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Differential Equations for Periods and Flat Coordinates in Two Dimensionsional Topological Matter Theories

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Abstract

We consider two dimensional topological Landau-Ginzburg models. In order to obtain the free energy of these models, and to determine the Kähler potential for the marginal perturbations, one needs to determine flat or 'special' coordinates that can be used to parametrize the perturbations of the superpotentials. This paper describes the relationship between the natural Landau-Ginzburg parametrization and these flat coordinates. In particular we show how one can explicitly obtain the differential equations that relate the two. We discuss the problem for both Calabi-Yau manifolds and for general topological matter models (with arbitary central charges) with relevant and marginal perturbations. We also give a number of examples.

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1. Introduction

Topological Landau-Ginzburg theories[†] are not only of interest in their own right, but they also determine the modular dependence of the Yukawa couplings in string theories [2]. The correlation functions of such topological models [3] are completely determined by a prepotential (or free energy), $\mathcal{F}$, and in particular there is a set of 'flat' (or 'special') [4-7] coordinates, t_i, $i = 1, \ldots, \mu$, in which the three point function can be written as

$$C_{ijk} = \frac{\partial^3 \mathcal{F}}{\partial t_i \partial t_j \partial t_k}, \qquad (1.1)$$

and for which

$$C_{ij}{}^m C_{klm} = C_{il}{}^m C_{kjm}. \qquad (1.2)$$

These coordinates are referred to as flat since the two point function, η_{ij}, is an invertible, t-independent matrix, providing a natural, flat metric on the space of chiral primary fields. One set of flat coordinates is provided by taking the t_i to be the coupling constants of the chiral primary perturbations about the underlying $N = 2$ superconformal field theory. In particular one has:

$$C_{ijk}(t) \equiv \left\langle \phi_i \, \phi_j \, \phi_k \, exp\left[\sum_\ell t_\ell \int d^2 z \, \phi_\ell^{(1,1)}(z, \bar{z}) \right] \right\rangle \qquad (1.3)$$

where $\phi_\ell^{(1,1)} \equiv G^-_{-\frac{1}{2}} \widetilde{G}^-_{-\frac{1}{2}} \phi_\ell$. One could, in principle, consider perturbations by any chiral primary field; however, for several reasons it is natural, and perhaps necessary [1], to restrict ones attention to relevant and marginal perturbations.

In string theory, only marginal perturbations are considered (relevant operators would generate space-time tachyons and thus are projected out). The three-point

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[†] By this we mean the topological, chiral primary sub-sector of two-dimensional $N = 2$ supersymmetric Landau-Ginzburg models, or equivalently, topologically twisted Landau-Ginzburg models [1].

functions, or structure constants, C_{ijk}, determine the Yukawa couplings of the low-energy effective field theory. In addition to this, the Kähler potential, K, of the Zamolodchikov metric is determined from the prepotential $\mathcal{F}$. To obtain K one first passes to homogeneous coordinates z^A, $A = 0, \ldots, \mu$, such that $t_i = z^i/z^0$; $i = 1, \ldots, \mu$ and views $\mathcal{F}(t_i)$ as a function of the z^A that is homogeneous of degree 2. That is, $\mathcal{F}(z^A) \equiv (z^0)^2 \mathcal{F}(z^A/z^0) = (z^0)^2 \mathcal{F}(t_i)$. One then has [8][4][*] :

$$K = -log \left(i\, \bar{t}^{\bar{\jmath}} \frac{\partial \mathcal{F}}{\partial t^j} - i\, t^j \frac{\partial \overline{\mathcal{F}}}{\partial \bar{t}^{\bar{\jmath}}} \right). \qquad (1.4)$$

In this paper we will consider topological (or $N = 2$ supersymmetric) theories that have a Landau-Ginzburg description [10]. One can obtain a set of coordinates for such a topological field theory simply by parametrizing the superpotential, W [3]. The problem is that these parameters are generally not the *flat* coordinates. One needs the flat coordinates to use (1.4). Moreover, for general coordinates the derivatives in (1.1) are covariant, making it difficult to determine $\mathcal{F}$ from C_{ijk}. However, once given a parametrization of W in terms of flat coordinates t_i, it is trivial to determine the structure constants $C_{ijk}(t)$ via simple polynomial multiplication modulo the vanishing relations:

$$\phi_i(t)\, \phi_j(t) \;=\; C_{ij}{}^k(t)\, \phi_k(t) \quad \text{mod} \quad \nabla W \equiv 0\,, \qquad (1.5)$$

where $\phi_i(t) \equiv -\frac{\partial}{\partial t_i} W(t)$ [3].

Thus our purpose will be to determine how these flat coordinates can be related to general parametrizations of the Landau-Ginzburg potential.

To date there have been several approaches to solving this problem. On Calabi-Yau manifolds the required coordinates can be related to the periods of the holomorphic 3-form evaluated on an integral homology basis [5] [4-12]. One can sometimes evaluate these periods explicitly as in [11]. One also knows that such periods

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* It appears [9] that K is not uniquely defined given flat coordinates that obey only (1.1). We believe that the 'correct' flat coordinates are those which obey (1.2) as well.

must satisfy a linear differential equation, and it turns out that there is an elementary algorithm for determining this differential equation directly from the Landau-Ginzburg superpotential W. (A brief exposition of this has already been given in [13].) This method has been well known to mathematicians for many years (see, for example, [14,15] [16]), but is apparently not well known in the physics community, and so we will give an exposition of the procedure, along with some examples, in section 2.

In section 3 we will discuss the relationship between flat coordinates and the differential equations of section 2, and derive in detail the flat coordinate of a family of $K3$ surfaces. We will also discuss the role of the duality group of the Landau-Ginzburg potential. In section 4 we will describe how the Calabi-Yau techniques can be generalized to general topological Landau-Ginzburg models. This time one considers periods of differential forms on the level curves of W, and then one shows that by choosing the gauge carefully, one can solve the consistency conditions (1.1) and (1.2). The basic method is also known in the mathematics literature and is an application of the work of K. Saito. A recent, rather brief exposition of this appeared in [6]. Our intention here is not only to simplify the exposition still further, but also to show that if one restricts to relevant and marginal perturbations then the calculations can be simplified. Indeed (contrary to the expectations expressed in [6]) it becomes relatively straightforward to solve topological models whose underlying conformal theory has $c > 3$.

2. Chiral Rings and Differential Equations for Periods

In this section we will, for simplicity, consider a d-dimensional (non-singular) hypersurface, V, defined by the vanishing of a homogenous polynomial, W, of degree ν in $\mathbb{CP}^{d+1}$ (the generalization to weighted projective spaces is elementary). We will denote the homogenous coordinates on $\mathbb{CP}^{d+1}$ by x^A, $A = 1, \ldots, d+2$. We will also consider W to be a function of the x^A and of some (dimensionless) moduli μ_i.

If the first Chern class of V vanishes then there is a globally defined, holomorphic d-form, Ω, on V. This form can be represented [15] [17,18] by

$$\Omega = \int_\gamma \frac{1}{W} \omega ; \qquad \omega = \sum_{A=1}^{d+2} (-1)^A x^A \, dx^1 \wedge \ldots \wedge \widehat{dx^A} \wedge \ldots \wedge dx^{d+2} , \qquad (2.1)$$

where γ is a small, one-dimensional curve winding around the hypersurface V. More generally, the integral

$$\Omega_\alpha = \int_\gamma \frac{p_\alpha(x^A)}{W^{k+1}} \omega , \qquad (2.2)$$

where $p_\alpha(x^A)$ is a homogenous polynomial of degree $k\nu$, represents a (rational) differential d-form. The form, Ω_α, is an element of $\bigoplus_{q=0}^{k} \bigwedge^{(d-q,q)}$. One finds [15][18] that Ω_α represents a non-trivial cohomology element in $F_k \equiv \bigoplus_{q=0}^{k} H^{(d-q,q)}(V, \mathbb{R})$ if and only if p_α is a non-trivial element of the local ring, $\mathcal{R}$, of W. If we take the p_α to be a basis for $\mathcal{R}$, then the corresponding forms, Ω_α, are a basis for the cohomology H^d. For the moment we will restrict our attention to these cohomologically non-trivial differential forms.

The set of periods of a differential form, Ω_α, is defined to be the integrals of Ω_α over elements of a basis of the integral homology of V. This also has a convenient representation:

$$\Pi_\alpha^{\ \beta} = \int_{\Gamma_\beta} \frac{p_\alpha(x^A)}{W^{k+1}} \omega , \qquad (2.3)$$

where Γ_β is a representative of a homology basis in $H_{d+1}(\mathbb{C}P^{d+1} - V, \mathbb{Z})$. The curve Γ_β may be thought of as a tube over the corresponding cycle in $H_d(V, \mathbb{Z})$.

From now on, we will fix Γ_β and consider the vector $\varpi_\alpha \equiv \Pi_\alpha^{\ \beta}$. Considered as a function of the moduli μ_i of W, the vector ϖ satisfies a regular singular, matrix

differential equation

$$\left[\frac{\partial}{\partial \mu_i} - A_i(\mu_j) \right] \varpi = 0 \qquad (2.4)$$

for some matrices A_i. (The complete set of solutions to this differential equation is in fact all of the columns of the period matrix $\Pi_\alpha^{\ \beta}$ [15] [19].)

Our purpose in this section is to give an elementary procedure that generates the differential equation directly from W. The key ingredient is a technical result established in [15]. That is, one considers a differential $(d-1)$-form of V defined by

$$\phi = \int_\gamma \frac{1}{W^l} \Big\{ \sum_{B<C} (-1)^{B+C} [x^B Y_C(x^A) - x^C Y_B(x^A)] dx^1 \wedge \ldots \wedge \widehat{dx^B} \wedge \ldots \wedge \widehat{dx^C} \wedge \ldots \wedge dx^{d+2} \Big\},$$
$$(2.5)$$

where the $Y_B(x^A)$ are homogenous of degree $l\nu - (d+1)$. One finds that

$$d\phi = \int_\gamma \frac{1}{W^{l+1}} \Big[l \Big(\sum_{A=1}^{d+2} Y_A \frac{\partial W}{\partial x^A} \Big) - W \Big(\sum_{A=1}^{d+2} \frac{\partial Y_A}{\partial x^A} \Big) \Big] \omega . \qquad (2.6)$$

Because this is an exact form, it provides us with a simple means of integrating by parts. Equivalently, if $p_\alpha(x^A)$ in (2.2) has the form $\sum Y_A \frac{\partial W}{\partial x^A}$ then, modulo exact forms, we have from (2.6)

$$\Omega_a \equiv \frac{1}{k} \int_\gamma \Big[\frac{1}{W^k} \sum_A \Big(\frac{\partial Y_A}{\partial x^A} \Big) \Big] \omega . \qquad (2.7)$$

One can iterate this procedure (if necessary) and so reduce the numerator until it lies in the local ring of W. Note that this procedure amounts to the most naive form of partial integration.

To derive the differential equation, simply differentiate under the integral to obtain

$$\frac{\partial \varpi_\alpha}{\partial \mu_i} = \int_\Gamma \Big[\frac{(\partial_{\mu_i} p_\alpha)}{W^{k+1}} - (k+1) \frac{p_\alpha (\partial_{\mu_i} W)}{W^{k+2}} \Big] \omega , \qquad (2.8)$$

and then partially integrate until all numerators have been reduced to elements

of the local ring of W. Expressing this reduced r.h.s. of (2.8) in terms of the Ω_α immediately yields (2.4).

If one is interested in the dependence of Ω_α on one particular modulus, μ_0, then one can reduce the first order system (2.4) to one linear, regular singular O.D.E. for $\varpi_1 \equiv \int \Omega$ of order equal to or less than the dimension, μ, of the local ring of W. Note that the order is often much less then μ. For example, if $p_\alpha(x^A)$ and $W(x^A; \mu_0)$ are invariant under some discrete symmetry, then the foregoing reduction procedure can generate only those Ω_β for which p_β is also invariant. Hence the order of the differential equation cannot be greater than the number of such invariant p_β's.

We conclude this section by calculating a couple of examples. First we consider the cubic torus, that is, we take $d = 1$, $(\mu_0 \equiv \alpha)$, and:

$$W(x^A) = \tfrac{1}{3}(x^3 + y^3 + z^3) - \alpha\, xyz \ . \tag{2.9}$$

Let $\varpi_1 = \int_\Gamma \frac{1}{W}\, \omega$ and $\varpi_2 = \int_\Gamma \frac{xyz}{W^2}\, \omega$. Then obviously $\frac{\partial}{\partial\alpha}\varpi_1 = \varpi_2$ and

$$\frac{\partial}{\partial\alpha}\varpi_2 = 2\int \frac{x^2 y^2 z^2}{W^3}\, \omega \ .$$

One now uses the identity:

$$(1 - \alpha^3)\, x^2 y^2 z^2 = xz^2 \left\{ \alpha^2 y \partial_z W + \alpha z \partial_x W + x \partial_y W \right\} \ , \tag{2.10}$$

and integrating by parts yields

$$(1 - \alpha^3)\frac{\partial}{\partial\alpha}\varpi_2 = \alpha \int \frac{z^3}{W^2}\, \omega + 2\alpha^2 \int \frac{xyz}{W^2}\, \omega \ .$$

Using $z^3 = z\partial_z W + \alpha xyz$ in the first term and again integrating by parts gives then

$$(1 - \alpha^3)\frac{\partial}{\partial\alpha}\varpi_2 = \alpha\varpi_1 + 3\alpha^2\varpi_2 \ ,$$

and hence

$$\frac{\partial}{\partial\alpha}\begin{pmatrix}\varpi_1 \\ \varpi_2\end{pmatrix} = \begin{pmatrix} 0 & 1 \\ \frac{\alpha}{(1-\alpha^3)} & \frac{3\alpha^2}{(1-\alpha^3)} \end{pmatrix}\begin{pmatrix}\varpi_1 \\ \varpi_2\end{pmatrix} \ .$$

Upon eliminating ϖ_2, this equation can be rewritten

$$\left[(1 - \alpha^3)\partial_\alpha^2 - 3\alpha^2\partial_\alpha - a \right]\varpi_1 = 0 \ . \tag{2.11}$$

This example can easily be generalized to the series of potentials,

$$W^{(N)}(x_A) = \frac{1}{N}\sum_{i=1}^{N} x_i^N - \alpha \prod_{i=1}^{N} x_i \ , \qquad N \geq 3 \ , \tag{2.12}$$

that describe $N = 2$ Landau-Ginzburg theories with central charge $c = 3(N - 2)$. Consider the following periods, which are associated with $(N - l - 1, l - 1)$-forms on $V^{(N)}$,

$$\varpi_l^{(N)} = (l - 1)! \int_\Gamma \frac{\prod_{i=1}^{N}(x_i)^{l-1}}{(W^{(N)})^l}\, \omega \ , \qquad l = 1, \dots, N - 1 \ .$$

We find an equation in a "Drinfeld-Sokolov" form:

$$\frac{\partial}{\partial\alpha}\varpi^{(N)}$$

$$= \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 1 \\ \frac{\alpha b_1^{(N)}}{(1-\alpha^N)} & \frac{\alpha^2 b_2^{(N)}}{(1-\alpha^N)} & \frac{\alpha^3 b_3^{(N)}}{(1-\alpha^N)} & \cdots & \frac{\alpha^{N-2}b_{N-2}^{(N)}}{(1-\alpha^N)} & \frac{\alpha^{N-1}b_{N-1}^{(N)}}{(1-\alpha^N)} \end{pmatrix} \cdot \varpi^{(N)} \ ,$$

where the coefficients are recursively defined:

$$b_l^{(N)} = l\, b_l^{(N-1)} + b_{l-1}^{(N-1)}\,, \qquad \text{for } l = 1, \ldots, N-2\,,$$

with $b_1^{(N)} = 1$ and $b_{N-1}^{(N)} = \frac{1}{2}N(N-1)$. The matrix equation yields

$$\left[(1 - \alpha^N)\frac{\partial^{N-1}}{\partial\alpha^{N-1}} - \Big(\sum_{l=1}^{N-1} b_l^{(N)}\alpha^l\,\frac{\partial^{l-1}}{\partial\alpha^{l-1}} \Big) \right] \varpi_1^{(N)} = 0\,. \qquad (2.13)$$

Note that due to the high degree of symmetry of the perturbation, the order of this equation is much less then the dimension of the corresponding local ring.

Under the substitutions $\alpha^N \to z^{-1}$ and $\varpi_1 \to z^{1/N}\varpi_1$, this equation transforms into the following generalized hypergeometric differential equation [20] with regular, singular points at $z = 0, 1, \infty$:

$$\left[\Big(z\frac{\partial}{\partial z}\Big)^{N-1} - z\Big(z\frac{\partial}{\partial z} + \frac{1}{N}\Big)\Big(z\frac{\partial}{\partial z} + \frac{2}{N}\Big) \cdots \Big(z\frac{\partial}{\partial z} + \frac{N-1}{N}\Big) \right] \varpi_1^{(N)} = 0\,. \quad (2.14)$$

For $N = 5$, this is identical to the equation that was discussed in [11]:

$$\left[z^3(1 - z)\frac{\partial^4}{\partial z^4} + (6 - 8z)z^2\frac{\partial^3}{\partial z^3} + (7 - \frac{72}{5}z)z\frac{\partial^2}{\partial z^2} + (1 - \frac{24}{5}z)\frac{\partial}{\partial z} - \frac{24}{625} \right] \varpi_1^{(5)} = 0\,. \qquad (2.15)$$

Equation (2.14) is solved [20] by

$$\varpi_1^{(N)} = {}_{N-1}F_{N-2}\left[\frac{1}{N}, \frac{2}{N}, \ldots, \frac{N-1}{N}; 1, 1, \ldots, 1; z \right]\,,$$

where

$$_AF_B\left[a_1, a_2, \ldots, a_A; b_1, b_2, \ldots b_B; z \right]$$
$$\equiv \frac{\prod_{k=1}^{B}\Gamma(b_k)}{\prod_{l=1}^{A}\Gamma(a_l)} \sum_{n=0}^{\infty} \frac{z^n}{n!} \frac{\prod_{l=1}^{A}\Gamma(a_l + n)}{\prod_{k=1}^{B}\Gamma(b_k + n)} \qquad (2.16)$$

is the generalized hypergeometric function. In direct generalization of the results of

[11], a complete set of linear independent solutions to (2.14) for $N > 3$ is given by

$$y_k = z^{-k/N}\, {}_{N-1}F_{N-2}\left[\frac{k}{N}, \frac{k}{N}, \ldots, \frac{k}{N}; \overbrace{\frac{k+1}{N}, \frac{k+2}{N}, \ldots, \frac{k+N-1}{N}}; z^{-1} \right] \quad (2.17)$$

($k = 1, \ldots, N - 1$), where the overbrace indicates that the entry with value equal to one is to be omitted.

The reduction method that we have described for obtaining the differential equation (2.4) works far more generally than for the class of potentials W discussed above. For example, the generalization to quasihomogenous spaces is straightforward. Moreover, one can apply these techniques to marginal deformations of more general Landau-Ginzburg models. One does not have to restrict to Landau-Ginzburg theories that have a sigma-model interpretation; one such generalization is obtained by formally combining theories with other ones so as to mimick the $c = 3d$ situation. Then one applies the results derived above and makes the trivial observation that so long as the marginal perturbations do not mix one component theory with another, this tensoring of theories is irrelevant for the determination of the differential equation. Hence we need not restrict to theories with $c = 3d$. As an illustration, consider

$$W = x^3 y + y^3 + z^3 - \alpha\, x^3 z\,, \qquad (2.18)$$

which describes a perturbation of an $N = 2$ theory with $c = \frac{11}{3}$ and $\mu = 14$. We find

$$\left[9(1 - \alpha^3)\frac{\partial^2}{\partial\alpha^2} - 24\alpha^2\frac{\partial}{\partial\alpha} - 4\alpha \right] \varpi_1 = 0\,. \qquad (2.19)$$

In general, for theories that have effectively one modulus, the order of the differential equation will be two if $3 \leq c < 6$, three if $6 \leq c < 9$, and so on.

In the next section, we will describe how the linear equations (2.4) for the periods are related to non-linear differential equations that determine the dependence of the flat coordinates on the moduli, μ_i. In section 4, we will further generalize the method to arbitrary marginal and relevant perturbations of generic Landau-Ginzburg potentials.

3. Non-Linear Equations, Duality and Monodromy Groups

To obtain the flat or 'special' coordinates on the moduli space of a Calabi-Yau manifold one expands the holomorphic $(3,0)$-form in a basis of *integral* cohomology [5][11]. That is, one introduces a symplectically diagonal basis, $\{\alpha_a, \beta^b\}$, of integral cohomology and writes the $(3,0)$-form as:

$$\Omega = z^a\, \alpha_a + \mathcal{G}_b\, \beta^b. \tag{3.1}$$

(The periods z^a and $\mathcal{G}_b$ are integral linear combinations of the entries of an appropriate row of the period matrix Π_α^β). Since Ω is only defined up to multiplication by an arbitrary function $f(z^a)$, the quantities z^a and $\mathcal{G}^b$ are only defined projectively. It was shown in [5][11][12] that the z^a define good projective coordinates on the moduli space, while the $\mathcal{G}_b$ satisfy $\mathcal{G}_b = \partial_b(z^a \mathcal{G}_a)$. It is thus the inhomogeneous coordinates $\zeta^a = z^a/z^0$ that constitute the required flat, or special coordinates, t_i. The crucial ingredient that leads to the flat coordinates is the choice of an *integral* cohomology basis. This, in a very strong sense, means that we are choosing a locally constant frame for the cohomology fibration over the space of moduli.

In the following, we will not restrict ourselves to 3-folds, but we will consider projective coordinates ζ^a that are provided by the expansion of the the holomorphic $(d,0)$-form on a general 'Calabi-Yau' manifold.

In section 2 we saw how to derive a linear system of equations (2.4) that is satisfied by the periods, z^a. Obviously, if one multiplies Ω by $f(z^a)$ then one will obtain a different set of equations, and thus (2.4) is not unique. The appropriate *invariant* equation is a *non-linear* system of equations for ζ^a that can be derived from the linear system for z^a.

For example, the linear, second order equation

$$\frac{d^2 z}{d\mu^2} + p(\mu)\frac{dz}{d\mu} + q(\mu)z = 0 \tag{3.2}$$

gives rise to the non-linear, Schwarzian differential equation:

$$\{\zeta; \mu\} = 2I, \qquad I \equiv q - \tfrac{1}{4}p^2 - \tfrac{1}{2}\frac{dp}{d\mu} \tag{3.3}$$

where $\zeta = z_1/z_2$ and z_i, $i = 1, 2$ are the two solutions of the linear equation (3.2). The Schwarzian differential operator on the left-hand-side of this equation is defined by:

$$\{\zeta; \mu\} \equiv \frac{\zeta'''}{\zeta'} - \frac{3}{2}\left(\frac{\zeta''}{\zeta'}\right)^2 \tag{3.4}$$

and satisfies:

$$\{y; x\} = -\left(\frac{dy}{dx}\right)^2 \{x; y\} \tag{3.5}$$

$$\{y; x\} = \left(\frac{dz}{dx}\right)^2 \Big[\{y; z\} - \{x; z\}\Big] \tag{3.6}$$

$$\{y; x\} \equiv 0 \quad \text{if and only if} \quad y = \frac{ax + b}{cx + d}, \tag{3.7}$$

for some $a, b, c, d \in \mathbb{C}$ and $ad - bc \neq 0$.

The quantity I in (3.3) is often referred to as the *invariant* of (3.2) since it is unchanged if one replaces (3.2) by the linear equation for $f(\mu)z(\mu)$ (where $f(\mu)$ is an arbitrary function).

Recall [21] that the 'duality group' Γ_W of the superpotential W consists of those transformations of the moduli that are induced through quasihomogeneous changes of the variables, x^A, that leave the form of the superpotential unchanged up to an overall factor. That is, if $W_0(x^B; \mu_i)$ is a quasihomogeneous potential then one seeks quasihomogeneous changes of variable $\hat{x}^A$ whose Jacobian, $det\left(\frac{\partial \hat{x}^A}{\partial x^B}\right)$, is constant, or at worst a function, $\Delta(\mu_i)$, of μ_i and for which:

$$W_0(\hat{x}^A; \mu_i) = h(\mu_i)^{-1}\, W_0(x^A; \hat{\mu}_i(\mu_i)), \tag{3.8}$$

where $\hat{\mu}_i$ is some function of μ_i. If one makes such a change of variables in the period integrals of Ω then the result is changed by an overall factor of $h(\mu_i)\Delta(\mu_i)$.

It follows that the linear system of differential equations must be covariant with respect to the duality group of the superpotential. That is, if $z(\mu_i)$ is a solution, then so is $\Delta(\mu_i)h(\mu_i)z(\bar\mu_i(\mu_i))$. The corresponding system of non-linear equations must be *invariant* with respect to this duality group. This duality covariance and invariance can be very instructive in understanding the properties of the linear and non-linear equations. Turning this around, it allows in principle to determine Γ_W from the differential equations. In particular, given a second order equation (3.2), one can often read off the solution of the associated Schwarzian equation in terms of triangle functions, $s(\alpha, \beta, \gamma)$. The parameters α, β and γ then determine the duality group of the superpotential. For triangle functions, this group can be thought of as being generated by reflections in the sides of hyperbolic triangles (with angles $\pi\alpha, \pi\beta$ and $\pi\gamma$) that cover the upper half-plane.

It is thus of interest to understand the relationship between the foregoing duality group Γ_W, the monodromy group Γ_M of the linear equations and the 'modular' group Γ of the surface. The integral homology basis undergoes an integral symplectic transformation when it is transported around singular points in the moduli space of the manifold. Consequently, the periods of the differential forms undergo just such symplectic transformations about these singular points. This is directly reflected in the monodromy around regular singular points of the solutions of the differential equations. The set of all such monodromies will generate a subgroup, Γ_M, of the 'modular group', Γ. The set of duality transformations Γ_W of the superpotential maps the surface back to itself and will thus extend the group Γ_M to an even larger subgroup of Γ. In some cases this extension is all of Γ, and then the duality group of W is $\Gamma_W = \Gamma/\Gamma_M$.

We are uncertain as to the general validity of this conclusion, but a simple illustration is provided by the cubic torus, (2.9). The non-linear system associated to (2.11) is given by the Schwarzian

$$\{\alpha; t\} \;=\; -\frac{1}{2}\frac{(8 + \alpha^3)}{(1 - \alpha^3)^2}\,\alpha(\alpha')^2 \;, \tag{3.9}$$

which is solved by the triangle function $\alpha(t) = s(\frac{1}{2}, \frac{1}{3}, \frac{1}{3}; J(t))$ [22][21]. The transformation properties of this function are well known; in particular, it is a modular form of $\Gamma(3) \equiv PSL(2, \mathbb{Z}_3)$, and this group is the monodromy group Γ_M of the differential equation (2.11). Both sides of (3.9) are invariant under $\Gamma = PSL(2, \mathbb{Z})$, which is the full modular group of the torus. The quotient, the tetrahedral group $\Gamma/\Gamma(3)$, is precisely the duality group Γ_W of the superpotential (2.9), [22][21].

As a further example of the non-linear systems of equations that one can derive for the flat coordinates, we will descibe in some detail the a very particular family of $K3$ surfaces. That is, we will consider the surface defined by the superpotential (2.12) for $N = 4$. Equation (2.13) becomes:

$$\left[(1 - \alpha^4)\frac{\partial^3}{\partial\alpha^3} - 6\alpha^3\frac{\partial^2}{\partial\alpha^2} - 7\alpha^2\frac{\partial}{\partial\alpha} - \alpha\right]\varpi_1^{(4)} \;=\; 0\;, \tag{3.10}$$

Before discussing the solution of this system, it is instructive to consider a general third order equation and see how one passes to the associated non-linear system and obtains the invariants. Our discussion will follow that of [23]. Consider the generic third order equation:

$$\varpi''' + 3p(\alpha)\,\varpi'' + 3q(\alpha)\,\varpi' + r(\alpha)\,\varpi = 0. \tag{3.11}$$

One starts by partially removing the freedom to multiply a solution by an arbitrary function of α. This is done by requiring the vanishing of the coefficient of the second derivative, and is accomplished by substituting $\varpi \equiv y e^{-\int p\,d\alpha}$. The differential equation then takes the form:

$$y''' + 3Q(\alpha)y' + R(\alpha)y \;=\; 0, \tag{3.12}$$

where

$$\begin{aligned} Q &= q - p^2 - p' \\ R &= r - 3pq + 2p^3 - p'' \;. \end{aligned} \tag{3.13}$$

Let y_1, y_2 and y_3 be solutions of (3.12) and define s and t by

$$s = \frac{y_2}{y_1}, \qquad t = \frac{y_3}{y_1}.$$

Substituting $y_2 = sy_1$ and $y_3 = ty_1$ into (3.12), and using the fact that y_1 is a solution, one obtains:

$$3s'y_1'' + 3s''y_1' + (3Qs' + s''')y_1 = 0$$
$$3t'y_1'' + 3t''y_1' + (3Qt' + t''')y_1 = 0. \tag{3.14}$$

If one now differentiates these two equations again, and eliminates y_1''' using (3.12) one obtains two more equations that are linear in y_1, y_1' and y_1''. These two equations, along with (3.14), provide four linear equations for the three non-trivial, independent unknowns y_1, y_1' and y_1'', and thus there are two indepedent 3×3 determinants that must vanish. The vanishing of these determinants gives two fourth order, non-linear equations[23] for s and t. Conversely, given a solution to these non-linear equations, one can eliminate y_1'' from the linear system described above to obtain a simple linear, first order equation for y_1, whose solution is:

$$y_1 = \left(s''t' - s't''\right)^{-\frac{1}{3}}.$$

The other solutions are then obtained from $y_2 = sy_1$ and $y_3 = ty_1$.

The actual non-linear system for s and t is fairly unedifying, but we will give it here for the sake of completeness. Define the following variables:

$$u_1 = s''t' - s't'' \qquad u_2 = s^{(3)}t' - s't^{(3)} \qquad u_3 = s^{(4)}t' - s't^{(4)}$$
$$v_1 = s^{(3)}t'' - s''t^{(3)} \qquad v_2 = s^{(4)}t'' - s''t^{(4)}, \tag{3.15}$$

where $s^{(i)} \equiv d^i s/d\alpha^i$, and introduce the differential operators:

$$D_1(s,t;\alpha) \equiv \frac{u_3 - 2v_1}{u_1} - \frac{4}{3}\left(\frac{u_2}{u_1}\right)^2$$
$$D_2(s,t;\alpha) \equiv \frac{9v_2}{u_1} - \frac{6u_2(u_3 + 4v_1)}{u_1^2} + 8\left(\frac{u_2}{u_1}\right)^3 . \tag{3.16}$$

The non-linear system may then be written:

$$D_1(s,t;\alpha) = 3Q(\alpha) \equiv I,$$
$$D_2(s,t;\alpha) = 27(\frac{\partial Q}{\partial \alpha} - R) \equiv J. \tag{3.17}$$

The operators D_1 and D_2 are invariant under fractional linear transformations:

$$D_1\left(\frac{a_2 + b_2 s + c_2 t}{a_1 + b_1 s + c_1 t}, \frac{a_3 + b_3 s + c_3 t}{a_1 + b_1 s + c_1 t}; \alpha\right) = D_1(s,t;\alpha)$$
$$D_2\left(\frac{a_2 + b_2 s + c_2 t}{a_1 + b_1 s + c_1 t}, \frac{a_3 + b_3 s + c_3 t}{a_1 + b_1 s + c_1 t}; \alpha\right) = D_2(s,t;\alpha) . \tag{3.18}$$

The right-hand-sides of (3.17) define the quantities I and J, which are called the *invariants* of the system.

To solve (3.10) one needs to use some more of the theory of reduced differential equations of the form (3.12)[23][*]. The form of (3.12) can be preserved by a combined rescaling and reparametrization. That is, one introduces a new parameter t and sets $y = \left(\frac{dt}{d\alpha}\right)^{-1}u$. Under this transformation the resulting differential equation has the form of (3.12), but with:

$$\alpha \to t, \qquad y \to u, \qquad Q \to \tilde{Q} \equiv \left(\frac{dt}{d\alpha}\right)^{-2}\left(Q - \frac{2}{3}\{t;\alpha\}\right)$$
$$R \to \tilde{R} \equiv \left(\frac{dt}{d\alpha}\right)^{-3}\left[\left(R - \frac{d}{d\alpha}\{t;\alpha\}\right) - 3\left(\frac{d^2t}{d\alpha^2}\right)\left(\frac{dt}{d\alpha}\right)\tilde{Q}\right] .$$

It is interesting to note that Q transforms precisely like an energy momentum tensor, and that the combination $W_3 \equiv R - \frac{3}{2}\frac{dQ}{d\alpha}$ transforms homogeneously, *i.e.*:

$$W_3 \to \tilde{W}_3 \equiv \left(\tilde{R} - \frac{3}{2}\frac{d\tilde{Q}}{dt}\right) = \left(\frac{dt}{d\alpha}\right)^{-3}W_3 .$$

It is precisely one of the classical W-generators [24]. One can fix the reparametriz-

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[*] A recent discussion of this subject may be found in [24].

ation invariance by requiring that $\bar{Q} = 0$, or

$$\{t;\alpha\} = \frac{3}{2} Q .$$ (3.19)

If one puts (3.10) in the form (3.12) one has:

$$Q = \frac{\alpha^2}{3} \frac{(\alpha^4 + 11)}{(1 - \alpha^4)^2}$$

$$R = \alpha \frac{11 + 36\alpha^4 + \alpha^8}{(1 - \alpha^4)^3} ,$$

From this one finds an extra bonus: $W_3 \equiv 0$, or $R = \frac{3}{2}Q'$. This means that when one passes to the equation for $u(t)$, one obtains $\frac{d^3 u}{dt^3} = 0$, whose solutions are 1, t and t^2. Therefore the solutions to (3.10) are:

$$\varpi = (1 - \alpha^4)^{-\frac{1}{2}} \left(\frac{dt}{d\alpha} \right)^{-1} u(t) ; \qquad u(t) = 1, t, t^2 ,$$ (3.20)

where $t(\alpha)$ is the solution of (3.19):

$$\{t;\alpha\} = \frac{3}{2} Q \equiv \frac{1}{2}\alpha^2 \left(\frac{\alpha^4 + 11}{(1 - \alpha^4)^2} \right) .$$ (3.21)

Finally, changing variables $z = \alpha^{-4}$ in (3.21) one obtains:

$$\{t;z\} = \frac{1}{2}\frac{1}{z^2} + \frac{3}{8}\frac{1}{(z-1)^2} - \frac{13}{32}\frac{1}{z(z-1)}.$$

The solution of this equation is given by a triangle function, $t(z) = s(0, \frac{1}{2}, \frac{1}{4}; z)$, which can, in turn, be re-expressed as the ratio of two solutions to the ordinary hypergeometric equation with parameters $\alpha = \frac{1}{8}$, $\beta = \frac{3}{8}$, and $\gamma = 1$. (The solution can, of course, also be expressed in terms of ratios of generalized hypergeometric functions (2.17).) We remark that the structure of the monodromy group of (3.10) is very similar to that of the quintic of [11], the difference being that all appearances of 5 in the formulae of [11] must be replaced by 4. This rule seems to hold for all N.

As we have seen, the modular dependence of the periods of this family of $K3$ surfaces could have involved a non-trivial W_3 invariant, but instead we found that W_3 vanished. In this sense, the structure is determined merely by the Virasoro algebra, that is, by the Schwarzian differential equation (3.19).

It would be very interesting to discover to what extent the higher dimensional surfaces defined by (2.12) might be similarly reduced, and to understand whether the appearance of such reduced W-algebras has any deeper meaning. In particular, note that the solutions of (3.10) are algebraically related (inspection of (3.20) shows that $\varpi_1 \varpi_3 = \varpi_2^2$), and as we have seen this is a consequence of the vanishing of W_3. It turns out that for the quintic, i.e. for (2.15), one also has $W_3 \equiv 0$ but $W_4 \neq 0$, and it is known that the $\mathcal{G}_b$ in (3.1) are homogenous functions of the z^a. Thus it appears that the vanishing of W-generators is closely connected to algebraic relations between the solutions. We hope to discuss these issues elsewhere.

4. Flat Coordinates for Generic Perturbations

We now wish to generalize the methods of section 2 to marginal and relevent perturbations of arbitrary topological Landau-Ginzburg field theories[*] . The basic problem is that general $N = 2$ superconformal theories have no obvious analogue of *integral* cohomology. As discussed in the previous section, it is this that leads one to flat coordinates for 'Calabi-Yau' spaces.

For general topological matter models, one can make a general ansatz for $\mathcal{F}$, or for W, in terms of the flat coordinates, and then evolve algebraic and differential equations from consistency conditions of the topological matter models [3] [25] [26]. In particular one requires that the C_{ijk} be given by (1.1) and that they satisfy (1.2). However, solving the system (1.2) is extremely laborious, except in the simplest cases.

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[*] Our discussion will follow, and extend, that of [6]. Flat coordinates in generic topological matter theories have also recently been discussed in [7].

Thus we like to obtain differential equations that determine the flat coordinates more directly from the superpotential. Let $W_0(x^A)$ be a quasihomogenous superpotential and $W(x^A; s_i)$ be a parametrization of a general, versal deformation of W_0 by elements ϕ_α of the chiral ring. The problem is to determine the relationship between the general coordinates s_i, and the flat coordinates t_i. Once the parametrization of W in terms of the t_i is known, the free energy $\mathcal{F}$ and all correlation functions can easily be computed.

We will regard $W(x^A; s_i)$ as a quasihomogeneous function of x^A and s_i, and thus the coupling constants s_i can be assigned dimensions. (We will adopt the convention that both W_0 and W have dimension equal to one.) Below we will actually consider only marginal and relevant perturbations, whose corresponding coupling constants will have vanishing or positive dimensions. This will lead to the major simplification that all quantities will have polynomial dependence on the coupling constants with positive dimension, and the only non-polynomial behavior will be via the marginal parameters.

The coupling constant associated to the constant term in $W(x^A; s_i)$ (i.e., the unique coupling constant of dimension one) will play a distinguished, important role and it will be denoted by s_1. The remaining coupling constants $s_2, s_3, \ldots$ will be denoted generically by s'. We will take

$$\widetilde{W}(x^A, s') = W(x^A, s_i) - s_1 \qquad (4.1)$$

as independent of s_1.

Let $\phi_\alpha(x^A; s')$, $\alpha = 1, \ldots, \mu$, be any (polynomial) basis for the chiral ring and consider integrals of the form

$$u_\alpha^{(\lambda)} = (-1)^{\lambda+1}\Gamma(\lambda+1)\int_\gamma \frac{\phi_\alpha(x^A; s')}{W^{\lambda+1}}dx^1 \wedge dx^2 \ldots \wedge dx^n , \qquad (4.2)$$

where the integral is taken over any compact homology cycle[†] γ in the set $\{x \in \mathbb{C}^n : W(x, s) \neq 0\}$. The gamma function and the factor of $(-1)^{\lambda+1}$ are introduced for later convenience. These integrals are related to the periods of differential forms on the level surfaces of W, [19]. They satisfy some important recurrence relations [27][6]:

$$\partial_{s_1}^k u_\alpha^{(\lambda)} = u_\alpha^{(\lambda+k)} , \qquad k \in \mathbb{Z} \qquad (4.3)$$

$$\partial_{s_i} u_\alpha^{(\lambda)} = \sum_{k=-1}^\infty \mathcal{B}_{i\ \alpha}^{(k)\ \beta}(s') u_\beta^{(\lambda-k)} \qquad (4.4)$$

$$s_1 u_\alpha^{(\lambda)} = -\sum_{k=0}^\infty \mathcal{A}_\alpha^{(k-2)\ \beta}(s') u_\beta^{(\lambda-k)} \qquad (4.5)$$

These recurrence relations are derived by the same procedure as that employed in section 2. Equation (4.3) is a trivial consequence of differentiation under the integral. Equation (4.4) is also obtained by differentiating under the integral, but in this second instance the numerator of the integrand is a polynomial $(\partial_{s_i}W)\phi_\alpha(x^A; s')$ which might need to be reduced. That is, by definition of the local ring of W this polynomial may always be rewritten in the following form:

$$(\partial_{s_i}W)\phi_\alpha(x^A; s') \equiv C_{i\alpha}^{\ \beta}\phi_\beta(x^A; s') + q_i^{(0)A}(x^A; s')\frac{\partial W}{\partial x^A}(x^A; s') \qquad (4.6)$$

for some polynomial $q_i^{(0)A}$. One now integrates by parts[*] to obtain

$$\partial_{s_i} u_\alpha^{(\lambda)} = C_{i\alpha}^{\ \beta}u_\beta^{(\lambda+1)} + (-1)^{\lambda+1}\Gamma(\lambda+1)\int \frac{(\frac{\partial}{\partial x^A}q_i^{(0)A}_\alpha)}{W^{\lambda+1}}dx^1 \wedge dx^2 \ldots \wedge dx^n .$$

[†] There are μ independent possible choices, but the choice is not important for the moment. The reader might find it helpful to consider the one-variable case, in which γ is some loop around some subset of the zeros of W.

[*] That is, one uses the fact that $0 \equiv \int \frac{\partial}{\partial x^A}\left(\frac{V^A}{W^{\lambda+1}}\right)dx^1 \wedge dx^2 \ldots \wedge dx^n$ for any vector V^A.

Once again one decomposes the numerator

$$\frac{\partial}{\partial x^A} q_{i\ \alpha}^{(0)A} \ = \ \mathcal{B}_{i\ \alpha}^{(0)\ \beta}(s')\,\phi_\beta(x^A;s') + q_{i\ \alpha}^{(1)A}(x^A;s')\frac{\partial W}{\partial x^A}(x^A;s')\,,$$

and integrates by parts. In this manner one may recursively compute $\mathcal{B}_{i\ \alpha}^{(k)\ \beta}(s')$, $k = -1, 0, 1, \dots$. Note that after every integration by parts the polynomial degree (in x^A) of the successive terms $(\frac{\partial}{\partial x^A}q_{i\ \alpha}^{(k)A})$ decreases by one unit, and hence this procedure must terminate after a finite number of steps. Also note that $C_{i\alpha}^{\ \beta} \equiv \mathcal{B}_{i\ \alpha}^{(-1)\ \beta}(s')$ are essentially the structure constants of the local ring.

Finally, equation (4.5) is obtained by taking s_1 inside the integral, and rewriting it as $s_1 \equiv W(x^A,s_i) - \widetilde{W}(x^A,s')$. The factor of W is cancelled immediately, while $\widetilde{W}(x^A,s')\phi_\alpha(x^A;s')$ is simplified by the identical, recursive reduction procedure described above.

It is very convenient to make a "Fourier transformation" in the s_1 variable. Specifically, it replaces $f(s_1)$ by $\int_{-\infty}^0 e^{s_1/z} f(s_1)\,ds_1$. This has the effect of sending $\partial_{s_1} \to z^{-1}$ and $s_1 \to -z^2\frac{d}{dz}$. Let $u_\alpha^{(\lambda)}(z;s')$ denote the transform of $u_\alpha^{(\lambda)}(s)$. Then equations (4.4) and (4.5) may be rewritten as a linear system:

$$\left(\partial_{s_i} - \sum_{k=-1}^\infty z^k \mathcal{B}_{i\ \alpha}^{(k)\ \beta}(s')\right) u_\beta^{(\lambda)}(z;s') \ = \ 0 \tag{4.7}$$

$$\left(\partial_z - \sum_{k=-2}^\infty z^k \mathcal{A}_\alpha^{(k)\ \beta}(s')\right) u_\beta^{(\lambda)}(z;s') \ = \ 0 \tag{4.8}$$

Observe that the $u_\beta^{(\lambda)}(z;s')$ are, by definition, covariant constant sections of a flat vector bundle whose connections are defined by $\mathcal{B}_i^{(k)}$ and $\mathcal{A}^{(k)}$.

To get more insight into why we make this construction, suppose that $\mathcal{B}_i^{(k)} \equiv 0$ for $k \geq 1$ and define $\mathcal{D}_i = \partial_{s_i} - B_i^{(0)}$, $C_{i\alpha}^{\ \beta} = \mathcal{B}_{i\ \alpha}^{(-1)\ \beta}(s')$. Then the flatness of the connection, or integrability of (4.7) implies

$$[\mathcal{D}_i - z^{-1}C_i\,,\mathcal{D}_j - z^{-1}C_j] \ = \ 0\,, \tag{4.9}$$

and separating out different orders in z, one gets the zero curvature equations[‡]

$$[\mathcal{D}_i\,,\mathcal{D}_j] \ \equiv \ \mathcal{D}_{[i}C_{j]} \ \equiv \ [C_i\,,C_j] \ \equiv \ 0\,. \tag{4.10}$$

Hence the connection $\mathcal{B}_i^{(0)}$ is flat, the structure constants $C_{i\alpha}^{\ \beta}$ commute and are covariantly constant. If we now arrange that the basis $\{\phi_\alpha\}$ of the chiral ring is, in fact, given by $\{\frac{\partial W}{\partial s_i}\}$, and let

$$\Gamma_{ij}^{\ k} \ = \ \mathcal{B}_{ij}^{(0)\,k}\,,$$

then one finds that $\Gamma_{ij}^{\ k} = \Gamma_{ji}^{\ k}$ and equation (4.10) implies that Γ is the flat coordinate connection we seek. We thus have solved the consistency conditions (1.1) and (1.2). The remainder of this section will essentially reduce the general problem to the foregoing simpler situation.

Because the connection defined by $\mathcal{A}$ and $\mathcal{B}$ is flat, one already knows that one can find a gauge transformation that will trivialize it. More precisely, because there might be non-trivial monodromy, one can find a matrix M such that[♮]

$$\begin{aligned}
\mathcal{A} &= (\partial_z M)M^{-1} + M\left(\frac{\tilde{A}(s')}{z}\right)M^{-1} \\
\mathcal{B}_i &= (\partial_{s_i}M)M^{-1}\,.
\end{aligned} \tag{4.11}$$

Thus we can gauge away all of $\mathcal{B}$ and almost all of $\mathcal{A}$. The problem is that the matrix M will in general involve all powers in z and $1/z$. Hence M will define a basis change involving $u_\alpha^{(\lambda+k)}$ for all $k \in \mathbb{Z}$. To control this, and indeed to preserve quasihomogeneity, we want to restrict ourselves to changes of basis that are upper triangular, that is, $u_\alpha^{(\lambda+k)}$ is only modified by addition of polynomials in s' and $u_\alpha^{(\lambda+l)}$ for $l \leq k$. This means that the change of basis must be analytic at $z = 0$.

[‡] Such equations have also been discussed in [5][7].

[♮] These matrices are analytic in s', but not in z, hence the form of the equation.

Now suppose that we can decompose M of equation (4.11) in the following manner:

$$M = g_0(z; s') g_\infty(\tfrac{1}{z}; s') , \qquad (4.12)$$

where g_0 is analytic at $z = 0$ and g_∞ is analytic at $z = \infty$. Now let $\mathcal{A}' \equiv g_0^{-1} A g_0 - g_0^{-1}(\partial_z g_0)$ and $\mathcal{B}'_i \equiv g_0^{-1} B_i g_0 - g_0^{-1}(\partial_z g_0)$. Then it is elementary to see that

$$\mathcal{A}' = \left[(\partial_z g_\infty) g_\infty^{-1} + g_\infty \left(\frac{A_0(s')}{z} \right) g_\infty^{-1} \right] \qquad (4.13)$$
$$\mathcal{B}'_i = (\partial_{s_i} g_\infty) g_\infty^{-1} .$$

Moreover, by modifying g_0 by multiplying by a suitable matrix, $h(s')$, one can further gauge away the z-independent term in $\mathcal{B}'$. Thus, provided that we can make the split in (4.12) there is a z-analytic gauge choice that has $\mathcal{A}^{(k)} \equiv \mathcal{B}_i^{(k)} \equiv 0, k \geq 0$.

The problem of finding the splitting (4.12) is called a Riemann-Hilbert problem, and is generically [28], but not always, solvable. Its solution is intimately connected with solving integrable models (see, for example, [29]). We have, in fact, a variational Riemann-Hilbert problem in that our matrices have parameters s'. This makes the problem much easier to address and it will be discussed further in the appendix. In particular, we show in the appendix that $g_0 = I + \mathcal{O}(s')$, where I is the identity matrix, and we will also show that g_0 is analytic in s' and preserves quasihomogeneity (i.e. the elements of the new basis have a well-defined scaling dimension).

To get flat coordinates, we need to make the restriction to marginal or relevant perturbations (of dimension less than or equal to one), which means $dim(s_i) \geq 0$. Let $v_\alpha(z; s')$ be the basis in which $\mathcal{A}^{(k)} \equiv \mathcal{B}^{(k)} \equiv 0$ for $k \geq 0$. Observe that if we restrict to $\phi_\alpha(x^A; s')$ of dimension strictly less than one, then the corresponding $u_\alpha^{(\lambda)}(s')$ have dimensions strictly less than $(\sum \omega_A) - \lambda$ (where ω_A is the weight of x^A). However, because the basis change has the form: $g_0 = 1 + \mathcal{O}(s')$, is analytic in z and s' and preserves quasihomogeneity, it follows that the v_α of dimension strictly less than $(\sum \omega_A) - \lambda$ are analytic, quasihomogenous combinations of s' and the $u_\alpha^{(\lambda)}(s')$.

In particular, such v_α do not involve any $u_\alpha^{(\lambda+k)}(s')$ for $k \neq 0$. Furthermore, the v_α of dimension equal to $(\sum \omega_A) - \lambda$ must be analytic, quasihomogenous combinations of s', the $u_\alpha^{(\lambda)}$, and

$$u'_0 \equiv u_0^{(\lambda-1)} \equiv (-1)^\lambda \Gamma(\lambda) \int \frac{1}{W^\lambda} dx^1 \wedge dx^2 \ldots \wedge dx^n . \qquad (4.14)$$

One of the v_α of dimension equal to $(\sum \omega_A) - \lambda$ must be (a dimensionless multiple of) $u_0^{(\lambda-1)} = z u_0^{(\lambda)}$, while the rest of the v_α must start with a $u_\alpha^{(\lambda)}$ term for which $\phi_\alpha(x^A; s')$ is a marginal operator. Let $\tilde{s}$ denote the (dimensionless) marginal parameters and let v'_α and $f(\tilde{s})$ be such that $\{ v'_\alpha, f(\tilde{s}) u'_0 \}$ forms a set of linearly independent $v_\alpha^{(\lambda)}$ of dimension less than or equal to $(\sum \omega_A) - \lambda$.

It follows from the foregoing that there is a quasihomogenous, analytic, invertible matrix $e_\alpha{}^j(s')$ and a set of functions $q_j(\tilde{s})$ such that

$$\frac{\partial}{\partial s_j} u'_0 = e_j{}^\alpha v'_\alpha - q_j(\tilde{s}) u'_0 \qquad (4.15)$$

and $q_j \equiv 0$ if $dim(s_j) > 0$. Next observe that

$$\begin{aligned} \frac{\partial^2}{\partial s_i \partial s_j} u'_0 &= (\partial_i e_j{}^\alpha) v'_\alpha + e_j{}^\alpha (\partial_i v'_\alpha) - q_j(\partial_i u'_0) - (\partial_i q_j) u'_0 \\ &= [\partial_i e_j{}^\alpha - q_j e_i{}^\alpha] v'_\alpha + z^{-1} e_j{}^\alpha C_{ia}{}^\beta v'_\beta + [q_i q_j - \partial_i q_j] u'_0 . \end{aligned}$$

By linear independence of the v'_α and u'_0 we have $\partial_{[i} q_{j]} \equiv 0$ and hence $q_j \equiv \partial_j q$ for some function $q(\tilde{s})$.

The foregoing combines to give us the following simple result. There is a 'universal' function $q(\tilde{s})$ of all the marginal (dimension zero) parameters such that the integral

$$u_0 = (-1)^\lambda \Gamma(\lambda) \int \frac{q(\tilde{s})}{W^\lambda} dx^1 \wedge dx^2 \ldots \wedge dx^n \qquad (4.16)$$

satisfies the following equation [*]

$$\frac{\partial^2}{\partial s_i \partial s_j} u_0 = C_{ij}{}^\alpha u_\alpha^{(\lambda+1)} + \Gamma_{ij}{}^k \left(\frac{\partial}{\partial s_k}\right) u_0 . \qquad (4.17)$$

Thus, the function $q(\tilde{s})$ is determined by requiring that (4.17) contains no term proportional to u_0 itself. The $C_{ij}{}^\alpha$ are just the structure constants of the chiral ring and $\Gamma_{ij}{}^k$ is the Gauss-Manin connection. Flat coordinates are determined by simply requiring that $\Gamma \equiv 0$ on the r.h.s. of (4.17).

In practice one takes the s_i to be the flat coordinates t_i, and considers a perturbation of the form

$$W(x;t) = W_0(x) + \sum \mu_i(t) m_i(x) , \qquad (4.18)$$

where $m_i(x)$ are monomials in the local ring (with degree less or equal than one), and the Landau-Ginzburg couplings, $\mu_i(t)$, are unknown functions to be determined. We note that it is elementary to explicitly write down the constraints implied by (4.17) since this only involves differentiating under the integral and integrating by parts, just like the reduction procedure described in section 2. The constraints take first the form of linear differential equations for the μ_i. One determines $q(\tilde{t})$ in terms of the $\mu_i(t)$ by requiring that the u_0 piece in (4.17) vanishes. Substituting for $q(\tilde{t})$ then turns the linear system into the associated non-linear system (e.g., into a Schwarzian differential equation) that determines the $\mu_i(t)$.

The function $q(\tilde{t})$ appears to be playing the rôle of a conformal rescaling of the vielbein. In particular we note that for the examples we computed, the function $q(\tilde{t})^{-2}$ is precisely the conformal factor that takes the Grothendieck metric of [7] to the flat metric.

For conformal theories with $c > 3$, there are chiral primary fields of dimension larger than one. With the restriction that we have made on the perturbed superpotential, we cannot write these irrelevant chiral primaries as $\frac{\partial W}{\partial s_i}$ for some s_i. Thus

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[*] Remember that W is only perturbed by marginal and relevant operators, i.e., $0 \leq dim(s_j) \leq 1$.

it might appear that these irrelevant chiral primaries play no role in determining the form of the equations that we derive from the procedure described above. This is not so. It is important to remember to pass first to the basis for *all* the chiral primaries in which one has $B_i^{(k)} = 0$ for $k \geq 0$, and then one must use this basis in calculating the separate terms in expressions like those on the right-hand-side of (4.17).

Finally we note that equations (4.17) and (4.10) can be recast in the familiar form

$$\tilde{\mathcal{D}}_i \varpi = 0 \qquad \text{and} \qquad [\tilde{\mathcal{D}}_i, \tilde{\mathcal{D}}_j] = 0 , \qquad (4.19)$$

where $\tilde{\mathcal{D}}_i = \partial_{s_i} + \Gamma_i + C_i \partial_1$ and $\varpi_i = \partial_i u_0$. The first equation is a generalization of the matrix differential equation (2.4) we discussed in section 2.

5. Examples Revisited

It is instructive to reconsider first the torus example (2.9) of section 2, but now with an additional, relevant perturbation:

$$W = \tfrac{1}{3}(x^3 + y^3 + z^3) - \alpha(t)\,xyz - s\,\beta_1(t)xy - \tfrac{1}{2}s^2\beta_2(t)z - \tfrac{1}{6}s^3\beta_3(t) . \qquad (5.1)$$

Here, t is a dimensionless, flat coordinate (the modular parameter of a torus), and s is a parameter of dimension $1/3$. The dependence of the Landau-Ginzburg coupling constants on the relevant perturbation parameter s is already fixed by its dimension, so we will have to determine only the dependence on the modular parameter. The two-parameter perturbation is certainly not the most general one (which was considered previously in [25]), but the extension is obvious. The specific perturbation we chose is however the most general one consistent with the $\mathbb{Z}_3 \times \mathbb{Z}_3$ symmetry generated by $(x, y, z, s) \to (\omega x, \omega^2 y, z, s)$ and $(x, y, z, s) \to (\omega x, \omega y, \omega z, \omega s)$.

Let $u_0 \equiv (-1)^\lambda \Gamma(\lambda) \int \frac{q(t)}{W^\lambda} dx^1 \wedge dx^2 \dots \wedge dx^n$. We want to solve for $\alpha(t)$ and $\beta_i(t)$ by requiring the connection Γ in (4.17) to be flat; this corresponds to the vanishing

of terms proportional to $u_\alpha^{(\lambda)}$ in this equation. In particular, we obtain

$$\frac{\partial^2 u_0}{\partial t^2} = \left(\frac{q''}{q}\right) u_0 + (-1)^{\lambda+1}\Gamma(\lambda+1)\int dx^1\wedge dx^2\ldots\wedge dx^n \frac{1}{W^{\lambda+1}} \times$$

$$\left\{ \left[2\frac{q'}{q} + \frac{\alpha''}{\alpha'}\right]\alpha' q\, xyz + s\left[2\frac{q'}{q} + \frac{\beta_1''}{\beta_1'}\right]\beta_1' q\, xy \right.$$

$$\left. + \frac{1}{2}s^2\left[2\frac{q'}{q} + \frac{\beta_2''}{\beta_2'}\right]\beta_2' q\, z + \frac{1}{6}s^3\left[2\frac{q'}{q} + \frac{\beta_3''}{\beta_3'}\right]\beta_3' q \right\}$$

$$+ (-1)^{\lambda+2}\Gamma(\lambda+2)\int dx^1\wedge dx^2\ldots\wedge dx^n \frac{1}{W^{\lambda+2}} \times$$

$$\left\{ (\alpha')^2 q\, x^2 y^2 z^2 + 2s\alpha'\beta_1' q\, x^2 y^2 z + s^2(\beta_1')^2 q\, x^2 y^2 + s^2\alpha'\beta_2' q\, xyz^2 + s^3[\beta_1'\beta_2' \right.$$

$$\left. + \frac{1}{3}\alpha'\beta_3']q\, xyz + \frac{1}{4}s^4(\beta_2')^2 q\, z^2 + \frac{1}{3}s^4\beta_1'\beta_3' q\, xy + \frac{1}{6}s^5\beta_2'\beta_3' q\, z + \frac{1}{36}s^6(\beta_3')^2 q \right\} .$$

$$(5.2)$$

We could obtain equations for q, α, β_i by considering all the different powers of s in this equation, but it is easier to just concentrate on the $s = 0$ pieces. We can thus use (2.10) and subsequently $z^3\big|_{s=0} = (z\partial_z W + \alpha xyz)\big|_{s=0}$ to integrate $\int \frac{x^2 y^2 z^2}{W^{\lambda+2}}$ by parts to reduce its degree. The vanishing of the connection Γ corresponds to the vanishing of the terms proportional to $\frac{1}{W^{\lambda+1}}$, i.e.

$$(-1)^{\lambda+1}\Gamma(\lambda+1)\int dx^1\wedge dx^2\ldots\wedge dx^n \frac{1}{W^{\lambda+1}}\left[2\frac{q'}{q} + \frac{\alpha''}{\alpha'} + \frac{3\alpha^2\alpha'}{(1-\alpha^3)}\right]\alpha' q\, xyz \equiv 0 .$$

This determines $q(t)$

$$q(t) = \left(\frac{1-\alpha(t)^3}{\alpha'(t)}\right)^{1/2} . \tag{5.3}$$

Integrating (5.2) (with $s = 0$) by parts, substituting (5.3) and requiring the vanishing of all terms that are proportional to u_0, we then indeed obtain directly the Schwarzian differential equation (3.9) for $\alpha(t)$,

$$\{\alpha; t\} = -\frac{1}{2}\frac{(8+\alpha^3)}{(1-\alpha^3)^2}\alpha(\alpha')^2 ,$$

which is associated to the linear equation (2.11). However, by using the methods derived in the foregoing section, we can now also solve for the couplings $\beta_i(t)$ of the

relevant perturbations. To obtain $\beta_1(t)$, it is easiest to consider the $s = 0$ piece of $\frac{\partial^2 u_0}{\partial s\partial t}$; by integrating by parts, we find the condition

$$\frac{1}{W^{\lambda+1}}\left[2\frac{q'}{q} + \frac{\beta_1'}{\beta_1} + \frac{2\alpha^2\alpha'}{(1-\alpha^3)}\right]\beta_1 q\, xy \equiv 0 ,$$

and this gives $\beta_1(t) = A(\alpha(t)')^{1/2}(1-\alpha(t)^3)^{1/6}$ (where A is an integration constant). Similarly, from the s independent piece of $\frac{\partial^2 u_0}{\partial s^2}$ we obtain

$$\frac{1}{W^{\lambda+1}}\left[\beta_2 + \frac{\beta_1^2\alpha}{(1-\alpha^3)}\right]q\, z \equiv 0 ,$$

which yields $\beta_2(t) = -A^2\alpha(t)\alpha(t)'(1-\alpha(t)^3)^{-2/3}$. Finally, for β_3 we consider the piece linear in s of $\frac{\partial^2 u_0}{\partial s^2}$, and use the identity

$$x^2 y^2 = \frac{1}{(1-\alpha^3)}\left\{\alpha xz\partial_z W + x^2\partial_y W + \alpha^2 xy\partial_x W \right.$$

$$\left. + 2s\alpha\beta_1 xyz + s\beta_1 x\partial_z W + s^2\beta_1^2 + \frac{1}{2}s^2\alpha^2\beta_2 xy\right\}$$

Partial integration gives

$$\int \frac{1}{W^{\lambda+1}}sq\left[\beta_3 + \frac{\beta_1^3}{(1-\alpha^3)}\right] \equiv 0$$

and thus determines $\beta_3(t)$. These results coincide with the expressions derived in [25]. One can also check that for the choice of α, β_i give above, the connection Γ is completely flattened.

To illustrate that our method may be used for theories with arbitrary central charges, reconsider the potential (2.18) with additional, relevant perturbations:

$$W = x^3 y + y^3 + z^3 - \alpha(t)x^3 z - s_1\beta_1(t)z - s_2\beta_2(t)x^2 ,$$

We find for the $\frac{1}{W^{\lambda+1}}$ piece in the $s_i = 0$ part of $\frac{\partial^2 u_0}{\partial t^2}$

$$\frac{1}{W^{\lambda+1}}\left[2\frac{q'}{q} + \frac{\alpha''}{\alpha'} - \frac{8}{3}\frac{\alpha^2\alpha'}{(1+\alpha^3)}\right]\alpha' q\, x^3 z ,$$

and this determines

$$q(t) = \alpha'(t)^{-1/2}[1 + \alpha(t)^3]^{4/9} .$$

The u_0 piece in (4.17) then vanishes if α satisfies the differential equation

$$\{\alpha; t\} = \frac{40}{9(1 + \alpha^3)^2}\alpha(\alpha')^2 .$$

This is precisely the Schwarzian form of the linear equation (2.19). Moreover, β_1 may be obtained from the $s_i = 0$ piece in $\frac{\partial^2 u_0}{\partial t \partial s_1}$: $\beta_1(t) = \alpha'(t)^{1/2}[1 + \alpha(t)^3]^{-1/3}$. Similarly, we find $\beta_2(t) = \alpha'(t)^{1/2}[1 + \alpha(t)^3]^{-1/9}$. Hence, we can compute the following term of the free energy:

$$\mathcal{F}(s_i, t) = \tfrac{1}{2}s_1 s_2{}^2 \alpha'(t)^{1/2}[1 + \alpha(t)^3]^{1/9} + \dots .$$

It is clear that we could compute the other terms of $\mathcal{F}$ in a similar way.

Acknowledgements

While this manuscript was in preparation we received the preprint [30] in which similar results to those of section 2 are derived.

APPENDIX

Our purpose here is to show why one can find a matrix $g_0(z, s')$ that is analytic in z and s' in the neighbourhood of $z = 0$, $s' = 0$ and which will gauge the potentials $\mathcal{A}^{(k)}$ and $\mathcal{B}^{(k)}$ to zero for $k \geq 0$. As was shown in section 3, it suffices to show that the matrix $M(z; s')$ defined in (4.11) can be factorized as in (4.12). The Birkhoff decomposition theorem [28] [31] implies that any matrix $M(z; s')$ can be decomposed according to:

$$M = g_0(z; s') \, \Lambda(z; s') \, g_\infty(\tfrac{1}{z}; s') , \qquad (A1)$$

where g_0 is analytic at $z = 0$, g_∞ is analytic at $z = \infty$ and $\Lambda(z; s')$ is a diagonal matrix whose entries are integral powers[*] of z. We need to show that $\Lambda = I$, where I is the identity matrix. More simply, it suffices to show that all the integral powers of z in Λ are, in fact, zero and hence Λ is z independent and can thus be absorbed into g_0 or g_∞. Since integers can only be continuous functions of s' by being constant, we can establish the desired result in a region about $s' = 0$ by simply showing that it is true at $s' = 0$.

Let $\phi_\alpha(x) = \phi_\alpha(x, s' = 0)$, and recall that $W_0(x) \equiv \widetilde{W}(x, s' = 0)$. By assumption $W_0(x)$ and $\phi_\alpha(x)$ are quasihomogeneous of weight 1 and of some weight λ_α.

---

[*] These integers are related to the Chern class of the relevant vector bundle over S^2.

respectively. It follows that $W_0 \equiv \sum_A \omega_A x^A \frac{\partial W_0}{\partial x^A}$ and hence

$$
\begin{aligned}
W_0(x)\,\phi_\alpha(x) &\equiv \sum_A \omega_A x^A \frac{\partial W_0}{\partial x^A}\,\phi_\alpha(x) \\
&= \sum_A \left[\frac{\partial}{\partial x^A}(\omega_A x^A W_0 \phi_\alpha) - \omega_A W_0 \phi_\alpha - \omega_A x^A W_0 \frac{\partial \phi_\alpha}{\partial x^A} \right] \\
&= \sum_A \frac{\partial}{\partial x^A}(\omega_A x^A W_0 \phi_\alpha) - \left[(\sum_A \omega_A) + \lambda_\alpha \right] W_0 \phi_\alpha .
\end{aligned}
$$

Therefore $W_0(x)\,\phi_\alpha(x)$ is a total derivative at $s' = 0$. It follows that $\mathcal{A}^{(k)}(s') \equiv 0$ for $k \geq 0$. Now take

$$
g_0(z, s') = I + \sum_j \sum_{k=0}^{\infty} z^k s'_j \mathcal{B}_j^{(k)}(s' = 0) + O((s')^2)
$$

where I is the identity matrix. This yields the desired gauge for all values of z but with $s' = 0$. As described above, the Birkhoff theorem then guarantees that it can be done in a region about $s' = 0$. Also note that $g_0(z, s' = 0) = I$.

To see that the solution can be made quasihomogeneously, consider the differential equation that needs to be solved:

$$
\begin{aligned}
g_0^{-1} \mathcal{A} g_0 - g_0^{-1} \partial_z g_0 &= z^{-2} P_{-2}(s') + z^{-1} P_{-1}(s') \\
g_0^{-1} \mathcal{B}_j g_0 - g_0^{-1} \partial_{s_j} g_0 &= z^{-2} Q_j(s')
\end{aligned}
$$

where P_{-1}, P_{-2} and Q_j are unknowns. This means that we must solve:

$$
\partial_{s_j} g_0 = g_0 \left[g_0^{-1} \mathcal{B}_j g_0 \right]_+ \tag{A2}
$$

where $[\ \ ]_+$ means: take only the non-negative powers of z in a power series expansion about $z = 0$. The fact that the system (A2) is integrable follows from the general observations above. (It could probably be proved more directly.) If g_0 is known to n-th order in s', then (A2) determines the $(n+1)$-st order term. Thus one

can evolve a power series in s'. (The convergence of this power series is guaranteed by the Cauchy-Kowalewski theorem.) It is elementary to see that quasihomogeneity is preserved order by order. Finally, it is of some interest to know over what size of patch one can find the required gauge. Generically one finds [32] that this gauge choice can be made in a large Schubert cell of the underlying Lie group. That is, one expects that $g_0(z; s')$ will become singular when one runs into a Weyl point of the underlying Lie group.

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