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Two-loop renormalization of multiflavor ϕ^3 theory in six dimensions and the trace anomaly

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We use the background-field method and the heat kernel to obtain all counterterms to two-loop order of conformally-coupled multiflavor ϕ^3 theory in six spacetime dimensions, defined in curved spacetime and with spacetime-dependent couplings. We also include spacetime-dependent mass terms for completeness. We use these results to write a general expression for the trace anomaly. With the use of Weyl consistency conditions we are able to show that the strong a-theorem for a certain natural candidate quantity $\tilde{a}$ is violated in this theory, and obtain a three-loop expression for the coefficient a of the Euler term in the anomaly.

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Contents

1. Introduction

Classical field theories that are invariant under scale and special conformal transformations generally fail to retain these symmetries once quantized. Famously there is an anomaly, that is, the trace of the stress-energy tensor does not vanish, signaling a violation of invariance by rescalings. The exception consists of a class of quantum field theories for which the trace vanishes, known as conformal field theories (CFTs). When this happens not only is scale invariance restored, but the theory is also symmetric under the full group of conformal transformations [1]. This occurs at the fixed points of the renormalization group (RG) flow.

It is also of interest to put the quantum field theory of interest on a curved background. When quantizing such a theory there are trace anomalies even at the fixed points of the RG flow of the corresponding flat-space theory [2]. These anomalies are given by a diffeomorphism-invariant local function involving derivatives of the metric. In d dimensions, there are a finite number of contributions of mass dimension d; for each there is a coefficient which is a function of the couplings. These coefficients are often of interest. Most notably, the coefficient a of the d-dimensional Euler

density is Cardy's proposed extension [3] of the central charge c of two-dimensional CFTs, whose monotonicity properties under RG flow were understood by Zamolodchikov [4].

These coefficients, some of which are the central charges of the theory, are well-understood at the fixed points but are also defined along the RG flow. In two dimensions, a suitable extension of c away from fixed points may be defined, called $\tilde{c}$, which is a function of the couplings, so one may speak of their values along the RG flow in a sensible fashion. It is this quantity, $\tilde{c}$, that has the interesting properties that it decreases monotonically along RG flows and is stationary at fixed points where it takes the numerical value of the central charge c of the CFT corresponding to the fixed point.

Given such remarkable properties of $\tilde{c}$, it is natural to ask whether such a quantity exists in the more physically interesting four-dimensional case. In fact, Weyl consistency conditions [5, 6] identify a quantity $\tilde{a}$ in even spacetime dimensions that make it the one possible candidate for a generalization of Zamolodchikov's $\tilde{c}$ to higher dimensions. In four dimensions it was shown by Jack and Osborn [7] that this quantity is stationary at fixed points where it reduces to the coefficient a of the Euler term. Moreover, using perturbation theory they showed that this quantity is monotonically decreasing towards the IR. More specifically, they gave an equation for the RG flow of $\tilde{a}$ that implies its monotonicity if a certain symmetric tensor, or "metric" in theory space parametrized by the couplings of the theory, is positive-definite. They then showed in an explicit perturbative calculation that this metric is in fact positive-definite for small couplings. More recently, positivity of this metric has been established in conformal perturbation theory [8, 9].

The extension of the quantity $\tilde{a}$ to six dimensions was computed by a set of the current authors in [6], and furthermore was shown to have a natural definition in any even-dimensional spacetime as a consequence of the Weyl consistency conditions and the existence of a generalization of the Einstein tensor, along with a metric on the space of couplings that is analogous to that of Jack and Osborn and Zamolodchikov. This generalization of $\tilde{a}$ is stationary at fixed points and reduces to a there. However, surprisingly, in [10] we showed by explicit computation in perturbation theory for a theory of scalars with a cubic self-coupling that the metric is negative-definite, and so $\tilde{a}$ monotonically increases in the flow out of the trivial UV fixed point. Adding to this surprise, in [9] it was found that in a model with two-forms in six dimensions the metric is positive-definite. It seems that, even in perturbation theory, there is no straightforward generalization of the a-theorem in six dimensions, at least as envisioned in the cases so far. As explained in [9], this may be attributed to the fact that in six dimensions the trace anomaly on a conformal manifold defines three independent symmetric tensors on the space of couplings, only one of which satisfies positivity properties. This positive-definite tensor is, however, not the tensor that appears in the RG equation for $\tilde{a}$, and thus the monotonicity of its flow remains undetermined. Contrary to this, in two and four dimensions there is a unique symmetric tensor with established positivity properties that also appears in the RG equation for $\tilde{c}$ or $\tilde{a}$.

It is not known beyond perturbation theory whether flows of $\tilde{a}$ in four dimensions are monotonic. However, there is another approach to the a-theorem that does not follow the previous lines of computation that uses unitarity of scattering processes in dilaton effective theories to establish positivity. Komargodski and Schwimmer have argued [11] without recourse to perturbation theory that the value of a on the UV fixed point is larger than that at the IR fixed point.[1] A similar argument considered the same question in six dimensions [13]; however it was not possible to reach a conclusion with the same methods as Komargodski and Schwimmer. Perhaps related in a general way to the difficulties encountered in [13], we note that the (massless) scalar model with cubic interactions investigated in [10] has only a single Gaussian (trivial) fixed point within the domain of validity of the perturbative calculation, so the difference between the values of a in the UV and IR cannot be contemplated. Non-perturbative CFTs are known to exist in six dimensions, but since in addition to being non-perturbative they are non-Lagrangian CFTs, little is known about flows between them.

As is clear from our discussion so far, the situation in six dimensions is significantly more complicated than that in two and four dimensions. We believe it would be useful to gain as much information as possible about the perturbative behavior of six-dimensional theories, beyond the computation of the quantity $\tilde{a}$. With this motivation in mind, we compute in this work, at two loops, the infinite part of the effective action and the trace anomaly in multiflavor ϕ^3 theory in six dimensions, including, for completeness of the analysis, the possibility that scale invariance is explicitly broken classically by a mass term. In addition to computing in a curved background (with a spacetime-dependent metric) we take the couplings to also have spacetime dependence. In effect this allows us to study the renormalization of correlators of operators that appear in the Lagrangian. With spacetime-dependent couplings counterterms proportional to derivatives of the couplings are required for finiteness. Correspondingly, the trace anomaly includes terms that contain derivatives not just of the metric but also of the couplings. The anomalies associated with these terms manifest themselves in the original model (with spacetime-independent metric and couplings) as coefficients of terms in the Greens' functions of composite operators (including the stress-energy tensor and its trace).

Given all these considerations, the focus of this paper is the Lagrangian

$$\mathscr{L} = \tfrac{1}{2}\big(\partial_\mu \phi_i\, \partial_\nu \phi_i\, \gamma^{\mu\nu} + (\xi_{ij} R + m_{ij})\phi_i \phi_j + h_i \phi_i\big) + \tfrac{1}{3!} g_{ijk}\phi_i \phi_j \phi_k\,, \tag{1.1}$$

of scalar fields ϕ^i, defined on a six-dimensional manifold with metric $\gamma^{\mu\nu}$, where the repeated lowercase Latin flavor indices are to be summed over regardless of their position. This Lagrangian is of interest because it is the only general interacting theory that has classical scale invariance

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[1]This is a statement of the weak version of the a-theorem in four dimensions, where it is only established that $a_{\mathrm{UV}} > a_{\mathrm{IR}}$. The strong version of the a-theorem states that $\tilde{a}$ decreases monotonically along the flow. The strongest version of the a-theorem states that this flow is a gradient flow [12].

(for the appropriate choice of ξ_{ij} and zero m_{ij} and h_i) in six dimensions.[2] Of course, one may consider theories that do not have Lagrangian descriptions in six dimensions, as mentioned above, but then the calculational methods of this paper are of no use and, typically, one must resort to holographic methods. With (1.1) we may proceed in the old-fashioned ways of perturbation theory and reliably calculate the quantities of interest order by order in g_{ijk}. This is the starting point of this paper but first we must establish how such computations are performed on curved backgrounds. We should note that our results are reported here with the choice $\xi_{ij} = \frac{1}{5}\delta_{ij}$ classically, with δ_{ij} the Kronecker delta, as found from the general result $\xi_{ij} = \frac{d-2}{4(d-1)}\delta_{ij}$ in d dimensions for conformal coupling of the scalar.

The main computational method used in this work was developed and applied to various cases in four dimensions by Jack and Osborn in [15–18]. The main ingredients are the background field method and the heat kernel in dimensional regularization. In ϕ^3 theory in six dimensions with a single scalar field and with spacetime-independent couplings, results for the two-loop effective action have been obtained in [19–21]; we have checked our results for some quantities against those listed in these references.

The layout of this paper is as follows. In the next section we describe in detail the (perhaps unfamiliar but very powerful) computational method of Jack and Osborn. In section 3 we describe briefly the Weyl consistency conditions in order to make contact between our computations of the effective action and the a-theorem. In section 4 we present our results for the infinite part of the effective action at two loops, and in section 5 we extract from those the two loop beta function and anomalous dimension. Finally, in section 6 we present results relevant to the a-theorem in six dimensions to three-loop order. Our conventions as well as various details and results needed for our computations are contained in three appendices.

2. Method of calculation

In this section we outline the method of calculation employed in this paper. For more details the reader is referred to [15–18], where such computations have been thoroughly explained and demonstrated. Until section 4 we assume for simplicity that no relevant parameters are present, for example $m_{ij} = 0$ and $h_i = 0$ in (1.1).

In this work we will study quantum field theories defined in spacetime dimension $d = D - \epsilon$, with D an integer, by a set of couplings g'^I and fields ϕ'. For our computations we will use dimensional regularization and make explicit the mass dimension of the renormalized parameters

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[2]One may object that the ϕ^3 theory is sick because of its potential, which is unbounded from below. However, within the context of perturbation theory, which is the scope of this paper, the ground state $\langle\phi(x)\rangle = 0$ is stable to fluctuations of $\phi(x)$ [14].

via

$$g'^I = \mu^{k^I \epsilon} g^I \,, \qquad \phi' = \mu^{\delta \epsilon} \phi \,, \tag{2.1}$$

for some numbers k^I and δ and where the index I labels the operators in the interaction Lagrangian, i.e. $I = (ijk)$ in (1.1).[3] Though we start the perturbative calculations with g'^I and ϕ', we will use (2.1) to express the resulting formulas in terms of the fields ϕ and the dimensionless couplings g^I. Then, with minimal subtraction, we have the bare parameters

$$g_0^I = \mu^{k^I \epsilon}(g^I + L^I(g)) \,, \qquad \phi_0 = \mu^{\delta \epsilon} Z^{1/2}(g) \phi \,, \tag{2.2}$$

with L^I and $Z^{1/2} - 1$ containing just poles in ϵ. In general $Z^{1/2}$ is a matrix to account for the multiple number of fields in (1.1). The beta function and anomalous dimension are given by

$$\hat{\beta}^I \equiv \mu \frac{dg^I}{d\mu} = -k^I g^I \epsilon + \beta^I \qquad \text{and} \qquad \hat{\gamma} \equiv \delta \epsilon + Z^{-1/2} \mu \frac{dZ^{1/2}}{d\mu} = \delta \epsilon + \gamma \,, \tag{2.3}$$

respectively, where β and γ are the quantum beta function and anomalous dimension respectively.

Now, in quantum field theory in flat spacetime, wavefunction and coupling renormalization are enough to render finite correlation functions involving fundamental fields. When correlation functions involving composite operators are included, further counterterms are necessary. A convenient way to deal with these is by introducing sources for the composite operators, and including counterterms proportional to spacetime derivatives on those sources. For operators that appear in the Lagrangian it is enough to take their couplings as spacetime-dependent sources, $g^I \to g^I(x)$, and introduce counterterms proportional to derivatives on $g^I(x)$ [7,22]. Finally, when a flat-space field theory is lifted to curved space with metric $\gamma_{\mu\nu}$, and the regularization procedure respects diffeomorphism invariance, new divergences proportional to the curvatures defined from $\gamma_{\mu\nu}$ appear, and thus further counterterms involving the curvatures are required for finiteness.

In [7] a systematic treatment of such effects was undertaken, and a general expression for the Lagrangian in the presence of the sources $\gamma_{\mu\nu}(x)$ and $g^I(x)$ was proposed, namely

$$\tilde{\mathscr{L}}_0 = \mathscr{L}_0 - \mu^{-\epsilon} \lambda \cdot \mathscr{R} + \mu^{-\epsilon} \mathscr{F} \,, \tag{2.4}$$

where $\lambda \cdot \mathscr{R}$ includes all field-independent counterterms, proportional only to curvatures and derivatives on $g^I(x)$, and $\mathscr{F} = \mathscr{F}(\phi)$ includes all field-dependent counterterms that also depend on curvatures and derivatives on $g^I(x)$. $\mathscr{L}_0$ is the bare Lagrangian, expressed in terms of g and ϕ with the use of (2.2), that contains terms that survive in flat space when the couplings are taken to be spacetime independent. It obeys the Callan–Symanzik equation

$$\left(\hat{\beta}^I \frac{\partial}{\partial g^I} - (\hat{\gamma}\phi) \cdot \frac{\partial}{\partial \phi} - \epsilon \right) \mathscr{L}_0 = 0 \,. \tag{2.5}$$

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[3]Note that the index carried by k of (2.1) is not subject to the summation convention.

The RGE one finds from (2.4) is

$$\left(\hat{\beta}^I \frac{\partial}{\partial g^I} - (\hat{\gamma}\phi) \cdot \frac{\partial}{\partial \phi} - \epsilon\right) \tilde{\mathscr{L}}_0 = \mu^{-\epsilon} \left(\beta_\lambda \cdot \mathscr{R} + \left(\hat{\beta}^I \frac{\partial}{\partial g^I} - (\hat{\gamma}\phi) \cdot \frac{\partial}{\partial \phi} - \epsilon\right) \mathscr{F}\right), \qquad (2.6)$$

which, by (2.4) and the Callan–Symanzik equation (2.5), requires

$$\left(\epsilon - \hat{\beta}^I \frac{\partial}{\partial g^I}\right) \lambda \cdot \mathscr{R} = \beta_\lambda \cdot \mathscr{R}, \qquad (2.7)$$

and similarly for the $\mathscr{F}(\phi)$ terms, though there is an additional derivative with respect to the fields. As explained in [7] and we will review in the following, the terms $\beta_\lambda \cdot \mathscr{R}$ defined by (2.7) contribute, among others, to the trace anomaly of the theory in curved space.

It is important to emphasize that in specific theories with possible relevant parameters like (1.1), the RGE (2.6) is incomplete. For example, it does not correctly reproduce higher-order poles in higher-loop computations, even if the relevant parameters are set to zero in the classical Lagrangian. This issue has been analyzed in detail in [7] for four-dimensional theories, and also in [9] for (1.1). While it does not affect our discussion below, it should be kept in mind.

2.1. Background field method

In this subsection we will give a brief overview of the background field method. We will present our expressions for the case of a single scalar field ϕ, although the generalization to multiple fields and fields with spin is well-known. Our motivation for using the background field method is that it allows us to compute perturbatively counterterms like $\lambda \cdot \mathscr{R}$ in (2.4) in a straightforward way.

In the background field method one simply computes the effective action starting from $\mathscr{L}_0$, which thus dictates the form of the counterterms. More specifically, we start by splitting the field ϕ into an arbitrary classical background part ϕ_b and a quantum fluctuation f,

$$\phi = \phi_b + f. \qquad (2.8)$$

We can also introduce a source J, and obtain the effective action $W[\phi_b, J]$ (the generating functional of connected graphs with implicit $\gamma_{\mu\nu}(x)$ and $g^I(x)$ dependence) after we integrate out f:

$$e^{W[\phi_b, J]} = \int Df \, e^{-\tilde{S}_0[\phi] + \int d^d x \sqrt{\gamma} \, J(x) f(x)}, \qquad \tilde{S}_0 = \int d^d x \sqrt{\gamma} \, \tilde{\mathscr{L}}_0, \qquad (2.9)$$

where γ is the determinant of the metric $\gamma_{\mu\nu}$, which is not to be confused with the anomalous dimension γ.

To continue, let us denote by $S^{(0)}$ the action without any counterterms. Then, we expand $S^{(0)}[\phi]$ in fluctuations,

$$S^{(0)}[\phi] = S^{(0)}[\phi_b] + \int d^d x \sqrt{\gamma} \left.\frac{\delta S^{(0)}}{\delta \phi}\right|_{\phi_b} f + \frac{1}{2} \int d^d x \sqrt{\gamma} \, f M f + S_{\text{int}}[f], \qquad (2.10)$$

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where $M = -\nabla^2 + d^2V/d\phi^2\big|_{\phi=\phi_b}$, with V the potential in $\mathscr{L}$. Then, by expanding (2.9) we find that, at the zeroth order,

$$W^{(0)}[\phi_b] = -S^{(0)}[\phi_b]\,, \tag{2.11}$$

and at the one-loop order (a superscript in parentheses indicates the loop order),

$$W^{(1)}[\phi_b] = -\tilde{S}_0^{(1)}[\phi_b] - \tfrac{1}{2}\ln\det M\,, \tag{2.12}$$

after we choose J appropriately in order to cancel terms linear in f, order by order in perturbation theory starting with (2.10), and subsequently perform in (2.9) the Gaussian integral over f. Here, $\tilde{S}_0^{(1)}$ contains poles in ϵ to cancel those in the $-\tfrac{1}{2}\ln\det M$ piece; in particular it contains the one-loop contributions to $Z^{1/2}$ and L of (2.2), which are chosen to absorb the associated infinities coming from $-\tfrac{1}{2}\ln\det M$ so that $W^{(1)}$ is finite. In addition, with the extension (2.4) it is clear from (2.12) that $\tilde{S}_0^{(1)}$ also contains the one-loop contribution to $\lambda\cdot\mathscr{R}$ that is given by the negative of the appropriate simple-pole part of $-\tfrac{1}{2}\ln\det M$:

$$\int d^d x \sqrt{\gamma}\,\mu^{-\epsilon}\lambda^{(1)}\cdot R \subset -(-\tfrac{1}{2}\ln\det M)^{\text{pole}}\,. \tag{2.13}$$

Then, from (2.7) and (2.13) we can evaluate $\beta_\lambda^{(1)}\cdot\mathscr{R}$. Of course, $-\tfrac{1}{2}\ln\det M$ also contains field-dependent terms that require the counterterms $\mathscr{F}^{(1)}$ for finiteness.

At higher loops the interaction term $S_{\text{int}}[f]$ in (2.10) is considered and vacuum bubble diagrams as well as diagrams with counterterm insertions are constructed.[4] The counterterms are of course fixed here by the previous loop order, i.e. by $\tilde{S}_0^{(1)}$. These diagrams can be evaluated in position space, using coincident limits of propagators according to the diagram topology. With these methods, which are explained thoroughly in the following, no loop integrations need to be performed. If we denote by $\mathscr{S}^{(2)}$ the contribution of all such diagrams, we find

$$W^{(2)}[\phi_b] = -\tilde{S}_0^{(2)}[\phi_b] + \mathscr{S}^{(2)}\,. \tag{2.14}$$

Again, finiteness of $W^{(2)}$ allows us to determine all counterterms in $\tilde{S}_0^{(2)}$. From the simple poles in $\lambda^{(2)}\cdot\mathscr{R}$ it is again straightforward to evaluate $\beta_\lambda^{(2)}\cdot\mathscr{R}$ using the RGE (2.7). Clearly these computations can be carried out order by order in perturbation theory.

2.2. Heat kernel

Using heat-kernel techniques the evaluation of $(-\tfrac{1}{2}\ln\det M)^{\text{pole}}$ and higher loop poles may be accomplished. A pedagogical explanation of the method may be found in [24]; we will mainly follow the procedure as laid out in [19], where it was used for single flavor ϕ^3 theory without

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[4]As explained in [23], one of the advantages of computations done in the background field method is that only vacuum bubble diagrams need be considered order by order in perturbation theory.

spacetime-dependent couplings. A nice review of the heat kernel method and its applications can be found in [25].

The object of central importance in the heat kernel method is the propagator function in the presence of a background field as presented in section 2.1. It obeys the identity

$$M_{ik} G_{kj}(x, x') = \delta_{ij} \delta^d(x, x'), \tag{2.15}$$

where the indices are the flavor indices of the theory (1.1), δ_{ij} is the Kronecker delta, the d-dimensional biscalar delta function is defined by

$$\int d^d x' \sqrt{\gamma'} \, \delta^d(x, x') \phi(x') = \phi(x), \qquad \gamma' = \gamma(x'), \tag{2.16}$$

and M_{ij} is the elliptic differential operator, evaluated at the point x, alluded to in the previous section and defined by (1.1) in our case of interest, having the general form

$$M_{ij} = -\delta_{ij} \nabla^2 + \left. \frac{\partial^2 V(\phi)}{\partial \phi^i \, \partial \phi^j} \right|_{\phi = \phi_b}. \tag{2.17}$$

The key to evaluating the determinant in the one-loop effective potential (2.12) and the higher order diagrams, which involve integrals over products of $G_{ij}(x, x')$, is to present $G_{ij}(x, x')$ in a way amenable to computation.

The heat kernel provides such amenities. First, we define the heat kernel $\mathscr{G}_{ij}$ by the equation

$$\left(\delta_{ik} \frac{\partial}{\partial t} + M_{ik} \right) \mathscr{G}_{kj}(x, x'; t) = 0, \tag{2.18}$$

with

$$\mathscr{G}_{ij}(x, x'; 0) = \delta_{ij} \delta^d(x, x'). \tag{2.19}$$

Formally, by virtue of (2.18), the heat kernel may then be written

$$\hat{\mathscr{G}}(t) = e^{-\hat{M}t} = \sum_n e^{-\lambda_n t} |\psi_n\rangle \langle \psi_n|, \tag{2.20}$$

with λ_n the eigenvalues of $\hat{M}_{ij}$ and the hats emphasizing that no particular basis of eigenstates $|\psi_n\rangle$ for the elliptic differential operator $\hat{M}_{ij}$ need be chosen (however, the position basis will recover our calculations). $G_{ij}(x, x')$ may be then written as

$$G_{ij}(x, x') = \int_0^\infty dt \, \mathscr{G}_{ij}(x, x'; t). \tag{2.21}$$

From the heat kernel $\mathscr{G}_{ij}(x, x'; t)$ the one-loop effective action is obtained through the well-established zeta-function method (elaborated in, e.g., [17] or [24]), which relates $-\frac{1}{2} \ln \det M$ to the heat kernel; to do so, an Ansatz for the form of the heat kernel must be given. This is suggested

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from the solution in flat space for the heat equation (2.18) and was given by DeWitt [26] for a small t expansion:

$$\mathscr{G}_{ij}(x, x'; t) = \frac{\Delta_{\mathrm{VM}}^{1/2}(x, x')}{(4\pi t)^{d/2}} e^{\sigma(x,x')/2t} \sum_{n=0}^{\infty} a_{n,ij}(x, x') t^n, \qquad a_{0,ij}(x, x) = \delta_{ij}, \qquad (2.22)$$

with $a_{n,ij}(x, x')$ the so-called Seeley–DeWitt coefficients and where $\sigma(x, x')$ is the biscalar distance-squared measure (called the geodetic interval by DeWitt),

$$\sigma(x, x') = \tfrac{1}{2} \left(\int_0^1 d\lambda \sqrt{\gamma_{\mu\nu} \frac{dy^\mu}{d\lambda} \frac{dy^\nu}{d\lambda}} \right)^2, \qquad y(0) = x, \ y(1) = x', \qquad (2.23)$$

with $y(\lambda)$ a geodesic. $\Delta_{\mathrm{VM}}(x, x')$ is another biscalar, called the van Vleck–Morette determinant, that describes the spreading of geodesics from a point, defined by

$$\Delta_{\mathrm{VM}}(x, x') = \gamma(x)^{-1/2} \gamma(x')^{-1/2} \det\left(-\frac{\partial^2}{\partial x^\alpha \partial x'^\beta} \sigma(x, x') \right). \qquad (2.24)$$

We shall suppress the x, x' dependence of σ, Δ_{VM}, and $a_{n,ij}$ henceforth. Now, the Ansatz (2.22) obeys (2.18) which yields the recursion relation

$$n a_{n,ij} + \partial_\mu \sigma \, \partial^\mu a_{n,ij} = -\Delta_{\mathrm{VM}}^{-1/2} M_{ik}\big(\Delta_{\mathrm{VM}}^{1/2} a_{n-1,kj}\big) \qquad \text{with} \qquad \partial_\mu \sigma \, \partial^\mu a_{0,ij} = 0, \qquad (2.25)$$

which allows us to compute the Seeley–DeWitt coefficients.

With the asymptotic expansion of the propagator via the heat kernel established in (2.22), its practicality in loop computations becomes evident. To elaborate on the comments above (2.22), at one loop one wishes to calculate the determinant in (2.12). This may accomplished by considering the so-called zeta function for the operator M_{ij},

$$\zeta_M(s) = \frac{1}{\Gamma(s)} \int_0^\infty dt \, t^{s-1} \int d^d x \sqrt{\gamma} \, \mathscr{G}_{ii}(x, x; t). \qquad (2.26)$$

This function is useful to define because then the log of the determinant may be computed by differentiating it with respect to s and sending s to zero, which may be seen by considering the formal definition of $\mathscr{G}_{ij}(t)$ in (2.20); this yields

$$-\ln \det M = \lim_{s \to 0} \frac{d\zeta_M}{ds} = \int_0^\infty \frac{dt}{t} \int d^d x \sqrt{\gamma} \, \mathscr{G}_{ii}(x, x; t). \qquad (2.27)$$

Given the formal definition in (2.20), the value of $\ln \det M = \sum_n \ln \lambda_n$ may computed with the equivalent of (2.26) for $\hat{\mathscr{G}}$, with $\mathrm{Tr}\,\hat{\mathscr{G}}(t) = \sum_n e^{-\lambda_n t}$. Explicitly evaluating ζ_M as a function of s, differentiating with respect to s and then taking the limit as $s \to 0$ reproduces the log of the determinant, formally, up to a minus sign.

Actually, the equality in (2.27) is only true up to the residue of a pole in s as $s \to 0$ and equation (2.27) is a bit misleading at face value. Following [17], $\lim_{s \to 0} \zeta'_M(s)$ is of the form

$$\lim_{s \to 0} \frac{d\zeta_M}{ds} = \lim_{s \to 0} \left(\int_0^\infty dt \, t^{s-1} \int d^d x \sqrt{\gamma} \, \mathscr{G}_{ii}(x, x; t) \; - \; \frac{P}{s} \right), \qquad (2.28)$$

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with P the residue of the integral inside the parentheses as $s \to 0$. However, in dimensional regularization this pole is displaced—this may be seen by noting the d dependence of the power series in t in (2.22). Hence, in dimensional regularization, P may be set to zero and we recover the ϵ-dependent determinant

$$(-\ln \det M)_{\text{dim. reg.}} = \lim_{s \to 0} \frac{d\zeta_M}{ds}\bigg|_{d=D-\epsilon} = \int_0^\infty \frac{dt}{t} \int d^d x \sqrt{\gamma}\, \mathscr{G}_{ii}(x,x;t)\,, \qquad (2.29)$$

with D the integer dimension of spacetime, justifying the assertion in (2.27). Its pole, which is the main interest to us, may then be calculated with (2.22) and by noting the coincident limits therein, where $x' \to x$. If D is even, then by expanding the series in (2.22) with $d = D - \epsilon$, it can be seen that the only piece that contains a pole in ϵ as $s \to 0$ is the $(D/2)$-th piece. Hence, the object of concern for the pole of the effective action is the coincident limit of the Seeley–DeWitt coefficient $a_{D/2, ii}$. In six dimensions, in particular, we then have

$$\left(-\tfrac{1}{2} \ln \det M\right)^{\text{pole}}_{\text{dim. reg.}} = \frac{\mu^{-\epsilon}}{64\pi^3} \frac{1}{\epsilon} \int d^d x \sqrt{\gamma}\, [a_{3, ii}](x)\,, \qquad (2.30)$$

where the $\mu^{-\epsilon}$ is inserted to preserve mass dimensions. The coincident limit of $a_{3, ii}(x, x')$, denoted in (2.30) by the brackets, may be found in appendix B, equation (B.12), and subsequently used to evaluate the one-loop counterterms of $\tilde{S}^{(1)}$ of (2.12).

The task is then to extend the relatively graceful computation of the one-loop effective action, *à la* (2.29), to higher loop order. When using the heat kernel method in the context of the background field method, two-loop and higher-order contributions to the effective action are encompassed entirely within the calculation of vacuum bubble diagrams. These are then evaluated in coordinate space by integrations over the spacetime points involved in the loop diagram of the products of Green's functions. It then becomes convenient, now specifying $d = 6 - \epsilon$, to express the Green's function through the expansion (2.22) and (2.21) which, after performing the integration, yields

$$\begin{aligned}
G_{ij}(x, x') &= G_0(x, x')\, a_{0, ij}(x, x') + G_1(x, x')\, a_{1, ij}(x, x') \\
&\quad + R_2(x, x')\, a_{2, ij}(x, x') + R_3(x, x')\, a_{3, ij}(x, x') + H(x, x')\,,
\end{aligned} \qquad (2.31)$$

where the $H(x, x')$ term does not contribute to UV divergences of the theory, i.e. do not have divergent behavior as $x' \to x$.[5] The utility of this expansion is that it allows extraction of the poles of higher-loop diagrams almost by inspection, once the $G_n(x, x')$ and $R_n(x, x')$ are computed with (2.21). For example, in the two-loop case, whose computation is detailed in section 4, the

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[5]It should be noted, however, that these still have IR-divergent behavior. Although it is not of interest in the calculation in this paper, it may be taken care of by considering the log of the ratio of the determinant of interest, equation (2.29), with determinant of the non-interacting operator $M_{ij}^{(0)} = -\delta_{ij}\nabla^2$. The ratio acts like a normalized version of (2.30) and removes uninteresting IR divergences. See [15, section 1] for details.

coincident limit is necessary to evaluate the contribution there, so the computation boils down to the coincident limits of the Seeley–DeWitt coefficients, tabulated in the appendices, times coincident limits of the $G_n(x, x')$ and $R_n(x, x')$, which are easily computed with knowledge of the coincident limits of $\sigma(x, x')$ and $\Delta_{\mathrm{VM}}(x, x')$, also tabulated in the appendices. Furthermore, although $G_{ij}(x, x')$ must be finite when $x \neq x'$ in 6 or $6 - \epsilon$ dimensions, products of the $G_n(x, x')$ and $R_n(x, x')$ may have poles in ϵ. In this two loop case the cubic product of $G_{ij}(x, x')$ is necessary, as evinced in Fig. 1 and (4.8), and the various products of the pieces of (2.31) give rise to the poles computed in section 4.

2.3. Trace anomaly

Now that the heat kernel method for the computation of the poles of the effective action has been established, we can proceed to the computation of the trace of the stress energy tensor defined by (1.1). To study the trace it is useful to promote the metric and couplings to spacetime dependent sources

$$\gamma^{\mu\nu} \to \gamma^{\mu\nu}(x), \qquad g^I \to g^I(x), \tag{2.32}$$

and subsequently promote the action of the theory to be diffeomorphism invariant. Then we can define the quantum stress-energy tensor and finite composite operators by functionally differentiating with respect to these sources:

$$T_{\mu\nu}(x) = 2\frac{\delta \tilde{S}_0}{\delta \gamma^{\mu\nu}(x)}, \qquad [\mathcal{O}_I(x)] = \frac{\delta \tilde{S}_0}{\delta g^I(x)}, \tag{2.33}$$

where functional derivatives are defined in d spacetime dimensions by

$$\frac{\delta}{\delta \gamma^{\mu\nu}(x)} \gamma^{\kappa\lambda}(x') = \delta_{(\mu}{}^{\kappa} \delta_{\nu)}{}^{\lambda} \, \delta^d(x, x'), \qquad \frac{\delta}{\delta g^I(x)} g^J(x') = \delta_I{}^J \delta^d(x, x'), \tag{2.34}$$

with $X_{(I}Y_{J)} \equiv \frac{1}{2}(X_I Y_J + X_J Y_I)$. With these definitions it is easy to see that

$$\gamma^{\mu\nu} T_{\mu\nu} = \epsilon \tilde{\mathscr{L}}_0 + \nabla_\mu I^\mu - (\Delta\phi) \cdot \frac{\delta}{\delta\phi} \tilde{S}_0, \tag{2.35}$$

where Δ is the canonical scaling dimension of ϕ and I^μ arises from variations of curvature-dependent terms in $\tilde{\mathscr{L}}_0$.

The trace anomaly may be viewed as the theory's response to the local Weyl rescalings

$$\gamma^{\mu\nu}(x) \to \big(1 + 2\sigma(x)\big)\gamma^{\mu\nu}(x), \qquad g^I(x) \to g^I(x) + \sigma(x)\hat{\beta}^I(x). \tag{2.36}$$

The scalar $\sigma(x)$ here is a variational parameter and should not be confused with the biscalar geodetic interval $\sigma(x, x')$ of the previous section. At the level of the generating functional we can implement these infinitesimal local Weyl transformations with the generators

$$\Delta_\sigma^{\mathrm{W}} = 2 \int d^d x \sqrt{\gamma}\, \sigma \gamma^{\mu\nu} \frac{\delta}{\delta\gamma^{\mu\nu}}, \qquad \Delta_\sigma^{\hat{\beta}} = \int d^d x \sqrt{\gamma}\, \sigma \hat{\beta}^I \frac{\delta}{\delta g^I}. \tag{2.37}$$

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With these definitions,

$$\Delta_\sigma^{\mathrm{W}} W = - \int d^d x \sqrt{\gamma}\, \sigma \gamma^{\mu\nu} \langle T_{\mu\nu} \rangle \,, \qquad \Delta_\sigma^{\hat{\beta}} W = - \int d^d x \sqrt{\gamma}\, \sigma \langle \hat{\beta}^I [\mathcal{O}_I] \rangle \,. \tag{2.38}$$

Now, it is easy to see that the term $\epsilon \tilde{\mathscr{L}}_0$ in (2.35) can be substituted with the use of (2.6), and so (2.35) can be written in the form

$$\gamma^{\mu\nu} T_{\mu\nu} - \hat{\beta}^I [\mathcal{O}_I] \supset -\mu^{-\epsilon} (\beta_\lambda \cdot \mathscr{R} - \nabla_\mu Z^\mu), \tag{2.39}$$

where Z^μ is the part of I^μ of (2.35) that contains field-independent terms.[6] In (2.39) we neglect field-dependent contributions besides those in $\beta^I [\mathcal{O}_I]$. Equivalently, we can write

$$\Delta_\sigma^{\mathrm{W}} W - \Delta_\sigma^{\hat{\beta}} W \supset \int d^d x \sqrt{\gamma}\, \sigma \mu^{-\epsilon} \beta_\lambda \cdot \mathscr{R} + \int d^d x \sqrt{\gamma}\, \partial_\mu \sigma \, \mu^{-\epsilon} Z^\mu. \tag{2.40}$$

Terms in the right-hand side of (2.39) have been computed in [7] for field theories in $d = 4$. In this work we will compute such terms for general multiflavor ϕ^3 field theories in $d = 6$. As we just saw, these computations give results on the various terms that appear in the consistency conditions derived from (2.40) [6].

Thus the relevant contributions to the trace of the stress energy tensor have their origin in the $\lambda \cdot \mathscr{R}$ terms, which are, in turn, obtained from the heat kernel methods of the previous section. The $\beta_\lambda \cdot \mathscr{R}$ terms are computed from the $\lambda \cdot \mathscr{R}$ terms by (2.7), and the Z^μ terms are obtained from the Weyl variation $\delta_\sigma(-\lambda \cdot \mathscr{R})$. One can also change the basis so that terms in the variation $\delta_\sigma(-\lambda \cdot \mathscr{R})$ that appear in Z^μ in one basis appear in $\beta_\lambda \cdot \mathscr{R}$ in another and vice-versa.

3. Weyl consistency conditions

The trace anomaly as presented in (2.40) is useful because it allows very powerful statements about the structure of the theory along the renormalization group flow to be made. These statements arise from the Weyl consistency conditions, a specific example of the Wess–Zumino consistency conditions [27] that constrain the form of a quantum anomaly based upon the algebra of the anomalous symmetry group.

Consider the two generators acting on the connected diagram generating functional W in (2.40). We may take the commutator of their actions on W for two different variational parameters σ and σ' and, because the Weyl rescalings are Abelian, obtain the Weyl consistency condition

$$\left[\Delta_\sigma^{\mathrm{W}} - \Delta_\sigma^{\hat{\beta}}, \, \Delta_{\sigma'}^{\mathrm{W}} - \Delta_{\sigma'}^{\hat{\beta}} \right] W = 0 \,. \tag{3.1}$$

Now, the terms in W, namely those coming from (2.4), have complicated transformations under (2.37) and so (3.1) imposes a set of non-trivial constraints and relations among these terms.

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[6]Note that the field-dependent part of I^μ is responsible for $\mathcal{O}_I \to [\mathcal{O}_I]$, for the difference between $\partial/\partial g^I$ and $\delta/\delta g^I$ is a total derivative.

In particular, as argued in [5], $\lambda \cdot \mathscr{R}$ must contain all terms that are diffeomorphism-invariant and, by simple power-counting, of mass dimension d that might arise in addition to the usual operators in $\mathscr{L}$ in (2.4) from the promotion of the metric and couplings to spacetime-dependent sources as in (2.32). In two dimensions there are precisely three terms that fit the bill, with a single resulting consistency constraint. In four dimensions there are sixteen candidates, with seven independent consistency conditions. In six dimensions there are ninety five candidates with many independent consistency constraint equations.

While the two- and four-dimensional cases are rather tractable and admit relatively simple interpretations of the consistency conditions, the six-dimensional case is significantly more complex. The $\beta_\lambda \cdot \mathscr{R}$ and Z^μ terms as well as the consistency conditions in six dimensions were categorized in [6]. There, it was established that the most interesting consistency condition found in two and four dimensions, the consistency condition governing the flow of a certain a-function along the renormalization group flow, survives in six dimensions. The key point is to identify the analogs of the terms relevant to this consistency condition from two and four dimensions. There they involve the Euler density in the specific spacetime dimensionality and, in four dimensions, terms involving the Einstein tensor. In six dimensions, in fact, the coefficient a of the six-dimensional Euler term E_6 is related to terms involving the generalization of the Einstein tensor, the so-called Lovelock tensor [28], in a way that is almost[7] completely analogous to the two- and four-dimensional settings.

To be clear, in six dimensions (2.40) takes the form [6]

$$(\Delta_\sigma^{\mathrm{W}} - \Delta_\sigma^\beta)W = \sum_{p=1}^{65} \int d^6x \sqrt{\gamma}\, \sigma\, (\beta_\lambda \cdot \mathscr{R})_p + \sum_{q=1}^{30} \int d^6x \sqrt{\gamma}\, \partial_\mu \sigma\, \mathscr{Z}_q^\mu\,. \tag{3.2}$$

The terms of interest to the aforementioned consistency condition are contained therein

$$(\Delta_\sigma^{\mathrm{W}} - \Delta_\sigma^\beta)W \supset \int d^6x \sqrt{\gamma}\, \sigma\Big(-aE_6 - b_1L_1 - b_3L_3 + \tfrac{1}{2}\mathcal{H}_{IJ}^1\, \partial_\mu g^I \partial_\nu g^J H_1^{\mu\nu}\Big)$$
$$+ \int d^6x \sqrt{\gamma}\, \partial_\mu \sigma\, \mathcal{H}_I^1\, \partial_\nu g^I H_1^{\mu\nu}\,, \tag{3.3}$$

where $L_{1,3}$, given in Appendix A, are dimension-six curvature terms whose coefficients $b_{1,3}$ vanish at fixed points, $H_1^{\mu\nu}$ is the Lovelock curvature tensor, given in (A.3), and the coefficients $\mathcal{H}_{IJ}^1$ and $\mathcal{H}_I^1$ are functions of the coupling constants. By varying each piece in (3.3) with (2.37) and applying the Weyl consistency conditions (3.1) one obtains the constraint equation

$$\partial_I \tilde{a} = \tfrac{1}{6}\mathcal{H}_{IJ}^1 \beta^J + \tfrac{1}{6}(\partial_I \mathcal{H}_J^1 - \partial_J \mathcal{H}_I^1)\beta^J\,, \tag{3.4}$$

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[7]In six dimensions there are curvature terms with coefficients called "vanishing anomalies" whose only analog in lower dimensions is the R^2 term in four dimensions. At fixed points they vanish (hence the name) and the a in six dimensions is completely analogous to lower dimensions, however away from fixed points they modify a to $\tilde{a}$ (to be discussed shortly) in a way distinct from two and four dimensions.

with

$$\tilde{a} = a + \tfrac{1}{6}b_1 - \tfrac{1}{90}b_3 + \tfrac{1}{6}\mathcal{H}_I^1\beta^I\,. \tag{3.5}$$

This equation is analogous to those found by Osborn in two and four dimensions [5].

By contracting with β^I on each side of (3.4) we arrive at six-dimensional equivalent of Zamolodchikov's theorem from two dimensions [4],

$$\beta^I\partial_I\tilde{a} = \tfrac{1}{6}\mathcal{H}_{IJ}^1\beta^I\beta^J\,. \tag{3.6}$$

A similar relation was shown to hold in any even-dimensional spacetime in [6]. Thus, if we can compute $\mathcal{H}_{IJ}^1$ in our theory and establish a definite sign, the monotonicity of the renormalization group flow of the theory can be established by way of (3.6).

All that remains now is to determine the terms in the consistency conditions from the computation of the two-loop effective potential. The following section sets itself to this task.

4. Poles of the effective action

In this section we present our results for the pole part of the effective action up to two loops. For completeness we will also include here a mass term in our theory as well as a term linear in ϕ, i.e. we will take our Lagrangian to be given by

$$\mathscr{L}(\phi, g, m, h, \gamma) = \tfrac{1}{2}\big(\partial_\mu\phi_i\,\partial_\nu\phi_i\,\gamma^{\mu\nu} + (\xi_{ij}R + m_{ij})\phi_i\phi_j + h_i\phi_i\big) + \tfrac{1}{3!}g_{ijk}\phi_i\phi_j\phi_k\,, \tag{4.1}$$

where m_{ij} and h_i have mass-dimension two and four respectively. Then, (2.4) is modified to

$$\tilde{\mathscr{L}}_0 = \mathscr{L}_0 - \mu^{-\epsilon}\lambda\cdot\mathscr{R} + \mu^{-\epsilon}\mathscr{F} + \mu^{-\epsilon}\mathscr{M} \tag{4.2}$$

with $\mathscr{F} = \mathscr{F}(\phi, g, \gamma)$ and $\mathscr{M} = \mathscr{M}(\phi, g, m, \gamma)$. As in (2.4), in (4.2) all quantities are bare quantities that may be written in terms of the renormalized quantities via (2.2). As stated in section 2.2, at one loop the effective action is related to the Seeley–DeWitt coefficient $a_{D/2, ij}$ with D the even integer spacetime dimension. To wit, in the theory of (4.1) in six dimensions, as in (2.30),

$$(-\tfrac{1}{2}\ln\det M)^{\mathrm{pole}} = \frac{1}{\epsilon}\frac{\mu^{-\epsilon}}{64\pi^3}\int d^dx\sqrt{\gamma}\,[a_{3, ii}](x)\,. \tag{4.3}$$

This result produces all terms in (4.2). Thus, at one loop, using the result from appendix B, discarding total derivatives, and choosing $\xi_{ij} = \big(\tfrac{1}{5} - \tfrac{\epsilon}{100}\big)\delta_{ij}$ from here on,[8] we can isolate each

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[8]The ϵ-dependent portion is necessary to maintain, in the absence of m_{ij} and h_i, classical conformal invariance of (4.1) in $6 - \epsilon$ dimensions. Although inconsequential at one loop, this ϵ-dependence is crucial at two loops. This has also been seen in higher-loop computations in four dimensions in [29]. We thank Hugh Osborn for bringing this issue to our attention.

piece of the one loop effective action according to (4.2) and (2.2). We find[9]

$$\lambda^{(1)} \cdot \mathscr{R} = \frac{1}{\epsilon}\frac{1}{64\pi^3} n_\phi \left(-\frac{1}{9072} E_6 + \frac{1}{540} I_1 - \frac{1}{3024} I_2 - \frac{1}{2520} I_3\right),\tag{4.4}$$

where n_ϕ is the number of scalar fields ϕ. The field-dependent counterterms at one loop are given by

$$Z_{ij}^{(1)} = -\frac{1}{\epsilon}\frac{1}{64\pi^3}\frac{1}{6}\,g_{ikl}g_{jkl}\,, \qquad L_{ijk}^{(1)} = -\frac{1}{\epsilon}\frac{1}{64\pi^3}\left(g_{imn}g_{jmp}g_{knp} - \frac{1}{12}(g_{ijl}g_{kmn}g_{lmn} + \text{permutations})\right),\tag{4.5}$$

and

$$\mathscr{F}^{(1)} = -\frac{1}{\epsilon}\frac{1}{64\pi^3}\frac{1}{6}\left(\frac{1}{30} F g_{ijj}\phi_i + \frac{1}{10} R\,g_{ikl}g_{jkl}\,\phi_i\phi_j + \frac{1}{2}\partial^\mu g_{ikl}\partial_\mu g_{jkl}\,\phi_i\phi_j + g_{ikl}\partial^\mu g_{jkl}\partial_\mu\phi_i\,\phi_j\right).\tag{4.6}$$

Finally, the mass-dependent counterterms are

$$\begin{aligned}
\mathscr{M}^{(1)} = -\frac{1}{\epsilon}\frac{1}{64\pi^3}\frac{1}{2}\Big(&\frac{1}{90} F m_{ii} + \frac{1}{30} R\,m_{ij}m_{ij}\\
&+ \frac{1}{15}((R - 5\nabla^2)m_{ij})\,g_{ijk}\,\phi_k + m_{ij}\,g_{ikl}g_{jkm}\,\phi_l\phi_m + m_{ij}m_{ik}\,g_{jkl}\,\phi_l\\
&+ \frac{1}{6}\partial^\mu m_{ij}\,\partial_\mu m_{ij} + \frac{1}{3}m_{ij}m_{jk}m_{ki}\Big).
\end{aligned}\tag{4.7}$$

At two loops in the background field method we must compute the relevant vacuum bubble diagrams. Thus we are led to consider the diagrams in Fig. 1, where in the diagram on the right $\otimes$ denotes the one-loop counterterm. Note that these are graphs in position space, and
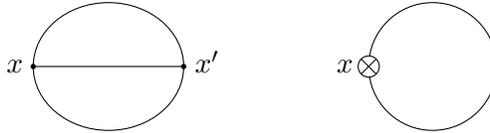

Fig. 1: The diagrams that need to be considered at the two-loop level.

that short distance singularities arise here from the coincident limit of products of position-space propagators. In particular, the left graph in Fig. 1 is given by

$$\bigcirc = \frac{1}{12}\mu^\epsilon \int d^d x\, d^d x'\sqrt{\gamma}\,\sqrt{\gamma'}\,g_{ijk}(x)\,g_{lmn}(x')\,G_{il}(x,x')\,G_{jm}(x,x')\,G_{kn}(x,x')\,,\tag{4.8}$$

with the factor of $\frac{1}{12}$ a symmetry factor from interchanging the Green's functions. The evaluation of this integral is straightforward, once the divergent parts of the products of the $G_n(x,x')$ from (2.31) are known; as these are listed in appendix C, we will not explicitly show the intermediate details of the cube of the propagator in (4.8) and simply include their contributions to the counterterms in (4.2) in the results listed below.

The graph on the right in Fig. 1 is slightly more complicated than the expression listed in [19] because of the spacetime dependence of the couplings. The counterterm insertion to the

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[9]This agrees with the results of [30], where similar computations were performed for fermions and two-forms.

propagator in that diagram is simply obtained from our one-loop results. More specifically, using our results (4.6) and (4.7), or, equivalently, the coincident limit of $a_{3,ii}(x,x')$ in (B.12), we obtain

$$\left(\bigotimes\!\!\bigcirc\right) = \frac{1}{\epsilon}\frac{1}{64\pi^3}\int d^dx\,d^dx'\,\sqrt{\gamma}\,\sqrt{\gamma'}\,\delta^d(x,x')\left[-\tfrac{1}{2}\left(g_{lmn}\phi_n + m_{lm} + (\xi_{lm} - \tfrac{1}{6}\delta_{lm})R\right)\right.$$
$$\left. + \tfrac{1}{12}\delta_{lm}\,\partial_x^\mu(\partial_{x'})_\mu\right]\left(g_{ikl}(x)g_{jkm}(x')\,G_{ij}(x,x')\right).$$
(4.9)

Here we have specified the spacetime point at which any derivatives are to be taken.

The UV divergences of these graphs are obtained from the divergences that arise from the products of propagators, discussed at the end of section 2.2. We use the results listed in appendix C, along with prudent integrations by parts, to reduce both (4.8) and (4.9) to poles in ϵ and coincident limits of the Seeley–DeWitt coefficients, which ultimately give explicit expression for the $-\mu^{-\epsilon}\lambda\cdot\mathscr{R}$, $\mu^{-\epsilon}\mathscr{F}(\phi)$, and $\mu^{-\epsilon}\mathscr{M}(m)$ pieces of (4.2). In particular, for the counterterm graph, we find[10]

$$\left(\bigotimes\!\!\bigcirc\right) = \frac{1}{\epsilon^2}\frac{\mu^{-\epsilon}}{(64\pi^3)^2}\int d^dx\sqrt{\gamma}\left[\left(-\tfrac{1}{2}g_{ikl}g_{jkm}\left(g_{lmn}\phi_n + m_{lm} + \left(\xi_{lm} - \tfrac{7}{36}\delta_{lm}\right)R\right)\right.\right.$$
$$\left. - \tfrac{1}{12}\partial^\mu g_{ikl}\,\partial_\mu g_{jkl}\right)\cdot 2\big[a_{2,ij}\big](x)$$
$$\left. + \tfrac{1}{12}g_{ikl}g_{jkl}\left(2\big[\nabla^2 a_{2,ij}\big](x) - (6-\epsilon)\big[a_{3,ij}\big](x)\right)\right].$$
(4.10)

Evaluting these graphs with the coincident limits listed in appendices B and C, we generate the terms as listed in (4.2) at the two loop level. For the curvature and derivatives on couplings terms, at two loops discarding total derivatives we have

$$\lambda^{(2)}\cdot\mathscr{R} = \tfrac{1}{\epsilon}\frac{1}{(64\pi^3)^2}\frac{1}{1620}\left(-\tfrac{1}{36}g_{ijk}g_{ijk}\left(I_1 - \tfrac{13}{4}I_2 - \tfrac{9}{8}I_3\right)\right.$$
$$+ \tfrac{3}{16}\left(\partial^\mu\nabla^2 g_{ijk}\,\partial_\mu\nabla^2 g_{ijk} - 4R^{\mu\nu}\nabla_\mu\partial_\nu g_{ijk}\nabla^2 g_{ijk} + R\nabla^2 g_{ijk}\nabla^2 g_{ijk}\right)$$
$$- \tfrac{1}{8}\left(H_1^{\mu\nu} + 30H_2^{\mu\nu} - 8H_3^{\mu\nu} - 10H_4^{\mu\nu} - 3H_5^{\mu\nu}\right)\partial_\mu g_{ijk}\partial_\nu g_{ijk}$$
$$\left. + \left(E_4 + \tfrac{49}{480}F + \tfrac{7}{400}R^2 - \tfrac{9}{80}\nabla^2 R\right)\partial^\mu g_{ijk}\partial_\mu g_{ijk}\right).$$
(4.11)

This result has been given in a conformally-covariant basis in [9]. Field-dependent counterterms at two loops are given by

$$Z_{ij}^{(2)} = -\tfrac{1}{\epsilon}\frac{1}{(64\pi^3)^2}\frac{1}{18}\left(g_{ikl}g_{jmn}g_{kmp}g_{lnp} - \tfrac{11}{24}g_{ikl}g_{jkm}g_{lnp}g_{mnp}\right),$$
(4.12)

$$L_{ijk}^{(2)} = -\tfrac{1}{\epsilon}\frac{1}{(64\pi^3)^2}\left(\tfrac{1}{4}\left(g_{ilm}g_{jnp}g_{kqr}g_{lnq}g_{mpr} + \tfrac{3}{2}g_{ilm}g_{jln}g_{kpq}g_{mpr}g_{nqr} - \tfrac{7}{12}g_{ilm}g_{jln}g_{kmp}g_{nqr}g_{pqr}\right)\right.$$
$$\left. - \tfrac{1}{36}\left(g_{ijl}g_{kmn}g_{lpq}g_{mpr}g_{nqr} - \tfrac{11}{24}g_{ijl}g_{kmn}g_{lmp}g_{nqr}g_{pqr} + \text{permutations}\right)\right),$$
(4.13)

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[10]This equation may be compared with (3.14b) of [19]. Note that our result reduces to this equation in the single spacetime-independent coupling case.

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which are relevant for the two-loop anomalous dimension and beta function, and

$$\mathscr{F}^{(2)} = -\frac{1}{\epsilon}\frac{1}{(64\pi^3)^2}\frac{1}{18}\Big(\Big(\frac{47}{2880}\,F\,g_{ijk}g_{klm}g_{jlm} - \frac{13}{48}\,R^{\mu\nu}\,g_{ijk}\,\partial_\mu g_{jmn}\,\partial_\nu g_{kmn}$$

$$+ \frac{1}{80}\,R\,g_{ijk}g_{jlm}\nabla^2 g_{klm} + \frac{101}{2400}\,g_{ijk}\partial^\mu g_{jlm}\,\partial_\mu g_{klm}$$

$$- \frac{3}{32}\,g_{ijk}\nabla^2 g_{jlm}\nabla^2 g_{klm} - \frac{7}{48}\,g_{ijk}\nabla^\mu\partial^\nu g_{jlm}\,\nabla_\mu\partial_\nu g_{klm}$$

$$- \frac{19}{48}\,g_{ijk}\partial^\mu\nabla^2 g_{jlm}\,\partial_\mu g_{klm} - \frac{1}{16}\,g_{ijk}g_{jlm}\nabla^2\nabla^2 g_{klm}\Big)\phi_i$$

$$+ \Big(\frac{23}{200}\,R\,g_{ikl}g_{jmn}g_{kmp}g_{lnp} - \frac{29}{600}\,R\,g_{ikl}g_{jkm}g_{lnp}g_{mnp}$$

$$+ \frac{1}{2}\,g_{klm}g_{knp}\partial^\mu g_{iln}\,\partial_\mu g_{jmp} - \frac{11}{48}\,g_{klm}g_{kln}\partial^\mu g_{imp}\,\partial_\mu g_{jnp}$$

$$- \frac{1}{2}\,g_{ikl}g_{kmn}\partial^\mu g_{jlp}\,\partial_\mu g_{mnp} + \frac{7}{4}\,g_{ikl}g_{kmn}\partial^\mu g_{jmp}\,\partial_\mu g_{lnp}$$

$$- \frac{11}{24}\,g_{ikl}g_{mnp}\partial^\mu g_{jkm}\,\partial_\mu g_{lnp} + \frac{3}{4}\,g_{ikl}g_{jmn}\partial^\mu g_{kmp}\,\partial_\mu g_{lnp}$$

$$- \frac{1}{16}\,g_{ikl}g_{jkm}\partial^\mu g_{lnp}\,\partial_\mu g_{mnp}\Big)\phi_i\phi_j$$

$$+ \Big(g_{jkl}g_{kmn}g_{lmp}\partial^\mu g_{inp} + \frac{7}{4}\,g_{ikl}g_{jmn}g_{kmp}\partial^\mu g_{lnp}$$

$$- \frac{11}{24}\,g_{jkm}g_{mnp}\partial^\mu(g_{ikl}g_{lnp}) - \frac{1}{2}\,g_{ikl}g_{jkm}g_{lnp}\partial^\mu g_{mnp}\Big)\phi_i\,\partial_\mu\phi_j\Big). \qquad (4.14)$$

The mass-dependent counterterms are given by

$$\mathscr{M}^{(2)} = -\frac{1}{\epsilon}\frac{1}{(64\pi^3)^2}\frac{1}{18}\Big(\Big(\frac{47}{2880}\,F\,g_{ikl}g_{jkl} - \frac{13}{48}\,R^{\mu\nu}\,\partial_\mu g_{ikl}\,\partial_\nu g_{jkl}$$

$$+ \frac{1}{80}\,R\,g_{ikl}\nabla^2 g_{jkl} + \frac{101}{2400}\,R\,\partial^\mu g_{ikl}\,\partial_\mu g_{jkl}$$

$$- \frac{3}{32}\,\nabla^2 g_{ikl}\nabla^2 g_{jkl} - \frac{7}{48}\,\nabla^\mu\partial^\nu g_{ikl}\,\nabla_\mu\partial_\nu g_{jkl} - \frac{1}{3}\,\partial^\mu g_{ikl}\,\partial_\mu\nabla^2 g_{jkl}$$

$$+ \frac{1}{10}\,R^{\mu\nu}\,g_{ikl}\,\partial_\mu g_{jkl}\,\partial_\nu - \frac{1}{20}\,R\,g_{ikl}\partial^\mu g_{jkl}\,\partial_\mu + \frac{1}{16}\,g_{ikl}\partial^\mu\nabla^2 g_{jkl}\,\partial_\mu$$

$$+ \frac{23}{100}\,R\,g_{ikl}g_{jkm}g_{lmn}\,\phi_n - \frac{29}{300}\,R\,g_{ikl}g_{jmn}g_{kmn}\,\phi_l$$

$$- \frac{7}{4}\,g_{ikl}\nabla^2 g_{jkm}g_{lmn}\,\phi_n + \frac{11}{24}\,g_{ikl}\nabla^2 g_{jmn}g_{kmn}\,\phi_l$$

$$+ \frac{1}{2}\,g_{ikl}g_{jmn}\nabla^2 g_{kmn}\,\phi_l$$

$$- \frac{1}{4}\,\partial^\mu g_{ikl}\,\partial_\mu g_{jkm}g_{lmn}\,\phi_n + \frac{5}{6}\,g_{ikl}\partial^\mu g_{jmn}\,\partial_\mu g_{mnk}\,\phi_l$$

$$- 2\,g_{ikl}\partial^\mu g_{jkm}g_{lmn}\,\phi_n\,\partial_\mu + \frac{5}{12}\,g_{ikl}\partial^\mu g_{jmn}g_{kmn}\,\phi_l\,\partial_\mu$$

$$+ \frac{1}{2}\,g_{ikl}g_{jmn}\partial^\mu g_{kmn}\,\phi_l\,\partial_\mu$$

$$- g_{ikl}g_{jkm}g_{lmn}\,\phi_n\nabla^2 + \frac{11}{24}\,g_{ikl}g_{jmn}g_{kmn}\,\phi_l\nabla^2$$

$$+ \frac{9}{4}\,g_{ikl}g_{jmn}g_{kmp}g_{lnq}\,\phi_p\phi_q + \frac{9}{8}\,g_{ikl}g_{jkn}g_{lpq}g_{npr}\,\phi_q\phi_r$$

$$+ \frac{9}{4}\,g_{ikl}g_{jmn}g_{kmp}g_{npq}\,\phi_l\phi_q - \frac{7}{8}\,g_{ikl}g_{jmn}g_{kpq}g_{mnp}\,\phi_l\phi_q$$

$$- \frac{7}{16}\,g_{ikl}g_{jmn}g_{kpq}g_{mpq}\,\phi_l\phi_n\Big)m_{ij}$$

$$+ \Big(\frac{23}{200}\,R\,g_{ikm}g_{jlm} - \frac{29}{600}\,R\,g_{imn}g_{kmn}\delta_{jl}$$

$$+ \frac{3}{4}\,\partial^\mu g_{ikm}\,\partial_\mu g_{jlm} - \frac{1}{16}\,\partial^\mu g_{imn}\,\partial_\mu g_{kmn}\delta_{jl}$$

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$$+ \tfrac{9}{4} g_{imn} g_{jkp} g_{lmp} \phi_n + \tfrac{9}{4} g_{ikm} g_{jln} g_{mnp} \phi_p - \tfrac{7}{16} g_{ikm} g_{jnp} g_{lnp} \phi_m$$

$$+ \tfrac{9}{8} g_{imn} g_{kmp} g_{npq} \phi_q \delta_{jl} - \tfrac{7}{8} g_{imn} g_{kpq} g_{mnp} \phi_q \delta_{jl} \big) m_{ij} m_{kl}$$

$$+ \big(\tfrac{7}{4} g_{ikm} \partial^\mu g_{jlm} - \tfrac{1}{2} g_{imn} \partial^\mu g_{kmn} \delta_{jl} - \tfrac{11}{24} \partial^\mu g_{imn} g_{kmn} \delta_{jl} \big) m_{ij} \partial_\mu m_{kl}$$

$$+ \big(\tfrac{1}{2} g_{ikm} g_{jlm} - \tfrac{11}{48} g_{imn} g_{kmn} \delta_{jl} \big) \partial^\mu m_{ij} \partial_\mu m_{kl}$$

$$+ \big(\tfrac{3}{4} g_{ikm} g_{jln} + \tfrac{9}{8} g_{ikp} g_{lmp} \delta_{jn} - \tfrac{7}{16} g_{ipq} g_{kpq} \delta_{jm} \delta_{ln} \big) m_{ij} m_{kl} m_{mn} \bigg). \qquad (4.15)$$

Although these terms are unsightly, they allow us to calculate the quantities of interest—they give us the complete, general trace anomaly on a curved background with spacetime dependent marginal sources ($g_{ijk}(x)$ and $\gamma_{\mu\nu}(x)$) via equation (2.39). Each of the terms presented in this section yields the relevant beta functions in the first part of (2.39) via equation (2.7). The second set of terms in (2.39), called $\nabla_\mu Z^\mu$, are obtained from a Weyl variation of the $\lambda \cdot \mathscr{R}$ terms, as seen in (2.40). Since the Weyl variation is non-trivial, we report the one- and two-loop contribution to Z^μ here, since it is required for the identification of terms necessary to the computation of $\tilde{a}$ in section 6.

At the one-loop level there are no contributions from the Weyl variation of (4.4) and so $Z^{(1)\mu} = 0$. At two loops the Weyl variation of (4.11) yields

$$Z^{(2)\mu} = \frac{1}{(64\pi^3)^2} \frac{1}{1620} \bigg(E_4 \, g_{ijk} \partial^\mu g_{ijk} + \tfrac{49}{480} F \, g_{ijk} \partial^\mu g_{ijk} + \tfrac{7}{400} R^2 \, g_{ijk} \partial^\mu g_{ijk} - \tfrac{9}{80} \nabla^2 R \, g_{ijk} \partial^\mu g_{ijk}$$

$$- \tfrac{1}{8} H_1^{\mu\nu} g_{ijk} \partial_\nu g_{ijk} - \tfrac{15}{4} H_2^{\mu\nu} g_{ijk} \partial_\nu g_{ijk} + H_3^{\mu\nu} g_{ijk} \partial_\nu g_{ijk}$$

$$+ \tfrac{5}{4} H_4^{\mu\nu} g_{ijk} \partial_\nu g_{ijk} + \tfrac{3}{8} H_5^{\mu\nu} g_{ijk} \partial_\nu g_{ijk}$$

$$+ \tfrac{3}{8} \nabla^\mu R^{\nu\rho} g_{ijk} \nabla_\nu \partial_\rho g_{ijk} - \tfrac{47}{40} \nabla^\mu R^{\nu\rho} \partial_\nu g_{ijk} \partial_\rho g_{ijk} + \tfrac{1}{4} \nabla^\nu R^{\mu\rho} \partial_\nu g_{ijk} \partial_\rho g_{ijk}$$

$$- \tfrac{29}{40} \partial^\nu R \, \partial^\mu g_{ijk} \partial_\nu g_{ijk} - \tfrac{43}{400} \partial^\mu R \, \partial_\nu g_{ijk} \partial^\nu g_{ijk}$$

$$+ \tfrac{3}{8} R^{\nu\rho} g_{ijk} \nabla^\mu \nabla_\nu \partial_\rho g_{ijk} + \tfrac{3}{8} R^{\mu\nu} g_{ijk} \partial_\nu \nabla^2 g_{ijk} - \tfrac{3}{8} R^{\nu\rho} \partial^\mu g_{ijk} \nabla_\nu \partial_\rho g_{ijk}$$

$$+ \tfrac{3}{8} R^{\mu\nu} \partial_\nu g_{ijk} \nabla^2 g_{ijk} - 4 R^{\mu\rho} \partial^\nu g_{ijk} \nabla_\nu \partial_\rho g_{ijk} - \tfrac{13}{5} R^{\nu\rho} \partial_\nu g_{ijk} \nabla^\mu \partial_\rho g_{ijk}$$

$$- \tfrac{3}{16} R \, g_{ijk} \partial^\mu \nabla^2 g_{ijk} - \tfrac{43}{80} R \, \partial^\mu g_{ijk} \nabla^2 g_{ijk} + \tfrac{121}{100} R \nabla^\mu \partial_\nu g_{ijk} \partial^\nu g_{ijk}$$

$$- \tfrac{1}{2} R^{\mu\nu\rho\sigma} \partial_\rho g_{ijk} \nabla_\nu \partial_\sigma g_{ijk}$$

$$+ \tfrac{3}{20} \nabla_\nu \partial_\rho g_{ijk} \nabla^\mu \nabla^\nu \partial^\rho g_{ijk} - \tfrac{3}{10} \nabla^\mu \partial^\nu g_{ijk} \partial_\nu \nabla^2 g_{ijk} + \tfrac{3}{16} \nabla^2 g_{ijk} \partial^\mu \nabla^2 g_{ijk}$$

$$+ \tfrac{9}{20} \partial_\nu g_{ijk} \nabla^\mu \partial^\nu \nabla^2 g_{ijk} - \tfrac{9}{16} \partial^\mu g_{ijk} \nabla^2 \nabla^2 g_{ijk}$$

$$+ \tfrac{3}{16} g_{ijk} \partial^\mu \nabla^2 \nabla^2 g_{ijk} \bigg).$$

$$(4.16)$$

It should be noted that the basis reported here is not identical to the $\mathscr{L}_q^\mu$ terms reported as in (3.2), which refers to the basis used in [6] written by some of the authors of the present work. However, as that basis is complete, the terms in (4.16) may be written in the $\mathscr{L}_q^\mu$ basis with repeated and judicious integrations by parts. For the purposes of the calculations in section 6 of

this paper, we did not find the basis referred to in (3.2) useful and were able to identify those terms required in equations (6.6a) and (6.6b) from the current presentation in (4.16).

From equations (4.4) to (4.7) and (4.11) to (4.15) we can extract the beta functions for the couplings and masses, the anomalous dimensions of the fields, and, perhaps most importantly, the quantity a (and $\tilde{a}$), as described in section 3, which is the analog of Zamolodchikov's celebrated c.

5. Beta functions and anomalous dimensions

As we have seen using background field and heat kernel methods the computation of Z_{ij} and L_{ijk} is easily done in position space and does not require the calculation of any integrals. With our results (4.5), (4.12) and (4.13) we can now compute the anomalous dimension γ_{ij} of ϕ_i and the beta function β_{ijk} to two-loop order.

The anomalous dimension is defined by

$$\gamma = -Z^{-1/2}\frac{dZ^{1/2}}{dt}\,, \qquad t = -\ln(\mu/\mu_0)\,, \tag{5.1}$$

where the RG time t is defined to increase as we flow to the IR. At one loop we find

$$\gamma^{(1)} = \frac{1}{64\pi^3}\frac{1}{12}-\!\!\bigcirc\!\!-\,, \tag{5.2}$$

where we use the diagram to denote the corresponding contraction of the couplings, i.e.

$$-\!\!\bigcirc\!\!- = g_{ikl}g_{jkl}\,. \tag{5.3}$$

The two-loop anomalous dimension is

$$\gamma^{(2)} = \frac{1}{(64\pi^3)^2}\frac{1}{18}\left(-\!\!\bigoplus\!\!- - \frac{11}{24}-\!\!\bigcirc\!\!\bigcirc\!\!-\right)\,. \tag{5.4}$$

For the case of a single field ϕ our results (5.2) and (5.4) reduce to the results of [31] (see also [19–21, 32]).

The beta function is defined by

$$\beta(g) = \mu\frac{dg}{d\mu} = -\frac{dg}{dt}\,. \tag{5.5}$$

At one loop we find

$$\beta^{(1)} = -\frac{1}{64\pi^3}\left(\succ\!\!\bigcirc\!\!- - \frac{1}{12}\succ\!\!\bigcirc\!\!-\right)\,, \tag{5.6}$$

where permutations of the free indices in the wavefunction-renormalization correction are understood, i.e.

$$\succ\!\!\bigcirc\!\!- = g_{ijl}g_{lmn}g_{kmn} + \text{permutations}\,. \tag{5.7}$$

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Eq. (5.6) reproduces the result of [31] (see also [19–21,32]) in the case of a single field ϕ. In that case $\beta^{(1)}$ has a negative sign, and hence the corresponding theory is asymptotically-free. The two-loop beta function is

$$\beta^{(2)} = -\frac{1}{(64\pi^3)^2}\frac{1}{2}\left(\bigotimes - \frac{7}{36} \; - \frac{7}{36} \; + \frac{1}{2} \; - \frac{1}{9} \; + \frac{11}{216} \; \right).$$

(5.8)

The first contribution to (5.8) is non-planar. For the seemingly asymmetric vertex corrections in (5.8) (second and third term) a symmetrization is understood; for example,

$$\text{represents} \quad + \quad + \quad .$$

(5.9)

In the single-field case (5.8) reproduces the result of [31] (see also [19–21][11]), which, just like $\beta^{(1)}$, is also negative.

The results presented here for the anomalous dimension and the beta function to two loops are found to agree with the results of [33], and can also be fully extracted from [34].

6. The metric in coupling space and the a-anomaly

In section 3, and in particular in equation (3.6), it was made apparent that there is an important piece of the $\beta_\lambda \cdot \mathscr{R}$ terms, called $\mathcal{H}^1_{IJ}$ in this paper and in [6],[12] that manifests itself as the coefficient of contact terms of certain correlation functions of the operators of the theory in flat spacetime and spacetime-independent $g_{ijk}(x)$ and $m_{ij}(x)$. This metric is important because it controls the behavior of a (or really, $\tilde{a}$) along the renormalization group flow; given the outstanding importance given to a (or its analogs) in two and four dimensions for its central role in characterizing quantum field theories there, its behavior in six dimensions gives insight into the universal features of quantum field theories in any dimension, possibly beyond the conventional Lagrangian description so ubiquitous in our understanding today.

In [10] a perturbative computation of the theory defined in the Lagrangian formalism by (1.1) yielded a surprising result for the value of $\mathcal{H}^1_{IJ}$ at the two loop level. One of the main purposes of this work is to give the details of that computation, along with other interesting results from the computation of the effective action.

In order to compute $\mathcal{H}^1_{IJ}$, we must first identify the corresponding piece in $\lambda \cdot \mathscr{R}$ so that we may use (2.7) to obtain the quantity of interest. There is no candidate in the one loop computation, $\lambda^{(1)} \cdot \mathscr{R}$. Thus, we must look for a two loop contribution in $\lambda^{(2)} \cdot \mathscr{R}$, where we do indeed find a

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[11] There is a typo in the relevant equation in [19].

[12] $\mathcal{H}^1_{IJ}$ was called χ_{IJ} in the two- and four-dimensional cases of [5] and in the six-dimensional case of [10].

candidate. We see from (4.11) that the relevant piece is

$$\lambda^{(2)} \cdot \mathscr{R} \supset -\frac{1}{\epsilon} \frac{1}{(64\pi^3)^2} \frac{1}{12960} H_1^{\mu\nu} \partial_\mu g_{ijk} \partial_\nu g_{ijk}. \tag{6.1}$$

From (6.1) and (2.40) we can immediately match to the term $\frac{1}{2}\mathcal{H}_{IJ}^1 \partial_\mu g^I \partial_\nu g^J H_1^{\mu\nu}$ in $\beta_\lambda \cdot \mathscr{R}$ and extract

$$\mathcal{H}_{IJ}^{1\,(2)} = -\frac{1}{(64\pi^3)^2} \frac{1}{3240} \delta_{IJ}, \tag{6.2}$$

where, as section 2, we use notation of [6] and denote $I = (ijk)$. Furthermore, performing a Weyl variation of (6.1) we find

$$\delta_\sigma(-\lambda^{(2)} \cdot \mathscr{R}) \supset \frac{1}{\epsilon} \frac{1}{(64\pi^3)^2} \frac{1}{6480} H_1^{\mu\nu} \hat{\beta}_{ijk} \partial_\nu g_{ijk} \partial_\mu \sigma, \tag{6.3}$$

and so

$$\mathcal{H}_I^{1\,(2)} = -\frac{1}{(64\pi^3)^2} \frac{1}{12960} g_I, \tag{6.4}$$

as also seen in (4.16). The result (6.2) is unambiguous and scheme-independent. As we observe, the leading, two-loop contribution to the metric is negative, and so the consistency condition (3.4) and its consequence (3.6) cannot possibly lead to a strong a-theorem for $\tilde{a}$.

Now, our theory has only the Gaussian fixed point in perturbation theory. Non-perturbatively there may be a non-trivial fixed point, but our results (6.2) and (6.4) cannot be used beyond perturbation theory. Nevertheless, as long as the flow of our theory can be described perturbatively, the quantity $\tilde{a}$ is monotonically increasing.

Another use of the consistency conditions is the evaluation of some quantities at higher loop orders. Regarding $\tilde{a}$, for example, we can use (3.4) with the results (5.6), (6.2), and (6.4) to obtain the three-loop contribution to $\tilde{a}$,

$$\tilde{a}^{(3)} = \frac{1}{(64\pi^3)^3} \frac{1}{77760} \left(\bigoplus - \frac{1}{4} \bigcirc\!\!\!\bigcirc \right). \tag{6.5}$$

Furthermore, from the consistency conditions (see [6] for the meaning of the various terms)

$$b_1 = \tfrac{1}{6}(\mathcal{F}_I - \tfrac{1}{2}\partial_I b_{14} - \tfrac{1}{2}\mathcal{I}_I^7)\beta^I, \tag{6.6a}$$

$$b_3 = (\mathcal{F}_I + \partial_I b_{13} - \partial_I b_{14} + \mathcal{I}_I^6 - \mathcal{I}_I^7)\beta^I, \tag{6.6b}$$

it is clear that at two loops $b_1^{(2)} = b_3^{(2)} = 0$. This, in conjunction with (3.4), (6.2) and (6.4), implies that $a^{(2)} = 0$. These results have been verified by our explicit computations (4.14). Now, at two loops we can use (4.14) and (4.16) to obtain

$$\mathcal{F}_I^{(2)} = \frac{1}{(64\pi^3)^2} \frac{1}{1080} g_I, \qquad b_{13}^{(2)} = b_{14}^{(2)} = 0, \qquad \mathcal{I}_I^{6\,(2)} = \mathcal{I}_I^{7\,(2)} = 0, \tag{6.7}$$

and so using (6.6) we can compute

$$b_1^{(3)} = \tfrac{1}{6}b_3^{(3)} = -\frac{1}{(64\pi^3)^3} \frac{1}{6480} \left(\bigoplus - \frac{1}{4} \bigcirc\!\!\!\bigcirc \right). \tag{6.8}$$

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With these results and using (3.4) with (5.6), (6.2), and (6.4) we find that the three-loop contribution to a is

$$a^{(3)} = \frac{1}{(64\pi^3)^3} \frac{1}{64\,800} \left(\bigcirc\!\!\!\!- - \frac{1}{4} \bigcirc\!\!\!\!\bigcirc \right).$$

(6.9)

This shows that, just like $\tilde{a}$, a increases in the flow out of the trivial UV fixed point in our theory.

There is one comment to be made about the value of the result in (6.5). It is a scheme-dependent quantity, in the sense that it is only defined modulo terms that are "exact" in the cohomology generated by the Weyl transformations $\Delta_\sigma^{\mathrm{W}} - \Delta_\sigma^{\beta}$, i.e. up to local additions to the original action whose variations shift quantities in (3.6). However, as shown in [6] in analogy with [5] these shifts are of the form $\delta\tilde{a} = z_{IJ}\beta^I\beta^J$ for z_{IJ} an arbitrary regular symmetric function of the couplings. Hence, at lowest order, and using (5.6), we have $\delta\tilde{a} \sim \mathcal{O}(g^6)$ which cannot possibly upset the conclusions of this section in perturbation theory. Moreover, equation (3.6) is of course unchanged by such shifts and in this sense is an invariant of the associated Weyl cohomology.

Appendix A. Conventions and basis tensors

In this work we define the Riemann tensor via

$$[\nabla_\mu, \nabla_\nu]A^\rho = R^\rho{}_{\sigma\mu\nu}A^\sigma\,,$$

(A.1)

and the Ricci tensor and Ricci scalar as $R_{\mu\nu} = R^\rho{}_{\mu\rho\nu}$ and $R = \gamma^{\mu\nu}R_{\mu\nu}$. We also commonly use the Weyl tensor defined in $d \geq 3$ by

$$W_{\mu\nu\rho\sigma} = R_{\mu\nu\rho\sigma} + \tfrac{2}{d-2}(\gamma_{\mu[\sigma}R_{\rho]\nu} + \gamma_{\nu[\rho}R_{\sigma]\mu}) + \tfrac{2}{(d-1)(d-2)}\gamma_{\mu[\rho}\gamma_{\sigma]\nu}R\,.$$

(A.2)

At mass dimension four we use the tensors

$$E_4 = \frac{2}{(d-2)(d-3)}\left(R^{\mu\nu\rho\sigma}R_{\mu\nu\rho\sigma} - 4R^{\mu\nu}R_{\mu\nu} + R^2\right),$$

$$F = W^{\mu\nu\rho\sigma}W_{\mu\nu\rho\sigma}, \qquad \frac{1}{(d-1)^2}R^2, \qquad \frac{1}{d-1}\nabla^2 R,$$

$$H_{1\mu\nu} = \frac{(d-2)(d-3)}{2}E_4\gamma_{\mu\nu} - 4(d-1)H_{2\mu\nu} + 8H_{3\mu\nu} + 8H_{4\mu\nu} - 4R^{\rho\sigma\tau}{}_\mu R_{\rho\sigma\tau\nu}, \qquad \text{(A.3)}$$

$$H_{2\mu\nu} = \frac{1}{d-1}RR_{\mu\nu}, \qquad H_{3\mu\nu} = R_\mu{}^\rho R_{\rho\nu}, \qquad H_{4\mu\nu} = R^{\rho\sigma}R_{\rho\mu\sigma\nu},$$

$$H_{5\mu\nu} = \nabla^2 R_{\mu\nu}, \qquad H_{6\mu\nu} = \frac{1}{d-1}\nabla_\mu\partial_\nu R.$$

A complete basis of scalar dimension-six curvature terms consists of [35]

$$K_1 = R^3, \qquad K_2 = RR^{\mu\nu}R_{\mu\nu}, \qquad K_3 = RR^{\mu\nu\rho\sigma}R_{\mu\nu\rho\sigma}, \qquad K_4 = R^{\mu\nu}R_{\nu\rho}R^\rho{}_\mu,$$

$$K_5 = R^{\mu\nu}R^{\rho\sigma}R_{\mu\rho\sigma\nu}, \qquad K_6 = R^{\mu\nu}R_{\mu\rho\sigma\tau}R_\nu{}^{\rho\sigma\tau}, \qquad K_7 = R^{\mu\nu\rho\sigma}R_{\rho\sigma\tau\omega}R^{\tau\omega}{}_{\mu\nu},$$

$$K_8 = R^{\mu\nu\rho\sigma}R_{\tau\nu\rho\omega}R_\mu{}^{\tau\omega}{}_\sigma, \qquad K_9 = R\nabla^2 R, \qquad K_{10} = R^{\mu\nu}\nabla^2 R_{\mu\nu}, \qquad K_{11} = R^{\mu\nu\rho\sigma}\nabla^2 R_{\mu\nu\rho\sigma},$$

$$K_{12} = R^{\mu\nu}\nabla_\mu\partial_\nu R, \qquad K_{13} = \nabla^\mu R^{\nu\rho}\nabla_\mu R_{\nu\rho}, \qquad K_{14} = \nabla^\mu R^{\nu\rho}\nabla_\nu R_{\mu\rho},$$

$$K_{15} = \nabla^\mu R^{\nu\rho\sigma\tau}\nabla_\mu R_{\nu\rho\sigma\tau}, \qquad K_{16} = \nabla^2 R^2, \qquad K_{17} = (\nabla^2)^2 R.$$

In $d = 6$ a convenient basis is given by

$$I_1 = \frac{19}{800}K_1 - \frac{57}{160}K_2 + \frac{3}{40}K_3 + \frac{7}{16}K_4 - \frac{9}{8}K_5 - \frac{3}{4}K_6 + K_8,$$

$$I_2 = \frac{9}{200}K_1 - \frac{27}{40}K_2 + \frac{3}{10}K_3 + \frac{5}{4}K_4 - \frac{3}{2}K_5 - 3K_6 + K_7,$$

$$I_3 = -\frac{11}{50}K_1 + \frac{27}{10}K_2 - \frac{6}{5}K_3 - K_4 + 6K_5 + 2K_7 - 8K_8$$
$$+ \frac{3}{5}K_9 - 6K_{10} + 6K_{11} + 3K_{13} - 6K_{14} + 3K_{15},$$

$$E_6 = K_1 - 12K_2 + 3K_3 + 16K_4 - 24K_5 - 24K_6 + 4K_7 + 8K_8,$$

$$J_1 = 6K_6 - 3K_7 + 12K_8 + K_{10} - 7K_{11} - 11K_{13} + 12K_{14} - 4K_{15}, \qquad \text{(A.4)}$$

$$J_2 = -\frac{1}{5}K_9 + K_{10} + \frac{2}{5}K_{12} + K_{13}, \qquad J_3 = K_4 + K_5 - \frac{3}{20}K_9 + \frac{4}{5}K_{12} + K_{14},$$

$$J_4 = -\frac{1}{5}K_9 + K_{11} + \frac{2}{5}K_{12} + K_{15}, \qquad J_5 = K_{16}, \qquad J_6 = K_{17},$$

$$L_1 = -\frac{1}{30}K_1 + \frac{1}{4}K_2 - K_6, \qquad L_2 = -\frac{1}{100}K_1 + \frac{1}{20}K_2,$$

$$L_3 = -\frac{37}{6000}K_1 + \frac{7}{150}K_2 - \frac{1}{75}K_3 + \frac{1}{10}K_5 + \frac{1}{15}K_6, \qquad L_4 = -\frac{1}{150}K_1 + \frac{1}{20}K_3,$$

$$L_5 = \frac{1}{30}K_1, \qquad L_6 = -\frac{1}{300}K_1 + \frac{1}{20}K_9, \qquad L_7 = K_{15},$$

where the first three transform covariantly under Weyl variations, and E_6 is the Euler term in $d = 6$. The J's are trivial anomalies in a six-dimensional CFT defined in curved space, and the first six L's are constructed based on the relation $\delta_\sigma \int d^6x\sqrt{\gamma}\,L_{1,\dots,6} = \int d^6x\sqrt{\gamma}\,\sigma J_{1,\dots,6}$.

In this paper we use the above basis for dimension-six curvature scalars, but, although it is not necessary, we define $I_{1,2,3}$ in general d, because of our use of dimensional regularization. More

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specifically, we define

$$I_1 = W^{\mu\nu\rho\sigma} W_{\tau\nu\rho\omega} W_\mu{}^{\tau\omega}{}_\sigma$$

$$= \frac{d^2 + d - 4}{(d-1)^2(d-2)^3} K_1 - \frac{3(d^2 + d - 4)}{(d-1)(d-2)^3} K_2 + \frac{3}{2(d-1)(d-2)} K_3 + \frac{2(3d-4)}{(d-2)^3} K_4$$

$$- \frac{3d}{(d-2)^2} K_5 - \frac{3}{d-2} K_6 + K_8 \,,$$

$$I_2 = W^{\mu\nu\rho\sigma} W_{\rho\sigma\tau\omega} W^{\tau\omega}{}_{\mu\nu}$$

$$= \frac{8(2d-3)}{(d-1)^2(d-2)^3} K_1 - \frac{24(2d-3)}{(d-1)(d-2)^3} K_2 + \frac{6}{(d-1)(d-2)} K_3 + \frac{16(d-1)}{(d-2)^3} K_4$$

$$- \frac{24}{(d-2)^2} K_5 - \frac{12}{d-2} K_6 + K_7 \,,$$

$$I_3 = W^{\mu\nu\rho\sigma} \left(\delta_\mu{}^\tau \nabla^2 + \frac{16}{d-2} R_\mu{}^\tau - \frac{4d}{(d-1)(d-2)} \delta_\mu{}^\tau R \right) W_{\tau\nu\rho\sigma}$$

$$+ 8 \nabla^\mu \nabla^\nu (W_\mu{}^{\rho\sigma\tau} W_{\nu\rho\sigma\tau}) - \frac{1}{2} \nabla^2 (W^{\mu\nu\rho\sigma} W_{\mu\nu\rho\sigma})$$

$$= -\frac{2(d^2 + d + 2)}{(d-1)^2(d-2)^2} K_1 + \frac{2(d^2 + 13d - 6)}{(d-1)(d-2)^2} K_2 - \frac{2(d-3)}{d-1} K_3 + \frac{4(d-10)}{(d-2)^2} K_4$$

$$- \frac{6(d-10)}{d-2} K_5 - \frac{1}{2}(d-10) K_7 + 2(d-10) K_8 - \frac{(d-3)(d-10)}{(d-1)(d-2)} K_9$$

$$- \frac{8(d-3)}{d-2} K_{10} + 2(d-3) K_{11} - \frac{(d-3)(3d-22)}{d-2} K_{13} + \frac{2(d-3)(d-10)}{d-2} K_{14}$$

$$+ \frac{1}{4}(d-2)(d-3) K_{15} + \frac{(d-3)(d-6)}{2(d-1)(d-2)} K_{16} \,. \tag{A.5}$$

These satisfy $\delta_\sigma I_{1,2,3} = 6\sigma I_{1,2,3}$ for any d for which they can be defined.

Appendix B. Coincident limits

Here we collect the coincident limits $x' \to x$ of the Seeley–DeWitt coefficients $a_{n,ij}(x, x')$ of (2.22) and the various functions (i.e. $\sigma(x, x')$ and $\Delta_{\mathrm{VM}}^{1/2}(x, x')$) needed therein to solve the recursion relation (2.25). Most of these results can be found in [19, 21][13], and [17], though in these works only the single coupling case was considered.

The fundamental quantities of interest on a curved background are the geodetic interval $\sigma(x, x')$, whose "equation of motion" is [26]

$$\tfrac{1}{2} \partial^\mu \sigma \, \partial_\mu \sigma = \sigma \,, \tag{B.1}$$

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[13]Reference [21] uses the opposite curvature convention as the one used in this work, (A.1). This must be taken into account when comparing expressions, as odd powers of curvature will have an extra relative minus sign.

and the van Vleck–Morette determinant $\Delta_{\mathrm{VM}}(x,x')$, which describes the rate at which geodesics coming from a point separate, follows from a corollary of the above equation for $\sigma(x,x')$:

$$\Delta_{\mathrm{VM}}^{1/2}\nabla^2\sigma + 2\partial^\mu\sigma\,\partial_\mu\Delta_{\mathrm{VM}}^{1/2} = d\Delta_{\mathrm{VM}}^{1/2}\,, \tag{B.2}$$

with d the spacetime dimension. Here we have abbreviated $\sigma(x,x')$ and $\Delta_{\mathrm{VM}}^{1/2}(x,x')$ as σ and $\Delta_{\mathrm{VM}}^{1/2}$, respectively, and will continue to do so throughout the rest of the appendix.

From these two equations we can construct the coincident limits of σ and $\Delta_{\mathrm{VM}}^{1/2}$. We will denote the coincident limit of a quantity X as $[X]$ for brevity. Throughout the rest of this appendix, to make the coincident limits as concise as possible, we use the semicolon notation for the covariant derivatives, e.g. $\nabla_\nu\partial_\mu\sigma = \sigma_{;\mu\nu}$. Now, since σ measures a distance, we clearly have

$$[\sigma] = 0\,. \tag{B.3}$$

Now using (B.1) and differentiating as many times as needed, we obtain the following limits:

$$[\sigma_{,\mu}] = 0\,, \qquad [\sigma_{;\mu\nu}] = \gamma_{\mu\nu}\,, \qquad [\sigma_{;\mu\nu\rho}] = 0\,,$$

$$[\sigma_{;\mu\nu\rho\sigma}] = -\tfrac{1}{3}(R_{\mu\rho\nu\sigma} + R_{\mu\sigma\nu\rho})\,,$$

$$[\sigma_{;\mu\nu\rho\sigma\tau}] = -\tfrac{1}{4}(R_{\mu\rho\nu\sigma;\tau} + R_{\mu\sigma\nu\rho;\tau} + \tau\leftrightarrow\sigma + \tau\leftrightarrow\rho)\,,$$

$$[\sigma_{;\rho}{}^{\rho}{}_{\sigma}{}^{\sigma}{}_{\mu\nu}] = \tfrac{4}{45}R_{\mu\rho}R^{\rho}{}_{\nu} + \tfrac{8}{45}R^{\rho\sigma}R_{\rho\mu\sigma\nu} - \tfrac{4}{15}R_{\mu}{}^{\rho\sigma\tau}R_{\nu\rho\sigma\tau} - \tfrac{2}{5}R_{\mu\nu;\rho}{}^{\rho} - \tfrac{6}{5}R_{;\mu\nu}\,,$$

$$[\sigma_{;\rho}{}^{\rho}{}_{\mu\nu\sigma}{}^{\sigma}] = [\sigma_{;\rho}{}^{\rho}{}_{\sigma}{}^{\sigma}{}_{\mu\nu}] + \tfrac{4}{3}(R^{\rho\sigma}R_{\rho\mu\sigma\nu} - R_{\mu}{}^{\rho}R_{\rho\nu})\,, \tag{B.4}$$

$$[\sigma_{;\mu\nu\rho}{}^{\rho}{}_{\sigma}{}^{\sigma}] = [\sigma_{;\rho}{}^{\rho}{}_{\mu\nu\sigma}{}^{\sigma}] - \tfrac{8}{3}(R^{\rho\sigma}R_{\rho\mu\sigma\nu} - R_{\mu}{}^{\rho}R_{\rho\nu}) - 2R_{\mu\nu;\rho}{}^{\rho} + 2R_{;\mu\nu}\,,$$

$$[\sigma_{;\mu}{}^{\mu}{}_{\nu}{}^{\nu}{}_{\rho}{}^{\rho}] = \tfrac{4}{5}(\tfrac{1}{3}R^{\mu\nu}R_{\mu\nu} - \tfrac{1}{3}R^{\mu\nu\rho\sigma}R_{\mu\nu\rho\sigma} - 2R_{;\mu}{}^{\mu})\,,$$

$$[\sigma_{;\mu}{}^{\mu}{}_{\nu}{}^{\nu}{}_{\rho}{}^{\rho}{}_{\sigma}{}^{\sigma}] = -\tfrac{152}{315}K_4 - \tfrac{176}{315}K_5 + \tfrac{296}{315}K_6 - \tfrac{44}{63}K_7 + \tfrac{80}{63}K_8 + \tfrac{3}{7}K_9 + \tfrac{148}{105}K_{10}$$
$$\qquad\qquad - \tfrac{12}{7}K_{11} - \tfrac{32}{35}K_{12} + \tfrac{2}{7}K_{13} + \tfrac{4}{7}K_{14} - \tfrac{9}{7}K_{15} - \tfrac{3}{14}K_{16} - \tfrac{18}{7}K_{17}\,.$$

The coincident limits of derivatives on $\Delta_{\mathrm{VM}}^{1/2}$ follow from (B.2):

$$[\Delta_{\mathrm{VM}}^{1/2}] = 1\,, \qquad [\Delta_{\mathrm{VM}}^{1/2}{}_{,\mu}] = 0\,, \qquad [\Delta_{\mathrm{VM}}^{1/2}{}_{;\mu\nu}] = \tfrac{1}{6}R_{\mu\nu}\,,$$

$$[\Delta_{\mathrm{VM}}^{1/2}{}_{;\mu\nu\rho}] = \tfrac{1}{12}(R_{\mu\nu;\rho} + R_{\rho\mu;\nu} + R_{\nu\rho;\mu})\,,$$

$$[\Delta_{\mathrm{VM}}^{1/2}{}_{;\rho}{}^{\rho}{}_{\mu\nu}] = \tfrac{1}{36}RR_{\mu\nu} - \tfrac{1}{15}R_{\mu}{}^{\rho}R_{\rho\nu} + \tfrac{1}{30}(R^{\rho\sigma}R_{\rho\mu\sigma\nu} + R_{\mu}{}^{\rho\sigma\tau}R_{\nu\rho\sigma\tau}) + \tfrac{3}{20}R_{;\mu\nu} + \tfrac{1}{20}R_{\mu\nu;\rho}{}^{\rho}\,,$$

$$[\Delta_{\mathrm{VM}}^{1/2}{}_{;\mu\nu\rho}{}^{\rho}] = [\Delta_{\mathrm{VM}}^{1/2}{}_{;\rho}{}^{\rho}{}_{\mu\nu}] + \tfrac{1}{3}(R_{\mu}{}^{\rho}R_{\rho\nu} - R^{\rho\sigma}R_{\rho\mu\sigma\nu})\,, \tag{B.5}$$

$$[\Delta_{\mathrm{VM}}^{1/2}{}_{;\mu}{}^{\mu}{}_{\nu}{}^{\nu}{}_{\rho}{}^{\rho}] = \tfrac{1}{216}K_1 - \tfrac{1}{60}K_2 + \tfrac{1}{60}K_3 + \tfrac{2}{189}K_4 + \tfrac{2}{63}K_5 - \tfrac{4}{63}K_6 + \tfrac{11}{189}K_7 - \tfrac{20}{189}K_8 - \tfrac{2}{105}K_9$$
$$\qquad\qquad - \tfrac{2}{21}K_{10} + \tfrac{1}{7}K_{11} + \tfrac{1}{7}K_{12} - \tfrac{1}{42}K_{13} - \tfrac{1}{21}K_{14} + \tfrac{3}{28}K_{15} + \tfrac{5}{84}K_{16} + \tfrac{3}{14}K_{17}\,.$$

We have only listed the limits that are needed to compute the coincident limits of the Seeley–DeWitt coefficients $a_{0,ij}$ up to $a_{3,ij}$. We also note that we have explicitly checked all limits with those found in the aforementioned references and find agreement.

Another quantity of interest, which is indirectly related to our calculations in this paper through the details laid out in appendix C, is $Y \equiv \Delta_{\mathrm{VM}}^{-1/2}\Delta_{\mathrm{VM};\mu}^{1/2}{}^{\mu}$. Knowledge of its coincident limits is necessary for the computation of (4.8). We find

$$[Y_{,\mu}] = \tfrac{1}{6}R_{,\mu}\,,$$

$$[Y_{;\mu\nu}] = -\tfrac{1}{15}R_{\mu}{}^{\rho}R_{\rho\nu} + \tfrac{1}{30}(R^{\rho\sigma}R_{\rho\mu\sigma\nu} + R_{\mu}{}^{\rho\sigma\tau}R_{\nu\rho\sigma\tau}) + \tfrac{3}{20}R_{;\mu\nu} + \tfrac{1}{20}R_{\mu\nu;\rho}{}^{\rho}\,,$$

$$[Y_{;\mu}{}^{\mu}] = -\tfrac{1}{30}(R^{\mu\nu}R_{\mu\nu} - R^{\mu\nu\rho\sigma}R_{\mu\nu\rho\sigma}) + \tfrac{1}{5}R_{;\mu}{}^{\mu}\,,$$

$$[Y_{;\nu}{}^{\nu}{}_{\mu}] = -\tfrac{1}{15}(R^{\nu\rho}R_{\nu\rho;\mu} - R^{\nu\rho\sigma\tau}R_{\nu\rho\sigma\tau;\mu}) + \tfrac{1}{5}R_{;\nu}{}^{\nu}{}_{\mu}\,, \qquad (\text{B.6})$$

$$[Y_{;\mu\nu}{}^{\nu}] = [Y_{;\nu}{}^{\nu}{}_{\mu}] + \tfrac{1}{6}R_{\mu}{}^{\nu}R_{,\nu}\,,$$

$$[Y_{;\mu}{}^{\mu}{}_{\nu}{}^{\nu}] = \tfrac{52}{945}K_4 + \tfrac{17}{315}K_5 - \tfrac{3}{35}K_6 + \tfrac{11}{189}K_7 - \tfrac{20}{189}K_8 - \tfrac{1}{126}K_9 - \tfrac{9}{70}K_{10}$$

$$+ \tfrac{1}{7}K_{11} + \tfrac{3}{70}K_{12} - \tfrac{1}{42}K_{13} - \tfrac{1}{21}K_{14} + \tfrac{3}{28}K_{15} + \tfrac{1}{252}K_{16} + \tfrac{3}{14}K_{17}\,.$$

Now we may proceed to the quantities that are directly related to the central computations of this paper, the Seeley–DeWitt coefficients $a_{n,ij}(x,x')$ that characterize the propagator's response to the curved background with metric $\gamma_{\mu\nu}(x)$. Restating its fundamental and defining condition, (2.25), we have

$$n a_{n,ij} + \partial_\mu\sigma\,\partial^\mu a_{n,ij} = -\Delta_{\mathrm{VM}}^{-1/2}M_{ik}\big(\Delta_{\mathrm{VM}}^{1/2}a_{n-1,kj}\big)\,, \qquad (\text{B.7})$$

with initial conditions

$$\partial_\mu\sigma\,\partial^\mu a_{0,ij} = 0 \qquad \text{and} \qquad [a_{0,ij}] = \delta_{ij}\,. \qquad (\text{B.8})$$

The limits of these coefficients depend on the elliptic differential operator of the form $M_{ij} = -\delta_{ij}\nabla^2 + X_{ij}$, with $X_{ij} = \left.\frac{\partial^2 V(\phi)}{\partial\phi^i\,\partial\phi^j}\right|_{\phi=\phi_b}$. In the case of the Lagrangian (4.2), we have

$$X_{ij} = m_{ij} + g_{ijk}\phi_k + \xi_{ij}R\,. \qquad (\text{B.9})$$

(Unless explicitly stated we will take ϕ to represent the background field ϕ_b for the sake of compressed notation.) Note that X_{ij} is symmetric, $X_{ij} = X_{ji}$. Then,

$$[a_{1,ij}] = \tfrac{1}{6}R\delta_{ij} - X_{ij}\,,$$

$$[\partial_\mu a_{1,ij}] = \tfrac{1}{2}\partial_\mu\big(\tfrac{1}{6}R\delta_{ij} - X_{ij}\big)\,,$$

$$[\nabla_\mu\partial_\nu a_{1,ij}] = \tfrac{1}{45}\big(\tfrac{1}{2}(R^{\rho\sigma}R_{\rho\mu\sigma\nu} + R_{\mu}{}^{\rho\sigma\tau}R_{\nu\rho\sigma\tau}) - R_{\mu}{}^{\rho}R_{\rho\nu} + \tfrac{3}{4}\nabla^2 R_{\mu\nu} + \tfrac{9}{4}\nabla_\mu\partial_\nu R\big)\delta_{ij} - \tfrac{1}{3}\nabla_\mu\partial_\nu X_{ij}\,,$$

$$[\partial_\mu\nabla^2 a_{1,ij}] = \tfrac{1}{60}(R^{\nu\rho\sigma\tau}\nabla_\mu R_{\nu\rho\sigma\tau} - R^{\nu\rho}\nabla_\mu R_{\nu\rho} - \tfrac{5}{3}R_{\mu}{}^{\nu}\partial_\nu R + 3\partial_\mu\nabla^2 R)\delta_{ij} - \tfrac{1}{4}\partial_\mu\nabla^2 X_{ij}\,,$$

$$[\nabla^2\partial_\mu a_{1,ij}] = [\partial_\mu\nabla^2 a_{1,ij}] + \tfrac{1}{2}R_{\mu}{}^{\nu}\partial_\nu\big(\tfrac{1}{6}R\delta_{ij} - X_{ij}\big)\,,$$

$$[(\nabla^2)^2 a_{1,ij}] = \big(\tfrac{16}{945}K_4 + \tfrac{13}{945}K_5 - \tfrac{19}{945}K_6 + \tfrac{11}{945}K_7 - \tfrac{4}{189}K_8 + \tfrac{19}{1260}K_9 - \tfrac{19}{630}K_{10} + \tfrac{1}{35}K_{11}$$

$$- \tfrac{1}{210}K_{12} - \tfrac{1}{210}K_{13} - \tfrac{1}{105}K_{14} + \tfrac{3}{140}K_{15} - \tfrac{19}{2520}K_{16} + \tfrac{3}{70}K_{17}\big)\delta_{ij}$$

$$+ \tfrac{4}{45}R^{\mu\nu}\nabla_\mu\partial_\nu X_{ij} + \tfrac{1}{10}\partial^\mu R\,\partial_\mu X_{ij} - \tfrac{1}{5}(\nabla^2)^2 X_{ij}\,.$$

$$(\text{B.10})$$

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for the relevant limits of $a_{1,ij}$. For $a_{2,ij}$ we have

$$[a_{2,ij}] = \tfrac{1}{180}(R^{\mu\nu\rho\sigma}R_{\mu\nu\rho\sigma} - R^{\mu\nu}R_{\mu\nu} + \tfrac{5}{2}R^2 + 6\nabla^2 R)\delta_{ij} - \tfrac{1}{6}(R + \nabla^2)X_{ij} + \tfrac{1}{2}X_{ik}X_{ij}\,,$$

$$[\partial_\mu a_{2,ij}] = \tfrac{1}{180}(R^{\nu\rho\sigma\tau}\nabla_\mu R_{\nu\rho\sigma\tau} - R^{\nu\rho}\nabla_\mu R_{\nu\rho} + \tfrac{5}{2}R\partial_\mu R + 3\partial_\mu\nabla^2 R)\delta_{ij}$$
$$- \tfrac{1}{12}\partial_\mu(RX_{ij}) - \tfrac{1}{12}\partial_\mu\nabla^2 X_{ij} + \tfrac{1}{3}(\tfrac{1}{2}X_{ik}\partial_\mu X_{kj} + \partial_\mu X_{ik}X_{kj})\,,$$

$$[\nabla^2 a_{2,ij}] = -(\tfrac{1}{540}K_2 - \tfrac{1}{540}K_3 - \tfrac{1}{1890}K_4 - \tfrac{1}{630}K_5 + \tfrac{1}{315}K_6 - \tfrac{11}{3780}K_7 + \tfrac{1}{189}K_8 - \tfrac{1}{1008}K_9 + \tfrac{1}{210}K_{10}$$
$$- \tfrac{1}{140}K_{11} - \tfrac{1}{140}K_{12} + \tfrac{1}{840}K_{13} + \tfrac{1}{420}K_{14} - \tfrac{3}{560}K_{15} - \tfrac{17}{3360}K_{16} - \tfrac{3}{280}K_{17})\delta_{ij}$$
$$- \tfrac{1}{90}(R^{\mu\nu\rho\sigma}R_{\mu\nu\rho\sigma} - R^{\mu\nu}R_{\mu\nu} + 6\nabla^2 R + 9\partial^\mu R\,\partial_\mu + 3R^{\mu\nu}\nabla_\mu\partial_\nu + 5R\nabla^2 + \tfrac{9}{2}(\nabla^2)^2)X_{ij}$$
$$+ \tfrac{1}{4}(\tfrac{1}{3}X_{ik}\nabla^2 X_{kj} + \nabla^2 X_{ik}X_{kj} + \partial^\mu X_{ik}\partial_\mu X_{kj})\,.$$

$$(\text{B.11})$$

Finally for $a_{3,ij}$ we have

$$[a_{3,ij}] = \tfrac{1}{7!}\big(\tfrac{35}{9}K_1 - \tfrac{14}{3}K_2 + \tfrac{14}{3}K_3 + \tfrac{8}{9}K_4 + \tfrac{8}{3}K_5 - \tfrac{16}{3}K_6 + \tfrac{44}{9}K_7 - \tfrac{80}{9}K_8 + 11K_9$$
$$- 8K_{10} + 12K_{11} + 12K_{12} - 2K_{13} - 4K_{14} + 9K_{15} + \tfrac{17}{2}K_{16} + 18K_{17}\big)\delta_{ij}$$
$$- \tfrac{1}{30}\big(\tfrac{1}{6}(R^{\mu\nu\rho\sigma}R_{\mu\nu\rho\sigma} - R^{\mu\nu}R_{\mu\nu} + \tfrac{5}{2}R^2)$$
$$+ \nabla^2 R + \partial^\mu R\,\partial_\mu + \tfrac{5}{6}R\nabla^2 + \tfrac{1}{3}R^{\mu\nu}\nabla_\mu\partial_\nu + \tfrac{1}{2}(\nabla^2)^2\big)X_{ij}$$
$$+ \tfrac{1}{12}\big(RX_{ik}X_{kj} + X_{ik}\nabla^2 X_{kj} + \nabla^2 X_{ik}X_{kj} + \partial^\mu X_{ik}\partial_\mu X_{kj}\big)$$
$$- \tfrac{1}{6}X_{ik}X_{kl}X_{lj}\,.$$

$$(\text{B.12})$$

It should be noted that all derivatives are evaluated inside the coincident limits, i.e. the coincident limits of derivatives on the quantities are to be taken, not the derivatives on the coincident limits of said quantities. Actually, it is not difficult to convert between the two—the relevant equation was given by Christensen [36] and was, in fact, necessary for the computation of (4.9).

Appendix C. Coincident limits and divergences of products of propagators

In this appendix we give the ϵ poles of the products of the propagator pieces $G_n(x,x')$ and $R_n(x,x')$ found in (2.31). As noted there, the full propagator in (2.15) is regular for any d at separate spacetime points. Explicitly,

$$G_0(x,x') = \frac{\Gamma\left(\tfrac{1}{2}d - 1\right)}{4\pi^{d/2}}\frac{\Delta_{\text{VM}}^{1/2}}{(2\sigma)^{d/2-1}}\,,$$

$$G_1(x,x') = \frac{\Gamma\left(\tfrac{1}{2}d - 2\right)}{16\pi^{d/2}}\frac{\Delta_{\text{VM}}^{1/2}}{(2\sigma)^{d/2-2}}\,,$$

$$R_n(x,x') = \frac{\Delta_{\text{VM}}^{1/2}}{4^{n+1}}\left(\frac{1}{\pi^{d/2}}\Gamma\left(\tfrac{1}{2}d - 1 - n\right)(2\sigma)^{n+1-d/2} - \frac{2}{\epsilon}\frac{(-1)^{n-1}\mu^{-\epsilon}}{\pi^3(n-2)!}(2\sigma)^{n-2}\right)\,,$$

$$(\text{C.1})$$

for $n = 2, 3$.

However, upon taking products with other Green's functions, as in (4.8), short-distance singularities will arise that will have poles in ϵ, in accordance with (C.1). The associated relations centrally depend on the coincident limit of inverse powers of σ in non-integer dimensions,

$$\frac{1}{\left(2\sigma(x,x')\right)^{\frac{1}{2}(d-\delta)}} \sim \frac{\mu^{-\delta}}{\delta} \frac{2\pi^{d/2}}{\Gamma\left(\frac{1}{2}d\right)} \delta^d(x,x'), \tag{C.2}$$

which is valid up to finite contributions as $x' \to x$. Here $\delta \propto \epsilon$ and the $\mu^{-\delta}$ factor is inserted to preserve dimensions. This dependence can be seen from the σ dependence in equations in (C.1). Varying powers of σ will arise depending on the product of propagators taken. A useful recursion relation can be obtained by differentiation and the use of (B.1) and (B.2). It reads

$$\left(\nabla^2 - Y\right) \frac{\Delta_{\text{VM}}^{1/2}}{\sigma^p} = p(2p + 2 - d) \frac{\Delta_{\text{VM}}^{1/2}}{\sigma^{p+1}}. \tag{C.3}$$

Equation (C.3) can be used to obtain the poles in products of propagators. For example, if we multiply (C.2) with $\Delta_{\text{VM}}^{1/2}$, act with $\nabla^2 - Y$ and use (C.3), we obtain

$$\frac{\Delta_{\text{VM}}^{1/2}}{(2\sigma)^{\frac{1}{2}(d-\delta)+1}} \sim \frac{\mu^{-\delta}}{\delta} \frac{\pi^{d/2}}{d\Gamma(\frac{1}{2}d)} (\nabla^2 - Y)\delta^d, \tag{C.4}$$

which is necessary for determining the ϵ poles in $(G_0 G_1 G_1)(x, x')$ in $d = 6 - \epsilon$.

Using these methods we can now list the relevant products as in [19]:

$$\begin{aligned}
(G_0 G_0)(x, x') &\sim \frac{\mu^{-\epsilon}}{64\pi^3} \frac{1}{\epsilon} \frac{1}{3} \nabla^2 \delta^d(x, x'), \\
(G_0 G_1)(x, x') &\sim \frac{\mu^{-\epsilon}}{64\pi^3} \frac{1}{\epsilon} \delta^d(x, x'), \\
(G_0 G_1 G_1)(x, x') &\sim \frac{\mu^{-2\epsilon}}{(64\pi^3)^2} \frac{1}{\epsilon} \frac{1}{6} \left(\nabla^2 + \tfrac{1}{6}R\right) \delta^d(x, x'), \\
(G_1 G_1 G_1)(x, x') &\sim \frac{\mu^{-2\epsilon}}{(64\pi^3)^2} \frac{1}{\epsilon} \frac{1}{2} \delta^d(x, x'), \\
(G_0 G_1 R_2)(x, x') &\sim \frac{\mu^{-2\epsilon}}{(64\pi^3)^2} \left(\frac{1}{\epsilon^2} + \frac{1}{4}\frac{1}{\epsilon}\right) \delta^d(x, x'), \\
(G_0 G_0 R_3)(x, x') &\sim -\frac{\mu^{-2\epsilon}}{(64\pi^3)^2} \left(\frac{1}{\epsilon^2} + \frac{1}{4}\frac{1}{\epsilon}\right) \delta^d(x, x'), \\
(G_0 G_0 R_2)(x, x') &\sim \frac{\mu^{-2\epsilon}}{(64\pi^3)^2} \left(\frac{1}{\epsilon^2} - \frac{1}{12}\frac{1}{\epsilon}\right) \frac{1}{3} \left(\nabla^2 + \tfrac{1}{6}R\right) \delta^d(x, x'),
\end{aligned} \tag{C.5}$$

where $\sim$ indicates that only the ϵ poles are considered on the right side. In these expressions powers of $\Delta_{\text{VM}}^{1/2}$ that appear in the propagator products are commuted through to the delta function and then we use $\Delta_{\text{VM}}^{1/2}(x, x') \delta^d(x, x') = \delta^d(x, x')$.

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For our purposes we also need to know the divergent behavior of the products $(G_0G_0G_1)(x, x')$ and $(G_0G_0G_0)(x, x')$. Their computation is performed after taking advantage of the fact that they appear under x, x'-integrals, and thus we can integrate by parts at will. For example, for $(G_0G_0G_1)(x, x')$ we need to find the poles in $\left(\Delta_{\text{VM}}^{1/2}\right)^3 / (2\sigma)^{5-3\epsilon/2}$. From (2.31) and (C.4), (C.3) we see that we need

$$\Delta_{\text{VM}}(\nabla^2 - Y)^2 \delta^d = \Delta_{\text{VM}} \left[(\nabla^2)^2 \delta^d - \nabla^2(Y\delta^d) - Y\nabla^2\delta^d + Y^2\delta^d \right]. \tag{C.6}$$

For the contribution of $(G_0G_0G_1)(x, x')$ to (4.8) we also have $g_{ikl}(x)g_{jkl}(x')a_{1,ij}(x, x')$ under the x, x'-integrals. Following the procedure that led to (C.5) we would now commute Δ_{VM} through the derivatives to the delta function. This is rather tedious, so we choose here to integrate the derivatives in (C.6) by parts instead. This way we are be able to do the x'-integral which will force the coincident limits of the various contributions that arise. The necessary results are then found in (B.5), (B.6) and (B.10).

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