G_i)}{\left(1+i\nu_i G_i\right)\left(\omega+\omega_{ec}\right)-i\nu_1 m_{ie}\left[\frac{G_i}{m_i}(\omega+\omega_{ec})-\frac{(1+i\nu_i G_i)}{m_e}\right]} . \qquad (36)$$

There is now no electron branch cut and therefore no region 3. $B_{31}=B_3=0$. Region 1 now includes what was region 3 .

D. Comparison of Kinetic Equation Result to Moment Equation Result

In view of the form of Eq. (35) we define

$$Y = 1 + (\omega - \omega_{ic} + i\,\nu_i)\,G_i$$

so that a similar expansion of G_i reduces Y to zero when only the first term of the expansion is retained. Expressing $n^2(k)$ in terms of Y leads to the equation

$$n^2 = 1 - \frac{\omega_p^{\ 2} - \frac{Y}{\omega}\left[\omega_{pi}^{\ 2}(\omega + \omega_{ec}) - \omega_{pe}^{\ 2}\ i\ \nu_i\right]}{(\omega-\omega_{ic})(\omega+\omega_{ec})+i\nu_1\omega+iY\left[\left(\nu_i-\frac{m_{ie}}{m_i}\nu_1\right)(\omega+\omega_{ec})+i\nu_i\nu_1\frac{m_{ie}}{m_e}\right]} \quad . \qquad (37)$$

Equation (37) agrees with Eq. (4) when $Y = 0$. An expansion of G_i as in Eq. (35) and the retention of the first two terms yields

$$Y \simeq -\frac{1}{2\mathcal{J}_i^{\ 2}} = \frac{-k^2\ a_i^{\ 2}}{2(\omega - \omega_{ic} + i\ \nu_i)^2} \quad .$$

Substitution into Eq. (37) yields

$$n^2 = 1 - \frac{\omega_p^{\ 2}\left[1 - \frac{k^2\ T_0}{(m_i + m_e)\omega(\omega - \omega_{ic} + i\nu_i)} + \frac{1}{2\mathcal{J}_i^{\ 2}}\right]}{\left[(\omega-\omega_{ic})(\omega+\omega_{ec})+i\nu_1\omega\right]\left[1 - \frac{k^2 T_0(m_e[\omega+\omega_{ec}]+im_{ie}\nu_1)}{m_i m_e(\omega-\omega_{ic}+i\nu_i)([\omega+\omega_{ec}][\omega-\omega_{ic}]+i\nu_1\omega)} + \frac{1}{2\mathcal{J}_i^{\ 2}}\right]}$$

which agrees with Eq. (14) when the numerator or denominator is expanded in the small parameter, $\frac{1}{2\phi_i^2}$, and terms of order $\frac{1}{(2\phi_i^2)^2}$ neglected.

A correspondence is thus seen between the asymptotic expansion of the function, G, and the use of moment equations.

We show in Appendix II that the moment equation approach with electron thermal effects ignored, leads to the same dielectric constant that appears in Eq. (36) when G_i of Eq. (36) is replaced by its asymptotic expansion and the cut in the k plane is chosen at arg k $=$ $\arg(\omega-\omega_{ic}+i\nu_i)$. For k not on this cut, the asymptotic expansion is

$$G_i \simeq \frac{-1}{(\omega-\omega_{ic}+i\nu_i)}\left[1+\frac{1}{2\phi_i^2}+\frac{3\cdot 1}{4\phi_i^4}+\cdots+\frac{(2m-1)(2m-3)\cdots 3\cdot 1}{(2\phi_i^2)^m}+\cdots\right] . \tag{37a}$$

Keeping higher moments in the moment approach is equivalent to keeping higher terms in this expansion. However, the use of the truncated moment equation precludes any knowledge of the existence of the branch cut, since these equations lead to the expression (37a) for $G_i(k)$ for all k. Since $G_i(k)$ appears to be single valued, no cut appears and B_{12} is non-existent.

We will now show that the evaluation of B(z) by the expression (27) using the first two terms of the expansion of G_i (that is, using Eq. (14) for n^2) leads to the previous result, Eq. (18) .

Using the definition of δ^2 above Eq. (18) we have

$$n_1^2(k)=n_2^2(k)=1-\frac{\omega_p^{\,2}\left[1-\dfrac{k^2T_0}{(m_i+m_e)\omega(\omega-\omega_{ic}+i\nu_i)}\right]}{\left[(\omega-\omega_{ic})(\omega+\omega_{ec})+i\nu_1\omega\right]\left[1-k^2/\delta^2\right]}$$

Therefore

$$B(z)=2B_b\left\{\frac{k_1\,e^{ik_1z}}{\left[\dfrac{d}{dk}\left(k^2-\dfrac{\omega^2}{c^2}\,n^2(k)\right)\right]_{k=k_1}}+\frac{k_2\,e^{ik_2z}}{\left[\dfrac{d}{dk}\left(k^2-\dfrac{\omega^2}{c^2}n^2(k)\right)\right]_{k=k_2}}\right\}$$

where k_1 and k_2 are the two solutions in the upper half plane of the equation $k^2-\frac{\omega^2}{c^2}\,n^2(k)=0$.

Now

$$k^2-\frac{\omega^2}{c^2}\,n^2(k)=(1-k^2/\delta^2)^{-1}\left[(1-k^2/\delta^2)(k^2-\omega^2/c^2)+\xi(1-k^2\tau)\omega^2/c^2\right]$$

where

$$\xi=\frac{\omega_p^{\,2}}{(\omega-\omega_{ic})(\omega+\omega_{ec})+i\,\nu_1\omega}$$

and

$$\tau = \frac{T_0}{\omega(m_i + m_e)(\omega - \omega_{ic} + i\,\nu_i)} .$$

Therefore

$$(k^2 - k_1^2)(k^2 - k_2^2) = k^4 - k^2\left[\delta^2 + (1 - \delta^2 \xi\, \tau)\omega^2/c^2\right] + \delta^2(1 - \xi)\omega^2/c^2 .$$

Thus

$$k_1^2 + k_2^2 = \delta^2 + (1 - \delta^2\, \xi\, \tau)\omega^2/c^2$$

and

$$k_1^2 k_2^2 = \delta^2(1 - \xi)\omega^2/c^2 .$$

Therefore

$$(-\xi + \delta^2\, \xi\, \tau)\delta^2\, \omega^2/c^2 = \delta^4 - \delta^2(k_1^2 + k_2^2) + k_1^2\, k_2^2 . \qquad (38)$$

Now

$$\frac{d}{dk}\left(k^2 - \frac{\omega^2}{c^2} n^2(k)\right) = \frac{2k}{(\delta^2-k^2)^2}\left(\delta^4-2k^2\delta^2+k^4-\delta^4\xi\tau\left(\omega^2/c^2\right)+\delta^2\xi\omega^2/c^2\right)$$

$$= \frac{2k}{(\delta^2 - k^2)^2}\left\{k^4-k_1{}^2k_2{}^2 - \delta^2[2k^2 - (k_1{}^2 + k_2{}^2)]\right\}$$

using Eq. (38). Therefore

$$B(z) = B_b\left[\frac{(\delta^2 - k_1{}^2)e^{ik_1z}}{k_2{}^2 - k_1{}^2} + \frac{(\delta - k_2{}^2)e^{ik_2z}}{k_1{}^2 - k_2{}^2}\right]$$

which agrees with Eq. (18).

The boundary condition $\overleftrightarrow{\Pi}(z = 0) = 0$, that led to Eq. (18) when the moment equations were used follows from the assumption of specular reflection.

Thus our expression (27) for B(z) contains the boundary conditions for all moments since it includes the assumptions: 1. specular reflection at the boundary; 2. no spatially growing solutions.

If one solved the problem using equations for the moments $\vec{u}$, $\overleftrightarrow{P}$, $\overleftrightarrow{\vec{Q}}, \cdots M^n$, one would find 2n solutions to the equation

$k^2 - \frac{\omega^2}{c^2} n^2(k) = 0$. Condition (2) would remove n solutions and condition (1), along with the requirement $B(z = 0) = B_b$ would determine the n coefficients of the remaining solutions. The result would be identical to that obtained by using expression (27) with G_i replaced by the first n terms of the expansion (37a).

The moment equations, then, may be incorrect for two reasons: First, B_{12}, the branch cut contribution which is unobtainable from the moment equations may be significant. Second, the expansion (37a) diverges for any finite $\mathcal{J}_i$. According to the theory of asymptotic expansions,[22] the best numerical approximation to G_i is obtained by the use of a finite number of terms of the expansion. The error is of the order of magnitude of the last term used. Therefore, the number of moments that should be retained for a quantitatively accurate result depends on the magnitude of

$$\mathcal{J}_i = \frac{\omega - \omega_{ic} + i\,\nu_i}{ka_i}$$

which is not known until the problem is solved, that is, $k(\omega)$ is found. The retention of too many moments leads to inaccurate results.

E. Solution

We shall solve the problem without expanding the function, G_i .

This necessitates a numerical solution for the wave fields in the following sense. The full solution is

$$B(z) = B_1 + B_2 + B_{12}$$

where B_1 and B_2 arise from singularities and B_{12} is the branch cut contribution. With the function, G_i, programmed for a computer, it is possible to use the computer to point out the position of the steepest descent contour and the existence and positions of singularities. The computer is then used to evaluate the residue of the integrand at the singularities and the value of B_{12}. We are then able to find the relative importance of the branch cut and the various singularities. With the electron thermal effects ignored and using the function

$Y^{\pm} = 1 + (\omega - \omega_{ic} + i\,\nu_i)G_i^{\pm}$ we have

$$B_{12} = \frac{2B_b\omega_{pi}^2\omega(\omega-\omega_{ic}+i\nu_i)(\omega+\omega_{ec})^2 k_s^2 \exp\left[-3\left(\frac{\omega-\omega_{ic}+i\nu_i}{k_s a_i}\right)^2\right]}{\sqrt{3}\,c^2 \quad X(Y^+(k_s))\left[k_s^2-\frac{\omega^2}{c^2}n^2(Y^+(k_s))\right]X(Y^-(k_s))\left[k_s^2-\frac{\omega^2}{c^2}n^2(Y^-(k_s))\right]} \qquad (39)$$

where

$$X(Y) = (\omega-\omega_{ic})(\omega+\omega_{ec})+i\nu_1\omega+iY\left[\nu_i(\omega+\omega_{ec})-m_{ie}\nu_1\left(\frac{\omega+\omega_{ec}}{m_i} - \frac{i\,\nu_i}{m_e}\right)\right] \qquad (40)$$

and

$$n^2(Y) = 1 - \frac{\omega_p^{\,2} - \frac{Y}{\omega}\left[\omega_{pi}^{\,2}(\omega + \omega_{ec}) - i\,\omega_{pe}^{\,2}\,\nu_i\right]}{X(Y)} . \qquad (41)$$

The "i" subscript on k_s has been dropped. We use Y^+ in region 1 and Y^- in region 2 when looking for zeros of the function

$$k^2 - \frac{\omega^2}{c^2}\, n^2(Y)$$

which correspond to the singularities of the integrand of Eq. (27) .

Since the boundary between regions 1 and 2 is the steepest descent contour, the contour's position in the k plane must be known in relation to the singularities. A singularity of the function

$$\frac{k\, e^{ikz}}{k^2 - \frac{\omega^2}{c^2}\, n^2(Y^+)}$$

makes no contribution to the solution if it occurs to the left of the contour in the k plane. Similarly, a singularity of the function

$$\frac{k\, e^{ikz}}{k^2 - \frac{\omega^2}{c^2} n^2(Y^-)}$$

makes no contribution to the solution if it occurs to the right of the contour. A preliminary approximation to the contour path is obtained by ignoring the effect of the relatively slowly varying g(k) in Eq. (31). The contour is then defined by

$$\text{Im } f(k) = \text{Im } f(k_s) \quad \text{(See Appendix I)}$$

where

$$f(k) = ikz - \left(\frac{\omega - \omega_{ic} + i\,\nu_i}{k\, a_i}\right)^2$$

and

$$k_s = \left[\frac{2i(\omega - \omega_{ic} + i\,\nu_i)^2}{z\, a_i^2}\right]^{1/3} .$$

Hence

$$z = \frac{2i(\omega - \omega_{ic} + i\,\nu_i)^2}{a_i^2\, k_s^3}$$

and

$$f(k) = -\left(\frac{\omega - \omega_{ic} + i\ \nu_i}{a_i}\right)^2 \left(\frac{2k}{k_s^{\ 3}} + \frac{1}{k^2}\right) .$$

Let

$$(\omega - \omega_{ic} + i\nu_i) = A\ e^{i\alpha}$$

and

$$k = \rho e^{i\theta} .$$

The contour is then defined by

$$\frac{\sin(2\alpha - 2\theta)}{\rho^2} + \frac{2\rho}{\rho_s^{\ 3}} \sin(2\alpha + \theta - 3\theta_s) = \frac{3}{\rho_s^{\ 2}} \sin(2\alpha - 2\theta_s)$$

where

$$k_s = \rho_s e^{i\theta_s} .$$

Now

$$\theta_s = \frac{\pi}{6} + \frac{2\alpha}{3}$$

so we have

$$\frac{\sin 2(\alpha - \theta)}{\rho^2} + \frac{2\rho}{\rho_s^3} \sin(\theta - \frac{\pi}{2}) = \frac{3}{\rho_s^2} \sin(\frac{2\alpha}{3} - \frac{\pi}{3}) .$$

As ρ goes to zero, the first term dominates. To keep it finite we must have θ approach α. As ρ becomes infinite, the second term dominates. To keep it finite we must have θ approach $\frac{\pi}{2}$. In the limit of infinite ρ, $\rho \cos \theta$, the real part of k, remains finite. We have

$$\rho \cos \theta = \frac{3}{2} \rho_s \sin (\frac{\pi}{3} - \frac{2\alpha}{3}) = \frac{3}{2} \rho_s \sin (\frac{\pi}{2} - \theta_s)$$

or

$$\rho \cos \theta = \frac{3}{2} \rho_s \cos \theta_s .$$

The contour has an asymptote at $k = \frac{3}{2} \mathrm{Re}\, k_s$. An example of the contour was sketched in Fig. 7.

Using the computer, the function g(k) is included in the determination of k_s and the contour. It is found that g(k) has no effect on the angular limits. That is

$$\lim_{\rho \to 0} \theta = \alpha$$

and

$$\lim_{\rho \to \infty} \theta = \frac{\pi}{2} \, .$$

However, the asymptote and the contour are affected by $g(k)$. We define

$$z_{near} = \left| \frac{50 \, a_i}{\omega - \omega_{ic} + i \, \nu_i} \right| .$$

At this value of z, we find that the contour path and asymptote are radically changed by the addition of $g(k)$ when the path passes close to a singularity. This is precisely the situation for which the path must be accurate and hence $g(k)$ must be retained. As z increases, the relative effect of $g(k)$ decreases.

For $z < z_{near}$, the steepest descent approximation becomes inaccurate. z_{near} is then the smallest distance from the boundary for which the steepest descent contour and the value of B_{12} are known to a reasonable degree of accuracy (about 10%) .

F. Results

For the parameters

$$n_0 = 3.5 \cdot 10^{14} \text{ cm}^{-3}$$
$$B_0 = 1.09 \cdot 10^4 \text{ gauss}$$

and

$$T_0 = 2 \cdot 10^4 \,{}^\circ K \text{ and } 3 \cdot 10^4 \,{}^\circ K$$

previously used, the results are very nearly identical to those obtained from the truncated moment equations. We find only two singularities, whose trajectories follow those outlined in Figs. 3 and 4. Defining

$$B_1 = a_1 B_b \, e^{ik_1 z}$$
$$B_2 = a_2 B_b \, e^{ik_2 z}$$

and

$$B_{12} = a_{12}(z) \, B_b \, ,$$

we find that the coefficients, a_1 and a_2, agree to 1% with those found from Eq. (18). The sum, $a_1 + a_2$, is unity to a good approximation. This means that B_{12} is negligible at $z = 0$ since

$$B_b = B_1(0) + B_2(0) + B_{12}(0)$$

requires

$$a_{12}(0) = 1 - a_1 - a_2 \, . \tag{42}$$

At z_{near}, $a_{12} \approx 10^{-20}$; z_{near} is quite small. Its maximum value is about 3 cm at $\omega = \omega_{ic}$. Thus B_{12} is completely negligible and the truncated moment equations (electron viscosity and ion heat flow neglected) previously used are adequate for these low temperatures.

The next case we consider is

$$T_0 = 10^5 \,^\circ K ,$$

where we found the criterion for validity of the truncated moment equations to be violated. At this temperature we still find two singularities, k_1 and k_2 . However, their trajectories, which appear in Fig. 8, show that their magnitudes are smaller than in the lower temperature case of Fig. 4. A third singularity, labeled k_3, also appears when $\Omega > 1.1$. Its trajectory is shown in Fig. 8 for $1.1 \leq \Omega \leq 1.2$. The trajectory is not carried to higher Ω because this singularity does not contribute to the solution. k_3 is a singularity of the function

$$\frac{ke^{ikz}}{k^2 - \frac{\omega^2}{c^2} n^2(Y^+)}$$

and it occurs to the left of the steepest descent contours for z_{near} and $z_{far} = 2z_{near}$, which are sketched in Fig. 8 for $\Omega = 1.2$. Only singularities of this function which lie to the right of the contour

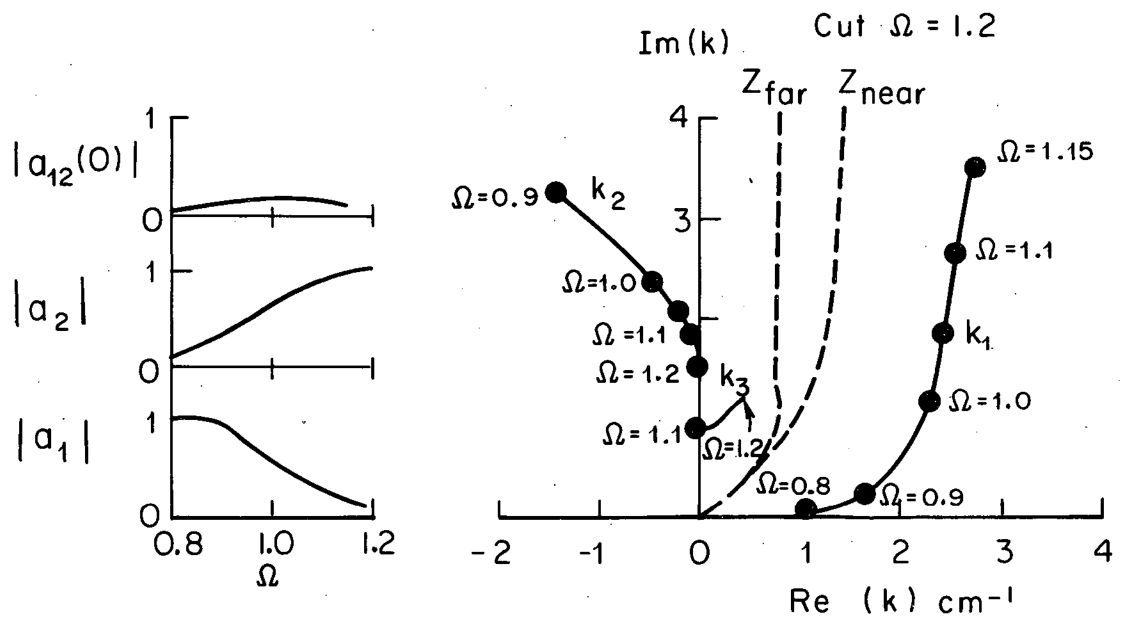

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Fig. 8. Trajectories of k_1 and k_2 and sample branch cuts. $T_0 = 10^5$ °K.

contribute to $B(z)$. Notice that the contour approaches the imaginary axis with increasing z . At some $z \gg z_{far}$, the steepest descent contour will be on the other side of k_3 at $\Omega = 1.2$. Then k_3 will be part of the complete solution. However, at this distance (≈10 meters) the contribution of this singularity to the solution is infinitesimal since its damping length is 1 cm.

We therefore have only two distinct waves from the singularities with exponential spatial dependence . The branch cut contribution, B_{12} , which was negligible at lower temperatures, is now on the threshold of importance. a_1 and a_2 are plotted in Fig. 8 along with $a_{12}(0)$. The maximum value of $a_{12}(0)$ occurs near resonance where $a_{12}(0) \approx 0.2$. Thus $B(z)$ may still be approximated by the two exponential solutions. However, the moment equations incorrectly describe these solutions. They must be obtained by the kinetic treatment.

The final case we consider is

$$T_0 = 5 \cdot 10^5 \, {}^\circ K \quad .$$

This case is representative of the low collision frequency regime where B_{12} is significant. We again find two singularities. Their trajectories are plotted in Fig. 9. A check of the results using a collisionless theory shows essentially the same results for k_1, k_2, a_1, a_2 and therefore $a_{12}(z = 0)$. At $T_0 = 5 \cdot 10^5 \, {}^\circ K$, we have

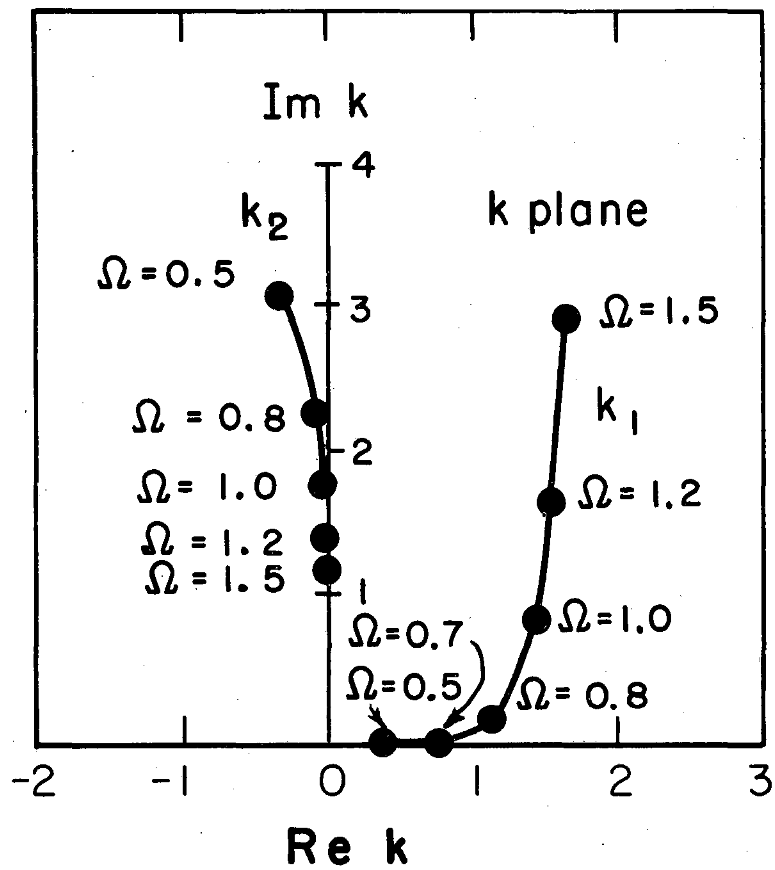

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Fig. 9. Trajectories of k_1 and k_2. $T_0 = 5 \times 10^5$ °K.

$$\nu_i = 7.7 \cdot 10^5 \text{ sec}^{-1} \ll \omega_{ic} = 5.24 \cdot 10^7 \text{ sec}^{-1} .$$

Thus collisions are negligible when $\omega \neq \omega_{ic}$. At resonance the function G_i in the index of refraction has the argument

$$\phi_i = \frac{i\nu_i}{ka_i} .$$

Since $|ka_i| = 1.7 \cdot 10^7 \text{ sec}^{-1}$ for both k_1 and k_2 near resonance we have $|\phi_i| = 0.045 \ll 1$. Now for small ϕ_i,

$$G_i^{\pm} = \pm \frac{i\sqrt{\pi}}{k\, a_i} - \frac{2}{k\, a_i} \left[\phi_i + O(\phi_i)^2 \right] .$$

The leading term is independent of ν_i . So if $\nu_i \ll \omega_{ic}$, or if $\nu_i \ll |a_i\, k_{resonance}|$, the dielectric constant, Eq. (36), may be replaced by the simpler result obtained in a collisionless theory,

$$n^2(k) = 1 + \frac{\omega_{pi}^2}{\omega} G_i(k) - \frac{\omega_{pe}^2}{\omega(\omega + \omega_{ec})} \qquad (43)$$

with very little change in the solution B_1 and B_2. The criterion for neglecting collisions at resonance is

$$\left| \frac{\nu_i}{k\, a_i} \right| \ll 1.$$

We now obtain the value of k at resonance. Replacing G by the first term of the small $\not{\phi}$ expansion we have

$$\frac{k^2 c^2}{\omega_{ic}^2} = n^2 = 1 \pm \frac{i\sqrt{\pi}\ \omega_{pi}^2}{\omega_{ic}\ k\, a_i} - \frac{\omega_{pe}^2}{\omega_{ic}(\omega_{ic} + \omega_{ec})} .$$

Ignoring $\omega_{ic} \ll \omega_{ec}$ in the third term we have

$$k^2 = \frac{\omega_{ic}^2}{c^2} + \frac{\omega_{ic}\, \omega_{pi}^2}{c^2} \left(\frac{\pm\, i\sqrt{\pi}}{k\, a_i} - \frac{1}{\omega_{ic}} \right)$$

since

$$\frac{\omega_{pe}^2}{\omega_{ec}} = \frac{\omega_{pi}^2}{\omega_{ic}} .$$

Since $\omega_{pi} \gg \omega_{ic}$, the first term is negligible compared to the third. The third term is not negligible compared to the second. It is about half the magnitude of the second term in this case. This indicates that the contribution of the electron current to the dielectric constant which produces this term should be kept even at ion cyclotron resonance. For the purpose of obtaining an order of magnitude approximation to k, we ignore it here and obtain

$$k^3 = \frac{\pm i \, \omega_{ic} \, \omega_{pi}^2 \sqrt{\pi}}{c^2 \, a_i}$$

so

$$|k_{1,2}|^3 \approx \frac{\omega_{ic} \, \omega_{pi}^2 \sqrt{\pi}}{c^2 \, a_i} \, .$$

Our criterion is then

$$\left| \frac{\nu_i}{k \, a_i} \right|^3 = \frac{c^2 \, \nu_i^3}{a_i^2 \, \omega_{ic} \, \omega_{pi}^2 \sqrt{\pi}} \approx \frac{n_0^2 (\ell n \, \Lambda)^3}{3 \, B_0 \, T_0^{11/2}} \ll 1 \qquad (44)$$

for ignoring collisions in obtaining B_1 and B_2.

This criterion is of no use for B_{12}. The branch cut contribution is heavily dependent on collisions. Near resonance we have

$$z_{near} = \frac{50\ a_i}{\nu_i} \tag{45}$$

and

$$k_s(z_{near}) = \frac{i\ \nu_i}{a_i\ (25)^{\frac{1}{3}}} \approx \frac{i\ \nu_i}{3\ a_i} .$$

Therefore

$$\not{\bar{\phi}}_i(k_s) = 3$$

at resonance. We then have $G_i(\not{\bar{\phi}}_i)$ replaced by its asymptotic value for large argument to obtain

$$G_i^{\pm} \approx \frac{-1}{i\ \nu_i} .$$

In contrast to the case of the singularities where $\left|\frac{\nu_i}{k\ a_i}\right| \ll 1$ and G_i is independent of ν_i, we have $\left|\frac{\nu_i}{k_s a_i}\right| > 1$ and G_i is now dependent on ν_i.

Thus at higher temperatures, where the criterion (44) is satisfied, the collisions must still be kept for the investigation of the branch cut contribution when this contribution is evaluated by the method of steepest descent.

For this low collision regime, the significance of the branch cut is further illustrated by the following occurrence. Referring to Fig. 9, we see that the trajectory of k_2 now remains near the imaginary axis throughout the frequency range of interest.

When $\omega < \omega_{ic}$, the branch cut is in the upper left quarter plane for all z and k_2 is on another sheet. k_2 therefore makes no contribution to the solution. When $\omega > \omega_{ic}$, the branch cut is in the upper right quarter plane for all z and k_2 contributes to the solution. Of course, the total solution varies continuously through this apparently discontinuous change in the results. Since B_{12} is evaluated at z_{near}, a distance at which the contribution of k_2 to the solution is less than 10^{-38} for $\omega \approx \omega_{ic}$, the presence or absence of k_2 is imperceptible. At $z = 0$, the presence or absence of k_2 is important since it has a coefficient, $a_2 \approx 0.7$ for $\omega \approx \omega_{ic}$. Since $a_1 \approx 0.65$ for $\omega \approx \omega_{ic}$, by Eq. (42) we must have $a_{12}(0) = 0.35$ for $\omega < \omega_{ic}$ and $a_{12}(0) = -0.35$ for $\omega > \omega_{ic}$. This illustrates the futility of attempting to attribute independence to each of the three terms B_1, B_2 and B_{12}. The existence of B_1 and B_2 as solutions and the value of B_{12} are wholly dependent on the choice of the position of the branch cut in the k plane.

Thus B_{12} is significant near resonance in the low collision frequency regime. However, the steepest descent approximation we have used does not give us the form of B_{12} at small z. z_{near} is of the order of meters for $T_0 = 5 \cdot 10^{5}\,^{\circ}K$ and $\omega \approx \omega_{ic}$. At this distance B_1 and B_2 are less than $10^{-38} B_b$ and $B_{12} \approx 10^{-16} B_b$. The steepest descent approximation determines B_{12} accurately only where it is small. This result has been useful at low temperatures ($T_0 \lesssim 10^{5}\,^{\circ}K$) where it demonstrated that B_{12} could be neglected compared to $B_1 + B_2$. At these higher temperatures however, it will be necessary to abandon the steepest descent approximation near resonance in order to study the behavior of B_{12} at reasonable distances from the boundary. A numerical integration of the complex integral in Eq. (30a) would have to be performed. We have not attempted this numerical analysis. In this low collision regime where $\nu_i \ll \omega_{ic}$, the effects of collisions may not be adequately represented by the Liboff collision model we have used. The reasons for this are given in the concluding section.

IV. SUMMARY, CONCLUSIONS AND SUGGESTIONS FOR FURTHER WORK

We have shown that collisional effects on the ion cyclotron wave allow the wave to be described in terms of a cold plasma theory (that is, via the moment equations with zero pressure tensor) when the criterion

$$\epsilon^{00} = \frac{\omega_{ic}\,\omega_p^{\,2}\,T_0}{\nu_1^{\,2}\,\nu_i\,c^2 m_{ie}} \approx \frac{3\cdot 10^3\,B_0\,T_0^{\,11/2}}{n_0^{\,2}(\ell n\ \Lambda)^3} < 1\ ,$$

is satisfied. The dimensions are:

T_0 , °Kelvin

n_0 , cm^{-3}

B_0 , gauss .

The thermal effects may be included solely through the components of the ion pressure tensor that lead to viscosity if the resultant waves have

$$\frac{|k\,a_i|}{\nu_i} < 1$$

at resonance. A coarse and pessimistic criterion for

$$\frac{|k\,a_i|}{\nu_i} < 1$$

is

$$\frac{|k\,a_i|}{\nu_i} \approx \epsilon^o = \frac{2\omega_{ic}\,\omega_p^{\,2}\,T_0}{\nu_1\,\nu_i^{\,2}\,c^2 m_i} \approx \frac{10^2\,B_0\,T_0^{\,11/2}}{n_0^{\,2}\,(\ln\Lambda)^3} \lesssim 1 .$$

If $\epsilon^o \approx 1$, the resultant waves might still have

$$\frac{|k\,a_i|}{\nu_i} < 1 .$$

If, on the other hand, the resultant waves have

$$\frac{|k\,a_i|}{\nu_i} \gtrapprox 1 ,$$

the moment equation approach must be abandoned. We have shown that under these conditions, the use of even higher moments than the pressure tensor is of no help. We proved that the addition of each higher moment is equivalent to keeping another term in an asymptotic expansion of the plasma dispersion function. Since the asymptotic expansion is invalid for

$$\frac{|k\,a_i|}{\nu_i} \gtrapprox 1 ,$$

we know that the moment expansion will be incorrect when

$$\frac{|k\,a_i|}{\nu_i} \gtrapprox 1 .$$

Using the kinetic approach we showed that the solution of a boundary value problem for the waves contains a new term which can be important near resonance at low collision frequencies. This term has the exponential dependence

$$\exp\left[-3\left(\frac{\omega - \omega_{ic} + i\,\nu_i}{2i\,a_i}\right)^{2/3}\right] z^{2/3}$$

at large z, which led Shafranov,[17] who discovered a similar term using a collisionless theory for the electron cyclotron wave, to call it the dominant term nearresonance . We have shown that this term is neither dominant nor negligible near resonance for low collision frequencies and that its value is negligibly small and strongly dependent on collisions at large z, where the

$$\exp\left[-3\left(\frac{\omega - \omega_{ic} + i\,\nu_i}{2i\,a_i}\right)^{2/3}\right] z^{2/3}$$

dependence is valid.

According to our kinetic model, collisions have no effect on waves with dependence e^{ikz} near resonance if $\nu_i \ll |k\,a_i|$. Using the collisionless theory to estimate k at resonance we find the criterion for neglecting collisions to be

$$\epsilon' = \frac{m_i\, c^2\, \nu_i^{\,3}}{2\sqrt{\pi}\, T_0 \omega_{ic} \omega_{pi}^{\,2}} \approx \frac{n_0^{\,2} (\ell n\, \Lambda)^3}{3\, B_0\, T_0^{\,11/2}} \ll 1 .$$

Fig. 10 is a logarithmic plot of the lines $\epsilon^{\infty} = 1$, $\epsilon^{0} = 1$, and $\epsilon' = 1$ as a function of density and temperature for the case $B_0 = 10^4$ gauss. The four cases we have treated are marked by circles.

The region below the line $\epsilon^{\infty} = 1$ consists of the values of n_0 and T_0 for which the thermal effects may be ignored. Between this line and the line $\epsilon^{0} = 1$, the thermal effects may be introduced by ion viscosity alone. Thus below $\epsilon^{0} = 1$, the moment equation approach with heat flow neglected is valid. Above the line $\epsilon^{0} = 1$, the kinetic treatment must be used. Above the line $\epsilon' = 1$, the collisions have no effect on the waves with e^{ikz} dependence according to the collisional model we use. In this region, the new term becomes important but is inadequately described without numerical analysis.

Further work will be necessary for the region above the line $\epsilon' = 1$. This region where numerical analysis will be necessary is also the region where the relaxation collision model of Liboff,[13] which we have used, may be insufficient for the description of collisional effects. J. P. Dougherty[23] has recently introduced a model Fokker-Planck equation for the collisions of a single species of particle. His model, though

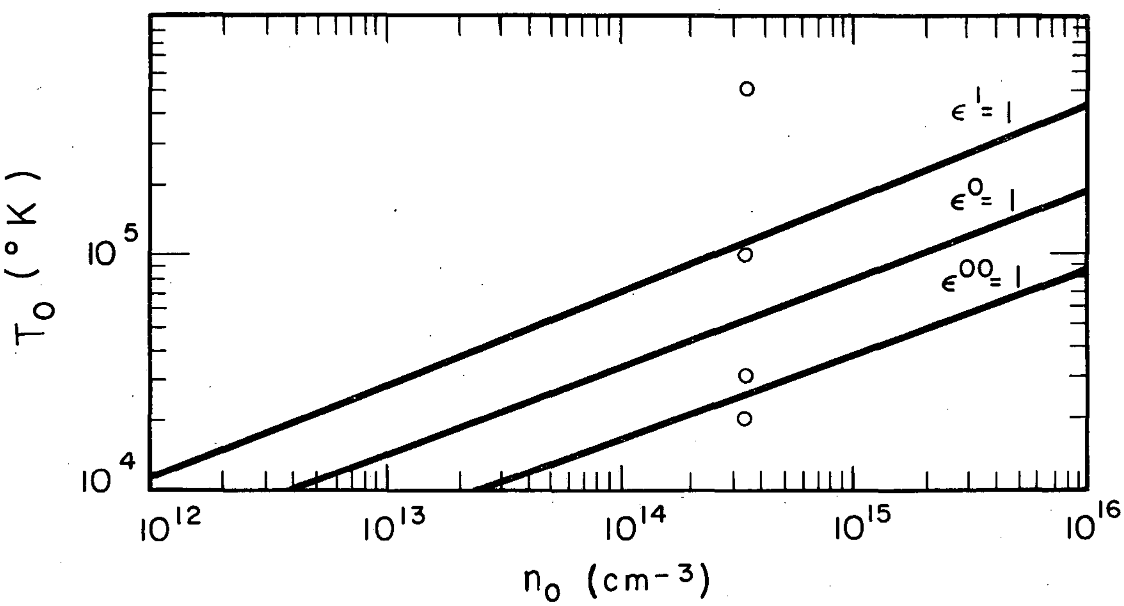

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Fig. 10. Regions of validity for methods of solution.

simpler than the integro-differential Fokker-Planck[8,9] equation previously mentioned, still requires numerical analysis. He shows that if $\nu_i \ll \omega_{ic}$ his model predicts larger effects of ion-ion collisions when applied to ionospheric radar scattering than does a simpler model of the form we use. When $\nu_i \gtrsim \omega_{ic}$, both models give similar results.

APPENDICES

Appendix I. Steepest Descent Method

Given the contour integral

$$I = \int_c dk e^{f(k)}$$

where the integrand vanishes at the end points of the contour, we first find a saddle point, k_s , of the function defined by

$$\left.\frac{df}{dk}\right)_{k=k_s} = 0 \quad .$$

We then expand the function in a Taylor series about the saddle point.

$$I = e^{f(k_s)} \int_c dk \exp\left[\frac{f''(k_s)}{2!}(k - k_s)^2 + \frac{f'''(k_s)}{3!}(k - k_s)^3 + \cdots\right] .$$

We deform the contour, c, to lie along the line of steepest descent through the saddle point. If the function is analytic along this line, the line is defined by the equation

$$\text{Im } f(k) = \text{Im } f(k_s) \quad .$$

The Cauchy-Riemann equations for an analytic function show that the gradient of the real part of $f(k)$ lies along this line. Thus $\text{Re } f(k)$ rises from a negative large value at the start of the contour to a maximum value at $k = k_s$ and then drops to a negative large value at the end of the contour. Let

$$\sigma^2 = -\left[\frac{f''(k_s)}{2!}(k - k_s)^2 + \frac{f'''(k_s)}{3!}(k - k_s)^3 + \cdots\right] \quad .$$

Then

$$I = e^{f(k_s)} \int e^{-\sigma^2} \frac{dk}{d\sigma} \, d\sigma \, .$$

The series expression for σ^2 can be inverted to obtain $\frac{dk}{d\sigma}$. The result is useful when the first term is dominant. The validity criterion then is

$$\left| \frac{f'''(k_s)}{\left[f''(k_s)\right]^{3/2}} \right| \ll 1 .$$

Neglecting $f'''(k_s)$ and higher terms, we find $\frac{dk}{d\sigma}$ independent of σ and the final result is

$$I = e^{f(k_s)} \frac{dk}{d\sigma} \int e^{-\sigma^2} d\sigma = \sqrt{\pi} \frac{dk}{d\sigma} e^{f(k_s)}$$

where the limits of integration have been extended to plus and minus infinity which is consistent with the approximation.

APPENDIX II

The Moment Equation Approach

We obtain an expression for the dielectric constant using the moment equations for the case of the left-handed circularly polarized wave.

Since the left-handed circularly polarized wave gives rise to a perturbed distribution of the form

$$f_i^{\,1} = f(v_\perp, v_z)\; e^{-i\phi}\, e^{i(kz-\omega t)} \;,$$

the general moment, $M^n_{ijk\cdots n}$, will have all components zero except those with one x or y subscript and the rest z.† Since the moments are symmetric we may consider simply $M^n_{zz\cdots zx}$ and $M^n_{zz\cdots zy}$. These two are not independent since $M^n_{zz\cdots zx} = i\, M^n_{zz\cdots zy}$.† Thus the information is contained in $\vec{M}_n = M_n(\hat{x} - i\,\hat{y})e^{i(kz-\omega t)}$. The full moment is then

$$\overset{\leftrightarrow}{\vdots}\!\!\overset{\leftrightarrow}{M}{}^n = \hat{z}\hat{z}\cdots\hat{z}\vec{M}_n + \hat{z}\hat{z}\cdots\vec{M}_n\hat{z} + \cdots + \vec{M}_n\,\hat{z}\hat{z}\cdots\hat{z} \,.$$

The subscript "n" on M_n denotes the rank of the moment. Using the Hermite polynomials we have the general moment equation for $n \geq 3$, Eq. (11c),

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† This is a result of the ϕ dependence of $f_i^{\,1}$.

$$\left[\left(\frac{\partial}{\partial t} + \nu_i + \omega_{ic}\hat{z}\times\right)\overset{\leftrightarrow}{\vdots}\overset{\leftrightarrow}{M}^{n}\right]_{zz\cdots(\hat{x}-i\hat{y})} + \frac{\partial}{\partial z}\left[\overset{\rightarrow}{\overset{\leftrightarrow}{\vdots}}\overset{\leftrightarrow}{M}^{n+1}\right]_{zzz\cdots(\hat{x}-i\hat{y})}$$

$$+ \frac{T_0}{m_i}(n-1)\frac{\partial}{\partial z}\left[\overset{\rightarrow}{\vdots}\overset{\leftrightarrow}{M}^{n-1}\right]_{z\cdots(\hat{x}-i\hat{y})} = 0$$

which yields

$$\left(\frac{\partial}{\partial t} + \nu_i + \omega_{ic}\ \hat{z}\times\right)\vec{M}_n + \frac{\partial}{\partial z}\vec{M}_{n+1} + \frac{T_0}{m_i}(n-1)\frac{\partial}{\partial z}\vec{M}_{n-1} = 0$$

or

$$(\omega - \omega_{ic} + i\ \nu_i)M_n = k\ M_{n+1} + \frac{k\ T_0(n-1)}{m_i}\ M_{n-1} .$$

Ignoring the moment, M_{s+1}, we have

$$M_s = \frac{k\ T_0(s-1)}{m_i(\omega-\omega_{ic}+i\nu_i)}\ M_{s-1} ,$$

$$M_{s-1} = \frac{k\ M_s}{(\omega - \omega_{ic} + i\ \nu_i)} + \frac{k\ T_0(s-2)}{m_i(\omega - \omega_{ic} + i\ \nu_i)}\ M_{s-2}$$

or

$$M_{s-1} = \frac{\dfrac{k\,T_0}{m_i(\omega-\omega_{ic}+i\nu_i)}\,M_{s-2}}{1 - \dfrac{k^2\,T_0\,(s-1)}{m_i(\omega-\omega_{ic}+i\nu_i)^2}} = \frac{\dfrac{k\,T_0(s-2)}{m_i(\omega-\omega_{ic}+i\,\nu_i)}\,M_{s-2}}{1 - \dfrac{(s-1)}{2\not{I}_i^{\,2}}}$$

then

$$M_{s-2} = \frac{\dfrac{(s-2)}{2\not{I}_i^{\,2}}\,M_{s-2}}{1 - \dfrac{(s-1)}{2\not{I}_i^{\,2}}} + \frac{k\,T_0(s-3)}{m(\omega-\omega_{ic}+i\nu_i)}\,M_{s-3}$$

or

$$M_{s-2} = \frac{\dfrac{k\,T_0(s-3)}{m(\omega-\omega_{ic}+i\,\nu_i)}\,M_{s-3}}{1 - \dfrac{\dfrac{(s-2)}{2\not{I}_i^{\,2}}}{1 - \dfrac{(s-1)}{2\not{I}_i^{\,2}}}}$$

and finally

$$M_3 = 2q_- = \cfrac{\dfrac{2k\,T_0\,M_2}{m_i(\omega-\omega_{ic}+i\nu_i)}}{1-\cfrac{\frac{3}{2\phi_i^2}}{1-\cfrac{\frac{4}{2\phi_i^2}}{1-\cfrac{\frac{5}{2\phi_i^2}}{1-\cdots\cfrac{}{1-\frac{(s-1)}{2\phi_i^2}}}}}} \; .$$

All the moments are retained in the limit of infinite s.

We replace the continued fraction by the quantity α. That is

$$q_- = \frac{\alpha\,k\,T_0\,M_2}{m_i(\omega-\omega_{ic}+i\nu_i)}$$

where

$$\alpha = \cfrac{1}{1-\cfrac{\frac{3}{2\phi_i^2}}{\cdots\cfrac{}{1-\frac{(s-1)}{2\phi_i^2}}}} \; .$$

Now $M_2 = \pi_{z-}$. We thus have an equation for q_- in terms of π_{z-} . Equation (13) is then

$$m_i \left[\omega-\omega_{ic}+i\frac{m_{ie}}{m_i}\nu_1 - \frac{k^2 T_0}{m_i(\omega-\omega_{ic}+i\nu_i)\left(1-\frac{\alpha}{\not{Z}_i^{\,2}}\right)}\right]u_{i-} - im_{ie}\nu_1 u_{e-} = ieE_- .$$

If we define

$$G = \frac{-\beta}{(\omega - \omega_{ic} + i\,\nu_i)}$$

where

$$\frac{1}{\beta} = 1 - \frac{k^2 T_0}{m_i(\omega - \omega_{ic} + i\,\nu_i)^2\left(1 - \frac{\alpha}{\not{Z}_i^{\,2}}\right)} ,$$

that is

$$\beta = \cfrac{1}{1 - \cfrac{\frac{1}{2\not{Z}_i^{\,2}}}{1 - \cfrac{\frac{2}{2\not{Z}_i^{\,2}}}{1 - \cfrac{3}{2\not{Z}_i^{\,2}} \cdots}}}$$

we have

$$\left[-\frac{m_i}{G} - i\nu_i m_i + i\nu_1 m_{ie}\right] u_{i-} - i m_{ie}\nu_1 u_{e-} = ie E_- .$$

The electron equation

$$-i m_{ie} \nu_1 u_{i-} + m_e(\omega + \omega_{ec}) u_{e-} = -i e E_-$$

then completes the problem to yield

$$n^2 = 1 + \frac{\frac{\omega_{pi}^2}{\omega}(\omega + \omega_{ec})G - \frac{\omega_{pe}^2}{\omega} (1 + i\nu_i G)}{(1+i\nu_i G)(\omega+\omega_{ec}) - i\nu_1 m_{ie}\left[\frac{G}{m_i}(\omega + \omega_{ec}) - \frac{(1 + i\nu_i G)}{m_e}\right]}$$

which agrees with Eq. (36) if $G = G_i$.

The question of the equality of G and G_i is a problem of the theory of continued fractions. The point here is that the approximate evaluation of the continued fraction by truncation of the series of moment equations can be related to the asymptotic expansion of the function G_i .

Neglecting the moment M_{s+2} , stops the continued fraction at the term,

$$1 - \frac{s}{2\phi_i^2} .$$

We then have

$$\beta \approx \beta_s = \cfrac{1}{1 - \cfrac{\cfrac{1}{2\phi_i^2}}{1 - \cfrac{\cfrac{2}{2\phi_i^2}}{\ddots \cfrac{}{1 - \cfrac{s}{2\phi_i^2}}}}} .$$

Using the expansion,

$$\frac{1}{1 - x} \approx 1 + x + x^2 + x^3 + \cdots$$

we obtain by induction,

$$\beta_s \approx 1 + \frac{1}{2\phi_i^2} + \frac{3}{4\phi_i^4} + \cdots \frac{(2s - 1)(2s - 3)\cdots 3 \cdot 1 \cdot 1}{(2\phi_i^2)^s} + 0\left(\frac{1}{\left(2\phi_i^2\right)^{s+1}}\right).$$

No terms in

$$\frac{1}{(2\bar{\phi}_i^{\,2})^{s+1}}$$

or higher are kept since the coefficient of these terms would be incorrect without the inclusion of the higher moments. The asymptotic expansion of G_i is[21]

$$G_i^{\pm} \simeq \frac{i\sqrt{\pi}}{ka_i}\gamma^{\pm} e^{-\bar{\phi}_i^{\,2}} - \frac{1}{(\omega-\omega_{ic}+i\nu_i)}\left[1+\frac{1}{2\bar{\phi}_i^{\,2}} + \frac{3}{4\bar{\phi}_i^{\,4}} + \cdots + \frac{(2s-1)(2s-3)\cdots 3\cdot 1}{(2\bar{\phi}_i^{\,2})^{s}} + \cdots\right]$$

where

$$\gamma^+ = 2, \quad \gamma^- = 0 \qquad \text{for } \operatorname{Im} \bar{\phi}_i < 0$$
$$\gamma^+ = 1, \quad \gamma^- = -1 \qquad \text{for } \operatorname{Im} \bar{\phi}_i = 0$$
$$\gamma^+ = 0, \quad \gamma^- = -2 \qquad \text{for } \operatorname{Im} \bar{\phi}_i > 0 \; .$$

The series terms agree with the expansion of β. The first term on the right-hand side of the expansion is non-zero if G_i is analytically continued across the real axis in the t plane. If we cut the k plane at $\arg k = \arg(\omega - \omega_{ic} + i\nu_i)$, G_i is not continued past the real axis and this term vanishes. Then

$$G_i \simeq \frac{-1}{(\omega - \omega_{ic} + i\,\nu_i)}\,\beta_{\infty} \; .$$

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